The Riemann hypothesis and GRH for Dirichlet L-functions

A readable account of the Lean proof in Levent Alpöge's archive

rh-lean-roadmap

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Contents

Introduction 4

I The five statements and the window criterion 7

1 Five statements and the critical line 8

2 How arithmetic windows enter the zero-free criterion 19

3 Zeros become poles 23

4 A Gaussian window excludes poles 28

II The arithmetic side: prime sums and auxiliary primes 32

5 Counting primes class by class 33

6 Reading off the target sum 38

7 Choosing the auxiliary primes q and d 43

8 Building the arithmetic host of a window 51

III Divisibility from one theta lift 60

9 Background: the Weil representation and theta lifts 61

10 The local kernel and the differential operator 66

11 Entry estimates: factorials against denominators 73

12 One lift on a lattice and the Hecke commutation 81

13 Induction along the chain 89

14 Global Frobenius-orbit parameters 93

IV Divisibility along the chain and division of lattices 102

15 What each layer records 103

16 From local lifts to bounds on whole columns 110

17 From columns to periods 120

18 Gluing row lattices across Hecke characters 130

19 The pairing after division 133

20 Slope change and the data for step 5 136

V Two bounds on one slope 144

21 The lattice after division 145

22 Prime sums and net gain 161

23 A slope bound for families of maps 178

24 The pointwise bound and its eight error terms 184

25 Four terms and the square-root bound 190

VI Input two: the window bound VWin 197

26 The window bound and its inputs 198

27 Three length bounds 217

28 The cyclotomic tower and a common window 220

29 Summing the plus component by classes 226

30 Automorphic counts and deformation rigidity 235

31 The exponent 401 and the downstream window 263

VII Input one: Green 264

32 Green’s classification and its use 265

VIII Input three: the witness for HypsAlt 276

33 A common witness for the 46 conditions 277

34 Theta correspondence, test vectors and evaluation 337

35 Covolume bounds 374

36 The definite side and local errors 420

37 Primes in short intervals, masses and counts 464

38 The chain length 477

IX Galois representations and local Langlands in the archive **479**

39 Galois representations attached to polarized cuspidal representations **480**

40 A weak local Langlands correspondence **497**

X Appendix: proofs **500**

41 Laplace derivatives exclude poles **1489**

Index of Lean declarations **1495**

Bibliography **1606**

Introduction

On 20 November 2026 Levent Alpöge announced a proof, found with Claude, of the Riemann hypothesis and of the generalized Riemann hypothesis for Dirichlet L-functions. The proof is a Lean 4 archive of 175,187 files and 17.4 million lines, without comments, checked by several independent kernels [1]; on 21 November 2026 the Lean FRO and the Mathlib Initiative certified in a joint statement that the statement and the proof of the Riemann hypothesis are correct, the statement being the one in Mathlib [2]. This book gives the argument of the archive as mathematics: every statement the proof needs, in the order in which the proof uses it, with a proof written for a reader and the location of the Lean declaration that proves it.

The theorem

The archive proves, for Mathlib's definitions of ζ and of $L(s,\chi)$:

- if $\zeta(s)=0$, $s\neq 1$ and s is not a negative even integer, then $\operatorname{Re} s=\frac{1}{2}$;
- for every Dirichlet character χ of any modulus, $L(s,\chi)\neq 0$ when $\operatorname{Re} s>\frac{1}{2}$ (for the principal character, $s=1$ is excluded, where Mathlib assigns a value to the pole);
- every zero of $L(s,\chi)$ with $0<\operatorname{Re} s<1$ lies on $\operatorname{Re} s=\frac{1}{2}$, for all χ and for primitive χ.

All five statements follow from one theorem: no Dirichlet L-function vanishes to the right of the critical line.

The argument in six steps

1. *The critical line* (Part I). For primitive χ the functional equation reflects zeros in $0<\operatorname{Re} s<\frac{1}{2}$ to $\frac{1}{2}<\operatorname{Re} s<1$, and imprimitive L-functions differ from primitive ones by Euler factors without zeros in $\operatorname{Re} s>0$. It suffices to show $L(s,\chi)\neq 0$ for $\operatorname{Re} s>\frac{1}{2}$ and primitive χ.
2. *A window criterion* (Part I). Fix a primitive character ψ of order o and let $\mathcal{X}$ be the o primitive characters induced by its powers. A Landau-type argument shows: if a weighted, centred prime sum $G_m(y)$ satisfies $G_m(y)\leq y^{\theta}$ on whole intervals $[Y,Y^{3/2}]$ for arbitrarily large Y, then no $L(s,\chi)$, $\chi\in\mathcal{X}$, vanishes for $\operatorname{Re} s>\theta$. Only a one-sided bound is needed, and only on a sequence of windows.
3. *The prime sum over Frobenius classes* (Part II). For each point x of a window the prime sum is distributed over the Frobenius classes of a finite abelian Galois group. Orthogonality of characters turns the count in each class into a sum over characters; averaging over a cyclic subgroup leaves the target family $\mathcal{X}$ together with three families of companion characters twisted by the quadratic characters of two auxiliary primes q,d. The auxiliary primes are chosen for each window by a mean-square estimate over candidate primes and a zero-density estimate, so that every twisted companion obeys a square-root bound on the whole window.
4. *Multiplicity from theta lifts* (Parts III and IV). The proof follows Stepanov's method for curves over finite fields, in which an auxiliary function vanishes to high order at many points. Over the integers the archive builds the multiplicity from a chain of about $\sqrt{x}$ theta lifts between definite unitary groups. Each lift applies a differential operator, given by the weights, to a local theta kernel; the factorials it produces, minus what the denominators of a formal-group expansion can cancel, give a computable lower bound for the divisibility of the periods by each prime, valid on a whole lattice. The input lattice

is then divided by these prime powers, and the pairing stays integral against a global lattice glued across Hecke characters.

5. *Two bounds on one slope* (Part V). Division raises the average slope of the lattices by a sum of logarithms of primes, which is the prime sum of step 3 up to a small loss. The size of the period maps at the complex places, through a slope inequality for the whole family of maps, bounds the same slope from above. This is Stepanov's inequality "multiplicity times number of points is at most the degree". The main terms on the arithmetic and geometric sides are the same number, and the chain length $\sqrt{x}$ puts the error at square-root size.
6. *Extra divisibility* (Part VI). Divisibility of special values that does not come from the chain is bounded by an estimate for the total length of Tate–Shafarevich-type groups over a family of CM subfields of cyclotomic fields.

The three inputs

In Lean, steps 3 to 6 start from three statements, proved elsewhere in the archive:

- Green: the classification of the cuspidal characters of $\mathrm{GL}_d(\mathbf{F}_p)$ by irreducible polynomials (Part VII);
- VWin: the window bound for Tate–Shafarevich-type groups (Part VI);
- HypsAlt: a structure of 46 fields asserting that one family of data satisfies all the conditions of the theta chain at once (Part VIII).

The proofs of VWin and HypsAlt use Galois representations attached to polarized cuspidal automorphic representations and a local Langlands correspondence; Part IX describes the forms in which the archive proves them.

How the book is written

- The arguments made for this proof are given with proofs. Results from the literature (prime number theorems in progressions, zero-density estimates, Siegel's mass formula, Galois representations, local Langlands, Kato's Euler system, foundations of étale cohomology) are stated in the form the archive uses, with references, and a sentence records how the archive's version compares with the published one.
- Statements are numbered by node identifiers such as 10.19, which group them by part. The Index of Lean declarations at the end of the appendix lists, for each statement by its number, the archive declarations that prove it, with their modules and line numbers. The archive often contains several versions of a result; the one cited is the version on the path of the final proof. The archive version is the one with SHA-256 `9631b4f0…c7fc`.
- Lean engineering (coercions, bookkeeping, routine inequalities) is absorbed into the proofs.
- The main text keeps every statement and the proofs of moderate length. Longer proofs are given in the appendix; in the main text an outline of each says how the proof goes and which results it uses.
- Chapter 4 proves the window criterion, Theorem 3.2, with a Gaussian kernel; this proof is shorter than the archive's and is not formalised. The archive's proof is given in Chapter 41 of the appendix. This is the only place where the main text departs from the route of the formal proof.
- Chapter 9 collects the standard facts on the Weil representation and theta lifts that Parts III, IV and VIII use.

Reading order

Part I and the first chapters of Parts II and V give the frame of the argument. Parts III, IV and V contain the new mechanism: the local divisibility of one theta lift, its accumulation along the chain and the slope

inequality that turns it into a bound for prime sums. Parts VI to IX treat the inputs.

About this text

The text was written by several AI models, including Claude, working from the Lean source. An independent author, who is not a specialist in this area, set the goals and the editorial rules (plain mathematical English, every statement tied to the Lean declaration that proves it) and organised the review: models working in isolation drafted and revised the chapters, and other models read anonymised copies of the whole book as referees over repeated rounds; every point they raised was settled against the Lean source. No human expert has reviewed the text. Where the text and the Lean archive differ, the archive is right; the proofs of 65 results, 64 in Part VIII and one in Part IX, are given in outline, each ending with the Lean declaration that proves it. The aim is a readable account for mathematicians; shorter versions of the same argument, down to one for readers without mathematics, are listed in the repository. Alpöge works at Anthropic; this text is independent of him and of Anthropic. The mathematics is Alpöge's archive; the exposition is ours.

Part I

The five statements and the window criterion

Chapter 1

Five statements and the critical line

Let $\mathbf{1}_N$ be the principal character modulo $N \geq 1$, and let χ^* be the primitive character inducing χ. Write $L(s,\chi)$ for the continued Dirichlet L-function. For modulus one it is $\zeta(s)$. With $G(s) = \pi^{-s/2}\Gamma(s/2)$, define the value at 1 by

$$L(1,\mathbf{1}_1) = \zeta(1) = \lim_{s\to 1}\left(\zeta(s) - \frac{1}{G(s)(s-1)}\right) = \frac{\gamma - \log(4\pi)}{2} < 0. \tag{1.1}$$

Here γ is Euler's constant. The constant term of ζ at 1 is γ, and $G(1) = 1$, $G'(1) = -(\gamma + \log(4\pi))/2$. Thus the subtraction in the display adds $G'(1)$ to the constant term. The sign follows from $\gamma < 1 < \log(4\pi)$ [5, §§5.4(ii), 25.2(ii)].

1.1 The five statements

Theorem 1.1 (Riemann hypothesis)**.** *Every zero s of ζ with $s \neq 1$ and $s \notin \{-2,-4,-6,\dots\}$ satisfies* $\operatorname{Re} s = 1/2$.

Proof. Apply Lemma 1.6 to Theorem 1.9. □

Theorem 1.2 (Explicit zeta statement)**.** *For every $s \in \mathbf{C}$,*

$$\zeta(s) = 0, \qquad \neg\exists n \in \mathbf{N} \;\; s = -2(n+1), \qquad s \neq 1 \quad\Longrightarrow\quad \operatorname{Re} s = \tfrac{1}{2}.$$

Here $0 \in \mathbf{N}$.

Proof. The integers $-2(n+1)$ for $n \in \mathbf{N}$ are the negative even integers excluded in Theorem 1.1. □

Theorem 1.3 (Nonvanishing for every character)**.** *Let χ be a complex Dirichlet character modulo $N \geq 1$ and let $s \in \mathbf{C}$. Then*

$$\operatorname{Re} s > \tfrac{1}{2}, \qquad (\chi \neq \mathbf{1}_N \text{ or } s \neq 1) \quad\Longrightarrow\quad L(s,\chi) \neq 0.$$

Proof. Apply Theorem 1.9. □

Theorem 1.4 (Zeros in the strip for every character)**.** *Let χ be a complex Dirichlet character modulo $N \geq 1$. Every zero s of $L(\cdot,\chi)$ with $0 < \operatorname{Re} s < 1$ satisfies* $\operatorname{Re} s = 1/2$.

Proof. Apply Lemma 1.7 to Theorem 1.9. □

Theorem 1.5 (Zeros in the strip for primitive characters)**.** *Let χ be primitive modulo $N \geq 1$. Every zero s of $L(\cdot,\chi)$ with $0 < \operatorname{Re} s < 1$ satisfies* $\operatorname{Re} s = 1/2$.

Proof. Apply Lemma 1.8 to Theorem 1.9. □

1.2 Reduction to a right half-plane

Write H for the assertion of Theorem 1.3.

Lemma 1.6 (The character of modulus one)**.** *The assertion H implies the Riemann hypothesis.*

Proof. Apply H to $L(s, \mathbf{1}_1) = \zeta(s)$ for $\operatorname{Re} s > 1/2$ and $s \neq 1$, then use Lemma 1.11. □

Lemma 1.7 (Reflection for an arbitrary character)**.** *Assume H. For every character χ modulo $N \geq 1$, a zero s of $L(\cdot, \chi)$ with $0 < \operatorname{Re} s < 1$ satisfies $\operatorname{Re} s = 1/2$.*

Proof. Lemma 1.12 gives $L(s, \chi^*) = 0$. Apply Lemma 1.8 to χ^*. □

Lemma 1.8 (Reflection for a primitive character)**.** *Assume H. For every primitive character χ modulo $N \geq 1$, a zero s of $L(\cdot, \chi)$ with $0 < \operatorname{Re} s < 1$ satisfies $\operatorname{Re} s = 1/2$.*

Proof. In the strip, H excludes $\operatorname{Re} s > 1/2$. If $\operatorname{Re} s < 1/2$, Lemma 1.13 gives a zero at $1 - s$ for χ^{-1}, also excluded by H. □

Theorem 1.9 (Common input to the five statements)**.** *The assertion H holds.*

Proof. Apply Theorem 1.23. □

Definition 1.10 (Two nonvanishing assertions)**.** Let $\mathcal{P}$ consist of pairs (N, ψ) with $N \geq 1$ and ψ primitive modulo N. Say that ψ is zero-free above $\theta \in \mathbf{R}$ if $L(s, \psi) \neq 0$ whenever $\operatorname{Re} s > \theta$. Write H_{prim} for the assertion that every $(N, \psi) \in \mathcal{P}$ is zero-free above $1/2$, including the value at 1.

Lemma 1.11 (Zeta reflection from the right)**.** *Suppose that $\zeta(s) \neq 0$ for $\operatorname{Re} s > 1/2$ and $s \neq 1$. Then the Riemann hypothesis holds.*

Proof. Suppose $\zeta(s) = 0$ and $\operatorname{Re} s < 1/2$. Put $t = 1 - s$. Since $\zeta(0) = -1/2$, one has $t \neq 1$ [5, §25.6]. The reflection formula at $\operatorname{Re} t > 1/2$ gives

$$\zeta(1 - t) = 2(2\pi)^{-t}\Gamma(t)\cos(\pi t/2)\zeta(t)$$

[4, §12.8, (12)][5, (25.4.1), §5.2(i)]. By the hypothesis and gamma nonvanishing, $\cos(\pi t/2) = 0$. Thus t is an odd integer greater than 1, so s is negative and even. □

Lemma 1.12 (Euler factors and the conductor)**.** *Let χ be a character modulo $N \geq 1$ with conductor f. If $\chi^* \neq \mathbf{1}_f$ or $s \neq 1$, then*

$$L(s, \chi) = L(s, \chi^*) \prod_{p \mid N} (1 - \chi^*(p)p^{-s}).$$

For any character φ and prime p, one has $|\varphi(p)p^{-s}| \leq p^{-\operatorname{Re} s} < 1$ if $\operatorname{Re} s > 0$. In particular $L(s, \chi) = 0$ in $0 < \operatorname{Re} s < 1$ implies $L(s, \chi^*) = 0$.

Proof. Compare Euler products on $\operatorname{Re} s > 1$ and continue on the stated domain [4, §12.10][5, (25.15.4)]. The factors at $p \mid f$ equal 1. Since $\varphi(p)$ is zero or a root of unity, the bound follows from $|p^{-s}| = p^{-\operatorname{Re} s}$. □

Lemma 1.13 (Reflection of primitive zeros)**.** *Let ψ be primitive modulo $N \geq 1$. Put $a_\psi = 0$ for even ψ and $a_\psi = 1$ for odd ψ. Write $\Gamma_{\mathbf{R}}(z) = \pi^{-z/2}\Gamma(z/2)$, with its meromorphic continuation. In the strip $0 < \operatorname{Re} s < 1$ put $\mathcal{L}_\psi(s) = \Gamma_{\mathbf{R}}(s + a_\psi)L(s, \psi)$. Then $L(s, \psi) = 0$ implies $L(1 - s, \psi^{-1}) = 0$.*

Proof. In this normalization the functional equation is

$$\mathcal{L}_{\psi^{-1}}(1 - s) = N^{s-1/2}W(\psi^{-1})\mathcal{L}_\psi(s), \qquad W(\psi) = i^{-a_\psi}N^{-1/2}\sum_{u \bmod N} \psi(u)e^{2\pi iu/N}.$$

For $N > 1$ use [4, §12.10, (16)]. For $N = 1$ use the completed zeta equation, with $W(\mathbf{1}_1) = 1$ [5, (25.4.3)–(25.4.4)]. The two gamma arguments have positive real part, so their values are finite and nonzero [5, §5.2(i)]. □

Lemma 1.14 (Passage to all characters)**.** *The assertion H_{prim} implies H.*

Proof. Fix N, χ, s as in H. If χ^* is principal, then χ is principal and $s \neq 1$. Lemma 1.12 expresses $L(s,\chi)$ as the product of the nonzero primitive value and nonzero Euler factors. □

Definition 1.15 (The five toric assertions)**.** Let E be CM, $F = E^+$ and $(\phi_i)_{i\in I}$ a CM type. Write D_F, D_E for the field discriminants, $f = |I| = [F:\mathbf{Q}]$, $k = 2s$ for an integer $s \geq 1$, and $n = kf$. For integers $\kappa_i \geq 0$ put $K_\kappa = \sum_i \kappa_i$ and $M = n + 2K_\kappa$. Write Nm for the absolute ideal norm. For a number field L, write Art_L for idèlic reciprocity normalized to send a uniformizer to geometric Frobenius modulo inertia. Arithmetic Frobenius acts on the residue field by $z \mapsto z^{Nv}$ at a prime v, and geometric Frobenius is its inverse. For a finite idèle t abbreviate $\mathrm{Art}_L((1,t))$ to $\mathrm{Art}_L(t)$.

A base conductor $\mathfrak{f}$ is a proper nonzero integral ideal of F. Each prime divisor splits in E/F, has exponent at most two and lies over an odd rational prime unramified in E with residue degree one at every place of E above it. For a finite set Q of split primes of F coprime to $\mathfrak{f}$, put $\mathfrak{f}_Q = \mathfrak{f}\prod_{\mathfrak{p}\in Q}\mathfrak{p}$ and $O_Q = \mathcal{O}_F + \mathfrak{f}_Q\mathcal{O}_E$. The characters χ are unitary, trivial on the idèles of F, with infinity component $\prod_i(\phi_i/\bar{\phi}_i)^{s+\kappa_i}$. At base level their conductor exponents above $\mathfrak{p}$ are those of $\mathfrak{f}$. At auxiliary level they may also have exponent at most one above Q.

Let $\mathbb{A}_{E,f}$ be the finite adeles and $\widehat{O}_Q$ the finite completion of O_Q. Put

$$\mathrm{Pic}(O_Q) = E^\times\backslash\mathbb{A}_{E,f}^\times/\widehat{O}_Q^\times, \qquad Y_y = E \cap y\widehat{O}_Q.$$

For a finite idèle y, let $|y|_{E,f}$ be the product of its normalized local absolute values. Define S_y by continuation to $\sigma = 0$ of

$$\sum_{t\in(Y_y\backslash 0)/\mathcal{O}_F^\times}\prod_i \bar{\phi}_i(t)^{\kappa_i}\phi_i(t)^{-k-\kappa_i}|\phi_i(t)|^{-2\sigma}.$$

The ordinary class term is $S_y/(\chi_f(y)|y|_{E,f}^{\mathrm{s}})$, where χ_f is the finite part of χ. The ordinary sum adds these terms, and the twisted sum multiplies the term at $[y]$ by $\psi([y])^{-1}$ for a character ψ of $\mathrm{Pic}(O_Q)$. For the positive integer d specified in each of items 1–3, put

$$C_0 = |D_F|^{(k-1)/2}\prod_i \Gamma(k+\kappa_i)(2\pi/\sqrt{d})^{K_\kappa}, \qquad x_\Omega(T) = \frac{C_0 T}{\Omega^M}, \qquad \beta_\Lambda(T) = x_\Omega(T)^{12} \quad (\Omega^{12} = \mathrm{disc}\,\Lambda).$$

Here $\Omega \neq 0$ and $\mathrm{disc}\,\Lambda$ is the Weierstrass discriminant of the lattice Λ. For an ordinary ideal class C, let $\zeta_F(z,C)$ be the continuation of the sum of $(\mathrm{Nm}\,\mathfrak{a})^{-z}$ over integral ideals in C. When all $z(C) = \zeta_F(1-k,C)$ are rational, put $e_\ell = \max_C \max(0, -v_\ell(z(C)/2^f))$, with zero contribution from a zero value and $v_\ell(\ell) = 1$.

The assertion $\mathscr{J}$ is the conjunction of the following statements.

1. Suppose $E/\mathbf{Q}$ is abelian, E/F is unramified at finite places, $2 \mid D_E$, $f \geq 2$, and $d > 4$ is prime. Suppose $w^2 = -d$ in E, $\phi_i(w) = i_{\mathbf{C}}\sqrt{d}$ for all i, and $\prod_i \phi_i(\xi) = 1$ for each root of unity ξ of E. Here $i_{\mathbf{C}}$ has positive imaginary part. For each $s \geq 1$, the partial zeta values $z(C) = \zeta_F(1-k,C)$ are rational for all classes C. For each κ, base conductor and base character use $\Lambda = \mathcal{O}_{\mathbf{Q}(\sqrt{-d})} \subset \mathbf{C}$. At a rational prime ℓ, let a_ℓ be the maximum base conductor exponent above ℓ. Put $c_\ell = a_\ell$ if $a_\ell > 0$ and $c_\ell = v_\ell(|D_E|)/(2f)$ otherwise. For each auxiliary set away from ℓ, every class term and every Picard-twisted sum T has algebraic $\beta_\Lambda(T)$, and at every embedding above ℓ,

$$v_\ell(\beta_\Lambda(T)) \geq -12b_\ell,$$
$$b_\ell = e_\ell + Ma_\ell + \begin{cases} 0, & c_\ell = 0, \\ nc_\ell + K_\kappa(2c_\ell + 1 + \mathbf{1}_{\ell=2} + \frac{1}{2}\mathbf{1}_{\ell=d}), & c_\ell > 0. \end{cases} \tag{1.2}$$

   Equivalently $\ell^r\beta_\Lambda(T)^j$ is integral above ℓ whenever $j \geq 1$, $r \in \mathbf{Z}$ and $r/j \geq 12b_\ell$.

2. Let K be imaginary quadratic of discriminant $-d < -4$, with a fixed embedding and integer lattice Λ_K. For each automorphism g of $\mathbf{C}$ fixing K, choose a finite idèle t_g, its inverse fractional ideal A_g, and the lattice Λ_g for that ideal, so that $j(\Lambda_g) = gj(\Lambda_K)$. The classes of A_g index the conjugate j-values bijectively. In every finite Galois $L/\mathbf{Q}$ containing K and $j(\Lambda_K)$ each class occurs $[L:\mathbf{Q}]/(2h_K)$ times

among the embeddings over K, where h_K is the class number. The choices work for every CM $E \supset K$ Galois over $\mathbf{Q}$ with trivial relative different over F, each type of size at least two extending the fixed embedding, all $s, \kappa_i \geq 1$, and all admissible conductors and ordinary sums at auxiliary level. For each such sum T, $\beta_{\Lambda_K}(T)$ is algebraic. Take any conjugate character χ' with the permuted infinity weights and the g-conjugate finite part, and any ordinary sum T' for it at the same auxiliary level. With $A = \operatorname{Nm} A_g$ and $l_\chi = \chi'_f(t_g)|t_g|^s_{E,f}$, using the image of t_g in the idèles of E, one has

$$\begin{aligned} g\beta_{\Lambda_K}(T) &= \frac{(C_0 A^{-K_\kappa} l_\chi^{-1} T')^{12}}{(\operatorname{disc} \Lambda_g)^M}, \\ |g\beta_{\Lambda_K}(T)| &= \frac{(C_0|T'|)^{12}}{(A^6|\operatorname{disc}\Lambda_g|)^M}. \end{aligned} \tag{1.3}$$

For two such sums with the same E, CM type, s, tuple κ, and base conductor $\mathfrak{f}$, and with $T_2 \neq 0$, the ratio is algebraic. For any corresponding conjugate sums, $T'_2 \neq 0$ and $g(T_1/T_2) = (l_{\chi_1}^{-1}T'_1)/(l_{\chi_2}^{-1}T'_2)$.

3. Let K be quadratic with $\delta^2 = -d$, $d > 0$, and a fixed complex embedding. There is $\Omega_0 \neq 0$ chosen before the CM extension, type or twist. For every such E/K unramified over F at finite places, type extending the fixed embedding, $s, \kappa_i \geq 1$, and Picard twist ψ of a base character at $\mathfrak{f}_Q$, the twisted sum T satisfies $x_{\Omega_0}(T) \in \overline{\mathbf{Q}}$. Fix a compatible embedding of an algebraic closure of K and an automorphism g of $\mathbf{C}$ fixing the embedded K. Take any finite idèle t satisfying $\operatorname{Art}_K(t)^{-1} = g|_{K^{\mathrm{ab}}}$. There is $\Omega_1 \neq 0$ common to all these sums. With $\psi' = g\psi$, T' the conjugate sum, $l_\chi = \chi'_f(t)|t|^s_{E,f}$, and $[t]$ the class of the image of t in $\operatorname{Pic}(O_Q)$,

$$g x_{\Omega_0}(T) = x_{\Omega_1}\big(|t|_{K,f}^{-K_\kappa}[l_\chi \psi'([t])]^{-1} T'\big). \tag{1.4}$$

4. For every CM E and $s \geq 1$, the partial zeta values $z(C) = \zeta_F(1-k, C)$ are rational, and

$$\begin{aligned} e_\ell &\leq f v_\ell(2) + \min_{A \geq 2,\ \ell \nmid A} v_\ell(A^n - 1), \\ \min_{A \geq 2,\ \ell \nmid A} v_\ell(A^n - 1) &= \begin{cases} 2 + v_2(n), & \ell = 2, \\ 1 + v_\ell(n), & \ell \text{ odd},\ \ell - 1 \mid n, \\ 0, & \ell \text{ odd},\ \ell - 1 \nmid n. \end{cases} \end{aligned} \tag{1.5}$$

For $n \geq 16$ the support is finite and $\sum_\ell e_\ell \log \ell \leq n \log(n+1)$.

5. Use the square-recursive size data of 34.49, the host, CM-type and arithmetic data of 34.46, and the level-character data and grids of 34.47. In this item, E/F is unramified at every finite place, $K = \iota_D(K_D) \subset \mathbf{C}$ denotes the image of the coefficient field of the point datum D, and t indexes the level tuples of the admissible system. There is $C_{F,0}$ such that for each ε and $C_F \geq C_{F,0}$ there are C'_0, x_0 valid for all $C \geq C'_0$ and every recursive size datum of 34.49 at $x \geq x_0$. Take a level-character datum and grid of 34.47 over 34.46 with reference index μ_0 and parameters $(1, C_F, b)$, its Weil system and a full choice of tests. In this datum and grid (34.46–34.47), at each level index t and grid position (i, j), the grid character $\chi_{t,i,j} = \chi_{0,i,j}\xi_t$ differs from the reference character $\chi_{0,i,j}$ at μ_0 by a Hecke character twist ξ_t of E (with $\xi_{\mu_0} = 1$) satisfying four conditions: (a) ξ_t is trivial on $\mathbb{A}_F^\times$, has finite order (hence trivial infinity component), and is tame at every finite place of E; (b) at each finite place v of E where ξ_t ramifies and $\chi_{0,i,j}$ is unramified, the residue field has prime order $q_v \mid Q_n(t)$ with $v \in W_t$ of depth $d_v \geq 1$, and the set of such places is stable under conjugation of E/F; (c) at each finite place v of E where both ξ_t and $\chi_{0,i,j}$ ramify, the residue field has prime order q_v, v is unramified over q_v, and $\chi_{0,i,j}$ has conductor exponent 2; and (d) for h equal to half the sum of the grid ranks at (i, j) and $(i+1, j+1)$, the fitted gap $\kappa_\tau(h) > 0$ of Lemma 34.43 (and 34.47) gives common $(\bar\tau/\tau)$-infinity exponent $m_\tau = \kappa_\tau(h) + h > h \geq e_0$ (angular exponent $-2m_\tau$ on Ψ) at every CM-type embedding τ, where e_0 is the lower range parameter below. Impose the following coefficient-field condition. The field K contains all finite local component values of the reference row characters, of every level twist and of every layer character. At places where the respective character is unramified, it contains the uniformizer value of the auxiliary character η divided by $\sqrt{Nw}$ and the uniformizer values of the three auxiliary characters $\chi_\Psi, \delta, \delta'$ of the level system. Here Nw is the residue cardinality. The field also

contains all eighth roots of unity. For each level index t, let $W_t = W_0 \cup Q(t)$, where W_0 is the base set of finite places and $Q(t)$ is the set specified by the level at t. For $w \in W_t$, put $d_w = 1$ off W_0 and let d_w on W_0 be the larger of one and the grid depth at the place below w. Require K to contain all $(Nw)^{d_w}$th roots of unity for $w \in W_t$, and $\sqrt{Nw}$ for every w in any W_t or in the set of places of the test choice. On each supported term, the additive phases with nonzero test and translated factors belong to K. In every coset expansion of least depth, the top and row jet values of every supported summand belong to K at every prescribed class and for all jet indices. At each pair of grid positions, require the ratio of the Hecke value at t to its reference value at μ_0 to lie in K whenever the latter is nonzero, at every integer from 1 to $1 + k_0 + m(t_{\mathrm{ch}} + r + 7)$. Here k_0 is the starting rank, m the chain length, t_{ch} the rank increment parameter and $r = k_0 + m - 1$. At every pair admitting a bottom triple, require a choice of such a triple and two nonzero periods, one for each side. For every t, finite adelic target point and monomial test for which the corresponding integral converges absolutely, its value divided by the period of its side lies in K. At each pair of grid positions let e_0 be the lower range parameter and let L_t, L_0 be the Hecke functions of the character at t and the reference character. Put $Q_n(t) = \prod_{w \in W_t \setminus W_0} Nw$. For every integer $1 - e_0 \le s \le 0$ there is $u \in K_D$, a unit outside the primes dividing $Q_n(t)$, such that

$$L_t(s)L_0(1-s) = \iota_D(u)L_t(1-s)L_0(s). \tag{1.6}$$

1.3 The common arithmetic input

Let $\mathcal{G}$ be Green's assertion of Definition 32.6. For each prime p, rank $r \ge 1$ and compatible label datum, it parametrizes irreducible cuspidal characters of $\mathrm{GL}_r(\mathbf{F}_p)$ by monic irreducible polynomials of degree r with nonzero constant term and gives the degrees and elliptic values.

Let $\mathcal{V}$ be the window assertion of Definition 26.3 with exponent 401 for every primitive ψ. Given $0 < \varepsilon < 1/52$ and a page function a as in 26.2, there are $J \ge 1$ and X_0 such that for each $X \ge X_0$ and each admissible pair q, d at scale $X^{1/J}$ one can choose $Y \ge X^{1/J}$ with $Y^2 \le X$. At each $x \in [Y, Y^2]$, for each field E specified by $(\psi, a(x), q, d)$ and all choices of 26.1, the lengths $s_{\ell,e}(E)$ are finite for every prime ℓ and $1 \le e < t = \lfloor x^{401} \rfloor$, and

$$\sum_{\ell \in S} \log \ell \sum_{1 \le e < t} s_{\ell,e}(E) \le \exp(x^{\varepsilon}/2) \qquad (S \text{ a finite set of primes}).$$

Let $\mathcal{D}(D, O, c, \beta, C, C_H)$ be the definite-side condition of 36.1. Here D, O are the point datum and reference realization, c, β are the selected cut and bundle, and C, C_H are the bounds. For each linked pair of sides it quantifies over compatible numerical parameters and coordinate submodules with division identifications agreeing with β. For every word it requires a factored column, metric frames and a height table satisfying the terminal and layer inequalities on the inverse image of the input lattice under the division identification.

A family satisfies the local-error estimate if there are $C, x_0 \in \mathbf{R}$ such that, at every point with $x \ge x_0$ where the prescribed cohomology lengths are finite for every prime and every $1 \le e < t$, inequality (36.17) holds in every layer whenever its weighted length series is summable. Write $\mathcal{N}(\mathcal{C})$ if, for every sufficiently small $\varepsilon > 0$, one can choose the size thresholds of Definition 36.10 so that every family in $\mathcal{C}$ of parameter ε meeting those thresholds at C_H satisfies the local-error estimate. The thresholds precede the choice of the family and C_H. Let $\mathcal{H}$ consist of the common data of Definition 33.6, with $\mathcal{D}$ as the definite-side condition and $\mathcal{N}$ as the condition on family classes. An element specifies a class $\mathcal{A}$ of families and selects a compatible bundle at each eligible point of each member. All the conditions of that definition use this selection. Let $W_3(\mathcal{A})$ be the subclass with the twisting, size and design conditions at parameter 3 specified in Definition 33.6. For each eligible family F_m in this subclass and each eligible C_H, one constant $C \ge 0$ and one point threshold give both the indefinite bounds and $\mathcal{D}$ for the selected bundles. Write $x_1(F_m)$ for the lower endpoint of the family data and $I_{F_m}(x)$ for the set indexing its points at x. Beyond the threshold for the period estimates, they hold at every $x > x_1(F_m)$ for all $i \in I_{F_m}(x)$. The local-error condition is $\mathcal{N}(W_3(\mathcal{A}))$.

1.3.1 Evaluation at CM lattices

Fix a rational prime ℓ and an embedding $\lambda : \overline{\mathbf{Q}} \cap \mathbf{C} \longrightarrow \mathbf{C}_\ell$. Write $v = v_\ell \circ \lambda$, with $v(0) = +\infty$, and put $R = \{z \in \overline{\mathbf{Q}} \cap \mathbf{C} : v(z) \geq 0\}$. A valuation bound includes algebraicity. For τ in the upper half-plane set $p = 2\pi i_{\mathbf{C}}$, $q = \exp(p\tau)$, and $\Delta = q\prod_{j\geq 1}(1-q^j)^{24}$. Use $j = q^{-1} + 744 + \cdots$.

Lemma 1.16 (Evaluation of a finite family of modular forms)**.** *Let $N \geq 1$ and $w \geq 0$ be integers. Let $(H_i)_{i\in I}$ be a finite family of holomorphic functions on the upper half-plane permuted by the weight-w slash action of $\mathrm{SL}_2(\mathbf{Z})$. Suppose they have convergent expansions in nonnegative powers of $q^{1/N}$ and all coefficients have valuation at least $-r$ for a common $r \in \mathbf{R}$. If $v(j(\tau)) \geq 0$ and $\omega^{12} = \Delta(\tau)$, then $v(H_i(\tau)/\omega^w) \geq -r$ for every i.*

Outline. Scale at valuation $t > r$ and take elementary symmetric functions of twelfth powers. Divide by powers of Δ and cancel cusp principal parts with j to get a monic equation over the valuation ring, then let t decrease to r. The complete proof is on p. 501. □

For a lattice Λ write $D(\Lambda) = g_2(\Lambda)^3 - 27g_3(\Lambda)^2$ and let $\wp_\Lambda$ be its Weierstrass function. The quasi-period homomorphism of the Weierstrass zeta function extends to $u \mapsto s_2(\Lambda)u + b\bar{u}$ on $\mathbf{C}$. For $\Lambda_\tau = \mathbf{Z}\tau + \mathbf{Z}$,

$$D(\Lambda_\tau) = p^{12}\Delta(\tau), \qquad s_2(\Lambda_\tau) = G_2(\tau) - \pi/\operatorname{Im}\tau,$$

where G_2 has constant term $\pi^2/3$ [7, Ch. 2].

Lemma 1.17 (The elliptic correction at a CM point)**.** *Let $d > 4$ be prime and suppose $w_0 = i_{\mathbf{C}}\sqrt{d}$ carries Λ into itself. Let $\Omega^{12} = D(\Lambda)$ and take a nonzero $P \in \mathbf{C}/\Lambda$ killed by a positive integer prime to ℓ. Then $U = (s_2(\Lambda) + \wp_\Lambda(P))/\Omega^2$ satisfies*

$$v(U) \geq -\begin{cases} 1, & \ell = 2,\ d \equiv 3 \pmod 4, \\ 2, & \ell = 2,\ d \equiv 1 \pmod 4, \\ 1/2, & \ell = d, \\ 0, & \text{otherwise.} \end{cases}$$

Outline. Use torsion expansions and 1.16 for integrality of $12X(P)$ and $X(P) - X(P')$. Cancel poles in the isogeny sum and integrate over a period to relate s_2 to kernel values, then use w_0 and, where needed, $1 \pm w_0$ to bound $v(U)$. The complete proof is on p. 501. □

Lemma 1.18 (CM coordinates for all orders of differentiation)**.** *Assume the elliptic main theorem of complex multiplication, the Hilbert class field theorem for j, Siegel–Klingen rationality with Deligne–Ribet integrality, and the partial zeta functional equation. Use the field, type, square root w, integer s, partial zeta values and base conductor in the first assertion of Definition 1.15. Take Q away from ℓ and a finite idèle y equal to one above ℓ. Put $Y = Y_y$, $a = a_\ell$, $c = c_\ell$, $e = e_\ell$ and $\Lambda = \mathcal{O}_{\mathbf{Q}(\sqrt{-d})} \subset \mathbf{C}$. For $|z_i| < 1$ put $u_i(t,z) = \phi_i(t) + z_i\overline{\phi_i(t)}$ and let $g(z)$ be $c_k = ((k-1)!)^f |D_F|^{(k-1)/2}$ times the sum of $\prod_i u_i(t,z)^{-k}$ over $(Y \setminus 0)/\mathcal{O}_F^\times$. At $k = 2$ insert $\prod_i |u_i(t,z)|^{-2\sigma}$, sum for $\operatorname{Re}\sigma > 0$ and continue to $\sigma = 0$.*

For each $\Omega^{12} = D(\Lambda)$ there are nonzero $a_i, b_i \in \mathbf{C}$, chosen before the derivative orders, such that for every $\kappa \in \mathbf{N}^I$ and any ordering of the derivatives,

$$\begin{aligned} v(a_i/\Omega) \geq -a, \qquad v\left(\frac{2\pi}{a_i b_i\sqrt{d}}\right) \geq -c - \tfrac{1}{2}\mathbf{1}_{\ell=d}, \\ v\left(\partial^\kappa g(0)\prod_i \frac{(b_i/a_i)^{\kappa_i}}{a_i^k}\right) \geq -e - (n + K_\kappa)c. \end{aligned} \tag{1.7}$$

Outline. Clear lattice denominators and correct by torsion to form modular families, then use Poisson summation, the partial zeta functional equation and normal convergence. Apply 1.16 at the CM point and 1.17 to the change of direction, then rescale for the bounds. The complete proof is on p. 502. □

Lemma 1.19 (Toric bounds, transport and reflection)**.** *The elliptic main theorem of complex multiplication, the Hilbert class field theorem for j, Siegel–Klingen rationality with Deligne–Ribet integrality, the partial zeta functional equation and the Hecke–Tate functional equation give $\mathscr{J}$ of Definition 1.15.*

Outline. Uses Siegel–Klingen, Deligne–Ribet, and Lemma 1.18 for toric denominator bounds, and elliptic complex multiplication, Lemma 1.17, and the Hilbert class field for transport. Local epsilon-factor formulas and Hecke–Tate functional equations give reflection. The complete proof is on p. 504. □

Definition 1.20 (The unitary and local theta inputs)**.** Let P and F be the following products of propositions. Each entry quantifies over the indicated fields, ranks, levels, measures and representations. A witness chosen in an entry has the scope of that entry.

The following entries occur in both products.

1. Over a nonarchimedean local field, smooth irreducibles of GL are admissible, Jacquet modules and parabolic induction preserve finite length, and normalized induction commutes with contragredients. The normalized product of two smooth irreducible unitarizable representations is irreducible. With normalized induction as multiplication, the direct sum of Grothendieck groups is polynomial over $\mathbf{Z}$ on the classes of segments.

2. For a unitary group the automorphic forms at fixed open level, finite-dimensional continuous compact type and infinitesimal character form a finite-dimensional space. Smooth uniformly moderate cuspidal forms have rapid decay. Convolution by a smooth compactly supported test is trace class on the cuspidal space for bi-invariant Haar measure. On a closed real matrix group such a test is a finite sum of convolutions $g_i * h_i$, with g_i supported in a prescribed identity neighborhood and h_i in the support of the test. An irreducible unitary adelic representation of a nondegenerate CM Hermitian group has a restricted tensor factorization with unramified unit vectors. For the special unitary group of a nondegenerate CM Hermitian space of dimension at least two, the rational points times the subgroup at the specified places are dense when that subgroup is noncompact.

3. For $\mathrm{U}(p,q)$ the Harish–Chandra parameters cover the discrete series with their compact types. Each permissible parameter gives a discrete series in the abstract group and in the matrix realization of its signature. For a parameter λ of rank a, the compact-dual normalization gives formal degree $\prod_{1\le i<j\le a}|\lambda_i-\lambda_j|/(j-i)$. The Harish–Chandra module assertions include analyticity of compact-finite vectors. A $(\mathfrak{g},K)$ map from the compact-finite vectors of an irreducible unitary representation to a smooth moderate globalization extends uniquely to an equivariant map on smooth vectors, with closed image there. A nonzero map controls each source Sobolev seminorm by a continuous target seminorm as in 36.9. An irreducible strongly continuous unitary representation of a locally compact group with regular two-sided Haar measure and one nonzero square-integrable coefficient has square-integrable coefficients and satisfies Schur orthogonality with one positive formal degree.

4. The prescribed restrictions of the Hecke splitting characters, with a standard additive character, give the Weil action for a CM unitary pair. For ranks a, b with $a \ge 1$ and with B anisotropic or $a+\mathrm{Witt}(B) < b$, the absolute theta integral is bounded by a continuous Schwartz seminorm. The theta kernel is integrable, and for $2a < b$ its mean equals the absolutely convergent Siegel sum with rational lattice of covolume one. The Witt index is the dimension of a maximal totally isotropic subspace. The doubling input gives meromorphic continuation and finite pole order for flat sections, the basic integral identity for $\mathrm{Re}\, s > a/2$, and continuation of the cuspidal zeta integral. For a characteristic-zero nonarchimedean field with ring of integers $\mathcal{O}$ and residue cardinality q, normalize $K = \mathrm{GL}_a(\mathcal{O})$ to volume one. Use the spherical section induced from the trivial character, normalized to have value one on the maximal compact subgroup of the doubled group. For an unramified representation with Satake parameters α and normalized zonal coefficient, there is s_0 such that for every $\mathrm{Re}\, s > s_0$ the spherical doubling integral converges absolutely and equals

$$L(\alpha, s+1/2)L(\alpha^{-1}, s+1/2)\prod_{j=1}^{a}(1-q^{-2s-j}).$$

Here $L(\alpha, z) = \prod_i (1-\alpha_i q^{-z})^{-1}$.

The additional entries of P are as follows.

1. Regular algebraic cuspidal representations of positive rank over a totally complex field are cohomological. Over a CM field the two conjugate infinitesimal weights are reflections about one half-integral centre. The polarized Galois assertion is that of 39.2, with the weights of 39.4. For above-ℓ comparison in conjugate self-dual rank $a \geq 2$ the polarization is $\chi_{E/F}^{a}$, where $\chi_{E/F}$ is the quadratic character of the CM extension. Choose the representation, weights and jump multisets before the prolongation and correspondence of 40.2. For each correspondence R of Definition 40.2, apply the following inertia test to the parameter $y = (\rho, N) = R(\pi_w \otimes |\det|_w^{(1-a)/2})$ at $w \mid \ell$. Require $N = 0$ and $\det(X - \rho(\sigma)) = (X-1)^a$ for every σ in wild inertia P_w. Write I_w for inertia and q_w for the residue cardinality. Put $d_w = 2$ if ℓ splits completely in E and either $a = 1$ or $\ell > 16a^2$, and put $d_w = 1$ otherwise. For every lift $\gamma \in I_w$ of a topological generator of I_w/P_w, require $z^{q_w - 1} = 1$ for each root z of $\det(X - \rho(\gamma))$ of multiplicity at least d_w. Under this test, the period parameter in Definition 39.6 has rank a, zero monodromy and the same inertia characteristic polynomials as y. For E/E^+ unramified at every finite place and the standard definite unitary group of rank $n \geq 1$, definite isobaric base change gives one family of components for all places as in 26.22. The GL entries give Jacquet–Shalika rigidity and fixed-level, compact-type and infinitesimal-character finiteness. For a unitary cuspidal representation over a totally complex number field, at each unramified place w with Satake parameters, every such parameter α satisfies $q_w^{-1/2} < |\alpha| < q_w^{1/2}$, where q_w is the residue cardinality [13].

2. The finite general linear group inputs are 32.6 and the regular formula of 32.5. At a nonarchimedean place use the double-coset filtration of the Jacquet module of an induced representation and irreducibility of products on disjoint unramified cuspidal lines. First-congruence-fixed vectors fixed by the integral unipotent radical inject into the Jacquet module. A first-congruence-fixed vector in an irreducible implies first-congruence-fixed vectors in every member of its cuspidal support. In characteristic zero generic unitarizable irreducibles are complementary products, and a standard ordering of multisegment factors gives an irreducible cosocle.

3. For each number field L and algebraic closure there is an open surjective global Artin homomorphism Art_L as in Definition 1.15, with kernel the closure of $L^\times$ times the archimedean identity component, geometric Frobenius normalization at finite places, and local inertia and decomposition compatibility on embedded local units and uniformizers. For this map each algebraic Hecke character has a unique continuous ℓ-adic realization with its prescribed infinity type. Above ℓ the character comparison gives the rank-one Weil–Deligne parameter for each compatible local component and every admissible correspondence of 40.2 with the specified period data. Continuous de Rham representations are potentially semistable. Integral crystalline lattices have prismatic realizations and equivariant maps lift uniquely to maps of crystals. Ray class fields realize ray groups. A character of ideals prime to the conductor, trivial on the ray-principal ideals, gives the unique open-kernel Galois character with the prescribed Frobenius values. At $v^n \| \mathfrak{m}$ its inertia image consists of the values on principal ideals generated by elements prime to v and congruent to one modulo $\mathfrak{m}/v^n$. The prime-existence entries are the finite Galois specialization and the open-normal-subgroup assertion of 36.24.

4. For a regular integral parameter translated by $(a-1)/2$, the discrete series has unique lowest compact type with weight $\lambda + \rho_n - \rho_c$, where ρ_n, ρ_c are the half-sums of the positive noncompact and compact roots of its chamber. The highest weights of the other compact types differ from the lowest compact-type weight by sums of positive roots in this chamber. The classification exhausts square-integrable irreducibles up to order-preserving relabeling within each sign. The analytic entry includes admissible coefficients and density for the Levi submodule. The two convergent Siegel–Weil fields are those of 34.17. The first uses $a < b \leq 2a$ and $a + \mathrm{Witt}(B) < b$. The second includes the volume and nonsingular Gram assertions. In the Schrödinger model a Fock vector in the convergent range has theta integral equal to the holomorphic Siegel value at the special point. The flat Siegel series continues with finite pole order.

5. For every prime $d > 3$ with $d \equiv 3 \pmod 4$, let $K = \mathbf{Q}(\sqrt{-d})$ and $F_0 = \mathbf{Q}(j(\mathcal{O}_K))$. There is a CM elliptic curve A/F_0 with $j(A) = j(\mathcal{O}_K)$ and good reduction away from d. At each finite place of F_0 there is a local minimal equation of discriminant a local unit times $-d^3$. For an imaginary quadratic

K of discriminant less than -4, let Λ be its integer lattice. For every positive integer n, the ray class field of (n) is

$$K\big(j(\Lambda),\, W_\Lambda(P) : 0 \neq P \in (\mathbf{C}/\Lambda)[n]\big), \qquad W_\Lambda(P) = \frac{g_2(\Lambda) g_3(\Lambda)}{D(\Lambda)}\, \wp_\Lambda(P).$$

The generator $j(\Lambda)$ can equivalently be divided by 1728. In particular at $n = 1$ the field is $K(j(\Lambda))$.

6. The nonendoscopic hypotheses in P have five factors. Its local part includes normalized induction and Jacquet finiteness, unitary and disjoint-product irreducibility, the geometric lemma, segment and square-integrable classification, the tempered product and generic unlinked criteria, multisegment classification, exhaustion and standard cosocle, the Moy–Prasad Jacquet isomorphism, local correspondence and converse, depth-zero induction, degree-zero Brauer induction, Chevalley units and the regular Green formula. Its global part includes decay and finiteness, cuspidal spectrum, Jacquet–Shalika, isobaric transfer, Gelfand–Kazhdan duality, polarized and dual local–global compatibility, purity and cohomologicality, reciprocity, ray fields, above-ℓ character and Weil–Deligne comparison, and potential semistability. Its cohomology part gives normalized Poitou–Tate, local and global finiteness and Euler characteristics, local duality and unramified orthogonality, Greenberg–Wiles, local finiteness, duality and Euler characteristic over the completed field, and local Bloch–Kato. The remaining factors are $\mathscr{J}$ of 1.15 and the Hecke–Tate functional equation.

7. The analytic and de Rham hypotheses in P include the Weil action, existence, analyticity and classification for discrete series, trace class, convolution and restricted tensor factorizations, the Weil–Siegel, doubling and unramified integral formulas, and partial Hecke continuation. Its de Rham count says that for a continuous rank-a representation de Rham at w, the filtration jumps at each compatible embedding inducing w have total multiplicity a. For the small-prime estimate, P includes the seven hypotheses $\mathscr{H}_2$ of Definition 36.15. The three further local cohomology entries of P are those of 36.14 for a finite discrete $\mathbf{Z}/\ell^r$-module with decomposition-group action and a valuation subring of the chosen algebraic closure.

The additional entries of F are as follows.

1. For every prime g and rank N there are polynomial-indexed finite-GL series characters, with the series law for $N > 0$. Over a finite field k there is a cuspidal irreducible in every rank $r \geq 1$. Let P_r be the mirabolic subgroup, consisting of matrices with last row $(0, \dots, 0, 1)$, and let U_r be the upper unitriangular subgroup. For every nontrivial additive character ψ_0 of k, the restriction of each irreducible cuspidal representation of $\mathrm{GL}_r(k)$ to P_r is $\mathrm{Ind}_{U_r}^{P_r} \psi_U$, where $\psi_U(u) = \psi_0(\sum_{i<r} u_{i,i+1})$. Weyl interlacing holds for the complex general linear and compact inclusions of successive ranks. For every $q \geq 1$ and every reduced class a mod q the least prime in the class is at most Cq^L for absolute $C, L > 0$.

2. The j-value of any ideal lattice of an imaginary quadratic field generates its Hilbert class field. For a finite idèle t with ideal $\mathfrak{b}$, the automorphism $\mathrm{Art}_K(t)^{-1}$ of the Hilbert class field carries the value at $\mathfrak{a}$ to that at $\mathfrak{b}^{-1}\mathfrak{a}$. The values form one orbit over the quadratic field. Nondegenerate Hermitian matrices of equal rank over a quadratic extension of a nonarchimedean field of characteristic different from two are congruent exactly when their determinant quotient is a norm. The entries include the local-to-global classification over a CM field and the Hasse principle over a quadratic extension of number fields. Quadratic class field theory gives the local characters, norm theorem, exact sequence, product formula and scalar extension compatibility.

3. For admissible nonarchimedean GL representations the Iwahori projection onto the maximal compact torus invariants of the upper-unipotent Jacquet module is bijective. Over characteristic zero the shuffle algebra on an ordered alphabet is polynomial on its Lyndon words. For the convergent Rallis formula use $a < b \leq 2a$ with B anisotropic or $a + \mathrm{Witt}(B) < b$, compatible doubled Weil data, a cuspidal source, vanishing lower lifts and a compact-finite doubled test. The lifts are square integrable and their Petersson pairing divided by target volume is the doubling zeta value of 34.17.

4. With finitely many Euler factors removed, the Hecke L-function of a unitary character outside the norm-power family is entire and nonzero on $\mathrm{Re}\, s \geq 1$. For the trivial character the corresponding assertion applies to $(s-1)\zeta^T(s)$, where T is the set of omitted places. At full level every nontrivial

unitary character, including the nontrivial norm powers, has continuation continuous and nonzero at one. There is an absolute $C > 0$ such that for every primitive Dirichlet character χ of conductor $q > 1$,

$$\left| \frac{L'}{L}(1,\chi) - \sum_{\substack{0<\operatorname{Re}\rho<1 \\ |\operatorname{Im}\rho|\le 1}} \frac{\operatorname{ord}_\rho L}{1-\rho} \right| \le C \log(3q). \tag{1.8}$$

The sum runs over distinct zeros with the indicated multiplicities.

5. The compact-finite subspace of an irreducible unitary realization is admissible, analytic, dense, Lie-stable and irreducible as a $(\mathfrak{g}, K)$-module. For $\mathrm{U}(p,q)$ the discrete-series entries give the lowest compact type with weight $\lambda+\rho_n-\rho_c$, equivalence for the same parameter, and equivalence after changing either frame of the same Hermitian matrix. Fix the complex structure for which the positive noncompact roots go from the negative to the positive signature block. The holomorphic lowest-weight property is required only when $\lambda_a > \lambda_b$ for every positive-sign index a and negative-sign index b. Globalization gives admissible closure and infinitesimal equivalence. Smooth uniformly moderate cusp forms and their invariant derivatives are bounded. At fixed open level, growth exponent A and decay order M, a finite set L of derivative words and a uniform constant bound the decay seminorm by the sum of their moderate-growth seminorms. Let d be the greatest word length in L, with $d = 0$ if L is empty. Write H for the group height and H_{q} for the quotient height. With D_l denoting differentiation along the word l, the estimate gives

$$\sup_g |\phi(g)| H_{\mathrm{q}}(g)^M \le C \sum_{|l|\le d} \sup_g \frac{|D_l\phi(g)|}{H(g)^A}.$$

Here the words have length at most d, including the empty word, and their letters are the complex-linear right derivatives in the matrix-unit directions at the infinite places. The hypothesis of uniform moderate growth is expressed by finiteness of these sums at every word-length cutoff for the fixed exponent A.

6. Over a nonarchimedean field of characteristic different from two, smooth irreducible Heisenberg representations with a prescribed nontrivial continuous central character exist and are equivalent. The two real metaplectic realizations give the twofold cover of the real symplectic group and its strongly continuous unitary and Schwartz actions intertwining Heisenberg translations. They supply the derived quadratic operators, the Gaussian coefficient identity and the Gaussian eigenline over the maximal compact. The general linear Howe decomposition applies to complex general linear and unitary inclusions. Kudla's splitting over a characteristic-zero nonarchimedean field gives a smooth Schrödinger action for the split unitary member. For a Levi element a, the action is

$$(\omega(a)\varphi)(x) = \chi(\det a)|N \det a|^{\dim V/2}\varphi(xa), \qquad H(w)\omega(g) = \omega(g)H(wg).$$

Here V is the partner space, N denotes the quadratic norm in the first formula, and $H(w)$ is the Heisenberg translation for the row w in the doubled matrix space. The action is isometric for every additive Haar measure on the Schrödinger matrix space. Fix an arithmetic datum D and a curve datum $(\mathfrak{c}, \delta)$ as in Definition 34.48(1). For the standard spaces E^N and $E^{N'}$ with identity Gram matrices and standard lattices, take NN' even, a nonzero theta vector e_0 in the standard-lattice theta space of this torus, and a compatible Weil action for e_0 as in Definition 34.48(2)(c). For rational unitary matrices γ, γ' acting through their adelic images, the transformation of every torus theta function $F(z)$ on $\prod_\tau M_{N'\times N}(\mathbf{C})$ sends $F(z)$ to $F(\gamma'^* z\gamma)$ (acting at each τ by $(\tau\gamma')^* z_\tau(\tau\gamma)$) times

$$\prod_\tau \tau(\det\gamma)^{-a_\tau}\tau(\det\gamma')^{-a'_\tau}, \qquad 2a_\tau = \epsilon' n'_\tau - N', \qquad 2a'_\tau = \epsilon n_\tau + N.$$

Here $\epsilon, \epsilon' \in \{\pm 1\}$ are the two Kudla split-place splitting-character sign exponents (as in Definition 34.48(2)(c)), and n_τ, n'_τ the splitting-character angular infinity exponents on $\tau/|\tau|$. For a CM unitary pair with compatible Hecke characters there is also an adelic Weil action, with a realization

for each good model. In the real polynomial oscillator model with compatible Cartan frames, stable subspaces containing the first member's coinvariant relations have bounded strict-chain length. The sum of two proper such subspaces is proper.

7. For a nondegenerate CM Hermitian space with Witt frame there are a compact subgroup K, a compact subset Ω of the modulus-one minimal parabolic, and a positive cone parameter giving the Iwasawa and rational Siegel-set coverings. The Siegel set $\mathfrak{S}$ is closed and has finite Haar volume. The conjugates $a^{-1}pa$ for $p \in \Omega$ and cone elements a stay in one compact set. For a fixed Haar measure there is C such that for every nonnegative f and measurable nonnegative h with $f(pak) \le h(t)$ on the Siegel set coordinates,
$$\int_{\mathfrak{S}} f(g)\,dg \le C \int_{\mathrm{cone}} h(t)\big(\delta_P(t)\textstyle\prod_i t_i\big)^{-1}\,dt.$$
Here t is the positive torus coordinate of a and δ_P is the parabolic modulus in these coordinates. There are C', A with $H(g) \le C'H(\gamma g)^A$ for every $g \in \mathfrak{S}$ and every rational γ, where H is the group height. Smooth moderate cusp forms decay on $\mathfrak{S}$ to every polynomial order in H.

Lemma 1.21 (Simplicity of external tensor products)**.** *Let $(A_i)_{i=1}^r$ be unital associative $\mathbf{C}$-algebras, and for each i let M_i be a simple left A_i-module of at most countable dimension over $\mathbf{C}$. Then $\mathrm{End}_{A_i}(M_i) = \mathbf{C}$, and the external tensor product $\bigotimes_{i=1}^r M_i$ is a simple left $\bigotimes_{i=1}^r A_i$-module. More generally, let $(A_v)_{v\in V}$ be a family of unital associative $\mathbf{C}$-algebras, let each M_v be a simple left A_v-module of at most countable dimension over $\mathbf{C}$, and for each v outside a finite subset $S_0 \subset V$ fix a nonzero vector $s_v \in M_v$. Then the restricted tensor product $\bigotimes'_{v\in V} M_v$ formed with respect to $(s_v)_{v\notin S_0}$ is simple under the restricted tensor algebra.*

Outline. Countable dimension and Schur's lemma force scalar commutants, so each algebra prescribes images of finite independent tuples. Isolating pure tensors gives simplicity for finite products, and passing to a common finite set $S \supseteq S_0$ yields the restricted case. The complete proof is on p. 508. □

Proposition 1.22 (Inputs for the unitary and local theta arguments)**.** *For every triple O of global and local inputs as in 39.1, both products P, F of Definition 1.20, as used in Theorem 33.1, hold.*

Outline. The proof verifies each component of P and F in Definition 1.20 from published automorphic, Galois, and theta theorems together with Theorem 34.16, Theorem 34.17, Lemma 26.22, Lemma 1.19, and Lemma 1.21. Direct arguments establish unramified doubling, cuspidal decay, trace class, Jensen–Blaschke bounds, Poisson theta invariance, and real Howe duality. The complete proof is on p. 508. □

Theorem 1.23 (Construction of the common input)**.** *Let $\mathcal{R}$ be the Galois representation assertion of Definitions 39.4–39.6, with the parameter laws of Definition 40.2. Then $\mathcal{R}$ implies H. In particular, H holds by Theorem 39.2.*

Outline. Obtain P, F by 1.22, the theta identity and tests by 34.7 and 34.41, and the bounds from 35.2 and 36.16. Use 33.1 to obtain an element of $\mathcal{H}$, then apply 1.24. The complete proof is on p. 514. □

Theorem 1.24 (Three inputs give all characters)**.** *Assume $\mathcal{G}$ and $\mathcal{V}$ and suppose $\mathcal{H}$ has an element. Then H holds.*

Proof. Apply Theorem 1.25 and Lemma 1.14. □

Theorem 1.25 (Three inputs give primitive nonvanishing)**.** *Assume $\mathcal{G}$ and $\mathcal{V}$ and suppose $\mathcal{H}$ has an element. Then H_{prim} holds.*

Outline. Use 32.64 with $\mathcal{G}$ and $\mathcal{H}$ for connectedness with exponents $(2, 0)$, then 24.1 for chain bounds under 26.1. For primitive ψ, use 8.21 with $\mathcal{V}$ to get hosts with $\lambda = 3/2$, then 2.2 for nonvanishing. The complete proof is on p. 514. □

Chapter 2

How arithmetic windows enter the zero-free criterion

Use the L-value convention of (1.1). Write $\mathcal{H}_\lambda(S)$ if, for every $Y_0 \in \mathbf{R}$, some $Y \ge Y_0$ satisfies $[Y, Y^\lambda] \subset S$. In each window condition we may take $Y \ge \max(Y_0, 1)$, since Y_0 is arbitrary.

2.1 The arithmetic interface

A family $\mathcal{F}$ at ε has scales $W(\mathcal{F})$ and objects $D_{x,i}$, $i \in I_{\mathcal{F}}(x)$. Write $x_1(\mathcal{F}) > 1$ for the threshold in the family data. Write $E_{\mathcal{L}}(\mathcal{F})$ if there are $C, x_0 \in \mathbf{R}$ such that, for every $x \ge x_0$ and $i \in I_{\mathcal{F}}(x)$, the host and top set of $D_{x,i}$ satisfy the arithmetic requirement of Definition 8.19 at $(\mathcal{L}, x, \varepsilon, C)$. This fixes the subgroup index at $o_{\mathcal{L}}$, represents the retained characters in the positive half of the host characters, and matches the designated character, when present, with the q-sign character. When no character is designated, the primitive representative of the q-sign lies outside the retained set. The top set must satisfy the coset counts (8.17) with tolerance $x^{-1+\varepsilon}$. For each primitive representative χ of an unretained positive host character or of a negative host character in Definition 8.19, the bounds are $|g_\chi(x)| \le x^{1/2+\varepsilon}$ and $|(L'/L)(1,\chi)| \le C \log x$. For a relation T on families and the real parameters below, write $\mathcal{T}_{e_D,e_L}(\mathcal{F})$ for

$$\exists K, c_{HH}, e_K, C_L, x_0 \in \mathbf{R}\ \forall x \ge x_0, \quad x \in W(\mathcal{F}) \Longrightarrow T(\mathcal{F}; e_D, e_L, K, c_{HH}, e_K, C_L, x).$$

The comparison $\mathcal{C}_{\mathcal{L}}$ asserts, for every $e_D, e_L \in \mathbf{R}$, $\varepsilon > 0$, and $\mathcal{F}$,

$$\begin{aligned} E_{\mathcal{L}}(\mathcal{F}) \wedge \mathcal{T}_{e_D,e_L}(\mathcal{F}) &\Longrightarrow \exists x_1\ \forall x \ge x_1, \\ x \in W(\mathcal{F}) &\Longrightarrow G_m^{\mathcal{L}}(x) \le x^{1/2+(e_D+e_L+7)\varepsilon}, \end{aligned} \tag{1}$$

with $G_m^{\mathcal{L}}$ from 2.5. Write $\mathcal{W}_{\mathcal{L}}(\lambda)$ for

$$\begin{aligned} &\exists e_D, e_L \in \mathbf{R}\ \exists \varepsilon_1 > 0\ \forall\, 0 < \varepsilon < \varepsilon_1\ \exists \mathcal{F}, \\ &E_{\mathcal{L}}(\mathcal{F}), \qquad \mathcal{H}_\lambda(W(\mathcal{F})), \qquad \mathcal{T}_{e_D,e_L}(\mathcal{F}). \end{aligned} \tag{2}$$

Definition 2.1 (Host conditions and the cyclic requirement)**.** At a scale $s = (x, \varepsilon, J)$ put $b = \exp(x^\varepsilon)$ and $\delta_1 = x^{-\varepsilon/8}$. Let E be the host field, $K = \mathbf{Q}(\sqrt{-d})$, and $N = Q_S q d$, where Q_S is the conductor parameter of the host. In its frame write $n = |A|$, $n_0 = |A_0|$ and $o = [A : A_0]$. For initial rank k and chain length m, put $r = k + m - 1$. The host conditions with constants $C_{T_2}, C_8, C_5, c_6, C_b$ are as follows.

The host has thickness constant C_{T_2}. Its thick set T_2 satisfies (8.15) with $C_T = C_{T_2}$, $4L_w|T_2| \le n_0$, $4L_w s < n_0$ for $s \in T_2$, and (8.16) with $L = L_w$, where

$$L_w = 10\lceil 405 \log x \rceil + 2\lceil 30/\varepsilon \rceil + 22.$$

Its top set $TT \subset A$ satisfies $\delta_1 n/2 \le |TT| \le 2\delta_1 n$ and the weighted bound (8.18) with constant C_b. The conductor and frame satisfy $Q_S \le C_8 n$ and $J, o \le C_8$. The auxiliary ideal in K is $\mathcal{O}_K$, and the host's splitting bound is at least $\lfloor x^{\varepsilon/(5J)} \rfloor$.

The level primes form the set $\{g_0, g_1, p, p', q_0\}$ and split in K. For $i = 0, 1$ one has $(kb)^8 \le g_i \le 2(kb)^8$, $g_i \equiv 1 \pmod N$, and there is $\pi_i \in \mathcal{O}_K$ with $\pi_i\bar\pi_i = g_i$ and $\pi_i \equiv 1 \pmod{N\mathcal{O}_E}$. The exceptional prime sets $\mathcal{B}_r(g_0)$ and $\mathcal{B}_r(g_1)$ of Theorem 8.12 are disjoint. The primes p, p' split completely in E and are 1 modulo $\operatorname{lcm}(15, N)$. The prime $q_0 \equiv 1 \pmod N$ is distinct from p and p'. All three are at most N^{C_8}.

There is one $Y \in [x^{1/J}, x]$ at which q, d satisfy the prime ranges, splitting conditions and bound on $\log N$ of Lemma 8.16, the six quadratic twists satisfy the conductor and zero-free box conditions there, and the real host characters whose conductor is divisible by q or d satisfy the conductor and box bounds of Theorem 8.13 at the heights specified before Theorem 8.17. For the primitive representatives of the two quadratic signs and their product, require

$$|g_\chi(x)| \le x^{1/2+\varepsilon}, \qquad |(L'/L)(1,\chi)| \le C_5 \log x.$$

For the quadratic character modulo d, the absolute value of the real part of its logarithmic derivative at 1 is at most x^ε. Every primitive host character satisfies (8.14) with constant c_6.

A *host requirement* is a condition on the scale, its frame, the arithmetic datum on that frame, a finite top set and an auxiliary rank $t \in \mathbf{N}$. At a host we use its own frame, arithmetic datum, top set and rank. The *length bound* at this host means the finiteness and the weighted inequality of Definition 26.1 for E, t and $B = \sqrt{b}$. For a primitive ψ, the *cyclic requirement* A_ψ is the conjunction of this length bound and the arithmetic requirement of Definition 8.19 for TT at $(\mathcal{L}_\psi, x, \varepsilon, 4)$, where $\mathcal{L}_\psi$ is the level of the powers of ψ from Definition 2.5.

Theorem 2.2 (Arithmetic input to the window criterion)**.** *Fix $\lambda > 1$. Assume the family selection principle stated below. Suppose that for every primitive character ψ of positive modulus there is a supply of hosts on λ-windows satisfying the cyclic requirement A_ψ of Definition 2.1. Then $L(s,\psi) \ne 0$ for every such ψ and $\operatorname{Re} s > 1/2$.*

A supply of hosts on λ-windows for a requirement A means that for each $0 < \varepsilon < 1/52$ there are real constants $J, C_8 \ge 1$, $C_5, c_6, C_b > 0$, $C_{T_2}, x_0 \in \mathbf{R}$, and a set $W \subset \mathbf{R}$ with $\mathcal{H}_\lambda(W)$ for which the following holds. For every $x \in W$ with $x \ge x_0$, one can choose rank data at $s = (x, \varepsilon, J)$ satisfying the three ranges for n, m, k in (8.1) and the identity $r + t = 8fr + 4 + T_{On_W}$ of Theorem 38.3, where $f = 2n$, $r = k + m - 1$, and t is the auxiliary rank. On these ranks there must be a host and a prime q_0 satisfying Definition 2.1 with the chosen constants, including the same thickness constant C_{T_2} at every point. The scale, frame, arithmetic datum, top set and auxiliary rank of this host must satisfy A.

For family selection use the arithmetic parameter families of the construction in Theorem 33.32. Their indexed models carry the design and lattice data with common parameter constants. In (1) and (2), take T to mean that at x one base model and one pair of integer weight functions admit a model at every translation u_0 in the host frame group, with the prescribed offset and those same top weights. The selected models have the same frame, m, f, q, d, and realise the global, local and arithmetic chain data with all the estimates in the conclusion of Theorem 24.1, with division, loss and height exponents at most e_D, e_L, c_{HH} and window constants K, e_K, C_L. For each $\lambda' > 1$ and request A, let $\mathcal{C}_{\lambda',A}$ be the families with λ'-windows satisfying A for every $x > x_1(\mathcal{F})$ and every $i \in I_\mathcal{F}(x)$. The family selection principle is the following implication for every such λ' and A. Suppose there is a supply of hosts for A on λ'-windows. Suppose also that each $\mathcal{F} \in \mathcal{C}_{\lambda',A}$ has a threshold x_V such that the length bound of Definition 2.1 holds at every $x \ge x_V$ and every $i \in I_\mathcal{F}(x)$, for the field, auxiliary rank and weight of that model. Then there are $e_D, e_L \in \mathbf{R}$ and $\varepsilon_1 > 0$ such that each $0 < \varepsilon < \varepsilon_1$ admits a family at ε in $\mathcal{C}_{\lambda',A}$ satisfying $\mathcal{T}_{e_D,e_L}$.

Outline. At each sufficiently large allowed scale, cyclicity gives the arithmetic condition of 8.19 with $C = 4$ and the length bound of 26.1. Family selection gives (2), 6.12 gives (1) at the same index, and 2.3 and 2.6 give the conclusion. The complete proof is on p. 515. □

Lemma 2.3 (Covering primitive characters by cyclic levels)**.** *Fix $\lambda > 1$ and a property Q of primitive characters. For every ψ satisfying Q, assume (1), the supply (2) at λ, and 2.6 for every window exponent > 1 at $\mathcal{L}_\psi$. Then $L(s,\psi) \ne 0$ whenever $Q(\psi)$ and $\operatorname{Re} s > 1/2$. Each ψ may use a containing level with these properties and its own $\lambda > 1$.*

Proof. Apply 2.4 to the assigned level. The cyclic level contains ψ, its own primitive reduction. □

Lemma 2.4 (Removing the exponent loss)**.** *Fix a level $\mathcal{L}$ of Definition 2.5, $\lambda \in \mathbf{R}$, a class of connected families indexed by $\varepsilon \in \mathbf{R}$ with underlying family data as above, and a host relation T. Assume (1), (2), and the displayed implication of Theorem 2.6 for this λ and every $0 < \theta < 1$. Then every character in $\mathcal{X}_{\mathcal{L}}$ has nonvanishing L-values throughout $\operatorname{Re} s > 1/2$.*

Outline. Choose $1/2 < \theta < \min(\operatorname{Re} s, 1)$ and ε with $1/2 + (e_D + e_L + 7)\varepsilon \le \theta$. On windows from (2) past the threshold in (1), comparison gives $G_m^{\mathcal{L}}(y) \le y^{\theta}$, so the window implication gives nonvanishing at s. The complete proof is on p. 515. □

2.2 Levels and positive weighted families

Definition 2.5 (Levels and cyclic levels)**.** A level has a finite kept set $\mathcal{X}_{\mathcal{L}}$ of primitive characters with positive moduli. The set contains the modulus-one character and is closed under inversion. The level also specifies $o_{\mathcal{L}} \in \mathbf{N}$ and may designate a real odd character $\xi \in \mathcal{X}_{\mathcal{L}}$. Put

$$w_{\mathcal{L}}(\chi) = \begin{cases} 1, & \chi = \xi \text{ when } \xi \text{ is specified,} \\ (3o_{\mathcal{L}} + 1)^{-1}, & \text{otherwise,} \end{cases}$$

$$G_m^{\mathcal{L}}(y) = G_z(y) + \sum_{\chi \in \mathcal{X}_{\mathcal{L}} \setminus \{1\}} w_{\mathcal{L}}(\chi) \operatorname{Re} g_{\chi}(y).$$

Use G_z, g_{χ} from 2.10. Write $\mathrm{ZF}_{\mathcal{L}}(\theta)$ if $L(s,\chi) \neq 0$ for every $\chi \in \mathcal{X}_{\mathcal{L}}$ and every $s \in \mathbf{C}$ with $\operatorname{Re} s > \theta$.

A subgroup Ξ of characters modulo $M \ge 1$ gives a level by primitive reduction, with $o_{\mathcal{L}} = |\Xi|$ and no distinguished character. Reduction preserves inversion and sends the principal character to modulus one, so the axioms hold. For $\Xi = \langle \psi \rangle$, write $\mathcal{L}_{\psi}$. If ψ has order o, each nonprincipal weight is $1/(3o+1)$.

Theorem 2.6 (The level window criterion)**.** *For a level $\mathcal{L}$, $\lambda > 1$ and $0 < \theta < 1$,*

$$\mathcal{H}_{\lambda}(\{y \in [1,\infty) : G_m^{\mathcal{L}}(y) \le y^{\theta}\}) \Longrightarrow \mathrm{ZF}_{\mathcal{L}}(\theta).$$

Proof. Apply 2.8 with $\delta = \lambda - 1$ and $C = 1$. The identities in 2.7 transfer the window bound and the nonvanishing conclusion. □

Lemma 2.7 (Realising a level as a positive weighted family)**.** *Every level $\mathcal{L}$ admits a family F of 2.9 with $G_m(F,y) = G_m^{\mathcal{L}}(y)$ for all $y \in \mathbf{R}$ and $\mathrm{ZF}(F;\theta) \iff \mathrm{ZF}_{\mathcal{L}}(\theta)$ for all $\theta \in \mathbf{R}$.*

Outline. Index F by $\mathcal{X}_{\mathcal{L}}$ with weight 1 at modulus 1 and $w_{\mathcal{L}}(\chi) > 0$ elsewhere for admissibility. Matching principal and nonprincipal terms via 2.10 identifies G_m, with identical nonvanishing indices. The complete proof is on p. 515. □

Theorem 2.8 (Pointwise power-window criterion)**.** *Let F be a positive weighted character family, $0 < \theta < 1$, $\delta > 0$, and $C \in \mathbf{R}$. If*

$$\forall Y_0 \in \mathbf{R}\ \exists Y \ge Y_0\ \forall y \in [Y, Y^{1+\delta}], \quad G_m(F,y) \le C y^{\theta}, \tag{3}$$

then $\mathrm{ZF}(F;\theta)$.

Proof. By 2.11, (3) gives the integral window condition. Apply 2.12. □

Definition 2.9 (Positive weighted character families)**.** A family F has a finite index set I, positive integer moduli N_i, complex Dirichlet characters χ_i modulo N_i, and weights $a_i > 0$, with $N_i = 1$ or $\chi_i \neq 1$ for each i (where $\chi_i = 1$ denotes the principal character modulo N_i). Characters may be imprimitive. For $\theta \in \mathbf{R}$, write $\mathrm{ZF}(F;\theta)$ if $L(s,\chi_i) \neq 0$ for every $i \in I$ and every $s \in \mathbf{C}$ with $\operatorname{Re} s > \theta$.

Definition 2.10 (Centred smoothed prime sums)**.** Let Λ be the von Mangoldt function and γ Euler's constant. For $y \in \mathbf{R}$, set

$$S_{\chi}(y) = \sum_{1 \le n \le \lfloor y \rfloor_+} \Lambda(n)\chi(n)\max(y/n - 1, 0),$$

$$G_{\chi}(y) = \begin{cases} S_{\chi}(y) - y\log y + (1+\gamma)y, & \chi = 1, \\ S_{\chi}(y) + yL'(1,\chi)/L(1,\chi), & \chi \neq 1. \end{cases}$$

Here $\chi = 1$ means that χ is the principal character modulo its modulus N. Put $G_m(F,y) = \sum_{i\in I} a_i \operatorname{Re} G_{\chi_i}(y)$. Here $\lfloor y\rfloor_+ = \max(\lfloor y\rfloor, 0)$. Use $\log y = \log|y|$ for $y<0$ and $\log 0 = 0$. Write $G_z = G_1$ at modulus one and $g_\chi = G_\chi$ for nontrivial primitive χ. Put $g_F(u) = G_m(F, e^u)$, $u \in \mathbf{R}$.

2.3 From pointwise bounds to integral windows

Lemma 2.11 (An integrable upper envelope)**.** *For $\theta, \delta \in \mathbf{R}$, let $\mathrm{WH}(F;\theta,\delta)$ mean that every $\eta > 0$ and $Y_0 \in \mathbf{R}$ admit $Y \ge Y_0$ and $R : \mathbf{R} \to \mathbf{R}$ with $y \mapsto R(y)y^{-\theta-1} \in L^1(J_Y)$, where $J_Y = [Y, Y^{1+\delta}]$, and*

$$\forall y \in J_Y, \quad 0 \le R(y), \quad G_m(F,y) \le R(y), \qquad \int_{J_Y} R(y) y^{-\theta-1}\, dy \le Y^\eta. \tag{4}$$

For every $\theta, C \in \mathbf{R}$ and $\delta > 0$, (3) implies $\mathrm{WH}(F;\theta,\delta)$.

Proof. Fix $\eta > 0, Y_0$. Choose a window in (3) with $Y \ge \max(Y_0, 1)$ and $|C|\delta \log Y \le Y^\eta$, possible since $(\log Y)/Y^\eta \to 0$. Set $R(y) = |C| y^\theta$ for $y > 0$ and $R(y) = 0$ for $y \le 0$. On J_Y, R is nonnegative and dominates $G_m(F,y)$. The integrand $R(y)y^{-\theta-1} = |C|/y$ is continuous there, and

$$\int_{J_Y} R(y) y^{-\theta-1}\, dy = |C| \log \frac{Y^{1+\delta}}{Y} = |C|\delta \log Y \le Y^\eta.$$

□

Theorem 2.12 (Integral window criterion for a character family)**.** *For a positive weighted character family F, $0 < \theta < 1$, and $\delta > 0$, one has $\mathrm{WH}(F;\theta,\delta) \Longrightarrow \mathrm{ZF}(F;\theta)$.*

Proof. By 3.1, the Laplace transform of g_F on $u > 0$ continues meromorphically to $\operatorname{Re} s > 0$ as Φ, with the pole data of 3.4. Its pole set Z_F contains all zeros of the family L-functions with positive real part. Apply 2.13 and then 3.2 to obtain $\operatorname{Re}\rho \le \theta$ for $\rho \in Z_F$. A zero s with $\operatorname{Re} s > \theta$ contradicts 3.3 if $\operatorname{Re} s \ge 1$, and the bound on Z_F if $\operatorname{Re} s < 1$. □

Lemma 2.13 (Logarithmic windows)**.** *For $g : \mathbf{R} \to \mathbf{R}$ and $\theta, \delta \in \mathbf{R}$, let $\mathrm{WH}_U(g;\theta,\delta)$ mean that every $\eta > 0$ and $U_0 \in \mathbf{R}$ admit $U \ge U_0$ and $R_U : \mathbf{R} \to \mathbf{R}$ with $u \mapsto R_U(u)e^{-\theta u} \in L^1(I_U)$, where $I_U = [U, (1+\delta)U]$, and*

$$\forall u \in I_U, \quad 0 \le R_U(u), \quad g(u) \le R_U(u), \qquad \int_{I_U} R_U(u) e^{-\theta u}\, du \le e^{\eta U}. \tag{5}$$

For every character family F, $\theta \in \mathbf{R}$, and $\delta > 0$, $\mathrm{WH}(F;\theta,\delta) \Longrightarrow \mathrm{WH}_U(g_F;\theta,\delta)$.

Outline. Choose $Y \ge \max(e^{U_0}, 1)$ with envelope R from the hypothesis, and set $U = \log Y$ and $R_U(u) = R(e^u)$ so exp maps I_U bijectively onto J_Y. Folland's change-of-variables theorem for $u \mapsto e^u$ yields integrability and the bound $e^{\eta U}$. The complete proof is on p. 516. □

Chapter 3

Zeros become poles

Throughout, characters have positive integer modulus. Fix a family $F = (\chi_i, a_i)_{i\in I}$ as in Definition 2.9 and use the sums of Definition 2.10. Put

$$g(u) = G_m(F, e^u) = \sum_{i\in I} a_i \operatorname{Re} G_{\chi_i}(e^u), \qquad \mathcal{M}H(s) = \int_1^\infty H(y) y^{-s-1}\, dy.$$

Put $D_\chi(s) = L'(s,\chi)/L(s,\chi)$ where defined. Use real logarithms in powers.

3.1 The analytic interface

Theorem 3.1 (Pole data for a character family)**.** *Choose the functions of 3.5 for χ_i, χ_i^{-1}. The sum*

$$\Phi(s) = \sum_{i\in I} \frac{a_i}{2}\big(\Phi_{\chi_i}(s) + \Phi_{\chi_i^{-1}}(s)\big)$$

gives pole data for g, Z_F as in 3.4.

Outline. Measurability and growth follow from 3.9, while 3.16 and 3.8 allow termwise integration on $\operatorname{Re} s > 2$. Applying 3.5 and 3.16 over finite I isolates Z_F and gives analyticity off Z_F, with residues from 3.17. The complete proof is on p. 516. □

Theorem 3.2 (Integral windows exclude poles)**.** *Let $g : \mathbf{R} \to \mathbf{R}$, $\Phi : \mathbf{C} \to \mathbf{C}$, and $Z \subset \mathbf{C}$ satisfy 3.4. Fix $0 < \theta < 1$ and $\delta > 0$. Assume the logarithmic window condition $\mathrm{WH}_U(g;\theta,\delta)$ of Lemma 2.13. Then $\operatorname{Re}\rho \le \theta$ for every $\rho \in Z$.*

Proof. This is Proposition 4.7, proved in Chapter 4. Chapter 41 gives an alternative proof by high-order derivatives of the Laplace transform. □

Lemma 3.3 (Nonvanishing on the closed half-plane)**.** *For $i \in I$ and $\operatorname{Re} s \ge 1$, $L(s,\chi_i) \ne 0$, including the assigned modulus-one value at 1.*

Proof. Davenport gives nonvanishing at regular points [33, Sections 4, 13–14]. At modulus one and $s = 1$, (1.1) gives $\zeta(1) = (\gamma - \log(4\pi))/2 \ne 0$. □

Definition 3.4 (Zero set and pole data)**.** Define

$$Z_F = \{\rho \in \mathbf{C} : \operatorname{Re}\rho > 0,\ \exists i \in I,\ L(\rho,\chi_i) = 0 \text{ or } L(\bar\rho,\chi_i) = 0\}.$$

For $g : \mathbf{R} \to \mathbf{R}$, $\Phi : \mathbf{C} \to \mathbf{C}$, and $Z \subset \mathbf{C}$, *pole data* require Borel measurability of g and some $C_0 \in \mathbf{R}$ with

$$|g(u)| \le C_0 e^{2u} \quad (u \ge 0), \qquad \Phi(s) = \int_0^\infty g(u) e^{-su}\, du \quad (\operatorname{Re} s > 2).$$

Every point of Z has positive real part. Φ is holomorphic on $\{s : \operatorname{Re} s > 0\} \setminus Z$. The set Z has no accumulation in $\operatorname{Re} s > 0$. Each $\rho \in Z$ admits $c_\rho \neq 0$ and h_ρ analytic near ρ with

$$\Phi(s) = \frac{c_\rho}{s-\rho} + h_\rho(s) \quad (0 < |s-\rho| \text{ sufficiently small}).$$

If ρ is real, c_ρ is a positive real number.

3.2 One character and the point 1

Lemma 3.5 (Continuation for one character)**.** *Let χ have modulus $N \geq 1$, with $N = 1$ or $\chi \neq 1$. There exists $\Phi_\chi : \mathbf{C} \to \mathbf{C}$ such that*

$$\Phi_\chi(s) = \int_0^\infty G_\chi(e^u) e^{-su}\, du \quad (\operatorname{Re} s > 2).$$

For each $\rho \neq 0$, Φ_χ is analytic at ρ if $L(\rho, \chi) \neq 0$, and there are $m \in \mathbf{N}$ and h analytic near ρ with

$$m = 0 \iff L(\rho,\chi) \neq 0, \qquad \Phi_\chi(s) = \frac{c_\chi(\rho)}{s-\rho} + h(s)$$

on a punctured neighbourhood of ρ. Here $c_\chi(\rho) = m/[\rho(1-\rho)]$ if $\rho \neq 1$, and $c_\chi(1) = 0$. At a zero, m is its multiplicity. The zeros of $L(\,\cdot\,, \chi)$ have no accumulation point in $\mathbf{C}$.

Outline. For $\chi \neq 1$, apply 3.7 with $b = 0$ to $L(\cdot, \chi)$ and obtain the integral from 3.12 and 3.8. For $\chi = 1$, apply 3.7 with $b = 1$ to $(s-1)\zeta(s)$ to match 3.14 via the Laurent expansion at 1, then apply 3.8. The complete proof is on p. 516. □

Lemma 3.6 (Local logarithmic derivative)**.** *Let f be analytic near ρ with a nonzero germ. Then $m = \operatorname{ord}_\rho f \in \mathbf{N}$, with $m = 0 \iff f(\rho) \neq 0$, and some h analytic at ρ satisfies $f'(s)/f(s) = m/(s-\rho) + h(s)$ on a punctured neighbourhood of ρ.*

Proof. This is the logarithmic derivative form of the standard local factorization theorem [32, Chapter 4]. □

Lemma 3.7 (Removing the apparent singularity)**.** *Let $f : \mathbf{C} \to \mathbf{C}$ be entire, $f(1) \neq 0$, and $b \in \mathbf{C}$. Write $D = f'/f$, with division by zero assigned the value zero here and in Φ_f (so $\Phi_f(0) = 0$), and put*

$$N(s) = D(s) - sD(1) + b(1-s), \qquad \Phi_f(s) = -\frac{N(s)}{s(s-1)} \quad (s \neq 1).$$

It extends analytically at 1, with $\Phi_f(1) = D(1) + b - D'(1)$. It is analytic where $s \neq 0$ and $f(s) \neq 0$. At a nonzero zero ρ of order m,

$$\Phi_f(s) = \frac{m}{\rho(1-\rho)(s-\rho)} + h_\rho(s)$$

on a punctured neighbourhood of ρ, with h_ρ analytic at ρ. The zeros of f are isolated.

Outline. Since $N(1) = 0$, $N(s)/(s-1)$ extends analytically to 1, and the identity theorem with $f(1) \neq 0$ isolates zeros. At a zero $\rho \neq 0, 1$, 3.6 expands D and splitting $-1/[s(s-1)]$ at ρ yields the simple pole with residue $m/[\rho(1-\rho)]$. The complete proof is on p. 516. □

3.3 The Mellin calculations

Lemma 3.8 (Exponential substitution)**.** *For $G : \mathbf{R} \to \mathbf{C}$ and $s \in \mathbf{C}$,*

$$\int_0^\infty G(e^u) e^{-su}\, du = \int_1^\infty G(y) y^{-s-1}\, dy,$$

where the Bochner integral has value zero for a nonintegrable integrand. Both integrands are absolutely integrable for $G = G_\chi$ and $\operatorname{Re} s > 2$.

Proof. The smooth inverse maps $y = e^u$ and $u = \log y$ preserve Lebesgue measurability. Change of variables for the absolute values [31, Theorem 2.47(a)] gives equivalence of integrability. In the integrable case the same substitution gives the identity. For G_χ, 3.9 gives the majorant $C_\chi e^{-(\operatorname{Re} s-2)u}$ when $\operatorname{Re} s > 2$. □

Lemma 3.9 (Measurability and quadratic growth)**.** *For every χ, $G_\chi : \mathbf{R} \to \mathbf{C}$ is Borel measurable, and some $C_\chi \in \mathbf{R}$ satisfies*

$$|G_\chi(y)| \le C_\chi y^2 \qquad (y \ge 1).$$

For $f : \mathbf{N} \to \mathbf{C}$ with $|f(n)| \le 1$,

$$\left| \sum_{1 \le n \le y} \Lambda(n) f(n)(y/n - 1) \right| \le 2y^2 \qquad (y \ge 1).$$

Proof. For $t \in \mathbf{R}$, write $t_+ = \max(t, 0)$. Since $0 \le \Lambda(n) \le n$ and $(y/n-1)_+ \le y/n$, each summand has norm at most y, and there are at most $2y$ terms. Use $0 \le \log y \le y$ to take $C_\chi = 3 + |1 + \gamma|$ for principal χ and $C_\chi = 2 + |D_\chi(1)|$ otherwise. On each Borel level set of $\lfloor y \rfloor_+$, the ramp is a finite sum of Borel functions, and the centring terms are Borel. □

Lemma 3.10 (Transform of a single ramp)**.** *For $n \in \mathbf{N}$, $n \ge 1$, and $s \in \mathbf{C}$ with $\sigma = \operatorname{Re} s > 1$,*

$$\int_1^\infty (y/n - 1)_+ y^{-s-1}\, dy = \frac{n^{-s}}{s(s-1)}, \qquad \int_1^\infty |(y/n - 1)_+ y^{-s-1}|\, dy = \frac{n^{-\sigma}}{\sigma(\sigma-1)}.$$

Proof. By [5, Equation (5.9.1), $\mu = \nu = 1$], the substitution $y = ne^u$ gives $n^{-s}((s-1)^{-1} - s^{-1})$ for the first integral. Taking the modulus replaces s by σ in the integrand. □

Lemma 3.11 (Termwise transformation)**.** *Let $f : \mathbf{N} \to \mathbf{C}$, $\operatorname{Re} s = \sigma > 1$, and suppose $\sum_{n \ge 1} |f(n)|\Lambda(n) n^{-\sigma} < \infty$. If $E : (1, \infty) \to \mathbf{C}$ is integrable, then*

$$\int_1^\infty \left(\sum_{1 \le n \le y} \Lambda(n) f(n)(y/n - 1) y^{-s-1} + E(y) \right) dy = \frac{\sum_{n \ge 1} f(n)\Lambda(n) n^{-s}}{s(s-1)} + \int_1^\infty E(y)\, dy.$$

Proof. By 3.10, the sum of the L^1-norms of the ramp terms is $\sum_{n \ge 1} |f(n)|\Lambda(n) n^{-\sigma}/[\sigma(\sigma-1)]$. Folland's Theorem 2.25 [31] gives the stated interchange. □

Lemma 3.12 (Transform for a nonprincipal character)**.** *For nonprincipal χ and $\operatorname{Re} s > 1$,*

$$\mathcal{M} G_\chi(s) = -\frac{D_\chi(s) - s D_\chi(1)}{s(s-1)}.$$

Proof. Logarithmic differentiation of the Euler product [4, Section 11.5, p. 231] gives

$$\sum_{n \ge 1} \chi(n)\Lambda(n) n^{-s} = -D_\chi(s).$$

In 3.11 take $E(y) = D_\chi(1) y^{-s}$ and use $\int_1^\infty y^{-s}\, dy = (s-1)^{-1}$. □

Lemma 3.13 (The logarithmic centring term)**.** *For $\operatorname{Re} s > 1$, the function $(\log y) y^{-s}$ is absolutely integrable on $(1, \infty)$, and*

$$\lim_{y \to \infty} (\log y) y^{1-s} = 0, \qquad \int_1^\infty (\log y) y^{-s}\, dy = \frac{1}{(s-1)^2}.$$

Proof. With $y = e^u$, equation (5.9.1) of [5] with $\mu = 1, \nu = 2$ gives $\int_0^\infty u e^{-(s-1)u}\, du = (s-1)^{-2}$. Since $\operatorname{Re}(s-1) > 0$, the integral converges absolutely and $u e^{-(s-1)u} \to 0$. □

Lemma 3.14 (Transform for the modulus-one character)**.** *For* 1 *modulo* 1 *and* $\operatorname{Re} s > 1$,

$$\mathcal{M}G_1(s) = -\frac{\zeta'(s)/\zeta(s) + 1/(s-1) + 1 - (1+\gamma)s}{s(s-1)}.$$

Proof. Here $G_1(y) = \sum_{1\le n\le y} \Lambda(n)(y/n-1) - y\log y + (1+\gamma)y$. Use $\sum_{n\ge1}\Lambda(n)n^{-s} = -\zeta'(s)/\zeta(s)$. Applying 3.11 to the ramp and 3.13 to the logarithmic term gives

$$\mathcal{M}G_1(s) = -\frac{\zeta'(s)/\zeta(s)}{s(s-1)} - \frac{1}{(s-1)^2} + \frac{1+\gamma}{s-1}.$$

The identity $s/(s-1) = 1 + 1/(s-1)$ gives the stated numerator. □

3.4 Conjugation and residues

Lemma 3.15 (Conjugation of the Hurwitz function)**.** *Let* $J(s) = \pi^{s/2}/\Gamma(s/2)$*, with the entire continuation of the reciprocal gamma function. Thus* J *is entire and* $J(1) = 1$*. Define* $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ *away from* $\{0, -2, -4, \dots\}$*, and assign* $\Gamma_{\mathbf{R}}(-2n) = 0$ *for* $n \in \mathbf{Z}_{\ge0}$*. For* $a \in \mathbf{R}/\mathbf{Z}$*, let* $a_* \in (0,1]$ *be its unique representative, and let* $\zeta(s, a_*)$ *be the classical Hurwitz zeta function. Define*

$$H_a(s) = \zeta(s, a_*) \quad (s \ne 1), \qquad H_a(1) = \lim_{\substack{s\to1\\ s\ne1}} \left(\zeta(s, a_*) - \frac{J(s)}{s-1}\right).$$

In particular, H_0 *is the Riemann zeta function with the assigned value at* 1 *used in Chapter 1. Put*

$$R_a(s) = \begin{cases} H_a(s) - J(s)/(s-1), & s \ne 1, \\ H_a(1), & s = 1. \end{cases}$$

Then R_a *is entire. For every* $s \in \mathbf{C}$*,*

$$H_a(\bar s) = \overline{H_a(s)}, \qquad \Gamma_{\mathbf{R}}(\bar s) = \overline{\Gamma_{\mathbf{R}}(s)}.$$

Outline. Subtracting $J(s)/(s-1)$ cancels the pole of $\zeta(s, a_*)$ at 1, making R_a entire. Gamma conjugation yields the $\Gamma_{\mathbf{R}}$ identity, and termwise conjugation for $\operatorname{Re} s > 1$ extends to $\mathbf{C}$ via R_a and the identity theorem. The complete proof is on p. 516. □

Lemma 3.16 (Inverse characters and conjugation)**.** *Let* $N \ge 1$*, let* χ *be a Dirichlet character modulo* N*, and use the Hurwitz functions of Lemma 3.15. At the pole of a principal character, assign the value*

$$L(1, \mathbf{1}_N) = \frac{1}{N} \sum_{j \in \mathbf{Z}/N\mathbf{Z}} \mathbf{1}_N(j) H_{j/N}(1).$$

For $N = 1$ *this is the assigned value* $H_0(1) = \zeta(1)$*. Here* j/N *denotes the class in* $\mathbf{R}/\mathbf{Z}$ *obtained by dividing any integer representative of* j *by* N*. With these point values, the finite Hurwitz expansion is*

$$L(s, \chi) = N^{-s} \sum_{j \in \mathbf{Z}/N\mathbf{Z}} \chi(j) H_{j/N}(s) \qquad (s \in \mathbf{C}).$$

For every $s \in \mathbf{C}$ *and* $y \in \mathbf{R}$*,*

$$L(\bar s, \chi^{-1}) = \overline{L(s, \chi)}, \qquad G_{\chi^{-1}}(y) = \overline{G_\chi(y)}.$$

Consequently, for every $\rho \in \mathbf{C}$*,*

$$L(\rho, \chi^{-1}) = 0 \quad \Longleftrightarrow \quad L(\bar\rho, \chi) = 0.$$

For nonprincipal χ*, also* $D_{\chi^{-1}}(1) = \overline{D_\chi(1)}$*.*

Outline. Grouping the Dirichlet series modulo N and continuing gives the Hurwitz expansion on $\mathbf{C}$. Conjugating via Lemma 3.15 yields the L-identity; differentiating at 1 gives $D_{\chi^{-1}}(1)$, and Definition 2.10 gives $G_{\chi^{-1}}$. The complete proof is on p. 517. □

Lemma 3.17 (Positive weighted multiplicities)**.** *For $\rho \in Z_F$, let $m_i(\rho), m_i^-(\rho)$ be the multiplicities at ρ for χ_i, χ_i^{-1}, respectively, with multiplicity zero at nonzeros. The sum $\sum_{i\in I} \frac{a_i}{2}(\Phi_{\chi_i} + \Phi_{\chi_i^{-1}})$ of the functions of 3.5 has residue*

$$c_\rho = \frac{S_\rho}{\rho(1-\rho)}, \qquad S_\rho = \sum_{i\in I} \frac{a_i}{2}\big(m_i(\rho) + m_i^-(\rho)\big) > 0.$$

Thus $c_\rho \neq 0$. If $\rho = \beta \in \mathbf{R}$, then $0 < \beta < 1$ and $c_\beta > 0$.

Proof. By 3.3 and 3.16, $0 < \operatorname{Re}\rho < 1$. Sum the expansions of 3.5 on a common punctured neighbourhood. By 3.5 and 3.16, an index i_0 witnessing $\rho \in Z_F$ contributes at least $a_{i_0}/2 > 0$ to S_ρ. The other terms are nonnegative. The denominator is nonzero, and for real $\rho = \beta$ it is positive. □

Chapter 4

A Gaussian window excludes poles

This chapter proves Theorem 3.2 with a Gaussian kernel. Chapter 41 gives a second proof by high-order derivatives of the Laplace transform; the two arguments share Lemmas 4.1 and 4.2.

We use the pole data (g, Φ, Z) of Definition 3.4. Write

$$H = \{s \in \mathbf{C} : \operatorname{Re} s > 0\}, \qquad B(z,r) = \{w \in \mathbf{C} : |w - z| < r\}, \qquad \overline{B}(z,r) = \{w \in \mathbf{C} : |w - z| \le r\}.$$

At $\rho \in Z$, denote the coefficient of $(s-\rho)^{-1}$ by c_ρ. It is nonzero, and it is positive real when ρ is real.

Lemma 4.1 (The absolute-convergence region)**.** *For every $\sigma_0 > 2$ there is $M \in \mathbf{R}$ such that $|\Phi(s)| \le M$ whenever $\operatorname{Re} s \ge \sigma_0$. More precisely, a growth bound $|g(u)| \le C_0 e^{2u}$ for $u \ge 0$ gives $M = C_0/(\sigma_0 - 2)$. Every $\rho \in Z$ satisfies $\operatorname{Re}\rho \le 2$.*

Proof. The Laplace majorant is $C_0 e^{-(\sigma_0-2)u}$ [231, Chapter II, §3]. If $\operatorname{Re}\rho > 2$, take $\sigma_0 = (2 + \operatorname{Re}\rho)/2$. Local boundedness gives $(s-\rho)\Phi(s) \to 0$. The pole expansion gives the limit $c_\rho \ne 0$. This is the bounded case of the removable-singularity criterion [32, Chapter IV, §3.1, Theorem 7]. □

Lemma 4.2 (Local finiteness)**.** *Let $Z \subset \mathbf{C}$. Suppose every $z \in H$ has a punctured neighbourhood disjoint from Z. For every $z \in \mathbf{C}$ and $r \in \mathbf{R}$ with $r < \operatorname{Re} z$, the set $Z \cap \overline{B}(z,r)$ is finite.*

Proof. This is the closed-disc case of the Bolzano–Weierstrass theorem [32, Chapter III, §1.4, Theorem 7]. □

Throughout, (g, Φ, Z) satisfies the pole data of Definition 3.4 with a constant $C_0 \ge 0$; thus g is real, Borel measurable, $|g(u)| \le C_0 e^{2u}$ for $u \ge 0$, and $\Phi(s) = \int_0^\infty g(u) e^{-su}\,du$ for $\operatorname{Re} s > 2$. Fix $0 < \theta < 1$. For $A > 0$, $U > 0$, $v \in \mathbf{R}$ and $s \in \mathbf{C}$ put

$$I_A(U,v) = \sqrt{\frac{A}{\pi U}} \int_0^\infty g(u)\, e^{-\theta u} e^{-A(u-v)^2/U}\,du, \qquad E_{U,v}(s) = \exp\Big((s-\theta)v + \frac{(s-\theta)^2 U}{4A}\Big).$$

The integral converges absolutely, its integrand being bounded by $C_0 e^{2u - A(u-v)^2/U}$. For $s = \sigma + it$ one has

$$|E_{U,v}(s)| = \exp\Big((\sigma-\theta)v + \frac{\big((\sigma-\theta)^2 - t^2\big)U}{4A}\Big), \qquad E_{U,v}(\bar s) = \overline{E_{U,v}(s)}. \tag{G}$$

Lemma 4.3 (Basic properties of the pole data)**.**

(a) $|\Phi(s)| \le C_0$ for $\operatorname{Re} s \ge 3$, and $\operatorname{Re}\rho \le 2$ for every $\rho \in Z$.

(b) Every compact $K \subset H$ meets Z in a finite set.

(c) If $\rho \in Z$, then $\bar\rho \in Z$ and $c_{\bar\rho} = \overline{c_\rho}$.

Proof. (a) is Lemma 4.1 with $\sigma_0 = 3$. (b) By compactness, finitely many open discs $B(z, r_z)$, $z \in K$, with $r_z = \frac{1}{2}\min_K \operatorname{Re} < \operatorname{Re} z$, cover K. Apply Lemma 4.2 to their closures.

(c) Put $Z^* = Z \cup \bar Z$. Since conjugation preserves H, no point of H is an accumulation point of Z^*. By (b), Z^* is countable, and $\Omega = H \setminus Z^*$ is open and invariant under conjugation. It is connected. Given

$p, q \in \Omega$, the paths $[p, w] \cup [w, q]$, with w on the perpendicular bisector of $[p, q]$ inside H, lie in H by convexity. Two of them meet only at p and q, and there are uncountably many, so one of them misses Z^*.

The function $\Psi(s) = \overline{\Phi(\bar{s})}$ is holomorphic on Ω. For $\operatorname{Re} s > 2$, since g is real, $\Psi(s) = \int_0^\infty g(u) e^{-su}\, du = \Phi(s)$. By the identity theorem, $\Phi = \Psi$ on Ω.

Let $\rho \in Z$ and choose $r > 0$ such that $B(\rho, r) \setminus \{\rho\} \subset \Omega$ and $\Phi = c_\rho/(s-\rho) + h_\rho$ there. For $0 < |s - \bar{\rho}| < r$ we get $\Phi(s) = \overline{c_\rho}/(s - \bar{\rho}) + \overline{h_\rho(\bar{s})}$, and the last term is bounded near $\bar{\rho}$. Hence Φ is unbounded near $\bar{\rho}$. Since Φ is holomorphic on $H \setminus Z$, this forces $\bar{\rho} \in Z$. Subtracting the expansion at $\bar{\rho}$ from this one, the function $(c_{\bar{\rho}} - \overline{c_\rho})/(s - \bar{\rho})$ is bounded near $\bar{\rho}$, so $c_{\bar{\rho}} = \overline{c_\rho}$. □

Lemma 4.4 (Gaussian representation)**.** *For $A, U > 0$ and $v \in \mathbf{R}$, the integral below converges absolutely and*

$$I_A(U, v) = \frac{1}{2\pi} \int_{\mathbf{R}} \Phi(3 + it)\, E_{U,v}(3 + it)\, dt.$$

Proof. Put $T = A/U$. For $b \in \mathbf{R}$, $\int_{\mathbf{R}} e^{-t^2/(4T)} e^{ibt}\, dt = 2\sqrt{\pi T}\, e^{-Tb^2}$. For real c, y the exponent $(c + it)y + (c + it)^2/(4T)$ equals $cy + c^2/(4T) - t^2/(4T) + it\big(y + c/(2T)\big)$. Hence

$$\frac{1}{2\pi} \int_{\mathbf{R}} \exp\Big((c + it)y + \frac{(c + it)^2}{4T}\Big) dt = \sqrt{T/\pi}\, e^{-Ty^2}.$$

Take $c = 3 - \theta$, $y = v - u$, $s = 3 + it$. The exponent becomes $(s - \theta)(v - u) + (s - \theta)^2 U/(4A)$; multiplying by $e^{-\theta u}$ gives, for every real u,

$$\sqrt{T/\pi}\, e^{-\theta u} e^{-T(u-v)^2} = \frac{1}{2\pi} \int_{\mathbf{R}} E_{U,v}(3 + it)\, e^{-(3+it)u}\, dt.$$

By (G), $t \mapsto |E_{U,v}(3+it)|$ is integrable. Moreover $|g(u) e^{-(3+it)u}| \le C_0 e^{-u}$ for $u \ge 0$, so $(u, t) \mapsto g(u) E_{U,v}(3 + it) e^{-(3+it)u}$ is integrable on $[0, \infty) \times \mathbf{R}$. Multiply by $g(u)$, integrate over $u \ge 0$, apply Fubini's theorem [31, Theorem 2.37], and use $\Phi(3 + it) = \int_0^\infty g(u) e^{-(3+it)u}\, du$. □

Lemma 4.5 (One-sided bound on a window)**.** *Let $\delta > 0$, $d = \min(\delta, 1)$, $A \ge 512/d^2$, $U \ge 1$, and $\eta > 0$. Let R_U satisfy (5) of Lemma 2.13 on $I_U = [U, (1 + \delta)U]$. If $v \le 2U$, $v - U \ge dU/4$ and $(1 + d)U - v \ge dU/4$, then*

$$I_A(U, v) \le \sqrt{\frac{A}{\pi U}}\, e^{\eta U} + \sqrt{2}\, C_0 e^{-4U}.$$

Proof. Put $T = A/U$, $W = [U, (1 + d)U] \subset I_U$, and $k(u) = \sqrt{T/\pi}\, e^{-T(u-v)^2} \in (0, \sqrt{T/\pi}\,]$.

On W we have $g(u) e^{-\theta u} k(u) \le \sqrt{T/\pi}\, R_U(u) e^{-\theta u}$. Indeed, if $g(u) \le 0$ the left side is at most 0; otherwise $0 < g(u) \le R_U(u)$. Since $R_U \ge 0$ on I_U,

$$\int_W g(u) e^{-\theta u} k(u)\, du \le \sqrt{T/\pi} \int_{I_U} R_U(u) e^{-\theta u}\, du \le \sqrt{T/\pi}\, e^{\eta U}.$$

Now let $u \ge 0$ with $u \notin W$, and put $r = |u - v|$; then $r \ge dU/4$. From $u \le v + r \le 2U + r$ and $\theta \ge 0$ we get $(2 - \theta)u \le 4U + 2r$. Also $Tr^2/4 \ge (A/U)(dU/4)^2/4 = Ad^2U/64 \ge 8U$ and $Tr/4 \ge Ad/16 \ge 32/d \ge 2$, so $Tr^2/4 \ge 2r$. Therefore

$$|g(u)| e^{-\theta u} e^{-Tr^2} \le C_0 e^{4U + 2r - Tr^2} \le C_0 e^{-4U} e^{-Tr^2/2}.$$

Since $\int_{\mathbf{R}} \sqrt{T/\pi}\, e^{-T(u-v)^2/2}\, du = \sqrt{2}$, the part of $I_A(U, v)$ outside W is at most $\sqrt{2}\, C_0 e^{-4U}$. Adding the two parts gives the claim. □

Lemma 4.6 (Residue expansion)**.** *Let $A \ge 1$, $R > 0$ with $R^2 \ge 9 + 32A$, and $a \in (\theta/2, \theta)$. Put $Q = \{\sigma + it : a \le \sigma \le 3,\ |t| \le R\}$ and assume $Z \cap \partial Q = \varnothing$. Then $P = Z \cap Q$ is finite and lies in the interior of Q. Moreover there is $K \ge 0$ such that, with $\kappa = \min\{2, (\theta - a)/2\}$, for all $U \ge 1$ and $v \in [U, 2U]$,*

$$\Big| I_A(U, v) - \sum_{\rho \in P} c_\rho E_{U,v}(\rho) \Big| \le K e^{-\kappa U}.$$

Proof. P is finite by Lemma 4.3(b). It lies in $\operatorname{Re} s \le 2$ by (a), hence in the interior of Q. Since Z is closed in H and misses ∂Q, Φ is holomorphic near ∂Q and on $Q^\circ \setminus P$. At $\rho \in P$ the function $\Phi E_{U,v}$ has a simple pole with residue $c_\rho E_{U,v}(\rho)$. The residue theorem [32, Chapter 4, §5.1] gives

$$\int_{3-iR}^{3+iR} \Phi E_{U,v}\, ds = 2\pi i \sum_{\rho\in P} c_\rho E_{U,v}(\rho) + \int_{a-iR}^{a+iR} + \int_{a+iR}^{3+iR} - \int_{a-iR}^{3-iR} \Phi E_{U,v}\, ds,$$

all along straight segments. Let $K_1 = \max_{\partial Q} |\Phi|$. Use (G) with $U \le v \le 2U$ and $|\sigma - \theta| \le 3$ on Q.

Horizontal edges ($|t| = R$). Here $(\sigma-\theta)v \le 6U$ and $(\sigma-\theta)^2 \le 9$, so $|E_{U,v}| \le \exp(6U + (9-R^2)U/(4A)) \le e^{-2U}$. Each edge contributes at most $3K_1 e^{-2U}$.

Left edge ($\sigma = a$). Here $0 < \theta - a < 1/2$ and $A \ge 1$, so $(a-\theta)v + (a-\theta)^2 U/(4A) \le -(\theta-a)U + (\theta-a)U/8$. Hence $|E_{U,v}| \le e^{-(\theta-a)U/2}$, and the edge contributes at most $2RK_1 e^{-(\theta-a)U/2}$.

Right tail ($\sigma = 3$, $|t| \ge R$). Here $|\Phi(3+it)| \le C_0$ and $|E_{U,v}| \le e^{-2U}\exp(-(t^2 - R^2)U/(4A))$, because $9 - t^2 \le -32A - (t^2 - R^2)$. From $t^2 - R^2 \ge 2R(|t| - R)$, $\int_{|t|\ge R} e^{-(t^2-R^2)U/(4A)}\, dt \le 4A/(RU) \le 4A/R$.

Split the integral of Lemma 4.4 at $|t| = R$, and note that $\frac{1}{2\pi}\int_{-R}^{R} \Phi E_{U,v}(3+it)\, dt = \frac{1}{2\pi i}\int_{3-iR}^{3+iR} \Phi E_{U,v}\, ds$. This gives the claim with $K = (6K_1 + 2RK_1 + 4AC_0/R)/(2\pi)$. □

Proposition 4.7 (The window criterion)**.** *Let $g : \mathbf{R} \to \mathbf{R}$, $\Phi : \mathbf{C} \to \mathbf{C}$, and $Z \subset \mathbf{C}$ satisfy Definition 3.4, fix $0 < \theta < 1$ and $\delta > 0$, and assume* $\mathrm{WH}_U(g; \theta, \delta)$ *of Lemma 2.13. Then* $\operatorname{Re}\rho \le \theta$ *for every $\rho \in Z$.*

Proof. Suppose $\rho_0 = \beta_0 + i\gamma_0 \in Z$ with $\beta_0 > \theta$.

Constants. Put $d = \min(\delta, 1)$ and $A = \max\{512/d^2,\ \gamma_0^2/(\beta_0 - \theta)\}$. Let $R_1 = 1 + |\gamma_0| + \sqrt{9 + 32A}$. By Lemma 4.3(b), the set $Z \cap \{\theta/2 \le \operatorname{Re} s \le 3,\ |\operatorname{Im} s| \le R_1\}$ is finite. Choose $R \in [R_1 - 1, R_1]$ different from every $|\operatorname{Im}\rho|$ there, and $a \in (\theta/2, \theta)$ different from every $\operatorname{Re}\rho$ there. Then $R > |\gamma_0|$, $R^2 \ge 9 + 32A$, and $Z \cap \partial Q = \varnothing$, because $\operatorname{Re}\rho \le 2 < 3$. Lemma 4.6 applies, and $\rho_0 \in P$ because $a < \theta < \beta_0 \le 2$.

Rates. For $\rho = \beta + i\omega \in P$ and $\lambda \in \mathbf{R}$ put $L_\rho(\lambda) = (\beta - \theta)\lambda + ((\beta - \theta)^2 - \omega^2)/(4A)$. By (G), for $v = \lambda U + h$,

$$E_{U,v}(\rho) = e^{UL_\rho(\lambda) + (\beta-\theta)h} e^{i\Theta}, \qquad \Theta = \omega(\lambda U + h) + (\beta - \theta)\omega U/(2A). \tag{R}$$

Each L_ρ is affine in λ, and $L_\rho \equiv L_{\rho'}$ if and only if $\beta = \beta'$ and $|\omega| = |\omega'|$. As P is finite, we may fix $\lambda \in (1 + d/3,\ 1 + 2d/3)$ at which no two non-identical L_ρ coincide. Let $M = \max_{\rho\in P} L_\rho(\lambda)$ and let P^* be the set of maximisers. All $\rho \in P^*$ share one real part β^* and one value $|\operatorname{Im}\rho| = \omega^* \ge 0$. By Lemma 4.3(c) and the symmetry of Q, either $P^* = \{\beta^*\}$, or $P^* = \{\rho^*, \overline{\rho^*}\}$ with $\rho^* = \beta^* + i\omega^*$, $\omega^* > 0$ and $c_{\overline{\rho^*}} = \overline{c_{\rho^*}}$.

Since $\lambda > 1$ and $A \ge \gamma_0^2/(\beta_0 - \theta)$,

$$M \ge L_{\rho_0}(\lambda) \ge (\beta_0 - \theta) - \gamma_0^2/(4A) \ge \tfrac{3}{4}(\beta_0 - \theta) > 0.$$

For $\beta \ge a$ we have $\lambda + (\beta-\theta)/(4A) > 1 - 1/(8A) > 0$. Hence $\beta \le \theta$ implies $L_\rho(\lambda) \le (\beta-\theta)\big(\lambda + (\beta-\theta)/(4A)\big) \le 0 < M$, and so $\beta^* > \theta$. Fix $\varepsilon > 0$ with $L_\rho(\lambda) \le M - \varepsilon$ for every $\rho \in P \setminus P^*$.

Moving the centre. Let $H_* = 0$ if $\omega^* = 0$ and $H_* = 2\pi/\omega^*$ otherwise. For $U > 0$, let $h_U = 0$ if $\omega^* = 0$. Otherwise let $h_U \in [0, H_*)$ be the unique number with $\arg c_{\rho^*} + \omega^*(\lambda U + h_U) + (\beta^* - \theta)\omega^* U/(2A) \in 2\pi\mathbf{Z}$. Put $v_U = \lambda U + h_U$.

Lower bound. By (R), the choice of h_U, and (G),

$$\sum_{\rho\in P^*} c_\rho E_{U,v_U}(\rho) = \begin{cases} c_{\beta^*} e^{MU}, & \omega^* = 0, \\ 2\operatorname{Re}\big(c_{\rho^*} E_{U,v_U}(\rho^*)\big) = 2|c_{\rho^*}| e^{MU + (\beta^*-\theta)h_U}, & \omega^* > 0. \end{cases}$$

This is a real number $\ge c_* e^{MU}$, where $c_* = c_{\beta^*} > 0$ (a real pole has positive residue, by 3.4) or $c_* = 2|c_{\rho^*}| > 0$. For $\rho \in P \setminus P^*$, (R) gives $|c_\rho E_{U,v_U}(\rho)| \le |c_\rho| e^{2H_*} e^{(M-\varepsilon)U}$.

Let $U \ge \max\{1, 3H_*\}$. Then $U \le v_U \le \frac{5}{3}U + H_* \le 2U$, so Lemma 4.6 applies. Since $I_A(U, v_U)$ and the sum over P^* are real,

$$I_A(U, v_U) \ge c_* e^{MU} - e^{2H_*}\Big(\sum_{\rho\in P} |c_\rho|\Big) e^{(M-\varepsilon)U} - Ke^{-\kappa U}.$$

Hence there is U_1 with $I_A(U, v_U) \ge \frac{1}{2} c_* e^{MU}$ for every $U \ge U_1$.

Upper bound. Let $U \geq \max\{1, 3H_*, 12H_*/d\}$. Then $v_U \leq 2U$, $v_U - U \geq (\lambda - 1)U > dU/4$, and $(1+d)U - v_U \geq (1 + d - \lambda)U - H_* > dU/3 - H_* \geq dU/4$.

Contradiction. Put $\eta = M/2$. Choose $U_0 \geq \max\{U_1, 1, 3H_*, 12H_*/d\}$ such that $\frac{1}{2}c_* e^{MU} > \sqrt{A/\pi}\, e^{MU/2} + \sqrt{2}\, C_0$ for all $U \geq U_0$. The hypothesis $\mathrm{WH}_U(g; \theta, \delta)$ gives $U \geq U_0$ and R_U satisfying (5). By Lemma 4.5 and the lower bound,

$$\tfrac{1}{2}c_* e^{MU} \leq I_A(U, v_U) \leq \sqrt{\tfrac{A}{\pi U}}\, e^{MU/2} + \sqrt{2}\, C_0 e^{-4U} < \tfrac{1}{2}c_* e^{MU},$$

a contradiction. Hence $\operatorname{Re}\rho \leq \theta$ for every $\rho \in Z$. □

Remark 4.8. The proof uses no growth bound for Φ in vertical strips: only $|\Phi| \leq C_0$ on $\operatorname{Re} s = 3$ and boundedness on the fixed compact set ∂Q. Positivity of residues is used only at the single real pole β^*.

Part II

The arithmetic side: prime sums and auxiliary primes

Chapter 5

Counting primes class by class

Fix a frame with finite abelian group A and distinguished element a_0, and put $A_0 = \langle a_0 \rangle$. The arithmetic data in Alpöge's archive [3] attach a Frobenius class and an inertia subgroup to each rational prime. Character orthogonality expresses the weighted prime-power count as a finite sum. After replacing the weights on thick positions by a common weight, we average over A_0. This bounds the real part of a weighted sum over the surviving characters.

5.1 The host and its local classes

Definition 5.1 (Host data used in the prime count)**.** For a frame ϕ, the host data H used in the prime count consist of the auxiliary primes and modulus below, together with a cyclotomic subfield and its Artin map satisfying the following conditions. The frame has $e \in \mathbf{N}$ and a surjective homomorphism $h : A \to (\mathbf{Z}/2^{e+2}\mathbf{Z})^\times/\{\pm 1\}$ whose restriction to A_0 is injective. Write $\mathcal{G}_\phi = (\{\pm 1\} \times \{\pm 1\}) \times A$, with the last factor written multiplicatively in host coordinates. The host supplies primes q, d and a positive integer Q_S such that

$$e \geq 2, \qquad q \equiv 3 \pmod 8, \qquad d \equiv 7 \pmod{16}, \qquad \left(\frac{-d}{q}\right) = 1,$$

$$2^{e+2} \mid Q_S, \qquad (Q_S, qd) = 1, \qquad \left(\frac{-d}{p}\right) = 1 \quad \text{for every odd prime } p \mid Q_S.$$

Set $\mathfrak{m} = Q_S qd$. Choose a primitive $\mathfrak{m}$th root of unity $\zeta_\mathfrak{m}$. The field data include a CM field E Galois over $\mathbf{Q}$, an embedding $E \hookrightarrow \mathbf{Q}(\zeta_\mathfrak{m})$, an isomorphism $\mathrm{Gal}(E/\mathbf{Q}) \simeq \mathcal{G}_\phi$, and an element $s_d \in E$ with

$$s_d^2 = -d, \qquad \sigma(s_d) = \sigma_d s_d \quad \text{if } \sigma \longleftrightarrow ((\sigma_q, \sigma_d), a).$$

Write $\mathrm{Art} : (\mathbf{Z}/\mathfrak{m}\mathbf{Z})^\times \to \mathcal{G}_\phi$ for the arithmetic ideal Artin homomorphism. At $p \nmid \mathfrak{m}$ this is the restriction of $\mathrm{Art}_\mathbf{Q}(t_p)^{-1}$, where the idèle t_p has component p at p and 1 elsewhere. If $\mathrm{Art}(u) = ((\sigma_q, \sigma_d), a)$, then $h(a) = [u \bmod 2^{e+2}]$. It is compatible with restriction of the cyclotomic automorphism $\zeta_\mathfrak{m} \mapsto \zeta_\mathfrak{m}^u$ and sends -1 to $((-1,-1),1)$. It also satisfies

$$u \equiv 1 \pmod{Q_S q} \implies \mathrm{Art}(u) \in \{1, \iota_d\}, \qquad \iota_d = ((1,-1),1),$$
$$u \equiv 1 \pmod{Q_S d} \implies \mathrm{Art}(u)^2 = 1.$$

The extension $E/\mathbf{Q}$ has degree $4|A|$. The Artin map is surjective, with d-coordinate (u/d) at u. For each prime $p \nmid \mathfrak{m}$, arithmetic Frobenius corresponds to $\mathrm{Art}(p)$ under the Galois isomorphism. More generally, let $\mathfrak{m}^{(p)}$ be the largest divisor of $\mathfrak{m}$ prime to p. The inertia subgroup at p is the Artin image of the units congruent to 1 modulo $\mathfrak{m}^{(p)}$. The Frobenius coset is the Artin image of the units congruent to p modulo $\mathfrak{m}^{(p)}$.

For the sums we use additive coordinates. Put

$$E_0 = (\mathbf{Z}/2\mathbf{Z})^2, \qquad G = E_0 \times A, \qquad \Phi = \{((v_d, v_q), h) : v_d = 0\}, \qquad f = |\Phi| = 2|A|.$$

Write $\epsilon_d = ((1,0),0)$, $c = ((1,1),0)$ and $\tau(t) = ((0,0),t)$. The character group $\widehat{G}$ consists of homomorphisms $G \to \mathbf{C}^\times$. Write $\mathbf{1}$ for the trivial character. Let $\chi_d((v_d,v_q),h) = (-1)^{v_d}$ and $\chi_q((v_d,v_q),h) = (-1)^{v_q}$.

Lemma 5.2 (Changing coordinates and characters)**.** *For every frame ϕ there is a group isomorphism*

$$T : \mathcal{G}_\phi \longrightarrow G, \qquad T((\sigma_q,\sigma_d),a) = ((\overline{\sigma_d},\overline{\sigma_q}),a), \qquad \overline{1} = 0, \quad \overline{-1} = 1.$$

Here the last coordinate is converted to additive notation. Put $c_\phi = ((-1,-1),1)$. For a host H, its Galois isomorphism sends complex conjugation to c_ϕ. We have $T(\iota_d) = \epsilon_d$ and $T(c_\phi) = c$. For $t \in A$, the host element $\tau_1(t) = ((1,1),t)$ satisfies $T(\tau_1(t)) = \tau(t)$. The kernel of the d-sign character is $\Psi_d = \{((\sigma_q,1),a) : \sigma_q \in \{\pm 1\},\ a \in A\}$. The map T carries Ψ_d onto Φ. Composition with T gives a bijection $\widehat{G} \to \widehat{\mathcal{G}_\phi}$, $\alpha \mapsto \theta_\alpha = \alpha \circ T$. Pullback sends the trivial character to the trivial character. At $((\sigma_q,\sigma_d),a)$, the pullbacks of χ_d and χ_q take the values σ_d and σ_q, respectively. Then θ_α is trivial on $\tau_1(A_0)$ exactly when α is trivial on $\tau(A_0)$. The set X_+ of such host characters with value 1 at ι_d therefore corresponds to the α trivial on $\tau(A_0)$ with $\alpha(\epsilon_d) = 1$.

Outline. Converting $\{\pm 1\}$ to $\mathbb{Z}/2\mathbb{Z}$, swapping signs, and writing A additively defines T, with element images and $\Psi_d \mapsto \Phi$ checked directly. Pullback gives the character bijection and matches the $\tau_1(A_0)$ and ι_d conditions. The complete proof is on p. 517. □

Let $I_p \subseteq G$ be the inertia subgroup, and choose a Frobenius representative $F_p \in G$. At a ramified prime its class is $F_p + I_p$.

Lemma 5.3 (The three local cases)**.** *For any host, let $S = \{q\} \cup \{p : p \text{ is prime and } p \mid Q_S\}$. Then $2, q \in S$, $d \notin S$, and*

$$\begin{array}{ll} p \text{ prime},\ p \notin S,\ p \neq d: & I_p = \{0\}, \\ p \in S: & I_p \subseteq \Phi, \quad F_p \in \Phi, \\ p = d: & I_d = \{0, \epsilon_d\}. \end{array}$$

Consequently every coset $iF_d + I_d$ contains exactly one element outside Φ.

Outline. Host conditions give trivial inertia outside $S \cup \{d\}$ and, with reciprocity, sign 1 for inertia and Frobenius in S. At d, the Chinese remainder theorem and Lemma 5.2 give $I_d = \{0, \epsilon_d\}$, with ϵ_d exchanging the halves. The complete proof is on p. 517. □

5.2 The exact prime-power identity

Let A be any finite abelian group. An arithmetic datum D consists of $q, d \in \mathbf{N}$ and, for each $n \in \mathbf{N}$, a subgroup $I_n \leq G$ and an element $F_n \in G$. The local hypotheses mean that d is prime and there is a finite $S \subseteq \mathbf{N}$ with $2, q \in S$, $d \notin S$, and the three local conditions displayed in Lemma 5.3. Write $[z]_+ = \max(z, 0)$ and use $a/0 = 0$ throughout.

Definition 5.4 (Local values, sums, and centring)**.** For $\alpha \in \widehat{G}$, a prime p, and $i \geq 1$, set

$$V_\alpha(p^i) = \frac{1}{|I_p|} \sum_{h \in I_p} \alpha(iF_p + h), \qquad v_\alpha(n) = \prod_{p \mid n} V_\alpha(p^{v_p(n)}) \quad (n \geq 1).$$

At $n = 1$ the product is empty. Take the empty product at $n = 0$ as well, so $v_\alpha(0) = v_\alpha(1) = 1$. Let $\ell : \widehat{G} \to \mathbf{C}$ be arbitrary. For every $y \in \mathbf{R}$ define

$$G_\alpha(y) = \begin{cases} G_z(y), & \alpha = \mathbf{1}, \\ \displaystyle\sum_{1 \leq n \leq y} \Lambda(n) v_\alpha(n)(y/n - 1) + y\ell_\alpha, & \alpha \neq \mathbf{1}. \end{cases}$$

Here G_z and the total real logarithm are as in Definition 2.10. A sum over $1 \leq n \leq y$ is empty for $y < 1$. For $t \in A$, define the partner map $r_t(g) = c + g + \tau(t)$. It maps $G \setminus \Phi$ bijectively onto Φ. Given $a \in \mathbf{R}$,

$w : G \to \mathbf{R}$, and $N \in \mathbf{N}$, put

$$\mathrm{ASK}_N(a,t,w) = \sum_{\substack{p \le N,\ p \text{ prime} \\ p \ne 2,q,d,\ F_p \notin \Phi}} 2\log p \sum_{\substack{1 \le i \le N \\ iF_p \notin \Phi}} [a/p^i - w(r_t(iF_p))]_+ + \log d \sum_{i=1}^{N} \sum_{\substack{g \in iF_d + I_d \\ g \notin \Phi}} [a/d^i - w(r_t(g))]_+.$$

For the same a, t, w, define the main term

$$\mathrm{Main}(a,w) = a\left(\log a - 1 - \gamma + \ell_{\chi_d} - \frac{1}{f}\sum_{\rho \in \Phi} \log w(\rho)\right)$$

and the centred character sum

$$\Delta(a,t,w) = \frac{1}{f}\sum_{g \notin \Phi} \sum_{\alpha \in \widehat{G}} \overline{\alpha(g)}\, w(r_t(g))\, G_\alpha\left(\frac{a}{w(r_t(g))}\right).$$

The value $V_\alpha(p^i)$ is unchanged when F_p is replaced by another representative of its inertia coset. Character orthogonality on I_p also gives

$$V_\alpha(p^i) = \begin{cases} \alpha(iF_p), & \alpha|_{I_p} = \mathbf{1}, \\ 0, & \alpha|_{I_p} \ne \mathbf{1}. \end{cases}$$

In particular $|v_\alpha(n)| \le 1$.

Lemma 5.5 (Orthogonality on an inertia coset)**.** *For every arithmetic datum D, every prime p, every integer $i \ge 1$, and every $g \in G$,*

$$\frac{1}{f}\sum_{\alpha \in \widehat{G}} \overline{\alpha(g)}\, v_\alpha(p^i) = \frac{2}{|I_p|}\, \mathbf{1}_{g \in iF_p + I_p}.$$

Under the local conditions of Lemma 5.3, the coefficient on the indicated coset is 2 at an unramified prime and 1 at d.

Proof. The product defining $v_\alpha(p^i)$ has the single factor $V_\alpha(p^i)$. Since $|\alpha(g)| = 1$, the left side is

$$\frac{1}{f|I_p|}\sum_{h \in I_p}\sum_{\alpha \in \widehat{G}} \alpha(iF_p + h - g).$$

The inner sum is $|G|$ when $iF_p + h = g$ and is 0 otherwise. There is one such h precisely when $g \in iF_p + I_p$. Finally $|G| = 2f$. □

Lemma 5.6 (Expansion of one layer)**.** *Assume the local conditions of Lemma 5.3. Let $a \in \mathbf{R}$, $t \in A$, and $w : G \to \mathbf{R}$ satisfy $w(\rho) > 0$ for $\rho \in \Phi$. Let $N \in \mathbf{N}$ satisfy*

$$a/w(\rho) \le N \qquad (\rho \in \Phi).$$

Then

$$\mathrm{ASK}_N(a,t,w) = \frac{1}{f}\sum_{g \notin \Phi}\sum_{\alpha \in \widehat{G}} \overline{\alpha(g)} \sum_{n=1}^{N} \Lambda(n) v_\alpha(n) [a/n - w(r_t(g))]_+.$$

The equality is in $\mathbf{C}$, with the real left side embedded in $\mathbf{C}$.

Outline. Expand into prime powers and use Lemma 5.5 to sum over the parts of inertia cosets outside Φ for $p^i \le N$. Primes in S vanish, d has weight $\log d$, and other primes contribute only at odd powers with $F_p \notin \Phi$. The complete proof is on p. 518. □

Theorem 5.7 (Exact balance for one layer)**.** *Under the hypotheses of Lemma 5.6, assume further that $a > 0$. For every choice of $\ell : \widehat{G} \to \mathbf{C}$,*

$$\mathrm{ASK}_N(a,t,w) - \mathrm{Main}(a,w) = \Delta(a,t,w).$$

Outline. Write each weighted sum as $WG_\alpha(a/W)$ plus a remainder in Lemma 5.6. On $\epsilon_d + \Phi$, orthogonality leaves the trivial and χ_d terms, and reindexing the trivial term by r_t gives $\mathrm{MAIN}(a,w)$. The complete proof is on p. 518. □

5.3 Averaging the thick contribution

Let A_0 be any subgroup of A. Define

$$P = \{\alpha \in \widehat{G} : \alpha(\tau(s)) = 1 \text{ for every } s \in A_0\}, \qquad e_1(t) = ((1,0),t).$$

For a finite subset $T_2 \subseteq A$, put $\delta = |T_2|/|A|$. For $b, y \in \mathbf{R}$ and $u, t \in A$, define the uniform thick sum

$$\Delta_{\mathrm{thick}}(u,t;y) = \frac{1}{f} \sum_{s \in T_2} \sum_{\alpha \in \widehat{G}} \alpha(-e_1(s+u-t))\, bG_\alpha(y).$$

Lemma 5.8 (Averaging over the subgroup)**.** *Let $s \in A_0$, $t \in A$, and $\alpha \in \widehat{G}$. Then*

$$\sum_{u \in A_0} \alpha(-e_1(s+u-t)) = \begin{cases} |A_0|\alpha(e_1(t)), & \alpha \in P, \\ 0, & \alpha \notin P. \end{cases}$$

Consequently, if $T_2 \subseteq A_0$, then for all $b, y \in \mathbf{R}$ and $t \in A$,

$$\sum_{u \in A_0} \Delta_{\mathrm{thick}}(u,t;y) = |A_0|\, b\, \frac{\delta}{2} \sum_{\alpha \in P} \alpha(e_1(t)) G_\alpha(y).$$

Outline. Using $-e_1(s+u-t) = e_1(t) + \tau(-s) + \tau(-u)$ factors out $\alpha(e_1(t))\alpha(\tau(-s))$ from $\sum_{u \in A_0} \alpha(\tau(-u))$, equal to $|A_0|$ on P and 0 off P. Summing $bG_\alpha(y)/f$ times this over $s \in T_2, \alpha \in \widehat{G}$ gives the sum. The complete proof is on p. 518. □

For $u \in A$, the thick positions form $\mathcal{T}_u = \{((0,1), s+u) : s \in T_2\} \subseteq \Phi$. Fix a scale $X' > 0$ and an error bound $\theta \geq 0$. The weight comparison uses the estimate

$$|G_\alpha(y)| \leq y\theta \qquad (\alpha \in \widehat{G},\ y \geq X'/3), \tag{5.1}$$

and a bound for changing a positive weight,

$$|vG_\alpha(a/v) - v'G_\alpha(a/v')| \leq \frac{3a}{\min(v,v')}|v - v'| \qquad (a, v, v' > 0,\ \alpha \in \widehat{G}). \tag{5.2}$$

Lemma 5.9 (Replacing the weights on thick positions)**.** *Let $T_2 \subseteq A$ and $u, t \in A$. Suppose*

$$x \geq 3, \quad b > 0, \quad b' > 0, \quad \eta \geq 0, \quad X' > 0, \quad \theta \geq 0, \quad b'X' \leq xb,$$
$$|a/b - x| \leq 1.$$

Assume (5.1) and (5.2), and suppose $w : G \to \mathbf{R}$ satisfies

$$\begin{aligned} &\rho \in \Phi : && w(\rho) > 0, \\ &\rho \in \mathcal{T}_u : && b \leq w(\rho) \leq b + \eta + 1, \\ &\rho \in \Phi \setminus \mathcal{T}_u : && w(\rho) \leq 2b'. \end{aligned}$$

Then

$$\mathrm{Re}\, \Delta_{\mathrm{thick}}(u,t;a/b) \leq \mathrm{Re}\, \Delta(a,t,w) + 2f\,(3(x+1)(\eta+1) + (x+1)b\theta)\,.$$

Outline. Use r_t^{-1} to locate thick positions and replace their weights by b using (5.2). Use $a/w(\rho) \geq X'/3$ and (5.1) to bound discarded terms, then sum absolute errors over positions and characters, normalize, and take real parts. The complete proof is on p. 519. □

Theorem 5.10 (From bounds on translates to an average bound)**.** *Assume the local conditions of Lemma 5.3. Let $A_0 \subseteq A$ be a subgroup, let $TT, T_2 \subseteq A$, and let $m, N \in \mathbf{N}$. Choose $a : A \to \mathbf{R}$ and write $a_t = a(t)$. Choose weights $w_{u,j} : G \to \mathbf{R}$ for $u \in A$ and $0 \leq j < m$. Suppose $T_2 \subseteq A_0$, $x \geq 3$, $b, b', X' > 0$, $\eta, \theta \geq 0$, and $b'X' \leq xb$. Assume $|a_t/b - x| \leq 1$ for every $t \in TT$. For each $u \in A_0$ and $0 \leq j < m$, assume the position bounds of Lemma 5.9 for $w_{u,j}$, and require $a_t/w_{u,j}(\rho) \leq N$ for every $t \in TT$ and $\rho \in \Phi$. Assume (5.1) and the two cost inequalities $2f(x+1)\theta \leq 1/x$ and $6f(x+1)(\eta+1) \leq b/x$. Define*

$$\operatorname{gap}(u) = \sum_{t \in TT} \sum_{j=0}^{m-1} \left(\operatorname{ASK}_N(a_t, t, w_{u,j}) - \operatorname{Re} \operatorname{Main}(a_t, w_{u,j})\right), \qquad Q = |TT|mb.$$

For any $\mathscr{E}, Y_* \in \mathbf{R}$, if $\operatorname{gap}(u) \leq \mathscr{E}$ for every $u \in A_0$ and

$$|TT| Y_* \leq \sum_{t \in TT} \operatorname{Re} \sum_{\alpha \in P} \alpha(e_1(t)) G_\alpha(a_t/b),$$

then

$$Q \frac{\delta}{2} Y_* \leq \mathscr{E} + \frac{2Q}{x}.$$

Outline. Use $\Psi(z) \leq 3z$ [34, Theorem 12] to prove (5.2) and apply 5.7 and 5.9 layerwise with the cost bounds. Sum using the gap bound, average over A_0 using 5.8, and apply the Y_* lower bound. The complete proof is on p. 519. □

Chapter 6

Reading off the target sum

Use the arithmetic data of 5.4 on $G = E_0 \times A$. For $\alpha \in \widehat{G}$ and $n \geq 1$, write $V_\alpha(n) = \prod_{p|n} V_\alpha(p^{v_p(n)})$ for the product of the local inertia averages defined there. Write 1 for the trivial character, so $G_1 = G_z$. For the subgroup $A_0 \leq A$, let P be the set of characters trivial on $\{0\} \times A_0$, as in 5.8, and put $e_1(t) = \epsilon_d + (0, t)$. For $K_p \subseteq \widehat{G}$ and $c \in \mathbf{R}$, set $w_c(\alpha) = 1$ when $\alpha|_{\{0\}\times A} = 1$, and $w_c(\alpha) = c$ otherwise. Define

$$G_m^A(y) = \sum_{\alpha \in K_p} w_c(\alpha) \operatorname{Re} G_\alpha(y), \qquad \Sigma_A(y) = \sum_{\alpha \in P \setminus K_p} |G_\alpha(y)|.$$

6.1 A common point of evaluation

We say that the prime sums satisfy the increment bounds with constant $C_d \geq 0$ if, for all $y, z \geq 1$,

$$|G_\alpha(y) - G_\alpha(z)| \leq |y - z|\big(\log \max(y, z) + C_d + |\ell_\alpha|\big) \qquad (\alpha \neq 1), \tag{6.1}$$

$$|G_z(y) - G_z(z)| \leq C_d |y - z|. \tag{6.2}$$

Lemma 6.1 (Moving one evaluation point)**.** *Assume the increment bounds (6.1) and (6.2) with $C_d \geq 0$. For $x \geq 3$, $|y - x| \leq 1$, and $\alpha \neq 1$,*

$$|G_\alpha(y) - G_\alpha(x)| \leq \log(x + 1) + C_d + |\ell_\alpha|, \qquad |G_\alpha(x)| \leq x(\log x + C_d + |\ell_\alpha|).$$

For the trivial character, $|G_z(y) - G_z(x)| \leq C_d$.

Proof. Since $y \geq 2$ and $\max(y, x) \leq x + 1$, (6.1) gives the first bound. By 5.4, $G_\alpha(1) = \ell_\alpha$ for $\alpha \neq 1$. Thus (6.1) at $x, 1$ gives

$$|G_\alpha(x)| \leq (x - 1)(\log x + C_d + |\ell_\alpha|) + |\ell_\alpha| \leq x(\log x + C_d + |\ell_\alpha|).$$

The trivial-character bound follows from (6.2). □

Lemma 6.2 (Comparison for each character)**.** *Let $TT \subseteq A$, $n_T = |TT|$, $x \geq 3$, and $|y_t - x| \leq 1$ for $t \in TT$. Assume the increment bounds (6.1) and (6.2) with $C_d \geq 0$. Let $K_p \subseteq \widehat{G}$, $c, L, R \in \mathbf{R}$, and $\rho, K_o \geq 0$. Suppose $1 \in K_p$, $\alpha(\epsilon_d) = 1$ for $\alpha \in K_p$, and*

$$\left|\sum_{t \in TT} \alpha(0, t) - c n_T\right| \leq \rho n_T \qquad (\alpha \in K_p,\ \alpha|_{\{0\}\times A} \neq 1),$$

$$|\ell_\alpha| \leq K_o \qquad (\alpha \in K_p,\ \alpha \neq 1),$$

$$|\ell_\alpha| \leq L \qquad (\alpha \in P \setminus K_p).$$

Require $C_d \leq R$ and

$$\log(x + 1) + C_d + K_o + \rho x(\log x + C_d + K_o) \leq R,$$

$$\log(x + 1) + C_d + L \leq R.$$

Set $v_\alpha = w_c(\alpha)\,\mathrm{Re}\,G_\alpha(x)$ *for* $\alpha \in K_p$, *and* $v_\alpha = -|G_\alpha(x)|$ *otherwise. Then*

$$n_T(v_\alpha - R) \le \sum_{t\in TT} \mathrm{Re}\big(\alpha(e_1(t))G_\alpha(y_t)\big) \qquad (\alpha \in P).$$

If $K_p \subseteq P$, *then* $\sum_{\alpha\in P} v_\alpha = G_m^A(x) - \Sigma_A(x)$.

Outline. Use 6.1 to sum the trivial bounds, control point shifts and character-sum errors for retained terms, and bound the remaining real parts using modulus-one values. If $K_p \subseteq P$, the terms sum to $G_m^A(x) - \Sigma_A(x)$. The complete proof is on p. 520. □

6.2 The geometric bound and the arithmetic comparison

Write $n = |A|$, $n_2 = |T_2|$, $Q = n_T mb$, and $\delta_2 = n_2/n$, with gap as in 5.10. For rank $k > 0$ and top-weight scale $a_E \ge 0$, put

$$\mathcal{B} = n_T a_E \left(1 + \frac{m^2}{k}\right) + m(2nmbx^\varepsilon) + m\frac{n_2 kbx^\varepsilon}{n(nx^{-\varepsilon})}. \tag{6.3}$$

Write $\mathrm{ASK}_D(N, a, t, w)$ for the sum $\mathrm{ASK}_N(a, t, w)$ of 5.4 with datum D. Write $\mathrm{ASK}_{\mathscr{D}}(u)$ for the prime sum attached to translation u in $\mathscr{D}$, and $\mathrm{Main}_{\mathscr{D}}(u)$ for the geometric main term of 24.3.

Theorem 6.3 (The bound at each translation)**.** *Let* $\mathscr{D}$ *be a datum of 24.12, with finite translation and top sets* $U, \mathcal{T}$, *finite nonempty word sets* $W(u)$ *for* $u \in U$, *and* $m, f \in \mathbf{N}$. *Fix arithmetic data* D, ℓ, *a cutoff* $N \in \mathbf{N}$, *sets* $TT, T_2 \subseteq A$, *and weights* $a_t = a(t)$ *and* $w_{s,j} : G \to \mathbf{R}$ *for* $t, s \in A$ *and* $0 \le j < m$. *Choose maps*

$$\mathrm{tr} : U \to A, \qquad e_T : \mathcal{T} \simeq TT, \qquad e_P : \{r \in \mathbf{N} : r < f\} \simeq \Phi.$$

For $\tau \in \mathcal{T}$, *require the geometric top weight at* τ *to equal* $a_{e_T(\tau)}$. *Write* $\omega_{u,j,r}$ *for the geometric weight at translation* u, *layer* j, *and index* r, *and require*

$$\omega_{u,j,r} = w_{\mathrm{tr}(u),j}(e_P(r)) \qquad (u \in U,\ 0 \le j < m,\ 0 \le r < f).$$

In the summand of the geometric main term of 24.3, require the two constants following $\log a_t - 1$ *to be* $-\gamma$ *and* $+\,\mathrm{Re}\,\ell_{\chi_d}$, *respectively, where* γ *is Euler's constant and* χ_d *is defined in 5.2. Assume*

$$\mathrm{ASK}_{\mathscr{D}}(u) = \sum_{t\in TT}\sum_{j=0}^{m-1} \mathrm{ASK}_D(N, a_t, t, w_{\mathrm{tr}(u),j}) \qquad (u \in U).$$

Let $\mathcal{C}$ *be the chain constants and* $X'', e_K, K \in \mathbf{R}$. *Assume* H1 *through* H11 *of 24.12 and the window hypotheses of 25.2 through its trace bound. Set* $C_{\mathrm{err}} = e_{\mathrm{Div}} + e_{\mathrm{Loss}} + 3$. *Let* x, ε *be the window parameters of* $\mathscr{D}$, b *its thick weight scale,* k *its starting rank, and* a_E *its top-weight scale, denoted by* a *in 25.2. Suppose*

$$B_0 \le 2nmbx^\varepsilon, \qquad N_c \le \frac{n_2 kbx^\varepsilon}{n(nx^{-\varepsilon})},$$
$$\log x \le x^\varepsilon, \qquad 200(1+K)^2(1+c_{HH})(1+e_K) \le x^\varepsilon,$$

where c_{HH} *is the height exponent in the chain constants* $\mathcal{C}$. *Then for every* $u \in U$,

$$\mathrm{gap}(\mathrm{tr}(u)) = \mathrm{ASK}_{\mathscr{D}}(u) - \mathrm{Main}_{\mathscr{D}}(u) \le x^{C_{\mathrm{err}}\varepsilon}\mathcal{B}.$$

Outline. By 24.12, the difference is at most $E_8(u)$ of 24.11, bounded by $x^{C_{\mathrm{err}}\varepsilon}\mathcal{B}$ via 25.2 and the B_0, N_c bounds. Reindexing via 25.3 matches the arithmetic main sum, giving gap by 5.10. The complete proof is on p. 520. □

For the family of 23.12, take $\varepsilon > 0$ sufficiently small and x above the common threshold. Its realisation in 24.1 uses $c_{HH} = C_H$ and supplies H9 of 24.12. The block Gram estimates of 23.9 pass to the word family

by the isometries J_v of 23.8. For each $u \in U$, let $\eta(u)$ and $e(u)$ be the frame parameters of $W(u)$. Apply 23.10 at the height $\mathcal{H} = x^{c_H} b$ of 24.1 to obtain

$$(\eta(u) + 4\sqrt{e(u)})\mathcal{H} \le (\bar{\eta} + 4\sqrt{\bar{e}})\mathcal{H} \le 6b.$$

For $M', K_o, C_E, c \in \mathbf{R}$, the character conditions on K_p and TT are

$$\begin{gathered} 1 \in K_p, \quad K_p \subseteq P, \quad \alpha(\epsilon_d) = 1 \ (\alpha \in K_p), \quad |P| \le M', \\ \left| \sum_{t \in TT} \alpha(0,t) - cn_T \right| \le M' x^{-1+\varepsilon} n_T \quad (\alpha \in K_p,\ \alpha|_{\{0\}\times A} \ne 1), \\ |\ell_\alpha| \le K_o \quad (\alpha \in K_p,\ \alpha \ne 1), \qquad |\ell_\alpha| \le C_E \log x \quad (\alpha \in P \setminus K_p). \end{gathered} \tag{6.4}$$

Montgomery and Vaughan give $\sum_{n \le u} \Lambda(n)/n = \log u + O(1)$ for $u \ge 2$ [35, §2.2, Theorem 2.7(a)].

Lemma 6.4 (Explicit bounds for the Mangoldt sum)**.** *For every integer $r \ge 1$,*

$$\log r - 1 \le \sum_{1 \le n \le r} \frac{\Lambda(n)}{n} \le \log r + \log 4 + 4.$$

Proof. Put $\Psi(r) = \sum_{1 \le n \le r} \Lambda(n)$. Sum $\log j = \sum_{n|j} \Lambda(n)$ for $1 \le j \le r$ to get $\log(r!) = \sum_{n \le r} \Lambda(n) \lfloor r/n \rfloor$. Thus

$$\log(r!) \le r \sum_{n \le r} \frac{\Lambda(n)}{n} \le \log(r!) + \Psi(r).$$

The exponential series gives $r^r/r! \le e^r$, and $r! \le r^r$. Rosser and Schoenfeld give $\Psi(r) < 1.03883r$ for $r > 0$ [34, Theorem 12]. Hence $\Psi(r) \le (\log 4 + 4)r$. Substitute and divide by r. □

Theorem 6.5 (Arithmetic comparison)**.** *Fix $C, C_E, K_o, c, M' \in \mathbf{R}$ and $\varepsilon > 0$, with $C \ge 0$ and $|c| \le 1$. For some x_1 depending on these constants, the following holds for $x \ge x_1$.*

Let A be a finite abelian group, $A_0 \le A$, $K_p \subseteq \widehat{G}$, and $S \subset \mathbf{N}$ finite. Let $m, N \in \mathbf{N}$ and take arithmetic data $D, \ell, TT, T_2, a_t, w_{s,j}$ as above. Assume d is prime, $2, q \in S$, and $d \notin S$. Require $I_d = \{0, \epsilon_d\}$ and $I_p = \{0\}$ for primes $p \notin S \cup \{d\}$. For every $p \in S$, require $I_p \subseteq \Phi$ and $F_p \in \Phi$.

Let $k > 0$ and $a_E \ge 0$. For real b, b', η, X', θ, require $b, b', X' > 0$, $\eta, \theta \ge 0$, and $b'X' \le xb$. Impose the weight, cost, and thin-value hypotheses of 5.10, including $a_t/w_{s,j}(\rho) \le N$ for every $s \in A_0$, $0 \le j < m$, $t \in TT$, and $\rho \in \Phi$. Require $T_2 \subseteq A_0$, (6.4), and

$$\begin{gathered} a_E \le 2xb, \qquad \tfrac{1}{2} n x^{-\varepsilon/8} \le n_T, \qquad x^{-\varepsilon} \le \delta_2, \\ x^{1/2-\varepsilon} \le m \le x^{1/2+\varepsilon/2}, \qquad xm \le k \le n^2 x^{1/2-\varepsilon}. \end{gathered} \tag{6.5}$$

If $\operatorname{gap}(s) \le x^{C\varepsilon}\mathcal{B}$ for every $s \in A_0$, then $G_m^A(x) \le x^{1/2+(C+3)\varepsilon} + \Sigma_A(x)$.

Outline. Bound increments using 5.4, 6.4, and the derivative of 2.10. Sum the bounds of 6.2 at scaled top points, use the threshold to control the error, and combine the gap bound with 5.10 to apply 25.12. The complete proof is on p. 520. □

6.3 Reading the characters on the host

Let ϕ be a frame and H a host satisfying 5.1. Use the arithmetic datum D obtained from its Artin map in the coordinates of 5.2. Let $\theta_\alpha = \alpha \circ T$, where T is the coordinate isomorphism of 5.2. For a host character θ, let $\pi_H(\theta)$ be the primitive Dirichlet character inducing $\theta \circ \mathrm{Art}$ on the units modulo $Q_S qd$, and put $\chi_\alpha = \pi_H(\theta_\alpha)$. A function ℓ is compatible if, for every $\alpha \ne 1$, each primitive Dirichlet character of positive modulus with values $V_\alpha(n)$ for all $n \ge 1$ has logarithmic derivative ℓ_α at 1.

Lemma 6.6 (Primitive representatives)**.** *The map π_H is injective and sends 1 to the primitive trivial character. If ℓ is compatible, then*

$$\ell_\alpha = \frac{L'(1,\chi_\alpha)}{L(1,\chi_\alpha)} \quad (\alpha \neq 1), \qquad G_\alpha(y) = g_{\chi_\alpha}(y) \quad (\alpha \in \widehat{G},\ y \in \mathbf{R}),$$

where $g_1 = G_z$ and g_χ for $\chi \neq 1$ is defined in 2.10.

Outline. Pull back via 5.2 and use conductor and inertia to identify primitive values and logarithmic derivatives. Artin surjectivity gives injectivity, the trivial term gives G_z, and the other formulas agree by 5.4 and 2.10. The complete proof is on p. 521. □

Lemma 6.7 (The number of surviving characters)**.** *Let A be the finite abelian group of the frame ϕ, let a_0 be its distinguished element, and put $A_0 = \langle a_0 \rangle$ and $o_\phi = |A|/\operatorname{ord}(a_0)$. Then $[A : A_0] = o_\phi$ and $|P| = 4o_\phi$. For any subgroup A_0 of a finite abelian group A,*

$$P \simeq E_0 \times \widehat{(A/A_0)}, \qquad |P| = 4[A : A_0].$$

Proof. The characters in P are constant on the fibres of $E_0 \times A \to E_0 \times (A/A_0)$, so pullback gives the isomorphism. For a finite abelian group B, $|\widehat{B}| = |B|$ [36, Theorems 7 and 9]. Apply this with $B = E_0 \times (A/A_0)$. The frame gives $o_\phi = |A|/\operatorname{ord}(a_0) = [A : A_0]$. □

Write $\mathcal{K}$ for the finite family of primitive Dirichlet characters retained at level Λ, and $o \in \mathbf{N}$ for the integer specified by the level. The family contains 1 and is closed under inversion. Its designated character χ_{sp}, when present, is a real odd member of $\mathcal{K}$. For $\chi \neq 1$ put $\kappa_\chi = L'(1,\chi)/L(1,\chi)$. Put $\kappa_1 = 0$. Set

$$c = (3o+1)^{-1}, \qquad K_H = \{\alpha : \chi_\alpha \in \mathcal{K}\}, \qquad K_o = \sum_{\chi \in \mathcal{K}} |\kappa_\chi|.$$

The retained host characters form the set $\pi_H^{-1}(\mathcal{K})$. Put $\iota_d = ((1,-1),1)$, the host element with additive coordinate ϵ_d. As in 8.19, let X_+ and X_- consist of the host characters trivial on A_0 that take the value $+1$ and -1, respectively, at ι_d. The pointwise bounds in 8.19 concern the set $\mathcal{B}_H(\mathcal{K}) = (X_+ \setminus \pi_H^{-1}(\mathcal{K})) \cup X_-$.

For $\alpha \in P \setminus K_H$, the host character θ_α is trivial on A_0 by 5.2. Since $\iota_d^2 = 1$, we have $\theta_\alpha(\iota_d) \in \{1,-1\}$. In the first case $\theta_\alpha \in X_+$ and χ_α lies outside $\mathcal{K}$ by the definition of K_H. In the second case $\theta_\alpha \in X_-$. Thus

$$\theta_\alpha \in \mathcal{B}_H(\mathcal{K}) \qquad (\alpha \in P \setminus K_H). \tag{6.6}$$

Lemma 6.8 (Coset shares give the required weights)**.** *Assume the arithmetic host conditions of 8.19 for level Λ, host H, and top set TT, with parameters $x > 0$ and $\varepsilon, C \in \mathbf{R}$. Let ℓ be compatible. Then the character conditions (6.4) hold with $K_p = K_H$, $c = (3o+1)^{-1}$, $M' = 4o$, $C_E = C$, and K_o as above.*

Outline. Identify retained characters by 8.19 and 6.6, and get $|P| = 4o$ by 6.7 and 8.19. Use quotient-character cancellation in the coset sums, the share errors in 8.19, and the bounds transferred via 6.6. The complete proof is on p. 521. □

Theorem 6.9 (The retained sum is the target sum)**.** *Let Λ be a level and suppose that H and TT satisfy the arithmetic host conditions of 8.19 for Λ at $x, \varepsilon, C \in \mathbf{R}$. Let ℓ be compatible. With $K_p = K_H$ and $c = (3o+1)^{-1}$, for every $y \in \mathbf{R}$,*

$$G_m^A(y) = G_{m,\Lambda}(y) := G_z(y) + \sum_{\chi \in \mathcal{K} \setminus \{1\}} w_\Lambda(\chi) \operatorname{Re} g_\chi(y).$$

Here $w_\Lambda(\chi) = 1$ for the designated χ_{sp} when present, and $w_\Lambda(\chi) = c$ otherwise. For a primitive Dirichlet character ψ of order o, let $\mathcal{X}$ consist of the primitive representatives of ψ^j for $0 \le j < o$. At the cyclic level of ψ, the formula gives

$$G_m(y) = G_z(y) + \frac{1}{3o+1} \sum_{\chi \in \mathcal{X} \setminus \{1\}} \operatorname{Re} g_\chi(y).$$

Outline. Use 5.2, 6.6, and 8.19 to identify K_H with $\mathcal{K}$, preserving weight 1 on the trivial term and any designated term and c elsewhere. Replace G_α by g_{χ_α} via 6.6 and reindex. The complete proof is on p. 522. □

6.4 The remaining characters and the final bound

Theorem 6.10 (The sum over the remaining characters)**.** *Under the hypotheses of 6.8,*

$$\Sigma_A(x) \le 4o\, x^{1/2+\varepsilon}, \qquad K_p = K_H.$$

Proof. For $\alpha \in P \setminus K_H$, (6.6) puts θ_α in the range of the pointwise bound of 8.19. Compatibility gives $G_\alpha(x) = g_{\chi_\alpha}(x)$ by 6.6. By 6.7 and the index condition in 8.19, $|P| = 4o$. Since $x > 0$,

$$\Sigma_A(x) = \sum_{\alpha \in P \setminus K_H} |g_{\chi_\alpha}(x)| \le |P \setminus K_H|\, x^{1/2+\varepsilon} \le 4o\, x^{1/2+\varepsilon}.$$

□

For the cyclic level of a primitive ψ of conductor M, take $0 < \varepsilon < 1/52$ and a sufficiently large window point $x \ge 1$ as in 8.20. Choose Y, q, d, e by 8.6, so that $Y \le x$ and (q, d) is a good pair in the sense of 7.24 for modulus M, exponent ε, and interval $[Y, x]$. Use these primes in the host of 8.3 and its completion in 8.20. For $\alpha \in P \setminus K_H$, 8.2 expresses χ_α on the units modulo $Q_S qd$ as a twist of a power ξ of ψ by the quadratic characters at q, d, with exponents $i, j \in \{0, 1\}$. The pair (i, j) is nonzero. Indeed, $M \mid Q_S$, so the zero pair would give $\chi_\alpha = \xi^{\mathrm{prim}} \in \mathcal{X}$ by 7.14, and hence $\alpha \in K_H$. Since $Q_S \mid \mathrm{lcm}(2^{e+2}, M)$, the twist identity holds on the units modulo $2Mqd$. The bound in 7.24 at $y = x$ gives

$$|g_{\chi_\alpha}(x)| \le x^{1/2+\varepsilon/2} \le x^{1/2+\varepsilon},$$

as in 8.4.

Definition 6.11 (Chain data at a host)**.** Let H be a raw host on a frame ϕ with group A, and let $TT \subseteq A$. We say that H and TT admit chain data at x, ε with bounds e_D, e_L, K, c_{HH}, e_K if the following requirements hold. There is a chain datum $\mathscr{D}$ with constants $\mathcal{C}$ and $X'' \in \mathbf{R}$, a translation map tr and bijections e_T, e_P as in 6.3, and a compatible ℓ. The data also include a cutoff $N \in \mathbf{N}$, a set $T_2 \subseteq A_0$, weights a, w as in 6.3, and real parameters η, X', θ. Require the window point and exponent of $\mathscr{D}$ to be x and ε, respectively. Write $c_{HH}(\mathcal{C})$ for the height exponent of this chain and require

$$e_{\mathrm{Div}} \le e_D, \qquad e_{\mathrm{Loss}} \le e_L, \qquad c_{HH}(\mathcal{C}) \le c_{HH}.$$

For the arithmetic datum of H, require the identifications, H1 through H11, and the bounds on B_0, N_c of 6.3. Impose the window hypotheses of 25.2 through its trace bound. In 6.5 use the same m, T_2, a, w, N. Take b, b' to be the thick and thin weight scales of $\mathscr{D}$, k its starting rank, and a_E its top-weight scale. Impose the weight, cost, thin-value, and cutoff conditions of 6.5 for the arithmetic datum of H, the same ℓ, and η, X', θ. The translation map covers A_0. The window hypotheses of 25.2 already give $m \le x^{1/2+\varepsilon/2}$ and $xm \le k$ in (6.5). Require the remaining five inequalities in (6.5), namely $a_E \le 2xb$, $\frac{1}{2} n x^{-\varepsilon/8} \le n_T$, $x^{-\varepsilon} \le \delta_2$, $x^{1/2-\varepsilon} \le m$, and $k \le n^2 x^{1/2-\varepsilon}$.

Theorem 6.12 (The target prime-sum bound)**.** *For every level Λ, fixed $e_D, e_L, K, c_{HH}, e_K, C \in \mathbf{R}$, and $\varepsilon > 0$, there exists $x_1 \in \mathbf{R}$ such that for all $x \ge x_1$, every frame ϕ, every host H satisfying 5.1, and every top set TT in the group of ϕ, the bound*

$$G_{m,\Lambda}(x) \le x^{1/2+(e_D+e_L+7)\varepsilon}$$

holds whenever H and TT satisfy the arithmetic host conditions of 8.19 for $\Lambda, x, \varepsilon, C$ and admit chain data in the sense of 6.11 at x, ε with bounds e_D, e_L, K, c_{HH}, e_K.

Outline. Take the chain from 6.11 and bound gaps via 25.2 and 6.3. With 5.3 and 6.8, apply 6.5, identify its left side by 6.9, and use 6.10 to bound the remaining sum, absorbing $1 + 4o$. The complete proof is on p. 522. □

Chapter 7

Choosing the auxiliary primes q and d

Fix a positive integer M, $0 < \varepsilon < 1/52$, and $A \geq 1$. For every sufficiently large Y, we choose primes q, d such that the bounds of Definition 7.24 hold throughout $[Y, Y^A]$ for every character modulo M and every $(i,j) \in \{0,1\}^2 \setminus \{(0,0)\}$. We first choose $q \equiv 3 \pmod 8$. For this q, we choose $d \equiv 7 \pmod{16}$ subject to congruences involving q and all small odd primes. At each stage we discard candidates with a large finite character sum, and candidates whose twisted L-function has a zero in a specified rectangle. A mean-square estimate controls the first set, and a zero-density estimate controls the second.

7.1 Candidate primes and the analytic inputs

Definition 7.1 (Candidate sets and finite twists)**.** For $U \in \mathbf{R}$ and $m_0, a_0 \in \mathbf{N}$, put

$$P(m_0, a_0, U) = \{p \text{ prime} : 1 \leq p \leq \lfloor U \rfloor_+,\ U^{1/2} < p,\ p \bmod m_0 = a_0 \bmod m_0\}.$$

Remainder modulo zero is the input itself. In particular $P = \varnothing$ for $U \leq 0$.

For $y, Z \in \mathbf{R}$, $q \in \mathbf{N}$, and $\kappa : \mathbf{N} \to \mathbf{C}$, define $c_y(n) = \Lambda(n) \min\{1, y/n\}$ for $n \geq 1$, and

$$S(Z, y, q, \kappa) = \sum_{1 \leq n \leq \lfloor Z \rfloor_+} c_y(n) \kappa(n) \left(\frac{n}{q}\right).$$

Here $\lfloor Z \rfloor_+ = \max\{0, \lfloor Z \rfloor\}$. All real upper limits in integer sums have this meaning. For general q, the symbol is the product of the Legendre symbols over its prime factors, counted with multiplicity, with value 1 at $q = 0$. The factor at 2 is 0 for even n and 1 for odd n. The selections use odd prime denominators.

For real α, Y, U, call (m_0, a_0, U) *admitted at* (α, Y) if

$$m_0 > 0, \qquad 8 \mid m_0, \qquad (a_0, m_0) = 1,$$
$$\frac{1}{4}(\log Y)^2 \log m_0 \leq \log U, \qquad Y^{\alpha/6} \leq \log U \leq Y^{\alpha/5} (\log Y)^2.$$

For a finite set $P \subset \mathbf{N}$ and a set $I \subseteq \mathbf{R}$, the *mean-square bound* on $P \times I$ with cutoff Z and constant C means that

$$\sum_{q \in P} |S(Z, y, q, \kappa)|^2 \leq C y \log y\, |P|$$

for every $y \in I$ and every $\kappa : \mathbf{N} \to \mathbf{C}$ with $|\kappa(n)| \leq 1$ for all n. For real arguments take $\log 0 = 0$ and $\log y = \log |y|$ for $y < 0$. For a real base y and real exponent r, use the real part of the principal complex power, with $0^0 = 1$ and $0^r = 0$ for $r \neq 0$.

For $Q \in \mathbf{R}$, let $\mathcal{R}(Q) = \{s \in \mathbf{C} : \operatorname{Re} s > 399/400,\ |\operatorname{Im} s| \leq Q\}$. Let $\mathcal{Z}(Q)$ consist of the Dirichlet characters χ of positive modulus N for which

$$\chi \neq 1, \qquad N \leq Q, \qquad L(s, \chi) \neq 0 \quad (s \in \mathcal{R}(Q)).$$

For such a nonprincipal character, use the centred sum

$$g_\chi(y) = \sum_{1 \le n \le y} \Lambda(n)\chi(n)\left(\frac{y}{n} - 1\right) + y\frac{L'}{L}(1,\chi).$$

The *analytic box* $\mathcal{B}(Q, C, \tau, Y, X)$ consists of the characters $\chi \in \mathcal{Z}(Q)$ for which $|(L'/L)(1,\chi)| \le C \log Y$ and $|g_\chi(y)| \le y^{1/2+\tau}$ for every $y \in [Y, X]$.

A primitive representative of a function $f : \mathbf{N} \to \mathbf{C}$ at a positive level L means a primitive character χ, of some positive conductor, such that $\chi(n) = f(n)$ whenever $(n, L) = 1$.

Set $z = Y^{\varepsilon/5}$ and $z_0' = z(\log Y)^2$. When a window exponent $K \ge 1$ has been fixed, the finite sums will have upper limit $Z = (Y^K)^{400}$.

Theorem 7.2 (Deuring–Heilbronn repulsion)**.** *There exist absolute constants $c_0, c_1, c_2 > 0$ and an integer $B \ge 0$ with the following property. Let $M_0 \ge 3$ be an integer, let $T \ge 2$, and put $\mathcal{L} = \log(M_0 T)$. Let χ_1, χ be characters modulo M_0. Suppose*

$$\chi_1^2 = 1, \quad \chi_1 \ne 1, \quad L(\beta_1, \chi_1) = 0, \quad 1 - \frac{c_1}{\mathcal{L}} < \beta_1 < 1.$$

If $L(\rho, \chi) = 0$, with the pole pair $(1,1)$ excluded, and

$$\operatorname{Re}\rho \ge 1 - c_0, \qquad |\operatorname{Im}\rho| \le T, \qquad (\chi, \rho) \ne (\chi_1, \beta_1),$$

then

$$1 - \operatorname{Re}\rho \ge \frac{c_2}{\mathcal{L}} \log\!\left(\frac{c_1}{(1-\beta_1)\mathcal{L}^B}\right).$$

Outline. Invoke the explicit zero-repulsion corollary of [37] at $T' = \max\{T, 4\}$ with $c_1 \le 1/10$, verify simplicity of the exceptional zero β_1 in the zero-free interval, and transfer the resulting Deuring–Heilbronn bound. The complete proof is on p. 523. □

Lemma 7.3 (The exceptional zero at a fixed modulus)**.** *There is an absolute $c > 0$ such that for every positive integer m there is at most one pair (χ, s), with χ a Dirichlet character modulo m, for which*

$$L(s,\chi) = 0, \qquad (\chi \ne 1 \text{ or } s \ne 1), \qquad \operatorname{Re} s \ge 1 - \frac{c}{\log(m(|\operatorname{Im} s| + 2))}.$$

If the pair exists, χ is real and nonprincipal, and s is a simple real zero.

Outline. McCurley's zero-free region at $k = 3m$ with $c = 1/50$ contains the stated region and allows at most one simple real zero of a real nonprincipal character. Injective induction from m to k and nonvanishing Euler factors on $\operatorname{Re} s > 0$ transfer this to modulus m. The complete proof is on p. 523. □

Theorem 7.4 (Exceptional characters across moduli)**.** *There is an absolute $c_{\mathrm{P}} > 0$ such that the following holds. Let $D \ge 3$, and let χ_i be nonprincipal real characters modulo positive integers $m_i \le D$, for $i = 1, 2$. If $L(\beta_i, \chi_i) = 0$ and $1 - c_{\mathrm{P}}/\log D < \beta_i < 1$, then*

$$\chi_1(n) = \chi_2(n) \qquad ((n, m_1 m_2) = 1).$$

Moreover, for every $\nu > 0$, there is $c_\nu > 0$ such that every real zero β of a nonprincipal real character modulo a positive integer m_0 satisfies $1 - \beta \ge c_\nu m_0^{-\nu}$.

Outline. Apply Page's exceptional-character theorem on the product modulus $m = m_1 m_2$ with constant $c_{\mathrm{P}} = c/3$ from Lemma 7.3, and descend equality of induced characters on units coprime to $m_1 m_2$ to equality of primitive representatives. The complete proof is on p. 523. □

Theorem 7.5 (Relative prime number theorem in progressions)**.** *There are absolute real constants c, A_0, C_0, with $c > 0$, such that, for every positive integer m_0 divisible by 8, one can choose either no exceptional pair or a pair (χ_1, β_1), where χ_1 is a character modulo m_0, satisfying*

$$\chi_1^2 = 1, \qquad \chi_1 \neq 1, \qquad L(\beta_1, \chi_1) = 0, \qquad 1 - \frac{1}{8 \log m_0} < \beta_1 < 1.$$

The choice of no exceptional pair means that no pair with these properties exists.

Put

$$\mathrm{li}_2(t) = \int_2^t \frac{du}{\log u}.$$

Let $\pi(t; m_0, a_0)$ count primes $p \le t$ with $p \equiv a_0 \pmod{m_0}$. For $(a_0, m_0) = 1$, define

$$H_{a_0}(t) = \begin{cases} \mathrm{li}_2(t)/\varphi(m_0), & \text{if there is no exceptional pair,} \\ \big(\mathrm{li}_2(t) - \chi_1(a_0)\,\mathrm{li}_2(t^{\beta_1})\big)/\varphi(m_0), & \text{otherwise.} \end{cases}$$

Then, whenever $t \ge 2$ and $\log t \ge C_0 \log m_0 \log(\log m_0 + 2)$, one has

$$|\pi(t; m_0, a_0) - H_{a_0}(t)| \le A_0 \left(e^{-c \log t / \log m_0} + e^{-\sqrt{c \log t}} \right) H_{a_0}(t).$$

Outline. Select any exceptional pair using the zero-free region of Lemma 7.3, and bound $\eta(m_0)$ below by Siegel's theorem. Bound $\psi(y, \chi)$ in the near range from the character estimate and in the far range from the smoothed explicit formula, zero-density bounds, and Theorem 7.2. Combine both ranges via character orthogonality, apply Abel summation, and absorb the remainders into a lower bound for $H_{a_0}(t)$. The complete proof is on p. 523. □

Lemma 7.6 (Size of every candidate set)**.** *For every $\alpha > 0$, there is $Y_0 = Y_0(\alpha)$ such that, for $Y \ge Y_0$, every progression admitted at (α, Y) satisfies $|P(m_0, a_0, U)| \ge U^{9/10}$.*

Outline. Use Theorem 7.4 to bound the exceptional subtraction in Theorem 7.5. Admissibility controls the error and leaves at least $U^{9/10}$ primes above $\sqrt{U}$. The complete proof is on p. 528. □

7.2 Quadratic cancellation and the mean square

Lemma 7.7 (One exceptional squarefree kernel)**.** *Fix $\alpha, A, B > 0$. For all sufficiently large Y, uniformly over progressions (m_0, a_0, U) admitted at (α, Y), there is $n_* \in \mathbf{N}$ such that*

$$\left| \sum_{q \in P(m_0, a_0, U)} \left(\frac{q}{s}\right) \right| \le Y^{-B} |P(m_0, a_0, U)| \tag{3}$$

for every odd squarefree integer s satisfying $1 < s \le Y^A$, $(s, m_0) = 1$, and $s \neq n_*$.

Outline. Theorem 7.4 and CRT allow at most one exceptional kernel. In Theorem 7.5, orthogonality cancels ordinary and mismatched terms, distance from 1 bounds any surviving term, and Lemma 7.6 controls the small-prime loss. The complete proof is on p. 528. □

For $N \in \mathbf{N}$, define

$$s_p(y, N) = \sum_{\substack{a \ge 1 \\ p^a \le N}} \Lambda(p^a) \min\{1, y/p^a\} \quad (p \text{ prime}),$$

and put $s_p(y, N) = 0$ for nonprime p. Let

$$R_0(y, m_0, N) = \sum_{\substack{p^a \le N \\ a \text{ even or } p \mid m_0}} \Lambda(p^a) \min\{1, y/p^a\}.$$

Each prime power in this last sum is counted once.

Lemma 7.8 (Expansion of the quadratic moment)**.** *Let P be a finite set of primes, let $8 \mid m_0$ and $(a_0, m_0) = 1$, and set $N = \lfloor Z \rfloor_+$. Suppose $y, \eta \geq 0$, $|\kappa(n)| \leq 1$ for all n, and $q \equiv a_0 \pmod{m_0}$, $q > N^2$ for every $q \in P$. Suppose $s_p(y, N) \leq B$ for every p, and suppose that for some $n_* \in \mathbf{N}$,*

$$\left| \sum_{q \in P} \left(\frac{q}{s}\right) \right| \leq \eta |P|$$

whenever

$$1 < s \leq N^2, \qquad s \text{ odd and squarefree}, \qquad (s, m_0) = 1, \qquad s \neq n_*.$$

Then

$$\sum_{q \in P} |S(Z, y, q, \kappa)|^2 \leq |P| \Big\{ \sum_{\substack{p \leq N \\ p \text{ prime}}} s_p(y, N)^2 + R_0(y, m_0, N)^2 + 2B^2 + 2BR_0(y, m_0, N) + \eta \left(\sum_{n=1}^{N} c_y(n) \right)^2 \Big\}.$$

Outline. Expand the square and use quadratic reciprocity to group pairs by the odd squarefree kernel of nn' with primes dividing m_0 removed. Count pairs with kernel 1 or the exceptional kernel separately, and apply cancellation to the rest. The complete proof is on p. 529. □

Lemma 7.9 (Prime-power coefficient estimates)**.** *There are absolute positive constants C_1, C_2, C_3, C_4 such that, for every $y \geq 3$ and $N \in \mathbf{N}$,*

$$s_p(y, N) \leq C_2 \log y \quad (p \in \mathbf{N}), \qquad \sum_{\substack{p \leq N \\ p \text{ prime}}} s_p(y, N)^2 \leq C_3 y \log y. \tag{10}$$

If $w \geq 1$ and every prime $p \mid m_0$ with $p \leq N$ satisfies $p \leq w$, then

$$R_0(y, m_0, N) \leq C_1(\sqrt{y} + w \log y). \tag{11}$$

For every $y \geq 0$ and $Z \geq 2$,

$$\sum_{1 \leq n \leq Z} c_y(n) \leq C_4 y \log Z. \tag{12}$$

One may take $C_2 = 3$.

Outline. Split prime-power sums at $p^a = y$ and bound the tails geometrically. Use Chebyshev's bound and partial summation for the squares, group even powers and primes dividing m_0 for R_0, and use Mertens's bound for the total mass. The complete proof is on p. 530. □

Theorem 7.10 (Uniform mean square over candidate primes)**.** *There is an absolute $C_{\mathrm{m}} > 0$ with the following property. Fix $0 < \alpha \leq 1$ and $K \geq 1$. For all sufficiently large Y, let (m_0, a_0, U) be admitted at (α, Y), and assume*

$$p \mid m_0,\ p \text{ prime} \quad \Longrightarrow \quad p \leq Y^{\alpha/5} \text{ or } p > (Y^K)^{400}. \tag{13}$$

Then, with $Z = (Y^K)^{400}$,

$$\sum_{q \in P(m_0, a_0, U)} |S(Z, y, q, \kappa)|^2 \leq C_{\mathrm{m}} y \log y \, |P(m_0, a_0, U)|$$

for every $Y \leq y \leq Y^K$ and every $\kappa : \mathbf{N} \to \mathbf{C}$ satisfying $|\kappa(n)| \leq 1$ for all n.

Outline. Insert the cancellation from Lemma 7.7 and the coefficient bounds from Lemma 7.9 into Lemma 7.8. Admissibility places the candidate primes above the cutoff needed for Lemma 7.8; the separately assumed gap bounds the prime divisors of m_0 that are at most Z. The complete proof is on p. 530. □

7.3 Bounds throughout a window and zero-density exclusions

Theorem 7.11 (Candidates exceeding the bound somewhere in a window)**.** *Fix $C \geq 0$, $\tau > 0$, and $\eta > 0$. There is $Y_0 = Y_0(C, \tau, \eta)$ such that, for $Y \geq Y_0$, the following holds for every real X, Z, finite sets $B \subseteq P \subset \mathbf{N}$, and function $\kappa : \mathbf{N} \to \mathbf{C}$ with $|\kappa(n)| \leq 1$ for all n. Assume the mean-square bound of Definition 7.1 on $P \times [Y, X]$ with cutoff Z and constant C. If every $q \in B$ has a point $y \in [Y, X]$ such that*

$$|S(Z, y, q, \kappa)| > \tfrac{1}{2} y^{1/2+\tau},$$

then $|B| \leq \eta |P|$.

Outline. The mean-square bound applies to $vS_q'(v)$ by restricting to $n > v$. On each dyadic block, use the fundamental theorem of calculus and Cauchy–Schwarz to bound the proportion exceeding the threshold by $O(T^{-\tau})$, then sum over the blocks. The complete proof is on p. 531. □

Theorem 7.12 (Primitive characters with a zero near 1)**.** *There is an absolute $C_{\mathrm{z}} > 0$ such that, for every $Q \geq 3$, every finite set $\mathcal{E}$ of primitive Dirichlet characters satisfying*

$$\operatorname{cond}(\chi) \leq Q, \qquad L(s, \chi) = 0 \text{ for some } s \in \mathcal{R}(Q)$$

has $|\mathcal{E}| \leq C_{\mathrm{z}} Q^{1/4}$. A character is counted once, regardless of how many zeros it has in the rectangle.

Outline. Multiply the partial sum of $L(\rho_\chi, \chi)$ by a Möbius mollifier and use partial summation to detect each zero on a dyadic block. Halász–Montgomery duality with the divisor bound and character-pair summation, summed over blocks, yields the bound. The complete proof is on p. 531. □

In the literature, the same conclusion also follows from the $4/5 \leq \sigma \leq 1$ clause of Montgomery's Theorem 12.2 [40, Theorem 12.2], which states that for $4/5 \leq \sigma \leq 1$ and $T \geq 2$,

$$\sum_{\substack{\chi \text{ primitive} \\ \operatorname{cond}(\chi) \leq Q}} N(\sigma, T, \chi) \ll (Q^2 T)^{2(1-\sigma)/\sigma} (\log QT)^{14},$$

where $N(\sigma, T, \chi)$ counts nontrivial zeros ρ with $\operatorname{Re}\rho \geq \sigma$ and $|\operatorname{Im}\rho| \leq T$. At $\sigma = 399/400$ and $T = Q$, this gives $O(Q^{6/399}(\log Q)^{14}) = O(Q^{1/4})$.

Lemma 7.13 (From bad characters to bad candidate primes)**.** *There is an absolute $C_{\mathrm{b}} \geq 0$ with the following property. Fix real numbers $Q \geq 3$ and F, and let B be a finite set of primes. Suppose that every $q \in B$ satisfies $q > F$ and admits a primitive character χ_q of conductor $N_q = f_q q$, where $f_q \in \mathbf{N}_{>0}$, $f_q \leq F$, and $N_q \leq Q$, whose L-function has a zero in $\mathcal{R}(Q)$. Then $|B| \leq C_{\mathrm{b}} Q^{1/4}$.*

Proof. Choose one witness (N_q, χ_q) for each q. The resulting map to primitive characters with their conductors is injective. Indeed, equality of two witnesses gives $f_q q = f_{q'} q'$. The prime q divides $f_{q'} q'$. If $q \mid f_{q'}$, then $q \leq f_{q'} \leq F$, contrary to $q > F$. Thus $q \mid q'$, so $q = q'$. Apply Theorem 7.12 to the image. □

Lemma 7.14 (Primitive twists and uniqueness of representatives)**.** *Let ξ be a primitive character of positive conductor f, and let q be an odd prime with $q \nmid f$. Then there is a nonprincipal primitive character χ of conductor fq such that*

$$\chi(n) = \xi(n) \left(\frac{n}{q}\right) \qquad (n \in \mathbf{N}). \tag{18}$$

Also, if primitive characters χ, χ' of positive conductors N, N' satisfy $\chi(n) = \chi'(n)$ whenever $(n, L) = 1$, for some positive integer L, then their conductors and characters agree.

Outline. The Chinese remainder theorem combines the primitive factors of coprime conductors into a nonprincipal character of conductor fq. For uniqueness, induce both representatives to $NN'L$ and descend their common character to $\gcd(N, N')$, then use primitivity. The complete proof is on p. 532. □

7.4 Recovering the centred prime sum

For a nonprincipal character χ of positive modulus, define

$$D_\chi = -\frac{L'}{L}(1,\chi), \qquad T_\chi(V) = D_\chi - \sum_{1\le n\le V} \frac{\Lambda(n)\chi(n)}{n}.$$

Lemma 7.15 (Tail estimate from a zero-free rectangle). *Fix $0 < \alpha \le 1$ and $K \ge 1$. There is $Y_0 = Y_0(\alpha, K)$ such that, for $Y \ge Y_0$, every character χ of positive modulus N satisfying*

$$Y^{\alpha/6} \le \log Q \le Y^{\alpha/4}, \qquad \chi \in \mathcal{Z}(Q)$$

obeys $\left|Y^K T_\chi((Y^K)^{400})\right| \le Y^\alpha$.

Outline. Shifts the integrated Perron contour for $\mathcal{S}_\chi(t)$ to $1 - 1/400 + \delta$ and bounds L'/L via convexity and Landau's Borel–Carathéodory lemma on the zero-free rectangle. Comparing the contour and finite-sum differences bounds $|T_\chi(V)|$. The complete proof is on p. 533. □

Lemma 7.16 (Logarithmic derivatives, including the constant 4). *There is an absolute $C_{\mathrm{L}} > 0$ such that, for every $0 < \alpha \le 1$, there is $Y_0 = Y_0(\alpha)$ such that, for every $Y \ge Y_0$ and every character χ of positive modulus,*

$$\left|\frac{L'}{L}(1,\chi)\right| \le C_{\mathrm{L}} \log Y \tag{22}$$

whenever $Y^{\alpha/6} \le \log Q \le Y^{\alpha/4}$ and $\chi \in \mathcal{Z}(Q)$.

More precisely, fix $0 < \varepsilon < 1/52$ and $A_1 > 0$. There is $Y_1 = Y_1(\varepsilon, A_1)$ such that, for $Y \ge Y_1$,

$$Y^{\varepsilon/6} \le \log Q \le A_1 Y^{\varepsilon/5} (\log Y)^2, \qquad \chi \in \mathcal{Z}(Q)$$

imply

$$\left|\frac{L'}{L}(1,\chi)\right| \le 4 \log Y. \tag{23}$$

Outline. Lemma 7.15 bounds the tail at Y^{400} and Mertens bounds the finite sum. For the explicit constant, the contour-shift tail bound established in the proof of Lemma 7.15 at cutoff $(\log Q)^{1000}$ bounds the tail, giving $1000 \log\log Q + O(1)$ and hence the $4 \log Y$ bound. The complete proof is on p. 534. □

Lemma 7.17 (From a finite sum to the analytic box). *There is an absolute $C > 0$ such that the following holds. Fix $0 < \alpha \le 1/4$, $\tau > 0$, and $K \ge 1$. For all sufficiently large Y, suppose χ is a character of positive modulus satisfying*

$$Y^{\alpha/6} \le \log Q \le Y^{\alpha/4}, \qquad \chi \in \mathcal{Z}(Q),$$

and suppose that, for some $q \in \mathbf{N}$ and $\kappa : \mathbf{N} \to \mathbf{C}$, $\chi(n) = \kappa(n)(n/q)$ for every $n \in \mathbf{N}$. If

$$|S((Y^K)^{400}, y, q, \kappa)| \le \tfrac{1}{2} y^{1/2+\tau} \qquad (Y \le y \le Y^K),$$

then $\chi \in \mathcal{B}(Q, C, \tau, Y, Y^K)$.

Outline. With $Z = (Y^K)^{400}$, splitting at $n = y$ gives $g_\chi(y) = -S(Z, y, q, \kappa) - y T_\chi(Z)$. Bounding $|y T_\chi(Z)| \le Y^\alpha \le \frac{1}{2} y^{1/2+\tau}$ via 7.15 and the triangle inequality gives the window bound on g_χ, plus the logarithmic derivative bound. The complete proof is on p. 535. □

7.5 The two selections

Theorem 7.18 (One selection stage). *There is an absolute $C > 0$ with the following property. Fix $0 < \alpha \le 1/4$, $\tau, \eta > 0$, $K \ge 1$, and $R \in \mathbf{N}$. There is $Y_0 = Y_0(\alpha, \tau, K, \eta, R)$ such that the following holds for $Y \ge Y_0$. Let (m_0, a_0, U) be admitted at (α, Y), and assume (13). Let $1 \le F \le \sqrt{U}$, and let Ξ be a finite set of primitive characters such that $|\Xi| \le R$ and $\mathrm{cond}(\xi) \le F$ for $\xi \in \Xi$. Then $P = P(m_0, a_0, U)$ is nonempty.*

Let $B \subseteq P$. Suppose that for every $q \in B$ there are $\xi \in \Xi$ and a positive integer L for which some primitive representative of $n \mapsto \xi(n)(n/q)$ at level L lies outside $\mathcal{B}(FU, C, \tau, Y, Y^K)$. Then $|B| \le \eta |P|$.

Outline. Use Lemma 7.6 and Lemma 7.13 to bound zero failures, and Theorems 7.10 and 7.11 to bound finite-sum failures over Ξ. For each remaining prime, 7.14 identifies the primitive twist and 7.17 gives the box bounds. The complete proof is on p. 535. □

Put

$$P_1(z) = P(8, 3, e^z), \qquad \mathfrak{m}(z, q) = q \prod_{\substack{3 \le p \le z \\ p \text{ prime}}} p.$$

Define

$$P_2(z, z', q) = \{d \text{ prime} : e^{z'/2} < d \le e^{z'}, \\ d \equiv 7 \pmod{16},\ \mathfrak{m}(z, q) \mid d + 1\}. \tag{29}$$

Definition 7.19 (Selection conditions for the two primes)**.** Fix a positive integer N_0 and real parameters C, τ, Y, X, z. The *first selection condition* on q requires that, for every character ξ modulo N_0, every primitive representative of $n \mapsto \xi(n)(n/q)$ at level $N_0 q$ belong to $\mathcal{B}(N_0 e^z, C, \tau, Y, X)$. For a given q and real z', the *second selection condition* on d requires that, for every character ξ modulo N_0 and each $a \in \{0, 1\}$, every primitive representative of $n \mapsto \xi(n)(n/q)^a(n/d)$ at level $N_0 qd$ belong to $\mathcal{B}(N_0 e^{z+z'}, C, \tau, Y, X)$.

Lemma 7.20 (First selection)**.** *There is an absolute $C_1 > 0$ with the following property. Fix $0 < \varepsilon < 1/52$, $\tau, \eta > 0$, $K \ge 1$, and $N_0 \in \mathbf{N}_{>0}$. There is $Y_0 = Y_0(\varepsilon, \tau, K, \eta, N_0)$ such that, for $Y \ge Y_0$, with $z = Y^{\varepsilon/5}$, the set $P_1(z)$ is nonempty. Every subset $B \subseteq P_1(z)$ whose members fail the first selection condition of Definition 7.19 on $[Y, Y^K]$ with constant C_1 satisfies $|B| \le \eta |P_1(z)|$.*

Outline. Apply Theorem 7.18 to primitive representatives modulo N_0 with $m_0 = 8$, $a_0 = 3$, $U = e^z$, and $F = N_0$. The parameters meet the hypotheses for large Y, and agreement on units modulo $N_0 q$ transfers the first selection condition. The complete proof is on p. 536. □

Lemma 7.21 (Second selection, uniformly in q)**.** *There is an absolute $C_2 > 0$ with the following property. Fix $0 < \varepsilon < 1/52$, $\tau, \eta > 0$, $K \ge 1$, and $N_0 \in \mathbf{N}_{>0}$. There is $Y_0 = Y_0(\varepsilon, \tau, K, \eta, N_0)$ such that the following holds for $Y \ge Y_0$. Put $z = Y^{\varepsilon/5}$. For every $z(\log Y)^2 \le z' \le K^2 z(\log Y)^2$ and every $q \in P_1(z)$, the set $P_2(z, z', q)$ is nonempty. Every subset $B \subseteq P_2(z, z', q)$ whose members fail the second selection condition of Definition 7.19 on $[Y, Y^K]$ with constant C_2 satisfies $|B| \le \eta |P_2(z, z', q)|$.*

Outline. Use CRT and the primorial bound to verify the progression and size conditions. Apply Theorem 7.18 to primitive representatives and their q-twists from Lemma 7.14, and transfer the second selection condition by agreement on units modulo $N_0 qd$. The complete proof is on p. 536. □

Theorem 7.22 (Proportions in the two selections)**.** *There is an absolute $C > 0$ such that, for fixed $0 < \varepsilon < 1/52$, $\tau, \eta > 0$, $K \ge 1$, and $N_0 \in \mathbf{N}_{>0}$, the following holds for all sufficiently large Y. Put $z = Y^{\varepsilon/5}$. For every real $X \le Y^K$ and $z(\log Y)^2 \le z' \le K^2 z(\log Y)^2$, there is $G_1 \subseteq P_1(z)$ such that*

$$|P_1(z)| > 0, \qquad |G_1| \ge (1 - \eta)|P_1(z)|.$$

Every $q \in G_1$ satisfies the first selection condition on $[Y, X]$ with constant C. For each such q, there is $G_2(q) \subseteq P_2(z, z', q)$ such that

$$|P_2(z, z', q)| > 0, \qquad |G_2(q)| \ge (1 - \eta)|P_2(z, z', q)|,$$

and every $d \in G_2(q)$ satisfies the second selection condition for this q on the same window with constant C.

Outline. Apply 7.20 and 7.21 using the larger of their constants and restrict from $[Y, Y^K]$ to $[Y, X]$. Defining G_1 and $G_2(q)$ as the complements of the failing candidates in $P_1(z)$ and $P_2(z, z', q)$ yields nonemptiness and both proportion bounds. The complete proof is on p. 537. □

Theorem 7.23 (A single pair satisfying both stages)**.** *There is an absolute $C > 0$ such that the following holds. Fix $0 < \varepsilon < 1/52$, $\tau > 0$, $K \geq 1$, and $N_0 \in \mathbf{N}_{>0}$. For every sufficiently large Y, every real $X \leq Y^K$, and every $z_0' \leq z' \leq K^2 z_0'$, where $z = Y^{\varepsilon/5}$ and $z_0' = z(\log Y)^2$, there are primes q, d with $N_0 < q < d$ satisfying*

$$\begin{aligned} q &\equiv 3 \pmod 8, & z/2 < \log q \leq z, \\ d &\equiv 7 \pmod{16}, & z'/2 < \log d \leq z', \\ p \mid d+1 &\quad \text{for every odd prime } p \text{ with } p \leq z \text{ or } p = q, \end{aligned} \tag{34}$$

and both selection conditions of Definition 7.19 on $[Y, X]$ with constant C.

Outline. Apply 7.22 with $\eta = 1/2$ so G_1 and $G_2(q)$ are nonempty, and pick $q \in G_1$ and $d \in G_2(q)$ to obtain the congruence, size, and box conditions. For large Y, $e^{z/2} > N_0$ and $z' \geq 2z$ ensure $N_0 < q < d$. The complete proof is on p. 537. □

7.6 The pair required by the arithmetic construction

Definition 7.24 (Window pairs and good pairs)**.** For real ε, Y, X, put $z = Y^{\varepsilon/5}$ and $z_0' = z(\log Y)^2$. A *window pair* (q, d) consists of primes satisfying (34) with $z' = z_0'$.

For $i, j \in \{0, 1\}$, put

$$f_{\xi,i,j}(n) = \xi(n) \left(\frac{n}{q}\right)^i \left(\frac{n}{d}\right)^j, \qquad H_j = 8Me^z e^{jz_0'}.$$

A window pair (q, d) is *good for modulus M on $[Y, X]$ with exponent ε* if $(M, qd) = 1$ and the following two conditions hold.

For every character ξ modulo M, every $(i, j) \in \{0, 1\}^2 \setminus \{(0, 0)\}$, and every primitive representative χ of $f_{\xi,i,j}$ at level $2Mqd$,

$$|g_\chi(y)| \leq y^{1/2+\varepsilon/2}, \qquad \left|\frac{L'}{L}(1, \chi)\right| \leq 4 \log y \quad (Y \leq y \leq X). \tag{35}$$

For every real primitive character ρ whose conductor divides $8M$, every $(i, j) \in \{0, 1\}^2 \setminus \{(0, 0)\}$, and every primitive representative χ of $f_{\rho,i,j}$ at level $2Mqd$,

$$L(s, \chi) \neq 0 \quad \big(\operatorname{Re} s > 399/400,\ |\operatorname{Im} s| \leq H_j\big). \tag{36}$$

Theorem 7.25 (Good auxiliary primes for every sufficiently large window)**.** *For every positive integer M, every $0 < \varepsilon < 1/52$, and every $A \geq 1$, there is Y_1 such that, for every $Y \geq Y_1$, there exists a good pair of primes (q, d) for modulus M on $[Y, Y^A]$ with exponent ε, in the sense of Definition 7.24.*

Outline. Use 7.23 for the window pair and second selection, and 7.14 for the twist in first selection. Induce to $8M$ for 7.24 and the centred sum in 7.1, using the boxes for zero-free rectangles and Lemma 7.16 for $4 \log y$. The complete proof is on p. 537. □

Chapter 8

Building the arithmetic host of a window

Fix a primitive Dirichlet character ψ of conductor $M \geq 1$. Write $\Xi = \langle \psi \rangle$, $o_\psi = |\Xi|$, and $\mathcal{X} = \{\xi^{\mathrm{prim}} : \xi \in \Xi\}$ for the retained family of the cyclic level $\mathcal{L}_\psi$ of Definition 2.5.

In the frame of 5.1, put $n = |A|$, $n_0 = |A_0|$, and $o_\phi = [A : A_0]$, where $A_0 = \langle a_0 \rangle$. For a scale $s = (x, \varepsilon, J)$ with $x > 1$, $0 < \varepsilon < 1/52$ and real $J \geq 1$, write $b = \exp(x^\varepsilon)$, $\delta_1 = x^{-\varepsilon/8}$ and $\varepsilon_z = \varepsilon/J$. The rank conditions on R at scale s require

$$\begin{gathered} x^{2+8\varepsilon} \leq n \leq 2x^{2+8\varepsilon}, \qquad \sqrt{x\delta_1} \leq m \leq 2\sqrt{x\delta_1}, \\ x^{9/2+10\varepsilon} \leq k \leq 2x^{9/2+10\varepsilon}. \end{gathered} \tag{8.1}$$

Here $f = 2n$, $r = k + m - 1$, and the auxiliary rank t satisfies $r + t = 8fr + 4 + T_O n_W$, with T_O, n_W as in 38.3.

8.1 The field and its characters

For a subgroup Ξ of the characters modulo M, let

$$B = (\mathbf{Z}/M\mathbf{Z})^\times / \operatorname{ann}(\Xi), \qquad \operatorname{ann}(\Xi) = \{u : \xi(u) = 1 \text{ for every } \xi \in \Xi\}.$$

Finite character duality gives $\widehat{B} = \Xi$. For $e \geq 0$, put $H_e = (\mathbf{Z}/2^{e+2}\mathbf{Z})^\times / \{\pm 1\}$ and $D_e = \operatorname{lcm}(2^{e+2}, M)$. Let $A^{(e)}$ be the image of $(\mathbf{Z}/D_e\mathbf{Z})^\times$ in $H_e \times B$. Choose $Q_S \mid D_e$ minimal among multiples of 2^{e+2} through which this map factors. In the frame at e, put $A = A^{(e)}$ and $A_0 = \ker(A^{(e)} \to B)$. Let a_0 generate A_0. For odd primes q, d coprime to Q_S and an integer a coprime to $Q_S qd$, let $\eta(a) \in A^{(e)}$ be the joint image of a, and let $w : \{\pm 1\} \to A^{(e)}$ take a sign to the joint image of that integer. Define the homomorphism $\mathrm{Art} : (\mathbf{Z}/Q_S qd\mathbf{Z})^\times \to (\{\pm 1\} \times \{\pm 1\}) \times A^{(e)}$ by

$$\mathrm{Art}(a) = \left(\left(\frac{a}{q}\right), \left(\frac{a}{d}\right), \eta(a) w\left(\left(\frac{a}{q}\right)\right) \right). \tag{8.2}$$

Lemma 8.1 (The order of the frame and the Artin kernel)**.** *For every $M \geq 1$ and subgroup Ξ of the Dirichlet characters modulo M, there are integers $1 \leq c \leq |\Xi|$ and e_0 such that $|A^{(e)}| = c2^e$ for $e \geq e_0$. At every e, $[A : A_0] = |\Xi|$.*

Take $\Xi = \langle \psi \rangle$. Suppose $e \geq 2$, the primes q, d satisfy

$$q \equiv 3 \pmod 8, \qquad d \equiv 7 \pmod{16}, \qquad \left(\frac{-d}{q}\right) = 1, \tag{8.3}$$

and $(M, qd) = 1$ and $(-d/\ell) = 1$ for every odd prime $\ell \mid M$. Then $M \mid Q_S$, and for the homomorphism Art

of (8.2) *and every integer* $a \geq 0$ *coprime to* $Q_S qd$,

$$\begin{aligned} \mathrm{Art}(a) = 1 \quad &\Longleftrightarrow \quad a \equiv 1 \text{ or } -1 \pmod{2^{e+2}}, \\ &\psi(a) = 1, \qquad \left(\frac{a}{q}\right) = \left(\frac{a}{d}\right) = 1. \end{aligned} \tag{8.4}$$

Outline. Project onto H_e and B to get the order and index, and use finiteness of B to stabilize C_e. The Artin kernel imposes both signs and the joint image, and primitivity gives $M \mid Q_S$. The complete proof is on p. 537. □

Lemma 8.2 (Characters trivial on the averaging subgroup)**.** *Under the hypotheses of the second part of 8.1, every host character* θ *trivial on* A_0 *has a representation*

$$\theta(\mathrm{Art}(a)) = \xi(a) \left(\frac{a}{q}\right)^i \left(\frac{a}{d}\right)^j, \qquad (a, Q_S qd) = 1, \tag{8.5}$$

where $\xi \in \Xi$ *and* $i, j \in \{0, 1\}$*. On the inertia generator* $\epsilon_d = ((1, -1), 1)$ *at* d*, one has* $\theta(\epsilon_d) = (-1)^j$*.*

Proof. The restriction of θ to A factors through $A/A_0 \simeq B$ and defines $\xi \in \widehat{B} = \Xi$. Factor

$$((s_q, s_d), z) = ((s_q, 1), w(s_q))\,((1, s_d), 1)\,((1, 1), z\, w(s_q)^{-1}).$$

The first two factors give the characters $s_q \mapsto s_q^i$ and $s_d \mapsto s_d^j$. Substitution into (8.2) gives (8.5). Evaluating the second sign factor at -1 gives $\theta(\epsilon_d) = (-1)^j$. □

For $B_0 \in \mathbf{N}$, impose the following field conditions at B_0. The first coordinate of $\mathrm{Art}(a)$ is (a/q), and E contains a square root of $-q$ on which each Galois automorphism acts by its first sign. For each prime $\ell \mid Q_S$ there must be a unit u modulo $Q_S qd$ with $u \equiv 1 \bmod Q_S qd/\ell$ and $\mathrm{Art}(u) \neq 1$. Require also $Q_S \leq n^2$, $o_\phi^{64} \leq 2^e$, $q \neq 3$, $(-d/p) = 1$ for odd primes $p \leq B_0$, and $4\ell \leq B_0$ for odd primes $\ell \mid Q_S$.

Theorem 8.3 (Constructing the field)**.** *For every primitive* ψ*, there are integers* $c \geq 1$ *and* e_0 *with the following property. Let* $e \geq e_0$ *and let* q, d *be primes satisfying* (8.3)*, with* $q \neq 3$ *and* $(M, qd) = 1$*. Suppose* $B_0 \geq 4M$ *and* $(-d/p) = 1$ *for every odd prime* $p \leq B_0$*. There are a frame* ϕ *at the prescribed exponent* e *and a host arithmetic datum* H *on* ϕ *with the prescribed primes* q, d*, such that* $n = c2^e$*,* $o_\phi = o_\psi$*, and*

$$Q_S \mid \mathrm{lcm}(2^{e+2}, M), \qquad Q_S \leq 4Mn. \tag{8.6}$$

The field conditions at B_0 *hold,* $M \mid Q_S$*, and the Artin kernel is given by* (8.4)*. Every host character trivial on* A_0 *has the twist representation and inertia value of Lemma 8.2. Every member of* $\mathcal{X}$ *is represented by a host character trivial on* A_0 *with value* 1 *on* ϵ_d*. The primitive representative of the* q*-sign character* $\theta_q((s_q, s_d), z) = s_q$ *lies outside* $\mathcal{X}$*.*

Outline. Use 8.1 for the frame order and take the Artin kernel's fixed field, using CRT and Gauss sums for 5.1. Apply 5.3 to factor the different, use minimality and the witness at 2 for local conditions, and use 8.1 and 8.2 for twists. The complete proof is on p. 538. □

Let X_+ and X_- denote the host characters trivial on A_0 with values 1 and -1 on ϵ_d, respectively. A character is *retained* if its primitive Dirichlet representative belongs to $\mathcal{X}$. Let $\mathcal{B}$ consist of the nonretained characters in X_+ and all characters in X_-.

Theorem 8.4 (A nonzero twist for every remaining character)**.** *Let* $\varepsilon > 0$*,* $x \geq 1$*, and* $Y \leq x$*. Suppose the twist representation and inertia value of Lemma 8.2 hold for* H*,* $M \mid Q_S$*, and every prime divisor of* Q_S *divides* $2M$*. Assume* (q, d) *is a good pair for modulus* M *on* $[Y, x]$ *with parameter* ε*, as in Definition 7.24. For every* $\theta \in \mathcal{B}$ *and every primitive Dirichlet representative* χ *of* θ*,*

$$|g_\chi(x)| \leq x^{1/2+\varepsilon}, \qquad \left|\frac{L'}{L}(1, \chi)\right| \leq 4 \log x.$$

Outline. Lemma 8.2 writes $\theta \circ \mathrm{Art}$ as a twist by $(a/q)^i(a/d)^j$, and membership in $\mathcal{B}$ rules out $(i, j) = (0, 0)$. Thus χ represents $f_{\xi,i,j}$ at level $2Mqd$, and Definition 7.24 at $y = x$ yields both bounds. The complete proof is on p. 538. □

8.2 Matching the two windows

For the local data of Definition 26.1, write $s_{p,j}(E)$ for the $\mathbf{Z}_p$-length of $\mathrm{Sha}_{p,j}(E)$. Write $V_E(t) \le B$ if, for every algebraic closure of E and every choice of those data, the lengths are finite for all primes p and $1 \le j < t$, and

$$\sum_{p \in S} \log p \sum_{j=1}^{t-1} s_{p,j}(E) \le B \quad \text{for every finite set of primes } S. \tag{8.7}$$

The field condition of Definition 26.2 requires an embedding $j : E \hookrightarrow \mathbf{Q}(\zeta_N)$ for some positive common multiple N of $2^{e+2}, M, q, d$. The automorphism $\zeta_N \mapsto \zeta_N^a$ fixes $j(E)$ pointwise exactly for the unit residues in (8.4). Call this the prescribed ψ-kernel.

For each $0 < \varepsilon < 1/52$ and page function e of 26.2, the cohomological window hypothesis of Definition 26.3 for ψ with exponent 401 gives $J \ge 1$ and X_0. If $X \ge X_0$ and (q, d) is a window pair at $X^{1/J}$ for ε in the sense of Definition 26.2, some $W \ge X^{1/J}$ satisfies $W^2 \le X$. For every $u \in [W, W^2]$ and every number field E with the prescribed ψ-kernel at $(e(u), q, d)$, we have $V_E(\lfloor u^{401} \rfloor) \le \sqrt{\exp(u^{\varepsilon})}$.

Lemma 8.5 (Rescaling a window)**.** *Let $\varepsilon > 0$, $c \ge 1$, $\lambda = c^{1/(2+8\varepsilon)}$, and $W \ge \max(1, \lambda^3, \lambda^{400})$. Put $Z = \lambda W$. For $Z \le x \le Z^{3/2}$, put $v = x/\lambda$. Then*

$$W \le v \le x \le W^2, \qquad \lfloor x^{400} \rfloor \le \lfloor v^{401} \rfloor, \qquad e^{v^{\varepsilon}/2} \le e^{x^{\varepsilon}/2}.$$

If $v^{2+8\varepsilon} \le 2^a \le 2v^{2+8\varepsilon}$, then $x^{2+8\varepsilon} \le c2^a \le 2x^{2+8\varepsilon}$.

Proof. We have $\lambda \ge 1$ and $x \ge \lambda W$, so $W \le v \le x$. Since $\lambda^3 \le W$, squaring gives $(\lambda W)^3 \le W^4$, and hence $x \le W^2$. Now $\lambda^{400} \le W \le v$ gives $x^{400} = \lambda^{400} v^{400} \le v^{401}$. Take floors. The exponential bound follows from $v \le x$. Multiply the bounds for 2^a by $\lambda^{2+8\varepsilon} = c$. □

Theorem 8.6 (A common analytic and cohomological window)**.** *Let $0 < \varepsilon < 1/52$ and $c \ge 1$. Assume the cohomological window hypothesis for ψ with exponent 401 and the pair-existence theorem 7.25. There is $J \ge 1$ such that, for every $y \in \mathbf{R}$, there is $Z \ge y$ for which every $x \in [Z, Z^{3/2}]$ admits $Y \in \mathbf{R}$ and $q, d, e \in \mathbf{N}$ satisfying*

$$\begin{gathered} x^{1/J} \le Y \le x, \qquad (q, d) \text{ is a good pair for } (M, \varepsilon) \text{ on } [Y, x], \\ x^{2+8\varepsilon} \le c2^e \le 2x^{2+8\varepsilon}. \end{gathered} \tag{8.8}$$

Every field with the prescribed ψ-kernel at (e, q, d) then satisfies

$$V_E(\lfloor x^{400} \rfloor) \le \sqrt{\exp(x^{\varepsilon})}. \tag{8.9}$$

Outline. Choose the page function by rounding up and apply 26.3 to a good pair supplied by 7.25. Use 8.5 to rescale the window, restrict the pair's interval, and transfer the bound by shortening the nonnegative sum for the data of 26.1. The complete proof is on p. 539. □

8.3 Primes for the level structure

Theorem 8.7 (A polynomial bound for a progression prime)**.** *There is an absolute constant $A \ge 1$ such that, for every integer $N \ge 2$, there is a prime p with $p \equiv 1 \pmod N$ and $p \le N^A$.*

Proof. Linnik's theorem follows from the relative prime-counting estimate of Thorner and Zaman [28, §1.2]. Apply it to the class 1 mod N. □

A coefficient family has a finite additive abelian group G and parameters $D \ge \max(4, |G|)$ and $D^{-1} \le \kappa \le 4$. Its coefficients $a_g(n), \lambda_g(n)$ are nonnegative for $g \in G$, $n \in \mathbf{N}$. Require $a_g(0) = 0$, $a_g(1) = \mathbf{1}_{g=0}$ and

$$\begin{aligned} \left| \sum_{1 \le n \le y} a_g(n) - \kappa y \right| &\le 10\sqrt{y} \quad (y \ge 1), \\ \left| \sum_{n} a_g(n) e^{-nt} - \kappa/t \right| &\le D^{10} t \quad (0 < t \le 1). \end{aligned} \tag{8.10}$$

With $\Lambda(n)$ the von Mangoldt function, require $\sum_g \lambda_g(n) \le 2\Lambda(n)$ and

$$a_g(n)\log n = \sum_{uv=n}\sum_{j\in G}\lambda_j(u)a_{g-j}(v) \qquad (n\ge 1).$$

For $\chi \in \widehat{G}$, put $a_\chi(n) = \sum_g \chi(g)a_g(n)$ and denote the trivial character by 1. For $\chi^2 = 1$, require the Dirichlet convolution $a_1 * a_\chi$ to be nonnegative and $(a_1 * a_\chi)(n^2) \ge a_1(n)$. Write $\widehat{\lambda}_\chi(n) = \sum_g \chi(g)\lambda_g(n)$, $\Psi_\chi(y) = \sum_{1\le n\le y}\widehat{\lambda}_\chi(n)$ and $\ell(t) = \log(D(|t|+2))$.

Lemma 8.8 (The possible exceptional zero)**.** *For a family as above let $F_\chi(s)$ denote the continuation to $\operatorname{Re} s > 1/2$ of $(s-1)^{\mathbf{1}_{\chi=1}}\sum_{n\ge1} a_\chi(n)n^{-s}$. There are absolute $0 < c_0 \le 1/20$, $a > 0$ and $0 < c_9 \le 1$ such that, in every family, the zeros of all the F_χ in $\operatorname{Re} s \ge 1 - c_0/\ell(\operatorname{Im} s)$ consist of at most one pair (χ,β). For this pair $\chi^2 = 1$ and β is a simple real zero with $3/4 \le \beta \le 1 - c_9 D^{-a}$.*

Outline. Derive the local logarithmic derivative by convolution and partial summation, and use trigonometric positivity and orthogonality to isolate a possible simple real zero of a real character. Theta estimates, positivity, and Cauchy's bound give the zero gap. The complete proof is on p. 539. □

Lemma 8.9 (A weighted estimate at one endpoint)**.** *Fix $0 < c_0 \le 1/20$. For any family and character χ, suppose the set Z_χ of zeros of F_χ in $\operatorname{Re} s \ge 1 - c_0/\ell(\operatorname{Im} s)$ is empty or consists of a simple real zero $\beta < 1$. For $u \ge 2$ and $\log D \le \sqrt{\log u}$,*

$$\left|\Psi_\chi(u) - \mathbf{1}_{\chi=1}u + \sum_{\beta\in Z_\chi}\frac{u^\beta}{\beta}\right| \le Cue^{-c_7\sqrt{\log u}}, \tag{8.11}$$

where $C, c_7 > 0$ depend only on c_0.

Outline. Use Chebyshev for bounded $\sqrt{\log u}$ and truncated Perron otherwise. Shift the contour using the local logarithmic derivative with the exceptional term removed, taking residues at the principal pole and any crossed exceptional zero and absorbing an uncrossed term. The complete proof is on p. 540. □

Lemma 8.10 (Principal-class weight in a short interval)**.** *There are absolute $A, c, \eta > 0$ such that for every family, $X \ge 1$, $\log D \le c\sqrt{\log X}$ and $e^{-c\sqrt{\log X}} \le h \le 1$ imply*

$$\sum_{X<n\le X(1+h)}\lambda_0(n) \ge \eta hX/D^A.$$

Outline. Use the zero region and gap from 8.8 in 8.9, apply orthogonality, and subtract at the two endpoints. Bound the exceptional integral by the gap and the endpoint errors by the hypotheses, then divide by $|G|$. The complete proof is on p. 541. □

For $d \equiv 3 \pmod 4$, call p a norm prime for (d, N) if p is prime and there exist integers u, v such that

$$4p = u^2 + dv^2, \qquad N \mid v, \qquad 2N \mid u - 2 - v. \tag{8.12}$$

Write $\mathcal{P}(d, N; X, X')$ for the norm primes for (d, N) in $(X, X']$.

Theorem 8.11 (Norm primes in a short interval)**.** *There are absolute $A, c, c' > 0$ as follows. Let $d \ge 100$ be prime with $d \equiv 3 \pmod 4$, $N \in \mathbf{N}$ with $N \ge 3$, and $X, h \in \mathbf{R}$ with*

$$X \ge 1, \qquad \log(4dN^2) \le c\sqrt{\log X}, \qquad e^{-c\sqrt{\log X}} \le h \le 1.$$

Then

$$|\mathcal{P}(d, N; X, X(1+h))| \ge \frac{c'hX}{(4dN^2)^A\log X}. \tag{8.13}$$

Outline. Use Minkowski's bound, lattice counts, Poisson summation, and ideal factorization to apply 8.10 to the principal ray class. Remove higher prime powers by Chebyshev's bound and read off the coordinates from generators congruent to 1 modulo N. The complete proof is on p. 541. □

For the separation of g_0, g_1, define

$$\mathcal{B}_r(g) = \{g\} \cup \bigcup_{j=1}^{r} \{\ell \text{ prime} : g^j < \ell^{4v_\ell(g^j-1)}\}.$$

Theorem 8.12 (Choosing the five level primes)**.** *There is an absolute $C_8 \geq 1$ such that the following holds for every $0 < \varepsilon < 1/52$ and all sufficiently large scales s. Suppose R satisfies the rank conditions of Theorem 38.3 at s, $N, d \in \mathbf{N}$, and $d \geq 100$ is prime with $d \equiv 3 \pmod 4$. Assume $N > 0$, $d \mid N$ and $\log N \leq 3x^{\varepsilon/5}(\log x)^2$. There are primes g_0, g_1, p, p', q_0 with*

$$(kb)^8 \leq g_i \leq 2(kb)^8, \quad g_i \text{ is a norm prime for } (d, N) \quad (i = 0, 1), \qquad g_0 \neq g_1, \quad \mathcal{B}_r(g_0) \cap \mathcal{B}_r(g_1) = \varnothing.$$

The primes p, p', q_0 are pairwise distinct and at most N^{C_8}. They satisfy $p \equiv p' \equiv 1 \pmod{15N}$, $q_0 \equiv 1 \pmod N$, and $p, p', q_0 \neq g_i$ for $i = 0, 1$.

Outline. Use 8.11 for a pool of norm primes and count roots modulo prime powers to exclude candidates whose sets intersect that of g_0. Apply 8.7 with moduli $15N$, $15Np$, and Npp' to get distinct smaller progression primes. The complete proof is on p. 542. □

8.4 Real characters and the thin positions

Extend the window-pair parameters of 7.24 to real Y. Put $a = \varepsilon/5$. For $Y > 0$, let $z = Y^a$. For $Y < 0$, let $z = |Y|^a \cos(\pi a)$. For $Y \neq 0$, put $z' = z(\log|Y|)^2$. At $Y = 0$, take $z = \mathbf{1}_{a=0}$ and $z' = 0$. Set $X_1(Y) = \exp z$, $X_2(Y) = \exp z'$ and $H_M(j) = 8MX_1(Y)X_2(Y)^j$ for $j \in \{0, 1\}$. Write $\mathcal{R}(T) = \{s \in \mathbf{C} : \operatorname{Re} s > 399/400, \ |\operatorname{Im} s| \leq T\}$. Let Q_{odd} be the odd part of Q_S.

Theorem 8.13 (Real-character boxes and the thin-position bound)**.** *Fix a raw host H on a frame ϕ, and denote its auxiliary primes by q, d. Let $M \geq 1$ be an integer and $\varepsilon, Y, X \in \mathbf{R}$. Suppose $Q_{\text{odd}} \mid M$ and every prime divisor of Q_S divides $2M$. If (q, d) satisfies the good-pair conditions of Definition 7.24 for (M, ε, Y, X), then every primitive real character χ of the host field whose conductor N_χ is divisible by q or d satisfies*

$$L(s, \chi) \neq 0 \quad (s \in \mathcal{R}(H_M(j))), \qquad N_\chi \leq 8Q_{\text{odd}}X_1(Y)X_2(Y)^j, \qquad j = \mathbf{1}_{d \mid N_\chi}.$$

In particular, these characters have no real zero $\beta > 399/400$.

Take $c_6 = 1$. For every $B \in \mathbf{R}$ and $0 < \varepsilon < 1/52$, there is $x_0(B, \varepsilon)$ such that the following holds for every $x \geq x_0$, every frame ϕ, and every raw host H on ϕ. Suppose $Q_{\text{odd}} \leq B$ and $\log(Q_S qd) \leq 2x^{\varepsilon/5}(\log x)^2$, and suppose every primitive real host character with conductor divisible by q or d has no real zero greater than $399/400$. Then every primitive host character χ satisfies

$$|G_\chi(y)| \leq y \exp(-c_6 x^{\varepsilon/8}) \qquad \left(y \geq \frac{\exp(x^{\varepsilon/3})}{3}\right), \tag{8.14}$$

where G_χ is as in 2.10, with $G_1 = G_z$.

Outline. Use 7.14 and 7.24 for the twist conductor and initial bounds. Combine real-zero exclusion with the classical region, bound the explicit formula's zero sum by the functional equation and zero counts, and use partial summation and 2.10 for tail decay. The complete proof is on p. 542. □

8.5 The thick set and the top set

In the multiplicative host coordinates, $\tau_2(a) = ((-1, 1), \iota(a))$, where ι identifies A with its multiplicative copy. The thick places are $\tau_2(u_0 + sa_0)$ for $s \in T_2$, with u_0 the host's offset in A.

Theorem 8.14 (A thick set without short relations)**.** *There is an absolute $C_T > 0$. Fix $0 < \varepsilon < 1/52$ and real C_o, C_L. For all sufficiently large x, let a frame ϕ and an integer $L \geq 2$ satisfy $L \leq C_L \log x$, $o_\phi \leq C_o$, and $x^{2+8\varepsilon}/2 \leq n \leq C_o x^{2+8\varepsilon}$. There is a finite set $T_2 \subset \mathbf{N}$ with $4Ls < n_0$ for $s \in T_2$, $4L|T_2| \leq n_0$, and*

$$|T_2| \geq \frac{n_0}{L} \exp\left(-C_T \sqrt{\log n_0 \log L}\right), \qquad |T_2| \geq x^{-\varepsilon} n. \tag{8.15}$$

For distinct $s_1, s_2, s_3 \in T_2$ and integers i, j with $0 < |i|, |j| \leq L$ and $i \neq j$,

$$j(s_2 - s_1) \neq i(s_3 - s_1). \tag{8.16}$$

Outline. Encode integer points on a sphere in base $4Ld$ so that a short relation forces three distinct points to be collinear. Choose the dimension and digit range for the size cutoff, then delete 0 and use the frame bounds for both counts. The complete proof is on p. 544. □

For $\gamma \in A$, take $\lambda_\gamma = 4/(3o_\phi + 1)$ on A_0 and $3/(3o_\phi + 1)$ elsewhere. The coset shares of TT have error at most ρ if

$$||TT \cap (\gamma + A_0)| - \lambda_\gamma |TT|| \leq \rho |TT| \qquad (\gamma \in A). \tag{8.17}$$

Put $X' = \exp(x^{\varepsilon/3})$, and let $h_p \in A$ be the host Frobenius coordinate for an unramified prime p. The inert-class primes of the host are those satisfying

$$p \nmid Q_S q d, \qquad \left(\frac{-d}{p}\right) = -1, \qquad \left(\frac{p}{q}\right) = -1.$$

Let $\mathcal{P}_s$ consist of these primes with $p < \lceil X' \rceil$ and $\operatorname{ord}(h_p) \geq x^{\varepsilon/4}$. Let $b_p(TT)$ count the members of TT whose h_p-orbit fails the good-orbit condition below. The required bound on these counts is

$$\sum_{p \in \mathcal{P}_s} \frac{\log p}{p} b_p(TT) \leq 4 C_b x^{-2} \delta_1 n \sum_{p \in \mathcal{P}_s} \frac{\log p}{p} \tag{8.18}$$

where TT is the top set of H.

For a selected point t, let t_0 be the first element of $t + \langle h_p \rangle$ in the frame's enumeration of A. Mark coordinate i when $t_0 + i h_p \in TT$. Define

$$H_\varepsilon = 2 + 8\varepsilon, \qquad K_\varepsilon = \lceil 8(H_\varepsilon + 2)/\varepsilon \rceil, \qquad R_\varepsilon = K_\varepsilon + 1, \qquad u = \lceil 1/\delta_1 \rceil,$$
$$\ell = (6R_\varepsilon + 1)u, \qquad M_0 = \left\lceil (H_\varepsilon + 3) \log x \, 10^{6R_\varepsilon + 1} \right\rceil, \qquad G' = (M_0 + 5)\ell,$$
$$N_1 = 1 + \lceil \max(6\mathrm{e} G' \delta_1, (H_\varepsilon + 3) \log_2 x) \rceil, \qquad L_2 = 2 \sup_{\substack{p \text{ prime} \\ p \text{ inert class}}} \left\lceil \log_p ((k x^{26})^4) \right\rceil.$$

On an orbit of length N_{orb}, the marked set must have at most K_ε points in every cyclic window of length L_2 and at most N_1 in every cyclic window of length G'. Divide the orbit into consecutive full tiles of length ℓ. A tile is marked when its odd-numbered cells of length u are empty and each even-numbered cell contains exactly one marked point, of even orbit coordinate. Every run of $M_0 + 1$ tiles, cyclically among the full tiles, must contain a marked tile.

The size, splitting and modulus conditions at Y require that q, d be primes satisfying (8.3) and

$$\sqrt{X_1(Y)} < q \leq X_1(Y), \qquad \sqrt{X_2(Y)} < d \leq X_2(Y),$$
$$\left(\frac{-d}{p}\right)_K = 1 \quad (p \text{ prime}, \ p \leq Y^{\varepsilon/5}), \qquad \log(Q_S q d) \leq 2 Y^{\varepsilon/5} (\log Y)^2.$$

The subscript K denotes the Kronecker symbol, so the splitting condition includes $p = 2$.

Theorem 8.15 (Top positions with the required shares)**.** *Fix $0 < \varepsilon < 1/52$ and $C_8 \in \mathbf{R}$. There are $C_b > 0$ and x_0 such that the following holds. Suppose R satisfies the rank conditions at $s = (x, \varepsilon, J)$ and the primes of H satisfy the size, splitting and modulus conditions at Y above. If*

$$x \geq x_0, \qquad J, o_\phi \leq C_8, \qquad Q_S \leq C_8 n, \qquad x^{1/J} \leq Y \leq x,$$

then some $TT \subset A$ satisfies $\delta_1 n/2 \leq |TT| \leq 2\delta_1 n$ and the coset share bound (8.17) with $\rho = x^{-1+\varepsilon}$. For this choice of top set, the weighted counts of points on bad orbits satisfy (8.18).

Outline. Sample independently at the two prescribed rates and use union bounds for crowded windows and independence across tiles to bound bad orbits conditional on selection. Combine the expected weighted bad count with the second moment of coset counts to choose the set. The complete proof is on p. 544. □

8.6 Completing a host over the chosen field

The pointwise bounds for the three sign characters require $|g_\chi(x)|$ to be at most $x^{1/2+\varepsilon}$ and $|(L'/L)(1,\chi)|$ to be at most $C\log x$ for every primitive character χ that agrees on units modulo $8qd$ with either sign character $(\cdot/q)$, $(\cdot/d)$ or with their product. For the six quadratic twists, take those three characters and their products with $\chi_8(n)=(2/n)_K$ of conductor 8. For each of the six, every primitive character χ that agrees with it on units modulo $8qd$ must be real, have conductor $N_\chi \le T$, and satisfy $L(s,\chi)\neq 0$ throughout $\mathcal{R}(T)$. Take $T=8X_1(Y)$ for the two twists involving q alone and $T=8X_1(Y)X_2(Y)$ for the four involving d.

Lemma 8.16 (The numerical conditions supplied by a good pair)**.** *Fix $0<\varepsilon<1/52$ and $C_0\in\mathbf{R}$. For all sufficiently large x, suppose R satisfies the rank conditions at s and H is a host on R with $J,o_\phi\le C_0$ and $Q_S\le C_0 n$. Here a host on R is a host datum of Definition 34.46 with auxiliary primes $q,d>4tr+1$, whose level set need not contain the auxiliary prime q_0. If $M\ge 1$, $x^{1/J}\le Y\le x$, and (q,d) is a good pair for (M,ε) on $[Y,x]$, then the size, splitting and modulus conditions at Y hold, the three sign characters obey the bounds with $C=4$, and the six twists obey the reality, conductor and box conditions, with $q,d>4tr+1$.*

Outline. Use 7.24 and $d\equiv 7 \pmod{16}$ for splitting, and rank and modulus bounds for the conductor budget. Identify the six twists by reciprocity and 7.14, apply 7.24 at M and $8M$ for bounds and boxes, and use the host for $q,d>4tr+1$. The complete proof is on p. 545. □

Put $N=Q_S qd$. The following conditions on a completed host at scale s use constants C_8,C_5,c_6,C_b and an auxiliary prime q_0. Require the size bounds and $\mathcal{B}_r$-disjointness of 8.12. Require the thick-set conditions of 8.14 at $L_w=10\lceil 405\log x\rceil+2\lceil 30/\varepsilon\rceil+22$, with C_T in (8.15) equal to the thickness constant chosen for the host. Require the top-set size bounds of Theorem 8.15 and the weighted orbit bound (8.18) with constant C_b. Require (8.14) with constant c_6 and the pointwise bounds for the three sign characters with $C=C_5$.

The primes p,p' must split completely in E, and all level primes must split in $K=\mathbf{Q}(\sqrt{-d})$. There must be $\pi_i\in\mathcal{O}_K$ with

$$\pi_i\overline{\pi_i}=g_i,\qquad \pi_i\equiv 1 \pmod{Q_S qd\,\mathcal{O}_E}\quad (i=0,1). \tag{8.19}$$

Require $p,p'\equiv 1 \pmod{\operatorname{lcm}(15,N)}$, $q_0\equiv 1 \pmod N$, $q_0\notin\{g_0,g_1,p,p'\}$, and require the set of level primes to be exactly $\{g_0,g_1,p,p',q_0\}$. For $\chi_d=(\cdot/d)$ require $|\operatorname{Re}(L'/L)(1,\chi_d)|\le x^{\varepsilon}$. Take $\mathfrak{c}_K=\mathcal{O}_K$, splitting bound at least $\lfloor x^{\varepsilon_z/5}\rfloor$, $J,o_\phi\le C_8$, and $Q_S\le C_8 n$. Require a single $Y\in[x^{1/J},x]$ for which the size, splitting and modulus conditions hold, the six quadratic twists satisfy the conditions preceding Lemma 8.16, and the host characters satisfy the boxes of 8.13 at height $H_{Q_{\mathrm{odd}}}(j)$ for the primitive real host characters with conductor divisible by q or d. Here $j=\mathbf{1}_{d\mid N_\chi}$ and the conductor is at most this height.

A choice record specifies a finite level set containing four distinct primes g_0,g_1,p,p', with $p,p'\equiv 1$ mod 15. Every member of the level set is coprime to Q_S. The record includes a nonzero integral ideal $\mathfrak{c}_K$ contained in no $m\mathcal{O}_K$ for an integer $m>1$. Each prime ideal factor has norm a rational prime that splits in K. The norm of $\mathfrak{c}_K$ is coprime to $2,3,5,q,d,Q_S$ and to each member of the level set. The remaining entries are finite sets $TT\subseteq A$ and $T_2\subseteq\mathbf{N}$, an offset $u_0\in A$, an integer $L_w\ge 2$ and a real constant C_{T_2}. They satisfy $4L_w s<n_0$ for every $s\in T_2$ and (8.16) with $L=L_w$.

Call the ranks R admissible for H_0 if the following holds for every $B_0\in\mathbf{N}$ and every choice of the data just described. Whenever H_0 satisfies the field conditions at B_0 preceding Theorem 8.3 and $q,d>4tr+1$, there is a host on R extending H_0 with those choices and splitting bound B_0.

Theorem 8.17 (Completing a fixed arithmetic datum)**.** *Fix $0<\varepsilon<1/52$ and $C_0\ge 1$. There are real constants C_{T_2},C_8,c_6,C_b,x_0, with $C_8\ge C_0$, $c_6>0$ and $C_b>0$, with the following property. Let R satisfy the rank conditions at $s=(x,\varepsilon,J)$ and be admissible for an arithmetic datum H_0 on its frame. Let $M\ge 1$ be an integer and $Y\in\mathbf{R}$. Suppose (q,d) is a good pair for (M,ε) on $[Y,x]$, the field conditions on H_0 hold*

at $B_0 = \lfloor Y^{\varepsilon/5} \rfloor$, *and*

$$x \geq x_0, \qquad J, o_\phi \leq C_0, \qquad Q_S \leq C_0 n, \qquad M \leq C_0,$$
$$Q_S \mid \operatorname{lcm}(2^{e+2}, M), \qquad x^{1/J} \leq Y \leq x.$$

There are a host H *extending* H_0 *with thickness constant* C_{T_2} *and a prime* q_0 *satisfying the completed-host conditions with constants* $C_8, 4, c_6, C_b$. *Its top set satisfies* (8.17) *with* $\rho = x^{-1+\varepsilon}$.

Outline. Bound $\log N$ via Lemma 8.16, then use Theorems 8.12 and 8.14 with admissibility to form a host. Theorem 8.15 supplies the top set, and Lemma 8.16 with Theorem 8.13 verifies the character bounds. The complete proof is on p. 545. □

Theorem 8.18 (Admissible ranks on a prescribed frame)**.** *For every* $0 < \varepsilon < 1/52$ *there is* x_0 *such that for* $s = (x, \varepsilon, J)$ *with* $x \geq x_0$, *a frame* ϕ *satisfying* $x^{2+8\varepsilon} \leq |A| \leq 2x^{2+8\varepsilon}$, *and any arithmetic datum* H_0 *on* ϕ, *there is a rank datum* R *on* ϕ *satisfying the rank conditions at* s *and admissible for* H_0.

Outline. Choose ranks by 38.3, choose a CM embedding using the field data, and identify $\mathrm{Gal}(E/K)$ by the trivial d-sign. Choose one prime of K per conjugate pair to factor the different for 8.3 and set the splitting bound to B_0. The complete proof is on p. 546. □

8.7 The arithmetic conditions and the window supply

Definition 8.19 (The arithmetic requirement at a host)**.** Let $\mathcal{L}$ be a level, ϕ a frame, and H a raw host on ϕ. Take $TT \subset A$ and $x, \varepsilon, C \in \mathbf{R}$. Let $S = \mathcal{X}_{\mathcal{L}}$ be the retained family of Definition 2.5, and let $K_H(S)$ be the set of host characters whose primitive representative belongs to S. Set $\mathcal{B}_H(S) = (X_+ \setminus K_H(S)) \cup X_-$. The arithmetic requirement for $\mathcal{L}$ at H consists of the following conditions. The index $[A : A_0]$ equals the level parameter $o_{\mathcal{L}}$, every $\chi \in S$ has a representative in X_+, and TT satisfies (8.17) with $\rho = x^{-1+\varepsilon}$. If the level designates a character, it must be the primitive representative of the q-sign character θ_q. If the level has no designated character, the primitive representative of θ_q must lie outside S. For every $\theta \in \mathcal{B}_H(S)$ and every primitive representative χ of θ, require $|g_\chi(x)| \leq x^{1/2+\varepsilon}$ and $|(L'/L)(1, \chi)| \leq C \log x$.

An arithmetic datum H_0 with field E satisfies the cyclic host requirement for ψ at scale s, top set TT and auxiliary rank t if its underlying field and Artin data satisfy the arithmetic requirement above for $\mathcal{L}_\psi$ with $C = 4$, and $V_E(t) \leq \sqrt{b}$.

Say that hosts satisfying a condition P exist on windows of exponent λ if for each $0 < \varepsilon < 1/52$ there are constants $C_{T_2}, C_8, C_5, c_6, C_b, x_0$, with $C_8 \geq 1$ and $C_5, c_6, C_b > 0$, a number $J \geq 1$, and a set $\mathcal{W} \subset \mathbf{R}$ such that

$$\forall y \in \mathbf{R}\ \exists Z \geq y, \qquad [Z, Z^\lambda] \subset \mathcal{W}. \tag{8.20}$$

For every scale $s = (x, \varepsilon, J)$ with $x \geq x_0$ in $\mathcal{W}$, there must be ranks R satisfying the rank conditions at s, a host H with thickness constant C_{T_2}, and a prime q_0 satisfying the completed-host conditions of Theorem 8.17 with constants C_8, C_5, c_6, C_b and the additional condition $P(C_5, s, R, H, q_0)$.

Theorem 8.20 (Assembling the data on one window)**.** *Fix a primitive* ψ *and assume the cohomological window hypothesis of Definition 26.3 for* ψ *with exponent* 401. *For every* $0 < \varepsilon < 1/52$ *assume the completion assertion of 8.17 for every* $C_0 \geq 1$. *Suppose also that for each* $0 < \varepsilon < 1/52$ *there is* $x_0(\varepsilon)$ *such that, at every scale* $s = (x, \varepsilon, J)$ *with* $x \geq x_0(\varepsilon)$, *every datum* H_0 *with* $M \mid Q_S$ *and* (8.4), *on a frame with*

$$x^{2+8\varepsilon} \leq n \leq 2x^{2+8\varepsilon}, \qquad o_\phi = o_\psi,$$

admits ranks on that frame satisfying the rank conditions at s *and admissible for* H_0. *Then hosts satisfying the cyclic host requirement of Definition 8.19 for* ψ *exist on windows of exponent* $3/2$, *with* $C_5 = 4$. *By 8.17, the conclusion holds under the window and rank-supply hypotheses.*

Outline. Theorem 8.6 with 7.25 supplies windows with good pairs, and Theorem 8.3 builds the frame and datum H_0. Rank supply, completion (via Theorem 8.17), and Theorem 8.4 then verify Definition 8.19. The complete proof is on p. 546. □

Theorem 8.21 (Arithmetic hosts on arbitrarily distant windows)**.** *Let ψ be a primitive Dirichlet character satisfying the cohomological window hypothesis of Definition 26.3 with exponent* 401*. For each $0 < \varepsilon < 1/52$, one choice of $J \geq 1$ and of the host constants works on arbitrarily distant intervals $[Z, Z^{3/2}]$. Every x in those intervals admits ranks satisfying the rank conditions at (x, ε, J) and a completed host with the chosen constants, $C_5 = 4$, and the cyclic host requirement of Definition 8.19 for its arithmetic datum, top set and auxiliary rank.*

Outline. Apply Theorem 8.20 with rank supply from Theorem 8.18, good pairs from Theorems 7.25 and 8.6, and completion from Theorem 8.17. The hosts inherit the character bounds of Lemma 8.16 and Theorems 8.13 and 8.4. The complete proof is on p. 547. □

Part III

Divisibility from one theta lift

Chapter 9

Background: the Weil representation and theta lifts

9.1 Lattice theta series

Theorem 9.1 (Weighted theta series)**.** *Let L be a positive definite even lattice of rank d, with bilinear form $(\ ,\ )$ and dual $L^\vee$. Let N_L be its level, the least positive integer such that $N_L(y,y)/2 \in \mathbf{Z}$ for every $y \in L^\vee$. If P is a homogeneous harmonic polynomial of degree j on $L \otimes \mathbf{R}$, then*

$$\theta_{L,P}(z) = \sum_{x \in L} P(x) \exp\bigl(\pi i z (x,x)\bigr), \qquad \operatorname{Im} z > 0,$$

is a holomorphic modular form of weight $d/2 + j$ on $\Gamma_0(N_L)$, with the lattice character and, in odd rank, the half-integral weight multiplier. It is cuspidal if $j > 0$. For even d the character is the Kronecker symbol $\bigl(\frac{(-1)^{d/2} \det L}{\cdot}\bigr)$, where $\det L$ is the Gram determinant. [44, Théorème, pp. 20-01–20-02][45, p. 519, equation (16)]

Outline. Check the harmonic differential equation $(E - \lambda)p = \Delta_Q p/(4\pi)$ for the weighted polynomial p, deduce the $\Gamma_0(N_L)$ modularity law from the theta transformation criterion, and verify holomorphy and cuspidality at every cusp for $j > 0$. The complete proof is on p. 547. □

The transformation $z \mapsto z + 1$ uses the evenness of L. For $z \mapsto -1/z$, Poisson summation on the cosets of L in $L^\vee$ gives the finite Fourier transform on $L^\vee/L$. Harmonicity makes the Fourier transform of the polynomial times the Gaussian another Gaussian times the same polynomial, with the extra factor of degree j in the weight. The coset series have nonnegative exponents, and their constant terms vanish for $j > 0$ because $P(0) = 0$. Summing an acted-on Schwartz function over rational points is the corresponding construction in the Weil representation.

9.2 The oscillator representation

Let k be a local field of characteristic zero and S a symplectic k-space with form $\langle\ ,\ \rangle$. Its Heisenberg group is $H(S) = S \times k$, with multiplication

$$(s,t)(s',t') = (s + s', t + t' + \tfrac{1}{2}\langle s, s' \rangle).$$

Fix a nontrivial unitary additive character ψ_0 of k.

Theorem 9.2 (Stone–von Neumann and Weil)**.** *There is, up to unitary equivalence, a unique irreducible unitary representation ρ_{ψ_0} of $H(S)$ with central character ψ_0. For $S = X \oplus X^\vee$ with $\langle (x,\xi),(y,\eta) \rangle = \eta(x) - \xi(y)$, its Schrödinger realization on $L^2(X)$ is*

$$(\rho_{\psi_0}(x,\xi;t)\phi)(u) = \psi_0\bigl(t + \xi(u) + \tfrac{1}{2}\xi(x)\bigr)\phi(u + x).$$

For each $g \in \mathrm{Sp}(S)$ a unitary intertwiner carrying $\rho_{\psi_0}(s,t)$ to $\rho_{\psi_0}(gs,t)$ is unique up to a scalar of modulus one. These intertwiners give a projective representation of $\mathrm{Sp}(S)$ and a genuine representation ω_{ψ_0} of its metaplectic double cover $\mathrm{Mp}(S)$. The nontrivial element of the covering kernel acts by -1. The operators preserve the Schwartz space $\mathcal{S}(X)$, with Schwartz–Bruhat functions at finite places. [47, 51]

Theorem 9.3 (Schrödinger operators)**.** *Use the preceding polarization. For a symmetric map $b : X \to X^\vee$, let $n_b(x,\xi) = (x, \xi - bx)$. For $a \in \mathrm{GL}(X)$ let $m_a(x,\xi) = (ax, \xi \circ a^{-1})$. These give the unipotent radical and Levi of the Siegel parabolic stabilizing $X^\vee$. In dual bases let $w(x,\xi) = (\xi, -x)$. The operators for these elements, with dv self-dual for ψ_0 and the coordinate pairing, are*

$$\omega_{\psi_0}(n_b)\phi(u) = \psi_0\big(\tfrac{1}{2}(bu)(u)\big)\phi(u),$$
$$\omega_{\psi_0}(\widetilde{m}_a)\phi(u) = c_{\psi_0}(\widetilde{m}_a)|\det a|_k^{-1/2}\phi(a^{-1}u),$$
$$\omega_{\psi_0}(\widetilde{w})\phi(u) = c_{\psi_0}(\widetilde{w})\int_X \phi(v)\psi_0(u\cdot v)\,dv.$$

The first line uses the canonical unipotent splitting. The scalars in the other two lines are the Weil-index factors for the chosen lifts. Changing a lift changes its scalar by a sign. [47, 51]

9.3 Unitary pairs and local theta lifting

A reductive dual pair consists of reductive subgroups that are mutual centralizers in a symplectic group. Let E/F be a CM extension with involution $z \mapsto \bar{z}$. For a Hermitian space V and a skew-Hermitian space W over E, the trace to F of the product of the two forms defines a symplectic form on $S = V \otimes_E W$. The tensor actions give the dual pair $(\mathrm{U}(V), \mathrm{U}(W))$; here U denotes the isometry group.

Theorem 9.4 (Splitting and local Howe duality)**.** *At a place v of F, let η_v be the quadratic character of $F_v^\times$ for E_v/F_v, trivial at a split place. Choose unitary characters χ_V, χ_W of $E_v^\times$ with restrictions $\eta_v^{\dim_E V}$ and $\eta_v^{\dim_E W}$ to $F_v^\times$, respectively. They give splittings over the two unitary groups in the circle extension obtained from the metaplectic cover by extending its central kernel $\{\pm 1\}$ to the unit circle. The splitting over $\mathrm{U}(V)$ uses χ_W, and that over $\mathrm{U}(W)$ uses χ_V. The resulting Weil representation is a representation of the product of the two unitary groups. [29, 50]*

For an irreducible admissible representation π of $\mathrm{U}(V)(F_v)$, the maximal quotient of this Weil representation on which $\mathrm{U}(V)(F_v)$ acts by copies of π has the form $\pi \otimes \Theta(\pi)$. At archimedean places this is understood on Harish-Chandra modules. If $\Theta(\pi) \neq 0$, its maximal semisimple quotient $\theta(\pi)$ is irreducible. Distinct π with nonzero small lifts have distinct small lifts. For the fixed pair and splitting data the correspondence is symmetric in the two groups. [6, 49, 50, 52]

Howe established the archimedean duality. Over nonarchimedean fields Waldspurger proved it in odd residue characteristic, and Gan and Takeda proved it for symplectic–orthogonal and unitary pairs over quadratic field extensions in arbitrary residue characteristic. At a split place the unitary pair becomes a general linear pair. At finite split places Howe duality in every residue characteristic follows from Mínguez, *Correspondance de Howe explicite : paires duales de type II*, Corollary 6.3. Definite real unitary groups are compact, and their types can be read in the Fock realization of the same Weil representation.

9.4 Adelic theta lifts and their periods

Write $\mathbb{A} = \mathbb{A}_F$ and $[G] = G(F)\backslash G(\mathbb{A})$. Choose a nontrivial character of $\mathbb{A}/F$ and global splitting characters on $E^\times\backslash\mathbb{A}_E^\times$ with the prescribed quadratic restrictions. With these data the restricted tensor product of the local Weil representations and the rational splitting give the kernel

$$K_\phi(g,h) = \sum_{x \in X(F)} (\omega(g,h)\phi)(x), \qquad \phi \in \mathcal{S}(X(\mathbb{A})),$$

where X is a rational Lagrangian in the ambient symplectic space. It is automorphic in g and h. For a cusp form f on $[G]$, the lift to the other group is the automorphic function

$$T_\phi f(h) = \int_{[G]} K_\phi(g,h)\overline{f(g)}\,dg.$$

The integral is absolutely convergent. For an irreducible cuspidal representation π, its lift is the span as $f \in \pi$ and ϕ vary. [47, 52]

Theorem 9.5 (Tower property)**.** *Fix the source and a target Witt tower $V_r = V_0 \oplus \mathbb{H}_E^r$, with V_0 anisotropic and $\mathbb{H}_E$ a hyperbolic plane. Use compatible splitting data. For $1 \le j \le r$, the constant terms along the parabolic stabilizing a j-dimensional isotropic subspace, restricted to the $\mathrm{U}(V_{r-j})$ factor of its Levi, span the lift to $\mathrm{U}(V_{r-j})$. The test function for this constant term is the restriction at zero in the variables of the removed isotropic part in a mixed Schrödinger model. Thus the first nonzero lift in the tower is cuspidal, and the lifts above it are nonzero. On an anisotropic target every automorphic lift is cuspidal. [53, 52, 57]*

Definition 9.6 (The doubled section condition)**.** For the norm of a lift from $\mathrm{U}(A)$ to $\mathrm{U}(B)$, the doubled group is $\mathrm{U}(A \oplus A^-)$, where A^- has the negative form. Its diagonal isotropic subspace has dimension $a = \dim_E A$. Write $b = \dim_E B$, and let r_B be the rational Witt index of B. The doubled Weil representation pairs this split group with $\mathrm{U}(B)$. For this fundamental-domain formula use the lattice realization of the oscillator representation. Write ψ_0 for the chosen character of $\mathbb{A}_F/F$. The lattice vectors satisfy $\phi(\gamma + z) = \psi_0(-\langle\gamma, z\rangle/2)\phi(z)$ for rational γ. In the convergent assertions below the theta kernel of a lattice vector is $K_\phi(g,h) = (\omega(g,h)\phi)(0)$. In Schrödinger coordinates this is the rational-point sum above. The antidiagonal is isotropic, so $w \mapsto \phi(w,-w)$ is periodic under the rational lattice. Let $\mathcal{F}$ be a fundamental domain for that lattice in the undoubled matrix space, with additive Haar measure ν. Put $\mathrm{ev}_0(\phi) = \int_{\mathcal{F}} \phi(w,-w)\,d\nu(w)$. The section at $s_0 = (b-a)/2$ is $f_\phi(g) = \mathrm{ev}_0(\omega_d(g,1)\phi)$. Denote its flat extension by $F_\phi(s)$. When the translates of ϕ under the standard maximal compact subgroup span a finite-dimensional space, $F_\phi(s)$ belongs to normalized Siegel induction from $\chi_B(\det t)^{-1}|\det t|^s_{\mathbb{A}_E}$, where t is the action of the Levi element on the diagonal isotropic subspace. The unnormalized exponent is $s + a/2$. The section condition requires, for every allowed doubled oscillator test, absolute integrability in the definition of ev_0, invariance of this functional under $\omega_d(1,h)$ for $h \in \mathrm{U}(B)(\mathbb{A})$, and smoothness of f_ϕ. For every element p_S of the Siegel parabolic with Levi coordinate t it requires $f_\phi(p_S g) = \chi_B(\det t)^{-1}|\det t|^{s_0+a/2}_{\mathbb{A}_E} f_\phi(g)$. For a vector finite under a chosen standard maximal compact subgroup it also requires membership in the indicated induced representation. Let $\mathcal{E}(g,s,F_\phi)$ be the Eisenstein series of this section. In the definite Schrödinger model of Theorem 34.16, use its functional l to form the section.

Theorem 9.7 (Siegel–Weil in the convergent and definite ranges)**.** *For the convergent assertion, take A, B over a common arithmetic point as in Theorem 34.17, with compatible doubled Weil data and an Iwasawa decomposition for the chosen standard maximal compact subgroup. Choose archimedean Cartan involutions on $A \oplus A^-$ and B, with the involution on $A \oplus A^-$ commuting with the archimedean action of that subgroup. Let ϕ be a doubled Schwartz vector. Suppose $a < b \le 2a$ and $b - r_B > a$. Suppose there is an integer $N \ge 0$ such that every product of N archimedean Fock annihilation operators for these involutions kills ϕ. Then the doubled theta integral converges absolutely and $\mathcal{E}(g,s,F_\phi)$ is holomorphic at $s = s_0$. With the rational additive covolume equal to one,*

$$\mathcal{E}(g,s_0,F_\phi) = \frac{1}{\mathrm{vol}[\mathrm{U}(B)]}\int_{[\mathrm{U}(B)]} K_\phi(g,h)\,dh,$$

where the kernel uses the doubled Weil representation [48, 56]. For the definite assertion, take $a, b \ge 1$ and use the good base, opposite-sign model conditions, doubling frame, standard maximal compact subgroup, Iwasawa height, and compatible doubling data of Theorem 34.16, with B as the space of common definite signature and A as the space being doubled. Suppose the translates of the doubled Schwartz vector ϕ under this compact subgroup span a finite-dimensional space. The Eisenstein series is holomorphic at s_0. Its value is the displayed average if $a < b$ and twice that average if $b \le a$ [57, Theorem 7.1 and §3.1].

Weil proved the first identity in the range $b > 2a$, and Yamana established it throughout the convergent range $b - r_B > a$ [56, Theorem 2]; this identity is used in 34.17 with the additional bound $b \le 2a$. For definite

B and $1 \le b \le a$, the point $s_0 = (b-a)/2 \le 0$ lies in the first-term range when $b < a$ and at the boundary when $b = a$; this case is the anisotropic unitary Siegel–Weil identity of Gan, Qiu and Takeda [57, Theorem 7.1 and §3.1] and is restated in Theorem 34.16. In general divergent ranges the regularized Siegel–Weil formula relates Laurent coefficients of the Siegel Eisenstein series to those of the regularized theta integral; Kudla and Rallis established the regularized first-term identity for orthogonal–symplectic pairs, and Gan, Qiu and Takeda proved the second-term identity and inner product formula including unitary pairs [55, 57].

Theorem 9.8 (Rallis inner product in the convergent unitary case)**.** *Use the compatible Weil, Iwasawa, Cartan, covolume-one and source Haar data of the inner-product assertion of Theorem 34.17, and take a Haar measure on* $\mathrm{U}(B)(\mathbb{A})$*. Suppose* $a < b \le 2a$*,* $b - r_B > a$*, and all lower lifts of the irreducible cuspidal representation* π *of* $\mathrm{U}(A)(\mathbb{A})$ *in the target Witt tower vanish on the specified smooth forms. Take* f_1, f_2 *in that smooth cuspidal realization and allowed oscillator tests* ϕ_1, ϕ_2 *as in Theorem 34.17, whose doubled vector* δ *is* K*-finite. In the lattice realization,* $\delta(x_1, x_2) = \phi_1(x_1)\overline{\phi_2(-x_2)}$*, so* $\mathrm{ev}_0(\delta) = \int_{\mathcal{F}} \phi_1(w)\overline{\phi_2(w)}\, d\nu(w)$*. To compare with Schrödinger coordinates, choose a rational polarization* $X \oplus X^\vee$*. The lattice transform is*

$$(\mathcal{Z}f)(x,\xi) = \psi_0(\langle x,\xi\rangle/2) \sum_{\lambda \in X(F)} f(x+\lambda)\psi_0(\langle \lambda,\xi\rangle).$$

Orthogonality on $X^\vee(\mathbb{A})/X^\vee(F)$*, followed by unfolding in* x*, identifies its fundamental-domain pairing with the full-space Schrödinger pairing. The normalized doubling polarization map carries the lattice tensor* δ *to the doubled Schrödinger vector whose value at zero is this pairing. Thus the fundamental-domain formula applies to the lattice vectors; in Schrödinger product coordinates the corresponding pairing is the integral over the full Lagrangian. For the doubling embedding* ι *use the convention in which the doubled kernel at* $\iota(g_1, g_2)$ *has the factor* $\chi_B(\det g_2)^{-1}$ *times the product of the first kernel and the conjugate of the second. Set*

$$Z(s) = \int_{[\mathrm{U}(A)]^2} \mathcal{E}(\iota(g_1,g_2), s, F_\delta)\, \overline{f_1(g_1)} f_2(g_2) \chi_B(\det g_2)\, dg_1\, dg_2,$$

initially in a right half-plane. Both lifts belong to $L^2([\mathrm{U}(B)])$*, and the constant Laurent coefficient of the continued integral satisfies*

$$\mathrm{CT}_{s=s_0}\, Z(s) = \frac{\langle T_{\phi_1} f_1, T_{\phi_2} f_2\rangle_{[\mathrm{U}(B)]}}{\mathrm{vol}[\mathrm{U}(B)]}.$$

[54, 57]

Theorem 9.9 (Seesaw identity)**.** *Let* (G_1, G_2) *and* (H_1, H_2) *be dual pairs with* $H_1 \subseteq G_1$ *and* $G_2 \subseteq H_2$*. Use compatible Weil data so that the two kernels restrict to a common kernel on* $[H_1] \times [G_2]$*. For forms* f *on* $[G_2]$ *and* u *on* $[H_1]$*, suppose the common kernel times* $\overline{f}\,\overline{u}$ *is absolutely integrable on that product. With the lift convention above,*

$$\int_{[H_1]} (T_{G_2 \to G_1} f)(h)\, \overline{u(h)}\, dh = \int_{[G_2]} (T_{H_1 \to H_2} u)(g)\, \overline{f(g)}\, dg.$$

The two sides use the same test vector under the common kernel identification. For background on the theta correspondence, see [46].

Proof. Write $K(h,g)$ for the common restriction of the two theta kernels to $[H_1]\times[G_2]$. By the lift convention, the left side is

$$\int_{[H_1]} \int_{[G_2]} K(h,g) \overline{f(g)}\, \overline{u(h)}\, dg\, dh.$$

The assumed absolute integrability permits Fubini's theorem. Reversing the order of integration gives

$$\int_{[G_2]} \left(\int_{[H_1]} K(h,g) \overline{u(h)}\, dh \right) \overline{f(g)}\, dg.$$

The inner integral is $(T_{H_1 \to H_2} u)(g)$, because the common kernel uses the same test vector and compatible Weil data. This is the right side of the stated identity. □

9.5 Use in the proof

Part III uses the Weil model for the dependence on a lattice indicator in 12.10 and the Hecke relation in 12.18. Taylor expansion of the corresponding theta sections in formal-group coordinates writes the local kernels as $\exp(-\Phi)U$ with the CM pairing phase and a factor integral after logarithmic substitution, as in 10.6 and 37.31. Part IV turns these kernel coefficients into lift bounds in 16.10 and whole-column bounds in 16.7. In Part VIII, Poisson summation reappears in 33.111, lower lifts and local matching enter 34.7 and 34.37, and the doubled identities give 34.16 and 34.17. The seesaw in 34.25 exchanges a lift against a period and supplies test functionals that separate the rows.

Chapter 10

The local kernel and the differential operator

Fix a prime p. For a multi-index n, write $|n| = \sum_x n_x$ and $n! = \prod_x n_x!$. Write $[Z^n]f$ for the coefficient of Z^n in f. Each series substituted for an indeterminate has zero constant term.

10.1 Local data and weight diagrams

Definition 10.1 (The Weyl weight lattice $H_j(L)$)**.** Let V be a finite set of places, and let $\iota : O \to K$ be an injective homomorphism from a discrete valuation ring O into a field K. Fix integers $k_1, t \in \mathbf{N}$ and put $N_j = k_1 + 1 + j(t+1)$ for $j \geq 0$. At each $v \in V$, let $n'_v \geq 2$ be even, let W_v be a discrete valuation ring with fraction field Q_v, and let σ_v be an automorphism of Q_v preserving W_v ($\sigma_v(W_v) = W_v$). Take an embedding $\iota_v : Q_v \to K$ and a ring map $j_v : W_v \to O$ satisfying $\iota_v(x) = \iota(j_v(x))$ for $x \in W_v$. Require j_v to be finite: O is a finitely generated W_v-module through j_v. For a local system at a prime p, fix $w \in O$ and require at each v that $p \geq 5$, that p generates the maximal ideal of W_v, that the residue field W_v/pW_v has $p^{n'_v}$ elements, that W_v is p-adically complete, that $\sigma_v(x) \equiv x^p \pmod{pW_v}$ for $x \in W_v$, and that $\sigma_v^{n'_v} = 1$. When $V \neq \varnothing$, require also that K is the fraction field of O, that O is complete, and that $w^2 = p$. Index the n'_v embeddings $\iota_v \circ \sigma_v^s$ by cyclic positions $s \in \{0, \ldots, n'_v - 1\}$. Fix nonincreasing initial weights $\lambda_{v,s} = (\lambda_{v,s,1} \geq \cdots \geq \lambda_{v,s,k_1+1}) \in \mathbf{Z}^{k_1+1}$ and shifts $w_{v,l}(s) \in \mathbf{Z}$ ($l \geq 1$) with $w_{v,1}(s) \geq -\lambda_{v,s,k_1+1}$ and $w_{v,l}(s) \leq w_{v,l+1}(s)$.

At level $j \geq 0$ and block (v, s), append $t+1$ copies of $-w_{v,b}(s)$ to $\lambda_{v,s}$ for each $b = 1, \ldots, j$ in order; if s is odd, reverse the resulting list of length N_j and negate its entries. Denote the last entry of the resulting nonincreasing list by $d_{j,v,s}$, and subtract $d_{j,v,s}$ from every entry to form a partition $\kappa_{j,v,s}$. In an $N_j \times N_j$ matrix $Y_{v,s}$ of indeterminates, let $H_{\kappa_{j,v,s}}(Y_{v,s})$ be the product of the initial principal minors of the column heights of $\kappa_{j,v,s}$. For rows $a, b \in \{1, \ldots, N_j\}$ and $r \geq 0$, define the divided-power operator $E_{a,b}^{(r)}$ on $O[Y_{v,s}]$ by extracting the T^r-coefficient of $P(Y_{v,s} + TE_{b,a}Y_{v,s})$, where $E_{b,a}$ is the matrix unit, so that the substitution adds T times row a to row b. Define the unframed Weyl lattice $\Delta_O(\kappa_{j,v,s}) \subset O[Y_{v,s}]$ as the O-span of finite words in the operators $E_{a,b}^{(r)}$ applied to $H_{\kappa_{j,v,s}}$.

For a tuple $L = (L_v)_{v \in V}$ of free W_v-lattices $L_v = G_v W_v^{N_j} \subset Q_v^{N_j}$ with $G_v \in \mathrm{GL}_{N_j}(Q_v)$, let $A_{v,s} = (\iota_v \circ \sigma_v^s)(G_v) \in \mathrm{GL}_{N_j}(K)$, and let $\Delta_O(\kappa_{j,v,s}; A_{v,s}) \subset K[Y_{v,s}]$ consist of the polynomials $F(Y_{v,s}) = F_0(A_{v,s}^{-1} Y_{v,s})$ with $F_0 \in \iota(\Delta_O(\kappa_{j,v,s}))$. The Weyl weight lattice $H_j(L) \subset K[(Y_{v,s})_{v,s}]$ is the O-span (via ι) of the products

$$\left(\prod_{v,s} (\det A_{v,s})^{-d_{j,v,s}} \right) \prod_{v,s} F_{v,s}(Y_{v,s}), \qquad F_{v,s} \in \Delta_O(\kappa_{j,v,s}; A_{v,s}).$$

Changing G_v on the right by $\mathrm{GL}_{N_j}(W_v)$ multiplies $A_{v,s}$ on the right by an element of $\mathrm{GL}_{N_j}(\iota(O))$, which has determinant in $\iota(O^\times)$ and preserves $\iota(\Delta_O(\kappa_{j,v,s}))$, so $H_j(L)$ depends only on L.

Definition 10.2 (Local systems and lattice bounds)**.** Let V be a finite set of places, O a DVR with uniformizer ϖ, and $\iota : O \to K$ a map to a field. At level j, let $\mathcal{M}_j$ be the K-space of polynomial-valued

functions on components and local-submodule tuples. Each component's level group acts coordinatewise via $\prod_{v\in V}\mathrm{GL}_{N_j}(Q_v)$ and on $K[(Y_{v,s})]$ by the determinant-twisted substitution of 10.1 (sending $H_j(L)$ onto $H_j(\gamma L)$), with character χ in $\iota(O^\times)$. Its O-submodule Λ_j consists of functions on lattice tuples with $F(L)\in H_j(L)$ and $F(\gamma L)=\chi(\gamma)\gamma F(L)$. Take K-linear lifts $K_j:\mathcal{M}_j\to\mathcal{M}_{j+1}$.

Fix $k_1,t,c,m,r,e_O\in\mathbf{N}$ and $d_G:\mathbf{N}\to\mathbf{N}$, with p associated to ϖ^{e_O}. Every lattice tuple at $0\le j\le m$ has finite stabilizer $\Gamma_{j,L}$ with $v_p(|\Gamma_{j,L}|)\le d_G(j)$. For $f\in\mathcal{M}_j$ and $g\in\mathbf{R}$, write $v_j(f)\ge g$ for $f\in\iota(\varpi)^{\lceil e_O g\rceil}\Lambda_j$.

At each $v\in V$ require $p\ge 5$ along with the bounds $m,r\ge 1$ and $2r\le k_1+1$. Use cyclic positions $s \bmod n'_v$ with $n'_v\ge 2$ even and marked set T_v. The integers $\lambda_{v,s,1}\ge\cdots\ge\lambda_{v,s,k_1+1}$ have absolute value at most c, except for the prescribed first and last entries $a_v(s),-b_v(s)$ at marked positions. The integer shifts satisfy

$$w_{v,1}(s)\ge\max\{c,-\lambda_{v,s,k_1+1}\},\qquad w_{v,l}(s)\le w_{v,l+1}(s)\quad(l\ge 1).$$

Definition 10.3 (Layer weights and remaining columns)**.** Suppress v. Put $N_j=k_1+1+j(t+1)$. The list $\nu_{j,s}$ starts with $\lambda_{s,i}+w_j(s)$ for $1\le i\le k_1+1$, followed by $t+1$ copies of $w_j(s)-w_b(s)$ for each $b=1,\dots,j$ in order. Set $\nu_{j,s,i}=0$ off this list. Put

$$\alpha_s=\mathbf{1}_{s\in T}(\lambda_{s,1}-\lambda_{s,2}),\qquad C_{j,s}=\nu_{j,s,1}-\alpha_s,\qquad L_{j,s}=\sum_i\nu_{j,s,i}-\alpha_s.$$

For $j\ge 1$, the rows form a partition by 10.2. Deleting α_s height-one columns leaves $C_{j,s}$ columns and $L_{j,s}$ boxes.

Definition 10.4 (Digit loss)**.** Set $\ell_p(z)=0$ for $z\le 0$ and $\ell_p(z)=1+\lfloor\log_p z\rfloor$ for $z>0$. For $1\le j\le m$, put $r_0=N_{j-1}$ and $r_1=N_j$. The digit loss is

$$\delta_j=\sum_s\big(r_1\ell_p(\alpha_s)+r_0r_1\ell_p(L_{j,s})\big)+\sum_{s\in T}\ell_p(\alpha_s)^2.$$

10.2 Formal parameters and admissible kernels

Definition 10.5 (Parameters and phase equivalence)**.** Let $W=W(\overline{\mathbf{F}}_p)$ and $K_0=\mathrm{Frac}(W)$. Write σ for Witt Frobenius. Set

$$\ell(T)=\sum_{b\ge 0}p^{-b}T^{p^{2b}},\quad f^\ell(X)=f((\ell(X_i))_i),\quad[X^n]\exp(f)=\sum_{k=0}^{|n|}\frac{[X^n]f^k}{k!}\quad(f(0)=0).$$

Write $f\simeq g$ for $f,g\in K_0[[X]]$ if $(f-g)(0)=0$ and $\exp((f-g)^\ell)\in W[[X]]$. Use f^ℓ also over extensions of K_0.

Definition 10.6 (The local theta condition)**.** The pairing laws for $b\in XYK_0[[X,Y]]$ are additivity in each variable modulo $\simeq$ and $b(\eta X,Y)\simeq b(X,\sigma(\eta)Y)$ for $\eta\in W$ with $\sigma^2\eta=\eta$. Put

$$E_v=\{0,\dots,n'_v/2-1\}\times\{0,\dots,N_j-1\}\times\{0,\dots,N_{j+1}-1\},\qquad E_A=\coprod_v E_v.$$

The field K_c contains K and K_0. Its valuation ring O_c contains W and satisfies $K\cap O_c=\iota(O)$. For each $z\in K_c^\times$ require $p^n/z\in O_c$ for some $n\in\mathbf{N}$, and if $z^{-1}\notin O_c$, require $z^a/p\in O_c$ for some $a\in\mathbf{N}$.

For each source L at level j and target L' at level $j+1$, including their components, the kernel data specify a set of frames from L to L'. In the arithmetic model a frame consists of a basis of each lattice at every place. For a frame ϕ, let $\Theta_{A,\phi}\in K_c[[X,Y]]$ be the generator on E_A and Θ_v its restriction by setting the other variables to zero. The theta condition on either series, with index set E, requires the pairing laws and a factorization with $\Phi\in K_0[[X,Y]]$,

$$\Theta=\exp(-\Phi)U,\qquad U^\ell\in O_c[[X,Y]],\qquad\Phi(-X,-Y)\simeq\Phi,\qquad\Phi\simeq\sum_{e\in E}b(X_e,Y_e).$$

At each place v (with $n'_v \geq 2$ even), choose trace-dual bases $(\omega_{v,r}, \omega^*_{v,r})$ of $W(\mathbf{F}_{p^{n'_v}})$ over $W(\mathbf{F}_{p^2})$, indexed by $0 \leq r < n'_v/2$, satisfying $\sum_{h=0}^{n'_v/2-1} \sigma^{2h}(\omega_{v,r}\omega^*_{v,r'}) = \delta_{r,r'}$. In the matrix variables $Z_{v,s,a,b}$ ($s \bmod n'_v$, $0 \leq a < N_j$, $0 \leq b < N_{j+1}$), set

$$L^X_{v,r,a,b} = \sum_{s \text{ even}} \sigma^s(\omega^*_{v,r}) Z_{v,s,a,b}, \qquad L^Y_{v,r,a,b} = \sum_{s \text{ odd}} \sigma^s(\omega_{v,r}) Z_{v,s,a,b},$$

and let $\mathcal{T}_j$ denote the unscaled linear substitution sending $(X_{v,r,a,b}, Y_{v,r,a,b})$ to $(L^X_{v,r,a,b}, L^Y_{v,r,a,b})$. The data include a matrix B_ϕ. In the arithmetic model its block at v, s sends Z to $A^{\mathsf{t}} Z B^{-\mathsf{t}}$ for even s and to $A^{-1} Z B$ for odd s, where A, B are the two frame matrices under $\iota_v \circ \sigma_v^s$. The fixed-coordinate series satisfies $\Theta^F_{j,L,L'} = (\mathcal{T}_j \Theta_{A,\phi})(B_\phi Z)$ for every ϕ.

Lemma 10.7 (Integral derivatives of the CM pairing)**.** *For every prime p, the pairing laws of 10.6 imply $m!n![X^m Y^n]b \in W$ for all $m, n \geq 0$.*

Outline. The product and chain rules give integral derivatives at zero for a phase equivalent to zero. Differentiate the additivity defects for exponents at least two and use the conjugation defect and the unit $\eta - \sigma(\eta)$ for $[XY]b$. The complete proof is on p. 548. □

Definition 10.8 (The rational weight space at level j)**.** At each (v, s), append $t+1$ copies of $-w_{v,l}(s)$ to the initial weight $\lambda_{v,s}$ for each $l = 1, \ldots, j$, in order. At odd s, reverse the resulting list and negate its entries. Subtract the last entry from every entry of the resulting weight to obtain a partition $\kappa_{v,s}$.

In an $N_j \times N_j$ matrix of variables for this block, take the polynomials over K that are homogeneous of degree $\kappa_{v,s,a}$ in column a for each a, and are invariant under adding a multiple of any earlier column to a later one. The level-j rational weight space is the K-span of products of one such polynomial from each block, in their separate matrix variables.

Definition 10.9 (The arithmetic differential operator)**.** Keep the data of Definitions 10.2–10.6. For $j \geq 0$, put $n = N_j$ and $n' = N_{j+1}$, write $Y = (Y_{v,s})_{v,s}$ and $Y' = (Y'_{v,s})_{v,s}$ for the $n \times n$ source and $n' \times n'$ target variable matrices, and introduce an $n \times n'$ matrix $d_{v,s}$ of symbol variables in each block (v, s), with $d = (d_{v,s})_{v,s}$. In block (v, s), put $c = \nu_{j+1,s,1}$, $e = \nu_{j+1,s,n}$, and $b = \sum_{a=1}^{n'} \nu_{j+1,s,a}$.

1. Define a K-algebra homomorphism from $K[Y]$ to $K_c[Y'][d]$ by substituting, in each block (v, s), the $n \times n$ initial-column submatrix $D_{v,s} = (d_{v,s} Y'_{v,s})_{:,1:n}$ for $Y_{v,s}$ when s is even, and $\operatorname{adj} T_{v,s}$ for $Y_{v,s}$ when s is odd, where $T_{v,s} = (\operatorname{adj} Y'_{v,s})_{\text{last } n \text{ rows}} d^{\mathsf{t}}_{v,s}$.

2. Form the rational monomial $\prod_{v,s} M_{v,s} = N_+/N_-$ by setting $M_{v,s} = (\det D_{v,s})^e$ for even s and $M_{v,s} = (\det Y'_{v,s})^{c-b} (\det T_{v,s})^{c-nc+b}$ for odd s, placing factors with nonnegative exponent in the polynomial numerator $N_+ \in K_c[Y'][d]$ and factors with negative exponent (with negated exponent) in the polynomial denominator $N_- \in K_c[Y'][d]$. Evaluating at $Y'_{v,s} = I_{n'}$ with $d^{\mathsf{t}}_{v,s}$ the inclusion of the first n rows (for s even) or last n rows (for s odd) gives $\det D_{v,s} = 1$ for even s, and $\det Y'_{v,s} = \det T_{v,s} = 1$ for odd s, so $N_- \neq 0$ in the integral domain $K_c[Y'][d]$ and multiplication by N_- is injective. Choose a K-linear left inverse $\Pi_j : K_c[Y'][d] \to K_c[Y'][d]$ to multiplication by N_-, and define the symbol $R_G(d) \in K_c[Y'][d]$ of $G \in K[Y]$ as Π_j applied to N_+ times the image of G under step (1).

3. Let $Z = (Z_{v,s,a,a'})_{v,s,\, 0 \leq a < n,\, 0 \leq a' < n'}$ be the kernel variables matching the entries of d. For $G \in K[Y]$ and $f \in K_c[[Z]]$, the arithmetic differential operator $\mathcal{D}_j : K[Y] \times K_c[[Z]] \to K_c[Y']$ is the K-bilinear map

$$\mathcal{D}_j(G, f) = \sum_{\substack{\alpha \in \operatorname{supp} R_G \\ \forall v,s:\ \sum_{a,a'} \alpha_{v,s,a,a'} = \sum_{a=1}^{n'} \nu_{j+1,s,a}}} [d^\alpha] R_G \, \alpha! [Z^\alpha] f.$$

Definition 10.10 (Admissibility of the kernel and lifts)**.** At each level $0 \leq j \leq m$, fix a finite set of component–lattice pairs containing one representative of each level-group orbit in each component. The kernel data give integer lengths $\ell_{v,j}(L)$ on tuples. A lower edge of index i at v decreases $\ell_{v,j}$ by i and preserves the other lengths. The data also give a K-bilinear polynomial-valued operator $\mathcal{D}_j(G, f)$ on the

ambient source polynomials and $K_c[[Z]]$. It depends on f through the coefficients of degree $\sum_a \nu_{j+1,s,a}$ in each block (v,s). Set

$$\Delta_v(l) = \sum_{s \text{ even}} w_{v,l}(s) - \sum_{s \text{ odd}} w_{v,l}(s),$$
$$D_j(L,L') = \prod_v p^{\Delta_v(j+1)(\ell_{v,j}(L)-\ell_{v,j+1}(L'))},$$
$$\mathcal{K}_{j,L,L'}(G) = D_j(L,L')\,\mathcal{D}_j(G,\Theta^F_{j,L,L'}).$$

For successive levels with $0 \le j < m$, the following conditions use lattice pairs as sources and targets. A framed condition applies to every frame unless it specifies a choice. The product evaluation and the operators y_v cited below are defined on these data, with $\Theta_{j,L,L',\phi}$ denoting $\Theta_{A,\phi}$ in the frame ϕ from L to L'. Write $V_j = K\Lambda_j$.

1. The prime-only clause is the Dieudonné–Dwork criterion for every odd prime, proved in Lemma 10.11. In every frame, the generator $\Theta_{A,\phi}$ and each restriction Θ_v satisfy the theta condition of Definition 10.6 on E_A and E_v, respectively. The fixed series b satisfies the pairing laws of Definition 10.6 for every choice of the place set, including the empty set.

2. For every column choice I, all coefficients of the polynomial in (12.9) (Chapter 12) at $\mathbf{i} = \mathbf{0}$ belong to O_c. The lift satisfies $K_j(V_j) \subseteq V_{j+1}$.

3. At each level $j < m$ there is a scalar $c_K \in \{0\} \cup O_c^\times$ such that, for every $F \in V_j$ and every target lattice pair L', over K_c one has the finite expansion

$$(K_jF)(L') = \sum_{L \in \mathcal{R}_j} \frac{c_K}{|\Gamma_{j,L}|} \mathcal{K}_{j,L,L'}(F(L)) \qquad (j+1 < m), \tag{10.1}$$

   where $\mathcal{R}_j$ contains one representative of each lattice orbit in each component, and at the terminal step $j+1 = m$,

$$(K_jF)(L') = \sum_x \frac{c_K}{w_x} \sum_y \mathcal{K}_{j,L_{x,y},L'}(F(L_{x,y})). \tag{10.2}$$

   At the terminal step the finite index sets and source pairs $L_{x,y}$ serve for all $F \in V_j$ and targets, as do the positive integer denominators w_x with $v_p(w_x) \le d_G(j)$. With these same terminal data, summing the kernel over source neighbours of index i gives the action of the source Hecke operator on F, as in (12.29) (Chapter 12), for every v and $i \ge 0$.

4. For each v and $j < m$ there are units $a, a' \in O^\times$ such that (12.26) (Chapter 12) expresses the fixed-coordinate theta series through the indicator of $xL_v \subseteq L'_v$ for every lattice pair L, L', using the given lengths. The index $\omega = \omega_{v,j}(L,L')$ of the additive map in this formula depends only on the two components and the lattice entries away from v. For each such index, Ξ_ω maps the additive group of rational-valued functions on the $N_{j+1} \times N_j$ matrices over the local field to $K_c[[Z]]$. The differential kernel is invariant under a change of source representative. For every element γ of the source level group and every G in the rational weight space of Definition 10.8,

$$\mathcal{K}_{j,\gamma L,L'}(\chi(\gamma)\gamma G) = \mathcal{K}_{j,L,L'}(G).$$

5. At each level $0 \le j \le m$, fix a place v and an edge index $i \ge 0$. For every K_c-valued function on these edges invariant under the simultaneous level-group action on both endpoints, summation by the source orbit and summation by the target orbit give the same result when each representative is weighted by the reciprocal of its stabilizer order, as in (12.28) (Chapter 12).

6. For an abstract operator $\mathcal{D}_j$, require the following frame condition. Let $0 \le j < m$, let L be a source component–lattice pair, and fix a target component and a target lattice tuple L'. For each v choose $0 \le i_v \le r$. The tuple L'' is in the same target component and satisfies either $i_v = 0$ and $L''_v = L'_v$, or $1 \le i_v$ and the lower-edge condition of index i_v from L'_v to L''_v. Put

$$N'_{v,j+1}(i) = \sum_{s \text{ even}} \sum_{a=1}^{i} \lambda_{v,s,a} + i \sum_{s \text{ odd}} w_{v,j+1}(s).$$

Use the coefficient lower-bound relation $\nu_c(z) \ge g$ of Definition 10.19; explicitly, it means $z^b p^{-a} \in O_c$ for every $a \in \mathbf{Z}$ and $b \in \mathbf{N}_{>0}$ with $a \le bg$. Let $P_{I,\mathbf{i}}$ denote the product-test symbol for the column choice I and edge tuple $\mathbf{i} = (i_v)_v$. Its differential evaluation is

$$\mathcal{D}_{P_{I,\mathbf{i}}}(f) = \sum_{\alpha} [Z^\alpha] P_{I,\mathbf{i}}\, \alpha! [Z^\alpha] f.$$

Extend the target polynomials to K_c. For $g \in \mathbf{R}$ put

$$H_{j+1}^{\ge g}(L') = \left\{ \sum_{a=1}^{h} t_a z_a : h \in \mathbf{N},\ t_a \in K_c,\ \nu_c(t_a) \ge g,\ z_a \in H_{j+1}(L') \right\}.$$

There must exist one frame ϕ from L to L'', with substitution matrix B_ϕ, such that for every $f \in K_c[[Z]]$, every $g \in \mathbf{R}$ and every $G \in H_j(L)$,

$$\begin{aligned} &\big(\forall I\ \forall n, \quad \nu_c\big([\tau^n]\mathcal{D}_{P_{I,\mathbf{i}}}(f)\big) \ge g\big) \\ &\qquad \Longrightarrow \quad \left(\prod_v p^{N'_{v,j+1}(i_v)}\right) D_j(L, L'')\mathcal{D}_j(G, f(B_\phi Z)) \in H_{j+1}^{\ge g}(L'), \end{aligned}$$

Lemma 10.17 proves this condition for the arithmetic operator of Definition 10.9.

7. The operators y_v of Definition 12.9 commute pairwise on $\mathcal{M}_0$, and each maps Λ_0 into itself. For every $0 \le j < m$ and every v, all coefficients in the single-place test of Definition 10.19 belong to O_c.

Lemma 10.11 (Dieudonné–Dwork criterion)**.** *Let p be odd and let $d \ge 0$. For $\Psi \in K_0[[X_1, \dots, X_d]]$ with $\Psi(0) = 0$,*

$$\exp(\Psi) \in W[[X]] \quad \Longleftrightarrow \quad (\sigma\Psi)(X_1^p, \dots, X_d^p) - p\Psi(X) \in pW[[X]].$$

Remark. For odd p this extends the one-variable criterion over $\mathbf{Q}_p$ to finitely many variables with Witt Frobenius. The one-variable criterion also holds at $p = 2$ [59, Lemma 1].

Outline. Legendre's formula identifies $p(X)W[[X]]$ with $1+p(X)W[[X]]$, rewriting the condition as $\varphi F/F^p \in 1 + p(X)W[[X]]$ for $F = \exp\Psi$. Necessity holds modulo p; sufficiency inducts on degree $h \ge 1$ by dividing F by its integral truncation below h. The complete proof is on p. 548. □

Definition 10.12 (An expansion with controlled denominators)**.** Let K'/K_0 be a field extension and R' a subring of K'. Use variables $Z_{s,a,b}$ indexed by $s \bmod n'$ with $0 \le a < r_0$ and $0 \le b < r_1$. Put

$$S_{s,a,b} = \sum_{l \ge 0} (-1)^l Z_{s-2l,a,b}^{p^{2l}}, \qquad H_u = \sum_{s,a,b} u_s S_{s,a,b}^p S_{s+1,a,b} \quad (u_s \in K').$$

For $f, H \in K'[[Z]]$, write $f \in \mathcal{C}_{R'}(H)$ if there are $U_i \in R'[[Z]]$ with $[Z^n]f = \sum_{i=0}^{|n|} p^{-i}[Z^n](H^i U_i)$ for every n. The normed-field version requires $\|[Z^n]U_i\| \le 1$ for all i, n.

10.3 Reduction of the phase

Theorem 10.13 (The explicit odd-power phase)**.** *Let $p \ge 5$ and let b satisfy the pairing laws of 10.6. There is $c \in W$ with $\sigma^2 c = c$ such that $\exp(b^\ell - G_c) \in 1 + (X,Y)W[[X,Y]]$, where*

$$A(X,Y) = \sum_{0 \le i \le l < \infty} \frac{X^{p^{2i}} Y^{p^{2l+1}}}{p^{i+l+1}}, \qquad G_c = cA(X,Y) - \sigma(c)A(Y,X).$$

For even $n' \ge 2$ and $r_0, r_1 \ge 0$, take bases ω_r, ω_r^* of $W(\mathbf{F}_{p^{n'}})$ over $W(\mathbf{F}_{p^2})$, indexed by $0 \le r < n'/2$, with $\sum_{h=0}^{n'/2-1} \sigma^{2h}(\omega_r \omega_{r'}^*) = \delta_{r,r'}$. In the variables of 10.12 set

$$L^X_{r,a,b} = \sum_{s \text{ even}} \sigma^s(\omega_r^*) Z_{s,a,b}, \qquad L^Y_{r,a,b} = \sum_{s \text{ odd}} \sigma^s(\omega_r) Z_{s,a,b}.$$

Let K'/K_0 contain π with $\pi^{p^2-1}=p$. Put $R_\pi = W[\pi]$ and $\mu(T)=\ell^{-1}(\pi T)/\pi$. Then $\mu \in T\mathbf{Z}[[T]]$ and $\pi^{(p+1)i}/i! \in R_\pi$ for $i \geq 0$. For every $\Phi \simeq \sum_e b(X_e, Y_e)$ with $e=(r,a,b)$ as above,

$$\exp(-\Phi(\pi L^X, \pi L^Y)) = \exp(-\pi^{p+1} H_u/p)V, \qquad V \in 1+(Z)R_\pi[[Z]], \qquad u_s = \begin{cases} -\sigma(c) & s \text{ even}, \\ c & s \text{ odd}. \end{cases}$$

Outline. Use Hazewinkel's lemma, root-of-unity averaging, and coefficient induction to identify the integral Frobenius difference with $D_\sigma(B-G_c)$ and apply 10.11. Trace duality gives H_u, and Legendre's formula makes the remaining exponential an integral unit. The complete proof is on p. 549. □

10.4 Kernel expansions

Theorem 10.14 (Single-place and simultaneous expansions)**.** *Assume 10.10 and fix $0 \leq j < m$. Fix a lattice pair and a frame. Let K'/K_c contain π with $\pi^{p^2-1}=p$, and let R' be a subring of K' containing $W[\pi]$ and O_c. Write Z_v for the variables at v and $Z=(Z_v)_v$.*

At each place one can choose $u_{v,s} \in R'$ so that $\Theta_v(\pi L_v^X, \pi L_v^Y) \in \mathcal{C}_{R'}(H'_v)$ whenever $H'_v = H_{u_v} + pG_v$ with $G_v \in (Z_v)R'[[Z_v]]$. For the generator one can choose all $u_{v,s} \in R'$ so that for every family $G_v \in (Z)R'[[Z]]$, with $H'_v = H_{u_v}(Z_v) + pG_v(Z)$, there are $U_{\mathbf{i}} \in R'[[Z]]$ for $\mathbf{i}: V \to \mathbf{N}$ such that

$$[Z^n]\Theta_A(\pi L^X, \pi L^Y) = \sum_{\substack{\mathbf{i}:V\to\mathbf{N} \\ i_v \leq |n| \ \forall v}} [Z^n]\left(U_{\mathbf{i}} \prod_v p^{-i_v} (H'_v)^{i_v} \right) \qquad \text{for every } n.$$

Outline. Use 10.6 and 10.5 for the zero phase at the empty set, then apply 10.13 place by place and absorb the phase correction into the integral cofactor. Expand using the factorial bound of 10.13 and the zero constant terms of the H'_v. The complete proof is on p. 550. □

10.5 The polynomial and its coefficients

Definition 10.15 (Products of column minors)**.** For $j \geq 0$ put $r_0 = N_j$, $r_1 = N_{j+1}$. Use $Z_{v,s,a,b}$ with $0 \leq a < r_0$, $0 \leq b < r_1$, and auxiliary variables $\tau_{v,s,h,a}$ with $h \in \mathbf{N}$. A column choice assigns $I_{v,s,c}: \mathbf{N} \to \{0, \dots, r_1 - 1\}$ to each column $0 \leq c < \nu_{j+1,s,1}$. In its block put $h_c = \#\{a : c < \nu_{j+1,s,a}\}$, with $1 \leq a \leq r_1$. For $\mathbf{i} = (i_v)_v \in \mathbf{N}^V$ set $q_c = \#\{h' < h_c : I_{v,s,c}(h') < i_v\}$ and

$$D_{v,s,c} = \det\left(\sum_{a<r_0} \tau_{v,s,h,a} Z_{v,s,a,I_{v,s,c}(h')} \right)_{h,h'<h_c}, \qquad P_{I,v} = \prod_{s,c} D_{v,s,c},$$

$$e_{v,s,c} = \begin{cases} q_c & s \text{ odd}, \\ \max\{\min(h_c, i_v) - q_c, 0\} & s \text{ even}, \end{cases} \qquad P_{I,\mathbf{i}} = \prod_{v,s,c} p^{e_{v,s,c}} D_{v,s,c}.$$

Thus $P_{I,\mathbf{0}} = \prod_v P_{I,v}$. At v the degree in each of Z and τ is the sum of column heights $\sum_s(\alpha_s + L_{j+1,s})$.

Lemma 10.16 (Contraction of a complementary column)**.** *Work over a field of characteristic zero. For a square matrix Y of size s, write $C_h(Y,z)$ for the matrix with first $s-h$ columns taken from Y and last h columns taken from the beginning of z. Fix heights $0 \leq h_k \leq s$, $1 \leq k \leq c$. Let H be the product of the initial principal minors of heights $s-h_k$. For polynomials in Y use the pairing*

$$[P,Q] = \sum_\alpha \alpha! [Y^\alpha] P [Y^\alpha] Q.$$

Let I_k be a list of $s-h_k$ distinct row indices, let E_{I_k} have the corresponding unit vectors as columns in the indicated order, and put $Q_I(Y) = \prod_k \det Y_{I_k,\,1:s-h_k}$. Then

$$\frac{\left[\prod_k \det C_{h_k}(Y,z), Q_I(Y)\right]}{[H,H]} = \prod_k \det[\, E_{I_k} \mid z_{:,1:h_k} \,].$$

The identity holds over the polynomial ring in the entries of z, including after a substitution that forces z to have deficient rank.

Outline. Row degrees reduce the normalized pairing with H to evaluation at the identity. Reverse and factor out the generic column matrix, use adjointness and expand along the unit columns, then specialize the resulting polynomial identity. The complete proof is on p. 550. □

Lemma 10.17 (A frame for simultaneous lower neighbours)**.** *Let $K = \mathrm{Frac}(O)$ and $K_c = \mathbb{C}_p$ with the given embeddings of K and K_0, where $K \hookrightarrow \mathbb{C}_p$ maps O into $\mathcal{O}_{\mathbb{C}_p}$. Use the arithmetic operator $\mathcal{D}_j$ of Definition 10.9. At v let W_v be the local Witt ring with residue field $\mathbf{F}_{p^{n'_v}}$ and uniformizer p. Index its embeddings into O by s. For $A = GW_v^n$ the arithmetic length is $s(A) = -\operatorname{ord}_p \det G$, independent of the basis since changes of basis have unit determinant.*

Let $0 \le j < m$ and let L, L' be component–lattice pairs at $j, j+1$. Fix $0 \le i_v \le r$ and take L'' in the target component with $pL'_v \subseteq L''_v \subseteq L'_v$ and $\dim_{W_v/p}(L'_v/L''_v) = i_v$. Let $D(L, L'')$ denote $D_j(L, L'')$ with the arithmetic lengths, and put

$$N'_{v,j+1}(i) = \sum_{s \text{ even}} \sum_{a=1}^{i} \lambda_{v,s,a} + i \sum_{s \text{ odd}} w_{v,j+1}(s).$$

There is a frame ϕ from L to L'' whose arithmetic substitution B_ϕ of 10.6 serves for all $f \in K_c[[Z]]$ and $g \in \mathbf{R}$. If every coefficient of $\mathcal{D}_{P_{I,\mathrm{i}}}(f)$ from 10.18 has valuation at least g for every I, then for every $G \in H_j(L)$,

$$\left(\prod_v p^{N'_{v,j+1}(i_v)}\right) D(L, L'')\, \mathcal{D}_j(G, f(B_\phi Z)) \in \left\{ \sum_{a=1}^{k} t_a z_a : k \ge 0,\ t_a \in K_c,\ v_p(t_a) \ge g,\ z_a \in H_{j+1}(L') \right\}.$$

Here $v_p(p) = 1$ and the target polynomials are extended to K_c.

Outline. Builds ϕ from compatible bases of L_v, L'_v, L''_v and evaluates the symbol of 10.9 on Weyl generators from 10.1. Framed pairings matching 10.15 give a unit scale, and integral Schur–Weyl duality yields the $H_{j+1}(L')$ sum. The complete proof is on p. 551. □

Definition 10.18 (Differential evaluation)**.** For $P \in \mathbf{Z}[\tau][Z]$ and $f \in K_c[[Z]]$ use

$$\mathcal{D}_P(f) = \sum_{\alpha} [Z^\alpha] P\, \alpha! [Z^\alpha] f = (P(\partial_Z) f)(0).$$

The equality follows from $(\partial_Z^\alpha f)(0) = \alpha! [Z^\alpha] f$. The same finite contraction applies to symbol polynomials with target polynomial coefficients.

Definition 10.19 (The single-place coefficient bound)**.** For $z \in K_c$ and $g \in \mathbf{R}$ define

$$\nu_c(z) \ge g \quad \Longleftrightarrow \quad \forall a \in \mathbf{Z}\ \forall b \in \mathbf{N}_{>0}, \quad a \le bg \implies z^b p^{-a} \in O_c.$$

For the ring of a real valuation v with $v(p) = 1$, this means $v(z) \ge g$ by density of the rationals. For $j \ge 0$ and $v \in V$, put $\Phi_{j,v} = \sum_{e \in E_v} b(X_{v,e}, Y_{v,e})$ in the full variable set. The single-place coefficient bound $\mathcal{L}_{v,j}(g)$ means that

$$\nu_c\big([\tau^n] \mathcal{D}_{P_{I,v}}\big(\mathcal{T}_j(\exp(-\Phi_{j,v}) U)\big)\big) \ge g$$

for every $U \in K_c[[X, Y]]$ with $U^\ell \in O_c[[X, Y]]$ and for all choices of I and auxiliary multi-index n. The condition at g implies the condition at every $g' \le g$.

Chapter 11

Entry estimates: factorials against denominators

The preceding chapter associates a differential polynomial to the target weight diagram and controls the denominators of the scaled theta kernel. The entry estimates bound their differential pairing at one step of the theta chain.

Fix a prime $p > 2$ and a positive integer ν. Put

$$S = \mathbf{Z}/(2\nu)\mathbf{Z}, \qquad \rho_p = \frac{1}{p^2-1}, \qquad \gamma_p = \frac{p}{p^2-1}.$$

The parity of a position is the parity of its representative in $\{0, \dots, 2\nu-1\}$. Addition to a position takes place in S. The constants satisfy

$$\sum_{l \geq 0} p^{-(2l+1)} = \gamma_p, \qquad \frac{1}{p-1} = \gamma_p + \rho_p. \tag{11.1}$$

For a nonarchimedean normed field K, put $\mathcal{O}_K = \{a \in K : |a| \leq 1\}$. A polynomial or power series is integral when its coefficients lie in $\mathcal{O}_K$. When $|p| = p^{-1}$, write $v_p(a) = -\log_p |a|$ for $a \neq 0$ and $v_p(0) = +\infty$. For $a \in K$ (with $a = 0$ allowed) and $x \in \mathbf{R}$, the bound $|a| \leq p^{-x}$ is equivalent to $v_p(a) \geq x$. All power series operations are formal.

11.1 The two local exponents

For the weight diagram at a place v in 10.3, take $2\nu = n'_v$, the number of positions. Identify each position with its residue class in S and let T be the image of the marked set T_v. Fix a layer $1 \leq j \leq m$, where m is the chain length. At s, let α_s count the distinguished height-one columns. The remaining columns number C_s and contain L_s boxes. Thus $\alpha_s = 0$ for $s \notin T$. Take $r_0 = N_{j-1}$ and $r_1 = N_j$, with N_j as in 10.3. In the entry estimates below, the marked set $T \subseteq S$ and the ranks $r_0, r_1 \in \mathbf{N}$ are arbitrary.

For $a \in \mathbf{N}$, set

$$\ell_p(a) = \begin{cases} 0, & a = 0, \\ 1 + \lfloor \log_p a \rfloor, & a > 0. \end{cases}$$

We use the error from 10.4 in the form

$$\delta = \sum_{s \in S} \big(r_1 \ell_p(\alpha_s) + r_0 r_1 \ell_p(L_s) \big) + \sum_{s \in T} \ell_p(\alpha_s)^2. \tag{11.2}$$

Definition 11.1 (The ordinary local exponent)**.** For $s \in S$, let

$$D_s = \big\{ (s' - s)_{\mathrm{rep}} : s' \in T,\ s' \text{ has parity opposite to } s \big\},$$

where $(s'-s)_{\mathrm{rep}}$ denotes the representative in $\{0,\dots,2\nu-1\}$. Define

$$g_A=\sum_{s\in T}\sum_{\substack{d\ge 1,\ d\ \text{odd}\\ d\notin D_s}}[\alpha_s p^{-d}-C_{s+d}]_+-\delta,\qquad [x]_+=\max(x,0). \tag{11.3}$$

For these weight data, $g_A=g_{0,v}(j)$.

Every $d\in D_s$ is the first positive distance from s to its endpoint. The sum omits this odd integer d and includes $d+2\nu$. The bound $[\alpha_s p^{-d}-C_{s+d}]_+\le\alpha_s p^{-d}$ gives convergence.

The identification with the local system follows by substituting

$$\alpha_s=\lambda_{s,1}-\lambda_{s,2},\quad C_s=\nu_{j,s,2}\quad(s\in T),\qquad \alpha_s=0,\quad C_s=\nu_{j,s,1}\quad(s\notin T).$$

The error (11.2) is then δ_j.

Definition 11.2 (The enhanced local exponent)**.** Write

$$T_{\mathrm{odd}}=\{s\in T: s\text{ is odd}\},\qquad T_{\mathrm{ev}}=T\setminus T_{\mathrm{odd}},\qquad A_{\mathrm{ev}}=\sum_{s\in T_{\mathrm{ev}}}\alpha_s.$$

Define

$$g_B=\gamma_p\sum_{s\in T}\alpha_s-\sum_{s\in T_{\mathrm{odd}}}\sum_{\substack{d\ge 1\\ d\ \text{odd}}}\min(\alpha_s p^{-d},C_{s+d})-\delta. \tag{11.4}$$

For these weight data, $g_B=g_v(j)$ and $A_{\mathrm{ev}}=\mathrm{alEven}(v)$.

The series converges by $0\le\min(\alpha_s p^{-d},C_{s+d})\le\alpha_s p^{-d}$.

11.2 Differential pairings and column bounds

For a finite variable set $\mathcal{V}$, a multi-index $n\in\mathbf{N}^{\mathcal{V}}$, a polynomial P, and a power series F, put

$$|n|=\sum_{x\in\mathcal{V}}n_x,\qquad n!=\prod_{x\in\mathcal{V}}n_x!,\qquad \langle P,F\rangle=\sum_n[z^n]P\,n!\,[z^n]F.$$

The sum in this differential evaluation (10.18) runs over the monomials of P. Write $\mathbf{1}_x$ for the unit multi-index at x.

Lemma 11.3 (Linear factors and Hasse derivatives)**.** *Let K be a field, and let $\mathcal{V},\mathcal{K}$ be finite sets. For directions $b_k\in K^{\mathcal{V}}$, define*

$$L_k(z)=\sum_{x\in\mathcal{V}}b_{k,x}z_x,\qquad D_b^{[\beta]}P=[t^\beta]P\left(z+\sum_{k\in\mathcal{K}}t_k b_k\right)\quad(\beta\in\mathbf{N}^{\mathcal{K}}).$$

For every polynomial P and power series F,

$$\left\langle P,\left(\prod_k L_k^{\beta_k}\right)F\right\rangle=\left(\prod_k\beta_k!\right)\left\langle D_b^{[\beta]}P,F\right\rangle. \tag{11.5}$$

For the integrality and degree assertions, $\mathcal{V}$ may be arbitrary. Let K be an ultrametric normed field. Integral P and integral directions give integral $D_b^{[\beta]}P$. If P is homogeneous of degree d, then the derivative is homogeneous of degree $d-|\beta|$ for $|\beta|\le d$, and is zero for $|\beta|>d$.

Outline. Check adjunction of linear multiplication and directional differentiation on monomials, then iterate and use the translation expansion to identify the iterated derivative with the Hasse derivative times the factorials. The expansion preserves integrality and gives the degree drop. The complete proof is on p. 553. □

Let Σ be a set of positions and use variables

$$z_{s,a,b}, \qquad s \in \Sigma, \quad 0 \le a < r_0, \quad 0 \le b < r_1.$$

For $c = 1, \dots, C$, choose a position s_c, a height $h_c \in \mathbf{N}$, a matrix $M_c = (M_{c,i,a})_{0 \le i < h_c,\ 0 \le a < r_0}$, and a map

$$I_c : \{0, \dots, h_c - 1\} \longrightarrow \{0, \dots, r_1 - 1\}.$$

Put

$$\Delta_c(z) = \det\left(\sum_{a < r_0} M_{c,i,a} z_{s_c,a,I_c(j)} \right)_{0 \le i,j < h_c}, \qquad Q_0(z) = \prod_{c=1}^{C} \Delta_c(z). \tag{11.6}$$

A height-zero determinant is 1. The maps I_c may repeat column labels. A direction at s with row vector $\xi \in K^{r_0}$ and column label b is

$$B(s, \xi, b)_{t,a,b'} = \begin{cases} \xi_a, & t = s,\ b' = b, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 11.4 (Vanishing when columns are exhausted)**.** *Let K be any field. For a finite family of directions $b_k = B(\mathrm{pos}(k), \xi_k, \mathrm{col}(k))$ indexed by $\mathcal{K}$, let $\beta \in \mathbf{N}^{\mathcal{K}}$. A direction is used when its multiplicity is positive. Each of the following conditions forces $D_b^{[\beta]} \Delta_c = 0$.*

1. *Some $\beta_k \ge 2$.*
2. *Distinct used directions k, k' have the same position and either $\xi_k = \xi_{k'}$ or $\mathrm{col}(k) = \mathrm{col}(k')$.*
3. *A used direction has position different from s_c, or has column outside $\operatorname{im} I_c$.*

Let $\mathcal{K}_0 \subseteq \mathcal{K}$ consist of directions at a single position s_0. If $D_b^{[\beta]} Q_0 \ne 0$, the following bounds hold.

$$\sum_{k \in \mathcal{K}_0} \beta_k \le \#\{c : s_c = s_0\} \quad \text{if all } \xi_k \ (k \in \mathcal{K}_0) \text{ are equal,} \tag{11.7}$$

and, for every $J \subseteq \{0, \dots, r_1 - 1\}$,

$$\sum_{k \in \mathcal{K}_0} \beta_k \le \sum_{c : s_c = s_0} \#(\operatorname{im} I_c \cap J) \quad \text{if } \mathrm{col}(k) \in J \text{ for } k \in \mathcal{K}_0. \tag{11.8}$$

Outline. Repeated labels give zero. Column multilinearity allows each available label in J at most once, and alternation allows a shared row vector at most once per determinant when the Hasse product formula distributes the directions. The complete proof is on p. 554. □

11.3 Four parts of the substituted kernel

On the cyclic position set S, the series from 10.12 is

$$S_{s,a,b}(z) = \sum_{l \ge 0} (-1)^l z_{s-2l,a,b}^{p^{2l}}, \qquad H(z) = \sum_{s \in S} \sum_{a < r_0,\ b < r_1} u_s S_{s,a,b}(z)^p S_{s+1,a,b}(z), \tag{11.9}$$

where $u_s \in \mathcal{O}_K$. Introduce top variables $t_{s,b}$ and remaining variables $y_{s,a,b}$. Let $\xi_s = (\xi_{s,a})_{a < r_0}$ be integral vectors. Make the substitution

$$z_{s,a,b} \longmapsto y_{s,a,b} + \mathbf{1}_{s \in T} \xi_{s,a} t_{s,b}. \tag{11.10}$$

Write $h(t, y)$ for the resulting series. The remaining degree of a monomial is its total degree in y.

For $E \in \mathbf{N}$, index the pure-top terms by

$$\phi = (v, l, l', a, b), \qquad v \in S, \quad 0 \le l, l' < E, \quad a < r_0, \quad b < r_1.$$

Set $s_\phi = v - 2l$, $t_\phi = v + 1 - 2l'$, and define

$$d^A_\phi = p^{2l+1}\mathbf{1}_{(s_\phi,b)} + p^{2l'}\mathbf{1}_{(t_\phi,b)}, \tag{11.11}$$

$$a^A_\phi = \mathbf{1}_{s_\phi,t_\phi\in T}\, u_v(-1)^{l+l'}\xi_{s_\phi,a}^{p^{2l+1}}\xi_{t_\phi,a}^{p^{2l'}}. \tag{11.12}$$

For the second family, let $\kappa = (s,l,b)$ with $s \in S$, $0 \le l < E$, $b < r_1$, and put

$$d^B_\kappa = p^{2l+1}\mathbf{1}_{(s,b)}, \tag{11.13}$$

$$a^B_\kappa = \mathbf{1}_{s\in T}(-1)^l u_{s+2l}, \tag{11.14}$$

$$L_\kappa(y) = \sum_{a<r_0} \xi_{s,a}^{p^{2l+1}} y_{s+2l+1,a,b}. \tag{11.15}$$

The exponents d^A_ϕ, d^B_κ index top monomials. The direction b_κ of L_κ is $B(s+2l+1, (\xi_{s,a}^{p^{2l+1}})_a, b)$.

Lemma 11.5 (The four-part decomposition)**.** *Assume K is nonarchimedean, $p > 2$ is prime, $|p| = p^{-1}$, and all $u_s, \xi_{s,a}$ are integral. Let $D, E \in \mathbf{N}$ satisfy $D < p^{2E}$. There are integral power series $Z_{\ge p}$ and Y such that every monomial of $Z_{\ge p}$ with nonzero coefficient has remaining degree at least p, and*

$$h(t,y) \equiv \sum_\phi a^A_\phi t^{d^A_\phi} + \sum_\kappa a^B_\kappa t^{d^B_\kappa} L_\kappa(y) + Z_{\ge p}(t,y) + pY(t,y) \pmod{\text{total degree } > D}. \tag{11.16}$$

The families and coefficients are given by (11.11)–(11.15).

Outline. Keep $l < E$ through degree D and use Frobenius modulo p to separate pure top terms, terms linear in y, and terms of y-degree at least p. Reindex and sum the linear terms to get L_κ, with remainder pY for integral Y. The complete proof is on p. 554. □

For a series h with zero constant term, write $g \in \mathcal{C}_p(h)$ if there are integral series U_i, $i \in \mathbf{N}$, with

$$[z^n]g = \sum_{i=0}^{|n|} p^{-i}[z^n](h^i U_i) \quad \text{for every multi-index } n. \tag{11.17}$$

This is the expansion class of 10.12. For finitely many source variables, let ϕ be a substitution by integral series with zero constant term. If $g \in \mathcal{C}_p(h)$, then $g\circ\phi \in \mathcal{C}_p(h\circ\phi)$. The witnesses are $U_i\circ\phi$, since substitution preserves integrality and lower bounds on total degree.

Lemma 11.6 (The factorial bound for a nonzero pairing)**.** *Let $\mathcal{T}, \mathcal{L}, \mathcal{A}, \mathcal{K}$ be finite sets, and let K be a nonarchimedean normed field with $|p| = p^{-1}$, where p is prime. Use variables t_τ, $\tau \in \mathcal{T}$, and y_x, $x \in \mathcal{L}$.*

Fix a total degree $D \in \mathbf{N}$. Suppose h has zero constant term, $g \in \mathcal{C}_p(h)$, and, in total degrees at most D,

$$h = \sum_{\phi\in\mathcal{A}} a^A_\phi t^{d^A_\phi} + \sum_{\kappa\in\mathcal{K}} a^B_\kappa t^{d^B_\kappa} L_\kappa(y) + Z_{\ge p} + pY.$$

Here $d^A_\phi, d^B_\kappa \in \mathbf{N}^{\mathcal{T}}$ and $L_\kappa(y) = \sum_x b_{\kappa,x} y_x$. The scalars a^A_ϕ, a^B_κ, the coordinates $b_{\kappa,x}$, and the series $Z_{\ge p}, Y$ are integral. Every nonzero monomial of $Z_{\ge p}$ has y-degree at least p.

Let $Q_0(y)$ be an integral homogeneous polynomial of degree d_0, and let $e \in \mathbf{N}^{\mathcal{T}}$ satisfy $|e| + d_0 = D$. If $J = \langle t^e Q_0(y), g\rangle \ne 0$, there exist multiplicities $m_\phi, \beta_\kappa \in \mathbf{N}$, an integer $i_2 \ge 0$, and a multi-index $n \in \operatorname{supp} D_b^{[\beta]} Q_0$ such that

$$m_\phi > 0 \Longrightarrow a^A_\phi \ne 0, \qquad \beta_\kappa > 0 \Longrightarrow a^B_\kappa \ne 0, \tag{11.18}$$

$$\sum_\phi m_\phi d^A_\phi + \sum_\kappa \beta_\kappa d^B_\kappa \le e, \qquad p i_2 \le |n|, \tag{11.19}$$

$$|J| \le p^{\sum_\phi m_\phi + \sum_\kappa \beta_\kappa + i_2} |e!| \left|\left(\sum_\kappa \beta_\kappa\right)!\right| |n!|. \tag{11.20}$$

The inequality between top multi-indices is coordinatewise.

Outline. Expand h^i by the multinomial formula and use 11.3 to move the linear factors to Hasse derivatives and collect $B!$. The selected monomials give the exponent bounds, pY cancels the matching denominators, and the ultrametric inequality gives the bound. The complete proof is on p. 555. □

11.4 The arithmetic contribution of the factorials

Lemma 11.7 (A factorial valuation bound)**.** *For every prime p and every $a \in \mathbf{N}$, the arithmetic valuation satisfies*

$$v_p(a!) \geq \frac{a}{p-1} - \ell_p(a). \tag{11.21}$$

Legendre's formula gives $v_p(a!) = (a - s_p(a))/(p-1)$, where $s_p(a)$ is the base-p digit sum [61, Ch. I, §2, Ex. 14, p. 7]. Substituting $s_p(a) \leq (p-1)\ell_p(a)$ in Legendre's formula gives the bound.

Lemma 11.8 (The net contribution of three factorial families)**.** *Let S be a finite set, $T \subseteq S$, p prime, and $r_0, r_1 \in \mathbf{N}$. Let $\alpha, L : S \to \mathbf{N}$, with $\alpha_s = 0$ outside T. Suppose nonnegative integer arrays*

$$e_{s,b}\ (b < r_1), \qquad a_{s,a,b}\ (a < r_0,\ b < r_1), \qquad i_{s,l}\ (l \in \mathbf{N})$$

and integers $i_0, i_2 \geq 0$ satisfy

$$\sum_{b<r_1} e_{s,b} = \alpha_s, \qquad \sum_{a,b} a_{s,a,b} \leq L_s, \tag{11.22}$$

$$i_{s,l} = 0 \quad \text{if } s \notin T \text{ or } l \notin E, \tag{11.23}$$

$$p^{2l+1} i_{s,l} \leq \alpha_s, \tag{11.24}$$

$$\sum_{s,a,b} a_{s,a,b} + \sum_{s\in S}\sum_{l\in E} i_{s,l} = \sum_s L_s, \qquad p i_2 \leq \sum_s L_s, \tag{11.25}$$

where $E \subseteq \mathbf{N}$ is finite. Define δ by (11.2), and set

$$\mathcal{F} = \sum_{s,b} v_p(e_{s,b}!) + \sum_{s\in S}\sum_{l\in E} v_p(i_{s,l}!) + \sum_{s,a,b} v_p(a_{s,a,b}!).$$

Then

$$\begin{aligned} \mathcal{F} - \rho_p \sum_{s\in S}(\alpha_s + L_s) - \left(i_0 + \sum_{s\in S}\sum_{l\in E} i_{s,l} + i_2 \right) \\ \geq \gamma_p \sum_{s\in T} \alpha_s - i_0 - \sum_{s\in T}\sum_{l\in E} i_{s,l} - \delta. \end{aligned} \tag{11.26}$$

Outline. Apply 11.7 to the three factorial families, controlling digit errors and the number of positive indices by the local degree bounds. Combine the degree identity with the local count relation and the upper bound on the second multiplicity family to obtain the estimate. The complete proof is on p. 555. □

11.5 A common estimate for every nonzero entry

Let K be an ultrametric normed field with $|p| = p^{-1}$, where $p > 2$ is prime, and let $\nu \geq 1$. Use variables $z_{s,a,b}$ with $s \in S$, $a < r_0$, $b < r_1$. Take the determinant product Q_0 of (11.6) with positions $s_c \in S$ and integral frame entries $M_{c,i,a}$.

Choose integers $e_{s,b} \geq 0$, supported on T, and integral vectors $\xi_s \in \mathcal{O}_K^{r_0}$. Define

$$P_{\mathrm{ent}}(z) = \prod_{s\in S,\ b<r_1} \left(\sum_{a<r_0} \xi_{s,a} z_{s,a,b} \right)^{e_{s,b}} Q_0(z), \tag{11.27}$$

$$\alpha_s = \sum_{b<r_1} e_{s,b}, \qquad C_s = \#\{c : s_c = s\}, \qquad L_s = \sum_{c:s_c=s} h_c, \tag{11.28}$$

$$D = \sum_s (\alpha_s + L_s). \tag{11.29}$$

Let H be (11.9), with integral u_s, and take $F \in \mathcal{C}_p(H)$. Under these entry hypotheses, the entry is $\mathcal{J} = \langle P_{\mathrm{ent}}, F \rangle$.

Theorem 11.9 (The common entry estimate)**.** *Under the entry hypotheses, either $\mathcal{J} = 0$, or there are an integer $E \geq 0$ and nonnegative multiplicities m_ϕ, β_κ for the families of 11.5 such that*

$$m_\phi > 0 \Longrightarrow s_\phi, t_\phi \in T, \qquad \beta_{(s,l,b)} > 0 \Longrightarrow s \in T, \tag{11.30}$$

$$\sum_\phi m_\phi d_\phi^A + \sum_\kappa \beta_\kappa d_\kappa^B \leq e, \tag{11.31}$$

$$D_b^{[\beta]} Q_0 \neq 0, \tag{11.32}$$

and

$$v_p(\mathcal{J}) \geq \gamma_p \sum_{s \in T} \alpha_s - \sum_\phi m_\phi - \sum_\kappa \beta_\kappa - \delta + \rho_p D. \tag{11.33}$$

Here b_κ is the direction in (11.15).

Outline. Use the chain rule to substitute on the series side, then apply 11.5 and 11.6 to obtain the multiplicities and a Hasse derivative monomial. Group linear factors by source and level, then apply 11.8 using factorial divisibility and the degree bounds. The complete proof is on p. 556. □

11.6 First distances and the ordinary gain

Lemma 11.10 (Charging losses to first distances)**.** *Fix $n, a, a', d \in \mathbf{N}$ and $s, s' \in \mathbf{Z}/n\mathbf{Z}$. For $n = 0$, interpret $\mathbf{Z}/0\mathbf{Z}$ as $\mathbf{Z}$. Suppose $a' < a$, $s + a = s' + a'$, and d is the least positive integer with $s + d = s'$. Then $d \leq a$.*

For the scalar estimate, take a finite subset T of $S = \mathbf{Z}/n\mathbf{Z}$, where $n \in \mathbf{N}$. Let $p > 1$ and δ be real and let $\alpha, C : S \to \mathbf{R}_{\geq 0}$. Give each s a finite set $D_s \subseteq \mathbf{N}$. Suppose nonnegative real arrays $i_{s,l}, \mathrm{ch}_{s,l}$ are supported in a common finite set of indices $l \in \mathbf{N}$ and satisfy

$$\mathrm{ch}_{s,l} = 0 \qquad \text{if } 2l + 1 \notin D_s, \tag{11.34}$$

$$\mathrm{ch}_{s,l} + i_{s,l} \leq \alpha_s p^{-(2l+1)} \qquad \text{if } s \in T,\ 2l + 1 \in D_s, \tag{11.35}$$

$$i_{s,l} \leq \min(\alpha_s p^{-(2l+1)}, C_{s+2l+1}) \qquad \text{if } s \in T,\ 2l + 1 \notin D_s. \tag{11.36}$$

If a real number V satisfies

$$V \geq \gamma_p \sum_{s \in T} \alpha_s - \sum_{s \in T, l} \mathrm{ch}_{s,l} - \sum_{s \in T, l} i_{s,l} - \delta,$$

then

$$V \geq \sum_{s \in T} \sum_{\substack{d \geq 1,\ d \text{ odd} \\ d \notin D_s}} [\alpha_s p^{-d} - C_{s+d}]_+ - \delta. \tag{11.37}$$

Outline. Minimality gives $d \leq a - a' \leq a$. The unused allowance is nonnegative at excluded odd distances and at least the positive part of the odd-distance weight minus C_{s+2l+1} at the others; summing these odd-distance terms (whose total is $\gamma_p \alpha_s$) and subtracting δ gives the bound. The complete proof is on p. 557. □

Theorem 11.11 (The ordinary entry bound)**.** *Under the entry hypotheses of 11.9, with D_s as in 11.1,*

$$|\mathcal{J}| \leq p^{-(g_A + \rho_p D)}. \tag{11.38}$$

Outline. For the multiplicities from 11.9, bound linear terms by source degree and target determinant count. Charge pure top terms to the first positive distance from the endpoint with larger shift, then use 11.10 to obtain the gain g_A in the valuation bound. The complete proof is on p. 557. □

11.7 Parity and the enhanced entry bounds

Theorem 11.12 (The second unmodified entry bound)**.** *Under the entry hypotheses,*

$$|\mathcal{J}| \le p^{-(g_B - A_{\mathrm{ev}} + \rho_p D)}. \tag{11.39}$$

Outline. For the multiplicities from 11.9, summing the even top degrees shows that the even-source and even-target counts together do not exceed A_{ev}. Bound the odd-source terms by source degree and target determinant count, then substitute into the common valuation estimate. The complete proof is on p. 558. □

Theorem 11.13 (The prescribed-column entry bound)**.** *Under the entry hypotheses, let $J \subseteq \{0, \dots, r_1 - 1\}$ be arbitrary. Define the nonnegative integer*

$$N_J = \sum_{\substack{s \text{ even} \\ b \notin J}} e_{s,b} + \sum_{\substack{s \text{ odd} \\ b \in J}} e_{s,b} + \sum_{c:\ s_c \text{ odd}} \#(\operatorname{im} I_c \cap J). \tag{11.40}$$

Then

$$|p^{N_J} \mathcal{J}| \le p^{-(g_B + \rho_p D)}. \tag{11.41}$$

Outline. For 11.9, sum top degrees at even labels outside J and odd labels in J, then use column bounds at odd targets to show N_J covers pure top and even-source terms. Bound odd-source terms at source and target and add N_J to the valuation bound. The complete proof is on p. 558. □

11.8 From entries to coefficients of the frame jet

At each s take $c_s \in \mathbf{N}$ columns of heights $h_{s,c} \in \mathbf{N}$, $h_{s,c} \le r_0$ for $0 \le c < c_s$. Choose maps

$$I_{s,c} : \{0, \dots, h_{s,c} - 1\} \longrightarrow \{0, \dots, r_1 - 1\}.$$

Use independent frame variables $\tau_{s,i,a}$, $s \in S$, $i, a < r_0$, and define

$$\mathcal{P}(\tau, z) = \prod_{s \in S} \prod_{c < c_s} \det \left(\sum_{a < r_0} \tau_{s,i,a} z_{s,a,I_{s,c}(j)} \right)_{0 \le i,j < h_{s,c}}. \tag{11.42}$$

For a power series $\Theta(z)$, its frame jet is the polynomial

$$\mathcal{J}_\Theta(\tau) = \sum_n n! \, [z^n]\Theta \, [z^n]\mathcal{P}(\tau, z). \tag{11.43}$$

This is the coefficientwise version of the operator in 10.15 and 10.18. For these diagrams put

$$\alpha_s = \begin{cases} \#\{c < c_s : h_{s,c} = 1\}, & s \in T, \\ 0, & s \notin T, \end{cases} \tag{11.44}$$

$$C_s = c_s - \alpha_s, \qquad L_s = \sum_{c < c_s} h_{s,c} - \alpha_s, \qquad D = \sum_s (\alpha_s + L_s) = \sum_s \sum_{c < c_s} h_{s,c}. \tag{11.45}$$

Use these counts in $\delta, g_A, g_B, A_{\mathrm{ev}}$.

Theorem 11.14 (Frame-jet coefficients of the unscaled kernel)**.** *Let K be a characteristic-zero nonarchimedean normed field. Fix a prime $p > 2$ with $|p| = p^{-1}$. Take $\pi \in K$ satisfying $\pi^{p^2-1} = p$. Take $\nu \ge 1$, $r_0, r_1 \in \mathbf{N}$, $u_s \in \mathcal{O}_K$, and H as in (11.9). Suppose $\Theta(\pi z) \in \mathcal{C}_p(H)$. For the diagrams and maps in (11.42), each coefficient $a_\eta = [\tau^\eta]\mathcal{J}_\Theta$ satisfies*

$$|a_\eta| \le p^{-g_A}, \tag{11.46}$$

$$|a_\eta| \le p^{-(g_B - A_{\mathrm{ev}})}. \tag{11.47}$$

For every integer $i \geq 1$, set

$$J_i = \{b : 0 \leq b < r_1,\ b < i\}, \qquad q_{s,c}(i) = \#(\operatorname{im} I_{s,c} \cap J_i),$$

and

$$M_i = \sum_{\substack{s \in S \\ s \text{ even}}} \sum_{c < c_s} \big(\min(h_{s,c}, i) - q_{s,c}(i)\big) + \sum_{\substack{s \in S \\ s \text{ odd}}} \sum_{c < c_s} q_{s,c}(i). \tag{11.48}$$

Then

$$|p^{M_i} a_\eta| \leq p^{-g_B}. \tag{11.49}$$

The differences in (11.45) *and* (11.48) *are nonnegative integers.*

Outline. Use the Gauss norm on $K(\tau)$ to bound frame coefficients and separate marked factors to apply 11.11 and 11.12. For prescribed columns, use $N_{J_i} \leq M_i$ and 11.13, then divide all bounds by $|\pi^D|$ to undo the degree-D scaling. The complete proof is on p. 558. □

For the target diagrams, 12.1 uses the ordinary coefficient bound. The bounds with exponents $g_B - A_{\mathrm{ev}}$ and g_B give the two estimates of 12.2.

Chapter 12

One lift on a lattice and the Hecke commutation

Fix a local system as in Definition 10.2 satisfying Definition 10.10 [3]. Write $\iota : O \to K$ for the coefficient map. For a uniformizer ϖ of O, put $e_O = \mathrm{ord}_\varpi(p)$. Put $k = k_1 + 1$ and $N_j = k + j(t+1)$. At level $0 \le j \le m$, the space $\mathcal{H}_j$ consists of polynomial-valued functions on components and tuples of submodules, with variables $X_{v,s,a,b}$ indexed by the places v, their embeddings s, and $1 \le a, b \le N_j$. A form in Λ_j is level-equivariant. At a component–lattice pair L it takes values in the weight lattice $H_j(L)$ of Definition 10.1, and it vanishes elsewhere. Put $V_j = K\Lambda_j$ and use

$$v_j(f) \ge g \quad \Longleftrightarrow \quad f \in \iota(\varpi)^{\lceil e_O g \rceil}\Lambda_j \qquad (g \in \mathbf{R}). \tag{12.1}$$

For the lifts K_j of Definition 10.2, put $K^{[0]} = 1$ and $K^{[b]} = K_{b-1}\cdots K_0$ for $1 \le b \le m$.

Let $\mathcal{P}$ be the finite set of local positions. At $v \in \mathcal{P}$, the discrete valuation ring W_v has uniformizer p and residue field of order $p^{n'_v}$. The embeddings are indexed in cyclic order by $s = 0, \dots, n'_v - 1$, where $n'_v \ge 2$ is even. Put $\eta_v = n'_v/2$. Write $\lambda_{v,s,1} \ge \cdots \ge \lambda_{v,s,k}$ for the initial weight and $w_{v,b}(s)$ for the level weights of Definition 10.2. If $\mathcal{P} \ne \varnothing$, then $p \ge 5$ and $1 \le r \le k/2$. At step $0 \le j < m$ use Definitions 10.2, 10.3, 11.1 and 11.2 to set

$$A_v = g_{A,v}(j+1), \qquad B_v = g_{B,v}(j+1), \qquad E_v = \sum_{\substack{s \in T_v \\ s \text{ even}}} \alpha_{v,s}, \qquad R_j(v) = \sum_{v' \ne v} A_{v'} - d_G(j). \tag{12.2}$$

Write T_v for the set of marked positions and $\alpha_{v,s}$ for the marked height-one column count of Definition 10.3. The bound on the p-valuation of the stabilizer orders is $d_G(j)$. We normalize $v_p(p) = 1$.

12.1 From the abstract kernel to the actual weights

Let $K_c \supseteq K$ and O_c be the kernel coefficient field and ring of Definition 10.6. The bound of Definition 10.19 is

$$\nu_c(z) \ge g \quad \Longleftrightarrow \quad z^b p^{-a} \in O_c \quad \text{for all } a \in \mathbf{Z},\ b \in \mathbf{Z}_{\ge 1} \text{ with } a \le bg. \tag{12.3}$$

When $e_O > 0$, $O_c \cap K = \iota(O)$ gives $p^{-1} \notin O_c$. For $z \ne 0$ set $v(z) = \sup\{a/b : z^b p^{-a} \in O_c,\ a \in \mathbf{Z},\ b \ge 1\}$. The rank-one condition applied to z, z^{-1} bounds this supremum above and below. The set of eligible rational exponents is downward closed, since $p^n \in O_c$ exactly for integers $n \ge 0$. Clearing denominators gives $v(xy) \ge v(x) + v(y)$. The valuation-ring dichotomy for exponents above $v(x)$ gives $v(x^{-1}) \ge -v(x)$. The product inequality applied to xx^{-1} and to $(xy)y^{-1}$ gives equality in both laws. The second rank-one condition gives $v(z) > 0$ for nonzero z in the maximal ideal. Thus $O_c = \{v \ge 0\}$ and $v(p) = 1$. On K we write $v_p = v|_K$ and identify O with $\iota(O) = O_c \cap K$. With $v(0) = +\infty$, (12.3) is equivalent to $v(z) \ge g$. Factoring out the summand of smaller value proves the ultrametric inequality. When $e_O = 0$, p is a unit of O_c and (12.3) is equivalent to $z \in O_c$.

Write $U^{\ell}(X) = U(\ell(X))$ for coordinatewise logarithmic substitution and call U integral after this substitution if $U^{\ell} \in O_c[[X]]$. For the pairing b of Definition 10.6, put $\Phi_v = \sum_{e \in E_v^{\mathrm{pair}}} b(X_{v,e}, Y_{v,e})$, with E_v^{pair} the pairing index set at v. Write $\mathcal{C}_j = \mathcal{T}_j$ for the substitution of Definition 10.6. With the determinant polynomial $P_{I,v}$ of Definition 10.15, put

$$\mathcal{J}_{I,v}(U) = P_{I,v}(\partial_Z)\mathcal{C}_j(\exp(-\Phi_v)U)|_{Z=0}\,. \tag{12.4}$$

The contraction leaves the auxiliary polynomial variables τ of Definition 10.18.

Theorem 12.1 (The ordinary coefficient bound for the actual weight diagram)**.** *For $v \in \mathcal{P}$, $0 \le j < m$, every U integral after logarithmic substitution, every column choice I, and every auxiliary multi-index n,*

$$\nu_c([\tau^n]\mathcal{J}_{I,v}(U)) \ge A_v. \tag{12.5}$$

Call $g \in \mathbf{R}$ an ordinary coefficient bound at (j,v) if $\nu_c([\tau^n]\mathcal{J}_{I,v}(U)) \ge g$ for every such U, I, n.

Outline. Delete the marked height-one columns from the diagram of Definition 10.3. Scale by π and use 10.13 and the phase of 10.12 to expand in $p^{-d}H_u^d$ with integral multipliers, then apply 11.14 for the norm bound p^{-A_v}. The complete proof is on p. 559. □

For $1 \le i \le r$, let $d_{s,c}(I,i) = \#\{a : 0 \le a < h_{s,c},\ I(s,c,a) < i\}$, with target labels starting at zero. The shifted factor of Definition 10.15 is

$$P_{I,v}^{(i)} = p^{\sum_{s,c}\epsilon_{s,c}(I,i)}P_{I,v}, \qquad \epsilon_{s,c}(I,i) = \begin{cases} d_{s,c}(I,i), & s \text{ odd}, \\ [\min(h_{s,c},i) - d_{s,c}(I,i)]_+, & s \text{ even}. \end{cases} \tag{12.6}$$

Theorem 12.2 (The shifted ordinary bound and the selected-column bound)**.** *For v, j, I, n, U as in Theorem 12.1,*

$$\nu_c([\tau^n]\mathcal{J}_{I,v}(U)) \ge B_v - E_v. \tag{12.7}$$

For $1 \le i \le r$,

$$\nu_c\Big([\tau^n]\ P_{I,v}^{(i)}(\partial_Z)\mathcal{C}_j(\exp(-\Phi_v)U)\Big|_{Z=0}\Big) \ge B_v. \tag{12.8}$$

Proof. Apply the second ordinary estimate of Theorem 11.14 to the diagram and scaled expansion of Theorem 12.1. Its exponent is $B_v - E_v$. A determinant with repeated target labels vanishes. Otherwise $d_{s,c}(I,i) \le \min(h_{s,c}, i)$ counts distinct labels below i. Thus (12.6) supplies the selected-column exponent in Theorem 11.14, which gives (12.8). □

12.2 Products of positions and the class expansion

Put $P_{I,\mathbf{i}} = \prod_v P_{I,v}^{(i_v)}$ for $\mathbf{i} = (i_v)_v$, $0 \le i_v \le r$, with $P_{I,v}^{(0)} = P_{I,v}$. For the framed series $\Theta = \Theta_{j,L,L',\phi}$ of Definition 10.10, put

$$\mathcal{J}_{I,\mathbf{i}}(\Theta) = P_{I,\mathbf{i}}(\partial_Z)\mathcal{C}_j(\Theta)|_{Z=0}\,. \tag{12.9}$$

Theorem 12.3 (Addition of the coefficient exponents)**.** *For $0 \le j < m$, every source and target lattice pair and every frame ϕ, there is a series U_1 integral after logarithmic substitution such that*

$$\Theta = \left(\prod_v \exp(-\Phi_v)\right) U_1. \tag{12.10}$$

If $q_v \in \mathbf{Q}$ satisfy

$$\begin{aligned} i_v = 0 &\implies q_v \text{ is an ordinary coefficient bound at } (j,v) \text{ or } q_v \le B_v - E_v, \\ i_v > 0 &\implies q_v \le B_v, \end{aligned} \tag{12.11}$$

then $\nu_c([\tau^n]\mathcal{J}_{I,\mathbf{i}}(\Theta)) \ge \sum_v q_v$ for every I, n.

Outline. Use phase equivalence in 10.10 to absorb the phase difference into an integral unit. Pull back logarithmic monomials by the inverse logarithm and contract by position, adding local bounds from 12.2 in products and preserving them in the finite sum. The complete proof is on p. 560. □

An index-i edge of full W_v-lattices A, B satisfies

$$pB \subseteq A \subseteq B, \qquad |B/A| = (p^{n'_v})^i. \tag{12.12}$$

At level $0 \le b \le m$ put

$$(T^{(b)}_{v,i} f)(c, \mathbf{L}) = \sum_{A \to_i L_v} f(c, \mathbf{L}[v \leftarrow A]). \tag{12.13}$$

Here $A \to_i L_v$ denotes an edge as above, and the operator on forms vanishes off lattice tuples. For $1 \le b \le m$ and $0 \le i \le r$ set

$$\begin{aligned} N'_{v,b}(i) &= \sum_{s \text{ even}} \sum_{a=1}^{i} \lambda_{v,s,a} + i \sum_{s \text{ odd}} w_{v,b}(s), \\ \widetilde{T}^{(b)}_{v,0} &= 1, \qquad \widetilde{T}^{(b)}_{v,i} = p^{N'_{v,b}(i)} T^{(b)}_{v,i} \quad (i > 0). \end{aligned} \tag{12.14}$$

For a list $v_1, \dots, v_a$ of distinct places containing the support of $\mathbf{i}$, set

$$\widetilde{T}^{(j+1)}_{\mathbf{i}} = \widetilde{T}^{(j+1)}_{v_1, i_{v_1}} \cdots \widetilde{T}^{(j+1)}_{v_a, i_{v_a}}. \tag{12.15}$$

Write $\ker_{j,L,L'}(G)$ for the differential kernel of Definition 10.10 applied to the source polynomial G.

Lemma 12.4 (The loss in summing lattice classes)**.** *Let $0 \le j < m$, $0 \le i_v \le r$, and use a list as in (12.15). Suppose that for every source and target lattice pair, every frame and every column choice I, all coefficients in (12.9) have lower bound $g \in \mathbf{R}$ in the sense of (12.3). Then for every $f \in \Lambda_j$,*

$$v_{j+1}(\widetilde{T}^{(j+1)}_{\mathbf{i}} K_j f) \ge g - d_G(j). \tag{12.16}$$

The common class factor c_K belongs to O_c and is a unit if nonzero. Each nonterminal denominator is a stabilizer order $|\Gamma_{j,L}|$, and each terminal denominator w_x is a positive integer. Both have p-valuation at most $d_G(j)$.

Outline. Use the class frames of 10.10 and 10.17 to put neighbours in the target fibre, then account for class denominators to get $g - d_G(j)$. Eliminate dependent vectors using coefficients of least uniformizer order and descend to K by a coordinate minor. The complete proof is on p. 560. □

Lemma 12.5 (Real exponents and the lattice endpoint)**.** *Let $0 \le j < m$, $0 \le i_v \le r$, and let $q_v \in \mathbf{R}$ satisfy (12.11). For a list as in (12.15) and every $f \in \Lambda_j$,*

$$v_{j+1}(\widetilde{T}^{(j+1)}_{\mathbf{i}} K_j f) \ge \sum_v q_v - d_G(j). \tag{12.17}$$

Proof. Put $s = \sum_v q_v$. For each $s' < s$, choose rationals $q'_v \le q_v$ with $\sum_v q'_v \ge s'$. Theorem 12.3 and Lemma 12.4 give the bound $s' - d_G(j)$. With $x = e_O(s - d_G(j))$, take $\delta = (x - \lceil x \rceil + 1)/(e_O + 1) > 0$. Then $\lceil x \rceil - 1 < x - e_O\delta \le \lceil x \rceil$. The bound at $s' = s - \delta$ therefore gives the stated lattice by (12.1). □

12.3 The three single-step estimates

Theorem 12.6 (The ordinary lift of an integral input)**.** *For $0 \le j < m$ and $f \in \Lambda_j$,*

$$v_{j+1}(K_j f) \ge \sum_v A_v - d_G(j). \tag{12.18}$$

For every lattice pair, frame, column choice I, and multi-index n, the coefficient $[\tau^n]\mathcal{J}_{I,\mathbf{0}}(\Theta)$ has bound $\sum_v A_v$. The integral-kernel hypothesis also gives

$$v_{j+1}(K_j f) \ge -d_G(j). \tag{12.19}$$

Outline. Take $i_v = 0, q_v = A_v$ by 12.1 and apply 12.5 to the empty list; tails in 11.1 give $A_v \in \mathbb{Q}$, so 12.3 gives the coefficient bound. At $\mathbf{i} = \mathbf{0}$, 10.10 gives bound 0 and 12.4 gives the integral-kernel bound. The complete proof is on p. 560. □

Theorem 12.7 (The distinguished-position bound for an ordinary lift)**.** *For $v \in \mathcal{P}$, $0 \le j < m$, and $f \in \Lambda_j$,*

$$v_{j+1}(K_j f) \ge B_v - E_v + R_j(v). \tag{12.20}$$

Proof. Set $q_v = B_v - E_v$ by Theorem 12.2 and $q_{v'} = A_{v'}$ for $v' \ne v$. Apply Lemma 12.5 with all indices zero. □

At level b and place v, let κ_s be the row obtained by appending to $\lambda_{v,s}$, for $l = 1, \dots, b$ in order, $t+1$ copies of $-w_{v,l}(s)$. The actual weight is κ_s at even s and its negative reverse at odd s. Put

$$\mathcal{A}_i(\kappa) = \sum_{s \text{ even}} \sum_{a=1}^{i} \kappa_{s,a} - \sum_{s \text{ odd}} \sum_{a=N_b-i+1}^{N_b} \kappa_{s,a}. \tag{12.21}$$

Theorem 12.8 (An output Hecke operator)**.** *Fix $v \in \mathcal{P}$. For $0 \le j < m$ and $1 \le i \le r$,*

$$\widetilde{T}_{v,i}^{(j+1)} \Lambda_{j+1} \subseteq \Lambda_{j+1}, \qquad v_{j+1}(\widetilde{T}_{v,i}^{(j+1)} K_j f) \ge B_v + R_j(v) \quad (f \in \Lambda_j). \tag{12.22}$$

At any level $0 \le b \le m$, for $0 \le i \le N_b$ one has $p^{\mathcal{A}_i(\kappa)} T_{v,i}^{(b)} \Lambda_b \subseteq \Lambda_b$.

Outline. Apply 12.5 with B_v at v and $A_{v'}$ elsewhere. Bound the rescaling exponent using row degrees and the determinant twist, tensor over embeddings, and sum over edges as in 12.4, using the even and odd row bounds for the level estimate. The complete proof is on p. 561. □

12.4 The operators and the neighbour count

Definition 12.9 (The input operator and the output polynomial)**.** Use (12.12), (12.13) and (12.14). At the initial weight put

$$\mathcal{A}_v = \sum_{s \text{ even}} \sum_{a=1}^{r} \lambda_{v,s,a} - \sum_{s \text{ odd}} \sum_{a=k-r+1}^{k} \lambda_{v,s,a}, \qquad y_v = p^{\mathcal{A}_v} T_{v,r}^{(0)}. \tag{12.23}$$

For $u_l, \mathfrak{m}_l \in K$, $u_l \ne 0$, $1 \le l \le b$, set

$$U_b = \prod_{l=1}^{b} u_l, \qquad \beta_{v,a} = p^{\eta_v (t-2a)} \ (0 \le a \le t), \qquad \mathcal{F}_{v,b} = (\mathfrak{m}_l \beta_{v,a})_{1 \le l \le b,\ 0 \le a \le t}. \tag{12.24}$$

With h_ℓ the complete homogeneous symmetric polynomial and $h_0 = 1$, put

$$\begin{aligned} c_{v,b,\ell} &= p^{N'_{v,b}(r) - N'_{v,b}(r-\ell) + \eta_v \ell (N_b - 2r + \ell)} h_\ell(\mathcal{F}_{v,b}) \quad (0 \le \ell \le r), \\ \widetilde{T}_{v,i}^{u,(b)} &= U_b^{-i} \widetilde{T}_{v,i}^{(b)}, \qquad Z_{v,b} = \sum_{\ell=0}^{r} (-1)^\ell c_{v,b,\ell} \widetilde{T}_{v,r-\ell}^{u,(b)}. \end{aligned} \tag{12.25}$$

Lemma 12.10 (The theta series as a function of a lattice indicator)**.** *Fix $v \in \mathcal{P}$ and $0 \le j < m$. Put $N = N_j$, $N' = N_{j+1}$, $\eta = \eta_v$ and $M = \mathrm{Mat}_{N' \times N}(\mathrm{Frac}\, W_v)$. For source and target lattice pairs L, L', set*

$$I_{L,L'}(x) = \mathbf{1}_{x L_v \subseteq L'_v}, \qquad s = \ell_{v,j}(L), \quad s' = \ell_{v,j+1}(L'), \quad d = N s' - N' s.$$

Here $\ell_{v,j}$ and $\ell_{v,j+1}$ are the integer lengths of Definition 10.10 at the source and target levels. A lower edge of index h decreases the length at its place by h and preserves the lengths at the other places. Replacing either v-entry preserves $\omega(L, L')$. There are $a, a' \in K^\times$ of valuation zero, common to all lattice pairs at v, j, and for each ω an additive map $\Xi_\omega : (M \to \mathbf{Q}) \to K_c[[Z_j]]$ with

$$\Theta_{L,L'}^{F} = a^s (a')^{s'} p^{-\eta d}\, \Xi_{\omega(L,L')}(I_{L,L'}). \tag{12.26}$$

Here $\Theta_{L,L'}^{F}$ is the theta series in the fixed coordinates Z_j. Lemma 12.15 gives the arithmetic charts and shows how the units and the representative law yield this identity on the selected chain.

Outline. In an abstract system, item 4 of 10.10 supplies the units and additive maps directly. For an arithmetic system, taking signed determinant lengths and the data outside v lets 12.15 choose the units and identify chart and representative values. The complete proof is on p. 561. □

Lemma 12.11 (Changing either lattice of the arithmetic section)**.** *Let E/K_0 be the host field over its imaginary quadratic subfield, and let Ψ consist of the embeddings of E inducing the chosen embedding of K_0. Write $L_n^0 = \mathcal{O}_E^n$. For full lattices A, B of the same rank, let $\mathfrak{d}(A,B)$ be the fractional ideal whose localization at a finite prime is generated by the determinant of a change from a basis of A to one of B. Put $c(A) = \chi(\mathfrak{d}(L_n^0, A))$. Here χ is a multiplicative character on norm-one fractional ideals of E satisfying $\chi((z)) = \prod_{\tau\in\Psi} \tau(z)^{-1}$ whenever $z \in E^\times$ has $z\bar{z} = 1$, and A is self-dual for the standard Hermitian form.*

For self-dual A, B of even ranks n, n', respectively, put

$$P(A,B) = \{(\delta\tau(X))_{\tau\in\Psi} : X \in \mathfrak{c}\mathcal{O}_E\, \mathfrak{b}^{-1} \operatorname{Hom}_{\mathcal{O}_E}(A,B)\}, \qquad C(A,B) = c(A)^{n'/2} c(B)^{-n/2}.$$

Here $\mathfrak{c}, \delta$ are the curve ideal and period multiplier, and $\mathfrak{b}\bar{\mathfrak{b}} = \mathfrak{D}_{E/K_0}$ as in Lemma 17.15. For period lattices P, Q write

$$I_{P,Q}\theta = [Q : P \cap Q]^{-1/2} \sum_{x\in Q/(P\cap Q)} W_x\theta,$$

where W_x is the theta translation including the semicharacter of Q. For the arithmetic sections $e_{A,B}$, the section $\theta_{A,B} = C(A,B)^{-1} e_{A,B}$ changes by $I_{P,Q}$ when either endpoint changes. Moreover, rational unitary changes g, g' of the two endpoints give $e_{gA,g'B}(Z) = e_{A,B}((g')^{-1} Z g)$, with the matrices read at the embeddings in Ψ.

Outline. Adapt the Weil moves (17.3) and (17.2) to the chosen character and divide by the new determinant factor. The ratio of the two determinant factors is invariant under rational unitary changes, so (17.1) gives the pullback. The complete proof is on p. 561. □

Definition 12.12 (Finite test of a decorated pair)**.** Fix a word, a twist, chart components at stages $i, i+1$, and their selected local Schwartz vectors. At a prime P of the word level, let l_P, l'_P be the source and target residue functionals on the corresponding lattices N_P, N'_P. Give Z the factor

$$\mathbf{1}_{\{ZN_P \subseteq N'_P,\ l'_P Z = l_P\}}.$$

At each type or twist prime P, read Z in the adapted source and target frames as $X = f'^{-1} Z f$. Put $q = |k(P)|$, and write ψ for the selected additive character. The factor at P is as follows.

At a matrix step put $a = N_i,\ b = N_{i+1},\ d = m-1-i,\ d' = d-1$. Use indices starting at zero. Complete X to a $(b+d') \times (a+d)$ matrix D. Its first a columns are $\binom{X}{0}$. In an extra column $a+j$, the free rows are all rows if $j = d'$, and the rows $l < b+j$ if $1 \le j < d'$. In all other extra-column entries put $\mathbf{1}_{l=b+j}$. Denote the set of free entries by F. Let A, B be the support and period depths, and set

$$D_{A,B} = \{q^{-A}u : 0 \le u < q^{A+B},\ u \in \mathbf{Z}\}.$$

For $t \in D_{A,B}^F$ let $D(t)$ be the completion with free entries t. Let $\Phi_{h,h'}$ be the selected vector transported by the source and target component matrices h, h'. In enlarged dimensions a_1, b_1, with local step character μ, valuation $v(q) = 1$ and chosen square root s_q of q, this transport is

$$\Phi_{h,h'}(D) = \frac{\mu(\det h)}{\mu(\det h')} s_q^{-b_1 v(\det h) + a_1 v(\det h')} \Phi(h'^{-1} D h).$$

The local factor is

$$\frac{q^{-B|F|}}{\operatorname{vol}(U)} \sum_{t\in D_{A,B}^F} \Phi_{h,h'}(D(t))\, \psi\left(-\sum_{j=0}^{d'-1} D(t)_{b+j,a+j+1}\right).$$

Here U is the intersection of the transported target level with the unipotent group fixing the first b coordinates pointwise, fixing the first extra basis vector and acting trivially on the graded lines of the extra-coordinate flag. Its Haar measure gives the stabilizer of the reference lattice in this unipotent group volume one.

At $i = m-2$, take a vector y' in the marked target coset and express it in the target frame. The local factor is $\Phi_{h,1}([X \mid y'])$. At $i = m-1$, write $X = \binom{X_0}{\eta}$, take y in the marked source coset in the source frame, and use

$$\Phi_{\mathrm{top}}(X_0)\psi(-\eta y)\begin{cases}\mathbf{1}_{\eta\in\mathcal{O}_P^a}, & P \text{ a type prime},\\ q^{-Ma}\mathbf{1}_{q^M\eta\in\mathcal{O}_P^a}, & P \text{ a twist prime}.\end{cases}$$

Here M is the twist depth, and Φ_{top} is the selected vector for the target with its last row removed. The product of all the factors, read in the fixed complex embedding, is $\xi_{y,y'}(Z)$ for the adelic pair y, y'. The vectors used to represent the two marked cosets are taken at their respective stages.

Lemma 12.13 (Class characters and the finite test in arithmetic charts)**.** *Fix the chart data of a constructed column chain, a word and a twist. Let $H \subset \mathbf{C}$ be its global coefficient field. Denote the compact level groups at consecutive components by U_j, U_{j+1} and their characters by λ_j, λ_{j+1}. Write μ_{j+1} for the step character on norm-one finite idèles and χ_I for the determinant-ideal character on these idèles. Put $b_l = \lfloor l(t+1)/2 \rfloor$. If level l has a component, then*

$$\chi_I(x)^{b_l} \in H^\times \qquad (x\bar{x} = 1).$$

Let S be the primes of E used for the finite test, and $c > 0$ its level. The test $\xi_{y,y'}(Z)$ at finite adelic points y, y' in fixed components depends on their components at the places below S. For rational unitary g, g', expressed in standard orthogonal coordinates, and $k \in U_j$, $k' \in U_{j+1}$, it satisfies

$$\xi_{gy,g'y'}(Z) = \xi_{y,y'}((g')^{-1}Zg),$$
$$\mu_{j+1}(\det k')\lambda_j(k)\,\xi_{yk,y'k'}(Z) = \mu_{j+1}(\det k)\lambda_{j+1}(k')\,\xi_{y,y'}(Z).$$

If $M = \mathfrak{c}\mathcal{O}_E\,\mathfrak{b}^{-1}\operatorname{Hom}(A,B)$ for the two kernel lattices, the test has the same value at Z_1, Z_2 whenever $Z_1 - Z_2 \in cM_Q$ at every $Q \in S$.

Outline. Use uniformizer values and the principal-ideal law to place the character powers in the coefficient field. Evaluate the local tests in transported frames using 12.12 and reindex the Fourier sums to verify congruence invariance and the two transformation laws. The complete proof is on p. 562. □

Lemma 12.14 (One finite operator for a family of lattice pairs)**.** *In the arithmetic charts of Lemma 12.13, fix the components at both ends and a finite family of self-dual pairs differing only at one split pair $P, \bar{P}$. Let S be the support of the finite test, disjoint from $\bar{S}$, and let $c > 0$ be its level. Assume $P, \bar{P} \notin S \cup \bar{S}$ and both are prime to c. The operation of applying the finite test to the arithmetic section is the restriction, at every pair in the family, of one complex linear operator on functions of the period variable. It takes entire arithmetic sections to functions with algebraic Taylor coefficients in the normalized differential coordinates of the curve.*

Outline. Use the Chinese remainder theorem and 12.13 to get one Fourier sum for every pair. Algebraicity follows from torsion translates, the fourth-power normalization 17.14, endpoint averages 12.11, and the test values in 12.12. The complete proof is on p. 563. □

Lemma 12.15 (The indicator identity on the arithmetic charts)**.** *At a permitted site of Definition 13.7, fix the selected chain on its prescribed filling. Use its compatible coefficient embeddings, the curve and parameter of Lemma 37.25, and the finite test of Lemma 12.13. Take a positive test level c and a test support disjoint from its conjugate. The split pair at each moving position lies off this support and away from the primes dividing c. For the selected even ranks and self-dual kernel lattices, the chart series has (12.26), with the length of a lattice GW_v^n at v given by $-\operatorname{ord}_p \det G$ and with ω recording both components and all entries away from v. Under a source level-group element γ with local chart matrices G_w, writing $\tau_{w,s}$ for the indicated embedding, $\chi_{\mathrm{lev}}(\gamma)$ for the local coefficient realization of the level character, and S_G for the source-row substitution by $\tau_{w,s}(G_w)^{\mathsf{T}}$ at even s and $\tau_{w,s}(G_w)^{-1}$ at odd s, it satisfies the source transformation*

$$\Theta^F_{\gamma L,L'} = \kappa(\gamma)^{-1}S_G\Theta^F_{L,L'}, \qquad \kappa(\gamma) = \chi_{\mathrm{lev}}(\gamma)\prod_{w,s}\tau_{w,s}(\det G_w)^{(-1)^s w_{w,j+1}(s)}\prod_w p^{-\Delta_w(j+1)\operatorname{ord}_w\det G_w}, \tag{12.27}$$

and hence agrees with the series transported from source representatives.

Outline. Compute the units from character valuations, then use finite Fourier sums and 12.11 to map indicators to volume-scaled sections. The common test and algebraic coefficients of 12.14 transfer relations to chart series, whose source law follows from 12.13. The complete proof is on p. 564. □

Lemma 12.16 (Evaluation at a fixed lattice pair)**.** *Use the notation of Lemma 12.10 and write s_w, s'_w for the lengths at w. Set $D(L,L') = \prod_w p^{\Delta_w(j+1)(s_w - s'_w)}$. For a source polynomial G and $b : M \to \mathbf{Q}$, define*

$$J_{L,L'}(b;G) = D(L,L')a^s(a')^{s'}p^{-\eta d}\mathcal{D}_j(G, \Xi_{\omega(L,L')}(b)),$$

where the differential evaluation $\mathcal{D}_j$ of Lemma 10.17 is K-linear in each variable. Then $\ker_{j,L,L'}(G) = J_{L,L'}(I_{L,L'};G)$. For finitely many $b_i : M \to \mathbf{Q}$ and $n_i \in \mathbf{N}$,

$$J_{L,L'}\Big(\sum_i n_i b_i; G\Big) = \sum_i n_i J_{L,L'}(b_i;G).$$

Proof. Substitute (12.26) in $D(L,L')\mathcal{D}_j(G, \Theta^F_{L,L'})$, the fixed-coordinate expression for the kernel. Additivity gives the multiplicity formula. □

Lemma 12.17 (Changing the endpoint in a weighted edge sum)**.** *Fix $v \in \mathcal{P}$ and $h \geq 0$. At $0 \leq j \leq m$, write $A \to_h B$ for an index-h edge at v with the same component and entries away from v. Put $w(A) = |\Gamma_{j,A}|$. For any K_c-valued function ϕ invariant under simultaneous level translation,*

$$\sum_{A \in \mathcal{R}_j} \frac{1}{w(A)} \sum_{B:\, A \to_h B} \phi(A,B) = \sum_{B \in \mathcal{R}_j} \frac{1}{w(B)} \sum_{A:\, A \to_h B} \phi(A,B). \tag{12.28}$$

For $j < m$, $f \in V_j$, and a target lattice pair L', the class data of Lemma 12.4 satisfy

$$\sum_x \frac{1}{w_x} \sum_y \sum_{L_{x,y} \to_h B} \ker_{j,B,L'}(f(L_{x,y})) = \sum_x \frac{1}{w_x} \sum_y \ker_{j,L_{x,y},L'}((T^{(j)}_{v,h} f)(L_{x,y})). \tag{12.29}$$

In the nonterminal case (where the lift is given by (10.1)), the sum (12.29) is indexed by representatives x with $w_x = w(L_x)$ and a single value of y.

Outline. Divide each orbit-pair sum by the stabilizer orders and invert the level elements to swap endpoints. Apply coefficientwise using 12.8 for the target fibre and 10.10 for equivariance and terminal neighbours, then use linearity to obtain the input Hecke sum. The complete proof is on p. 567. □

Lemma 12.18 (Neighbour counting and the one-step Hecke relation)**.** *Let W be a discrete valuation ring with uniformizer p. Its fraction field is Q_W, and its residue field has order $q = p^{n'}$. Let N, g, i be nonnegative integers. Take full lattices $A \subset Q_W^N$, $B \subset Q_W^{N+g}$ and a linear map $x : Q_W^N \to Q_W^{N+g}$. Then*

$$\begin{aligned} &\#\{B_1 : pB \subseteq B_1 \subseteq B,\ |B/B_1| = q^i,\ xA \subseteq B_1\} \\ &\quad = \sum_{\ell=0}^{\min(i,g)} q^{(i-\ell)(g-\ell)} \begin{bmatrix} g \\ \ell \end{bmatrix}_q \#\{A_1 : A \subseteq A_1 \subseteq p^{-1}A,\ |A_1/A| = q^{i-\ell},\ xA_1 \subseteq B\}. \end{aligned} \tag{12.30}$$

Here $\begin{bmatrix} g \\ \ell \end{bmatrix}_q$ is the Gaussian binomial coefficient. For $v \in \mathcal{P}$ put $\mathcal{E}^{(b)}_{v,i} = p^{-\eta_v i(N_b - i)} T^{(b)}_{v,i}$. There are $u_l, \mathfrak{a}_l \in K^\times$, $1 \leq l \leq m$, of valuations $0, \Delta_v(l)$, respectively, such that for $0 \leq j < m$, $i \geq 0$, $f \in V_j$,

$$\mathcal{E}^{(j+1)}_{v,i} K_j f = u^i_{j+1} K_j \left(\sum_{\ell=0}^{i} e_\ell((\mathfrak{a}_{j+1}\beta_{v,a})^t_{a=0}) \mathcal{E}^{(j)}_{v,i-\ell} f \right). \tag{12.31}$$

Here e_ℓ is elementary symmetric, with $e_0 = 1$ and $e_\ell = 0$ for $\ell > t+1$.

Outline. Count graphs with prescribed intersection and image for the Gaussian convolution. Evaluate the indicator identity via 12.10 and 12.16, use 12.17 for Hecke sums preserving V_j by 12.8, then rewrite Gaussian coefficients and collect powers. The complete proof is on p. 567. □

Definition 12.19 (The exact commutation exponent)**.** For $v \in \mathcal{P}$ and $b \geq 1$, put

$$P_v^{\mathbf{Z}}(b) = \sum_{s \text{ odd}} \left(r w_{v,b}(s) + \sum_{a=k-r+1}^{k} \lambda_{v,s,a} \right) + \eta_v r b(t+1). \tag{12.32}$$

Define $P_v(b)$ by the same sum in $\mathbf{R}$, with last term $n'_v r b(t+1)/2$. Since $n'_v = 2\eta_v$, the image of $P_v^{\mathbf{Z}}(b)$ in $\mathbf{R}$ is $P_v(b)$.

12.5 Commutation along a prefix

Theorem 12.20 (The prefix Hecke commutation and its integral coefficients)**.** *For each $v \in \mathcal{P}$ there are $u_l, \mathfrak{m}_l \in K^\times$, $1 \leq l \leq m$, with*

$$v_p(u_l) = 0, \qquad v_p(\mathfrak{m}_l) = \Delta_v(l), \tag{12.33}$$

such that for $1 \leq b \leq m$ and $f \in V_0$,

$$Z_{v,b} K^{[b]} f = p^{P_v^{\mathbf{Z}}(b)} K^{[b]}(y_v f). \tag{12.34}$$

The coefficients satisfy

$$c_{v,b,\ell} \in O \quad (0 \leq \ell \leq r), \qquad v_p(c_{v,b,r}) \geq E_v. \tag{12.35}$$

At $b = j+1$, every $f \in \Lambda_j$ satisfies

$$v_{j+1}(Z_{v,j+1} K_j f) \geq B_v + R_j(v). \tag{12.36}$$

Outline. Normalize and iterate 12.18, then extract the coefficient of z^r using the reciprocal product. Use 10.2 for integrality and extra identity-term valuation, then apply 12.8 to positive-index terms and 12.7 to the identity term. The complete proof is on p. 568. □

Chapter 13

Induction along the chain

Let X be a local system at a prime p as in 10.2, with finite place set $\mathcal{P}$ and chain length m. The weight conditions give $m \geq 1$ if $\mathcal{P} \neq \varnothing$. Let K be the coefficient field and O the coefficient DVR with uniformizer ϖ and coefficient map $\iota : O \to K$. Write $(p) = (\varpi^e)$ in O, where $e = e_O \geq 0$.

Let $\Lambda_j \subset M_j$ be the integral lattice in the space of forms at level j, and write $K\Lambda_j$ for its K-span. For the lifts K_j, put

$$K^{[0]} = 1, \qquad K^{[b]} = K_{b-1} \cdots K_0 : M_0 \longrightarrow M_b.$$

Write $\nu_j(F) \geq h$ for $F \in \varpi^{\lceil eh \rceil}\Lambda_j$, and $\nu_K(a) \geq h$ for $a \in \varpi^{\lceil eh \rceil}O$, with scalars mapped into K by ι. Write $\nu_K(a) = h$ when h is the greatest bound for a. If $e > 0$ and $K = \mathrm{Frac}(O)$, the scalar relation uses the valuation normalized by $\nu_K(p) = 1$. Write $d_j = d_G(j)$ for the stabilizer loss and $[h]_+ = \max(h, 0)$.

For $v \in \mathcal{P}$ and $1 \leq b \leq m$, write $A_v(b) = g_{A,v}(b)$ and $B_v(b) = g_{B,v}(b)$ as in 11.1–11.2. Let $P_v(b)$ be the real image of the integer exponent $P_v^{\mathbf{Z}}(b)$ of 12.19. For $s \in T_v$, put $\alpha_{v,s} = \lambda_{v,s,1} - \lambda_{v,s,2}$ as in 10.3, and set $E_v = \sum_{s \in T_v,\ s \text{ even}} \alpha_{v,s}$.

Fix kernel data Y for X. The lifts K_j and input operators y_v are K-linear. So are the normalized output operators $\widetilde{T}_{v,i,b}$ of 12.9. The y_v commute. Write $\mathcal{L}_{v,j}(h)$ for the single-place bound of Definition 10.19: for every series allowed there and every column choice, each frame coefficient z of the differentiated local kernel at (j, v) satisfies $z^b p^{-a} \in O_c$ whenever $a \in \mathbf{Z}$, $b \in \mathbf{N}_{>0}$, and $a \leq bh$.

For each place, choose sequences $(u_{v,a})_{a=1}^m$ and $(\mathfrak{m}_{v,a})_{a=1}^m$ in K. For $1 \leq b \leq m$, use the coefficients $c_{v,b,\ell}$ of (12.25) in 12.9 for these sequences. Put $U_v(b) = \prod_{a=1}^b u_{v,a}$, and let r be the system parameter. The operator of 12.9 is

$$Z_{v,b} = \sum_{\ell=0}^{r} (-1)^\ell c_{v,b,\ell} U_v(b)^{-(r-\ell)} \widetilde{T}_{v,r-\ell,b}, \qquad \widetilde{T}_{v,0,b} = 1.$$

At a fixed layer $1 \leq b \leq m$, the single-place relation is

$$Z_{v,b} K^{[b]} F = p^{P_v^{\mathbf{Z}}(b)} K^{[b]} y_v F \qquad (v \in \mathcal{P},\ F \in K\Lambda_0). \tag{13.1}$$

13.1 Several places and the input lattice

For an ordered list $L = (v_1, \ldots, v_a)$, put

$$Y_L = y_{v_1} \cdots y_{v_a}, \qquad Z_{L,b} = Z_{v_1,b} \cdots Z_{v_a,b}.$$

The rightmost operator acts first, and empty products are identities. For a subset C of the places in L, let $Z_{C,b} = Z_{L|_C,b}$, where $L|_C$ retains the order in L. For a set C of places use $Y_C = \prod_{v \in C} y_v$, which is well defined by commutativity.

Lemma 13.1 (Simultaneous operators at distinct places)**.** *Let L be a list of distinct places. Fix $1 \leq b \leq m$. Suppose $p \neq 0$ in K and every y_v preserves $K\Lambda_0$. Assume (13.1) at this layer. Then for $F \in K\Lambda_0$,*

$$Z_{L,b} K^{[b]} F = p^{\sum_{v \in L} P_v^{\mathbf{Z}}(b)} K^{[b]} Y_L F. \tag{13.2}$$

For the estimate, fix $0 \le j < m$. Assume $K = \mathrm{Frac}(O)$. At this step, assume the joint coefficient estimate of 12.3 under (12.11) and the coefficient-to-lattice implication of 12.4. Suppose that for every $v \in \mathcal{P}$,

$$u_{v,a} \in O^\times \quad (1 \le a \le m), \qquad \nu_K(c_{v,j+1,\ell}) \ge 0 \quad (0 \le \ell \le r), \qquad \nu_K(c_{v,j+1,r}) \ge E_v.$$

Let $q_v \in \mathbf{R}$ for every $v \in \mathcal{P}$. Suppose $q_v \le B_v(j+1)$ for $v \in L$ and $\mathcal{L}_{v,j}(q_v)$ holds for $v \notin L$. Then every $F \in \Lambda_j$ satisfies

$$\nu_{j+1}\big(Z_{L,j+1} K_j F\big) \ge \sum_{v \in \mathcal{P}} q_v - d_j. \tag{13.3}$$

Outline. Induct on the places using the single-place identity. Expand the output, lower the bound by E_v at zero indices, and apply the real-exponent argument of 12.5 with 12.3 and 12.4, restoring the subtracted amounts with the scalar coefficient. The complete proof is on p. 569. □

Definition 13.2 (The lattice allowing enhanced estimates)**.** For $V \subseteq \mathcal{P}$ and $S : \mathcal{P} \to \mathbf{R}$, let $I(S,V) \subseteq M_0$ be the O-span of the vectors

$$y_{v_1}^m \cdots y_{v_a}^m \left(\iota(\varpi)^{\sum_{v \in V \setminus C} \lceil e S_v \rceil} F_0\right), \qquad C \subseteq V, \quad F_0 \in \Lambda_0, \tag{13.4}$$

where $(v_1, \dots, v_a)$ ranges over enumerations of C without repetition.

Commutativity gives $y_{v_1}^m \cdots y_{v_a}^m = Y_C^m$.

13.2 Prefix induction and the maximum at each layer

For Lemma 13.3 and Theorem 13.4, assume $K = \mathrm{Frac}(O)$ and $p \ne 0$ in K. Require $y_v(\Lambda_0) \subseteq \Lambda_0$ for every v. Assume the joint coefficient estimate and coefficient-to-lattice implication used in Lemma 13.1 at every step. Use sequences satisfying (13.1) at every layer and the unit and coefficient bounds of Lemma 13.1 at every step.

Lemma 13.3 (Prefix estimate for an arbitrary selection rule)**.** *Let L be a list of distinct places. Let $G^+, G^0 : \mathcal{P} \times \mathbf{N} \to \mathbf{R}$ satisfy, for $0 \le j < m$,*

$$G_v^+(j+1) \le B_v(j+1) \quad (v \in L), \qquad \mathcal{L}_{v,j}\big(G_v^0(j+1)\big) \quad (v \in \mathcal{P}).$$

For each $0 \le j < m$, choose a subset C_j of the places in L and put

$$\beta_j = \sum_{v \in C_j} G_v^+(j+1) + \sum_{v \notin C_j} G_v^0(j+1) - d_j - \sum_{v \in C_j} P_v(j+1). \tag{13.5}$$

Then, for every $0 \le b \le m$ and $F \in \Lambda_0$,

$$\nu_b\big(K^{[b]} Y_L^b F\big) \ge \sum_{j=0}^{b-1} \beta_j. \tag{13.6}$$

Outline. Induct on the prefix length, commuting the complementary operators into the integral input. Use 13.1 to replace the selected operators and divide out the resulting power of p, then apply the ceiling inequality to gain β_b. The complete proof is on p. 569. □

Theorem 13.4 (Chain bound with external numerical estimates)**.** *Fix $V \subseteq \mathcal{P}$ and $S : \mathcal{P} \to \mathbf{R}$. Let $G^+, G^0, P^* : \mathcal{P} \times \mathbf{N} \to \mathbf{R}$. Suppose*

$$\begin{aligned} G_v^+(b) \le B_v(b), \qquad & P_v(b) \le P_v^*(b) && (v \in V,\ 1 \le b \le m), \\ \mathcal{L}_{v,j}\big(G_v^0(j+1)\big) & && (v \in \mathcal{P},\ 0 \le j < m), \\ \sum_{b=1}^{m} [G_v^+(b) - P_v^*(b) - G_v^0(b)]_+ \le S_v & && (v \in V). \end{aligned}$$

Then every $F \in I(S,V)$ satisfies

$$\nu_m(K^{[m]}F) \geq T, \tag{13.7}$$

where

$$T = \sum_{b=1}^{m} \left(\sum_{v \in V} \max\{G_v^+(b) - P_v^*(b), G_v^0(b)\} + \sum_{v \notin V} G_v^0(b) \right) - \sum_{j=0}^{m-1} d_j. \tag{13.8}$$

Outline. Apply 13.3 to each generator of 13.2, selecting places in C with $G_v^+ - P_v^* \geq G_v^0$. Use $P_v \leq P_v^*$ for the bound on C, cover rounded shortfalls off C with the uniformizer powers, and extend by linearity. The complete proof is on p. 569. □

13.3 The scalar budget and the local theorem

Lemma 13.5 (Budget for enhancement)**.** *Fix $p, n', k_1, t, m, c, r \in \mathbf{N}$ and $T \subseteq \{0, \dots, n'-1\}$. Let the functions*

$$a, b : \{0, \dots, n'-1\} \to \mathbf{Z}, \quad \lambda : \{0, \dots, n'-1\} \times \{1, \dots, k_1+1\} \to \mathbf{Z}, \quad w : \mathbf{N} \times \{0, \dots, n'-1\} \to \mathbf{Z}$$

satisfy the standard weight conditions of 10.2. In particular $p \geq 5$ is prime, $n' \geq 2$ is even, $r \geq 1$, $2r \leq k_1+1$, and $m \geq 1$. With the gains $g_A(j), g_B(j)$ of 11.1–11.2 and the price $P(j)$ of 12.19, set

$$S = \sum_{j=1}^{m} [g_B(j) - P(j) - g_A(j)]_+, \qquad \gamma_p = \frac{p}{p^2-1}.$$

Then $0 \leq S \leq m\gamma_p \sum_{s \in T} \alpha_s$.

Outline. Dominance and monotonicity make capacities in 10.3 nonnegative, so 11.1, 11.2, and 10.4 give $g_A(j) \geq -\delta_j$ and $g_B(j) \leq \gamma_p \sum_{s \in T} \alpha_s - \delta_j$. Since $P(j) \geq 0$ by 12.19, summing over j gives the bound. The complete proof is on p. 570. □

Theorem 13.6 (Chain bound for the original local parameters)**.** *Assume $\mathcal{L}_{v,j}(A_v(j+1))$ for every $v \in \mathcal{P}$ and $0 \leq j < m$. Assume also that for every choice of G^+, G^0, P^*, S, V satisfying the four conditions of Theorem 13.4, inequality (13.7) holds for every $F \in I(S,V)$. For each v, set*

$$S_v = \sum_{b=1}^{m} [B_v(b) - P_v(b) - A_v(b)]_+.$$

Then $S_v \leq m\gamma_p \sum_{s \in T_v} \alpha_{v,s}$, and for every $V \subseteq \mathcal{P}$ and $F \in I(S,V)$,

$$\nu_m(K^{[m]}F) \geq \sum_{b=1}^{m} \left(\sum_{v \in V} \max\{B_v(b) - P_v(b), A_v(b)\} + \sum_{v \notin V} A_v(b) \right) - \sum_{j=0}^{m-1} d_j. \tag{13.9}$$

Proof. Apply 13.5 at each place. In the assumed external bound, take $G_v^+ = B_v$ and $G_v^0 = A_v$. Set $P_v^* = P_v$. The local coefficient condition holds by hypothesis, and the three inequalities of 13.4 hold with equality. Substitution into (13.8) gives (13.9). □

Definition 13.7 (Permitted pairs and the prescribed initial model)**.** Fix the host of 5.1, a word u with level prime T_u, and a finite coefficient place w of residue characteristic p. Call (u, w) permitted if $p \geq 5$ is of inert class (14.1), lies outside the host level, does not divide T_u, and the second component of arithmetic Frobenius at every host prime above p has even order. At a permitted pair write $G_{u,w}$ for the arithmetic datum's initial model: it gives the coefficient map $O \to K$ with $\omega \in O$, $\omega^2 = p$, local places and weights, finite component set, each component's level group acting on local lattices with $O^\times$-valued character, and the nonnegative integer comparing the two integral weight lattices.

Theorem 13.8 (The local-system theorem)**.** *For every prime p, each local system X with kernel data Y satisfying Definition 10.10 has the following eleven properties.*

The first is the basic lift bound $-d_j$ of 12.6. The next two are the single-place and all-place product expansions of 10.14. The fourth and fifth are the ordinary lift bound $\sum_v A_v(j+1) - d_j$ and the full differentiated-kernel coefficient bound $\sum_v A_v(j+1)$ of 12.6. The sixth is the family of ordinary local coefficient bounds of 12.1.

The seventh is the distinguished-place bound of 12.7 for the unmodified lift. The eighth asserts that, for $1 \le i \le r$, $\widetilde{T}_{v,i,j+1}$ preserves Λ_{j+1} and satisfies the bound of 12.8 on $K_j\Lambda_j$. For each v, the ninth assertion supplies sequences $u_{v,a}, \mathfrak{m}_{v,a}$ such that, for $1 \le a \le m$,

$$\nu_K(u_{v,a}) = 0, \qquad \nu_K(\mathfrak{m}_{v,a}) = \sum_{s \text{ even}} w_{v,a}(s) - \sum_{s \text{ odd}} w_{v,a}(s),$$

where the sums run over the local positions $0 \le s < n'_v$. For these sequences it gives the prefix identity and coefficient estimates of 12.20 and the bound

$$\nu_{j+1}(Z_{v,j+1}K_jF) \ge B_v(j+1) + \sum_{v' \ne v} A_{v'}(j+1) - d_j \qquad (v \in \mathcal{P},\ 0 \le j < m,\ F \in \Lambda_j).$$

The tenth assertion gives the two bounds in 13.6. The eleventh gives (13.7) for every $F \in I(S,V)$ whenever the four conditions of Theorem 13.4 hold. For every choice of local weights satisfying the hypotheses of Lemma 13.5, the enhancement budget also satisfies $S \le m\gamma_p \sum_{s \in T} \alpha_s$.

At each permitted pair of Definition 13.7, the eleven assertions hold for every admissible system at $p = \operatorname{char} k(w)$ with initial model $G_{u,w}$.

Outline. Use Definition 10.6 for the fraction field and 10.10 for lifts in 12.20. Use 13.4 with 12.3–12.4 for $e > 0$, iterate 12.6 for $e = 0$, combine 12.1 with the chain bound for 13.6, and use 13.5 for the budget. The complete proof is on p. 570. □

Chapter 14

Global Frobenius-orbit parameters

Fix arithmetic data D and size data L in Alpöge's construction [3]. Let X consist of the coset representatives and offsets of 14.1–14.2, a real share $\kappa_I(p, n', S, \mathbf{a}, \mathbf{b})$ and an exponent $e_p \in \mathbf{N}$. Here $p, n' \in \mathbf{N}$, $S \subseteq I_{n'} = \{i \in \mathbf{N} : i < n'\}$, and $\mathbf{a}, \mathbf{b} : I_{n'} \to \mathbf{Z}$. Write H for the additive group A of the frame in 5.1, $n = |H|$ and $f = 2n$. Let k be the starting rank, m the number of lifts and $t+1$ the rank increase per lift. The rank after j lifts is $N_j = k + j(t+1)$, and f is the number of embedding indices. Let $TT \subseteq H$ be the host's set of top positions. Write a_s, b_s for the two integer weights prescribed by L at $s \in TT$, extended by zero on $H \setminus TT$.

Let $x > 1$ and $0 < \varepsilon < 1/52$ be the window parameters. Set

$$b = \exp(x^{\varepsilon}), \quad a_* = xb, \quad X' = \exp(x^{\varepsilon/3}), \quad b' = \frac{a_*}{X'}, \quad w'' = \frac{b'}{xn}, \quad X'' = \frac{a_*}{w''}.$$

Let $\Delta > 0$ be one quarter of the spacing between consecutive entries of the lower interlacing row of the arithmetic Harish–Chandra parameter. Put $S = 4k\Delta$ and $c = 4S$. Let $\eta > 0$ be the offset width in 14.2, and put $p_0 = k^2 x^{26}$, $\delta_1 = x^{-\varepsilon/8}$ and $[y]_+ = \max(y, 0)$. The top-weight window and $k \geq 5$ give

$$a_* \leq a_s \leq a_* + \frac{a_*}{kx^{13}} \leq \frac{6}{5} a_*, \qquad b_s \geq b' \qquad (s \in TT). \tag{14.1}$$

14.1 Orbits and global weight diagrams

Definition 14.1 (Frobenius orbits and marked positions)**.** Use the host isomorphism $\mathrm{Gal}(E/\mathbf{Q}) \simeq G = \{\pm 1\}^2 \times H$ of 5.1, writing the last coordinate additively. For each rational prime p, let $\mathcal{F}_p \subseteq G$ be the set of images of arithmetic Frobenius elements at primes of E above p. Fix $g_p \in \mathcal{F}_p$ and let $h = h_p$ be its H-coordinate.

Write q, d, Q_S for the host parameters. For $p \nmid Q_S q d$, take the first sign coordinate of the Artin image of $p \bmod Q_S q d$. At the other primes set this sign to zero. For $p \nmid Q_S$ with $p \neq q, d$ and $\left(\frac{-d}{p}\right) \neq 1$, the signal and inert classes have signs 1 and -1, respectively. With ord denoting additive order, set

$$(\Delta_p, n_p) = \begin{cases} (2h, 2\,\mathrm{ord}(2h)), & p \text{ of signal class}, \\ (h, \mathrm{ord}(h)), & \text{otherwise}. \end{cases}$$

For $a, s \in H$, X supplies a representative $\beta_a(s) \in s + \langle a \rangle$ with $\beta_a(s + a) = \beta_a(s)$. Put

$$\mathcal{R}_p = \beta_{\Delta_p}(H), \qquad T(t_0) = \{s \in TT : \beta_{\Delta_p}(s) = t_0\}, \qquad N_v(t_0) = |T(t_0)|.$$

For $s \in H$ and $e \in \mathbf{N}$, the labelled partner is

$$z_e(s) = \left(\mathbf{1}_{\{p \text{ of signal class},\ e \text{ odd}\}}, s + eh\right) \in \{0, 1\} \times H.$$

Call s even on its orbit if $\beta_{\Delta_p}(s) + ih = s$ for some even integer $0 \leq i < n_p$.

Definition 14.2 (Global capacities and logarithmic error)**.** A position $(0, s)$ is marked if $s \in TT$. On the second sheet the thick positions are $(1, u_0 + ia_0)$ for $i \in T_2$, where a_0 is the distinguished element of 5.1, $T_2 \subset \mathbf{N}$ is the host's thick index set and $u_0 \in H$ is its chosen translation. The remaining positions are thin. For $j \in \mathbf{N}$, X supplies offsets $o_j(z)$ in

$$[b'/x, b'/x + \eta], \qquad [b, b + \eta + 1], \qquad [w'', w'' + \eta]$$

at marked, thick, and thin positions, respectively. Define

$$W_j(e, s) = \mathbf{1}_{\{e=0,\ s\in TT\}} b_s + o_j(e, s), \qquad \kappa_j = \frac{k + (2j-1)(t+1)}{2}, \qquad B_j(z) = W_j(z) - \kappa_j.$$

At layer j put $\bar{\alpha}_s = a_s - 2c$, $\bar{C}_{s,e} = B_j(z_e(s)) + c$, and

$$z_p = \log_p(2a_*), \qquad \bar{\delta}(t_0) = K_{\delta,1} n_p N_m^2 z_p + K_{\delta,2} N_v(t_0) z_p^2.$$

The real constants $K_{\delta,1}$ and $K_{\delta,2}$ weight the period and marked-position terms in this error. Write K for the family of global real constants and K_{Pr} for its price coefficient in (14.4).

Definition 14.3 (Ordinary global gain)**.** For $s, s' \in H$, let $d_p(s, s')$ be the least $e \geq 1$ with $z_e(s) = (0, s')$, taking $d_p(s, s') = \infty$ if there is no such e. Let $\mathcal{E}_p(s)$ consist of the finite odd distances $d_p(s, s')$ for $s' \in TT \backslash \{s\}$. Define

$$g_0(t_0, j) = \sum_{s\in T(t_0)} \sum_{\substack{e\geq 1,\ e \text{ odd} \\ e\notin \mathcal{E}_p(s)}} \left[\frac{\bar{\alpha}_s}{p^e} - \bar{C}_{s,e}\right]_+ - \bar{\delta}(t_0).$$

Definition 14.4 (Enhanced global gain)**.** Let $T^V(t_0)$ consist of the marks in $T(t_0)$ that are even on their orbit in the sense of 14.1. Put $\gamma_p = p/(p^2-1) = \sum_{e\geq 1,\ e \text{ odd}} p^{-e}$. Define

$$g_U(t_0, j) = \gamma_p \sum_{s\in T(t_0)} \bar{\alpha}_s - \sum_{s\in T(t_0)\backslash T^V(t_0)} \sum_{\substack{e\geq 1 \\ e \text{ odd}}} \min\left(\frac{\bar{\alpha}_s}{p^e}, \bar{C}_{s,e}\right) - \bar{\delta}(t_0).$$

Both gains are defined for $t_0 \in H$ and $j \in \mathbf{N}$, with nonsummable series assigned sum zero. For $C \geq 0$,

$$\max\{[u/p^e - C]_+, |\min(u/p^e, C)|\} \leq |u| p^{-e}. \tag{14.2}$$

If $\bar{C}_{s,e} \geq 0$ for every $s \in T(t_0)\backslash T^V(t_0)$ and every positive odd e, (14.2) gives absolute convergence. Summing $v - \min(v, C) = [v - C]_+$ over these indices gives

$$g_U(t_0, j) + \bar{\delta}(t_0) = \gamma_p \sum_{s\in T^V(t_0)} \bar{\alpha}_s + \sum_{s\in T(t_0)\backslash T^V(t_0)} \sum_{\substack{e\geq 1 \\ e \text{ odd}}} \left[\frac{\bar{\alpha}_s}{p^e} - \bar{C}_{s,e}\right]_+. \tag{14.3}$$

14.2 Prices and selected orbits

Put $h_G = 2 + 8\varepsilon$, $K_G = \lceil 8(h_G + 2)/\varepsilon \rceil$, $r_G = K_G + 1$, and $M_{\mathrm{Pr}} = w'' + \eta + c + N_m$.

Definition 14.5 (Embedding and filling identifications)**.** The host labels the embedding indices τ bijectively by the elements of G with second sign $+1$. Write $\ell(\tau)$ for this label and $\ell(e, s) = (((-1)^e, 1), s)$ for the label of a position on sheet $e \in \{0, 1\}$.

An *embedding identification* gives, for each word u and coefficient place w permitted by 13.7, a bijection from the local positions (v, i) to the embedding indices. The base weight at (v, i) has k entries and equals the arithmetic level-zero weight for u at the corresponding embedding. Let $p = \operatorname{char} k(w)$. For every prime of E above p and every arithmetic Frobenius there, let $g \in G$ be its image. The period at each v is the order of the H-coordinate of g. Advancing i by one multiplies the embedding label by g and by the image of complex conjugation.

A *filling identification* for these bijections requires the local filling to have length m, parameter t and row bound at most c. The edge rank is r_G for $p < p_0$ and 1 for $p \ge p_0$. The position map $t_0 \mapsto v(t_0)$ is injective on $\mathcal{R}_p$. A local position with label $\ell(1,s)$ is unmarked and has shift $B_j(1,s)$ at every layer $1 \le j \le m$. For $t_0 \in \mathcal{R}_p$, the period at $v(t_0)$ is $\operatorname{ord}(h_p)$. The orbit coordinate at this representative is $\iota_{t_0} : \mathbf{Z}/n_p\mathbf{Z} \to H$, $\iota_{t_0}(i) = t_0 + ih_p$. Write $\iota(i)$ when t_0 is fixed. Its image is the coset of t_0, and equality of two values means that the indices differ by a multiple of $\operatorname{ord}(h_p) = n_p$. At index i put $s = \iota(i)$. The label is $\ell(0,s)$, the shift is $B_j(0,s)$, and the position is marked exactly when $s \in TT$. At such a mark the two local weight parameters are a_s, b_s, so the extreme base-weight entries are $a_s, -b_s$. A *choice of ties* for the orbit data and a local model Y is a choice of the embedding bijections at all permitted pairs for which the filling identification above holds. Call Y with these identifications a *tied model*. The quantifier over ties ranges over all such choices of bijections.

Definition 14.6 (Size bounds at a window)**.** Use the following notation for the two sets of numerical hypotheses in this chapter:

$$B = x^{\varepsilon}, \quad \theta = \varepsilon/100, \quad U = x^{\theta}, \quad \nu = 2n\delta_1, \quad R_0 = n/B, \quad r = k + m - 1, \quad N = N_m, \quad M = Q_S qd.$$

Let Λ be the real size parameter for the bottom-weight window, $\delta_2 = |T_2|/n$ the thick density and L_w the length used to exclude short relations in T_2. Put

$$a_0' = \frac{kB}{m\delta_2}, \qquad l_* = \frac{mb}{nt^2(N+t)B^2 + kn^2(S+t+B)}, \qquad H_0 = \frac{\eta}{m}.$$

Write d_E for the root discriminant of E, $\varepsilon_0 = \varepsilon/J$ for the host exponent with prescribed $J \ge 1$, and C_t for the prescribed exponent in $t \le x^{C_t}$. Let χ_{-d} be the quadratic character modulo d. Write $b_i(\tau), e_i(\tau)$, $0 \le i < m$, for the two rows of layer parameters at τ; these $b_i(\tau)$ are distinct from the bottom weights b_s on TT. Let p_1^*, p_2^* denote the host's two auxiliary level primes congruent to 1 mod 15, and let $\mathcal{W}$ be the set of words.

The following bounds are common to the size and parameter hypotheses:

$$k \le \sqrt{x}\, nR_0, \quad \sqrt{k/m} \le R_0, \quad \max\{m^2B, m\sqrt{x}\log x\} \le k,$$
$$\log f \le \sqrt{m/2-1}, \quad (\log f)^2 \le m^2B, \quad M^{36}t^{150} \le \sqrt{b}, \quad x^{\varepsilon_0/6} \le \log q, \quad |\operatorname{Re}(L'/L)(1,\chi_{-d})| \le B,$$
$$|TT| \le \nu, \quad b_s \le (1 + \nu B\Lambda/k)b' \quad (s \in TT), \quad 4S \le \Lambda b'/k, \quad Nm^{10} \le \Delta, \quad \log\Delta \le x^{\varepsilon/4}/2, \quad kS \le w'',$$
$$t \le \eta \le b/(t^3fk^3), \quad \log\eta \le x^{\varepsilon/4}, \quad t\eta \le w''/x, \quad m(t+1)\eta \le w'', \quad m(S+\eta) \le w''B,$$
$$N\eta \le \delta_2 mw'', \qquad \eta^2 rB \le \delta_2 m(w'')^2, \qquad r(b'+S)(\log x)a_0' \le \delta_2 mb,$$
$$b' + S + \eta + b/l_* < l_*, \qquad 2\eta + t + 8b/l_* + 7 < l_*,$$
$$\max\{12b/l_* + 2t + 8,\ ktB,\ 4N^3 + 2t + 6,\ 2(r+mt)^3 + t + 3,\ Nm^{10}\} \le H_0.$$

At every embedding require $b_{m-1}(\tau) \le 2b_0(\tau)$. For all embeddings τ, τ' and $0 \le i, i' < m$, require $e_i(\tau) \le X''e_{i'}(\tau')$. Also require $\log p_a^* \le B$ for $a = 1, 2$ and $|\mathcal{W}| \le 2^{k-3}$.

The *size bounds* impose the common bounds and the following additional conditions. Each of n, m, k lies between U^{-1} and U times the respective quantity

$$x^{2+8\varepsilon}, \qquad \sqrt{x\delta_1}, \qquad x^{9/2+10\varepsilon}.$$

For the positive constant C_Γ in the size data, normalized by $\log\Gamma_u = C_\Gamma m^5 fB$ for the prescribed column parameter Γ_u, require

$$\max\{Um^5fB^2,\ 8C_\Gamma m^5 fB\} \le k, \quad Um^6B \le k\log k, \quad Um^3B\log d_E \le k(\log k)^2,$$
$$U\log\log b \le rf\log r, \quad UB \le f\log k, \quad f \le \min\{m^2n, r^2\}, \quad 1 \le t \le Ux^{C_t}.$$

For $a = q, d$ require $x^{\varepsilon_0/5-\varepsilon_0/100} \le \log a \le x^{\varepsilon/5}U$. The remaining bounds are

$$1 \le d_E \le Q_S\sqrt{qd}/2, \quad U^{-1}n\delta_1 \le |TT|, \quad B \le \Lambda \le UB, \quad 4\nu B\Lambda/k + U(\log x)\Lambda/k \le x^{-2},$$
$$B^{-1} \le \delta_2, \quad L_w \le (\log x)^2, \quad \max\{Ur^{19}f^5B,\ Ukm^{9/2}\} \le H_0, \quad UmN \le \Delta,$$
$$Umk\Delta \le w'', \quad 4Uk^2\Delta \le \Lambda b', \quad Um\log(d_E/2) \le k, \quad \sqrt{b} \le w''.$$

For the two fixed primes require $(kb)^8 \le g_i \le 2g_j$ and $\log g_i \le UB$ for $i,j \in \{0,1\}$. The host's prescribed splitting bound for odd primes is at least 11.

Proposition 14.7 (The global price and its two bounds)**.** *Define*

$$\Pi_p(t_0,j) = \begin{cases} \displaystyle\sum_{s \in T(t_0)\setminus T^V(t_0)} (B_j(0,s) - b_s) + K_{\mathrm{Pr}} n_p M_{\mathrm{Pr}}, & p \ge p_0, \\ r_G\big(N_v(t_0)(2b'+c) + n_p M_{\mathrm{Pr}}\big), & p < p_0. \end{cases} \tag{14.4}$$

Assume the size bounds of Definition 14.6. Put $R = 2n\delta_1 x^{\varepsilon}\Lambda/k$. The size clauses give

$$\begin{gathered} \Lambda \ge x^{\varepsilon}, \qquad b_s \le (1+R)b' \quad (s \in TT), \qquad 4R + x^{\theta}(\log x)\Lambda/k \le x^{-2}, \\ 1 \le t \le \eta \le b/(t^3 f k^3), \qquad m(t+1)\eta \le w'', \qquad kS \le w'', \end{gathered} \tag{14.5}$$

and consequently

$$b' \le b_s \le \frac{5}{4}b' \qquad (s \in TT). \tag{14.6}$$

Let Y satisfy the embedding and filling identifications of Definition 14.5, and take a word u and place w permitted by 13.7, with $p = \operatorname{char} k(w)$. For $t_0 \in \mathcal{R}_p$, let $P_v(j)$ be the exchange exponent of 12.19 at $v = v(t_0)$. If $p \ge p_0$ and $K_{\mathrm{Pr}} \ge 1$, the Hecke rank is 1 and $P_v(j) \le \Pi_p(t_0,j)$ for $1 \le j \le m$. If $p < p_0$, $r_G < k$, and $x \ge 2$, the rank is r_G and the same bound holds.

Outline. Identify the orbit via 14.5 and use the exchange sum of 12.19. For large primes, evaluate marked terms and bound unmarked terms by 14.2 and 10.2, while for small primes use the size bounds at marks and bound the rows after the first. The complete proof is on p. 570. □

Put $n' = \operatorname{ord}(h)$, $S_{t_0} = \{0 \le i < n' : t_0 + ih \in TT\}$, and

$$\begin{gathered} u_1 = \lceil 1/\delta_1 \rceil, \qquad \ell = (6r_G+1)u_1, \qquad M_0 = \lceil (h_G+3)(\log x)10^{6r_G+1} \rceil, \\ G' = (M_0+5)\ell, \qquad N_{\mathrm{win}} = \lceil \max\{6\mathrm{e}G'\delta_1, (h_G+3)\log_2 x\} \rceil + 1. \end{gathered} \tag{14.7}$$

Let $L_1(p) = \lceil \log_p((kx^{26})^4) \rceil$ and $L_2 = 2\sup_{p \text{ of inert class}} L_1(p)$, with the supremum over primes. Call t_0 good if $S_{t_0} \subseteq \mathbf{Z}/n'\mathbf{Z}$ satisfies the window and tile conditions preceding Theorem 8.15 with $(K_\varepsilon, R_\varepsilon, u, N_1) = (K_G, r_G, u_1, N_{\mathrm{win}})$ and the parameters of (14.7).

Definition 14.8 (Selection and counted gain)**.** Let $\mathcal{G}_p$ be the good representatives in $\mathcal{R}_p$. Write $K_{\kappa,II}$ for the small-prime share coefficient and K_Λ, K_ρ, K_N for the global coefficients in the bandwidth formula below. Set $\Sigma_p = m\sum_{t_0 \in \mathcal{G}_p} \kappa_v(t_0)$, where

$$\kappa_v(t_0) = \begin{cases} \min\{1, K_{\kappa,II}\,|S_{t_0}| r_G^2 k x^{-26}\}, & p < p_0, \\ \kappa_I\big(p, n', S_{t_0}, (a_{t_0+ih})_i, (b_{t_0+ih})_i\big), & p \ge p_0. \end{cases}$$

At an inert-class prime, $N_v(t_0) = |S_{t_0}|$ for $t_0 \in \mathcal{R}_p$. A candidate is an inert-class prime $p < X'$ with $n' \ge x^{\varepsilon/4}$. For $p \ge p_0$ impose one further condition. Every positive integral power of p must lie outside the closed interval $[X'(1-\rho_B), X'(1+\rho_B)]$, where

$$N_* = 2n\delta_1, \qquad \mathcal{L} = K_\Lambda N_{\mathrm{win}}^2 \log a_*, \qquad \rho_B = \frac{4N_* x^{\varepsilon}\mathcal{L}}{k} + \frac{K_\rho K_N (\log x)\mathcal{L}}{k}.$$

Set

$$V_p = \begin{cases} \mathcal{G}_p, & p \text{ a candidate and } \Sigma_p \le x^{-\varepsilon}, \\ \varnothing, & \text{otherwise.} \end{cases} \tag{14.8}$$

For $1 \le j \le m$, the counted gain is

$$g_{p,j}^{\mathrm{count}} = \begin{cases} 0, & (p \text{ of inert class and } h = 0) \text{ or } p > X'', \\ \displaystyle\sum_{t_0 \in V_p} \max\{g_U(t_0,j) - \Pi_p(t_0,j), g_0(t_0,j), 0\} & \\ \qquad + \displaystyle\sum_{t_0 \in \mathcal{R}_p \setminus V_p} [g_0(t_0,j)]_+, & \text{otherwise.} \end{cases} \tag{14.9}$$

14.3 Identifying the global data

Definition 14.9 (Filling compatibility)**.** At a finite coefficient-field place w with $p = \operatorname{char} k(w)$, call the fillings compatible if the following identities hold. Use the labels $\ell(\tau)$ and $\ell(z)$ of Definition 14.5. For $1 \le j \le m$, let $w_j(\tau)$ be entry $j-1$, numbered from zero, of the exponent row of the arithmetic Harish–Chandra parameter at τ. Then

$$W_j(z) = w_j(\tau) \quad (1 \le j \le m,\ \ell(\tau) = \ell(z)), \tag{14.10}$$

and

$$\omega_{j,p}(s,e) = W_j(z_e(s)) \qquad (j \in \mathbf{N},\ s \in TT,\ e \in \mathbf{N}). \tag{14.11}$$

Here $\omega_{j,p}(s,e)$ is the partner weight prescribed by L. Regard the prescribed set $T_w \subseteq TT$ of top positions at w as a subset of H, and require

$$T_w = \begin{cases} TT, & p \text{ of signal class or } p \ge X', \\ \{s \in TT : \beta_{\Delta_p}(s) \in V_p\}, & \text{otherwise.} \end{cases} \tag{14.12}$$

Let $\mathcal{S}_L$ be the prescribed finite support of the size data and $g_j(w)$ their gain before primes are discarded. If p lies in neither the inert nor the signal class, require $g_j(w) = 0$ at every layer and $\tau_w = I_w = 0$. For $w \in \mathcal{S}_L$ and p of inert or signal class require

$$g_j(w) = g_{p,j}^{\mathrm{count}} \qquad (1 \le j \le m). \tag{14.13}$$

Write τ_w and $I_w = \mathrm{io}_w$ for the exponent and index budgets prescribed at w, and set $\mathrm{io}_p = I_P(p)$. Require

$$\tau_w = \tau_P(p), \qquad I_w = I_P(p), \qquad \Lambda = \mathcal{L}. \tag{14.14}$$

Both budgets vanish if $V_p = \varnothing$. Otherwise put

$$\begin{aligned} S_v(t_0) &= \left\lceil \sum_{j=1}^{m} [g_U(t_0,j) - \Pi_p(t_0,j) - [g_0(t_0,j)]_+]_+ \right\rceil, \\ \tau_P(p) &= e_p + m \sum_{t_0 \in V_p} \big([\lceil t_v(t_0) \rceil]_+ + D_p + 1 \big), \\ I_P(p) &= 2\Sigma_p \max_{t_0 \in V_p} S_v(t_0). \end{aligned}$$

Use $(t_v(t_0), D_p) = (t_{II}, D_{II})$ for $p < p_0$ and (t_I, D_I) for $p \ge p_0$, from 14.10 with length $n' = n_p$, marks S_{t_0}, and weights $\mathbf{a} = (a_{t_0+ih})_i$, $\mathbf{b} = (b_{t_0+ih})_i$.

Lemma 14.10 (The two local presentations)**.** *In this lemma use $A = a_*$, $r = r_G$, $z = \log_p A$, $q_p = p^{G'}$, $c_0 = 2b' + 2c$, and $U = 40n'\delta_1$. Let K_1, K_2 be the coefficients of the two error terms in t_I below, with the same K_2 in c_6. Write $K_{\mathrm{DI}}, K_{\mathrm{DII}}, K_{\mathrm{tII}}$ for the coefficients in D_I, D_{II}, t_{II}, respectively. For N marks on a cycle of length n', the bounds and costs are*

$$\begin{aligned} D_I &= \lceil K_{\mathrm{DI}} U x^{26} (N_{\mathrm{win}}^5 + k^3) z \rceil_+, \\ D_{II} &= \lceil K_{\mathrm{DII}} U (N_{\mathrm{win}}^3 + r^5 + k^2) z + 2rn'c + rUc_0 \rceil_+, \\ t_{II} &= K_{\mathrm{tII}} r N (\mathcal{L}A/(pk) + Ap^{-L_1(p)} + c_0 + kq_p) + 2rn'c, \\ c_6 &= 8(D_I + 1) + K_2 U k z + 4Uc + 2n'c, \\ t_I &= J + K_1 N N_{\mathrm{win}} (A/p^2 + \mathcal{L}A/(pk)) + K_2 N (q_p c_6 + kq_p + kz + c) + 2n'c. \end{aligned}$$

Here $\lceil y \rceil_+ = [\lceil y \rceil]_+$. Each odd mark y contributes $\min(b_y, a_{y-d_y}/p^{d_y})$ to J, where d_y is the least positive distance back to an even mark within one period. Its contribution is zero if there is no even mark. Parity uses representatives $0, \dots, n'-1$.

Prescribe the index exponents by $e_p = \lceil C_e f k^2 \log_p a_* \rceil_+$ *for all* $p \in \mathbf{N}$*, with a fixed* $C_e \in \mathbf{R}$*. At* $p = 0, 1$ *take the logarithm to base* p *to be zero. For every* $p, n' \in \mathbf{N}$*,* $S \subseteq I_{n'}$ *and* $\mathbf{a}, \mathbf{b} : I_{n'} \to \mathbf{Z}$*, prescribe* $\kappa_I(p, n', S, \mathbf{a}, \mathbf{b})$ *as the infimum of the nonnegative* κ *passing the following test for every eligible model and dominant base weight. This infimum passes the test. The test takes* $r', t_*, D_0, c_* \in \mathbf{N}$ *with* $4(D_0 + 1) \le c_*$*. For this share use*

$$(r', t_*, D_0, c_*) = (1, \lceil t_I \rceil_+, D_I, 8(D_I + 1)).$$

The models use complete DVRs W *with uniformizer* $p \ge 5$ *prime, residue order* $p^{n'}$*,* $n' \ge 2$ *even, and Frobenius lift* σ *with* $\sigma^{n'} = 1$*. Take all finite extensions* $W \to O$ *of complete DVRs with compatible fraction-field inclusion, uniformizer* ω *satisfying* $\omega^2 = p$*, residue order* $p^{2n'}$*, and square roots in* O *of all rational integers. A dominant base weight* $\lambda_{s,a}$ *has* k *entries at each* s*, numbered from zero. At* $s \in S$ *its extremes are* $a_s, -b_s$*. The other entries at a mark and all entries off* S *have absolute value at most* $\max(0, \lfloor c \rfloor)$*. If there is no eligible model, the share is zero.*

Use $\lambda^{\mathrm{act}}_{s,a} = \lambda_{s,a}$ *at even* s *and* $\lambda^{\mathrm{act}}_{s,a} = -\lambda_{s,k-1-a}$ *at odd* s*. Let* V *be the tensor product over* $\mathrm{Frac}(O)$ *of the dual Weyl spaces for these weights. For a full* W*-lattice* $\Lambda \subset \mathrm{Frac}(W)^k$*, let* H_Λ *be the integral dual Weyl tensor lattice on its Frobenius translates over* O*. Products of the tableau bases in transported frames, with their determinant twists, form a field basis of* V *and span* H_Λ *over* O*. Indeed two frames of the same* O*-lattice differ by* $U \in \mathrm{GL}_k(O)$*. Substitution by* U *and* U^{-1} *preserves the integral column-invariant polynomials, and the determinant twist for* U *is a unit. The intersection over frames in each dual Weyl factor therefore equals its value in any one frame.*

A rank-r' *edge* $e : \Lambda' \to \Lambda$ *is a pair of full* W*-lattices with* $p\Lambda \subseteq \Lambda' \subseteq \Lambda$ *and* $\dim_{W/pW}(\Lambda/\Lambda') = r'$*. Form* H_e *on* $\Lambda'_O + \omega\Lambda_O$ *at each translate, where the subscript denotes extension to* O*. Put*

$$A_{r'} = \sum_{s=0}^{n'-1} \sum_{0 \le a < \min(r', k)} \lambda^{\mathrm{act}}_{s,a}, \qquad R_e = \omega^{-A_{r'}} H_e, \qquad R'_e = \omega^{-A_{r'}} H_e^\vee,$$

where $^\vee$ *denotes the* O*-linear dual lattice. For every target* Λ *there must be a finite list* $(e_i, F_i)_{i=1}^M$*, with repetitions allowed, such that* $e_i : \Lambda_i \to \Lambda$ *and*

$$F_i \in \omega^{-t_*} H_{\Lambda_i} \cap \bigcap_{e : \Lambda_i \to *} R_e, \qquad F_i \in \omega^{c_*} R_e \quad (e : \Lambda_i \to *,\ e \ne e_i).$$

All edges have rank r'*. For some* $h_0 \in \mathbf{N}$ *with* $h_0 \ge (1 - \kappa) \dim V$*, there must be* $B \in O^{h_0 \times M}$ *and an* O*-linear map*

$$U_0 : O^{h_0} \longrightarrow \omega^{-t_*} H_\Lambda^\vee \cap \bigcap_{e : * \to \Lambda} R'_e$$

such that

$$\sum_{i=1}^{M} B_{ji} \, (U_0 \xi)(p^{A_{r'}} F_i) = p^{D_0} \xi_j \qquad (\xi \in O^{h_0},\ 1 \le j \le h_0).$$

Proof. Substitute (14.7) into the local budgets. The share is the infimum s of the passing nonnegative fractions. The empty list with $h_0 = 0$ passes at 1, so $s \in [0, 1]$. For each model, weight and target, put $q = (1 - s) \dim V$. If $q > 0$, choose a passing κ with $(1 - \kappa) \dim V > \lceil q \rceil - 1$. Its witness has integer rank at least $\lceil q \rceil$, so it passes at s. For $q = 0$ use the empty list. □

Definition 14.11 (Arithmetic parameter conditions)**.** Use the notation and the common bounds of Definition 14.6. The *parameter conditions* specify positive real constants in the following inequalities. A repeated constant has the same value throughout. First require

$$x^{2C_t + 5} < \min(q, d), \qquad C_k m^5 f B^2 \le k, \qquad C_6 m^6 B \le k \log k, \qquad C_f B \le f \log k,$$
$$\log\log b \le c_L r f \log r, \qquad 4\nu B\Lambda/k + C_\rho (\log x)\Lambda/k \le x^{-2}, \qquad 4C_{\Delta G} k^2 \Delta \le \Lambda b'.$$

For each integer $l \ge 11$ some $j \in \mathbf{N}$ satisfies $a'_0 \le (l - 1)^j$ and $2j + 2 \le L_w$. Require $L_w \le C_L \log x$ and $j_0 \le C_j \log x$, where j_0 is the prescribed jet depth.

Write $u_i(\tau), v_i(\tau)$ for the upper and lower interlacing rows at τ. If $\ell(\tau) = \ell(0, s)$ for $s \in TT$, require $b_0(\tau) = b_s + \lfloor b'/x \rfloor$. At every other embedding require

$$u_0(\tau) - v_0(\tau) \le 2S, \qquad u_{k-1}(\tau) = v_{k-2}(\tau) - \tfrac{1}{2}.$$

For every embedding require, over the indicated row indices,

$$\begin{gathered} |b_i(\tau) - e_j(\tau)| \ge 2b/l_* + t/2 + 3 \quad (0 \le i, j < m), \\ |e_i(\tau) - b_i(\tau) - H_0/2| \le 2 \quad (0 \le i < m), \\ H_0 \le b_{i+1}(\tau) - b_i(\tau) \le H_0 + 1 \quad (0 \le i < m - 1). \end{gathered}$$

Each consecutive difference $v_i(\tau) - v_{i+1}(\tau)$ (with $v_i(\tau) - v_{i+1}(\tau) = 4\Delta$), the sum $v_{k-2}(\tau) + b_0(\tau)$, and each consecutive difference $b_{i+1}(\tau) - b_i(\tau)$ must exceed $C_{\mathrm{gap}} k m^4 + N m^{10}$.

Let η_F be the parameter in the division data governing the size of the retained fraction. Require $\eta_F = C_{\eta F} x^{-C_F}$ and

$$\begin{gathered} \max\{k^{C_w}, b^{c_w}\} \le w'', \\ \max\{k^{C_\Delta}(C_\Delta m)^3/\eta_F^{C_\Delta},\ C_\Delta m N,\ k^{C_\Delta}(C_\Delta m^4 B)^4,\ (NfB)^{C_{\mathrm{count}}}\} \le \Delta, \\ \max\{C_r r^{19} f^5 B,\ (NfB)^{C_{\mathrm{count}}},\ C_h k m^{9/2},\ 120 r m^{3/2},\ 16 m N\} \le H_0, \\ t + 2\big(k + m(t + r + 7)\big)^3 + 1 \le H_0, \qquad f N^{C_N} \le H_0, \\ (4f + 10) r \le C_W r\big(f + f^2 (k-1)^3 B^2/m\big). \end{gathered}$$

There is a real $Y \ge (kb)^8$ with $N^4 M^{C_Y} \le Y^{1/4}$, $\log Y \le C_{YL} B$, and $Y \le g_i \le 2Y$ for $i = 0, 1$. Both g_i are 1 mod M. For each g_i there is a $\pi_i \in \mathcal{O}_E \cap \mathbf{Q}(\sqrt{-d})$ with $\pi_i \bar{\pi}_i = g_i$. Both p_a^* split completely in E, are 1 mod 15, and are distinct from each g_i. Every host level prime is of split quadratic class.

For the prescribed conductor ideal $\mathfrak{c}$ of the imaginary quadratic field require $\log N\mathfrak{c} \le B$ and $N\mathfrak{c} \le C_{\mathfrak{c}} d$. For every root of unity $\zeta \in E$ and each $0 \le i < m$, require

$$\prod_{\tau} \tau_E(\zeta)^{e_i(\tau)} = 1.$$

Here τ_E is the host embedding labelled by $\ell(\tau)$ and $e_i(\tau)$ is integral.

Proposition 14.12 (The index between the two integral weight lattices)**.** *Let O be the coefficient DVR of a permitted model, with fraction field F and residue characteristic p. For a dominant integral weight $\lambda = (\lambda_0, \dots, \lambda_{k-1})$, put $\nu_i = \lambda_i - \lambda_{k-1}$ and $l = \nu_0$. In the polynomial ring over O on a k by k matrix Y, let A_λ consist of polynomials homogeneous of degree ν_j in column j and invariant under adding an indeterminate multiple of column i to column j for $i < j$. Let h_λ be the product of initial principal minors, one of size equal to the height of each column of ν. Let D_λ be the O-span of successive coefficients of elementary row substitutions on h_λ, with the empty word included. These are the dual Weyl and Weyl lattices before the determinant twist. For $k \ge 5$, put $d_p(\lambda) = \binom{k}{2} \lfloor \log_p(l + k - 1) \rfloor$. Then*

$$p^{d_p(\lambda)} A_\lambda \subseteq D_\lambda \subseteq A_\lambda. \tag{14.15}$$

The inclusions hold after common transport to a lattice frame and the determinant twist for λ.

For the arithmetic specialization assume the parameter conditions of Definition 14.11 and the embedding identification of Definition 14.5. At a permitted model these identify the pairs (v, s) of an orbit and a position on it with the f embedding indices and match the base weights with the arithmetic level-zero weights. Let M_{na} and M be the level-zero function lattices of 10.2 with fibres given by the dual Weyl and Weyl tensor lattices, respectively. If $C_e \ge 1$ and $e_p = \lceil C_e f k^2 \log_p a_* \rceil_+$, then

$$p^{e_p} M_{\mathrm{na}} \subseteq M \subseteq M_{\mathrm{na}}. \tag{14.16}$$

Outline. Compare ranks by tableau reduction and 10.17 and bound denominators of the projectors that produce a scalar. Transport and twist by 10.17, then sum weight-spread bounds with R from 14.7 to get the lattice inclusions of 10.2. The complete proof is on p. 571. □

14.4 The numerical comparison

At $v = v(t_0)$ in a permitted tied model use $\iota(i)$ of 14.7. Write $\alpha_i, C_i, L_i, \delta_v$ for the column count, capacity, remaining box count and digit error of 10.3–10.4, and $g_{A,v}(j), g_{B,v}(j)$ for the gains of 11.1–11.2.

Theorem 14.13 (Comparison when $z_p \geq 1$)**.** *Assume the size bounds of Definition 14.6, $K_{\delta,1} \geq 5$, $K_{\delta,2} \geq 4$, $K_{\mathrm{Pr}} \geq 1$, $r_G < k$ and $x \geq 2$. For every choice of ties, permitted pair (u, w), $t_0 \in \mathcal{R}_p$, and $1 \leq j \leq m$, the condition $z_p \geq 1$ implies*

$$g_U(t_0, j) \leq g_{B,v}(j), \qquad g_0(t_0, j) \leq g_{A,v}(j), \qquad P_v(j) \leq \Pi_p(t_0, j). \tag{14.17}$$

Outline. Compare counts and capacities via 10.2, 10.3 and match exclusions via 11.1. Compare positive parts termwise using 11.2 at odd marks, subtract digit errors bounded via 14.2 and 10.4, and use 14.7 for the price. The complete proof is on p. 573. □

Theorem 14.14 (Zero counted gain when $z_p < 1$)**.** *Assume the parameter conditions of Definition 14.11, nonnegativity of the global constants K, and $K_{\mathrm{Pr}} \geq 1$. For every finite coefficient-field place w with $p = \operatorname{char} k(w)$, $t_0 \in \mathcal{R}_p$, and $1 \leq j \leq m$,*

$$z_p < 1 \quad \Longrightarrow \quad g_U(t_0, j) - \Pi_p(t_0, j) \leq 0, \qquad g_0(t_0, j) \leq 0. \tag{14.18}$$

Outline. For $z_p < 1$, the ordinary summands vanish by 14.2 and the enhanced gain is at most $N_v(t_0)$ by 14.4. The price is at least $N_v(t_0)$ by 14.1 and marked prices for large primes and by size bounds for small primes, so both counted gains vanish. The complete proof is on p. 574. □

Definition 14.15 (Twisted chains and their comparison)**.** For each word u and coefficient place w satisfying the permission conditions of 13.7, the local datum Y specifies a filling with levels and a chain, a position map v, and an $\mathcal{O}$-submodule V^{ram} of the local function space. Write h_g for these permission hypotheses. A family of twisted chains for Y assigns levels on this filling and a chain on those levels to each tuple μ at each permitted pair (u, w). At the tuple μ_w attached to w, this pair equals the pair in Y. Write Y_μ for Y with the levels and chain replaced by this pair. Conversely, given levels and chains for all tuples on the fixed fillings, use the pairs at μ_w as the local data at w. The given family is then a family of twisted chains for these data.

Let ι be the comparison datum with maps ι_{u,w,h_g} from the arithmetic initial spaces to the local models. Call ι a *tied arithmetic comparison* for these data if the following three conditions hold. In the first condition it suffices that there exist one choice of ties.

1. The embedding and filling identifications of Definition 14.5 hold with the same bijections at every permitted pair.
2. At each permitted (u, w, h_g), the underlying set of V^{ram} is the K_f-linear span of the image under ι_{u,w,h_g} of the arithmetic ramified submodule. Here K_f is the local coefficient field.
3. For $1 \leq j \leq m$ and every w, the stored $F_j(w)$ equals the value in 15.10.

For each tuple μ, let $Q'(\mu)$ be the finite set of residue characteristics excluded by the division data, and fix the twisted forms T. A family of twisted chains satisfies the *terminal comparison* when it has the following valuation property, proved under the stage and lattice hypotheses of Lemma 16.18, at each permitted pair (u, w) and each used tuple μ with $p_w = \operatorname{char} k(w) \notin Q'(\mu)$. The used tuples are the reference tuple, the two distinguished twists, and the tuples $\mu_{w'}$ attached to coefficient places w'. For each arithmetic initial vector x and $s \in \mathbf{R}$, if the lift of $\iota_{u,w,h_g}(x)$ along Y_μ has terminal valuation at least s, the corresponding column has a rational representative with terminal lattice valuation at least s. Use the reference column and lattice at the reference tuple and those of T at the other tuples, as in 16.1.

14.5 Substitution into the integral theta chain

Theorem 14.16 (Global exponents in the local chain)**.** *Fix D, L, X, Y, K with the embedding and filling identifications of Definition 14.5, and fix a pair (u, w) permitted by 13.7, with $p = \operatorname{char} k(w)$. Assume*

(14.17) *for every choice of ties and permitted pair with* $z_p \ge 1$, *and* (14.18) *for every coefficient-field place with* $z_p < 1$. *Both comparisons range over all* $t_0 \in \mathcal{R}_p$ *and* $1 \le j \le m$.

Fix a complementary budget $\mathrm{low} \in \mathbf{R}$. *Assume there is a standard system as in 10.2 on the local model at* (u, w), *with the prescribed filling, levels and chain and the trace-dual Witt bases of 10.13. Use its uniformizer* ϖ *of* $\mathcal{O}$, *exponent* $e_{\mathcal{O}}$ *with* $(p) = (\varpi^{e_{\mathcal{O}}})$, *and stabilizer bounds* $d_G(j)$. *Assume the system has kernel data as in 10.19 and that* $\mathcal{O} \to K_f$ *is injective. Require the ordinary coefficient bound of 12.1 and the conditional chain inequality of 13.4. Also require the coefficient bound of 10.19 with exponent zero at every position and every step* $0 \le j < m$, *as in 10.10. Assume there are real numbers* $F_n(v, j)$ *giving the coefficient bound of 10.19 at step* $j - 1$ *and position* v *for* $v \notin v(\mathcal{R}_p)$ *and* $1 \le j \le m$, *with*

$$\mathrm{low} \le \sum_{j=1}^{m} \sum_{v \notin v(\mathcal{R}_p)} F_n(v, j) - \sum_{j=0}^{m-1} d_G(j). \tag{14.19}$$

Let $K^{[m]} = K_{m-1} \cdots K_0$ *and let* Λ_m *be the terminal integral lattice. For* $V' \subseteq V_s \subseteq \mathcal{R}_p$ *and* F *in the initial integral lattice* M, *put* $y_{V'}F = (\prod_{t_0 \in V'} y_{v(t_0)}^{m})F$, *in the prescribed finite-set order of 13.2. Write* $v_{\Lambda_m}(z) \ge h$ *for* $z \in \varpi^{\lceil e_{\mathcal{O}} h \rceil}\Lambda_m$. *Then*

$$\begin{aligned} v_{\Lambda_m}\big(K^{[m]} y_{V'} F\big) \ge \mathrm{low} + \sum_{j=1}^{m} \Bigg(& \sum_{t_0 \in V'} \max\{g_U(t_0, j) - \Pi_p(t_0, j), g_0(t_0, j), 0\} \\ & + \sum_{t_0 \in V_s \setminus V'} [g_0(t_0, j)]_+ + \sum_{t_0 \in \mathcal{R}_p \setminus V_s} [g_0(t_0, j)]_+ \Bigg). \end{aligned} \tag{14.20}$$

Outline. Combine bounds in 10.19 and use global gains and price or zero inputs in 13.4 according to z_p. Use injectivity to preserve generation in 13.2, sum layers with stabilizer losses, scale and round up, and split $\mathcal{R}_p \setminus V'$ into $V_s \setminus V'$ and $\mathcal{R}_p \setminus V_s$. The complete proof is on p. 574. □

14.6 Passage to the global service condition

Lemma 14.17 (Hecke images for the chosen twisted chains)**.** *Take a member at or above the threshold for the first closed condition of Theorem 16.21, at* $x > x_1$. *Assume the error hypotheses of 15.12 for this member. Assume the comparisons of 14.13 at every permitted model with* $\log_p(2a_*) \ge 1$ *and those of 14.14 at every coefficient place with* $\log_p(2a_*) < 1$. *Choose the levels and twisted chains supplied by that condition, with its matching admissible systems and its complementary bounds at* $\sigma = 1$. *For every used twist* μ, *every embedding and filling identification for its local data as in 14.15, and every permitted pair* (u, w) *with* $p_w \notin Q'(\mu)$, *inequality* (14.20) *holds for every indicated subset of representatives and every initial integral vector. Its complementary budget is the first quantity in* (16.6) *at* $\sigma = 1$.

Outline. Use the first clause for the complementary budget, 13.8 for the ordinary and chain bounds, and 10.10 for the zero bound. Apply 14.16 using the fraction field and the comparisons in both prime ranges. The complete proof is on p. 575. □

Suppose the fillings are compatible at every finite place in the sense of 14.9 and the discarded set of 15.14 contains every inert-class prime for which h_p is nonzero of odd order. For $p = \mathrm{char}\, k(w)$ and $1 \le j \le m$, Lemma 17.24 bounds $\mathrm{gainD}_j(w)$ above by the gain in (16.7). Theorem 17.28 uses this inequality together with the size conditions of 16.20 and both column bounds of 16.19 at normalization 1, with their stated hypotheses.

Part IV

Divisibility along the chain and division of lattices

Chapter 15

What each layer records

Fix chain data D of initial rank k and even length m, with size data L [3]. Each lift increases the rank by $t_{\mathrm{ch}}+1$, where t_{ch} is odd. Let d be the host discriminant prime and g_0, g_1 the two fixed auxiliary level primes of the host data. Let $x>1$ be the window point and $0<\varepsilon<1/52$ its exponent in the size datum. Put $b=\exp(x^{\varepsilon})$. At a finite place w of the coefficient field K, let p be the characteristic of the residue field and $e_w=v_w(p)$ the ramification index, with v_w normalized to take value one on a uniformizer. In a formula by cases, use the first applicable case. Let $\mathcal{C},\mathcal{S}$ be the inert and signal classes of 14.1. The chain identifies its embedding indices with the elements of $G=\{\pm1\}^2\times A$ whose second sign is $+1$. For $z=(\epsilon,t)\in Z=\{0,1\}\times A$, let $\tau(z)$ be the index corresponding to $(((-1)^{\epsilon},1),t)$. Write $w_j(z)=w_j(\tau(z))$ for extension weight $j-1$, where $1\le j\le m$. Set $X'=e^{x^{\varepsilon/3}}$.

Definition 15.1 (The orbit base value)**.** For integers $p\ge 2$, $r\ge 1$ and capacities $d_s\in\mathbf{R}_{\ge 0}$ on $\mathbf{Z}/r\mathbf{Z}$, let $\mathcal{P}$ consist of arrays $x_{s,i}\in\mathbf{R}_{\ge 0}$ for positive odd $i<r$ with $\sum_i(p^i x_{s,i}+x_{s-i,i})\le d_s$. Put

$$F_{p,r}(\mathbf{d})=\frac{p}{p^2-1}\sum_s d_s-\sup_{x\in\mathcal{P}}\sum_{s,i}x_{s,i}.$$

Use the filling compatibility of 14.9 for the chain weights. For a prime p, take h_p, n_p from 14.1. Set $P_p(\epsilon,t)=(\epsilon+\mathbf{1}_{p\in\mathcal{S}},t+h_p)$ with the first coordinate modulo 2, and put

$$F_{\mathrm{orb}}(j,p,z)=F_{p,n_p}\Big(\big((k-1)w_j(P_p^s z)\big)_{s\in\mathbf{Z}/n_p\mathbf{Z}}\Big).$$

Lemma 15.2 (Attainment and change of orbit origin)**.** *For $p\ge 2$, $r\ge 1$ and nonnegative capacities, the supremum in 15.1 is attained. For the filled capacities, $F_{\mathrm{orb}}(j,p,P_p^a z)=F_{\mathrm{orb}}(j,p,z)$ for every $a\ge 0$.*

Outline. Use the periods from 14.1 and the weight bounds from 14.2 to obtain nonnegative capacities and maximize flow on the compact feasible set. Changing the orbit origin cyclically permutes the data, preserving the capacity sum and maximum flow. The complete proof is on p. 575. □

Lemma 15.3 (Duality and changing the demands on an orbit)**.** *For fixed p,r and a capacity vector D, write $F(D)=F_{p,r}(D)$ as in Definition 15.1, and put $\gamma=p/(p^2-1)$. For integers $p\ge 3$, $r\ge 1$ and capacities $d_s\ge 0$, the maximum flow in 15.1 equals the minimum of $\sum_s d_s y_s$ over $0\le y_s\le 1$ with $p^i y_s+y_{s+i}\ge 1$ for every positive odd $i<r$. Every minimizer satisfies*

$$\sum_{s:\,y_s\ge 2/p} d_s\le p^{-1}\sum_s d_s.$$

Suppose $p\ge 5$ is prime, $r\ge 2$ is even, $T\subseteq\mathbf{Z}/r\mathbf{Z}$, $k\in\mathbf{N}$ with $k\ge 3$, $w_s>0$ and $0\le e_s\le 4w_s$. Set $W_s=(k-1)w_s$ and $D_s=W_s+e_s-2w_s\mathbf{1}_T(s)$. Then

$$F(D)\ge F(W)-\frac{11}{p}\sum_s w_s.$$

Outline. Dualize the loads of 15.1 using attainment from 15.2, and use the gap identity to cap the dual at 1 and force incoming saturation at $y_s \geq 2/p$. Test the dual at $W + e$ and use flow monotonicity to bound the two losses. The complete proof is on p. 575. □

Lemma 15.4 (Weights of the letter logarithm and of a phase)**.** *Let K' be a characteristic-zero nonarchimedean normed field, and let p be an odd prime with $0 < |p| < 1$. Normalize the additive valuation by $v(p) = 1$. Put $\rho = 1/(p^2 - 1)$ and $q_* = 1/(p - 1)$, and give cZ^n weight $v(c) + \rho|n|$. For a self-map P of a finite index set, the substitution*

$$L_i(Z) = \sum_{h \geq 0} p^{-h} Z_{P^h i}^{p^{2h}}$$

has a formal inverse g with zero constant terms. Every term of g_i has weight at least ρ.

Let $r \geq 2$ be even and take $P(s) = s - 2$ on $\mathbf{Z}/r\mathbf{Z}$. Let $\beta \in K_0[[Z_s : s \in \mathbf{Z}/r\mathbf{Z}]]$, where $K_0 \subseteq K'$ is a subfield with $v(K_0^\times) \subseteq \mathbf{Z}$. Let $X = (Z_s)_{s \text{ even}}$ and $Y = (Z_s)_{s \text{ odd}}$. Fix a primitive $(p^r - 1)$-st root $\zeta \in K'$. Assume that β is invariant under $Z_s \mapsto \zeta^{(-1)^s p^s} Z_s$, with s represented in $[0, r-1]$, and that every monomial in its support has positive degree in each parity block and bidegree different from $(1, 1)$. For a copy X_1 of X, put

$$\Delta_X \beta = \beta(X + X_1, Y) - \beta(X, Y) - \beta(X_1, Y),$$

and define $\Delta_Y \beta$ by splitting Y in the same way. Suppose $\exp(\Delta_X \beta)$ and $\exp(\Delta_Y \beta)$ have integral coefficients and constant one after substitution by L in each copy. Every nonzero term of β has weight at least $q_* - 1$. Each term of weight below q_* has the form

$$cZ_s^{p^u} Z_t^{p^v}, \qquad s, t \text{ of opposite parity}, \quad u, v \in \mathbf{Z}_{\geq 0}, \quad (u, v) \neq (0, 0), \quad s + u \equiv t + v \pmod r.$$

Outline. Invert the letter logarithm by degree and use integrality and torus invariance to bound both coboundary weights below by q_*. Binomial splitting gives the bound $q_* - 1$ for exceptional pairs of powers, with $s + u \equiv t + v \pmod r$ by torus invariance. The complete proof is on p. 576. □

Proposition 15.5 (Cyclic and one-letter bounds)**.** *Let $p \geq 5$ be prime and $N_0, N_1 \in \mathbf{N}$ with $N_0 < N_1$. On the even cycle $\mathbf{Z}/r\mathbf{Z}$, $r \geq 2$, take Young diagrams ν_s with at most N_0 rows. Put*

$$\ell_{s,ab} = \sum_{h \geq 0} \frac{Z_{s-2h,ab}^{p^{2h}}}{p^h}.$$

Suppose a formal kernel over $\mathbf{C}_p$ has an expansion

$$\Theta(\xi \ell) = U \exp\left(\epsilon \sum_{a,b} \beta(\ell_{\bullet,ab}) \right), \qquad |\xi|_p = 1, \quad \epsilon = \pm 1, \quad |[Z^n]U|_p \leq 1$$

for every exponent n, with (a, b) ranging over the $N_0 N_1$ matrix entries. Use the phase hypotheses of 15.4 over $\mathbf{C}_p$ with $K_0 = \widehat{\mathbf{Q}_p^{\mathrm{nr}}}$, the completed maximal unramified extension of $\mathbf{Q}_p$. Its valuation is integer-valued on nonzero elements, since each such element has an approximation of the same valuation in a finite unramified extension.

Let A_s, B_s be the first two row lengths, with missing rows of length zero. For $T \subseteq \mathbf{Z}/r\mathbf{Z}$ put $D_s = |\nu_s| - A_s \mathbf{1}_T(s)$, where $|\nu_s|$ counts boxes. Put $c_s = B_s$ on T and $c_s = A_s$ elsewhere. All distances e in the following formulas are positive odd integers. For $s \in T$ set

$$h_s = \inf\{e : s + e \in T\}, \qquad M_s = \max_{p^e \leq |\nu_{s-e}|} c_{s-e},$$

with empty values ∞ and zero, respectively. Put

$$H_s = \sum_{e < h_s} (A_s p^{-e} - c_{s+e})_+ + \sum_{e > h_s} A_s p^{-e},$$
$$J_s = \sum_{e \neq h_s} ((A_s - B_s) p^{-e} - c_{s+e})_+.$$

For each diagram column of height h, choose a strictly increasing map $\{1,\dots,h\}\to\{1,\dots,N_1\}$ and form P_I by 10.15. For a real size bound $a\ge p$ with $|\nu_s|\le a$ for all s and $2\log_p a+2\le N_0-1$, each coefficient $[\tau^\mu]\mathcal{D}_{P_I}(\Theta)$ of 10.18 has valuation at least

$$F(D)+\sum_{s\in T}(H_s-\gamma M_s)-rN_0N_1(\log_p a+1),$$

where F,γ are as in 15.3. If the capacities D_s have the form in the comparison of 15.3, one may replace $F(D)+\sum_{s\in T}H_s$ by $F(W)-11\sum_s w_s/p+\sum_{s\in T}J_s$.

For one letter, let ν have at most N_0 rows. Use $\ell_{ab}=\sum_{h\ge0}Z_{ab}^{p^{2h}}/p^h$ and assume the same form of expansion with phase $(\epsilon/2)\sum_{a,b}\beta(\ell_{ab},\xi'\ell_{ab})$, where $|\xi'|_p=1$ and β satisfies the phase hypotheses of 15.4 at $r=2$. For $a\ge p$ and $|\nu|\le a$, each frame coefficient of the evaluation 10.18 for such column choices has valuation at least

$$\rho|\nu|-N_0N_1(\log_p a+1),\qquad \rho=\frac{1}{p^2-1}.$$

Outline. Use 11.14 for the frame and 10.18 for row degrees, then 15.4 and the dual of 15.3 to raise phase weights to q_*, using period two for one letter. Apply Legendre and 11.7 for digit loss and the dual to compare with $F(W)$. The complete proof is on p. 576. □

Definition 15.6 (The two integer weight rows)**.** Fix an embedding type τ. Number coordinates from zero. The interlacing coordinates are integers $v_0,\dots,v_{k-2}$. Write $b_0,\dots,b_{m-1}$ and $e_0,\dots,e_{m-1}$ for its two sequences of auxiliary integer weights, with $e_{j-1}=w_j(\tau)$. These b_i are indexed weights at τ. The scale b is common to all types. Put $t=t_{\mathrm{ch}}$ and

$$(s_0,\dots,s_{k+m-2})=(v_0,\dots,v_{k-2},-b_0,\dots,-b_{m-1}).$$

For $1\le j\le m$, put $P=k+j-1$, $Q=m-j$. The prescribed rows at layer j are

$$\kappa^{\mathrm{big}}_{j,\tau,i}=s_i+e_{j-1}-(P-1-i)-(t+1)/2\quad(0\le i<P),$$
$$\kappa^{\mathrm{small}}_{j,\tau,c}=e_{j-1}-b_{j+c}+(t+1)/2+c\qquad(0\le c<Q).$$

The weight compatibility condition requires equality with these rows coordinate by coordinate. Both rows are integral since t is odd. When $j=m$ the second row is empty.

Lemma 15.7 (Spacing of the auxiliary weights)**.** *At every type the weight parameters of Definition 15.6 satisfy*

$$e_i-b_i\ge(t+3)/2\quad(0\le i<m),\qquad b_{i+1}-e_i\ge(t+3)/2\quad(0\le i<m-1).$$

In particular $e_{i+1}-e_i\ge t+3$. The initial column coordinates $u_0,\dots,u_{k-1}$ are half-integers in $\mathbf{Z}+1/2$ with $u_i>v_i>u_{i+1}$ and $u_{k-1}>-b_0$.

Proof. The two displayed inequalities and the strict interlacing are the defining inequalities on the parameters at an embedding. Apply the first inequality at $i+1$ and the second at i. Adding them cancels b_{i+1}. □

Lemma 15.8 (The two row weights and the column degrees)**.** *Write $t=t_{\mathrm{ch}}$, $N_l=k+l(t+1)$, and number coordinates from zero. Fix an embedding index ρ. The Harish–Chandra parameters at ρ consist of half-integers $U_0,\dots,U_{k-1}$ for the word, integers $v_0,\dots,v_{k-2}$ for the common row, and integers b_q,e_q for $0\le q<m$. Their separation conditions are*

$$U_i>v_i>U_{i+1},\qquad U_{k-1}>-b_0,\qquad e_q-b_q\ge(t+3)/2,\qquad b_{q+1}-e_q\ge(t+3)/2,$$

where a condition involving $q+1$ is imposed for $q+1<m$. The entries v_i,b_q,e_q and the two endpoints of U are independent of the word.

For $1\le j\le m$, put $l=j-1$, $p=k+j-1$, $w=w_{j\rho}=e_l$, and let z be the concatenation

$$(v_0,\dots,v_{k-2},-b_0,\dots,-b_{m-1}).$$

The two signed row tuples are

$$\kappa^{\mathrm{big}}_{j\rho i} = z_i + w - (p-1-i) - (t+1)/2 \quad (0 \le i < p), \qquad \kappa^{\mathrm{small}}_{j\rho c} = -b_{j+c} + w + (t+1)/2 + c \quad (0 \le c < m-j).$$

The first tuple is weakly decreasing with entries at least one. The second is weakly decreasing with entries at most minus one.

The column degrees at this layer form a weakly decreasing tuple of positive integers of length N_l. In order, they are

$$\nu_{uj\rho i} = U_i + e_l - N_l - t/2 + i \quad (0 \le i < k), \qquad \nu_{uj\rho,k+q(t+1)+d} = e_l - e_q - (l-q)(t+1) \quad (0 \le q < l,\ 0 \le d \le t).$$

Outline. Simplify the row shifts using the oddness of t, then use interlacing and half-gap bounds for row order and endpoint signs. Substitute the source parameters to obtain the column head and tail, and use the same bounds for their order and positivity. The complete proof is on p. 579. □

Lemma 15.9 (Weights of a selected word)**.** *Take a point of the construction of 33.32 (Part VIII, Chapter 33) beyond its numerical cutoff. Use the top set TT and top-position weights $a_s, b_T(s)$ ($s \in TT$) of Definition 14.2. Let $n = |A|$, $L = x^{\varepsilon}$, $X' = \exp(x^{\varepsilon/3})$, $b = e^L$, $a = xb$, $b' = a/X'$, $w'' = b/(nX')$. With the exponents of 33.27 put*

$$\delta = \lceil x^{c_D} \rceil, \qquad h_0 = 2\lceil x^{c_h} \rceil, \qquad S_c = 4k\delta, \qquad \eta = mh_0.$$

At each ρ there is an integer B_ρ such that

$$b_q = B_\rho + qh_0, \quad e_q = B_\rho + (q+1/2)h_0, \quad v_i = 4\delta(k-2-i).$$

At a marked position $\rho = (0,s)$, $s \in TT$, one has $B_\rho = b_T(s) + \lfloor b'/x \rfloor$. At a thin position $B_\rho = \lceil w'' \rceil$. At a thick position $B_\rho = \lceil b \rceil + d_\rho$, where the integer d_ρ is zero except possibly at one designated thick position and $0 \le d_\rho < |\mu(E)|$. Here $\mu(E)$ is the group of roots of unity of the host field.

If $\xi_i \in \{0,1\}$ is bit i of the word, then $U_{i+1} = v_i - 1/2 - \xi_i\delta$ for $0 \le i < k-2$. Its endpoints are

$$(U_0, U_{k-1}) = \begin{cases} (a_s + (k-1)/2, -b_T(s) - (k-1)/2), & \rho = (0,s),\ s \in TT, \\ (v_0 + S_c + 1/2, -1/2), & \text{otherwise.} \end{cases}$$

In particular the two half-gaps of 15.8 are both $h_0/2$, and each actual tail degree in block $q < j-1$ is $(j-1-q)(h_0-t-1)$.

Outline. Take the common base and endpoints from 33.32, use the bit formula for positive gaps, and use 33.27 and 38.3 for marked endpoint inequalities. Combine the rank and exponent bounds for $h_0 \ge t+3$ and use 15.8 for tail degrees. The complete proof is on p. 579. □

Definition 15.10 (The base term of a layer)**.** Put $n = |A|$ and $X'' = nxX'$. For the integer row tuples $\kappa^{\mathrm{big}}_{j,\tau}, \kappa^{\mathrm{small}}_{j,\tau}$ of lengths $k+j-1, m-j$, respectively, use the row compatibility of Definition 15.6 whenever the prescribed weights are required. Set

$$F_d(j) = -\tfrac{k-1}{2} \sum_{t \in A} \min\{w_j(0,t), w_j(1,t)\},$$

$$D^{\mathrm{Fock}}_j = \sum_{\tau} \left(\sum_a |\kappa^{\mathrm{big}}_{j,\tau,a}| + \sum_a |\kappa^{\mathrm{small}}_{j,\tau,a}| \right), \quad F_g(j) = -D^{\mathrm{Fock}}_j/2.$$

Write $F^L_j(w)$ for the base term that the size datum assigns to layer j at w. The base-term condition requires $F^L_j(w) = F_j(w)$ for every $1 \le j \le m$ and every finite place w, using the first applicable case in

$$F_j(w) = \begin{cases} F_g(j), & p \in \{g_0, g_1\}, \\ F_d(j), & p = d, \\ \displaystyle\sum_{\tau} (k-1) w_j(\tau)/(p^2-1), & p \in \mathcal{C},\ h_p = 0,\ p \le X'', \\ \displaystyle n_p^{-1} \sum_{z \in Z} F_{\mathrm{orb}}(j,p,z), & p \in \mathcal{C} \cup \mathcal{S},\ p \le X'', \\ 0, & \text{otherwise.} \end{cases}$$

In particular, the size datum satisfies $F_j^L(w) = 0$ whenever $p > X''$ and $p \notin \{g_0, g_1, d\}$.

Lemma 15.11 (Counting the orbit contribution)**.** *If T contains exactly one representative of each P_p-orbit in Z, then $n_p^{-1} \sum_{z \in Z} F_{\mathrm{orb}}(j,p,z) = \sum_{z \in T} F_{\mathrm{orb}}(j,p,z)$. The host prime d has the ramified class corresponding to d.*

Proof. By 14.1 each orbit has length n_p. Away from the signal class a return takes exactly a multiple of the order of h_p. In the signal class it takes an even number $2a$ of steps, with $a(2h_p) = 0$. Lemma 15.2 makes the summand constant on each of these orbits. At $p = d$ the tests for a prime dividing Q_S and for $p = q$ both fail, since $d \nmid Q_S$ and $q \neq d$. The next test, $p = d$, gives the stated class. □

Definition 15.12 (The loss charged to a column)**.** Let $M_u > 0$ be the stabilizer denominator of column u and u_0 the reference word. Put $\delta_{u,w} = v_p(M_u)$, where $v_p(p) = 1$. Write T_u for the level prime of u, E for the host field, $F = E^+$ for its real subfield, and $\tau(y)$ for the image of $y \in E$ in K at embedding index τ. Use Q_S, q, d of 14.1 and the fixed auxiliary primes g_0, g_1. The datum X specifies π_i for $i = 0, 1$ and $\pi_{T,u}$ for each u, all in $\mathcal{O}_E \cap \mathbf{Q}(\sqrt{-d})$, with

$$\pi_i \bar{\pi}_i = g_i, \qquad \pi_{T,u} \bar{\pi}_{T,u} = T_u, \qquad \pi_{T,u} \equiv 1 \pmod{Q_S q d (g_0 g_1)^2 \mathcal{O}_E}.$$

It injects the local level indices a of u into the primes $v_u(a)$ of F above T_u. At a prime v of F let F_v be the completion and $\mathfrak{p}_v$ its prime ideal in $\mathcal{O}_F$. The local level data specify a splitting $E \otimes_F F_v \simeq F_v \times F_v$ for each u, i and $v \mid g_i$. For each u and $v \mid T_u$ they specify a finite module $\mathcal{M}_{u,v}$ over the subring of $E \otimes_F F_v$ generated by $\mathcal{O}_E \otimes 1$ and $1 \otimes \mathcal{O}_{F_v}$. The column data supply the seen flag $\epsilon_u(w) \in \{0, 1\}$ and an embedding index $\tau_u(w, a)$ for each local level index a.

For real constants $c_g, c_{\mathrm{dig}}, c_6$, the local loss conditions are the following seven requirements.

1. $c_g \geq 10$, $c_{\mathrm{dig}} \geq 3$ and $c_6 \geq 12$.
2. For every u, i and $v \mid g_i$, the first component of $\bar{\pi}_i \otimes 1$ under the specified splitting belongs to the maximal ideal of $\mathcal{O}_{F_v}$.
3. For each u, a and $v = v_u(a)$, every integral lift of $\pi_{T,u} \otimes 1$ in the indicated subring annihilates $\mathcal{M}_{u,v}$.
4. For each u and $w \mid T_u$, $\epsilon_u(w) = 1$ exactly when $v_w(\tau(\pi_{T,u})) > 0$ for every embedding index τ.
5. For each u, $w \mid T_u$ and a, put $\tau = \tau_u(w, a)$. For every $z \in \mathcal{O}_F$, $z \in \mathfrak{p}_{v_u(a)} \iff v_w(\tau(z)) > 0$.
6. For every word u and tuple μ, the twisted chain has denominator M_{u_0}. This integer is divisible by the stabilizer order of every class at every stage $0 \leq i \leq m$ of each twisted chain.
7. Write $E_{u,j}^L(w)$ for the loss that the size datum assigns to word u at layer j and place w. Require $E_{u,j}^L(w) = E_{u,j}(w)$ for every word u, every $1 \leq j \leq m$ and every finite place w, with $E_{u,j}(w)$ given below.

Put $N_i = k + i(t_{\mathrm{ch}} + 1)$, $f = 2|A|$ and $a_* = xb$. Use the parameter $\eta > 0$ from the weight intervals of Definition 14.2. The row scale in the orbit error term is $S_c = k(v_0 - v_1)$, where $v_0 - v_1$ is the common spacing of consecutive interlacing coordinates. For $0 \leq i < N_{j-1}$ let $s_{u,\tau,i}$ be weight i of column u at τ. For $i < k$ use its initial weight. For $i = k + \ell(t_{\mathrm{ch}} + 1) + r$, $0 \leq r \leq t_{\mathrm{ch}}$, use $-w_{\ell+1}^u(\tau) + (t_{\mathrm{ch}} - 2r)/2$, where $w_{\ell+1}^u(\tau)$ is extension weight ℓ of column u at τ. Put

$$\nu_{u,j,\tau,i} = s_{u,\tau,i} + w_j(\tau) - \frac{N_{j-1} + N_j - 1 - 2i}{2}, \qquad A_{u,j,\tau} = \sum_{i < N_{j-1}} \nu_{u,j,\tau,i}.$$

For fixed u, j, write $A_{\epsilon,t}, \nu_{\epsilon,t,i}$ for these quantities at $\tau(\epsilon, t)$. Put

$$\begin{aligned} G_{i,u,j,w} &= F_g(j) + e_w^{-1} \sum_{\tau} A_{u,j,\tau} v_w(\tau(\bar{\pi}_i)) + c_g f (N_m^2 + m^2 \eta), \\ c_{u,j,t} &= A_{0,t} + \mathbf{1}_{t \in T} T (\nu_{0,t,1} - \nu_{0,t,0} + \nu_{1,t,0}), \\ D_j^* &= F_d(j) + \tfrac{1}{2} \max_u \sum_{t \in A} c_{u,j,t}, \\ C_{j,p}^* &= c_{\mathrm{dig}} f N_m^2 \log_p a_* + \frac{\sum_\tau w_j(\tau)}{p^2 - 1}. \end{aligned}$$

Set $Z_A=\{(\epsilon,t)\in Z : p\notin\mathcal{S},\ \epsilon=0\}$ and $Z_{BC}=Z\setminus Z_A$. Put $Z_{\mathrm{top}}=Z_{BC}\cap(\{0\}\times TT)$, $\gamma(p)=p/(p^2-1)$, and let $K_{\delta,1},K_{\delta,2}$ be the digit-loss constants of Definition 14.2. Define

$$\begin{aligned}R_{BC}(j,p)=\gamma(p)\#Z_{\mathrm{top}}(b+\eta+1+3S_c)+\frac{c_6}{p}\sum_{z\in Z_{BC}}w_j(z)\\ +\#Z_{BC}\,N_{j-1}N_j(\log_p(4n_pN_ma_*)+1)\\ +K_{\delta,1}\#Z_{BC}\,N_m^2\log_p(2a_*)+K_{\delta,2}\#Z_{\mathrm{top}}(\log_p(2a_*))^2.\end{aligned}\tag{15.1}$$

Define

$$I^*_{j,p}=\max\left\{0,n_p^{-1}\sum_{z\in Z_A}F_{\mathrm{orb}}(j,p,z)\right\}+R_{BC}(j,p)+X^*_{j,p}.$$

Here $X^*_{j,p}=(k-1)p^{-3}\sum_{z\in Z_{BC}}w_j(z)+c_6fN_m^2\log_p a_*$ if $p\in\mathcal{C}$ and h_p has odd order greater than 1, and is zero otherwise.

Put $\mathrm{Top}=\{\tau(0,t):t\in TT\}$. For integral highest weights $\lambda^u_{i,\tau,r}$ put

$$W_{u,\tau}=\max_{i,r}\left|\lambda^u_{i,\tau,r}-\lfloor i(t_{\mathrm{ch}}+1)/2\rfloor\right|$$

over $0\le i\le m$ and $0\le r<N_i$, except for $r=0$ on Top. Use zero for an empty maximum. Set

$$T^*_{u,w}=\epsilon_u(w)\big(2\sum_{\tau}W_{u,\tau}+fN_m+|\mathrm{Top}|(t_{\mathrm{ch}}+1)(m+1)/4\big).$$

The first applicable case gives the loss

$$E_{u,j}(w)=\begin{cases}T^*_{u,w}, & p=T_u,\\ \delta_{u,w}+G_{i,u,j,w}, & p=g_i,\ i=0,1,\\ \delta_{u,w}+D^*_j, & p=d,\\ \delta_{u,w}+C^*_{j,p}, & p\in\mathcal{C},\ h_p=0,\ p\le X'',\\ \delta_{u,w}+I^*_{j,p}, & p\in\mathcal{C}\cup\mathcal{S},\ p\le X'',\\ \delta_{u,w}, & \text{otherwise.}\end{cases}$$

Lemma 15.13 (The common stabilizer denominator)**.** *Under condition (6) of 15.12, every word has $M_u=M_{u_0}$, and every twisted chain has this denominator.*

Proof. Condition (6) gives the assertion for each twisted chain. The twist by the reference tuple μ_0 gives the original chain of u. Its denominator M_u therefore equals M_{u_0} by condition (6) of 15.12. □

The ordinary and enhanced column bounds of 16.19 apply at $p\notin\{T_u,g_0,g_1\}$ for used twists μ with $p\notin Q'(\mu)$. The enhanced bound also requires matching local systems and the terminal estimate. For $p=g_i\neq T_u$, see 17.21 and 17.23.

Definition 15.14 (The gain retained after selection)**.** Use ρ_B,Σ_p and candidacy from 14.8. Put $o_p=\mathrm{ord}(h_p)$ and define B_p by

$$\begin{aligned}B_p\iff\ & o_p<x^{\varepsilon/4}\\ &\vee\big(p\ge k^2x^{26}\ \wedge\ \exists i\in\mathbf{N}_{>0},\ |p^i-X'|\le\rho_BX'\big)\\ &\vee\big(p\text{ is a candidate}\ \wedge\ \Sigma_p>x^{-\varepsilon}\big).\end{aligned}$$

For $C_r\in\mathbf{R}$, define a set $\mathcal{N}_{C_r}$ of rational primes by

$$\begin{aligned}p\in\mathcal{N}_{C_r}\iff\ & p>X''\ \vee\ p\notin\mathcal{C}\cup\mathcal{S}\\ &\vee\big(p\in\mathcal{C}\ \wedge\ ((o_p>1\wedge o_p\text{ is odd})\vee(p<X'(1+C_r\rho_B)\wedge B_p))\big).\end{aligned}$$

The prime selection condition requires the set $\mathcal{D}$ of discarded primes to equal $\mathcal{N}_{C_r}\setminus\{T_u:u\text{ is a word}\}$. At a place w above p, the retained gain $\mathrm{gainD}_j(w)$ is zero if $p\in\mathcal{D}$. Otherwise it equals the original gain assigned by the size datum to layer j at w.

Lemma 15.15 (Gain at the exceptional primes)**.** *If $p = T_u$ for any word, the retained gain equals the original gain. At $p = g_0, g_1$ both gains vanish. At a supported place of inert or signal class satisfying the filling identities of 14.9, the original gain is the prime gain in that definition for $1 \le j \le m$.*

Proof. Membership in the discarded set requires $p \neq T_v$ for every word v. Thus a level prime uses the second branch of 15.14. The size datum prescribes zero original gain at each fixed auxiliary prime, and either branch then gives zero. The last assertion is the gain identity of 14.9 on its specified support. □

Definition 15.16 (The net budget of a column)**.** Set

$$\mathrm{sg}(u, w) = \sum_{j=1}^{m} \big(\mathrm{gainD}_j(w) + F_j^L(w) - E_{u,j}^L(w)\big).$$

Let S_L denote the finite support specified by the size datum. Its original gain, stored base term and stored loss vanish at every $w \notin S_L$, for every layer and word. If p_w lies in neither the inert nor the signal class, the raw gain vanishes at every layer and both budgets vanish.

Lemma 15.17 (Value and support of the budget)**.** *For every u and $w \notin S_L$ one has $\mathrm{sg}(u, w) = 0$. When the base and loss identifications of 15.10 and 15.12 hold, the budget at any w is $\sum_{j=1}^{m}(\mathrm{gainD}_j(w) + F_j(w) - E_{u,j}(w))$.*

Proof. Outside S_L, either branch defining the retained gain gives zero because the original gain vanishes there. The other two summands vanish by the support conditions. For the second assertion substitute the two identifications into the finite sum of 15.16. □

Chapter 16

From local lifts to bounds on whole columns

Let p be the characteristic of the residue field at a finite place w of K. Normalize $v_w(K^\times) = \mathbf{Z}$ and put $e_w = v_w(p)$. For a word u, let T_u be its level prime and $\delta_{u,w} = v_p(M_u)$ its stabilizer loss. The chain of $m \geq 2$ lifts has spaces $V_{u,i}$ with lattices $\Lambda_{u,i}$ and complex embeddings $\iota_{u,i}$. Superscripts μ denote twists, and e_0^μ identifies the initial lattices. The reference index is μ_0. The indices in use are $\mu_0, \mu_*, \mu_*', \mu_w$, with excluded sets $Q'(\mu)$.

Put $a_* = xb$ and $f = 2|A|$. Write $t_{\mathrm{ch}} + 1$ for the increase in rank at each lift, so that the rank at stage i is $N_i = k + i(t_{\mathrm{ch}} + 1)$. The even starting rank and odd t_{ch} give even N_i. Use the weights $w_j(z)$, base terms $F_j(w)$ and cutoff X'' of 15.10. Write $E^{\mathrm{ev}}_{u,j}(w)$ for the explicit loss given by the case formula of 15.12, with the constants specified there and in 14.2 [3]. The error hypotheses of 15.12 impose its bounds on the error constants, its level and place conditions, the common stabilizer denominator, and the identity $E_{u,j}(w) = E^{\mathrm{ev}}_{u,j}(w)$ for every u, w and $1 \leq j \leq m$.

16.1 Valuation bounds and the input lattice

Definition 16.1 (Bounds for vectors and columns)**.** Write $\Lambda_{(w)} = \mathcal{O}_{K,(w)}\Lambda$. For $s \in \mathbf{R}$ put $\Lambda_w[s] = \{az : z \in \Lambda_{(w)},\ a \in K,\ a = 0 \text{ or } v_w(a) \geq e_w s\}$. Let $\mathcal{C}_{u,\mu,w}(s)$ consist of those $x \in V_{u,0}$ whose complex column is the image of a rational vector in $\Lambda_w[s]$, with Λ the terminal lattice of the column. At μ_0 the column is $C_u(\iota_{u,0}x)$ and the lattice is $\Lambda_{u,m}$. At $\mu \neq \mu_0$ they are $C^\mu_u(\iota^\mu_{u,0}e^\mu_0 x)$ and $\Lambda^\mu_{u,m}$. The embedding of $e^\mu_0 x$ is the image of $\iota_{u,0}x$ under the initial complex twist isomorphism.

At (u, w), let h_g be the permitted-pair conditions of 13.7 ($p \geq 5$ inert-class outside the host level, $p \nmid T_u$, even second Frobenius order above p). Write $(\mathcal{O}, \mathcal{K})$, $\omega \in \mathcal{O}$ ($\omega^2 = p$), and M for the local pair (O, K) of 13.7 (reserving K for the global field), uniformizer root, and input lattice. Use R_p (14.1) and $W_p = V_p$ (14.8). At the top weights a_L, b_L of 14.3–14.7, set

$$A_{t,j} = [g_{0,t}(j)]_+, \qquad H_{t,j} = \max\{g_{U,t}(j) - \mathrm{Pr}_t(j), A_{t,j}\}, \qquad [a]_+ = \max(a, 0).$$

Let $Y_U = \prod_{t \in U} y^m_{\iota(t)}$ for $U \subseteq R_p$, in the model's order, with the neighbour y_v of 12.9 and the position map ι. Put $S_t = \lceil \sum_{j=1}^m (H_{t,j} - A_{t,j}) \rceil$ and $n(B) = \sum_{t \in W_p \setminus B} S_t$.

Definition 16.2 (The lattice supporting the enhanced estimate)**.** In a permitted model, let V^{ram} be its ramified submodule and τ_p the budget of 14.9. Set

$$N_P = \mathrm{span}_{\mathcal{O}}\{Y_B(p^{n(B)}F) : B \subseteq W_p,\ F \in M\} \cap p^{[\lceil \tau_p \rceil]_+} M \cap V^{\mathrm{ram}}. \tag{16.1}$$

For the rational ramified subspace $V^{\mathrm{ram}}_{u,0}$, put $S^{\mathrm{ram}}_{u,w} = (\Lambda_{u,0})_{(w)} \cap V^{\mathrm{ram}}_{u,0}$. Let $\mathcal{N}_{u,w}$ consist of the rational initial vectors x for which $c_{u,w,h_g}(x) \in N_P$ for some permitted model with initial comparison c_{u,w,h_g}, together with all of $S^{\mathrm{ram}}_{u,w}$ when $W_p = \varnothing$.

Lemma 16.3 (The rational lattice below the enhanced lattice)**.** *At a permitted (u, w) use the tied arithmetic comparison $c = c_{u,w,h_g}$ into the space $\mathscr{P}$ of component-indexed polynomial functions of lattice tuples. Put $A = \mathcal{O}_{K,(w)}$, $V = V_{u,0}$, $L_w = (\Lambda_{u,0})_{(w)}$ and $V^r = V^{\mathrm{ram}}_{u,0}$. The coefficient embedding $i : K \hookrightarrow \mathcal{K}$ has $i^{-1}(\mathcal{O}) = A$. The comparison is i-semilinear with $\mathcal{K}$-independent images of a K-basis. It gives $M = \mathcal{O}c(L_w)$ and $V^{\mathrm{ram}} = \mathcal{K}c(V^r)$, with the latter viewed as an $\mathcal{O}$-submodule of $\mathscr{P}$.*

Write $R_w = L_w \cap V^r$ and, for $W_p \neq \varnothing$, $J_w = \{z \in V^r : c(z) \in N_P\}$. Then $\mathcal{O}c(R_w) = M \cap V^{\mathrm{ram}}$, $\mathcal{O}c(J_w) = N_P$, and for $a = \sum_{t \in W_p} S_t$, $b_0 = [\lceil \tau_p \rceil]_+$,

$$p^{a+b_0} R_w \subseteq J_w \subseteq p^{b_0} R_w. \tag{16.2}$$

The member, u and w determine these modules.

Outline. Transport the class polynomials to lattice tuples and use the tableau degree bound for integral scaling in 14.12. Retract to K, commute intersections with flat base change, and compare basis coordinates to get $J_w = C_w$. The complete proof is on p. 579. □

Lemma 16.4 (An individual edge in the neighbour sum)**.** *Fix integers $q_v \geq 0$ in a permitted model with $\omega^2 = p$. For base and intermediate fibres H_L and H_a (on $S_{\mathcal{O}} + \omega L_{\mathcal{O}}$, 14.10), put $A_v = \mathcal{A}_v$, $R_a = \omega^{-A_v} H_a$, $R'_a = \omega^{-A_v} H_a^\vee$ (integral duals). Define*

$$U_L = \omega^{-q_v} H_L \cap \bigcap_{a : L \to *} R_a, \qquad P_L = \left(\omega^{-q_v} H_L^\vee \cap \bigcap_{a : * \to L} R'_a \right)^\vee.$$

Then

$$\omega^{q_v} U_L \subseteq H_L \subseteq U_L, \qquad \omega^{q_v} H_L \subseteq P_L \subseteq H_L, \qquad p^{A_v} R_a \subseteq P_L \quad (a : * \to L). \tag{16.3}$$

Let L_B be the equivariant function lattice with product fibre $\mathscr{L}_B$ given by P on B and H elsewhere. For $v \in B$, replace P at v by U to obtain $L^+_{B,v}$ and $\mathscr{L}^+_{B,v}$. Assume trivial level stabilizers and fix a full lattice tuple $\mathbf{L}$, so every $\mathbf{L}_v$ is a lattice. Denote the equivariant orbit function with value ξ at $\mathbf{L}$ in component i by $\delta_{i,\mathbf{L}}(\xi)$, and write $\mathbf{L}[S] = \mathbf{L}[v \leftarrow S]$. Then $y_v L^+_{B,v} \subseteq L_B$. If $S \to L_v$, $\xi \in \mathscr{L}^+_{B,v}(\mathbf{L}[S])$, and $p^{A_v}\xi \in p^E \mathscr{L}_B(\mathbf{L}[L'])$ for every other edge $S \to L'$, where $E \in \mathbf{N}$, then

$$y_v \delta_{i,\mathbf{L}[S]}(\xi) \equiv \delta_{i,\mathbf{L}}(p^{A_v}\xi) \pmod{p^E L_B}. \tag{16.4}$$

Outline. Use the half-step inclusions in the fibres of 14.10 and the defining dual pairing to prove integrality. In the orbit function, all other edges vanish modulo p^E, and the trivial stabilizer gives unique transport of the chosen edge. The complete proof is on p. 580. □

Proposition 16.5 (Colength of the enhanced lattice)**.** *In a permitted model put $R = M \cap V^{\mathrm{ram}}$, $W = W_p$, $e = v_{\mathcal{O}}(p)$ and $d_M = \mathrm{rk}_{\mathcal{O}} M$. Assume finitely many level classes with trivial stabilizers, commuting neighbours preserving M, M_{na}, saturation of R in M, and the following inclusions for the prescribed lattice exponent $b_p \in \mathbf{N}$,*

$$p^{b_p} M_{\mathrm{na}} \subseteq M \subseteq M_{\mathrm{na}}. \tag{16.5}$$

In the arithmetic specialization of 14.12, the orbit positions index the f embeddings and their k weight entries agree with the level-zero weights. Under the parameter bounds of that proposition, its comparison of Weyl and dual Weyl lattices gives these inclusions with $b_p = [\lceil C_e f k^2 \log_p a_* \rceil]_+$ for $C_e \geq 1$. Saturation follows from the ramified subspace comparison of 16.3.

Assume that the position map is injective on W and identify W with its image. At each $v \in W$ require the presentation of 14.10 with rank r, $t_* = q_v = [\lceil t_v \rceil]_+$, $D_0 = D_p$ and share κ_v. Its off-edge test is $F_i \in \omega^{c_*} R_{a'}$ at every other edge from the source of a_i, with $c_* \in \mathbf{N}$ and $c_* \geq 4(D_p + 1)$ when $W \neq \varnothing$. The test at $c_* = 8(D_p + 1)$ suffices. The tied models use $r = r_G$ for $p < p_0$ and $r = 1$ otherwise. If $m \geq 1$ and $m \sum_{v \in W} \kappa_v \leq 1/2$, then

$$N_P \subseteq p^{\tau_p} R, \qquad \mathrm{length}_{\mathcal{O}}(R/N_P) \leq e d_M (\tau_p + \mathrm{io}_p)$$

for the budgets $b_p = e_p$, $\tau_p = \tau_P(p)$, and $\mathrm{io}_p = I_P(p)$ of 14.9.

Outline. Use Smith bases, 16.4, 14.10, and Nakayama's lemma to bound the number of exceptional tensors. Descending induction bounds the tails of the Smith exponents, and summing these bounds with saturation and the budgets of 14.9 gives the result. The complete proof is on p. 581. □

Definition 16.6 (The two column budgets)**.** For $\sigma \in \mathbf{R}$, denote the base and enhanced column budgets by $B^{0}_{u,w}(\sigma)$ and $B^{+}_{u,w}(\sigma)$. With $g^{\mathrm{s}}_{p,j}$ the gain used in the column bound, put

$$B^{0}_{u,w}(\sigma) = \sum_{j=1}^{m} (F^{L}_{j}(w) - E^{\mathrm{ev}}_{u,j}(w)) + \sigma\delta_{u,w}, \qquad B^{+}_{u,w}(\sigma) = B^{0}_{u,w}(\sigma) + \sum_{j=1}^{m} g^{\mathrm{s}}_{p,j}. \tag{16.6}$$

Let $\mathcal{E}$ be the set of primes dividing $N_{\mathrm{cond}} g_0 g_1$, where N_{cond} is the host conductor. Let $\mathcal{O}_{\mathrm{odd}}$ be the inert-class primes at which $h_P \neq 0$ has odd additive order. Denote the counted gain of 14.8 by $g^{\mathrm{c}}_{p,j}$. Then

$$g^{\mathrm{s}}_{p,j} = \begin{cases} g^{\mathrm{c}}_{p,j}, & p \text{ inert-class or signal, } p \notin \mathcal{E} \cup \mathcal{O}_{\mathrm{odd}}, \\ 0, & \text{otherwise.} \end{cases} \tag{16.7}$$

The counted gain of 14.8 vanishes for $p > X''$ or inert class with $h_P = 0$. Elsewhere it is

$$g^{\mathrm{c}}_{p,j} = \sum_{t \in W_p} H_{t,j} + \sum_{t \in R_p \setminus W_p} A_{t,j}. \tag{16.8}$$

16.2 Hecke images and their integral span

Use the local data Y and the common identification of orbit positions with embeddings in 14.15(1), with the filling, edge ranks, marked positions and weights specified there. Let W assign levels and a chain on this filling to every tuple μ at every permitted (u, w). Its levels and chain at μ_w must agree with those of Y. Write Y_μ for Y with the levels and chain assigned by W at μ. With $K^{[m]}_{\mu}$ the local composite and ϖ a uniformizer of $\mathcal{O}$, let $\mathcal{L}_\mu(s)$ consist of the local inputs F satisfying

$$K^{[m]}_{\mu} F \in \varpi^{\lceil v_{\mathcal{O}}(p) s \rceil} \Lambda^{\mathrm{loc}}_{\mu,m}. \tag{16.9}$$

Theorem 16.7 (The whole-chain estimate on Hecke images)**.** *Fix Y, W, σ and assume the error hypotheses of 15.12. For each used μ, identification for Y_μ as in 14.15(1), and permitted (u, w, h_g) with $p \notin Q'(\mu)$, assume a matching admissible system and kernel of 13.8 on the prescribed model, filling, levels and chain. Write $d_G(i)$ for its stabilizer losses and $\mathcal{V}$ for the local positions, with representative injection $\iota : R_p \to \mathcal{V}$. Require $f_{v,j} \in \mathbf{R}$ such that the differentiated coefficients at step $j-1$ and position v satisfy the lower bound $f_{v,j}$ of 10.19 for $v \in \mathcal{V} \setminus \iota(R_p)$ and $1 \le j \le m$, and*

$$B^{0}_{u,w}(\sigma) \le \sum_{j=1}^{m} \sum_{v \notin \iota(R_p)} f_{v,j} - \sum_{i=0}^{m-1} d_G(i). \tag{16.10}$$

For every tie for Y and permitted place with $\log_p(2a_*) \ge 1$, assume the comparisons of 14.13 for $t \in R_p$, $1 \le j \le m$,

$$g_{U,t}(j) \le g^{\mathrm{loc}}_{B,\iota(t)}(j), \qquad g_{0,t}(j) \le g^{\mathrm{loc}}_{A,\iota(t)}(j), \qquad P^{\mathrm{loc}}_{\iota(t)}(j) \le \mathrm{Pr}_t(j). \tag{16.11}$$

At every coefficient-field place with $\log_p(2a_*) < 1$, assume the comparisons of 14.14 for the same t, j,

$$g_{U,t}(j) - \mathrm{Pr}_t(j) \le 0, \qquad g_{0,t}(j) \le 0. \tag{16.12}$$

Then for each of the indicated twists, ties and permitted places outside $Q'(\mu)$, every $V' \subseteq V \subseteq R_p$ and $F \in M$ satisfy

$$Y_{V'} F \in \mathcal{L}_\mu \left(B^{0}_{u,w}(\sigma) + \sum_{j=1}^{m} \left(\sum_{t \in V'} H_{t,j} + \sum_{t \in V \setminus V'} A_{t,j} + \sum_{t \in R_p \setminus V} A_{t,j} \right) \right). \tag{16.13}$$

Outline. The matching system supplies 13.8 and 13.4, while item 7 of 10.10 gives the exponent-zero bound of 10.19 everywhere. Carrying the tie and comparisons to the twist, apply 14.16 with lower exponent $B^{0}_{u,w}(\sigma)$ and split at V. The complete proof is on p. 582. □

Lemma 16.8 (Extension from generators to N_P)**.** *Under 16.7, for each used μ, identification for Y_μ as in 14.15(1), and permitted (u, w, h_g) with $p \notin Q'(\mu)$, every $F \in N_P$ satisfies*

$$F \in \mathcal{L}_\mu \left(B^{0}_{u,w}(\sigma) + \sum_{j=1}^{m} g^{c}_{p,j} \right). \tag{16.14}$$

The lattice N_P is common to these twists.

Outline. Apply the Hecke-image bound with $V = W_p, V' = B$, multiply by $p^{n(B)}$, and use $\sum_j H_{t,j} \le \sum_j A_{t,j} + S_t$ on omitted representatives to dominate $\sum_j g^{c}_{p,j}$. Extend to N_P by $\mathcal{O}$-linearity, with common defining data across twists. The complete proof is on p. 582. □

Lemma 16.9 (Testing a local form lattice at its classes)**.** *Fix a word u, a used tuple μ, and $0 \le i < m$. Let w have residue characteristic p outside $\{T_u, g_0, g_1\} \cup Q'(\mu)$. Put $A = \mathcal{O}_{K,(w)}$, and fix an embedding $\iota_w : K \hookrightarrow \mathbf{C}_p$ inducing w.*

Use the constructed twisted column with its compatible class charts, prescribed class-pair unfoldings and source jet data. Choose compatible local lattice frames at stages i and $i+1$ with matrix entries in $\iota_w(K)$. At a class representative, express the value of a form in this frame, with its determinant twist. Remove the determinant factor to obtain its polynomial part. In the source frame include the full-height columns prescribed by the source shift. A reading of this value is its normalized Weyl pairing with a product of flag minors of the resulting diagram, one minor for each column, as in 10.17. Use the target diagram and target determinant twist for the readings at stage $i+1$. A rational class value means the evaluation of a weight polynomial over K, with the prescribed determinant twist.

At either stage, a complex class function belongs to the image of the localized form lattice if and only if it belongs to the complex space of forms at that stage and each of its class values has a rational representative all of whose readings have absolute value at most one. For a rational target representative every reading has the absolute value of an element of $\iota_w(K)$.

Outline. Choose rational frames by approximation and use 10.17 to test integrality on twisted flag-minor products. Project and polarize to recover readings in the coefficient field, then clear the denominators at the other finite places by an integer that is a unit at w. The complete proof is on p. 583. □

Lemma 16.10 (From kernel entries to a lift)**.** *Fix u and a used twist μ. A place w is level-free when $p \notin \{T_u, g_0, g_1\}$ and $p \notin Q'(\mu)$. Fix such a place. At $0 \le i < m$ let ν_z be the entry diagram at z, with at most N_i rows, and Θ the kernel series for a pair of classes at $i, i+1$. A column choice I selects h increasing target indices from $\{1, \dots, N_{i+1}\}$ for each diagram column of height h. Let $G_{\nu,I}$ be the product of minors of $X_z Z_z$ on the first h rows and the chosen columns, as in 10.15. An entry bound B means that the absolute values of all coefficients of $P_I = G_{\nu,I}(X, \partial_Z)\Theta|_{Z=0}$ are at most p^{-B}, for every pair and I.*

Suppose the diagrams and series agree with the kernel datum, the class-pair scalars have norm one, the pair maps preserve evaluated rational spaces, and the lift carries the complex space of forms at stage i into that at stage $i+1$. Use the local class readings of 16.9. Assume the denominator condition of 15.12, and use $\sigma = 0$, or $\sigma = 1$ with trivial initial stabilizers. An entry bound $B \in \mathbf{R}$ gives a rational lift of every integral input with target lattice bound

$$B - \big(1 - \sigma \mathbf{1}_{\{i=0\}}\big)\delta_{u,w}. \tag{16.15}$$

At μ_0 the same bound holds for the reference lift.

Outline. Cauchy–Binet and straightening bound coefficients, and 16.9 bounds pair terms and rounds exponents. Clear denominators by 15.12, apply the target test of 16.9, and transport by stage isomorphisms, using trivial stabilizers at $\sigma = 1$. The complete proof is on p. 584. □

16.3 Orbit bounds and ordinary places

Let P be the shift of period n_P from 15.1. The entry datum $\mathscr{E}$ gives diagrams $\nu_{u,i,z}$ with at most N_i rows, prescribed by 15.12 for $i < m$, and a series Θ over $\mathbf{C}_p$ for each u, μ, w, i and consecutive class pair. The letter data at w consist of a scalar ξ_w and, for each position z, a scalar $\epsilon_{w,z}$ and a series $\beta_{w,z}$ in letters indexed by $\mathbf{Z}/n_P\mathbf{Z}$. A *legal representative set* at p is a subset $R \subseteq \{0,1\} \times A$ such that for each position z there is exactly one $r \in R$ with $P^e r = z$ for some integer $e \geq 0$. Thus it contains one point of each orbit of the shift in Definition 15.1. For every such R, admissibility requires $|\xi_w|_p = 1$, signs $\epsilon_{w,r} = \pm 1$ for $r \in R$, and the four letter conditions of 15.4 for each $\beta_{w,r}$. Its coefficients lie in $\widehat{\mathbf{Q}_p^{\mathrm{nr}}}$, and it is invariant under $Z_s \mapsto a^{(-1)^s p^s} Z_s$ for $a^{p^{n_P}-1} = 1$. Every monomial with nonzero coefficient has positive degree in both parity blocks and bidegree different from $(1,1)$. The exponentials of the two additive coboundaries have integral coefficients and constant term one after letter-logarithm substitution, as in 15.4.

Use $\ell_{z,ab}(Z) = \sum_{h \geq 0} p^{-h} Z_{P^{-2h}z,ab}^{p^{2h}}$, with a, b indexing the $N_i N_{i+1}$ matrix entries. An even cyclic expansion has the form of 15.5, with $\Theta(\xi_w \ell)$ equal to an integral multiplier times the exponential of the sum of the phases from R. At inert class with $h_P = 0$, each kernel instead has a common unit ξ_0 scaling ℓ on the left, and its own units ξ_z, signs ϵ_z and two-letter phases β_z satisfying the coefficient-field, root-of-unity invariance, support and coboundary conditions of Lemma 15.4 at period two. The exponent on the right is $\sum_z (\epsilon_z/2) \sum_{a,b} \beta_z(\ell_{z,ab}, \xi_z \ell_{z,ab})$. For inert class of nonzero odd order, the required expansion is

$$\Theta(\xi \ell(Z)) = U(Z) \exp\left(\sum_{r \in R} \frac{\epsilon_r}{2} \sum_{a,b} \beta_r((\ell_{P^s r,ab}(Z))_{s \in \mathbf{Z}/(2n_P)}) \right), \tag{16.16}$$

with U integral, $|\xi|_p = 1$, signs ϵ_r, and a finite set R of positions. For each kernel the phases satisfy the conditions of 15.4 at period $2n_P$, over one field $K_0 \subseteq \mathbf{C}_p$ with integer-valued v_p and for one primitive $(p^{2n_P} - 1)$-st root.

Definition 16.11 (Admissible orbit data)**.** Call the orbit data admissible if the following five requirements hold and, under the common denominator condition, (16.15) holds for every used μ, level-free w, $0 \leq i < m$ and $B \in \mathbf{R}$.

1. At every signal prime and every even-order inert prime require $p \geq 5$ and even $n_P \geq 2$. Every inert-class prime is at least five and $g_0, g_1 \neq d$.

2. At every level-free place on the signal and even-order inert branches, each legal R has admissible letter data. For every class pair at $i < m$ of even rank product the cyclic expansion uses the same $\xi_w, \epsilon_{w,r}, \beta_{w,r}$, for each R. At level-free places on the zero-shift and nonzero odd-order inert branches, require the respective expansions above for every such class pair.

3. Require the three estimates of 15.5 for all admissible data with its size and rank hypotheses, in the following form with constants C_6, C. Multiply each dimension and logarithm error by C. In the refined cyclic bound use $C_6 p^{-1} \sum_s w_s$ for the weight loss, and keep $\gamma(p) = p/(p^2 - 1)$ for the reverse-partner loss. Also require, at every inert-class w with $5 \leq p \leq X''$, $h_P \neq 0$ and even $n_P \geq 2$, the bound $A_{t,i+1}$ for every $u, i < m, t \in R_p$ and every local kernel with the expansion at $(0,t)$ for the specified letter data there. Proposition 37.26 supplies this last hypothesis for the selected arithmetic family.

4. At every level-free place with $p \neq d$, require entry exponent zero for even rank product. At level-free places of discriminant class $p = d$, require (37.34) for every class pair at $i < m$.

5. For every $u, i < m, z$, put $j = i + 1$, $w_z = w_j(z)$, $\nu_z = \nu_{u,i,z}$, and number the rows from one. Set $\mathrm{cap}_z = \nu_{z,2}$ on the top and $\nu_{z,1}$ elsewhere. Require

$$\begin{gathered} w_z > 0, \quad |\nu_z| \leq a_*^2, \quad 2\log_5(a_*^2) + 2 \leq k - 1, \quad X'' \leq a_*, \quad a_* > 1, \\ 0 \leq |\nu_z| - (k-1)w_z \leq 4w_z \quad \text{off the top}, \\ 0 \leq |\nu_z| - \nu_{z,1} - (k-3)w_z \leq 4w_z, \quad \nu_{z,1} \geq 2w_z \quad \text{on the top}, \\ \mathrm{cap}_z \leq b + \eta + 1 + 3S_0, \qquad \mathrm{cap}_z \leq B_j(z) + c_G, \qquad \bar{\alpha}_t \leq \nu_{(0,t),1} - \nu_{(0,t),2} \quad (t \in TT). \end{gathered} \tag{16.17}$$

Here $B_j(z) = \mathrm{fill}_j(z) - \kappa_j$, $\kappa_j = (k + (2j-1)(t_{\mathrm{ch}}+1))/2$, and the filling and $\bar{\alpha}_t$ are as in 14.2–14.9; put $c_G = c = 4S$ in the notation of 14.2. Here η is the weight-window width and $S_0 = S_c$ is the row-scale parameter of 15.12 (with orbit period $n_P = n_p$). The constants satisfy

$$C \geq 0, \quad C_6 \leq c_6, \quad K_{\delta,1}, K_{\delta,2} \geq 0, \quad 3C \leq \min(1 + K_{\delta,1}, c_{\mathrm{dig}}). \tag{16.18}$$

Let $g_{0,S}(p,j) = \sum_{t \in R_p} A_{t,j}$ except at $p \in \mathcal{O}_{\mathrm{odd}}$, where it is zero. On the signal and even-order inert branches the representatives are respectively $\{0\} \times R_p$ and $\{0,1\} \times R_p$; put $g_{0,r,j} = A_{t,j}$ for $r = (0,t)$ and $g_{0,r,j} = 0$ for $r = (1,t)$. Use the errors of 15.12. Write BC for kind-BC positions, TT_{BC} for their top positions, and $F_{\mathrm{Orb},j}(p,z)$ for the orbit base of 15.1–15.10. Let $\mathrm{err}_{r,j}$ be the following error restricted to the orbit of r (summing over kind-BC orbits to $R_{BC}(j,p)$ of (15.1)):

$$\begin{aligned} &\gamma(p)\#TT_{BC}(b+\eta+1+3S_0) + \frac{c_6}{p}\sum_{z\in BC} w_j(z) \\ &\quad + \#BC\, N_{j-1}N_j(\log_p(4n_P N_m a_*)+1) + K_{\delta,1}\#BC\, N_m^2 \log_p(2a_*) + K_{\delta,2}\#TT_{BC}(\log_p(2a_*))^2. \end{aligned} \tag{16.19}$$

Proposition 16.12 below records the two restriction laws, the two addition laws and the four remaining orbit estimates; the one-letter kernel bound in (4) also uses item (3) above.

Proposition 16.12 (The orbit entry estimates)**.** *Assume the cyclic and one-letter estimates with constants C_6, C in Definition 16.11(3), the conditions (16.17)–(16.18), and the discriminant expansion (37.34) at level-free places in the discriminant class. In (2)–(5), u is any word, $0 \leq i < m$, $j = i+1$, and $\rho(p) = 1/(p^2-1)$. At inert primes let F_{inert} be the base term of 15.10 at j,p.*

1. *For every $i \in \mathbf{N}$ and prime $p \geq 5$ with even $n_P \geq 2$, the cyclic expansion restricts to each orbit for every representative system and admissible letter datum. For $p \geq 5$ and $n_P = 1$ the one-position expansion restricts to each position. For $i < m$ uniform local exponent bounds add to a full kernel bound in each case.*

2. *For $5 \leq p \leq X''$ with even $n_P \geq 2$, of signal class or inert class with $h_P \neq 0$, each kind-BC representative r has exponent at least $g_{0,r,j} + F_{\mathrm{Orb},j}(p,r) - \mathrm{err}_{r,j}$. This holds for every unit ξ, sign, phase satisfying the four letter conditions and kernel with the corresponding local expansion.*

3. *At a level-free place above d, every class pair has entry exponent at least $-\frac{1}{2}\sum_t c_{u,j,t}$, using the contraction counts of 15.12.*

4. *At inert-class primes $5 \leq p \leq X''$ with $h_P = 0$, one has $F_{\mathrm{inert}} - C^*_{j,p} \leq B_C$, where*

$$B_C = \rho(p)\sum_z |\nu_{u,i,z}| - CfN_iN_{i+1}(\log_p(a_*^2)+1).$$

Every kernel with the one-position expansion has exponent at least B_C.

5. *At nonzero odd-order inert primes $5 \leq p \leq X''$, each kernel with (16.16) has exponent at least*

$$L_{\mathrm{odd}} = \frac{1}{n_P}\sum_{z\in BC} F_{\mathrm{Orb},j}(p,z) - \left(\frac{k-1}{p^3} + \frac{8}{p}\right)\sum_{z\in BC} w_j(z) - 3fN_m^2 \log_p a_*.$$

The large-error condition of 15.12 gives $F_{\mathrm{inert}} - I^*_{j,p} \leq L_{\mathrm{odd}}$.

Outline. Factor evaluations by 10.18 and apply cyclic bounds with 14.3 and 15.5, or the one-letter bound at zero shift. Use 15.12 and the integral residual series at the discriminant, and at odd order raise weights from 15.3 and apply 15.4. The complete proof is on p. 584. □

Lemma 16.13 (Adding step bounds along a chain)**.** *In either chain let $K^{[r]}$ be the first r lifts and V_i, Λ_i, ι_i its rational data. If $n \leq m$ and step $i < n$ has bound $b_i \in \mathbf{R}$ on every vector of Λ_i, then for $x \in (\Lambda_0)_{(w)}$ and $0 \leq r \leq n$ there is $y_r \in V_r$ with*

$$\iota_r(y_r) = K^{[r]}(\iota_0 x), \qquad y_r \in (\Lambda_r)_w[\textstyle\sum_{i<r} b_i]. \tag{16.20}$$

Proof. The vectors of a given bound form an $\mathcal{O}_{K,(w)}$-module by factoring the scalar of least valuation in a sum. Thus step bounds extend to localized inputs. Start at $y_0 = x$. Given $y_r = az$, apply the next bound to z and multiply its representative by a. Linearity gives the correct embedded output and the scalar valuations add. □

Lemma 16.14 (Four consequences of the orbit estimates)**.** *Assume the admissibility conditions of Definition 16.11 and the conclusions of 16.12. In (i)–(iii) assume the base-term identity of 15.10 and the error hypotheses of 15.12, and take a used μ with $p \notin Q'(\mu)$ and $x \in S^{\mathrm{ram}}_{u,w}$.*

(i) For $p \neq T_u$, $p \notin \mathcal{E}$, $p \leq X''$, and p signal or inert-class with $h_P \neq 0$, one has

$$x \in \mathcal{C}_{u,\mu,w}\left(B^0_{u,w}(\sigma) + \sum_j g_{0,S}(p,j)\right). \tag{16.21}$$

(ii) For $p \neq T_u$, $p \notin \mathcal{E}$, $p \leq X''$, and p inert-class with $h_P = 0$, the column exponent is at least

$$\sum_j (F_j(w) - C^*_{j,p} - \delta_{u,w}) + \sigma\delta_{u,w}. \tag{16.22}$$

(iii) For $p \neq T_u$ in the discriminant class, the column exponent is at least

$$\sum_j (F_j(w) - D^*_j - \delta_{u,w}) + \sigma\delta_{u,w}. \tag{16.23}$$

(iv) Under the denominator condition, at $p \notin \{T_u, d, g_0, g_1\}$ each integral input at $0 \leq i < m$ has a lift bound of

$$-\big(1 - \sigma\mathbf{1}_{\{i=0\}}\big)\delta_{u,w}. \tag{16.24}$$

This holds for the reference chain and each used nonreference twist outside $Q'(\mu)$.

Outline. Use the kind-A bound and 16.12(1)–(2) at signal and even-order inert primes, and (5), (4), (3) at odd-order inert, zero-shift inert, and discriminant primes with errors from 15.12. Lift by (16.15) and sum by Lemma 16.13. The complete proof is on p. 586. □

Lemma 16.15 (Adding the bounds at the remaining primes)**.** *For the primes and twists of Lemma 16.14(iv), and $x \in S^{\mathrm{ram}}_{u,w}$, one has*

$$x \in \mathcal{C}_{u,\mu,w}(-(m-\sigma)\delta_{u,w}). \tag{16.25}$$

Proof. Start the twisted column at $e^{\mu}_0 x$ and apply Lemma 16.13 to sum (16.24), obtaining (16.25). □

Lemma 16.16 (The base column bound at an ordinary place)**.** *Assume the base and error identities and 16.14–16.15. At $p \notin \{T_u, g_0, g_1\}$, for any used μ with $p \notin Q'(\mu)$ and $x \in S^{\mathrm{ram}}_{u,w}$,*

$$x \in \mathcal{C}_{u,\mu,w}(B^0_{u,w}(\sigma)). \tag{16.26}$$

Proof. At the discriminant prime the error is $\delta_{u,w} + D^*_j$, so use (16.23). Inert-class and signal primes are coprime to the conductor. For these primes up to X'', apply (16.22) when the inert shift vanishes and (16.21) otherwise. At remaining places, $p \neq d$, and 15.10–15.12 give $F_j(w) = 0$, $E^{\mathrm{ev}}_{u,j}(w) = \delta_{u,w}$. Apply (16.25). □

Definition 16.17 (Arithmetic chart conditions at positive levels)**.** Fix a word, a twist and a permitted place w over p. Put $A_w = \mathcal{O}_{K,(w)}$ and write $\jmath : K \hookrightarrow \mathcal{K}$ for the coefficient embedding. At each stage $1 \leq j \leq m$, let V^{rat}_j be the twisted rational form space, $\Lambda^{\mathrm{rat}}_{j,(w)}$ its localized lattice and L^{loc}_j the lattice of integral local forms.

In each chart require the following comparison data. The Hermitian form is nondegenerate and has identity Gram matrix in the chosen frame. The local splittings above p exchange their two factors under conjugation. The component representatives are unitary and have identity component at the specified places above p. Every finite adelic unitary point has a presentation $\gamma y_a k$ with γ rational unitary, y_a a component

representative and k unitary and in the prescribed level away from p. The exceptional places are away from p. A chart stabilizer acts through a rational unitary matrix whose conjugate by y_a belongs to the level away from p. In the chosen frame its local matrix gives the prescribed action on lattice tuples.

The position labels identify the weights and the embeddings on each block. At position s the embedding is the s-th Frobenius translate of the local embedding, with conjugation on the host-field argument for odd s. The local embeddings restrict on the real host field to the completion maps, and the local field identifications carry the specified valuation rings onto the completion integer rings. A local unitary matrix whose matrix and inverse are integral in the chosen frame belongs to the local level. The level character and the twisting character at class representatives have values in the complex image of K. For corresponding stabilizer and level elements, the local stabilizer character times the image under $\jmath$ of the level character is one.

Require w to belong to the domain of the chart's integral lattice test. Thus $x \in \Lambda^{\mathrm{rat}}_{j,(w)}$ exactly when, at every class and in every host-field basis of its dual lattice at p, the class value is represented by a weight polynomial in the prescribed A_w-lattice, with the frame and determinant twist of 16.9. The scalar normalizing this twist is the complex image of an A_w-unit. For each full tuple representing the class, its compared value belongs to the local Weyl fibre exactly when the normalized class value passes the integral test.

Finally, in each component a finite set of full lattice tuples meets every orbit under the level group, and

$$\dim_K V_j^{\mathrm{rat}} = \mathrm{rk}_{\mathcal{O}} L_j^{\mathrm{loc}} \qquad (1 \le j \le m).$$

Lemma 16.18 (Descent of a bound on the terminal vector)**.** *Let c be the input comparison, T the twisted forms and $L_C(\mu, u, w, h_g)$ levels and chains on the filling. For each used μ and permitted (u, w, h_g) with $p \notin Q'(\mu)$, suppose that at each stage j there is a map c_j from the twisted rational space to the local form space, semilinear for the coefficient embedding. Require $c_0(e_0^{\mu} x) = c(x)$ and $c_{j+1}(y) = K_j^{\mathrm{loc}} c_j(z)$ whenever y represents the complex lift of z at $j < m$. For every rational terminal vector y and real s, assume that a bound s on $c_m(y)$ in the local terminal lattice gives the same bound on y in the twisted rational lattice. Each twisted lift on a rational input has a rational representative. Assume the reference identification of the initial isomorphism at μ_0. For those indices and all $x \in V_{u,0}$ and $s \in \mathbf{R}$,*

$$c_{u,w,h_g}(x) \in \mathcal{L}_{\mu}(s) \implies x \in \mathcal{C}_{u,\mu,w}(s). \tag{16.27}$$

For the arithmetic chart comparisons, assume the positive-level chart conditions of Definition 16.17, rationality of the polynomial representations in the identified embedding blocks, and bases of the localized rational lattices. Then the terminal descent holds.

Outline. In the charts of Definition 16.17, use rational frames, flat base change, and coefficient-field retraction to identify the local lattice with the extended rational lattice. Descend the bound by coordinate valuations and transport it by the terminal isomorphism. The complete proof is on p. 586. □

Theorem 16.19 (The complete column estimates at ordinary places)**.** *Fix $\sigma \in \mathbf{R}$ and assume the terminal comparison (16.27) for W. The column theorem uses the two specializations $\sigma = 0$ and $\sigma = 1$. Under the error hypotheses of 15.12, assume the systems and comparisons of 16.7. Assume 16.14–16.15 with their respective hypotheses. For every used twist and permitted model, assume the terminal estimate of 13.4 for every matching admissible system and kernel. This assumption also covers $p \in Q'(\mu)$.*

At $p \notin \{T_u, g_0, g_1\}$ and for a used μ with $p \notin Q'(\mu)$, the filling and embedding identifications, the ramified-span equality and the base-term identity of 14.15(1)–(3), together with the error hypotheses of 15.12, give, for all $x \in V_{u,0}$,

$$x \in \mathcal{N}_{u,w} \implies x \in \mathcal{C}_{u,\mu,w}(B^{+}_{u,w}(\sigma)). \tag{16.28}$$

At the same places and twists, the base-term identity of 15.10 and the error hypotheses of 15.12 give, for $x \in S^{\mathrm{ram}}_{u,w}$,

$$x \in \mathcal{C}_{u,\mu,w}(B^{0}_{u,w}(\sigma)). \tag{16.29}$$

The served gain vanishes under any of

$$p \in \mathcal{E}, \quad p \text{ outside the inert and signal classes}, \quad (p \text{ inert-class and } h_P = 0), \quad p > X'', \quad p \in \mathcal{O}_{\mathrm{odd}}. \tag{16.30}$$

For every permitted model and used μ with $p \notin Q'(\mu)$, and each identification for Y_{μ} as in 14.15(1), the error hypotheses of 15.12 give the local bound of 16.8.

Outline. Use 16.16, 16.6, and 16.7–16.8 for ordinary, zero-gain, and local bounds. Transport ties and descend by (16.27), and handle the empty set by 14.15(3) or (16.21) according to served gain. The complete proof is on p. 587. □

16.4 Uniform constants and service for a family

Let $\mathfrak{F}_\varepsilon$ denote the parameter families of 2.2 at exponent ε. Every family $\mathcal{F} \in \mathfrak{F}_\varepsilon$ carries the built-in structural normalizations that all global constants K of Definition 14.2 are nonnegative and that $K_{\mathrm{Pr}} = 1$, $K_{\delta,1} = 5$, $K_{\delta,2} = 4$, $C_e = 1$, $c_6 = 12$, $c_{\mathrm{dig}} = 3$, and $C_r = 1$. Fix $\mathcal{F} \in \mathfrak{F}_\varepsilon$, with members $D_{x,i}, L_{x,i}$ for $i \in I_x$ and design and size hypotheses for $x > x_1$. For each member put $n = |A|$, $Y = X''(n)$, and write $\overline{F}_{A,j}(p) = n_P^{-1} \sum_{z \in Z_A} F_{\mathrm{Orb},j}(p, z)$ for the contribution from the kind-A positions Z_A of 15.12. The uniform stabilizer bound means that there are $C_D, x_D \in \mathbf{R}$ such that $\log M_u \le C_D f N_m \log(f N_m)$ for every $x \ge x_D$, $i \in I_x$ and word u.

Lemma 16.20 (Uniform size constants)**.** *Put $\theta = \varepsilon/100$, $z = w''(n)$, $h = t_{\mathrm{ch}} + 1$ and let Δ be the spacing parameter for the shifted weights. The size conditions give*

$$x^{2+8\varepsilon-\theta} \le n \le x^{2+8\varepsilon+\theta}, \qquad x^{9/2+10\varepsilon-\theta} \le k \le x^{9/2+10\varepsilon+\theta},$$
$$t_{\mathrm{ch}}\eta \le z/x, \qquad mh\eta \le z, \qquad kS_0 \le z, \qquad N_m m^{10} \le \Delta, \qquad 2\log\Delta \le x^{\varepsilon/4}.$$

Use also $z \le b'/x$ from the family scales.

Suppose $\mathcal{F}$ satisfies the uniform stabilizer bound. There are $C \ge 0$ and $x_0 \in \mathbf{R}$ such that (16.31)–(16.33) hold simultaneously for every $x \ge x_0$ with $x > x_1$ and every $i \in I_x$. For all $u, 1 \le j \le m$ the first two bounds are

$$\sum_{t \in A} c_{u,j,t} \le 2\#TT\, b + 2fkb', \tag{16.31}$$

$$\log M_u \le C f N_m \log(f N_m). \tag{16.32}$$

For $1 \le j \le m$ and every finite set S of inert-class or signal primes $p \le Y$, excluding zero-shift inert primes,

$$\sum_{p \in S} \log p\, (I^*_{j,p} - [\overline{F}_{A,j}(p)]_+) \le C f b \log Y. \tag{16.33}$$

Outline. Interlacing and weight windows bound the contractions of 15.12, and the stabilizer hypothesis bounds denominators. Apply Rosser–Schoenfeld to the harmonic error sum and use $p \equiv \pm 1 \pmod{4q_e}$ at odd Frobenius order to bound the cubic sum. The complete proof is on p. 587. □

A property P is closed on the family class $\mathscr{C}$ if

$$\begin{gathered}\exists \varepsilon_1 > 0 \quad \forall\, 0 < \varepsilon \le \varepsilon_1 \quad \exists \mathcal{K}_{\mathrm{size}} \quad \forall \mathcal{F} \in \mathfrak{F}_\varepsilon,\ C_H \in \mathbf{R}, \\ \mathcal{F} \in \mathscr{C} \ \wedge\ (\mathcal{F}, C_H) \text{ meets the thresholds } \mathcal{K}_{\mathrm{size}} \implies P(\mathcal{F}, C_H).\end{gathered} \tag{16.34}$$

The preliminary constants have a fixed lower threshold. The middle threshold may depend on the outer constants. The final threshold may depend on both the outer and the middle constants, with C_H included among the latter. Finite conjunctions use the smallest ε_1 and the componentwise largest thresholds. Let $P_{x,i}(C)$ mean that (16.31)–(16.33) hold with constant C, and that the enhanced and ordinary column estimates of 16.19 hold at $\sigma = 1$ under their respective hypotheses. Say that the family satisfies the uniform column and size bounds if

$$\exists C \ge 0\ \exists x_0 \in \mathbf{R}\ \forall x \in \mathbf{R}, \qquad (x \ge x_0 \wedge x > x_1) \implies \forall i \in I_x,\ P_{x,i}(C). \tag{16.35}$$

Theorem 16.21 (Uniform service for the family)**.** *Assume that the uniform stabilizer bound and each of the following two conditions are closed on $\mathscr{C}$ in the sense of (16.34).*

First, beyond a family threshold each member admits levels and chains $LC(\mu)$ on its filling. Form Y' by taking the levels and chain at μ_w for each w, and let W use the pair at μ for each twist as in 14.15. Require the terminal comparison (16.27). Under the error hypotheses of 15.12, for every used μ, identification for

Y_μ as in 14.15(1) and permitted (u, w, h_g) with $p \notin Q'(\mu)$, require the matching system and kernel of 16.7 and the complementary bounds of (16.10) at $\sigma = 1$.

Second, beyond a family threshold each member admits kernel data E. For every μ and every level-free (u, μ, w), the class-pair scalars have norm one at each $i < m$. For each used μ at such a place, the pair maps carry evaluated rational vectors to evaluated rational vectors. Each reference or used twisted lift sends the complex form space at i into that at $i+1$. In each of these chains every class at stage zero has trivial stabilizer. Choose a common system of letter data $\xi_w, \epsilon_{w,z}, \beta_{w,z}$ as above. Use the weights and E for the diagrams and series of $\mathscr{E}$. Require the expansion, jet integrality, discriminant and shape conditions of Definition 16.11(2), (4) and (16.17). Require the bound without a neighbour operator of item 3 for the specified data, for every inert-class w with $5 \le p \le X''$, $h_P \neq 0$, even $n_P \ge 2$, every $u, 0 \le i < m, t \in R_p$, and every kernel with the prescribed local expansion at $(0, t)$.

Then the uniform column and size bounds (16.35) are closed on $\mathscr{C}$. The conclusion also holds for a prescribed selection under the same closed stabilizer and twisted-chain conditions. In the second closed condition, require that for every family in $\mathscr{C}$, every x at or above its selection threshold with $x > x_1$, and every $i \in I_x$, the selected kernel satisfies the stated requirements.

Outline. Use 16.20 and splitting for 14.13–14.14, and 13.8 for 13.4 within 16.7–16.8. Use 16.10, 16.12, 16.14–16.15, and 16.19 for column bounds while preserving 14.15 and 16.2. The complete proof is on p. 588. □

The period estimate of 17.5 gives exponent $e_w(s - \delta_{u,w})$ from a column bound s. Theorems 17.25 and 17.26 apply it to (16.28) and (16.29), respectively.

Chapter 17

From columns to periods

Fix the arithmetic member D, its size data L and its twist exclusion data D_x as in 14.15 [3]. Write K for its coefficient field, $\iota : K \hookrightarrow \mathbf{C}$ for the chosen embedding and $\mathcal{O}$ for its ring of integers. For a finite place w, write $p_w, \mathfrak{p}_w, e_w$ for its residue characteristic, prime ideal and ramification index. Normalize v_w by $v_w(p_w) = e_w$, and put $\mathcal{O}_{(w)} = \mathcal{O}_{K,\mathfrak{p}_w}$. For a word u, let T_u be its level prime, Δ_u its stabilizer denominator and $\delta_{u,w} = v_{p_w}(\Delta_u)$. Here v_p on positive integers is the exponent of p. Let E be the host CM field with quadratic subfield $K_d = \mathbf{Q}(\sqrt{-d})$. The $f = [E : K_d]$ types τ are those of 15.12. The chain has m lifts and ranks $N_i = k + i(t+1)$, with t the odd chain parameter.

Fix the twisted forms T. The used tuples are the reference tuple μ_0 and the selected tuples μ_w, with $p_w \notin Q'(\mu_w)$, where $Q'(\mu)$ is the finite exclusion set attached to μ in 14.15. At the reference tuple this set is empty. Write $\Lambda^{\mu}_{u,j}$ for the lattice at stage j of the μ-chain and $\jmath^{\mu}_{u,j}$ for its complex embedding. Superscript 0 denotes the reference chain. Let V_u be the rational ramified input space of the word u, let $j_u : V_u \hookrightarrow V^0_{u,0}$ be its initial inclusion, and let N_u be its input lattice. Write $M_{H,u} = M^{\mathrm{ram}}_u = j_u^{-1}(\Lambda^0_{u,0})$ for the original ramified lattice. For $\phi \in V_u$, write $P^{\mu}_u(\phi, H)$ for the column period at μ tested against H, with complex input the image of ϕ under the ι-semilinear realization of V_u. Lemma 17.4 expresses this period in terms of j_u. An admissible test lies in $\widetilde{M}_\mu \cap V_0(\mu)$, where $\widetilde{M}_\mu$ is the test lattice and $V_0(\mu)$ is the image of the self-adjoint projector on the test space at μ given by the central idempotent of the type. As in 15.16 put

$$S_{u,w} = \mathrm{sg}(u, w), \qquad B_{u,w} = \sum_{j=1}^{m} (F_j(w) - E_{u,j}(w)).$$

For an $\mathcal{O}$-submodule Λ, write $\Lambda_{(w)} = \mathcal{O}_{(w)}\Lambda$.

17.1 Localizing the terminal pairing

Lemma 17.1 (A pairing bound survives localization)**.** *Let Λ be an $\mathcal{O}$-submodule of a K-space V, let $g : V \to \mathbf{C}$ be ι-semilinear, and fix $b \in \mathbf{R}$ and a finite place w. If every $x \in \Lambda$ has $g(x) = \iota(a_x)$ with $a_x \in K$ and $a_x = 0$ or $v_w(a_x) \geq b$, then the same holds on $\Lambda_{(w)}$.*

Proof. For $z = \sum_\nu c_\nu x_\nu$ with $c_\nu \in \mathcal{O}_{(w)}$, semilinearity gives $g(z) = \iota(\sum_\nu c_\nu a_{x_\nu})$. Each nonzero summand has valuation at least b. □

Let $\mathcal{T}^{\mu}_u(z, H)$ be the pairing obtained by evaluating and lowering $\jmath^{\mu}_{u,m} z$ and testing against H pulled back to the chain test space. The reference point hypothesis is item (1) of 33.77 for every word. Evaluation is injective on class labels and induces bijections on their stabilizers. Evaluation followed by lowering takes the terminal forms into the point space and has a rational realization carrying the terminal lattice into the point lattice. This realization satisfies (33.17) at every infinite place, and the complex map satisfies the squared-norm bound there stated. The twisted point hypothesis requires an integral rational realization of evaluation followed by lowering for each word and each used $\mu \neq \mu_0$, as in item (2) of 33.77. Use the common stabilizer denominator of 15.12: the denominator of every twisted chain equals Δ_{u_0} for the distinguished word u_0, and

every class stabilizer order at every stage of every such chain divides it. Under this condition the reference period hypothesis requires, for each word, point-lattice vector and admissible test, one representative $y \in K$ of the point period with $y = 0$ or $v_w(y) \geq -e_w\delta_{u,w}$ at every finite w. Under the same denominator condition the twisted period hypothesis requires one representative for each such vector and test at each used $\mu \neq \mu_0$, with the bound simultaneous over all w for which $p_w \notin Q'(\mu)$. These are items (3) and (4) of 33.77.

Lemma 17.2 (The terminal denominator)**.** *Assume the common stabilizer denominator and the reference and twisted point and period hypotheses above. Fix u, w and take $\mu = \mu_0$ or a used $\mu \neq \mu_0$ with $p_w \notin Q'(\mu)$. For $z \in (\Lambda^\mu_{u,m})_{(w)}$ and admissible H at μ, there is $a \in K$ with*

$$\iota(a) = \mathcal{T}^\mu_u(z, H), \qquad a = 0 \text{ or } v_w(a) \geq -e_w\delta_{u,w}.$$

Proof. For an integral terminal vector, the rational point map represents its evaluated and lowered vector. Apply the period hypothesis to its point-lattice image, then 17.1 in the terminal variable. □

Lemma 17.3 (Class denominators on the constructed columns)**.** *Assume the common stabilizer denominator of 15.12, with each point stabilizer injecting into the corresponding terminal stabilizer. Suppose that for every word u and used tuple μ, evaluation followed by lowering on the μ-column is given in local class coordinates at each finite place w by an integral interlacing map, multiplied on μ_0 under reference frame changes by inverse values of a terminal level character of finite order taking values in $\iota(K)$. Suppose that at each finite place w the local terminal lattice spans the terminal K-space, the point embedding into the complex point space is injective and characterizes vectors of the localized point lattice by w-integrality of their local class coordinates, and the point lattice is the intersection of its localizations across finite places. Suppose further that every admissible test H is an $\mathcal{O}$-linear combination of row generators whose rational row lifts have class polynomials in the w-local Weyl dual of the point lattice at every finite w for μ_0 and at every w with $p_w \notin Q'(\mu)$ for other used μ, and that the point period is the Petersson class sum weighted by the reciprocal point stabilizer orders. Then for every word and used tuple, the rational point map takes the terminal lattice into the point lattice. For a point-lattice vector x and admissible H, the point period has a representative $y = \sum_c b_c/h_c \in K$, where h_c is the point stabilizer order at class c. For μ_0 at every w, and for other used μ at every w with $p_w \notin Q'(\mu)$, all b_c and $\Delta_u y$ are integral at w.*

Outline. Integral interlacing and root-of-unity frame factors extend linearly from the terminal lattice and glue to the point lattice. Pairing with w-local Weyl-dual row lifts gives w-integral b_c, and $h_c \mid \Delta_u$ makes $\Delta_u y$ integral at w. The complete proof is on p. 589. □

A column has bound s at w in the sense of 16.1 if its terminal image is represented by az, with $z \in (\Lambda^\mu_{u,m})_{(w)}$ and $a \in K$ either zero or of valuation at least $e_w s$. For $\mu \neq \mu_0$, first transport the initial reference vector by the comparison of 16.1, then apply the twisted lifts.

Lemma 17.4 (The two evaluations of a period)**.** *Write $j : V_u \to V^0_{u,0}$ for the inclusion of the rational ramified subspace. Let E_0 be the identity on reference initial class functions and E_μ the initial class-function comparison for $\mu \neq \mu_0$. Write C^μ for the composite of the lifts on the corresponding chain. For a complex terminal class function Y, let $t^\mu(Y, H)$ be its evaluation followed by lowering and pairing with the pullback of H to that chain. Then*

$$P^\mu_u(\phi, H) = t^\mu\big(C^\mu E_\mu \jmath^0_{u,0}(j\phi), H\big).$$

If $Y = \jmath^\mu_{u,m}(z)$, the right terminal pairing is $\mathcal{T}^\mu_u(z, H)$.

Outline. Apply the complex input identification to recover $\jmath^0_{u,0}(j\phi)$, then substitute into the reference and twisted definitions. On an embedded rational terminal vector this is the pairing of 17.2. The complete proof is on p. 589. □

Theorem 17.5 (From a column bound to a period bound)**.** *Assume the hypotheses of 17.2. Fix a word u, a finite place w, and $\phi \in V_u$. Take $s \in \mathbf{R}$ and a used μ with $p_w \notin Q'(\mu)$. If the μ-column from the initial vector $j\phi$ of 17.4 has bound s, then for every admissible H at μ there is $y \in K$ with*

$$\iota(y) = P^\mu_u(\phi, H), \qquad y = 0 \text{ or } v_w(y) \geq e_w(s - \delta_{u,w}).$$

Outline. Write the column image as $\jmath^\mu_{u,m}(az)$ with $v_w(a) \geq e_w s$ by 16.1 and 17.4, and apply 17.2 to z, H for $y_0 \in K$ with $v_w(y_0) \geq -e_w\delta_{u,w}$. Linearity in 17.4 gives $y = ay_0$, and adding valuations yields the bound. The complete proof is on p. 589. □

17.2 The level prime

Lemma 17.6 (The budget above T_u)**.** *Suppose the level primes belong to the split class and the base terms and losses have the values in 15.10 and 15.12. If $p_w = T_u$, then the local size parameter $\tau(w)$ of 14.9 vanishes and, for $1 \le j \le m$,*

$$g_j^{\mathrm{raw}}(w) = F_j(w) = 0, \qquad E_{u,j}(w) = T^*_{u,w}, \qquad S_{u,w} = -mT^*_{u,w},$$

where $g_j^{\mathrm{raw}}(w)$ is the gain before screening in 15.14 and $T^*_{u,w}$ is the level-prime loss defined in 15.12.

Proof. The level prime T_u belongs to the split class and is distinct from both fixed auxiliary primes. Outside the inert and signal classes the size data give $\tau(w) = g_j^{\mathrm{raw}}(w) = 0$ by 15.16. The split class and the fixed-prime exclusion give $F_j(w) = 0$ by 15.10. Apply 15.12 to the loss and 15.14 to the screened gain. □

Lemma 17.7 (The level prime exceeds the stabilizer cutoff)**.** *Write t for the odd chain parameter and $N_j = k + j(t+1)$ for the rank after j lifts. The rank conditions on a member are*

$$2 \le m, \qquad m+3 \le k, \qquad f \le k, \qquad k+m-1+5 \le t, \qquad 3m < t.$$

Its size condition at $x > x_1$ includes

$$(Q_S q d)^{36} t^{150} \le \sqrt{b}.$$

If the level prime of a word u satisfies $b^{m/2} \le T_u$, then $2fN_m + 1 < T_u$.

Proof. The rank conditions give $t \ge 7$, $m \le t$ and $f \le k \le t$. Thus

$$N_m \le t^2 + 2t, \qquad 2fN_m + 1 \le 2t^3 + 4t^2 + 1 \le 7t^3 < t^5.$$

The host conductor factor Q_S is positive. Indeed, the level prime is coprime to Q_S, whereas its greatest common divisor with zero would be $T_u > 1$. Since q, d are primes, $Q_S q d \ge 1$. The displayed size condition now gives $t^{300} \le b$. In particular $b \ge 1$, so $t^5 \le b \le b^{m/2} \le T_u$. □

Definition 17.8 (The design class)**.** Use the arithmetic families of 2.2, with size and column data as in 16.21. For a family class $\mathcal{C}$ and $C_0 \in \mathbf{R}$, write $\mathcal{D}(\mathcal{C}, C_0)$ for the subclass with the following design conditions. All point cutoffs are uniform in the member index.

The family has the good twists used in the design construction of Chapter 33 and satisfies its choice conditions with C_0, some $C_H \in \mathbf{R}$ and positive constants C_5, c_6, C_b. These are the lower bound on the thick set, the bounds on the number of top positions and on bad orbits, the estimates for the three quadratic characters and the thin character sums, and the two bounds on word sizes and word count. In the latter two bounds the required inequalities are

$$C_0^{-1} \le \frac{d_u \Gamma^2}{D_*} \le C_0 \quad \text{for every word } u, \qquad C_0 \Gamma^2 x^{C_H} \le \#\{\text{words}\},$$

where d_u is the dimension attached to the word, D_* the reference dimension scale and $\Gamma = \exp(C_\Gamma m^5 f L_0)$, with L_0 the logarithmic scale of the member.

Beyond a family cutoff, for every member and every word u, each prime divisor of Δ_u is at most $2fN_m+1$, and

$$b^{m/2} \le T_u, \qquad \log T_u \le C_T m L_0, \qquad \log \Delta_u \le C f N_m \log(f N_m).$$

Here C_T is the family's level-size constant, and one $C \in \mathbf{R}$ works for the family.

For a class $\mathcal{A}$, write $\mathcal{A}_{\mathrm{ch}}$ for the subclass used in the construction before 33.1. It requires $C_{YL} \le 9$, the zero-free boxes and quadratic-prime conditions of 37.4, tame distinctness at both fixed primes, and eventual divisibility of every Δ_u by the following convention:

$$\Delta_u \mid M(2fN_m), \qquad M(a) = \prod_{\substack{p \le a+1 \\ p \text{ prime}}} p^{\sum_{j \ge 0} \lfloor a/(p^j(p-1)) \rfloor}.$$

At a fixed prime g, tame distinctness requires the tame parameters to be squares and the list consisting of 1, the parameters and their inverses to have distinct entries. Their prime-to-ℓ parts must also remain distinct for each prime $\ell \ne g$ outside the exceptional set $\{\ell : \exists\, 1 \le d \le r,\ g^d < \ell^{4v_\ell(g^d-1)}\}$, where r is the rank parameter. The tame, box and quadratic-prime conditions hold eventually at every member.

Lemma 17.9 (Eventual coprimality of the level prime)**.** *Fix a family class $\mathcal{C}$ and $C_0, \varepsilon \in \mathbf{R}$. For every arithmetic family $\mathfrak{F}$ of parameter ε in the design class $\mathcal{D}(\mathcal{C}, C_0)$ of Definition 17.8, there is $x_0 \in \mathbf{R}$ such that $T_u \nmid \Delta_u$ for every word of every member $D_{x,i}$ with $x \geq x_0$, $x > x_1$ and $i \in I_x$, where x_1 is the family's initial threshold and I_x its index set at x. This eventual assertion is closed on $\mathcal{D}(\mathcal{C}, C_0)$ in the sense of (16.34).*

Outline. Taking $x_0 = \max(x_s, x_T)$ from the uniform stabilizer and level cutoffs lets 17.7 give $T_u > 2fN_m + 1$, excluding $T_u \mid \Delta_u$ by primality of T_u. Setting the ε-cutoff to 1 and size thresholds to 0 in 16.21 gives closedness. The complete proof is on p. 590. □

Proposition 17.10 (One input lattice for a word)**.** *Fix a word u of an arithmetic member. Write $j = j_u$ and let $V^r = j(V_u) \subseteq V^0 = V^0_{u,0}$. Put $L^0 = \Lambda^0_{u,0}$ and $S^{\mathrm{ram}}_{u,w} = (L^0)_{(w)} \cap V^r$. Define $\mathcal{O}_{(w)}$-submodules $B_w \subseteq V_u$ as follows.*

At a permitted place with $W_{p_w} \neq \varnothing$, take $B_w = j^{-1}J_w$ using 16.3. Equivalently $x \in B_w$ if $c(jx) \in N_P$ for the comparison map and enhanced lattice of that model. The permitted prime conditions in 16.2 certify the model at the given u, w.

At $p_w = T_u$, let $I^{\mathrm{top}}_{u,w}$ consist of the level indices v with top type $\tau_w(v)$ when w is seen in the sense of 15.12. Take $I^{\mathrm{top}}_{u,w} = \varnothing$ otherwise. The upward operator U_v on zeroth columns sums the values at the upper neighbors indexed by residue vectors of marking one, with the values pulled back by the frame maps. Put

$$U = \prod_{v \in I^{\mathrm{top}}_{u,w}} U_v, \qquad b_v = -\lambda^u_{0,\tau_w(v),k-1}, \qquad a_0 = m \sum_{v \in I^{\mathrm{top}}_{u,w}} b_v,$$

where the product is in the level order and the weight coordinates start at zero. Let $N_{\lambda,u,w}$ consist of rational $y \in V^0$ for which

$$j^0_{u,0}(y) = T_u^{a_0} U^m j^0_{u,0}(z) \quad \text{for some } z \in S^{\mathrm{ram}}_{u,w}.$$

At these places put $B_w = j^{-1}N_{\lambda,u,w}$. At the remaining places put $B_w = (M_{H,u})_{(w)}$. A permitted prime does not divide T_u, so the first two prescriptions have disjoint domains.

The intersection $N_u = \bigcap_{w<\infty} B_w$ is a full $\mathcal{O}$-lattice in V_u and $(N_u)_{(w)} = B_w$ at every finite place. Away from the level prime one has

$$(N_u)_{(w)} \subseteq (M_{H,u})_{(w)} \qquad (p_w \neq T_u).$$

There is equality when $p_w \neq T_u$ and $W_{p_w} = \varnothing$.

Outline. Use 16.3 at selected places and invert the upward operators by 21.9 at level places to get commensurable lattices. The exceptions are finite by 14.8, so glue by 18.4. The complete proof is on p. 590. □

Remark 17.11. The normalized local colength is

$$\ell_{u,w} = \frac{1}{e_w} \operatorname{length}_{\mathcal{O}_{(w)}} \left(\left(M_{H,u} / (N_u \cap M_{H,u}) \right)_{(w)} \right).$$

Some integer $n_0 > 0$ (from the proof of 17.10) kills the quotient. Away from T_u, the denominator localizes to $(N_u)_{(w)}$; at T_u it is the displayed intersection.

Theorem 17.12 (Periods above the level prime)**.** *Assume the hypotheses of 17.2, the level-prime identities of 17.6 and the description $(N_u)_{(w)} = j_u^{-1}N_{\lambda,u,w}$ of 17.10 at every $w \mid T_u$. Suppose $T_u \nmid \Delta_u$ for every word. Assume both sharp column estimates of Proposition 17.20 below: scaling the reference column by $\iota(c)$ makes it integral for $v_w(c) \geq e_w m T^*_{u,w}$, and each used nonreference column outside its exclusion set has bound $-mT^*_{u,w}$, in both cases on $N_{\lambda,u,w}$. For u, w with $p_w = T_u$, $\phi \in N_u$ and admissible H at μ_w, there is $y \in K$ with*

$$\iota(y) = P^{\mu_w}_u(\phi, H), \qquad y = 0 \text{ or } v_w(y) \geq e_w S_{u,w}.$$

Outline. Using $n_0 = mT^*_{u,w} \in \mathbb{N}$ from 15.12, apply 17.20 (with $c = T_u^{n_0}$ at μ_0) to get column bound $-n_0$ on $j_u\phi$. Apply 17.5 with $\delta_{u,w} = 0$, and use 17.6 over the m lifts to identify $S_{u,w} = -n_0$. The complete proof is on p. 591. □

Lemma 17.13 (Binomial moments and simultaneous approximation)**.** *Let p be prime, $r \ge 0$, and $b_\nu(c) = \prod_{j=1}^{r} \binom{c_j}{\nu_j}$ for $\nu \in \mathbf{N}^r$ and $c \in \mathbf{Z}_p^r$. Given $A_\nu \in \mathcal{O}_{\mathbf{C}_p}$, there is a unique continuous $\mathbf{C}_p$-linear functional μ on $C(\mathbf{Z}_p^r, \mathbf{C}_p)$ with $\mu(b_\nu) = A_\nu$. Its norm is at most one. For any finite family g_i of continuous functions and any $\eta > 0$, there are finitely many $c \in \mathbf{N}^r$ and coefficients $d_c \in \mathcal{O}_{\mathbf{C}_p}$, common to the family, with*

$$\left| \mu(g_i) - \sum_c d_c g_i(c) \right|_p < \eta \qquad \text{for every } i.$$

Outline. Finite differences give the binomial basis norm identity, and Mahler interpolation gives density, so the moment functional extends with norm at most one. Approximate all tests by step functions on one residue partition. The complete proof is on p. 591. □

Lemma 17.14 (Standard isometry and the two determinant factors)**.** *Use the host E/K_d of 5.1, the CM type Σ, and the normalized curve lattice $\Lambda = \delta\mathfrak{c}$ of 21.22. Put $f = [E : K_d]$. In a coordinate product put*

$$H(z,w) = \frac{\sum_i z_i \overline{w}_i}{\operatorname{area}(\Lambda)}, \quad B(z,w) = \operatorname{Im} H(z,w), \quad \alpha(z) = \exp\left(\frac{\pi i \sqrt{d}}{2} H(z,z) \right).$$

Define $U_y F(z) = F(z+y)\exp(-\pi H(z,y) - \pi H(y,y)/2)$ and $W_y = \alpha(y) U_y$. For commensurable self-dual period lattices P, Q with their indicated semicharacters, write

$$I_{P,Q} F = [Q : P \cap Q]^{-1/2} \sum_{y \in Q/(P \cap Q)} W_y F$$

on P-fixed sections. The square root is positive real. Let $P_1 = \{(\delta\tau(x))_{\tau\in\Sigma} : x \in \mathfrak{c}\mathcal{O}_E\mathfrak{b}^{-1}\}$, where $\mathfrak{b}\bar{\mathfrak{b}} = \mathfrak{D}_{E/K_d}$. Let Θ be the one-entry theta section with the fourth-power normalization described below. For every odd prime p split in K_d and prime to $N\mathfrak{c}$, and every field isomorphism $\jmath : \mathbf{C} \simeq \mathbf{C}_p$, there is an isometry g and a scalar $e \in \mathbf{C}$ such that $Q = g(P_1^4)$ agrees with $P = \Lambda^{4f}$ locally at p and

$$\Theta^{\otimes 4}(z) = e\,(I_{P,Q}\theta_\Lambda^{\otimes 4f})(gz), \qquad |\jmath(e)|_p = 1.$$

For even ranks a, b, put $F_0 = \Theta^{\otimes ab}$ on the standard matrix lattice. Under a unitary source change γ, its pullback equals its normalized lattice average times $(\prod_\tau \tau(\overline{\det\gamma}))^{b/2}$. Under a unitary target change γ', the factor is $(\prod_\tau \tau(\det\gamma'))^{a/2}$. The pullback in both variables is $F(z) \mapsto F(\gamma'^* z \gamma)$, with matrices read at the types.

Outline. Use four squares and Cayley approximation to match the lattices at p. Gaussian integration and finite Gauss sums compare pullbacks with theta averages, giving a p-adic unit normalization and, by repetition across rows or columns, the endpoint determinant factors. The complete proof is on p. 591. □

Lemma 17.15 (Integral moments of the matrix theta vector at a split prime)**.** *Let K_d and the selected CM type Σ of E be those of the host of 5.1, and fix a field isomorphism $\jmath : \mathbf{C} \simeq \mathbf{C}_p$. Suppose $p \neq 2$ splits in K_d. Let $\mathfrak{c}$ be the nonzero curve ideal, of norm prime to $2dp$, and $\Lambda = \delta\mathfrak{c} \subset \mathbf{C}$ its period lattice with the discriminant normalization of 21.22. Write $\mathfrak{b}\bar{\mathfrak{b}} = \mathfrak{D}_{E/K_d}$ for the host factorization of the relative different. For self-dual lattices L, L' in the standard Hermitian spaces E^a, E^b with a, b even, put*

$$M(L,L') = (\mathfrak{c}\mathcal{O}_E)\mathfrak{b}^{-1}\operatorname{Hom}_{\mathcal{O}_E}(L,L'), \qquad \Lambda(L,L') = \{(\delta\tau(x))_{\tau\in\Sigma} : x \in M(L,L')\}.$$

Let $\omega = \Omega_0/\jmath(\delta_0)$, where δ_0 is the discriminant root of the principal curve lattice and Ω_0 its normalized split period. For a holomorphic function F and a list $v = (v_1, \dots, v_r)$ of period directions, put

$$A_\nu(F; v) = \jmath\left(\left[\prod_{j=1}^{r} \binom{D_{\jmath^{-1}(\omega)v_j}}{\nu_j} \right] F(0) \right), \qquad \binom{D}{n} = \frac{D(D-1)\cdots(D-n+1)}{n!}.$$

Here D_v denotes directional differentiation and $\binom{D}{0} = 1$. Suppose $|A_\nu|_p \le 1$ for the normalized even theta of Λ and its prime-to-p torsion translates, for all lists in Λ and all $\nu \in \mathbf{N}^r$. Then the same bound holds for lists

in $\Lambda(L,L')$ and prime-to-p torsion translates of the character-normalized matrix theta $e_{L,L'}$ constructed in the proof (with $P(L,L')=\Lambda(L,L')$ and a multiplicative character χ on norm-one ideals satisfying $\chi((z))=\prod_{\tau\in\Sigma}\tau(z)^{-1}$ for $z\bar z=1$), which satisfies, for rational unitary γ,γ' with $P_{\gamma,\gamma'}F(z)=F(\gamma'^{*}z\gamma)$ and self-dual lattices L_1,L'' of ranks a,b,

$$P_{\gamma,\gamma'}e_{L,L'}=e_{\gamma L,\gamma' L'}, \tag{17.1}$$

$$e_{L,L''}=\chi(\det(L',L''))^{-a/2}I_{P(L,L'),P(L,L'')}e_{L,L'}, \tag{17.2}$$

$$e_{L_1,L'}=\chi(\det(L,L_1))^{b/2}I_{P(L,L'),P(L_1,L')}e_{L,L'}. \tag{17.3}$$

For each list the derivatives in the scaled directions are power moments of a $\mathbf{C}_p$-valued measure of norm at most one on $\mathbf{Z}_p^r$.

Outline. Apply 17.14 to the fourth tensor power to get integrality for the standard matrix theta vector from the elliptic case. Prove lattice-move equivariance via 17.14 and 5.1, align L,L' above p by unitary maps, and get the measure from 17.13. The complete proof is on p. 593. □

Lemma 17.16 (Translation of the normalized elliptic theta section)**.** *Let d be the host prime, $d\equiv 7 \bmod 16$, and let the odd prime p split in K_d. On the principal curve with lattice $\mathcal{O}_{K_d}$, let θ be the even theta section with $\theta(0)=1$ and zero at the nonzero half-period $1/2$. Use U,W,H of 17.14 for this lattice. For a complex analytic germ F, write $\widehat{F}$ for its Taylor expansion transported by $\jmath:\mathbf{C}\simeq\mathbf{C}_p$ and composed with $\omega\log(1+Z)$, in the normalized ordinary multiplicative chart. Write $S_\zeta f(Z)=f(\zeta(1+Z)-1)$ on bounded series when $|\zeta-1|_p<1$.*

Suppose $s\ge 1$, $p\nmid s$, $st\in\mathcal{O}_{K_d}$, $n\ge 0$, and

$$y=sv/p^n,\qquad v\in\mathcal{O}_{K_d},\qquad |\jmath(\bar v)|_p\le p^{-2n}.$$

Put $\zeta_y=\jmath(\exp(-2\pi i\ \mathrm{Im}\,H(1,y)))$. Then

$$\zeta_y^{p^n}=1,\qquad \widehat{W_yU_t\theta}=S_{\zeta_y}\widehat{U_t\theta}.$$

Outline. Normalize the multiplicative chart by the period pairing. The elliptic theta quotient gives integral coefficients and a functional equation forcing the translation ratio to be one, and elliptic translation extends the identity to $U_t\theta$. The complete proof is on p. 595. □

Lemma 17.17 (One measure for all allowed matrix translations)**.** *Let $M=M(N,N')$ and $e=e_{N,N'}$ be as in 17.15 for the constructed family. Fix a prime-to-p translate t, and choose $N_0>0$ prime to p with N_0t a period. Use the embedding of rational matrices into the analytic period space in 17.15. Let $(\beta_j)_{j=1}^r$ in M be a K_d-basis that generates the period module locally at both primes over p. Put $x(c)=\sum_j c_j\beta_j$ for $c\in\mathbf{N}^r$. Write*

$$[y,x]=\exp(-2\pi iB(x,y)),\qquad M_x(\nu)=\prod_{\tau,k,l}\bigl(u_\tau\iota_w(\tau(x^\dagger)_{kl})\bigr)^{\nu_{\tau kl}},$$

where $x^\dagger$ is the Hermitian adjoint. Choose $\jmath:\mathbf{C}\simeq\mathbf{C}_p$ so that $\iota_w=\jmath\circ\iota$ induces w, with ι the coefficient embedding fixed at the start of the chapter. Read all type coordinates, differential generators and periods through these compatible embeddings. Then u_τ is the image of the differential generator at type τ divided by the ideal-curve period in $\mathbf{C}_p$. Write ∂_c^ν for the coordinate derivative with the differential generators inserted. If v is the quotient of the conjugate level factors with valuation -1 in the chosen split component and $+1$ in the other, then there is a single measure μ_t of norm at most one such that for every $y\in N_0vM$ and every jet index ν,

$$\jmath\bigl(\partial_c^\nu(W_yW_te)(0)\bigr)=\jmath(\alpha(t))\int_{\mathbf{Z}_p^r}\jmath([y,x(c)])M_{x(c)}(\nu)\,d\mu_t(c). \tag{17.4}$$

The integrand denotes the continuous extension from $\mathbf{N}^r$. The same statement holds for a self-dual companion pair with its own fixed t.

Outline. Show prime-to-p shifts preserve the Gauss norm via the special-fibre valuation and 17.16, so Lemma 17.15 applies to U_te. Expand e in an orthogonal frame, push forward the measure of Lemma 17.13, and transport W_y using 17.16. The complete proof is on p. 596. □

Lemma 17.18 (The two finite quotients in the split-level moment calculation)**.** *Let $p = \varpi\bar{\varpi}$ be a split level prime as in Lemma 17.17. Write $M = M(N, N')$ for its period module. At every prime $P \mid \varpi$ let the target have surjective residue markings. Replace the target lattice there by the kernels of the markings, at their conjugates by the Hermitian duals of these kernels, and keep all other completions. Denote the resulting period module by M' and put $I = M \cap M'$. Let B be the half-period module obtained by multiplying M by pg at the primes over ϖ and the row primes, by $(pg)^{-1}$ at their conjugates, and by 1 elsewhere. Here g is supported on the rational row primes, which differ from p. Let c_A be supported on the same row primes, and let $F = fa$, where a is the source rank and f is the number of primes over ϖ. The residue fields there have order p.*

Put $H_S = \mathrm{Hom}_S(N, N')$, $Z_0 = pgH_S$, $r = c_A g\bar{\varpi}$. The curve and different factors in M are units at the test primes. Then

$$[M : I] = [M' : I] = p^{\ell}, \quad 0 \le \ell \le F, \qquad |rI/(rI \cap Z_0)| = [I : \varpi M].$$

If $n \mid [M : M \cap B]$ and $J = rI/(rI \cap Z_0)$, then $v_p(n) \le \ell + v_p(|J|)$. In $B/(B \cap M)$ the annihilator of rI under $e^{2\pi i B_H(-,-)}$, where $B_H = \mathrm{Im}\, H$, has prime-to-p part the image of pB and p-part the image of $M' \cap B$. The latter has order p^{ℓ}.

Outline. Use the local marking equations, the preimage of Z_0, and the chain of lattice indices to obtain the index and valuation bounds. Duality identifies the p-part of the annihilator, and integrality identifies its prime-to-p part. The complete proof is on p. 598. □

Lemma 17.19 (The point kernel at the two halves of the split level)**.** *Use the constructed theta family with rational jets and unramified level characters. Fix a used tuple μ, a word u and a stage $0 \le i < m$. Write $p = T_u = \varpi\bar{\varpi}$, $a = N_i$, $b = N_{i+1}$ and $s = (b-a)/2 = (t+1)/2$ in this lemma. Fix decorated points d, d' with self-dual lattices N, N' of ranks a, b, markings l_P, l'_P at $P \mid \varpi$ and adapted frames at the selected row primes S_{row}. Write $\mathrm{Hom}_S(N, N')$ for rational E-linear maps integral on the local lattices at the primes in S. Put*

$$S = S_{\mathrm{row}} \cup \{P : P \mid \varpi\}, \quad q_0 = \prod_{P \in S_{\mathrm{row}}} |k(P)|, \quad c_A = q_0^{A_{\mathrm{row}}}, \quad g = q_0^{B_{\mathrm{row}}}, \quad c_B = pg,$$

where $k(P)$ is the residue field and $A_{\mathrm{row}}, B_{\mathrm{row}}$ are the support and period exponents of the row selection. The row test $\Xi_0 : \mathrm{Hom}_E(E^a, E^b) \to K$ has support in $c_A^{-1}\,\mathrm{Hom}_{S_{\mathrm{row}}}(N, N')$, is integral at $w \mid p$, and is invariant under $g\,\mathrm{Hom}_{S_{\mathrm{row}}}(N, N')$. The set S is disjoint from its conjugate. By 37.21 there are f primes over ϖ, each of residue field $\mathbf{F}_p$, and p is a unit at the row primes.

For the decorated source d, let $L_d \subseteq N$ be the sublattice cut out by $l_P = 0$ at every $P \mid \varpi$, and write $L_d^{\vee}$ for its Hermitian dual. Define $L_{d'} \subseteq N'$ using the target markings l'_P, and write $L_{d'}^{\vee}$ for its dual. Let $S_w(M)$ denote the local Weyl value lattice on M with the weight at its stage. The ordinary source condition uses $S_w(L_d^{\vee})$, which is the condition on N at a seen place. Inserting all target level indices at such a place gives $S_w(L_{d'})$. For the point kernel $k(d, d')$ with row test Ξ_0, assume that the decorated source and target lattices are standard at every $P \mid \varpi$ in the fixed level coordinates. Then

$$\begin{aligned} p^{fa - \sum_\tau h'_\tau} k(d, d') S_w(L_d^{\vee}) &\subseteq S_w(L_{d'}) \qquad (w \text{ seen}), \\ k(d, d') S_w(L_d^{\vee}) &\subseteq p^{fs} S_w(L_{d'}^{\vee}) \quad (w \text{ unseen}), \end{aligned}$$

where h'_τ is the last coordinate of the target point weight λ'_τ. The size hypothesis is $2\,\mathrm{spread}(\lambda'_\tau) < p$ at every type, where spread is the largest minus the smallest coordinate plus b. With data away from P fixed, the upper-neighbor relation $e \uparrow_P e'$ at $P \mid \varpi$ gives

$$\sum_{d' \uparrow_P e'} k(d, e') = p^s \sum_{e \uparrow_P d} k(e, d').$$

Outline. Bounds coset theta jet tables via 17.15, 17.13, 17.17, and 17.18, and passes to Weyl lattices via Jantzen's sum formula. Unfolding neighbor sums onto a common lattice using 17.15 and 17.16 proves the neighbor identity. The complete proof is on p. 599. □

Proposition 17.20 (The sharp column exponent at the level)**.** *On the column construction of* (33.1) *and Definition 34.46 (using reciprocal-stabilizer class sums, rational jets and size conditions of 17.19,* $T_u > 2fN_m + 1$*, weight signs* $b_v \geq 0$*, and level-zero twist-lattice compatibility), fix* u, w *with* $p_w = T_u$*. For* $x \in N_{\lambda,u,w}$ *and* $c \in K^\times$ *with* $v_w(c) \geq e_w m T^*_{u,w}$*, the reference column of* x *scaled by* $\iota(c)$ *lies in its terminal local lattice image. For used* $\mu \neq \mu_0$ *with* $p_w \notin Q'(\mu)$*, the twisted column of* x *has bound* $-mT^*_{u,w}$*.*

Outline. Use 17.19 with unit stabilizer orders to commute upward sums through lifts and iterate to $mT^*_{u,w}$ by 15.12. Transport the input of 17.10 through the level-zero identification and read the twisted bound by 16.1. The complete proof is on p. 604. □

17.3 The two fixed primes

Fix local error data X_e. With the notation of 15.12 put

$$\nu_{i,u,j,w} = e_w^{-1} \sum_\tau A_{u,j,\tau} v_w(\tau(\bar\pi_i)), \qquad \mathcal{E}_G = f\big(mN_m(N_m + t) + m^3\eta\big).$$

The two fixed-prime hypotheses use the same constant $C_1 \in \mathbf{R}$. For $p_w = g_i \neq T_u$, the tuple bound requires, for every used μ, $\phi \in M_u^{\mathrm{ram}}$ and $H \in \widetilde{M}_\mu$, a representative $y \in K$ of $P_u^\mu(\phi, H)$ with $y = 0$ or $v_w(y)/e_w \geq -\sum_j \nu_{i,u,j,w} - C_1\mathcal{E}_G$. At the same places the budget inequality is $B_{u,w} \leq -\sum_j \nu_{i,u,j,w} - C_1\mathcal{E}_G$.

Lemma 17.21 (The bound on the original lattice at g_i)**.** *For* $p_w = g_i$*,* $i \in \{0, 1\}$*, one has* $S_{u,w} = B_{u,w}$*. If* $p_w \neq T_u$ *and the fixed-prime hypotheses hold, then for* $\phi \in M_u^{\mathrm{ram}}$ *and* $H \in \widetilde{M}_{\mu_w}$ *there is* $y \in K$ *representing* $P_u^{\mu_w}(\phi, H)$ *with* $y = 0$ *or* $v_w(y) \geq e_w B_{u,w}$*.*

Proof. By 15.15 the screened gain vanishes above g_i. Apply the tuple bound at μ_w and the budget inequality at w to obtain $v_w(y)/e_w \geq B_{u,w}$ for $y \neq 0$. □

Proposition 17.22 (The common constant at the two fixed primes)**.** *On the row construction of* (33.1)*, Lemma 33.33, and Definition 34.46, assume for each used* μ *that one CM realization (induced CM type, abelian unramified host extension, trivial auxiliary ideal, chosen primes, prescribed infinity weights, Weil character, discriminant twelfth root, theta character and differential generators) realizes the period comparison for every word, identifying the rational column and row lattices and pairing with the complex period, giving (1) rational normalized jets of the prescribed multidegree on the summation support at every positive divisor of* $c_A c_B$*, and (2) the coupled last-two-step coset decoration (*$m = m_2 + 2$*) along each word chain. Put* $D_{u,j} = \sum_\tau A_{u,j,\tau}$*. Suppose for* $p = g_0, g_1$ *and all* u*,* $1 \leq j \leq m$ *that*

$$2fN_m + 1 < p, \qquad 2D_{u,j} + 2 < p, \qquad D_{u,j} + N_m < p.$$

Then the tuple bound holds with $C_1 = 1$*. If the base terms have the values in 15.10 and the local error data satisfy 15.12, the fixed-prime budget inequality holds for* $C_1 \geq 0$ *with* $2C_1 \leq c_g$*, the fixed-prime error coefficient of 15.12. For* $0 < \varepsilon \leq 1/100$*, a selected family (under the same realization and 15.10–15.12 hypotheses, with middle size threshold* 2*) satisfies both bounds with* $C_1 = 1$ *beyond a cutoff uniform in the member index.*

Outline. Reduces the Hurwitz jet bound to elliptic theta translates via 17.15 and 17.14–17.17 by formal-group root extraction, using Weyl/Howe–Umeda/Jantzen simplicity for integral comparison. Combines 17.20–17.19 for $C_1 = 1$, then 15.10–15.12 for the budget and family bounds. The complete proof is on p. 606. □

Theorem 17.23 (Periods above the fixed primes)**.** *Assume the two fixed-prime inequalities above 17.21 and the inclusion* $(N_u)_{(w)} \subseteq (M_{H,u})_{(w)}$ *away from* T_u*. Suppose that for every word, tuple and test, multiplying the complex column input by any* $c \in \mathbf{C}$ *multiplies its period by* c*. If* $p_w = g_i \neq T_u$*, then for every* $\phi \in N_u$ *and* $H \in \widetilde{M}_{\mu_w}$ *there is* $y \in K$ *representing* $P_u^{\mu_w}(\phi, H)$ *with* $y = 0$ *or* $v_w(y) \geq e_w S_{u,w}$*.*

Proof. By the localized inclusion choose $\rho \in \mathcal{O} \setminus \mathfrak{p}_w$ with $\rho\phi \in M_u^{\mathrm{ram}}$. Apply 17.21. The complex input for $\rho\phi$ is $\iota(\rho)$ times that for ϕ, so division by ρ gives a representative for ϕ. Its valuation is unchanged since $v_w(\rho) = 0$. □

17.4 All remaining finite places

Use the filling and lattice comparison data of 16.2–16.19. Write $G_{p,j}$ for the gain in (16.7).

Lemma 17.24 (Comparison of the two gains)**.** *If the filling identities of 14.9 hold at every finite place and the discarded set $\mathcal{D}$ of 15.14 contains every inert-class prime p with h_p of odd order greater than one, then* $\mathrm{gainD}_j(w) \le G_{p_w,j}$ *for every w and $1 \le j \le m$.*

Outline. Nonnegativity $G_{p,j} \ge 0$ from 14.8 covers all places where the screened or raw gain vanishes. At remaining inert and signal places outside $\mathcal{D}$, the filling identity of 14.9 and excluding odd-order inert primes match both gains to the counted gain. The complete proof is on p. 612. □

Substituting the loss formula of 15.12 into (16.6) at normalization 1 gives the base and enhanced column exponents

$$s^{\mathrm{base}}_{u,w} = B_{u,w} + \delta_{u,w}, \qquad s^{\mathrm{enh}}_{u,w} = \delta_{u,w} + \sum_{j=1}^{m} (G_{p_w,j} + F_j(w) - E_{u,j}(w)).$$

Theorem 17.25 (The enhanced period bound away from T_u, g_0, g_1)**.** *Assume the point and period hypotheses of 17.5, the local error conditions of 15.12 and the three compatibility conditions of 14.15 on the local fillings, the ramified space and the base terms. Assume the enhanced column estimate (16.28) at normalization 1, the localized inclusion of 17.10 away from T_u, and the gain comparison of 17.24. Assume that every w with $W_{p_w} \neq \varnothing$ and $p_w \neq T_u$ satisfies the model conditions of 16.2. For every such place assume the selected-place description of 17.10,*

$$\phi \in (N_u)_{(w)} \quad \Longleftrightarrow \quad c_{u,w}(j_u\phi) \in N_{P,u,w} \qquad (\phi \in V_u).$$

Here $c_{u,w}$ is the comparison map and $N_{P,u,w}$ the enhanced lattice of the permitted model in 16.2. If $p_w \notin \{T_u, g_0, g_1\}$, then for $\phi \in N_u$ and admissible H at μ_w there is $y \in K$ representing $P_u^{\mu_w}(\phi, H)$ with $y = 0$ or $v_w(y) \ge e_w S_{u,w}$.

Outline. Use the local lattice conditions of 17.10 to apply the enhanced column estimate, then bound its exponent below by $S_{u,w} + \delta_{u,w}$ using 17.24. With the denominator of 15.12, 17.5 subtracts $\delta_{u,w}$ in passing to the period. The complete proof is on p. 612. □

Theorem 17.26 (The complementary bound on the original lattice)**.** *Assume the point and period hypotheses of 17.5, the base-term formula of 15.10, the local error conditions of 15.12 and the ordinary column estimate (16.29) at normalization 1. If $p_w \notin \{T_u, g_0, g_1\}$, then for $\phi \in M_u^{\mathrm{ram}}$ and admissible H at μ_w there is $y \in K$ representing $P_u^{\mu_w}(\phi, H)$ with $y = 0$ or $v_w(y) \ge e_w B_{u,w}$.*

Proof. The input image lies in $S^{\mathrm{ram}}_{u,w}$, so (16.29) gives exponent $s^{\mathrm{base}}_{u,w} = B_{u,w} + \delta_{u,w}$. With the common stabilizer denominator of 15.12, Theorem 17.5 gives exponent $B_{u,w}$ for the period. □

17.5 The all-place conclusion

Lemma 17.27 (Rounding the valuation budget)**.** *Suppose that for every word u, every finite place w, every $\phi \in N_u$, and every admissible H at μ_w, there is a representative $y \in K$ of $P_u^{\mu_w}(\phi, H)$ with $y = 0$ or $v_w(y) \ge e_w S_{u,w}$. The same representative satisfies $y = 0$ or $v_w(y) \ge \lfloor e_w S_{u,w} \rfloor$.*

Proof. The floor is at most $e_w S_{u,w}$. □

The complementary period bound requires that for every u, w with $p_w \neq T_u$, every $\phi \in M_u^{\mathrm{ram}}$ and every admissible H at μ_w, there be a representative $y \in K$ of $P_u^{\mu_w}(\phi, H)$ with $y = 0$ or $v_w(y) \ge e_w B_{u,w}$.

Theorem 17.28 (Divisibility for fixed arithmetic data)**.** *Fix D, L and the local data for the size and column estimates in 16.21. Assume the point and period hypotheses of 17.2, the local error conditions of 15.12 and the compatibility conditions of 14.15. For some $C \in \mathbf{R}$ assume the size estimates of 16.20 with constant C and both column estimates of 16.19 at normalization 1. Require the level-prime hypotheses of 17.12, the two fixed-prime inequalities with one $C_1 \in \mathbf{R}$, and the scalar homogeneity of the column period in 17.23. Assume*

the localized inclusion away from T_u in 17.10, the permitted-model and selected-lattice hypotheses of 17.25, and the filling and screening hypotheses of 17.24. Then the rounded bound of 17.27 holds on N_u at every finite place, and the complementary period bound holds on M_u^{ram} away from T_u.

Proof. For N_u, apply 17.12 above T_u and 17.23 above the fixed primes distinct from T_u. Apply 17.25 elsewhere. Lemma 17.27 gives the rounded bound at all finite places. For M_u^{ram}, apply 17.21 at fixed primes away from T_u. At the remaining places away from T_u, the base-term identity in 14.15 and the ordinary column estimate in 16.19 allow us to apply 17.26. □

17.6 Uniformity in the selected family

For an arithmetic family $\mathfrak{F}$ of parameter ε, an assertion holds eventually if there is $x_0 \in \mathbf{R}$ such that it holds for every $x \geq x_0$ with $x > x_1$ and every $i \in I_x$. A family meets the thresholds $(\mathcal{K}, C_H)$ when its constants satisfy the staged lower bounds of Definition 36.10. Use the closedness quantifiers of that definition.

Theorem 17.29 (A common threshold for the family)**.** *Let $\mathfrak{F}$ be an arithmetic family of parameter $\varepsilon > 0$. Assume that each of the reference and twisted point hypotheses and each of the reference and twisted period hypotheses of 17.2 holds eventually. Suppose that for one $C \geq 0$ the size estimates of 16.20 with constant C and both column estimates of 16.19 at normalization 1 hold eventually. Assume also that both sharp level-prime column bounds of 17.20 hold eventually for all words, and that eventually $T_u \nmid \Delta_u$ for all words. Assume that for this family there is one $C_1 \in \mathbf{R}$ for which both fixed-prime hypotheses hold eventually. Then the rounded and complementary period bounds of 17.28 hold eventually with a common cutoff.*

For closedness on a class $\mathcal{C}$, count the two sharp column bounds as one input and the existence of C_1 for both eventual fixed-prime bounds as one input. If these eight inputs are closed, the conjunction of the two eventual conclusions is closed.

Outline. Take the largest cutoff and verify 16.2 using 14.8. Use 17.10, 15.10–15.14, 16.19, and 17.6 to apply 17.28, with closedness by the finite conjunction rule of 36.10. The complete proof is on p. 612. □

Theorem 17.30 (The selected family has divisible periods)**.** *For each family in $\mathcal{A}$, fix a cutoff and, at every member beyond it with $x > x_1$, a compatible twist realization as in 33.6, including its rows, comparison maps, self-dual host and Weil equivariance. Call this choice the twist selection. Let $\mathcal{D}_3 = \mathcal{D}(\mathcal{A}_{\mathrm{ch}}, 3)$, where $\mathcal{A}_{\mathrm{ch}}$ is the character subclass in Definition 17.8. Assume the eventual point and period hypotheses and both sharp-column hypotheses of 17.29 are closed on $\mathcal{D}_3$. Assume the existence of a common $C_1 \in \mathbf{R}$ for the two eventual fixed-prime hypotheses is closed there. On $\mathcal{D}_3$ assume both closed hypotheses of the selection form of 16.21: eventual existence of compatible twisted chains with the local system and complementary bounds, and the orbit and jet conditions there stated for the kernel of each selected member beyond its selection cutoff.*

Then there is $\varepsilon_1 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_1$ there are staged lower thresholds $\mathcal{K}$ with the following property. For every arithmetic family $\mathfrak{F} \in \mathcal{D}_3$ of parameter ε and every $C_H \in \mathbf{R}$ for which the family meets $(\mathcal{K}, C_H)$, eventually for each word u, each finite place w, each $\phi \in N_u$ and each admissible H at μ_w, there is $y \in K$ with

$$\iota(y) = P_u^{\mu_w}(\phi, H), \qquad y = 0 \text{ or } v_w(y) \geq \left\lfloor e_w \sum_{j=1}^{m} \big(\mathrm{gainD}_j(w) + F_j(w) - E_{u,j}(w)\big) \right\rfloor .$$

All period data are those of the member $D_{x,i}, L_{x,i}$.

Outline. Use 17.8 and 16.21 for closedness of 16.20 and 16.19, and 17.9 for coprimality. Apply 17.29 and take the closedness quantifiers in order, using the constructed rows of 33.77 and 33.34. The complete proof is on p. 613. □

Chapter 18

Gluing row lattices across Hecke characters

18.1 Characters and the participating family

Definition 18.1 (Allowed finite-order characters)**.** Fix a number-field extension E/F and a coefficient field K. Let $S \subseteq \mathbf{N}$. With $\mathbb{A}_{F,f}$ the finite adele ring of F, put

$$\mathcal{A} = E \otimes_F \mathbb{A}_{F,f}, \qquad \mathcal{C} = \mathrm{Hom}(\mathcal{A}^\times, K^\times).$$

Let $R \subseteq \mathcal{A}$ be the subring generated by the images of $\mathcal{O}_E$ and $\prod_{v<\infty} \mathcal{O}_{F_v}$. For an ideal $J \subseteq R$, let $U(J)$ be the image in $\mathcal{A}^\times$ of $\ker(R^\times \to (R/J)^\times)$.

Let v range over the nonzero prime ideals of $\mathcal{O}_F$. Write $Nv = \#(\mathcal{O}_F/v)$ and let $p(v)$ be the residue characteristic. Set

$$U_v = \{a \in \mathcal{A}^\times : a_{v'} = 1 \text{ for every } v' \neq v\}, \qquad I_v = R^\times \cap U_v, \qquad W_v = U(\sqrt{p(v)R}) \cap U_v.$$

Here $a_{v'}$ denotes the projection to $(E \otimes_F F_{v'})^\times$. We identify $R^\times$ with its image in $\mathcal{A}^\times$.

Call v good for S if there are distinct primes $\mathfrak{P}, \mathfrak{P}'$ of $\mathcal{O}_E$ such that

$$v\mathcal{O}_E = \mathfrak{P}\mathfrak{P}', \qquad Nv \text{ is prime}, \qquad Nv \notin v^2, \qquad p(v) \notin S.$$

In the third condition, Nv is viewed as an element of $\mathcal{O}_F$.

Let $\mathcal{C}_{\mathrm{fin}}$ consist of the finite-order characters $\tau \in \mathcal{C}$ such that $\tau(e \otimes 1) = 1$ for every $e \in E^\times$ and $U(NR) \subseteq \ker \tau$ for some positive integer N. Define the subgroup $\mathcal{T}(E/F, K, S)$ of $\mathcal{C}_{\mathrm{fin}}$ by requiring triviality on the base ideles $1 \otimes a$, $a \in \mathbb{A}_{F,f}^\times$, and on every W_v. Require also trivial restriction to I_v whenever v fails the good-place condition. The reference character is 1.

Definition 18.2 (The family used by the division construction)**.** Let K be a number field and $TU \subseteq \mathcal{T}(E/F, K, S)$ a finite subset. The row and column types form $\mathcal{T}_U = \{1\} \cup TU$. The division kernel datum gives a bijection $e : T \to \mathcal{T}_U$ from its index set T. Put $t_0 = e^{-1}(1)$ and write τ_t for the underlying character of $e(t)$.

For the arithmetic family, let U be the chosen set of labels and take $TU = \{\tau_a : a \in U\}$, where τ_a is the finite character of the label a. The reference label has finite character 1, and the map $a \mapsto \tau_a$ is injective. The reference label therefore supplies the unique label for t_0. For each $t \neq t_0$, the character τ_t belongs to TU and has a unique preimage in U. Write $\theta(t)$ for the resulting label and $\sigma : K \hookrightarrow \mathbf{C}$ for the arithmetic embedding. Character compatibility gives

$$\chi_{\theta(t)}(a) = \sigma(\tau_t(a_f))$$

for every idele a of E, with finite component $a_f \in \mathcal{A}^\times$.

18.2 The common row lattice

Put $\mathcal{O} = \mathcal{O}_K$. For a finite place w of K, write $\mathfrak{p}_w$ for its prime ideal. Localize each $\mathcal{O}$-submodule L inside its ambient K-space by

$$L_{(w)} = \mathcal{O}_{\mathfrak{p}_w} L = \{x : ax \in L \text{ for some } a \in \mathcal{O} \setminus \mathfrak{p}_w\}.$$

A full lattice is a finitely generated $\mathcal{O}$-submodule whose K-span is the ambient space.

Definition 18.3 (Local choice and global intersection)**.** Fix $r \in \mathbf{N}$. For each $t \in T$, take $n_t, d_t \in \mathbf{N}$ and an $\mathcal{O}$-submodule $M_t \subseteq K^{n_t}$. Choose K-linear maps

$$\pi_t : K^{n_t} \longrightarrow K^{d_t}, \qquad \iota_t : K^{d_t} \longrightarrow K^r.$$

Put $\Lambda_t = \pi_t(M_t)$ and $L_t = \iota_t(\Lambda_t)$. Choose $t_w \in T$ at each finite place and put

$$RR = \bigcap_{w<\infty} (L_{t_w})_{(w)} \subseteq K^r.$$

Theorem 18.4 (Gluing finitely many local changes)**.** *Fix $r \in \mathbf{N}$. Let L_0 and $L(w)$ be full lattices in K^r, with $L(w)$ given at each finite place w. Suppose $(L(w))_{(w)} = (L_0)_{(w)}$ outside a finite set. Then $G = \bigcap_w (L(w))_{(w)}$ is a full lattice and $G_{(w)} = (L(w))_{(w)}$ at every finite place.*

This is the existence and local recovery part of Voight's lattice correspondence specialized to $\mathcal{O}$ and K^r [73, Theorem 9.4.9]. The inverse in that correspondence takes the intersection of the local lattices [73, Lemma 9.4.6].

Proposition 18.5 (Recovery of the chosen row lattice)**.** *In Definition 18.3, suppose each Λ_t is full and each ι_t is surjective. Suppose $\{w : t_w \neq t_0\}$ is finite. Then RR is a full lattice and*

$$RR_{(w)} = (\iota_{t_w}(\Lambda_{t_w}))_{(w)} \qquad (w < \infty).$$

Proof. The images of generators of Λ_t generate L_t, and $KL_t = \iota_t(K\Lambda_t) = K^r$. Apply 18.4 with $L_0 = L_{t_0}$ and $L(w) = L_{t_w}$. □

Lemma 18.6 (The type at conjugate places)**.** *Use the arithmetic column family of 18.2. Let* Tw *be the image in the finite places of K of a finite conjugation-stable set W of arithmetic places. Write $\bar{w}$ for the conjugate of w under the arithmetic involution of K and p_w for the residue characteristic of w. Let δ, δ' be the auxiliary Hecke characters, each of order 15 on the local units at every place of E above its respective rational prime $p_\delta, p_{\delta'}$. For $w \in$ Tw take Hecke characters ψ_w unramified above both primes with $\psi_{\bar{w}} = \psi_w$. Put $\xi_w = \delta'\psi_w$ if $p_w = p_\delta$ and $\xi_w = \delta\psi_w$ otherwise. Assume the full arithmetic column family contains ξ_w for every $w \in$ Tw. First put $U = \{a : \chi_a = \xi_w \text{ for some } w \in \text{Tw}\}$ and $TU = \{\tau_a : a \in U\}$. Form both coordinate systems of 18.2, including their index sets, bijections, lattices and maps, with this chosen TU.*

Define a selector $w \mapsto t_w$ in both coordinate systems as follows: for each $w \in$ Tw, choose a label a with $\chi_a = \xi_w$ and set $t_w = e^{-1}(\tau_a)$ via 18.2 (so $\theta(t_w) = a$ and $\chi_{\theta(t_w)} = \xi_w$); for $w \notin$ Tw, set $t_w = t_0$. In each system suppose every Λ_t is full and every ι_t is an isomorphism. Then

$$t_{\bar{w}} = t_w, \qquad w \in \text{Tw} \iff t_w \neq t_0.$$

In either system, fix w and put $t = t_w$. Then $RR_{(z)} = (L_t)_{(z)}$ for $z = w, \bar{w}$.

Outline. Choose t_w by ξ_w via 18.2 on Tw and take t_0 elsewhere. Conjugation invariance, injectivity and order 15 give $t_{\bar{w}} = t_w \neq t_0$ on Tw, and 18.5 recovers both localizations of L_{t_w}. The complete proof is on p. 613. □

18.3 Passing the period bound to coordinate rows

Let V_μ be the complex row space for the arithmetic label μ. Put $V_0(\mu) = \mathrm{im}(e_{\mathrm{op},\mu})$ for the idempotent row operator $e_{\mathrm{op},\mu}$. Let $A \subseteq \mathcal{O}$ be the image of the arithmetic coefficient ring. The test set $\widetilde{M}_\mu$ is the $\sigma(A)$-span of a finite set $\mathcal{G}_\mu$.

Lemma 18.7 (Integral decomposition of test vectors)**.** *Fix μ and suppose*

$$\sigma(a)y \in \widetilde{M}_\mu \qquad (a \in \mathcal{O},\ y \in \widetilde{M}_\mu).$$

For every $x \in V_0(\mu)$ admitting an expression

$$x = \sum_{v \in \mathcal{G}_\mu} \sigma(c_v)v, \qquad c_v \in \mathcal{O},$$

there are a finite set $S_x \subseteq \widetilde{M}_\mu \cap V_0(\mu)$ and coefficients $b_y \in \mathcal{O}$ such that

$$x = \sum_{y \in S_x} \sigma(b_y)y.$$

Proof. Scalar stability and closure under sums give $x \in \widetilde{M}_\mu$. Take $S_x = \{x\}$ and $b_x = 1$. □

For a finite place w and $s \in \mathbf{R}$, put

$$\mathcal{V}_{w,s} = \{0\} \cup \{a \in K^\times : v_w(a) \geq s\}.$$

Here $v_w(a)$ is the exponent of $\mathfrak{p}_w$ in $a\mathcal{O}$. For $a = b/c \in K^\times$ with $b, c \in \mathcal{O}$, both v_w and the additive valuation used in 17.27 give $\mathrm{ord}_{\mathfrak{p}_w}(b) - \mathrm{ord}_{\mathfrak{p}_w}(c)$. The set $\mathcal{V}_{w,s}$ is an $\mathcal{O}$-submodule. It is the fractional ideal $(\mathfrak{p}_w\mathcal{O}_{\mathfrak{p}_w})^{\lceil s \rceil}$ of the discrete valuation ring $\mathcal{O}_{\mathfrak{p}_w}$.

Proposition 18.8 (Period bounds on the coordinate row lattice)**.** *Fix a column u. Let $N_u \subseteq V_u$ be the input lattice of Proposition 17.10 in the rational ramified input space of u, and let $E_u : K^{a_u} \to V_u$ be the column bridge. For each $t \in T$, let $J_t : K^{n_t} \to V_{\theta(t)}$ be the row bridge. Write $P_{u,t}(\phi, H) \in K$ for the coordinate bilinear pairing. As in Chapter 17, $P_u^\mu(v, y)$ denotes the complex period obtained by pairing the image of $v \in V_u$ under the complex embedding j_N with the row $y \in V_\mu$ at label μ. Assume that for every $t \in T$ the identity*

$$\sigma(P_{u,t}(\phi, H)) = P_u^{\theta(t)}(E_u\phi, J_tH)$$

holds for all $\phi \in K^{a_u}$ and $H \in K^{n_t}$. Assume also that for every $t \in T$ and $H \in M_t$,

$$J_tH \in V_0(\theta(t)), \qquad J_tH = \sum_{v \in \mathcal{G}_{\theta(t)}} \sigma(c_v)v \quad \text{for some } c_v \in \mathcal{O}.$$

Fix a type t, a finite place w, a threshold $s \in \mathbf{R}$ and a subset $N \subseteq K^{a_u}$. Suppose the integral decomposition in the conclusion of Lemma 18.7 holds for $\theta(t)$ and, for every $x \in N$ and $y \in \widetilde{M}_{\theta(t)} \cap V_0(\theta(t))$, there is $a \in \mathcal{V}_{w,s}$ with

$$\sigma(a) = P_u^{\theta(t)}(E_ux, y).$$

Then

$$P_{u,t}(\phi, H) \in \mathcal{V}_{w,s} \qquad (\phi \in \mathrm{span}_{\mathcal{O}} N,\ H \in M_t).$$

Let μ_0 be the reference label and $\mu_\star, \mu'_\star$ the two auxiliary labels. Let μ_w be the arithmetic label used in the period bound of 17.28. Assume the column bridge satisfies $\theta(t_w) = \mu_w$ at every finite place w. Suppose the integral decomposition in the conclusion of Lemma 18.7 holds for every $\mu \in \{\mu_0, \mu_\star, \mu'_\star\} \cup \{\mu_w : w < \infty\}$. If $\mathcal{N}_u \subseteq K^{a_u}$ is an $\mathcal{O}$-submodule with $E_u(\mathcal{N}_u) \subseteq N_u$ and the periods on N_u satisfy the rounded valuation bound of Theorem 17.28 at every finite place, then

$$P_{u,t_w}(\phi, H) = 0 \quad \text{or} \quad v_w(P_{u,t_w}(\phi, H)) \geq s_{u,w}, \qquad s_{u,w} = \lfloor e_w \mathrm{sg}(u, w) \rfloor$$

for every finite w and for all $\phi \in \mathcal{N}_u$ and $H \in M_{t_w}$. Here e_w is the ramification index of w over its rational residue characteristic and $\mathrm{sg}(u, w)$ is the budget $S_{u,w}$ of 17.27.

Outline. The integral decomposition and bridge identity give the period over $\mathcal{V}_{w,s}$; extend by linearity using 17.28. The complete proof is on p. 613. □

Chapter 19

The pairing after division

Let K be the coefficient number field and put $\mathcal{O} = \mathcal{O}_K$. For a finite place w, write $\mathfrak{p}_w$ for its prime ideal and $v_w : K^\times \to \mathbf{Z}$ for the normalized additive valuation. For an $\mathcal{O}$-submodule M of a K-vector space, use the localization

$$M_{(w)} = \{x : \text{there is } t \in \mathcal{O} \setminus \mathfrak{p}_w \text{ with } tx \in M\}.$$

A lattice means a finitely generated $\mathcal{O}$-submodule spanning its ambient K-vector space. For $R \subset W$ put $R^\vee = \{\lambda \in W^* : \lambda(R) \subset \mathcal{O}\}$.

19.1 The divided column module

Fix the arithmetic data D and the size data L. For each word u and finite place w, let

$$S_{u,w} = \mathrm{sg}(u, w), \qquad s_{u,w} = \lfloor e_w S_{u,w} \rfloor, \qquad e_w = \text{the ramification index of } w \text{ over } \mathbf{Q}.$$

Definition 15.16 gives the budget $S_{u,w}$, and Lemma 17.27 passes from the period bound with threshold $e_w S_{u,w}$ to the bound with integer threshold $s_{u,w}$. By Lemma 15.17, both quantities vanish outside the finite set S_L of Definition 15.16. For each word u, let V_u be its column space over K and $N_u \subset V_u$ the input $\mathcal{O}$-submodule specified by L.

Definition 19.1 (Division ideal and divided submodule)**.** For each word u, define

$$I_u = \prod_{w \in S_L} \mathfrak{p}_w^{-s_{u,w}}, \qquad I_u N_u = \mathrm{span}_{\mathcal{O}}\{a\phi : a \in I_u,\ \phi \in N_u\}.$$

The division condition for (D, L) is the equality $N'_u = I_u N_u$ for every u, where $N'_u \subset V_u$ is the $\mathcal{O}$-submodule specified by the division data of D for the word u.

At a place with $s_{u,w} > 0$, this operation permits additional denominators of order $s_{u,w}$. A negative $s_{u,w}$ imposes additional divisibility.

Proposition 19.2 (Realization of local shifts)**.** *Let $N \subset K^r$ be a lattice, with $r \in \mathbf{N}$. Fix a finitely supported integer function $w \mapsto \tau_w$. Put $I = \prod_w \mathfrak{p}_w^{-\tau_w}$. Then IN is a lattice. For every w there exists $a_w \in K^\times$ with $v_w(a_w) = -\tau_w$, and for every such choice*

$$(IN)_{(w)} = (a_w N)_{(w)}.$$

Moreover, any $\mathcal{O}$-submodule $N' \subset K^r$ with these localizations equals IN.

More generally, if J is any nonzero fractional ideal and $a \in K^\times$ satisfies $v_w(a) = v_w(J)$, where $v_w(J)$ is the exponent of $\mathfrak{p}_w$ in J, then $(JN)_{(w)} = (aN)_{(w)}$ for every $\mathcal{O}$-submodule $N \subset K^r$.

Local principality follows from the discrete valuation ring criterion [74, Lemma 10.120.17, Tag 034X]. Voight gives the local criterion and reconstruction for lattices over a Dedekind domain [73, Corollary 9.4.7 and Theorem 9.4.9]. We specialize reconstruction to prescribed shifts over $\mathcal{O}_K$ and obtain uniqueness among all submodules. For uniqueness, apply [74, Lemma 10.23.1, Tag 00HN] to $(N'+IN)/N'$ and $(N'+IN)/IN$.

For arbitrary $\mathcal{O}$-submodules N, N' and any integer function τ, call N' a *shift of N by τ* if for every w there is $a_w \in K^\times$ of valuation $-\tau_w$ such that $N'_{(w)} = (a_w N)_{(w)}$.

Lemma 19.3 (The arithmetic and coordinate shifts agree)**.** *Fix u and a K-linear isomorphism $e : K^r \to V_u$. Let $N = e^{-1}(N_u)$ and let $N^c \subset K^r$ be an $\mathcal{O}$-submodule with $e(N^c) = N'_u$. Assume the division condition of Definition 19.1. Let g_u be the real-valued function on finite places given by $g_u(w) = s_{u,w}$. For any real-valued function g on the finite places, put $d_w = \lfloor g(w) \rfloor$. When d has finite support, write $N(g) = (\prod_{d_w \neq 0} \mathfrak{p}_w^{-d_w})N$. For $g = g_u$ one has $d_w = s_{u,w}$ at every w, and*

$$N^c = N(g_u) = I_u N.$$

In particular N^c is the shift of N by $w \mapsto s_{u,w}$.

Proof. Since $s_{u,w}$ is an integer, the w-factor of the shift ideal is $\mathfrak{p}_w^{-s_{u,w}}$. Their product is I_u by Definition 19.1. By K-linearity, $e(I_u N) = I_u e(N) = I_u N_u = N'_u = e(N^c)$. Since e is injective, $I_u N = N^c$. At w, a power of a local uniformizer gives $a_w \in K^\times$ with $v_w(a_w) = -s_{u,w} = v_w(I_u)$. Proposition 19.2 then gives $N^c_{(w)} = (a_w N)_{(w)}$. □

19.2 Cancellation and local integrality

Lemma 19.4 (Cancellation on a shifted column)**.** *Fix a finite place w and $s \in \mathbf{Z}$. Let $N, N' \subset K^r$, $r \in \mathbf{N}$, be $\mathcal{O}$-submodules with $N'_{(w)} = (aN)_{(w)}$ for some $a \in K^\times$ of valuation $-s$. Let $f : K^r \to K$ be K-linear. If*

$$\forall \psi \in N, \qquad f(\psi) = 0 \quad \text{or} \quad v_w(f(\psi)) \geq s,$$

then for every $\phi \in N'$ one has $f(\phi) = 0$ or $v_w(f(\phi)) \geq 0$.

Proof. For $\phi \in N'$, the equality of localizations gives $t \in \mathcal{O} \setminus \mathfrak{p}_w$ and $\psi \in N$ such that $t\phi = a\psi$. Hence

$$t f(\phi) = a f(\psi), \qquad v_w(t) = 0.$$

If $f(\psi) = 0$, then $f(\phi) = 0$. If $f(\phi) \neq 0$, both sides are nonzero and

$$v_w(f(\phi)) = v_w(a) + v_w(f(\psi)) - v_w(t) = -s + v_w(f(\psi)) \geq 0.$$

□

Lemma 19.5 (From a local row pairing to global integers)**.** *Fix w, integers $n, h \geq 0$, an $\mathcal{O}$-submodule $M \subset K^n$, and K-linear maps $\pi : K^n \to K^h$, $\iota : K^h \to K^h$. Let $b : K^h \to K$ and $P : K^n \to K$ be K-linear. Fix $\zeta \in K^\times$ with $v_w(\zeta) = 0$. Suppose $P(F) = \zeta\, b(\iota\pi F)$ for every $F \in K^n$. For any $c \in \mathbf{Z}$, if $P(F) = 0$ or $v_w(P(F)) \geq c$ for every $F \in M$, then*

$$x \in (\iota\pi M)_{(w)} \quad \Longrightarrow \quad b(x) = 0 \text{ or } v_w(b(x)) \geq c.$$

For the global assertion, fix h, a K-linear functional $b : K^h \to K$, and an $\mathcal{O}$-submodule $R \subset K^h$. Suppose the local hypotheses hold at every w with this b, source dimension n_w, and $c = 0$. If $R_{(w)} = (\iota_w \pi_w M_w)_{(w)}$ for every w, then $b(R) \subset \mathcal{O}$.

Outline. Writing $tx = \iota\pi F$ with $t \in \mathcal{O} \setminus \mathfrak{p}_w, F \in M$ gives $P(F) = \zeta t b(x)$, so $v_w(\zeta) = v_w(t) = 0$ gives $v_w(b(x)) \geq c$ if $b(x) \neq 0$. Globally, $v_w(b(x)) \geq 0$ everywhere makes $(b(x))$ integral, so $b(x) \in \mathcal{O}$. The complete proof is on p. 613. □

19.3 The pairing into the common dual

Theorem 19.6 (Integral pairing after division)**.** *Fix $r, h \in \mathbf{N}$ and a K-linear map $B : K^r \to (K^h)^*$. Take $\mathcal{O}$-submodules $N, N' \subset K^r$ and $R \subset K^h$. For each finite place w, fix $n_w \in \mathbf{N}$ and an $\mathcal{O}$-submodule $M_w \subset K^{n_w}$. Let*

$$\pi_w : K^{n_w} \to K^h, \quad \iota_w : K^h \to K^h, \quad P_w : K^r \times K^{n_w} \to K$$

be K-linear maps and a K-bilinear pairing, respectively. Suppose $\zeta_w \in K^\times$ has $v_w(\zeta_w) = 0$ and

$$P_w(\phi, F) = \zeta_w B(\phi)(\iota_w \pi_w F) \quad (\phi \in K^r,\ F \in K^{n_w}), \qquad R_{(w)} = (\iota_w \pi_w M_w)_{(w)}.$$

Let $\sigma_w \in \mathbf{R}$ and $\tau_w \in \mathbf{Z}$ be arbitrary functions of w. Suppose N' is a shift of N by τ, with $\tau_w \le \lceil \sigma_w \rceil$ at every w. Assume at each w that

$$\phi \in N,\ F \in M_w \quad \Longrightarrow \quad P_w(\phi, F) = 0 \text{ or } v_w(P_w(\phi, F)) \ge \sigma_w.$$

Then $B(N') \subset R^\vee$.

Proof. Fix w and $F \in M_w$. Each nonzero value of $f_F(\phi) = P_w(\phi, F)$ on N has integer valuation, hence

$$v_w(f_F(\phi)) \ge \lceil \sigma_w \rceil \ge \tau_w.$$

Apply Lemma 19.4 to f_F. Fix any $\phi \in N'$. The lemma gives $P_w(\phi, F) = 0$ or $v_w(P_w(\phi, F)) \ge 0$ for every w and $F \in M_w$. Lemma 19.5 with $b = B(\phi)$ gives $B(\phi)(R) \subset \mathcal{O}$. □

Theorem 19.7 (The selected characters give one dual lattice)**.** *Use the column record and the row data of Proposition 18.5, with selected index t_w and common coordinate row lattice $R^c \subset K^h$. Put $A_w = \iota_{t_w} \pi_{t_w}$, where π_t is the row map of Definition 18.3. Proposition 18.5 gives $R^c_{(w)} = (A_w M_{t_w})_{(w)}$. For a column index i, let $B^c_i : K^{a_i} \to (K^h)^*$ be its coordinate column map. The column record gives*

$$P_{i,t_w}(\phi, F) = \zeta_{i,t_w} B^c_i(\phi)(A_w F), \qquad \zeta_{i,t_w} \in K^\times, \quad v_w(\zeta_{i,t_w}) = 0.$$

Let $N, N' \subset K^{a_i}$ be $\mathcal{O}$-submodules and let $\sigma_w \in \mathbf{R}$, $\tau_w \in \mathbf{Z}$. Suppose N' is a shift of N by τ, with $\tau_w \le \lceil \sigma_w \rceil$ at every w. Assume at each w that

$$\phi \in N,\ F \in M_{t_w} \quad \Longrightarrow \quad P_{i,t_w}(\phi, F) = 0 \text{ or } v_w(P_{i,t_w}(\phi, F)) \ge \sigma_w.$$

Then $B^c_i(N') \subset (R^c)^\vee$.

For the arithmetic application, assume $\theta(t_w) = \mu_w$ at every finite place w, with the arithmetic labels of Definition 18.2. For each arithmetic word u, let $i(u)$ be the coordinate column index assigned to u by the division data, and put $B^c_u = B^c_{i(u)}$ and $a_u = a_{i(u)}$. Let $B_u : V_u \to W^*$ be the period map and $RR \subset W$ the common row lattice in the arithmetic row space W. Fix K-linear isomorphisms $E_u : K^{a_u} \to V_u$ and $E_R : K^h \to W$, and an $\mathcal{O}$-submodule $N^c_u \subset K^{a_u}$. Assume

$$E_u(N^c_u) = N'_u, \qquad E_R(R^c) = RR, \qquad B_u(E_u x)(E_R y) = B^c_u(x)(y) \quad (x \in K^{a_u},\ y \in K^h).$$

Under the division condition of Definition 19.1 and the hypotheses of Proposition 18.8 for E_u and column $i(u)$, one has $B_u(N'_u) \subset RR^\vee$.

Outline. Apply 19.6 with $R = R^c$, $B = B^c_i$, $M_w = M_{t_w}$, and $\pi_w = A_w$ for the coordinate claim. Setting $N = E_u^{-1}(N_u)$, obtain the pairing bound from 18.8 and the shift N^c_u from 19.3, then transport the inclusion via E_u and E_R. The complete proof is on p. 614. □

Chapter 20

Slope change and the data for step 5

Write $n_K = [K : \mathbf{Q}]$ and $\mathcal{O} = \mathcal{O}_K$. At a finite place w, write $\mathfrak{p}_w$ for its prime ideal and p_w for its residue characteristic. Normalize the valuation v_w by $v_w(\mathfrak{p}_w) = 1$. Put $e_w = v_w(p_w)$ and $\ell_w = \log N\mathfrak{p}_w/n_K$. Use $S_{u,w} = \mathrm{sg}(u, w)$ from 15.16 and $s_{u,w} = \lfloor e_w S_{u,w} \rfloor, N_u, I_u, N'_u$ from 18.8–19.1. The column modules N_u, N'_u use the same basis and frames. Write S for the prescribed finite set of places outside which the local gains, base terms and losses vanish, as in 15.16.

20.1 The exact change of slope

Let V be a finite-dimensional space over a totally complex number field K, with basis b and invertible frames X_v at infinity. Use one embedding per infinite place. The real realization $\rho_{b,X}(y)$ consists of the real and imaginary parts of $X_v^{-1} v([y]_b)$. The norm at v is the Euclidean norm in this frame. For a nonzero finitely generated $\mathcal{O}$-submodule $M \subset V$, put

$$\mu_{b,X}(M) = 2 \operatorname{slope}(\rho_{b,X} M) - \log 2 + \frac{\log |\operatorname{disc} K|}{n_K},$$

where slope is minus the log covolume in the real span divided by the $\mathbf{Z}$-rank. Let $\mu_{\max}(M)$ be the supremum of $\mu(L)$ over all nonzero $\mathcal{O}$-submodules $L \subseteq M$, with induced metrics. A full lattice spans its ambient K-space. Its dual $M^\vee = \{f \in V^\vee : f(M) \subset \mathcal{O}\}$ has the dual metrics.

Lemma 20.1 (Multiplication by an integral ideal)**.** *For M as above and a nonzero integral ideal $\mathfrak{b}$,*

$$\mu_{b,X}(\mathfrak{b}M) = \mu_{b,X}(M) - \frac{2}{n_K} \log N\mathfrak{b}.$$

Lemma 20.2 (Multiplication by a fractional ideal)**.** *Let I be a nonzero fractional ideal and $M \subset V$ an $\mathcal{O}$-submodule. If $N = IM$ is nonzero and finitely generated, then so is M, and*

$$\mu_{b,X}(N) = \mu_{b,X}(M) - \frac{2}{n_K} \log N(I),$$

where $N(I) \in \mathbf{Q}_{>0}$ is the absolute fractional-ideal norm.

Proof. For 20.1, $\mathfrak{b}M \subseteq M$ has the same K-rank r and index $N\mathfrak{b}^r$, and the realization is injective. Hence the covolume is multiplied by $N\mathfrak{b}^r$, while the $\mathbf{Z}$-rank is $n_K r$. For 20.2, multiplication by a nonzero $a \in I$ embeds M in N, so M is finitely generated. Write $I = d^{-1}J$ with $0 \neq d \in \mathcal{O}$ and J integral. Then $dN = JM$. Apply 20.1 to both sides and use $N(I) = N(J)/N(d\mathcal{O})$. □

Lemma 20.3 (Norm of the division ideal)**.** *For every word u, $I_u \neq 0$ and*

$$\log N(I_u) = -\sum_{w \in S} s_{u,w} \log N\mathfrak{p}_w.$$

Proof. The nonzero factors $\mathfrak{p}_w^{-s_{u,w}}$ of 19.1 have norms $(N\mathfrak{p}_w)^{-s_{u,w}}$. Multiply and take logarithms. □

Theorem 20.4 (Slope difference in the arithmetic data)**.** *Assume $N'_u = I_u N_u$ for every word u, as in 19.1. For any word u with N'_u nonzero and finitely generated and with invertible frames,*

$$\Delta\mu_u := \mu(N'_u) - \mu(N_u) = 2\sum_{w\in S} \ell_w s_{u,w}.$$

Proof. Apply 20.2 to $N'_u = I_u N_u$ and substitute 20.3. □

For a full lattice $N \subset K^d$ over a totally complex field put

$$\operatorname{lcv}_X(N) = \frac{2}{n_K}\log\Big(2^{n_K d/2}|\operatorname{disc} K|^{-d/2}\operatorname{covol}_X(N)\Big),$$

with Euclidean covolume in the frames. Whenever the localizations $A_{(w)}, A'_{(w)}$ are full lattices in the same local space, let $\operatorname{idx}_w(A, A')$ be the valuation of the determinant of a basis of $A'_{(w)}$ relative to one of $A_{(w)}$. A shift of N by $\tau_w \in \mathbf{Z}$ at w means $N'_{(w)} = a_w N_{(w)}$ with $a_w \in K^\times$ and $v_w(a_w) = -\tau_w$.

Theorem 20.5 (Slope difference in coordinates)**.** *Let K be a number field with involution and $N \subset K^d$ a full lattice, $d \geq 0$. If a submodule $N' \subset K^d$ is a shift of N by τ_w at w, then $\operatorname{idx}_w(N, N') = -d\tau_w$. For totally complex K and invertible frames X_v, $\mu_X(N) = -\operatorname{lcv}_X(N)/d$ if $d > 0$. If N' is a full lattice and the shift holds at every finite place, then for $d > 0$ the shifts have finite support and*

$$\operatorname{lcv}_X(N') - \operatorname{lcv}_X(N) = -2d\sum_w \ell_w\tau_w, \qquad \mu_X(N') - \mu_X(N) = 2\sum_w \ell_w\tau_w.$$

For $d = 0$ both normalized log covolumes are zero.

Proof. The local determinant is a_w^d. For full lattices put $N_0 = N \cap N'$. The quotients N/N_0 and N'/N_0 are finite: finite generation and the common K-span give nonzero denominators $a, b \in \mathcal{O}_K$ with $aN \subseteq N'$ and $bN' \subseteq N$. Thus the local length differences are supported at only finitely many places. When $d > 0$, $\operatorname{idx}_w(N, N') = -d\tau_w$ gives the same finite-support assertion for the shifts. The covolume ratio is $[N : N_0]/[N' : N_0]$. The local length difference is $\operatorname{idx}_w(N, N')$, so the normalized log covolume difference is $2\sum_w \ell_w \operatorname{idx}_w(N, N')$. The real rank $n_K d$ gives the slope formula. In dimension zero the covolume and its normalizing factor are both 1. □

20.2 Rounding and the rational primes

Put $A_u = \sum_{w\in S} 2\ell_w e_w S_{u,w}$.

Lemma 20.6 (Rounding error)**.** *Under the hypotheses of 20.4,*

$$|\Delta\mu_u - A_u| \leq 2\sum_{\substack{w\in S\\ S_{u,w}\neq 0}} \ell_w.$$

Proof. By 20.4, the difference sums $2\ell_w(\lfloor e_w S_{u,w}\rfloor - e_w S_{u,w})$. Each summand vanishes for $S_{u,w} = 0$ and has absolute value at most $2\ell_w$. □

Lemma 20.7 (Total weight above a rational prime)**.** *For a number field K and a rational prime p, let W_p be the places above p. With $f_w = [\mathcal{O}/\mathfrak{p}_w : \mathbf{F}_p]$,*

$$\sum_{w\in W_p} e_w f_w = n_K, \qquad \sum_{w\in W_p} e_w \ell_w = \log p.$$

This is the ramification degree formula [76, Remark 15.113.6, Tag 09E8] for $\mathbf{Z}_{(p)}$ in $K/\mathbf{Q}$, with $N\mathfrak{p}_w = p^{f_w}$ in the weighted form.

Lemma 20.8 (Support of the gain and loss in a parameter family)**.** *Fix a parameter family at ε as in 16.20. Write x_1 for its threshold and I_x for the set indexing its members at x. Take any $x > x_1$ and $i \in I_x$. Use X'' from 14.8 and the stabilizer denominator M_u from 15.12. At each place w with $p_w > X''$, the raw gain vanishes for every $1 \le j \le m$. If also $p_w \notin \{T_u, g_0, g_1, d\}$ and $p_w \nmid M_u$, then $E_{u,j}(w) = 0$. We call the simultaneous vanishing of the raw gain and the loss for every u, j, w satisfying these three prime conditions the support condition above X''.*

Proof. The raw gain vanishes outside S and outside the inert and signal classes. For $w \in S$ with p_w of inert or signal class the filling equality of 14.9 gives the prime gain of 14.8, which vanishes above X''. In the formula for the loss $E_{u,j}(w)$ in 15.12, the exceptional terms occur only at $p_w \in \{T_u, g_0, g_1, d\}$. The fixed inert branch and the remaining inert-or-signal branch require $p_w \le X''$. The remaining term is $v_{p_w}(M_u) = 0$. □

Theorem 20.9 (Rational-prime bound for the error)**.** *Retain the hypotheses of 20.4. Assume $X'' \ge 0$ and the support condition above X'' of 20.8. If $p_w > X''$, $p_w \notin \{T_u, g_0, g_1, d\}$ and $p_w \nmid M_u$, then $S_{u,w} = 0$. Moreover,*

$$|\Delta\mu_u - A_u| \le 2\big(X'' \log 4 + \log T_u + \log g_0 + \log g_1 + \log d + \log M_u\big).$$

Outline. Outside P_u, the budget in 15.16 gives $S_{u,w} = 0$. Apply 20.6 and 20.7, then bound the product of small primes by binomial induction and include the exceptional primes and prime divisors of M_u. The complete proof is on p. 614. □

Definition 20.10 (Size bounds for the auxiliary primes)**.** At scale $x > 1$ with $0 < \varepsilon < 1/52$ and window parameter $J \ge 1$, put $\varepsilon_z = \varepsilon/J$, $\theta = \varepsilon/100$ and $b = \exp(x^{\varepsilon})$. Let χ_d be the quadratic character of $\mathbf{Q}(\sqrt{-d})$, and put $D_1 = \mathrm{Re}(L'(1,\chi_d)/L(1,\chi_d))$. The bounds for the host primes q, d are

$$x^{\varepsilon_z/6} \le \log q, \qquad x^{19\varepsilon_z/100} \le \log p \le x^{\varepsilon/5+\theta} \quad (p = q, d),$$
$$|D_1| \le x^{\varepsilon}, \qquad 1 \le \mathrm{rd}(E) \le \tfrac{1}{2} Q_S \sqrt{qd},$$

where E is the host field and Q_S is the host modulus of 14.1. For the two fixed primes require, for all $i, j \in \{0, 1\}$,

$$(kb)^8 \le g_i \le 2g_j, \qquad \log g_i \le x^{\theta+\varepsilon}.$$

Theorem 20.11 (Uniform error in the family parameters)**.** *Assume the auxiliary-prime bounds of Definition 20.10 and the support condition above X'' of 20.8. For every u, assume $N'_u = I_u N_u$ as in 19.1, with the invertible column frames of the arithmetic datum. Suppose every N'_u is finitely generated and has positive K-rank. Let $C_T, C_{\mathrm{stab}} \in \mathbf{R}$ satisfy $\log T_u \le C_T X''$ and $\log M_u \le C_{\mathrm{stab}} X''$ for every u. Then*

$$|\Delta\mu_u - A_u| \le 2(\log 4 + 3 + C_T + C_{\mathrm{stab}})X'' \qquad \text{for every } u.$$

Proof. The positive rank of each N'_u makes it nonzero, as required in 20.4. With $\theta = \varepsilon/100$ the size assumptions give $\log g_i \le x^{\theta+\varepsilon}$ for $i = 0, 1$ and $\log d \le x^{\varepsilon/5+\theta}$. Since $0 < \varepsilon < 1/52$ and $x > 1$, both powers are at most $x \le X'' = X' x n$, where $X', n \ge 1$. Apply 20.9. □

Lemma 20.12 (The index of a projected theta lattice at one place)**.** *Work over the local ring at a finite place v, with valuation $v(\cdot)$ and index $[B : B']_v = \mathrm{idx}_v(B, B')$. For a subspace V put $B_V = B \cap V$. Let a finite group P act by ρ on an n-dimensional K-space. Let $U \le P$ have order h and index s, with character $\psi : U \to \mathcal{O}^\times$. For each $g \notin U$ assume some $u \in U \cap gUg^{-1}$ satisfies $\psi(u) \ne \psi(g^{-1}ug)$. Let W be the ψ-eigenspace. For an idempotent e with image V_0, put $Z = W \cap V_0$ and $d_Z = \dim Z$.*

Take full lattices $\Theta_1 \subseteq \Theta \subseteq B$, with B, Θ stable under P. For an integer $E \ge 0$ and $r \in \mathbf{R}$ assume

$$[B_V : (\Theta_1)_V]_v \le E \dim V + r \qquad \text{for every subspace } V. \tag{20.1}$$

Suppose $\Theta \subseteq \sigma\Theta_K$ for a full lattice Θ_K and $\sigma \in K^\times$, and take a lattice Θ_0 with $h(\Theta_K)_W \subseteq \Theta_0 \subseteq W$. Take a module Θ_J with $e\Theta_0 \subseteq e\Theta_J$ and full lattices $e\Theta_J, X_3, A$ in Z with $X_3 \subseteq A$ and

$$[B_Z : X_3]_v \le [B_Z : e\Theta_J]_v + e_v\xi, \qquad \xi \in \mathbf{R}. \tag{20.2}$$

Then

$$[B_Z : A]_v \le d_Z\{E - v(\sigma) + v(h)\} + \frac{r + nv(h)}{s} + e_v\xi.$$

Outline. Use the subspace index bound and character averaging to bound $s[B_Z : B'_Z]_v$ by $r + nv(h)$ via maps with composite h. Scale B'_Z into $e\Theta_J$ and compare indices using $X_3 \subseteq A$. The complete proof is on p. 614. □

20.3 The row and column data

The estimates of 21.12 and 21.28 use the row data specified here and supply bounds for Theorem 20.16. Fix a totally complex field K with involution $a \mapsto \bar{a}$. Choose a subfield $K_0 \subseteq K$ such that for every embedding $\sigma : K \hookrightarrow \mathbf{C}$ and $a \in K_0$, $\sigma(\bar{a}) = \overline{\sigma(a)}$. At the chosen infinite place v_0 require the identity on all of K. Use common types t with base type t_0. Let $\mathcal{I}$ be the finite spectral index set and index the layers by $1 \le j \le m$. On each definite side, reference and selected, let M_t be the test lattice, P_t the surjective quotient projection and $\Lambda_t = P_t M_t$. Write T_{jt} for the layer lattice with Hermitian matrix H_{jt} and frames $X_{jt,v}$. All these lattices are full. The test matrices $H_{V,t}$ have entries in K_0 and are positive definite at every infinite place. They induce the quotient metrics. Every $X_{jt,v}$ is invertible, and

$$(X^{-1}_{jt,v_0})^* X^{-1}_{jt,v_0} = v_0(H_{jt}).$$

At t_0 the quotient type subspaces form a direct sum spanning the quotient, and the layer type subspaces are in direct sum.

Definition 20.13 (Reference rows and selected columns)**.** Fix the selected pairings $P_{u,t}$ and column frames $X_{\mathrm{c},v}$ for each word u of dimension d_u. Fix a row subspace $\mathcal{R} \subseteq \mathcal{H}$ of a complex inner-product space $\mathcal{H}$. Take complex-linear maps $P^{\mathrm{per}}_u : \mathbf{C}^{d_u} \to \mathcal{H}$. The common real data are $F_{0,j,w}, l_{3,j}, l_{6,j}, E_{1,j}, h_{\mathrm{r}}, \lambda_{\mathrm{r}}, \delta$, with $F_{0,j,w}$ the reference base term. Choose the normalized cut-exponent bound $J_0(t, w, j)$ of 21.28 for every type, finite place and layer.

On each side choose a selector $w \mapsto t_w$ with $t_{\bar{w}} = t_w$, where $\bar{w}$ is the conjugate place. The set $\mathrm{Tw} = \{w : t_w \ne t_0\}$ is finite. Write $J_{0,j,w} = J_0(t_w, w, j)$ in the index bounds. The base quotient has positive dimension. Choose a row space K^d with frames $X_{E,v}$, spectral subspaces E_α and maps ι_t from the definite quotient spaces. For each layer choose a space K^{d_j} with frames $X_{W,j,v}$, subspaces $W_{j,\alpha}$, an $\mathcal{O}$-submodule M_j and maps θ_{jt} from the definite layer spaces. Take real factors $\kappa_v, \lambda_{jt\alpha v}$ and real local errors e^0_{wj}. Denote the reference and selected row dimensions by D_R, D_{c}.

On the reference side H_{j,t_0} has entries in K_0 and induces the definite layer metrics at every infinite place. The type subspaces at t_0 span each layer. All reference row and layer frames are invertible. Each M_j is full and the maps ι_t, θ_{jt} are isomorphisms. At t_0 these maps send the definite type subspaces onto $E_\alpha, W_{j,\alpha}$, respectively, with $\dim W_{j,\alpha} = \dim E_\alpha$. The factors $\kappa_v, \lambda_{jt\alpha v}$ are positive. For $t = t_0$ or $v = v_0$, the pullbacks of the row and layer Hermitian forms under ι_t, θ_{jt} equal the definite forms times $\kappa_v \prod_j \lambda_{jt\alpha v}$ and $\lambda_{jt\alpha v}$, respectively, on pairs whose first argument lies in the preimage of the indicated spectral subspace.

Require $\mathrm{lcv}_{X_{W,j}}(M_j)/D_R \le l_{3,j}$ in each reference layer. With $A_{jt} = \theta_{jt}(T_{jt})$ and $e'_{jw} = e_w e^0_{wj}$, impose the three inequalities of 36.8 on the reference data. Require nonnegativity and (36.12) at each place in Tw. Impose (36.14) in each layer. Under the hypotheses of 36.13, the window bound gives constants C'_N, x_0 such that, for every family member at $x \ge x_0$ and every layer j, the weighted sum η_j of the local cut errors over the twisted places in S satisfies $\eta_j \le D_0 C'_N x^\varepsilon (B_0 + N_c)$, where D_0 is the reference row dimension. Apply row 3 of 36.7 beyond the selection and row cutoffs with the common row constant of 36.8. Then $e'_{jw} = R_w/m$.

On the selected side the row frames are invertible. The maps ι_t are bijective and their type images at t_0 are the E_α. Require positive factors and the same rule for the row metric. Put $\lambda_{j,\alpha,v} = \lambda_{j,t_0,\alpha,v}$ and require

$$l_{6,j} \le \log \lambda_{j,\alpha,v} \qquad (j, \alpha, v). \tag{20.3}$$

For each word take a K-linear map $B^{\mathrm{c}}_u : K^{d_u} \to (K^{D_{\mathrm{c}}})^\vee$ and factors $\zeta_{ut} \in K^\times$ with $\zeta_{u,t_0} = 1$ and $v_w(\zeta_{u,t_w}) = 0$. The definite evaluation is

$$P_{u,t}(\phi, F) = \zeta_{ut} B^{\mathrm{c}}_u(\phi)(\iota_t P_t F).$$

At v_0 take a conjugate-linear isometry s_E from the complex row coordinates onto $\mathcal{R}$, of dimension D_{c}. Require a row-frame isometry V and, for each u, a column-frame isometry U_u and $a'_u \in \mathbf{C}^\times$ such that

$$(B^{\mathrm{c}}_u)_{v_0}(x)(y) = a'_u \langle s_E(Vy), P^{\mathrm{per}}_u(U_u x)\rangle.$$

The inner product is conjugate-linear in its first argument.

The selectors and factors $\kappa_v, \lambda_{jt\alpha v}$ agree across the two sides. Off Tw, both $F_{0,j,w}$ and

$$\mathrm{idx}_w(M_j, A_{j,t_w}) + \mathrm{idx}_{\bar{w}}(M_j, A_{j,t_w}) + \mathrm{idx}_w(T_{j,t_w}, T_{j,t_w}^{\vee_{H_{j,t_w}}})$$

vanish in every reference layer. Here $\vee_H$ is the lattice dual for the K-valued Hermitian form H, with integral pairings. Let R_0, R_{c} be the glued row lattices of 18.5, formed from $\iota_{t_w}\Lambda_{t_w}$ on their respective sides. Write $\langle\cdot\rangle_v$ for the mean over infinite places. Put

$$\bar{\kappa} = \langle \log \kappa_v \rangle_v, \qquad \mu_0 = \mu(R_0^\vee) - \bar{\kappa}, \qquad \mu_C = \mu(R_{\mathrm{c}}^\vee).$$

Require $\mu_{\max}(R_0) \le h_{\mathrm{r}} - \bar{\kappa}$. For $0 \le \delta \le 1/3$, require the passage inequalities of Lemma 20.18 below for $R_1 = R_{\mathrm{c}}$, $\lambda = \lambda_{\mathrm{r}}$ and every $U \ge \mu_{\max}(R_0)$.

In the arithmetic construction use on both sides the index sets, lattices and metrics attached to the choice of U, TU in 18.6; that lemma gives the selector condition. The first row of 36.7 gives passage beyond the selection and row cutoffs.

Definition 20.14 (Definite row and layer conditions at denominator s)**.** Let s be a positive integer. The distinguished embedding takes the involution on K to complex conjugation. The set of tuple labels is finite. Each retained K-valued character of the row group G is the character of an irreducible representation over $\overline{K}$.

Impose the following conditions on the row space for every tuple and on the space for every layer and tuple. The coefficient pairing is nondegenerate and Hermitian, and each class stabilizer preserves it. At a class c, write A_c for its full coefficient lattice, $A_c^\vee$ for its dual under this pairing and $\Delta_c \subseteq A_c$ for its specified stabilizer-stable submodule. Require

$$sA_c^\vee \subseteq \Delta_c, \qquad s\Delta_c \subseteq A_c^\vee.$$

Forms take stabilizer-invariant values at the classes. Their pairing is the sum of the coefficient pairings weighted by the reciprocal stabilizer orders. Its Gram matrix in the chosen basis is positive definite at every infinite place. The Gram entries and the matrices of G in this basis belong to the subfield of $a \in K$ satisfying $\sigma(\bar{a}) = \overline{\sigma(a)}$ for every embedding σ. The group preserves the class-sum pairing and both modules of forms defined by taking values in A_c and in Δ_c, respectively.

The ordinary and inert operators together commute pairwise and commute with G. Each operator has split characteristic polynomial over K. The place maps for the ordinary and inert operators are each injective. At each ordinary place P, its norm parameter q_P is positive. The block length is at least one, the list of prescribed blocks is nonempty, and each block parameter $\nu(P)$ is nonzero at every ordinary place.

Definition 20.15 (Division inputs for a word)**.** Use the reference rows and selected columns of Definition 20.13. Fix a word u and the numerical data R of 21.26. Write β_j for the row coefficient at layer j and $w_{j,\rho}$ for the weight at position ρ. Put

$$H = H^* = H_P + \sum_j (H_{B,j} - H_{X,j}), \qquad R_* = \text{the right-hand side of 21.26.}$$

Require $d_u > 0$, invertible column frames at infinity and a full lattice $N \subset K^{d_u}$. Take $\sigma_w \in \mathbf{R}$ with finitely supported floors $\tau_w = \lfloor \sigma_w \rfloor$.

Start the selected column chain of 21.12 in frames $X_{\mathrm{c},v}$. Its maps K_i have invertible source and target frames, and $i(j)$ assigns a step to each layer. The terminal map P_u^{term} goes from the last space, with frames $X_{\mathrm{last},v}$, to the base test dual. Its composite with the chain is P_{u,t_0}. Choose invertible base test frames $X_{0,v}$ with $(X_{0,v}^{-1})^* X_{0,v}^{-1} = v(H_{V,t_0})$. Use the height table of 36.1 for R under the rank, scale and constant conditions of 21.17. Write $h_P = A$ and $k_{j,v} = s_{j,v}$ for its terminal and layer entries. Use the coefficient maps and weights of 36.1. Let a_i count the coefficient groups at step i, let h_i^{cl} bound their cardinalities, and let $t_{\max}(i,v)$ bound the squared norms of the transitions from step i to step $i{+}1$ at v. Put $C_{i,v} = a_i a_{i+1} h_i^{\mathrm{cl}} h_{i+1}^{\mathrm{cl}} t_{\max}(i,v)$. Require

$$\begin{aligned} \|P_u^{\mathrm{term}}\|^2_{X_{\mathrm{last},v} \to X_{0,v}^\vee} \le e^{h_P}, \qquad \log C_{i,v} \le \sum_{j:i(j)=i} k_{j,v} \quad (i,v), \\ \sum_j H_{X,j} \le \sum_j l_{6,j}. \end{aligned} \tag{20.4}$$

For the coefficient bound use the convention $\log 0 = 0$.

For every reference type t and finite place w, choose a basis of Λ_t over the semilocal ring at $w, \bar{w}$. Write $G_{t,w}$ for the quotient Hermitian Gram determinant in this basis as in 21.27, and $G_w = G_{t_w,w}$. Take a finite $S_y \supseteq \mathrm{Tw}$ with $v_w(G_w) = 0$ off S_y. Put $J_{j,w} = \mathrm{idx}_w(T, T^{\vee H_{j,t_w}})$ for the reference layer $T = T_{j,t_w}$. Choose $L_G, E_3, E_{2,j}, g_w, \eta_{w,j}$ satisfying

$$\begin{gathered} v_w(G_w)/D_R \le g_w \quad (w \in S_y), \qquad \sum_{w \in S_y} \ell_w g_w \le L_G, \\ J_{j,w}/D_R \le e_w J_{0,j,w} + \eta_{w,j}/D_R \quad (w \in \mathrm{Tw},\ 1 \le j \le m), \\ \sum_{w \in \mathrm{Tw}} \ell_w \eta_{w,j}/D_R \le E_{2,j} \quad (1 \le j \le m), \qquad \sum_{w \in S_y} 2\ell_w e_w/p_w^2 \le E_3. \end{gathered} \tag{20.5}$$

Set

$$F_* = \sum_j \sum_{w \in \mathrm{Tw}} \ell_w e_w F_{0,j,w}, \qquad B_* = \sum_j l_{3,j} + 2F_* + L_G + \sum_j (E_{1,j} + E_{2,j}) + E_3.$$

Choose $E_{l3}, Q_{\mathrm{pass}}$ with

$$\begin{gathered} \sum_j l_{3,j} \le \sum_{j,\rho} \beta_j w_{j,\rho} \log \Omega + E_{l3}, \qquad E_b := E_{l3} + L_G + \sum_j (E_{1,j} + E_{2,j}) + E_3 \le c_e m x^{\varepsilon}(B_0 + N_c), \\ 0 \le \delta, \qquad 1 \le Q_{\mathrm{pass}}, \qquad 3\delta Q_{\mathrm{pass}} \le 1, \\ B_* + h_{\mathrm{r}} + 28\lambda_{\mathrm{r}} \le Q_{\mathrm{pass}}, \qquad \lambda_{\mathrm{r}} \le \lambda_0, \qquad 28 \le C_0. \end{gathered} \tag{20.6}$$

In the three-sided construction impose the conditions of Definition 20.14 with denominator s. Assume that in every layer the transferred labels of any two distinct base spectral types differ in an ordinary Hecke polynomial or an inert eigenvalue at a place of that layer. Require a stable selection contained in the prescribed blocks. The selected side supplies the chain bounds by 36.4 and the height comparison by 21.21. On the reference side take $S_y = \mathrm{Tw} \cup \{w : v_w(G_{t_0,w}) \ne 0\}$. In a fixed K-basis the nonzero Gram determinant has finite valuation support. The lattice generated by that basis and Λ_{t_0} agree locally off a finite set by commensurability. A semilocal change of basis with determinant c changes the Gram valuation by $v_w(c) + v_{\bar{w}}(c)$. Outside the finite exceptional set and its conjugate this change is zero. Thus S_y is finite and $v_w(G_w) = 0$ off S_y.

Use the positive integer denominators s, d_g and the reference layer denominators $d_{j,t} \mid d_g$ of 21.28. For the row group G assume $\log |G| \le \lambda_0$. At twisted places require the cut exponent $J_{jw}^{\mathrm{cut}} \le e_w J_0(t_w, w, j)$ and the cut lengths a_{jw} with $\sum_{w \in \mathrm{Tw}} \ell_w a_{jw} \le D_R b_j$ in each layer. By the local bounds of 21.28, take $g_w = v_w(|G|s)$ and $\eta_{w,j} = a_{jw} + D_R(v_w(d_{j,t_w}) + v_w(s))$. For a positive integer q, 20.7 gives $\sum_w \ell_w v_w(q) = \log q$ by factoring q into rational primes. Consequently,

$$\sum_{w \in S_y} \ell_w g_w \le \log(|G|s) \le \lambda_0 + \log s, \qquad \sum_{w \in \mathrm{Tw}} \ell_w \eta_{w,j}/D_R \le b_j + \log d_g + \log s.$$

These right-hand sides give L_G and $E_{2,j}$ in (20.5). The denominator $p_w/\sqrt{e_w}$ gives the summand $2\ell_w e_w/p_w^2$, so 21.28 gives $E_3 = 1$. Under 21.30, take $Q_{\mathrm{pass}} = Q_0$ for sufficiently large selected members.

Theorem 20.16 (Four conclusions in coordinates)**.** *Use 20.13 and 20.15. Write $B = B_u^{\mathrm{c}}$ with operator norms in the column and dual selected row frames. At every finite w suppose $P_{u,t_w}(\phi, F) \in \mathcal{V}_{w,\sigma_w}$ for all $\phi \in N$ and all selected $F \in M_{t_w}$, where $\mathcal{V}_{w,\sigma_w}$ consists of zero and elements of valuation at least σ_w. Let N^{σ} be the shift by τ_w from 19.2. Then $B(N^{\sigma}) \subseteq R_{\mathrm{c}}^{\vee}$, and there are $c_v > 0$ with $\|B\|_v^2 \le c_v$ and $\langle \log c_v \rangle_v = H - \bar{\kappa}$. Moreover,*

$$\mu_C - \bar{\kappa} + H - 2F_* \le R_*, \qquad \mu_{X_{\mathrm{c}}}(N^{\sigma}) - \mu_{X_{\mathrm{c}}}(N) = 2\sum_w \ell_w \tau_w.$$

Outline. Use 19.2, 19.7, and 20.5 for fullness, inclusion, and slope. Adjust the log norm bounds from 21.17 and 21.12 to mean $H - \bar{\kappa}$, then use 21.28 and the row budgets and apply 21.26 with the adjusted slope. The complete proof is on p. 614. □

Lemma 20.17 (Identification of the row base term)**.** *Use the finite support S in the arithmetic sum of 21.3. Let ι send each arithmetic place w to the place of the row coefficient field represented by $\mathfrak{p}_w$. Use this bijection and the layer bijection $j \mapsto a(j)$ to $\{1,\dots,m\}$. Assume the reference row is untwisted off* Tw*. If*

$$F_{0,j,y} = F_{a(j)}(\iota^{-1}y) \qquad \text{for all } y, j, \tag{20.7}$$

then $2F_* = \mathrm{Fo}$. On the fixed arithmetic scale (20.7) holds for the weight tuples of 15.8 and the finite base terms of 15.10.

Outline. Use 15.8, 14.1, and 14.9 to match weights and Frobenius orbits, hence the capacities of 15.1 and base terms of 15.10. Sum over the common support via the place and layer bijections, using the doubled weights of 21.3. The complete proof is on p. 615. □

Lemma 20.18 (Passage between character levels)**.** *Let K be totally complex. Let a finite group G of order M act unitarily at infinity on a metrized K-space and preserve a full lattice. Assume the retained irreducible types have scalar endomorphisms and distinct characters. Their central projectors must sum to the identity. For $i = 0, 1$, take a linear K-valued character χ_i on a subgroup of order h_i. In type s let their multiplicities be $1, n_s$, respectively, with*

$$(1-\delta)\bar{n} \le n_s \le (1+\delta)\bar{n}, \qquad \bar{n} > 0, \quad 0 \le \delta \le 1/3. \tag{20.8}$$

Let P_i be the ambient lattice cut by the χ_i-space. Assume the row embedding identifies R_i with a nonzero lattice between h_iP_i and P_i, in the ambient frames multiplied by $\sqrt{h_i}$. For $U \ge \mu_{\max}(R_0)$ and $\lambda \ge \log M$,

$$\begin{aligned} \mu_{\max}(R_1) &\le U + 14\lambda, \\ \mu(R_1^\vee) &\le \mu(R_0^\vee) + 3\delta\big(\mu(R_0^\vee) + U + 28\lambda\big) + 28\lambda. \end{aligned}$$

Outline. Compare row slopes with character cuts using 20.1 and clear projector denominators. Use integral isotypic maps and their adjoints to bound maximum slopes and compare degrees, then use multiplicity bounds for the degree deficit and duality for the mean slope. The complete proof is on p. 615. □

Theorem 20.19 (Return to the original period data)**.** *Fix the arithmetic data and local budgets of 15.16, and constants $C, C_{\mathrm{abs}} \in \mathbf{R}$. For each arithmetic word u choose its selected column and an input of 20.15 for a coordinate lattice $\mathcal{N}_u \subset K^{d_u}$, numerical data R and $\sigma_w = s_{u,w}$ under the place identification. Use the bridges E_u, J_t and period identity of 18.8. For each selected type t and $F \in M_t$ require $J_tF \in V_0(\theta(t))$ and an integral expression for J_tF in the generators $\mathcal{G}_{\theta(t)}$ via the coefficient embedding. Require $\theta(t_w) = \mu_w$ at every finite place, where μ_w is the arithmetic label of 17.28.*

The column isometry E_u identifies the shift of $\mathcal{N}_u$ by the floors of σ_w with N'_u. A row isometry E_R identifies R_{c} with RR. The pairing identity of 19.7 reads $B_u(E_u\phi)(E_Ry) = B_u^{\mathrm{c}}(\phi)(y)$. Assume the period bound of 17.28: for every u, w, every $\phi \in N_u$ and every $F \in \widetilde{M}_{\mu_w} \cap V_0(\mu_w)$, the period of ϕ against F is the image of some $y \in K$ under the coefficient embedding, with $y = 0$ or $v_w(y) \ge s_{u,w}$. Assume also $N'_u = I_uN_u$ for every u, as in 19.1. For each of the reference and two auxiliary labels, and each label μ_w, assume the integral decomposition of 18.7: every vector in $V_0(\mu)$ in the $\mathcal{O}$-span of $\mathcal{G}_\mu$ is a finite $\mathcal{O}$-linear combination of vectors in $\widetilde{M}_\mu \cap V_0(\mu)$, with scalars acting through the coefficient embedding.

Put $a = xe^{x^\varepsilon}$, $n_T = \#TT$ and $L_a = \log(a\,\mathrm{rd}(E))$, with $\mathrm{rd}(E)$ the root discriminant of the host field. For top positions $t \in TT$ with weights a_t, put

$$\mathrm{MAIN}(u) = \sum_{t \in TT} \big[m(\varphi(a_t) - a_t \log\Omega) - a_t \log T_u\big], \qquad \varphi(z) = z\log z - z \ (z>0), \quad \varphi(0) = 0.$$

For the coordinate bounds $H_u, R_{*,u}$ of 20.15 and the recorded arithmetic height bound h_u, assume

$$H_u - \bar{\kappa} = h_u, \qquad 2F_* = \mathrm{Fo}, \qquad R_{*,u} \le \mathrm{MAIN}(u) + Cn_TaL_a + Cm(B_0+N_c)x^{C_{\mathrm{abs}}\varepsilon}. \tag{20.9}$$

On the fixed scale the middle equality follows from 20.17 with the tuples of 15.8 and the base terms of 15.10. Then for every word u,

$$\begin{aligned} B_u(N'_u) \subseteq RR^\vee, \qquad B_u \ne 0 \implies h(B_u) := \langle \log\|B_u\|_v^2\rangle_v \le h_u, \\ \mu(RR^\vee) + h_u - \mathrm{Fo} \le \mathrm{MAIN}(u) + Cn_TaL_a + Cm(B_0+N_c)x^{C_{\mathrm{abs}}\varepsilon}. \end{aligned}$$

Outline. Identify the shift ideal by 15.16, 19.1, 19.2 and 20.3, then bound and transfer valuations by 18.7, 17.28, 18.8 and 20.16. The complete proof is on p. 616. □

Part V

Two bounds on one slope

Chapter 21

The lattice after division

Fix $x > 1$, $0 < \varepsilon < 1/52$, and a translation u_0 with finite word set $W(u_0)$. The host CM field is E, with $f = [E : \mathbf{Q}]/2$ column positions ρ. The coefficient field K is a totally complex number field with a distinguished infinite place v_0, given by an embedding σ_0, and a ring involution $a \mapsto \bar{a}$ satisfying $\sigma_0(\bar{a}) = \overline{\sigma_0(a)}$; the field K itself need not be a CM field. For a finite place w of K, write $\mathfrak{p}_w, p_w, e_w$ for its prime ideal, residue characteristic and ramification index. Let S be the finite support of the local data, and put

$$\ell_w = \frac{\log N\mathfrak{p}_w}{[K : \mathbf{Q}]}, \qquad c_w = 2e_w\ell_w.$$

Write $\langle\cdot\rangle_\infty$ for the mean over infinite places. In the metrics of Part IV, a nonzero map B has height $h(B) = \langle \log \|B\|_v^2 \rangle_\infty$. Slope differences use twice the normalized Arakelov convention. The height $h(B)$ is twice the archimedean contribution to the height of a linear map in [77, §2.6].

For a word u, let N_u^0 denote the input lattice N_u of 19.1 and let I_u be the fractional ideal of 21.5. Set $N_u' = I_u N_u^0$. Let M_u be the Hermitian lattice of column forms for u before the first lift. Write $D_{\Gamma,u}$ for the stabilizer denominator denoted by M_u in 15.12, so $\delta_{u,w} = v_{p_w}(D_{\Gamma,u})$. The K-linear column inclusion identifies the word space with the subspace of the zeroth column killed by all marked diamond sums. Let $M_{H,u}$ be the inverse image of M_u under this inclusion, with the metric of N_u^0. If V_u, W are the spans of N_u', RR, the period map is $B_u : V_u \to W^\vee$. Put $C^\vee = RR^\vee = \{\ell \in W^\vee : \ell(RR) \subseteq \mathcal{O}_K\}$ and $\mu_C = \mu(C^\vee)$.

Use

$$L = x^\varepsilon, \quad b = e^L, \quad a = xb, \quad B_0 = fmbL, \quad L_a = \log(a\,\mathrm{rd}E), \quad n_T = \#TT, \quad S_a = \sum_{t \in TT} a_t.$$

Here $\mathrm{rd}E$ is the root discriminant. The set TT consists of the top positions, with integer top weights $a \le a_t \le 2a$. The chain has m lifts and starts at rank k. Its ranks are $N_j = k + j(t_{\mathrm{ch}} + 1)$ for $0 \le j \le m$, with t_{ch} as in 15.12. Write T_u for the word's level prime, and use the auxiliary size b' and prime cutoff X'' of the window.

21.1 Prime powers and finite-place budgets

Definition 21.1 (The required prime-power sum)**.** For the host group A, set

$$G = (\mathbf{Z}/2\mathbf{Z})^2 \times A, \qquad \Phi = \{((e_1, e_2), t) \in G : e_1 = 0\}, \qquad r_t(g) = ((1,1),0) + g + ((0,0),t).$$

Let F_p be the chosen Frobenius element and $I_p \le G$ the inertia subgroup. The exceptional primes are $q_1 = q$ and d. For $N \in \mathbf{N}$, $a_t \in \mathbf{R}$ and $w_{u_0 j} : G \to \mathbf{R}$, put $[z]_+ = \max(z, 0)$ and

$$A_{tj}^{(N)} = \sum_{\substack{p \le N,\ p \text{ prime} \\ p \notin \{2, q_1, d\} \\ F_p \notin \Phi}} 2 \log p \sum_{\substack{1 \le i \le N \\ iF_p \notin \Phi}} \left[\frac{a_t}{p^i} - w_{u_0 j}(r_t(iF_p)) \right]_+ + \log d \sum_{i=1}^{N} \sum_{\substack{g \in iF_d + I_d \\ g \notin \Phi}} \left[\frac{a_t}{d^i} - w_{u_0 j}(r_t(g)) \right]_+ .$$

Then $\mathrm{ASK}(u_0) = \sum_{t \in TT} \sum_{j=0}^{m-1} A_{tj}^{(N)}$, with j denoting layer $j+1$.

Lemma 21.2 (Cutoff for a positive partner weight)**.** *In Definition 21.1, if the partner weight is at least one and $a_t \le 2a$, every strictly positive summand has both its prime and its exponent at most $\lceil 2a \rceil$.*

Proof. The first binary coordinate forces i odd in the ordinary-prime sum. A positive term with partner weight $w \ge 1$ requires $p^i < a_t/w$, with $p = d$ in the exceptional sum. Thus $N = \lceil 2a \rceil$ includes all positive terms for such weights. □

Definition 21.3 (Local multiplicities and their global totals)**.** Use the gain g_{jw}, base term F_{jw} and loss E_{ujw} of layer $1 \le j \le m$ from Parts III and IV. Set $\mathrm{gainD}_j(w) = 0$ when p_w belongs to the discarded set $\mathcal{D}$ of 15.14, and $\mathrm{gainD}_j(w) = g_{jw}$ elsewhere. Put

$$\mathrm{sg}(u, w) = \sum_j (\mathrm{gainD}_j(w) + F_{jw} - E_{ujw}).$$

For the containment budget τ_w and excess colength io_w of 14.9, define

$$\mathrm{Ga} = \sum_w c_w \left(\sum_j \mathrm{gainD}_j(w) - \tau_w \right), \qquad \mathcal{T} = \sum_w c_w \tau_w,$$
$$\mathrm{Fo} = \sum_{j,w} c_w F_{jw}, \qquad \mathrm{Ee}_u = \sum_{j,w} c_w E_{ujw}, \quad \mathrm{IO} = \sum_w e_w \ell_w \, \mathrm{io}_w.$$

The place sums are over S.

Lemma 21.4 (Sum of the local budgets)**.** *The totals* Ga, Fo *are common to the words, and* $\sum_w c_w \mathrm{sg}(u, w) = \mathrm{Ga} + \mathrm{Fo} - \mathrm{Ee}_u + \mathcal{T}$.

Proof. The gain, containment and base terms in Definition 21.3 are independent of the word. Interchange the finite layer and place sums and add back $\mathcal{T}$. □

Definition 21.5 (The ideal prescribed by period valuations)**.** At w use the selected character μ_w, row lattice $\widetilde{M}_{\mu_w}$, test subspace $V_0(\mu_w)$ and column inclusion j_N of Part IV. Normalize v_w to be one on uniformizers. The unrounded bound requires that for every $\phi \in N_u^0$ and $F \in \widetilde{M}_{\mu_w} \cap V_0(\mu_w)$, the column period at character μ_w, evaluated on $j_N \phi$ and F, be the image under the chosen embedding of some $y \in K$ with $y = 0$ or $v_w(y) \ge e_w \mathrm{sg}(u, w)$. Put

$$s_{uw} = \lfloor e_w \mathrm{sg}(u, w) \rfloor, \qquad I_u = \prod_{w \in S} \mathfrak{p}_w^{-s_{uw}}.$$

By 19.1, the ideal of Definition 21.5 gives $N_u' = I_u N_u^0 = \mathrm{span}_{\mathcal{O}_K}\{a\phi : a \in I_u,\ \phi \in N_u^0\}$ for every word u. The unrounded bound gives the threshold s_{uw}. 17.28 gives the rounded bound and 18.8 extends it to coordinate rows. The shift of 19.2 is N_u' by 19.3. For the pairing in 19.6, 19.7 gives $B_u(N_u') \subseteq RR^\vee$.

21.2 The two slopes of the divided lattice

Lemma 21.6 (The exact change of mean slope). *Let $N \subseteq M$ be lattices of rank $d > 0$ in the same Hermitian space over K, with M finitely generated over $\mathcal{O}_K$. Write $\lambda_w = \mathrm{length}_w(M_{\mathfrak{p}_w}/N_{\mathfrak{p}_w})$. Then*

$$\mu(N) = \mu(M) - \frac{2}{d}\sum_w \ell_w \lambda_w.$$

If $N_u^0 \subseteq M_{H,u}$ have quotient supported on S, write $\ell_{uw}^{\mathrm{col}} = e_w^{-1}\,\mathrm{length}_{\mathcal{O}_{K,\mathfrak{p}_w}}((M_{H,u}/N_u^0)_{\mathfrak{p}_w})$ for its normalized local length. For $d_u = \mathrm{rk}_K N_u' > 0$,

$$\mu(N_u') = \mu(M_{H,u}) + 2\sum_{w\in S} \ell_w s_{uw} - \frac{2}{d_u}\sum_{w\in S} \ell_w e_w\, \ell_{uw}^{\mathrm{col}}.$$

Proof. The covolume ratio is $\prod_w (N\mathfrak{p}_w)^{\lambda_w}$, by the cardinalities of the local composition factors. Divide its negative doubled logarithm by $d[K:\mathbf{Q}]$. Apply the formula to $N_u^0 \subseteq M_{H,u}$ and use 20.2 and 20.3 for the twist by I_u. □

Lemma 21.7 (A local comparison for the maximum slope). *Suppose $N_u^0, M_{H,u}$ span the same space of dimension $d_u > 0$, and $(N_u^0)_{\mathfrak{p}_w} \subseteq \mathfrak{p}_w^{k_w}(M_{H,u})_{\mathfrak{p}_w}$ at every finite place, with $k_w \in \mathbf{N}$ supported on S. Then*

$$\mu_{\max}(N_u') \le \mu_{\max}(M_{H,u}) + 2\sum_{w\in S} \ell_w(s_{uw} - k_w).$$

Proof. For $J = \prod_{w\in S}\mathfrak{p}_w^{k_w}$, the hypotheses give $N_u' \subseteq I_u J M_{H,u}$ with its induced metric. Apply 20.2 after dividing each sublattice by $I_u J$. Take the supremum. □

Lemma 21.8 (Numerical conditions at a split level). *Fix a word u. Put $T = T_u$, $N = N_m$, $a = xb$, and let $a_{\max}$ be the maximum of zero and the leading marked weights of this word. For $0 \le j \le m$, write $\lambda_{j,\tau,i}$, $0 \le i < N_j$, for the integer highest weight at type τ. The level conditions on these weights are*

$$\begin{gathered} 2fN + 1 < T, \qquad 4a_{\max} < T, \\ 2(\lambda_{j,\tau,i} - \lambda_{j,\tau,h} + N_j) < T \quad (0 \le j \le m,\ 0 \le i, h < N_j), \\ -\lambda_{0,\tau,k-1} \ge 0 \qquad (\tau \text{ marked}). \end{gathered}$$

They hold whenever $x > 1$, $m \ge 4$, $1 \le f \le N$, $3N \le b$, $24x \le b$, $0 < b' \le b$, $T \ge b^{m/2}$, and the weights satisfy

$$a_{\max} \le 2xb, \qquad -(2b + b') \le \lambda_{j,\tau,i} \le 2xb + (b')^2/64, \qquad \lambda_{0,\tau,k-1} \le 0 \quad (\tau \text{ marked})$$

with the last inequality required only at the marked types and the two-sided bound required at every type.

Proof. We have $T \ge b^2$, $b > 24$ and $2fN + 1 \le 2N^2 + 1 < 9N^2 \le b^2$. Also $4a_{\max} \le 8xb \le b^2/3$. At any two indices of the same weight the two-sided estimate gives

$$2(\lambda_{j,\tau,i} - \lambda_{j,\tau,h} + N_j) \le 4xb + b^2/32 + 8b \le (1/6 + 1/32 + 1/3)b^2 < b^2.$$

Here $N_j \le N \le b$. The final sign condition is the bottom-entry hypothesis. □

Lemma 21.9 (Cost at the level prime). *Use the split level models with the trivial class stabilizers of 37.21 and the weight inequalities of 21.8. Use the compatible symmetric column system and the finite-order level-zero character with values in the embedded coefficient field of item 5 of 33.77. At the level prime use the local lattice prescription of 17.10. For a selected word at type ρ, the first k integral weight coordinates are $U_i - (k-1)/2 + i$, numbered from zero. Lift $q+1$ appends $t_{\mathrm{ch}} + 1$ copies of $s_{q+1} - w_{q+1,\rho}$, with s_j as in 21.24. At a mark $\rho = (0, s)$ the last initial coordinate is $-b_s$. Assume $a_{\max} \le 2a$, $k \le N_m \le b'$, $T_u - 1 > 2f$, and, at every active marked type, $b_v \le 2b'$ and $W_{u,\tau} \le 3b'$ in the notation of 15.12, where $b_v = -\lambda_{0,\tau,k-1}$.*

Put $T = T_u$, $H = T - 1$, let d be the dimension of the common kernel of the marked diamond sums in the zeroth column, and let $d_M = \dim_{\mathbf{C}} M^0_{u,\mathbf{C}}$ be the zeroth column dimension. Then $d_M > 0$, $d > 0$, and

$$d = d_M(1 - 1/H)^f, \qquad \mu(M_{H,u}) = \mu(M_u) - \frac{f \log H}{H - 1}, \qquad \mu_{\max}(M_{H,u}) \le \mu_{\max}(M_u).$$

The containment of 21.7 holds at $w \mid T$ with exponent zero. For every finite $F \subseteq \{w : w \mid T\}$,

$$\sum_{w \in F} \frac{2\ell_w e_w \, \ell^{\mathrm{col}}_{uw}}{d} \le 36 n_T m(a/k + b') \log T.$$

Outline. Theorem 37.21 and pro-g_i kernels at two primes show the diamond action is free, yielding the rank and slope formulas. The up and down operators map between the local lattices S_0 and B_v, preserve the common diamond kernel, and satisfy $U_v D_v = T^{k-1} I$ there by an incidence identity. Additivity of one-step colengths over ordered rounds, Bernoulli's inequality, and Lemma 20.7 give the final bound. The complete proof is on p. 617. □

Remark 21.10. In the selected family, the endpoint bounds of 21.34 and $s_j \le w_{j\rho} \le 2b$ of 21.24 give $|\lambda_{j,\rho,i}| \le 2a$, hence $4a_{\max} \le 8a$ and $2(\lambda_{j,\rho,i} - \lambda_{j,\rho,h} + N_j) \le 8a + 2N_m$, while 37.10 and 38.3 give $m \ge 4$ and $T_u \ge b^{m/2} \ge b^2 > \max\{8a + 2N_m, 2fN_m + 1\}$ eventually.

Theorem 21.11 (Mean and maximum slopes after division)**.** *Fix u and use the comparisons of 21.6–21.9, with $d = d_u > 0$, $d_M > 0$, $T_u > 2$, $k_w = e_w \tau_w$. Put $\delta = 1 - d/d_M$, $S_T = \{w \in S : p_w = T_u\}$, and*

$$\sigma'_w = s_{uw}/e_w, \quad R_u = \sum_{w \in S} c_w(\mathrm{sg}(u,w) - \sigma'_w), \quad P_u = \sum_{w \in S_T} c_w \, \ell^{\mathrm{col}}_{uw}/d.$$

Assume $\tau_w \in \mathbf{N}$, $\mathrm{io}_w \ge 0$, $\tau_w = 0$ on S_T, and $\ell^{\mathrm{col}}_{uw} \le d_M(\tau_w + \mathrm{io}_w)$ off S_T. Suppose

$$\mathcal{T} \le b^2, \quad 2f/T_u \le 1/2, \quad 4fb^2/T_u + f \log(T_u - 1)/(T_u - 2) \le 1,$$
$$P_u \le c_1 n_T m(a/k + b') \log T_u, \quad R_u \le c_1 X'', \quad c_1 \ge 4, \quad B_0, \mathrm{IO} \ge 0.$$

For $J_u = \mathrm{Ga} + \mathrm{Fo} - \mathrm{Ee}_u$ and $\mathrm{Lo}_u = c_1 \mathrm{IO} + c_1 n_T m(a/k + b') \log T_u + c_1 X'' + 1$, one has

$$\mu(M_u) + J_u - \mathrm{Lo}_u \le \mu(N'_u), \qquad \mu_{\max}(N'_u) \le \mu_{\max}(M_u) + J_u + c_1 m B_0.$$

Outline. The exact mean comparison of 21.6 and 21.9 subtracts the rounding and colength losses. Bernoulli's inequality bounds the rank loss, and the bound on $\mathcal{T}$ controls the additional weighted colength off the level prime. For the maximum slope, use the containment of 21.7 and the nonnegativity of the rounding loss. The complete proof is on p. 618. □

21.3 The height of a chain

Theorem 21.12 (The archimedean chain estimate)**.** *Let q map the test space to its Hermitian quotient Q, and let $\iota : Q \to W$ be the row isomorphism at μ_0. Suppose the spectral spaces Q_p, $p \in I$, span Q, with I finite and nonempty. For $z \in Q_p$, $y \in Q$ suppose*

$$\langle \iota z, \iota y \rangle_v = c_{pv} \langle z, y \rangle_v, \qquad c_{pv} = \kappa_v \prod_{j=1}^{m} \lambda_{jpv}, \qquad \kappa_v, \lambda_{jpv} > 0. \tag{21.1}$$

Suppose $B_u \ne 0$ and $(\iota q)^\vee B_u$ factors as kernel maps K_i followed by P^{term}_u, with layer j assigned to $i(j)$. At each infinite place v, assume

$$\|P^{\mathrm{term}}_u\|^2_v \le e^{h_P}, \qquad \|K_i\|^2_v \le \exp\left(\sum_{j : i(j) = i} h_{K,jv} \right).$$

Then

$$\log\|B_u\|_v^2 \le z_v = h_P + \sum_j h_{K,jv} - \log\kappa_v - \min_{p\in I}\sum_j \log\lambda_{jpv}.$$

If $\sum_j L_{6,j} \le \langle \min_{p\in I}\sum_j \log\lambda_{jpv}\rangle_\infty$, *then*

$$h(B_u) \le \langle z_v\rangle_\infty \le h_P + \sum_j \langle h_{K,jv}\rangle_\infty - \sum_j L_{6,j} - \langle\log\kappa_v\rangle_\infty.$$

In the tables of 36.1–36.4 and the realization of 36.7, take the common size constant C_{sz}. *At an embedding* $\sigma : K \hookrightarrow \mathbf{C}$, *the row metric scalar is* $\kappa_{[\sigma]}$, *where* $[\sigma]$ *is the infinite place of* σ. *Let* x_1 *be the family cutoff. At each* $x > x_1$ *and each index* $i \in I_x$, *the construction assigns every word* u *its height allowance* h_u *by the equality below. Whenever* $B_u \neq 0$, *the period satisfies*

$$h(B_u) \le h_u = H_{P,u} + \sum_j H_{K,uj} - \sum_j L_{6,j} - \langle\log\kappa_v\rangle_\infty.$$

Use the width S_c, *the block allowance* η, *and the degrees* $\nu_{uj\rho i}$ *of 15.8–15.12. Put* $n_V = k + m - 1 + t_{\mathrm{ch}}$, $v_E = \mathrm{rd}E/2$, *and* $\varphi(t) = t\log|t| - t$ *for* $t \neq 0$, *with* $\varphi(0) = 0$. *For the quadratic character* χ_d, *put* $D_1 = \mathrm{Re}(L'(1,\chi_d)/L(1,\chi_d))$ *and* $\log\Omega = \gamma + \log(\pi/v_E) - D_1$. *The budgets are*

$$\begin{aligned}
H_{P,u} &= -S_a\log T_u + C_{\mathrm{sz}}\big(n_T b' N_m + f(m\eta + S_c)N_m + fN_m^2\big)L_a,\\
H_{K,uj} &= \sum_\rho K_{uj\rho} + C_{\mathrm{sz}} f N_m^2 (L_a + m^3 L),\\
K_{uj\rho} &= \sum_{i=0}^{N_{j-1}-1} \big(\varphi(\nu_{uj\rho i}) - \nu_{uj\rho i}\log\Omega\big),\\
L_{6,j} &= \sum_\rho \log\mathrm{Hook}(a_{j\rho}^+) - C_{\mathrm{sz}} f n_V^2 L', \qquad L' = \log(b\,\mathrm{rd}E\, g_0 g_1 n_V).
\end{aligned}$$

Here g_0, g_1 *are the fixed theta primes, and* $a_{j\rho,i}^+ = \kappa_{j,\rho,i}^{\mathrm{big}}$ *of 15.10 for* $0 \le i < k + j - 1$. *For a partition* α *with* p *parts including trailing zeros,*

$$\log\mathrm{Hook}(\alpha) = \sum_{i=0}^{p-1}\log\Gamma(\alpha_i + p - i) - \sum_{0\le i<l<p}\log(\alpha_i - \alpha_l + l - i).$$

Outline. The spectral row weights bound the smallest squared singular value of the dual row map. Multiply the terminal and kernel norm bounds from 21.17, then average the logarithms and use the spectral lower bound of 21.21. The complete proof is on p. 619. □

Lemma 21.13 (The last branching map and the class sum)**.** *Let* $N = n + 1$. *At each type* τ *let* λ, σ *be dominant integral weights of ranks* N, n *with common determinant shift and* $\lambda_i \ge \sigma_i \ge \lambda_{i+1}$. *Put*

$$A_i = \sigma_i - \lambda_{i+1}, \quad B_i = \lambda_i - \sigma_i, \quad D_{0,\tau} = \sum_i A_i, \quad n_\tau = |\lambda| - |\sigma| = \lambda_1 - D_{0,\tau},$$

where bars denote sums of signed entries. In the Fock variables let P_i *be the leading minor of size* i *and* Z_i *the same minor with row* i *replaced by row* N. *After removing the determinant shift put*

$$h_\lambda = \prod_i P_i^{A_i + B_i}, \quad \xi = \prod_i P_i^{A_i} Z_i^{B_i}, \quad \kappa_\tau = \|\xi\|_F^2 / \|h_\lambda\|_F^2.$$

The Fock metric has orthogonal monomials of squared norm $\|X^\alpha\|_F^2 = \prod_s \alpha_s!$. *Set*

$$c_\tau = \prod_{i=1}^{n}\prod_{v\in I_i\setminus\{\sigma_i\}}\prod_{j=1}^{n}(v - i - \sigma_j + j), \qquad I_i = [\lambda_{i+1}, \min(\lambda_i, \lambda_{i+1} + D_{0,\tau})] \cap \mathbf{Z}.$$

With $|t_\tau|^2 = T > 0$, *the squared branching norm in unit coefficient frames is* $C(T) = T^{-\sum_\tau n_\tau}\prod_\tau c_\tau^2\kappa_\tau$. *In inverse-stabilizer class metrics the associated period on* ψ, F *has absolute square at most* $C(T)\|\psi\|^2\|F\|^2$.

Outline. Extract the last-row coefficient and use multiplicity-one branching to identify the highest vector. Tensor the adjoint norm identities, scale by the CM generator of 15.12, and apply Cauchy–Schwarz to the coefficient pairings and weighted class sum. The complete proof is on p. 620. □

Lemma 21.14 (Vanishing of the constant terms of the level kernel)**.** *Let E/F be the CM extension of the selected realization, and let V, W be its Hermitian spaces of dimensions n (definite) and r with $\mathcal{O}$-unimodular Gram matrices J_V, J_W in the prescribed split frames. Suppose at one marked split place the finite Schwartz function is supported on matrices $X \in \mathrm{Mat}_{r,n}(\mathcal{O})$ whose first r columns are invertible modulo the maximal ideal. For the reference theta kernel, every lift from the definite class space has zero constant term along every proper rational parabolic of $\mathrm{U}(W)$.*

Outline. Average along a rational isotropic line and use Heisenberg covariance under transvections to eliminate nonconstant Fourier modes. Residue-field modulations at the marked split place eliminate the constant mode, so the parabolic constant term vanishes. The complete proof is on p. 620. □

Lemma 21.15 (The spherical integral for the self-dual lattice vector)**.** *Assume the unramified spherical GL_r doubling identity of 1.20. Let w be a finite unmarked place of F in the reference tuple, let $Q_w = Nw$, and give the stabilizer K_w of the self-dual lattice in $\mathrm{U}(W)(F_w)$ volume one. Put $n = r + t$, $t \geq r + 5$, and $u = (1 + t)/2$. Let $\phi^\circ_w(g)$ be the Weil coefficient of the lattice vacuum divided by its value at the identity, in the convention linear in the translated vector. Let F_1 be a K_w-fixed simultaneous Hecke eigenvector in a unitary representation with untwisted base-change Satake multisets B_P, $P \mid w$. Let ℓ be a linear functional bounded on its translates. Write $R(g)$ for translation, χ_P for the splitting character of V evaluated at a uniformizer at P, and $\epsilon_w = 1$ in the split case, -1 in the inert case. Then, with*

$$d_w = \prod_{h=t+1}^{n} (1 - \epsilon_w^h Q_w^{-h})^{-1},$$

one has the absolutely convergent identity

$$\sum_{gK_w} \phi^\circ_w(g)\, \overline{\ell(R(g)F_1)} = \frac{1}{d_w} \prod_{P \mid w} \prod_{\beta \in B_P} (1 - \chi_P^{-1}\beta(NP)^{-u})^{-1}\, \overline{\ell(F_1)}. \tag{21.2}$$

Outline. Match the split and inert Cartan vacuum coefficients to the spherical doubling section. Apply the unramified doubling formula of Eischen, Harris, Li and Skinner to the Hecke eigenvector, using the trivial spherical representation to majorize the coset sum. The complete proof is on p. 622. □

Lemma 21.16 (The full theta lift and its projection at a coefficient embedding)**.** *Fix $1 \leq j \leq m$ at μ_0 in a normalized primitive realization with the standard definite lattice, an admissible choice of member parameters, and the extremal vectors and level action of 37.24. Assume the rapid-decay, Siegel–Weil, doubling, finite-length and formal-degree inputs of 1.20. The twist-prime set at μ_0 is empty. Each marked prime P is prime to $Q_S q d\, N(\mathfrak{c}_K)$, where Q_S is the excluded-prime product and q, d are the auxiliary primes. The host datum specifies $\mathfrak{c}_K$ as a nonzero integral ideal of $\mathcal{O}_{K_d}$. Put $r = k + m - 1$, $t = t_{\mathrm{ch}}$, $n = n_V = r + t$. Assume $t \geq r + 5$, $k \geq 5$, $r \leq 2k$, $n \geq 8$, $g_0, g_1 \geq (kb)^8$. At each real label the shift satisfies $0 < e_j \leq 2b$ and the cloud weights are in $[0, b]$. The auxiliary entries satisfy $|b_l - e_j| \leq b$.*

Let $\sigma : K \hookrightarrow \mathbf{C}$ agree with the distinguished embedding on K_d. At ρ, let τ be the label induced by σ, and put $p_\rho = k + j - 1$, $q_\rho = m - j$. With indices starting at zero, the two partitions are

$$a^+_{\sigma,\rho,i} = \kappa^{\mathrm{big}}_{j,\tau,i} \quad (0 \leq i < p_\rho), \qquad a^-_{\sigma,\rho,i} = -\kappa^{\mathrm{small}}_{j,\tau,q_\rho-1-i} \quad (0 \leq i < q_\rho).$$

Fix the conjugate splitting characters, the prescribed lowest-type kernel and principal level at the marked places on the rank-r side. Let Y_σ be the multiplicity space of the prescribed discrete series in cusp forms at this level. For a system c of spherical eigenvalues off the marked places, put $Y_\sigma(c) = \bigcap_A \ker(A - c(A)I)$, over spherical double-coset operators A. There is a full lift T to Y_σ, and its projection $S = eT$ by the retained-type projector is the prescribed projected lift. For $Y_\sigma(c) \neq 0$ one has

$$TT^* z = l_c z \quad (z \in Y_\sigma(c)), \qquad \log l_c \geq \sum_\rho \log\big(\mathrm{Hook}(a^+_{\sigma,\rho})\,\mathrm{Hook}(a^-_{\sigma,\rho})\big) - 8fn^2L', \qquad l_c > 0.$$

In particular, for $H_j^+ = \sum_\rho \log \mathrm{Hook}(a_{j\rho}^+)$ *and* $D_j = H_j^+ - 8fn^2L'$, *one has* $l_c \geq e^{D_j}$ *and* $\|Sx\|^2 \geq e^{D_j}\|x\|^2$ *for* $x \perp \ker S$.

Outline. Construct the cuspidal lift using 21.14. Unfold its adjoint pairing by 34.16, use 21.15 and the hook formula 21.12 to compute the scalar, then bound the mass, Euler and formal-degree factors and compress to the retained types. The complete proof is on p. 623. □

Lemma 21.17 (Terminal and kernel budgets)**.** *Let* $\mathcal{P}$ *be the set of types in the height table of 36.1. Assume* $N = N_m \geq k+m,\ k,m \geq 1,\ |\mathcal{P}| \leq f,\ 1 \leq L \leq L_a$ *and the table's lower bounds on* $C_h = C_{\mathrm{sz}}$. *Its terminal and layer entries satisfy* $A \leq H_{P,u},\ \langle s_{jv}\rangle_\infty \leq H_{K,uj}$. *In the frames of 36.4,*

$$\|P_u^{\mathrm{term}}\|_v^2 \leq e^A, \qquad \|K_i\|_v^2 \leq \exp\left(\sum_{j:i(j)=i} s_{jv}\right).$$

Outline. Sum the terminal table bounds with the level term kept exact and use 21.13 in the selected evaluation. Average the kernel table using 21.22 and 21.12, then apply weighted Cauchy–Schwarz to the frame coefficients. The complete proof is on p. 627. □

Lemma 21.18 (The coefficient of a definite class)**.** *Fix a layer of the reference tuple of 37.24. Let* E *be the CM field of the unitary pair,* $L_0 = \mathcal{O}_E^n$ *the standard definite lattice, and* $r = k+m-1$ *the rank on the other side. For a definite class* a, *let* L_a^h *be the lattice of its adelic representative. The lattice used in the theta section is* $L_a = \overline{L_a^h}$, *with conjugation in the standard coordinates. Choose its clearing integer* $N_a > 0$, *so that* $N_aL_0 \subset L_a \cap L_0$, *and put*

$$m_a = [L_a \cap L_0 : N_aL_0]^r, \qquad Z_a = [L_a : N_aL_0]^r, \qquad I_a = [L_a : L_a \cap L_0]^r.$$

At each moving split pair choose the half specified by the representative. Let d_a *be its determinant on those halves, extended by one elsewhere. Write* ϵ_a *for the coefficient of this class in the algebraic theta section, before the lattice average and the class measure. If* $\widetilde{\sigma}$ *is the embedding of the comparison field over* σ, *and* $\chi_{W,\sigma}$ *is the splitting character on the target side, then*

$$\widetilde{\sigma}(\epsilon_a)\,\frac{Z_a}{m_a} = \chi_{W,\sigma}(d_a)^{-1}\sqrt{I_a}.$$

Outline. Use split elementary divisors and the odd rank from Theorem 38.3 to compute the lattice index and the rational factor in the normalization of 37.24. Evaluate the determinant value under the embedding and substitute into the class coefficient. The complete proof is on p. 628. □

Lemma 21.19 (The rational projected lift and its analytic realizations)**.** *At the reference tuple of 37.24, fix layer* j, *with zero-based index* $j-1$. *Let* X *be its* K*-space of definite class forms and* $Y = K^{n_W}$ *the coordinate space of its rational cusp frame. Write* j_σ, J_σ *for their semilinear realizations, and use the subscript* 0 *for the distinguished embedding. With base phase one there is a unique* K*-linear map* $\theta : X \to Y$ *satisfying* $J_0\theta = S_0j_0$. *At each embedding* σ *agreeing with the distinguished embedding on* K_d, *there are a unitary component multiplication* U_σ, *a central level translation* g_σ, *and a scalar* $|\xi_\sigma| = 1$ *such that*

$$U_\sigma J_\sigma \theta x = \xi_\sigma g_\sigma S_\sigma j_\sigma x \qquad (x \in X). \tag{21.3}$$

Here S_σ *is the projected lift for the conjugate splitting characters and the prescribed extremal tensor.*

Outline. Project the rational class expansion of 37.24 using 21.16. At each embedding, match Fourier weights by central translation, vacuum coefficients by 21.18, and extremal polynomials up to phase by Schur's lemma, then apply the projector. The complete proof is on p. 629. □

Lemma 21.20 (The two Gram forms on a rational spectral part)**.** *Fix a layer* $1 \leq j \leq m$ *at the reference tuple* μ_0 *of the primitive realization in 36.7. Use the phase-one lift* $\theta : X \to Y$ *of 21.19 and the sizes of 21.16. Write* $Z_j = K^{d_j}$ *for the reference layer coordinates and* $B : Z_j \to X$ *for their inclusion as definite class forms by the orthogonal section of the layer quotient. Let* N *be the order of the reference level subgroup.*

A spherical Hecke system is carried by $p \in I$ if its dual Hecke space on the initial row is nonzero and the reference row adjoint takes that space into the complex span of the initial inert type indexed by p. Let W_{jp} be the sum, at the distinguished embedding, of the retained layer spaces of the systems carried by p. Put

$$Z_{jp} = \{x \in Z_j : S_0 j_0 B x \in W_{jp}\}.$$

Here j_0 realizes a definite class form and S_0 is the projected lift. In particular Z_{jp} is a K-space.

For $v \neq v_0$, let σ represent v. Use the fixed rational class-form basis (e_a) of the reference realization, with each class weighted by the reciprocal of its stabilizer order, and let H be its Gram matrix. Its realization compatibility is

$$\langle j_\tau e_a, j_\tau e_b \rangle = \tau(H_{ab})$$

for every embedding τ. All coordinate columns in the following formula use this basis. Under a change of basis P, the realized Gram matrix at τ is $\overline{\tau(P)}^{\,t}\,\tau(H)\tau(P)$. Define

$$M_v(x,x') = \frac{1}{N}\,\overline{\sigma(Bx)}^{\,t}\sigma(H)\sigma(Bx'), \qquad y_v(x) = \frac{1}{\sqrt{N}}\,J_\sigma \theta B x.$$

The paired row at j,p,v consists of the two forms $\langle y_v(x), y_v(x')\rangle$ and $M_v(x,x')$, with $x \in Z_{jp}$ and $x' \in Z_j$. Call the part occurring at v if $M_v(x,x) \neq 0$ for some $x \in Z_{jp}$.

Suppose that $j_\tau B Z_j \subseteq (\ker S_\tau)^\perp$ for each embedding τ agreeing with the distinguished embedding on K_d. On an occurring part, take an ordered K-basis of Z_{jp} and let b be its first member with $M_v(b,b) \neq 0$. Then

$$\lambda_{jpv} = \frac{\|y_v(b)\|^2}{M_v(b,b)} > 0, \qquad \langle y_v(x), y_v(x')\rangle = \lambda_{jpv} M_v(x,x') \quad (x \in Z_{jp},\ x' \in Z_j).$$

This determines the same scalar for every choice of basis. On an empty part both forms vanish on $Z_{jp} \times Z_j$, and the selected scalar is $\lambda_{jpv} = \exp L_{6,j}$ at the fixed scale of 21.12.

Outline. Use 36.7 to isolate the rational Hecke part. The inert vacuum transfer 34.18 and the split triangular relations give a common target Hecke system, so the scalar of 21.16 relates the two Gram forms by adjunction. The complete proof is on p. 634. □

Lemma 21.21 (Subtraction on the base spectrum)**.** *Use a normalized primitive realization with the separated types and level-one Hecke decomposition of 33.74. Assume the hypotheses of 21.16 and the class expansion and level action of 37.24. Use the definite block conditions of Definition 20.14. Equip the totally complex coefficient field K with its ring involution $a \mapsto \bar a$ compatible with complex conjugation at the distinguished infinite place v_0 (so $\sigma_0(\bar a) = \overline{\sigma_0(a)}$ at the base embedding σ_0, and $M^* = \bar M^{\,t}$), and let $K_0 = \{a \in K : \tau(\bar a) = \overline{\tau(a)}$ for all $\tau : K \hookrightarrow \mathbf{C}\}$ be its maximal CM subfield. For the surjective reference block quotient P, let H_1 be the block Gram matrix, E_1 its orthogonal section and $G_1 = E_1^* H_1 E_1$. Require H_1, G_1 to have entries in K_0, with $\sigma(H_1) > 0$ at every embedding. At the representative σ of each infinite place v, let X_v be the orthonormal frame of the quotient metric induced by $\sigma(P)$ from $\sigma(H_1)$. Require $(X_v^{-1})^* X_v^{-1} = \sigma(G_1)$. Use the conjugate pairing on these rational quotient coordinates. At an embedding representing a nonbase place, the realized squared norm on a rational vector in an occurring block is its primitive scalar times the corresponding diagonal value of that pairing. Take the layer sizes from 21.16. Take phase one for the rational reference lift and $C_{\mathrm{sz}} \geq 8$. Use the spaces W_{jp} of 21.20. At v_0 take the prescribed scalar of $S_0 S_0^*$ on W_{jp} when this space is nonzero and put $\lambda_{jpv_0} = e^{L_{6,j}}$ when it is zero. At $v \neq v_0$ take λ_{jpv} from the rational part and its Gram forms as in 21.20. Then*

$$L_{6,j} \leq \log \lambda_{jpv}, \qquad \sum_j L_{6,j} \leq \min_{p \in I} \sum_j \log \lambda_{jpv}$$

at every infinite v. The left sum equals $\sum_j H_{X,j}$ in 21.26.

Outline. Compares $L_{6,j}$ of 21.12 with D_j at v_0. Uses Definition 20.14 and the quotient frame to show $\tau(B^*) = \tau(B)^*$, transports kernel orthogonality to all $v \neq v_0$ for $\lambda_{jpv} \geq e^{D_j}$, and sums over j. The complete proof is on p. 636. □

Lemma 21.22 (Metric of a CM generator)**.** *Let $K_d = \mathbf{Q}(\sqrt{-d})$ have discriminant $-d < -4$. Take a nonzero integral $\mathcal{O}_{K_d}$-ideal $\mathfrak{c}$ of norm prime to $2d$, and put $\Lambda = \delta\mathfrak{c}$, with $\delta^{12} = \operatorname{disc}(\mathfrak{c})$ its Weierstrass discriminant. Let $M \subseteq \mathbf{C}$ be a number field containing $g_2(\Lambda), g_3(\Lambda)$ and the prescribed images of E. If $s_\tau \in M^\times$ generates the host differential ideal at τ over $\mathcal{O}_M$, then*

$$\frac{1}{[M:\mathbf{Q}]} \sum_{\sigma : M \hookrightarrow \mathbf{C}} \log \frac{\operatorname{area}(\Lambda_\sigma)}{\pi |\sigma s_\tau|^2} = \gamma + \log(\pi / v_E) - D_1,$$

where Λ_σ has invariants $\sigma g_2(\Lambda), \sigma g_3(\Lambda)$.

Outline. Use Chowla–Selberg and the equal class multiplicities of 1.19 to average log areas. Remove the unit term by the product formula, subtract twice the mean log norm of the differential generator, and use the relative discriminant identity for 21.12. The complete proof is on p. 638. □

21.4 Combining the divisor and height budgets

Let $J, \mathcal{P}$ be finite with $\#J = m$, $\#\mathcal{P} \le f$, and $\mathcal{U} \subseteq \mathcal{P}$ of size n_T. Use $x > 1$, $\varepsilon > 0$, $L = x^\varepsilon$, $b = e^L$, $a = xb$, $B_0 = fmbL$, and take $r_d \ge 1$, $\log r_d, \log x \le L$, $L_a = \log(a r_d)$. Assume

$$\begin{gathered} N_c \ge 0, \quad m, f, k \ge 1, \quad k + m \le n_*, \\ S_c, t_c, \eta, b', w'', \lambda_0, n_V, L_p \ge 0, \quad n_V \le n_*, \\ m\lambda_0,\ n_*^2,\ m^3 n_*^2 L,\ m\eta n_* L,\ k n_* \le b, \\ kS_c,\ m(t_c+1)\eta \le w'', \quad n_* w'' \le b, \quad b' n_*,\ mkb' \le a, \\ 1 \le C_0 \le c_e, \quad 0 \le C_h \le C_0, \quad C_h L_p \le C_0 L_a. \end{gathered}$$

Fix $r \in \mathbf{N}$, real $\log T, \log\Omega$ with $|\log\Omega| \le L_a$, and weights a_ρ. Use φ from 21.12. For j, ρ take row entries $\nu_{j\rho i}$ for $0 \le i < r$ and log hooks $\log H_{j\rho}$. Put $I_\rho = \mathbf{1}_{\mathcal{U}}(\rho)$. Assume $0 \le w_{j\rho} \le 2b$, $|\beta_j - k| \le m$ with $\beta_j = (k + j - 1) - (m - j)$ in arithmetic layer numbering, and

$$\begin{aligned} \sum_i \varphi(\nu_{j\rho i}) - \log H_{j\rho} &\le I_\rho \varphi(a_\rho) + e_{j\rho}, \\ \sum_i \nu_{j\rho i} &= I_\rho a_\rho + k w_{j\rho} + \Delta_{j\rho}, \\ e_{j\rho} &\le C_0 (w_{j\rho} + S_c + n_*^2 + m(t_c+1)\eta) L_a, \\ |\Delta_{j\rho}| &\le C_0 (w_{j\rho} + k(n_* + S_c) + m(t_c+1)\eta + I_\rho k b'). \end{aligned}$$

Set

$$\begin{aligned} H_P &= -\sum_{\rho \in \mathcal{U}} a_\rho \log T + C_h (n_T b' n_* + f(m\eta + S_c) n_* + f n_*^2) L_a, \\ H_{B,j} &= \sum_{\rho, i} (\varphi(\nu_{j\rho i}) - \nu_{j\rho i} \log\Omega) + C_h f n_*^2 (L_a + m^3 L), \\ H_{X,j} &= \sum_\rho \log H_{j\rho} - C_h f n_V^2 L_p, \qquad H^* = H_P + \sum_j (H_{B,j} - H_{X,j}), \\ \mathcal{D}(T) &= \sum_{\rho \in \mathcal{U}} \big(m(\varphi(a_\rho) - a_\rho \log\Omega) - a_\rho \log T \big). \end{aligned}$$

Lemma 21.23 (An upper bound for the extension weights)**.** *At a selected point with the weights of Lemma 15.9, write $e_q(\rho)$ for the extension weight at position ρ, $0 \le q < m$. For all sufficiently large x, uniformly in the position,*

$$e_q(\rho) \le e_{m-1}(\rho) \le 2b.$$

Outline. Use the increasing weights in 15.9 and bound their last term at each kind of position. The totient bound on roots of unity and the size estimates 38.3 and 36.2 absorb the thick-position adjustment into half the step. The complete proof is on p. 638. □

Lemma 21.24 (Wordwise numerical conditions)**.** *With even integer $k \ge 2$, write $t = t_{\mathrm{ch}}$, $N = N_m$, $n_V = k+m-1+t$, $r_d = \mathrm{rd}\, E$, $A_* = a r_d$, $L_a = \log A_*$, $L_p = \log(b r_d g_0 g_1 n_V)$. For $1 \le j \le m$ set*

$$s_j = \{k + (2j-1)(t+1)\}/2, \qquad B = m(t+1)\eta.$$

For a fixed word use the rows of 15.8. The wordwise numerical conditions on $N_c, \lambda_0, C_0, c_e, C_h, S_w, C_1$ are the following scalar and coordinate bounds:

$$\log r_d, \log x \le L, \quad |\log \Omega| \le L_a, \quad N_c, \lambda_0 \ge 0, \quad m\lambda_0, N^2, m^3N^2L, m\eta NL, kN \le b,$$
$$kS_c, B \le w'', \quad Nw'' \le b, \quad b'N, mkb' \le a,$$
$$1 \le C_0 \le c_e, \quad 0 \le C_h \le C_0, \quad C_h L_p \le C_0 L_a, \quad 2 \le C_1, \quad 2S_w \le C_1 S_c, \quad C_1 + 1 \le C_0,$$
$$s_j \le w_{j\rho} \le 2b, \quad w_{j\rho} + S_w \le A_*, \quad a_s + w_{j,(0,s)} \le A_* \quad (s \in TT),$$
$$\nu_{uj\rho i} \ge 0 \quad (i < N_{j-1}), \qquad \sum_{k \le i < N_{j-1}} \nu_{uj\rho i} \le B, \qquad U_i > v_i > U_{i+1} \quad (i < k-1).$$

Here and below the U_i, v_i are taken at the position in question. Off the marked set the endpoint conditions are

$$U_{k-1} + (k-1)/2 \ge 0, \qquad U_0 - (k-1)/2 \le S_w.$$

At a mark put $a_s = U_0 - (k-1)/2$ and $b_s = -U_{k-1} - (k-1)/2$ as in 15.9. The conditions there are $a_s, b_s \ge 0$, $v_{k-2} + (k-2)/2 \ge 0$, and $v_0 - (k-2)/2 \le S_w$.

These conditions hold for every word of the selected family for all sufficiently large x. Take $C_h = C_{\mathrm{sz}} \ge 0$, $S_w = (1 + |C_c|)S_c$, $C_1 = 2(1 + |C_c|)$, where $C_c = 2$ in 33.3. Take C_0 sufficiently large in terms of the fixed constants and $c_e \ge C_0$. Here

$$N_c = |T_2| k b L^2 / n^2, \qquad \lambda_0 = \log M,$$

with T_2 the thick host labels, $n = |A|$, and M the order of the finite row group.

Outline. Use the rank and size bounds 38.3 and 36.2 and count matrices to control the scalar conditions. Use 14.2 and 21.23 for the weight range and 15.8 and 15.9 for the degree, tail and interlacing bounds. The complete proof is on p. 639. □

Lemma 21.25 (Cancellation of the kernel factorials against the row hook)**.** *Fix a word and impose the numerical conditions of 21.24. At layer j and position ρ, put $I_\rho = 1$ if $\rho = (0, s)$ with $s \in TT$, and $I_\rho = 0$ otherwise. Write $a_\rho = a_s$ at a mark and $a_\rho = 0$ elsewhere. Set $\alpha_i = \kappa^{\mathrm{big}}_{j\rho i}$, $H_{j\rho} = \mathrm{Hook}(\alpha)$, and abbreviate $\nu_{uj\rho i}$ to ν_i. Define the two real remainders by*

$$e_{j\rho} = \sum_{i < N_{j-1}} \varphi(\nu_i) - \log H_{j\rho} - I_\rho \varphi(a_\rho), \qquad \Delta_{j\rho} = \sum_{i < N_{j-1}} \nu_i - I_\rho a_\rho - k w_{j\rho}.$$

They satisfy

$$e_{j\rho} \le C_0(w_{j\rho} + S_c + N^2 + B)L_a,$$
$$|\Delta_{j\rho}| \le C_0\big(w_{j\rho} + k(N + S_c) + B + I_\rho k b'\big).$$

Extending the degree tuple by zeros up to length N preserves both sums. Thus all four row hypotheses preceding 21.26 hold for the selected words with $r = n_* = N$.

Outline. Remove the shifts of 15.8 and bound the hook product of 21.12 by row factorials. Telescope the paired differences, retain the marked leading term, and sum the shifted degrees for the second remainder using 21.24. The complete proof is on p. 639. □

Lemma 21.26 (The divisor slope and the height budget)**.** *For the general arrays specified above, suppose that the four row bounds hold. For the selected arrays use 21.24 and 21.25. Suppose also that*

$$\widetilde{\mu}_C \le \mu_0 + L_{\mathrm{pass}}, \qquad \mu_0 \le \sum_{j,\rho} \beta_j w_{j\rho} \log \Omega + 2F_* + E_b,$$
$$L_{\mathrm{pass}} \le 1 + C_0 m \lambda_0, \qquad E_b \le c_e m L (B_0 + N_c).$$

Then

$$\widetilde{\mu}_C + H^* - 2F_* \le \mathrm{rhs} := \mathcal{D}(T) + (50C_0 + 10) n_T a L_a + 60 c_e m L (B_0 + N_c).$$

Outline. Insert the degree identity and the remainders of 21.25, collect the marked and terminal terms into $\mathcal{D}(T)$, and sum the residual allowances. Use 21.28 and (20.6) for the reference divisor bound. The complete proof is on p. 640. □

Lemma 21.27 (The two local indices of a change of row type)**.** *Use the reference row system of 20.16, with base type t_0 and glued lattice R_0. Write $L_t = \iota_t \Lambda_t$ for the image of the quotient lattice, $A_{jt} = \theta_{jt} T_{jt}$ for the layer image in the space of M_j, and $\sharp$ for Hermitian duality. For commensurable lattices put*

$$[A : B]_w = \operatorname{length}_w(A/(A \cap B)) - \operatorname{length}_w(B/(A \cap B)).$$

Here the totally complex field K carries the ring involution $a \mapsto \bar{a}$ satisfying $\sigma_0(\bar{a}) = \overline{\sigma_0(a)}$ at the distinguished infinite place v_0. Let $\bar{w}$ be the finite place of K pulled back from w under $a \mapsto \bar{a}$ (so $v_{\bar{w}}(a) = v_w(\bar{a})$), and let $G_{t,w} \in K$ represent the quotient Gram determinant at v_0 in a semilocal basis at $w, \bar{w}$. Define

$$C_w(t) = v_w(G_{t,w}) + \sum_j \big([M_j : A_{jt}]_w + [M_j : A_{jt}]_{\bar{w}} + [T_{jt} : T_{jt}^{\sharp}]_w \big).$$

For the type t_w of 18.6,

$$[L_{t_0} : R_0]_w + [L_{t_0} : R_0]_{\bar{w}} = C_w(t_w) - C_w(t_0).$$

Outline. Express the local indices by determinant valuations in semilocal bases. Factor the row determinant ratio into layer ratios and a common power by 20.13, then divide the two type identities and take valuations at conjugate places. The complete proof is on p. 641. □

Lemma 21.28 (The reference row divisor)**.** *Take the well-formed reference side of 20.16 and the selector of 18.6. Use the untwisted identities with the reference-row clauses of 36.8. Take the bounds from (20.5). Write $D_C = \operatorname{rk} R_0 > 0$, $\bar{\kappa}_0 = \langle \log \kappa_{\mathrm{base}} \rangle_\infty$. Let* lcv *be log covolume in the slope normalization and $l_{3,j}$ bound $\operatorname{lcv}(M_j)/D_C$. With $F_*, L_G, E_{1,j}, E_{2,j}, E_3$ of 20.16,*

$$\mu(R_0^\vee) - \bar{\kappa}_0 \le \sum_j l_{3,j} + 2F_* + L_G + \sum_j (E_{1,j} + E_{2,j}) + E_3.$$

Outline. Use 21.27, exterior Cauchy–Schwarz, and the product formula to cancel the base terms. Bound the rest by 36.8 and (20.5) using the denominator, cut, and weights of 20.16, 36.13, and 20.7, then sum by parts. The complete proof is on p. 641. □

21.5 The four conclusions of division

Theorem 21.29 (The division package)**.** *Use the coordinate lattices and valuation bounds of 20.16 for N_u^0, N_u', with $s_{uw} = \lfloor \sigma_w \rfloor$ of finite support. Take H^*,* rhs *from 21.26 and the factorization and norm bounds of 21.12, with mean log budget after spectral subtraction and before metric correction at most H^*. Put $\bar{\kappa} = \langle \log \kappa_v \rangle_\infty$ and $\mu_0 = \mu(R_0^\vee) - \langle \log \kappa_{\mathrm{base}} \rangle_\infty$, bounded above by B_* from 21.28. Use the size and μ_0 bounds of 21.26 and assume*

$$\delta_0 \ge 0, \quad Q_0 \ge 1, \quad 3\delta_0 Q_0 \le 1, \quad B_* + H_0 + 28\lambda \le Q_0, \quad 0 \le \lambda \le \lambda_0, \quad C_0 \ge 28,$$
$$\mu(RR^\vee) - \bar{\kappa} \le \mu_0 + 3\delta_0(\mu_0 + H_0 + 28\lambda) + 28\lambda.$$

The four conclusions of 20.16 hold for B_u, N_u', RR with $H = H^*$ and $R_* =$ rhs.

Outline. Apply 21.26 with $\widetilde{\mu}_C = \mu_C - \bar{\kappa}$ after bounding passage cost by $1 + C_0 m \lambda_0$, and shift and exponentiate z_v of 21.12 to mean $H^* - \bar{\kappa}$. Then apply 20.16, using 18.8 and 21.5 for $\sigma_w = s_{uw}$. The complete proof is on p. 642. □

Lemma 21.30 (The unit budget for row passage)**.** *Use the selected family's error bounds (20.6). Put $g = \min(g_0, g_1)$, $r = k + m - 1$, with M the finite row group order, and assume $4fr \le g$. Set*

$$A_3 = \sum_{j,\rho} \beta_j w_{j\rho} \log \Omega, \quad B = A_3 + \mathrm{Fo} + c_e m L(B_0 + N_c), \quad Q_0 = \max\{1, B + x^{C_H} b + 28 \log M\}.$$

For each retained character χ, let $n(\chi)$ be the multiplicity of the selected subgroup character and put $\bar{n} = \prod_w q_w^k$, with q_w the residue size at a selected place. For fixed family constants and sufficiently large x, $|n(\chi)/\bar{n} - 1| \le \delta_0 = 8fr/g$. With $H_0 = x^{C_H} b$, $\lambda = \log M$, one has

$$B_* + H_0 + 28\lambda \le Q_0, \qquad 1 \le Q_0 \le 3b^2, \qquad 3\delta_0 Q_0 \le 1.$$

For $P = 1 + f + k + m + t_{\mathrm{ch}} + N_m + L$, one has $\mathrm{Fo} \le 256P^9 b$.

Outline. Use 32.49 and 32.51 to bound multiplicities by subset counts and 20.17 for the row budget. Bound passage by the sizes, inert terms, and grouped Mertens prime sum of 21.26, 15.10, 15.1, and 20.7. The complete proof is on p. 642. □

Definition 21.31 (Pointwise division conditions)**.** Use the selected row lattice RR in the row space W, the divided word lattices N'_u and the periods B_u of 20.19. Put $d_u = \operatorname{rk} N'_u > 0$, $\mu_C = \mu(RR^\vee)$, $S_a = \sum_{t \in TT} a_t$, and

$$s_{\mathrm{fr}} = \frac{\sum_u d_u(\mu(N'_u) - h_u)}{\sum_u d_u}, \qquad H_u = h_u + S_a \log T_u,$$
$$\mathcal{D}(T) = m \sum_{t \in TT} \big(\varphi(a_t) - a_t \log \Omega\big) - S_a \log T, \quad E_{\mathrm{div}} = c_{\mathrm{Div}}\big(n_T a L_a + m(B_0 + N_c) x^{e_{\mathrm{Div}}\varepsilon}\big).$$

Here the heights and φ are as in 21.12. The pointwise division conditions with constants $c_{\mathrm{Div}}, e_{\mathrm{Div}}, C_H$ are the following.

The lattice RR is finitely generated over $\mathcal{O}_K$ and spans W. The frame satisfies $s_{\mathrm{fr}} + \mu_{\max}(RR) \le x^{C_H} b$. For each word u and layer j, the raw gain g_{jw} and loss E_{ujw} vanish at every w with $p_w > X''$, $p_w \notin \{T_u, g_0, g_1, d\}$ and $p_w \nmid D_{\Gamma,u}$.

The complex period on rational vectors is the distinguished embedding of the K-valued pairing. Write $\mathcal{B}_u(\phi, F)$ for the inner product of the realized row vector F with the period image of the realized column vector ϕ, in that order. There is a common row isometry V and, for each u, a column isometry U_u and a scalar $a_u \in \mathbf{C}^\times$ such that $B_u^{\mathbf{C}}(\phi, F) = a_u \mathcal{B}_u(U_u\phi, VF)$.

For every word u,

$$B_u(N'_u) \subseteq RR^\vee, \qquad B_u \ne 0 \implies h(B_u) \le h_u, \qquad \mu_C + h_u - \mathrm{Fo} \le \mathcal{D}(T_u) + E_{\mathrm{div}}.$$

Finally, for every word u, layer j, and second word u',

$$\left| \sum_{w \in S} c_w E_{ujw} - S_a \log(g_0 g_1) \right| \le c_{\mathrm{Div}} B_0, \qquad |H_u - H_{u'}| \le c_{\mathrm{Div}}(n_T a L_a + m B_0).$$

Theorem 21.32 (Division data for the actual period family)**.** *Fix a selected family in the range of ε and with the ordered thresholds of 21.33. Use full divided column lattices of positive rank and a finitely generated row lattice spanning W. Assume the rational row and column isometries of 20.19 carry the coordinate lattices onto these lattices and preserve their pairings. Assume the rationality and complex isometries for the period in Definition 21.31, with a common row isometry for all words.*

Use the finite support and fixed weight tuples of 20.8 and 20.17, and the coordinate valuation and lattice hypotheses of 20.16. For every word use its selected level T_u, the definite tables and frame factorizations of 36.1–36.4, and the primitive reference realization with the Hecke, metric and size hypotheses of 21.17 and 21.21. Give it the height allowance of 21.12 on the same positions and layers. Assume the wordwise size and interlacing hypotheses of 21.24–21.25, the reference covolume and Gram bounds of 21.28, the paired-index and cut hypotheses of 36.8 and 36.13, and the numerical passage hypotheses of 21.30. Assume also the row-passage hypotheses of 20.18 (the finite group action, scalar endomorphisms with distinct characters, and intermediate lattices $h_i P_i \subseteq R_i \subseteq P_i$) and the level-zero row-lattice bound $U \ge \mu_{\max}(R_0)$ of 34.69 (applied at height exponent a_{row} as in 21.33 for the maximum-slope bound on RR, and at C_H for $H_0 = x^{C_H} b$), which together with the multiplicity bound of 21.30 and the choice of C_H in 21.33 supply both the maximum-slope bound for RR in 21.33 and the passage inequality $\mu(RR^\vee) - \bar{\kappa} \le \mu_0 + 3\delta_0(\mu_0 + H_0 + 28\lambda) + 28\lambda$ of 21.29.

For the frame clause via 21.33, assume the functional, row, level and stabilizer hypotheses of 34.68, 34.69 and 36.7, together with the hypotheses of 21.11 and 21.39: namely, the local comparisons and containments

of 21.6–21.9 and 22.37, the integrality and colength conditions $\tau_w \in \mathbf{N}$, $\mathrm{io}_w \geq 0$, $\tau_w = 0$ on S_T, and $\ell^{\mathrm{col}}_{uw} \leq d_M(\tau_w + \mathrm{io}_w)$ off S_T, the bounds on $\mathcal{T}$, T_u, P_u and R_u in 21.11, the counting window of 21.35, the column models beyond a common threshold, and $|D_1| \leq x^{\varepsilon}$ eventually. Take the hypotheses of 21.41 (the seven error conditions of 15.12, the eight clauses of 21.40, and equality of stored and evaluated losses) and the size bounds of 21.42. All these data belong to the same selected member.

Let c_E be the constant of 21.41, choose C_H as in 21.33, and take $e_{\mathrm{Div}} \geq 1$ and $c_{\mathrm{Div}} \geq \max(50C_0 + 10, 60c_e, c_E, 20 + 6/\varepsilon)$. Beyond a common cutoff, every selected member satisfies the pointwise division conditions of Definition 21.31.

Outline. Applies 21.29 to the coordinate pairing via 36.1, 21.17, 21.21, and 21.28, transferring bounds through 20.19. Supplies the remaining clauses of 21.31 from 20.8, 21.33, 21.41, and 21.42. The complete proof is on p. 643. □

Lemma 21.33 (Choosing the height exponent)**.** *Take $0 < \varepsilon \leq \varepsilon_1$ in the chapter's range for sufficiently small $\varepsilon_1 > 0$. Use the selected class with the functional and row hypotheses of 34.68, 34.69 and 36.7, and the level and stabilizer bounds. Apply 21.11, 21.12, 21.39 and 21.41 with full divided lattices of positive rank. Fix the outer constants c_{out}, including the chain bound C_t. Set $M = \max(C_t, 5) + 3$. With $F_0 = \max(a_{\mathrm{row}}, 1) + 1$ for the row-slope exponent $a_{\mathrm{row}} = a_{\mathrm{row}}(\varepsilon, c_{\mathrm{out}})$ given by the middle-list threshold $\mathcal{K}_{\mathrm{mid}}(c_{\mathrm{out}})$ of the closed row assertion 34.69 (conjoined with $C_{\mathrm{sz}} \geq 0$) in Definition 36.10, with the final-list threshold of 34.69 evaluated using a_{row} in place of C_H in the middle list, take $C_H \geq 1 + \max\{9M + 2, F_0\}$ (note $4M + 1, 9M + 1 \leq 9M + 2$ for $M \geq 8$). Take the thresholds of 36.10 for the row and size estimates. Assume every preliminary constant meets the preliminary lower bound, every middle constant (including C_H) meets the middle bound evaluated at c_{out}, and every final constant meets the final bound evaluated at the concatenated outer and middle lists. Then there is an $x_0 > x_1$ such that $s_{\mathrm{fr}} + \mu_{\max}(RR) \leq x^{C_H} b$ for every $x \geq x_0$ and every $i \in I_x$.*

Outline. Bound row slopes by 34.69 and 20.18, and divided slopes by 21.11 and 21.39, summing 14.8 and 14.9 via 20.8 and 20.7. Average with heights from (36.10), 36.1–36.5, and 15.7. The complete proof is on p. 644. □

21.6 The auxiliary prime and the original lattice

Use the starting volume of Lemma 37.9 and the genus masses of 37.5. Set

$$d^{\star}_u = \mathrm{mass}_k \, \mathrm{idx}_{\Pi}(g_0, g_1) \prod_{\rho} \dim V(u, \rho), \qquad D^{\star} = \mathrm{vol}(Y_0) \, \mathrm{idx}_{\Sigma}(g_0, g_1) \prod_{\rho} d_{\Sigma,\rho}.$$

With the local indices of 33.31, put $\mathrm{idx}_{\Pi}(g_0, g_1) = \prod_{i=0}^{1} I_c(g_i)^f$ and $\mathrm{idx}_{\Sigma}(g_0, g_1) = \prod_{i=0}^{1} I_r(g_i)^f$. Write $V(u, \rho)$ for the column representation and $d_{\Sigma,\rho}$ for the row spectral degree. Let C_{Γ} be the family's size constant for the count and put $\Gamma_u = \exp(C_{\Gamma} m^5 f L)$, the same for every word. Set $X_u = (D^{\star}/d^{\star}_u)^{1/(fk)}$, $T^{\star}_u = X_u \Gamma_u^{-2/(fk)}$ and $\ell_g = \log(g_0 g_1)$.

Lemma 21.34 (Separated parameters for the count)**.** *At all sufficiently large selected members put $C_Y = C_0 = \max(5, C_{YL} + 1)$, $r = k + m - 1$, $v_E = \mathrm{rd}\, E/2$. Use the zero-based indices of 15.9. The scalar conditions for the count are*

$$\pi \leq v_E, \quad \log(2v_E), \log(2k) \leq L, \quad f, k \leq x^{C_0},$$
$$m^2 L \leq k, \quad 44 \leq m \leq \min(k, x), \quad (kb)^8 \leq g_i, \quad \log g_i \leq C_Y L.$$

The interlacing row has spacing 4δ, $\delta \geq 1$, and belongs to $[0, S_c]$. The interior entries of U are the bit choices in 15.9. For all words $U_0 - U_{k-1} \leq 2a$, $U_0 - v_0 \geq S_c$. Off the marks $U_0 - v_0 \leq 2S_c$ and $U_{k-1} = v_{k-2} - 1/2$. At the marks $U_0 \leq 2a$ and $-a \leq U_{k-1} \leq -S_c$. Finally,

$$0 < b_q < e_q \leq (1 + 1/x) b_q, \qquad e_{m-1} \leq 2b, \qquad S_c \leq b_0/x,$$
$$e_q < b_{q+1}, \qquad b_{q+1} - b_q \geq r^{19} f^5 L / C_0 \quad (0 \leq q < m - 1).$$

Outline. Use 36.2 and 8.12 for size and prime bounds and 15.9 for the word endpoints. Compare the step with the base for the relative weight bound and with the rank scales for the gap, using 21.23 for the final weight. The complete proof is on p. 644. □

Theorem 21.35 (Three counting bounds for the auxiliary primes)**.** *Suppose* $1 \le L \le L_a$, $k \ge 3$, $f \ge 1$, $m \ge 1$, $\Gamma_u > 1$ *is common to the words with* $\log \Gamma_u/(fk) \le 1$, *and* $T_u^\star \le T_u \le (1 + 1/(fk))T_u^\star$. *Write* $\mathcal{W} = \sum_{j=1}^{m} f^{-1} \sum_\rho \log(w_{j\rho} v_E/\pi)$. *For a common* C *satisfying*

$$\mathcal{W} + m\ell_g - C(L + m^2 L/k) \le \log X_u \le CmL, \qquad |\log X_u - \log X_{u'}| \le C \log k, \tag{21.4}$$

put $c_{\mathrm{Count}} = \max(C, 0) + 2$ *and* $\lambda_5 = \mathcal{W} + m\ell_g - c_{\mathrm{Count}}(L_a + m^2 L_a/k)$. *Then*

$$\lambda_5 \le \log T_u \le c_{\mathrm{Count}} mL, \qquad |\log T_u - \log T_{u'}| \le c_{\mathrm{Count}} \log k.$$

For an admissible family with 37.6 and 37.7, the mass and spectral estimates give common c_{Count}, x_0 *on all members at* $x \ge x_0$.

Outline. Use 21.34 and 37.8 to compare row and column dimensions. Combine the volume and mass formulas 37.9 and 37.5 with the wordwise Vandermonde comparison, then transfer the bounds from X_u to T_u using 37.11. The complete proof is on p. 644. □

Lemma 21.36 (Uniform level ratios in the spectral row)**.** *Use the rank, numerical and weight hypotheses of the selected spectral row of 33.78, with column-count factor* $C_0 \ge 1$ *and* $C_\ell \ge 40$, *with* C_ℓ *evaluated for this family as in Theorem 33.78. Beyond the rank and mass thresholds, every ordered pair of words and every type* ρ *satisfy*

$$T_u/T_{u'} \le C_0 e^{7/4}, \qquad \Delta_\rho(u') \le (C_0 e^{7/4})^k \Delta_\rho(u).$$

The distinct-word row uses $C_0 = 3$.

Outline. Expand the log Vandermonde ratio, telescope the linear terms, and bound the remainder and edges using 21.9 at the edge. Compare count products from 37.21 and (A.152) using 37.5 and shared row bounds, then take the kf-th root. The complete proof is on p. 646. □

Definition 21.37 (Column model)**.** For a word u with zeroth-layer weight λ_0 and level T_u, a *column model* over the coefficient field L consists of a nonempty finite set $\mathcal{C}$ of zeroth-layer classes, embeddings $\rho_\tau : E \to L$ indexed by the CM type labels τ, rank-k Hermitian $\mathcal{O}_L$-lattices $Q_{c,\tau}$ ($c \in \mathcal{C}$) that are isometric ρ_τ-base changes of rank-k Hermitian $\mathcal{O}_E$-lattices $\Lambda_c \subseteq \Lambda_c^\vee$ with $[\Lambda_c^\vee : \Lambda_c] = T_u^{2f}$, and an injective L-linear coordinate map from the zeroth-layer space $M^0_{u,L}$ into $\mathcal{C}$-tuples of polynomials over L in variables indexed by (τ, i, j) $(0 \le i, j < k)$. Its image consists of all tuples whose c-component lies in the tensor product of highest-weight Weyl polynomial spaces for λ_0. At each finite place w of L, membership in the localized lattice $M_{u,w}$ is equivalent to each shifted coordinate polynomial (under any local bases of $Q_{c,\tau}$) lying in the integral Weyl span at w; at each infinite place v, the squared Hermitian norm equals the sum over $c \in \mathcal{C}$ of the squared Fock norms in the Hermitian frames of $Q_{c,\tau}$ at v, each divided by the product over τ of the squared Fock norm of the highest-weight vector for λ_0.

Lemma 21.38 (Slopes before summing the column weights)**.** *For a word with a column model (Definition 21.37) and* $v_E = \mathrm{rd}(E)/2 \ge 2$, *write the integral highest weight at* ρ *as* $\lambda_i = \nu_i + t_\rho$, $1 \le i \le k$, *with* $\nu_1 \ge \cdots \ge \nu_k = 0$. *Put* $d_\rho = \sum_i \nu_i$, $s_\rho = \nu_1$, $l = \log T_u$. *Then*

$$\mu(M_u) \ge -\sum_\rho \big[(d_\rho/k + t_\rho)l + d_\rho \log k + k^2 \log(s_\rho + k)\big],$$

$$\mu_{\max}(M_u) \le \sum_\rho \big[4d_\rho \log(kv_E) + k^2 \log(s_\rho + k) + |t_\rho| l\big].$$

Outline. Compute the mean slope using the dual index, determinant character, Cauchy–Binet, and the hook formula of 21.12. Bound maximum slope using transference for short columns, the product formula for the twist, and coordinate minors before taking the orthogonal sum. The complete proof is on p. 647. □

Theorem 21.39 (Mean and maximum slopes of the original lattice)**.** *For every word and column model, the zeroth-layer lattice* $Q_{u,c,\rho}$ *at a class* c *and column embedding* ρ *has slope* $-\log T_u/k$. *Suppose an admissible family satisfies 21.35 and has column models beyond a common threshold. Assume also that* $|D_1| \le x^\varepsilon$ *eventually. Then for common* $c_{\mathrm{Slope}} \ge 0, x_0$ *and every member at* $x \ge x_0$, *every word satisfies*

$$-\mu(M_u) \le c_{\mathrm{Slope}} n_T a L_a (1 + m/k), \qquad \mu_{\max}(M_u) \le c_{\mathrm{Slope}} n_T a L_a.$$

Outline. Apply 21.38 to the zeroth-layer lattice and sum the degree, determinant, and weight terms, retaining the leading top entries. Absorb the remaining terms by growth bounds and use the level estimate for m/k in the mean slope. The complete proof is on p. 648. □

Definition 21.40 (The second complement at a point)**.** Use the weights and losses of 15.10 and 15.12. For a word u, layer $1 \le j \le m$ and label τ, let $A_{u,j,\tau}$ sum the entries of the layer diagram, and put $A_{uj} = \sum_{\tau} A_{u,j,\tau}$. Use the Fock degree D_j^{Fock} of 15.10 and write $w_{j\tau} = w_j(\tau)$. Let c'_g be the fixed-prime error coefficient. Define

$$R_j = 2\sum_{\tau} w_{j\tau} + C_{\deg}\big(f(kS_c + mt_{\mathrm{ch}}\eta + kN_m) + |TT|b'/x\big), \qquad H_g = c'_g f(N_m^2 + m^2\eta).$$

Let $n_{iujw} = e_w^{-1}\sum_{\tau} A_{u,j,\tau} v_w(\tau(\bar{\pi}_i))$ be the normalized norm contribution, and let $E_{u,w}^{T} = T_{u,w}^{*}$ be the level loss of 15.12. Its full allowance E_u^{π} is the parenthesis of $T_{u,w}^{*}$, used in 17.20. For each signal or residue-class-zero prime $p \le X''$, let $J_{j,p}^{*} = C_{j,p}^{*}$ if p is in residue class zero and $h_p = 0$, and $J_{j,p}^{*} = I_{j,p}^{*}$ otherwise, with the two losses as in 15.12.

For real constants $C_{\deg}, C_g, C_T, C_{\mathrm{stab}}, C_d, C_I$, the point predicate requires each word and layer to satisfy the following clauses at $i = 0, 1$.

1. $T_u \neq g_i$.
2. On the complete fibre, $\sum_{w|g_i} c_w n_{iujw} = A_{uj}\log g_i$.
3. $\big|A_{uj} - D_j^{\mathrm{Fock}} - S_a\big| \le R_j$.
4. $(R_j + 2v_{g_i}(D_{\Gamma,u}) + 2H_g)\log g_i \le C_g B_0$.
5. For every finite set F of places over T_u,
$$0 \le \sum_{w\in F} c_w E_{u,w}^{T} \le E_u^{\pi}\log T_u \le C_T B_0.$$
6. $2\log D_{\Gamma,u} \le C_{\mathrm{stab}} B_0$.
7. $2\log d\,|D_j^{*}| \le C_d B_0$.
8. For every finite set P of signal or residue-class-zero primes $p \le X''$,
$$\sum_{p\in P} 2\log p\,|J_{j,p}^{*}| \le C_I B_0.$$

For a family with cutoff x_1, require that there exist six real constants $C_{\deg}, C_g, C_T, C_{\mathrm{stab}}, C_d, C_I$, with $C_g \ge 0$, and a real x_0 such that all eight clauses hold for every $x \ge x_0$ with $x > x_1$ and every $i \in I_x$. The hypothesis in 33.6 requires this property on W_3 with the quantifiers of 36.10. For each sufficiently small $\varepsilon > 0$, one choice of thresholds works for every family in W_3 and every C_H satisfying the threshold inequalities.

21.7 Comparing words

Lemma 21.41 (The layerwise E-comparison)**.** *Use the seven error conditions of 15.12 and the eight clauses of 21.40 on the selected family above their common cutoff. The stored loss is the evaluated loss and the fixed primes are outside the level fibre. With $c_E = 2C_g + C_T + C_{\mathrm{stab}} + C_d + C_I$, each word and layer satisfies*

$$\left|\sum_{w\in S} c_w E_{ujw} - S_a \ell_g\right| \le c_E B_0, \qquad \sum_{\substack{w\in S\\ p_w\notin\{g_0,g_1\}}} |c_w E_{ujw}| \le c_E B_0.$$

Consequently $|\mathrm{Ee}_u - mS_a\ell_g| \le mc_E B_0$.

Outline. Combine 15.12, 15.10, and 20.7 to evaluate the fixed-prime sums. Apply 21.40 to bound their deviations, the level contribution, and the remaining terms grouped by prime, then sum over layers. The complete proof is on p. 648. □

Lemma 21.42 (The difference of adjusted heights)**.** *For the heights of 21.12 put* $H_u = h_u + S_a \log T_u$*. For every pair of words of the fixed member,*

$$H_u - H_{u'} = \sum_{j=1}^{m} \sum_{\rho} (K_{uj\rho} - K_{u'j\rho}).$$

Use the column degrees of 15.8 and suppose that the following size bounds hold. For selected members they follow from 21.24 and 36.2, since $S_c + w_{j\rho} \le 3b$*.*

$$S_c + w_{j\rho} \le 6x^6 b, \qquad kS_c \le b, \qquad |D_1| \le L, \qquad 0 \le \log \mathrm{rd} E \le L.$$

Then $|H_u - H_{u'}| \le (20 + 6/\varepsilon)(n_T a L_a + m B_0)$*.*

Outline. Add $S_a \log T_u$ to the heights of 21.12 and cancel common terms. Use 15.8 to bound the varying head degrees, then integrate $\log s - \log \Omega$ between them and sum using $kS_c \le b$. The complete proof is on p. 649. □

Chapter 22

Prime sums and net gain

At a window point x of Alpöge's construction [3], let $n_T = |TT|$ count the top positions and m the layers. Put $Q = n_T m b$, with $b = e^{x^{\varepsilon}}$ and $0 < \varepsilon < 1/52$. The common weights of 22.44 give bounds of order Q for the deficit $\mathrm{ASK} - \mathrm{Ga}$ and the index IO of 21.1–21.3.

22.1 The deficit at a rational prime

Let J be the layer set and P_s the finite set of ordinary primes. For $j \in J$ and $t \in TT$ let A_{jtp} be the contribution before $2\log p$ and D_{jt} the contribution at d, including $\log d$. Write

$$\mathrm{ASK} = 2\sum_{j,t,p\in P_s} \log p\, A_{jtp} + \mathrm{ASK}_d, \qquad \mathrm{ASK}_d = \sum_{j,t} D_{jt},$$

where the first sum is over $J \times TT \times P_s$.

Lemma 22.1 (Decomposition over rational primes)**.** *Give a finite set of places W weights λ_w with $\sum_{w|p}\lambda_w = \log p$ for $p \in P_s$. Suppose the gains and deductions have constant values g_{jp} and τ_p on each fiber and vanish off P_s. Set*

$$\mathrm{Ga} = 2\sum_{w\in W} \lambda_w \left(\sum_j g_{j,p(w)} - \tau_{p(w)}\right), \qquad \Delta_p = \sum_{j,t} A_{jtp} - \sum_j g_{jp} + \tau_p.$$

Then

$$\mathrm{ASK} - \mathrm{Ga} = 2\sum_{p\in P_s} \log p\, \Delta_p + \mathrm{ASK}_d. \tag{22.1}$$

Let x_1 be the starting scale of the admissible family F_m. For $x \ge \max(x_1+1, 3)$, the model sums use the signal and inert-class primes of 14.1 up to $B = \max(\lfloor a + a/(kx^{13})\rfloor, \lfloor X''\rfloor)$, with the scales of (22.12). In a consistent presentation, the model sum equals ASK_N of 21.1 for any integer cutoff $N \ge a_t/w_j(\rho)$ for every $t \in TT$, $j = 0, \dots, m-1$, and $\rho \in \Phi$.

Outline. Group places by prime using residue degrees, ramification indices and fiber constancy from 14.9. The bounds in 22.4 and 22.43 remove ordinary terms above B, and the matching places in 22.43 identify the weights at each eligible cutoff. The complete proof is on p. 649. □

Lemma 22.2 (Covered positions and discarded primes)**.** *Partition P_s into served primes S and discarded classes P_1, P_2, P_h. At $p \in S$ suppose $TT_p \subseteq TT$ and*

$$\sum_{j\in J,\ t\in TT_p} (A_{jtp} - \ell_p) - m r_p \le \sum_j g_{jp} - \tau_p. \tag{22.2}$$

If gains and deductions vanish at discarded primes, then

$$\mathrm{ASK} - \mathrm{Ga} \le 2\sum_{p\in S} \log p \left(\sum_{j,\ t\notin TT_p} A_{jtp} + m|TT_p|\ell_p + m r_p\right) + 2\sum_{p\in P_1\cup P_2\cup P_h} \log p \sum_{j,t} A_{jtp} + \mathrm{ASK}_d. \tag{22.3}$$

Proof. Use (22.2) in Δ_p at served primes and the vanishing hypothesis at discarded primes, then apply (22.1). □

In the arithmetic model P_h consists of discarded primes above X''. Among the remaining discarded primes, P_2 consists of the candidates of 14.8 with $\Sigma_p > \theta = x^{-\varepsilon}$. The others form P_1.

22.2 The finite-sum loss bound

Write $[u]_+ = \max(u, 0)$ and $\log_p a = \log a / \log p$. With nonnegative constants $C, f, S_k, \ell_x, \mathcal{L}, Q_1, Q_2$, an independent real size parameter N_0, positive a, k, K_0, and a real function L_1, put

$$F_p = Cf\{N_0^2 \log_p a + (\log_p a)^2\}, \quad C_p = CS_k \log_p(2a),$$
$$R_p = \begin{cases} Q_1 & p < p_0, \\ Q_2 & p \ge p_0, \end{cases} \qquad O_p = C\ell_x \left(\frac{\mathcal{L}a}{pk} + \begin{cases} ap^{-L_1(p)} & p < p_0, \\ a/p^2 & p \ge p_0. \end{cases} \right) \tag{22.4}$$

Theorem 22.3 (The losses from all prime classes)**.** *Assume 22.1 and 22.2, $a, X' \ge 1$, $X'' \ge 2$, $X''w'' = a$, $p_0 \ge 3$, $d \ge 4$, and $0 \le a_t \le a(1+\vartheta)$ with $0 \le \vartheta \le 1/2$. Let*

$$A_{jtp} = \sum_{i \in I_{jtp}} [a_t/p^i - w_{jtpi}]_+, \qquad D_{jt} = \log d \sum_{i \in I^d_{jt}} [a_t/d^i - w^d_{jti}]_+. \tag{22.5}$$

The exponent sets are finite, with i positive odd and $w_{jtpi} \ge w''$ in the ordinary sum and $i \ge 1$, $w^d_{jti} \ge 0$ in the ramified sum. Suppose

$$\begin{gathered} p \le X'' \ (p \in S), \quad p > X'' \ (p \in P_h), \\ r_p \le F_p, \qquad \ell_p \le C_p + \mathbf{1}_{\{p<X'\}}(R_p + O_p) \quad (p \in S), \\ K_0 \le p^{L_1(p)-1} \quad (p \in P_s, \ p < p_0). \end{gathered} \tag{22.6}$$

Then $\mathrm{ASK} - \mathrm{Ga} \le E_{\mathrm{fl}} + E_{\mathrm{cp}} + E_{\mathrm{pr}} + E_{\mathrm{or}} + E_{\mathrm{out}} + E_1 + E_2 + E_h + E_d$, *where*

$$\begin{aligned} E_{\mathrm{fl}} &= 2mCfX''\{N_0^2 \log a + 2(\log a)^2\}, \\ E_{\mathrm{cp}} &= 2CS_k \log(2a)\, mn_T X'', \\ E_{\mathrm{pr}} &= 2\log 4\, mn_T(p_0 Q_1 + X'Q_2), \\ E_{\mathrm{or}} &= 2C\ell_x a\, mn_T \left\{ \left(\frac{\mathcal{L}}{k} + \frac{1}{K_0} \right)(\log X' + \log 4) + \frac{2(\log(p_0 - 1) + 1)}{p_0 - 1} \right\}, \\ E_{\mathrm{out}} &= 4ma \sum_{p \in S} \frac{\log p}{p} |TT \setminus TT_p|, \\ E_i &= 4mn_T a \sum_{p \in P_i} \frac{\log p}{p} \quad (i = 1, 2), \\ E_h &= 2mn_T \log(2X'')\{a\vartheta^2 + w''\vartheta\}, \\ E_d &= 2mn_T a \frac{\log d}{d}. \end{aligned} \tag{22.7}$$

Outline. Bound losses by geometric series and counts of served primes and covered positions, keeping only the first power per short interval above X''. Use binomial and factorial counts for the prime sums and integrate the inverse-square tail to bound price and orbit errors. The complete proof is on p. 649. □

Lemma 22.4 (Lower bound for an ordinary partner)**.** *In the model of 14.1–14.2, the weight at position z in lift ℓ is $B_\ell(z) + \kappa_\ell$. It is at least $w'' = b'/(xn)$ for every lift and every $z \in \{0,1\} \times H$. Each ordinary exponent i in (22.5) is odd and has weight $B_\ell(z_i(t)) + \kappa_\ell$.*

Proof. The marked, thick, and thin lower bounds in 14.2 are $b'/x, b, w''$, respectively, and $w'' \le b'/x \le b$. The partner formula and exponent restriction are those of 14.1 and 14.9. □

22.3 Exceptional primes and exceptional positions

Lemma 22.5 (Small order and narrow prime-power bands)**.** *Let $e, J_0 \ge 1$ be integers, $Y_0 \ge 1$, $X' \ge 2$, $p_0 \ge 8$, and $0 \le \rho \le 1/2$. Write $h_e(s)$ for the class of an odd integer in $(\mathbf{Z}/2^{e+2}\mathbf{Z})^\times/\langle -1\rangle$. Suppose every prime in a finite set T is odd and satisfies at least one of*

$$p \le Y_0, \quad \operatorname{ord}(h_e(p)) \le J_0, \qquad p \ge p_0, \quad X'(1-\rho) \le p^i \le X'(1+\rho) \quad \text{for some } i \ge 1.$$

Then, with $C_{\mathrm{ord}} = 12$,

$$\sum_{p\in T} \frac{\log p}{p} \le C_{\mathrm{ord}} J_0 \frac{\log^2(2^e Y_0)}{2^e} + \lfloor \log X' \rfloor (4\rho + p_0^{-1}) \log(2X'). \tag{22.8}$$

For P_1 under these hypotheses with $Y_0 = \max(Y, X'')$ and $J_0 = \lfloor x^{\varepsilon/4} \rfloor$, write $E_1 \le E_{\mathrm{ord}} + E_{\mathrm{band}}$ for (22.8) multiplied by $4mn_T a$, in the order of its two terms.

Outline. Use successive differences of squares to put small-order classes in two residue classes and bound their contribution by a harmonic sum. For each band exponent, take roots of the interval endpoints, count the integers, and sum the resulting contribution bounds. The complete proof is on p. 650. □

For candidate primes $\mathcal{P}_c$ write

$$R_W = \sum_{p\in\mathcal{P}_c} \frac{\log p}{p} (2\mathbf{1}_{\{\Sigma_p > \theta\}} + 4\Sigma_p). \tag{22.9}$$

Lemma 22.6 (Bad positions, second-class primes, and local indices)**.** *For each prime p in a finite set $\mathcal{P}_R$, choose a set $U_p \subseteq TT$ and put $U_p = \varnothing$ for $p \notin \mathcal{P}_R$. Suppose $TT \setminus TT_p \subseteq U_p$ at every served prime and $\sum_{p\in\mathcal{P}_R} (\log p/p)|U_p| \le R_{\mathrm{out}}$. Then $E_{\mathrm{out}} \le 4amR_{\mathrm{out}}$. If the candidate shares are nonnegative and $P_2 \subseteq \{p \in \mathcal{P}_c : \Sigma_p > \theta\}$, then $E_2 \le 2amn_T R_W$. If*

$$\mathrm{io}_p \le 2\Sigma_p \frac{2mn_T a}{p} \qquad (p \in \mathcal{P}_c), \tag{22.10}$$

then, for arbitrary real shares,

$$\sum_{p\in\mathcal{P}_c} \log p \,\mathrm{io}_p \le amn_T R_W. \tag{22.11}$$

The last two implications hold with any $T \ge 0$ in place of n_T throughout and $R \ge R_W$ on the right.

Proof. Apply the bad-position count to E_{out}. For E_2 use the indicator in (22.9). Summing (22.10) gives (22.11) since the indicator terms are nonnegative. □

22.4 Losses on the scale Q

For $x \ge 1$, $0 < \varepsilon \le 1$ put

$$\begin{gathered} z = x^\varepsilon, \quad u = x^{\varepsilon/3}, \quad b = e^z, \quad a = xb, \quad X' = e^u, \quad b' = a/X', \quad f = 2n, \quad w'' = b'/(xn), \quad X'' = xnX', \\ \delta_1 = x^{-\varepsilon/8}, \quad \theta = x^{-\varepsilon}, \quad \ell_x = \log x, \quad S_k = S_c + k + \eta, \quad Q_1 = Cr_G b' \ell_x, \\ Q_2 = C\{b'/x + \eta + S_c + (w'' + t_0)\log x/\delta_1\} + b_0. \end{gathered} \tag{22.12}$$

For $\nu \ge 0$, $K_s \ge 1$ the size conditions are

$$\begin{gathered} x^2 \le 2^{e_2} \le n \le x^\nu, \quad x^2 \le k \le x^\nu, \quad 0 \le N_0, t_0, r_G \le x^\nu, \quad x^4 + 1 \le p_0 \le x^\nu, \\ 0 \le \mathcal{L} \le x^{2\varepsilon}, \quad 0 \le S_c, \eta \le e^{x^{\varepsilon/4}}, \quad 0 \le \rho \le x^{-2}, \quad 0 \le \vartheta \le x^{-1}, \\ K_0, d \ge x^2, \quad 0 \le C, C_{\mathrm{ord}} \le K_s, \quad 1 \le Y \le 2X', \quad 0 \le b_0 \le e^{z/2}, \end{gathered} \tag{22.13}$$

with e_2, p_0, d integers. Assume

$$100K_s x^{4\varepsilon} \le x, \qquad 100K_s x^{4\nu+4} \le e^u, \qquad 100K_s x^{4\nu+4} e^{u+x^{\varepsilon/4}} \le e^{z/2}. \tag{22.14}$$

Set $\sigma_x = 100K_s x^{4\varepsilon-1}$.

Theorem 22.7 (Joint loss bounds in a size window)**.** *Take $C_{\mathrm{ord}} = 12$, $TT \neq \varnothing$, and P_s consisting of odd primes. Assume (22.12)–(22.14). For each family member υ assume the hypotheses of 22.3 and 22.6, with common nonnegative shares and bad-position predicate, and 22.5 for P_1 with $e = e_2$, $Y_0 = \max(Y, X'')$ and order $< x^{\varepsilon/4}$ in the small-order case. If*

$$R_{\mathrm{out}} \le n_T x^{-2+\varepsilon}, \qquad R_W \le x^{-9}, \tag{22.15}$$

then, for $Q = n_T|J|b$,

$$\begin{aligned} &\mathrm{ASK}_\upsilon - \mathrm{Ga}_\upsilon \le (2C\log 4 + 13\sigma_x)Q \le (3C+13)Q, \\ &\sum_{p\in\mathcal{P}_c} \log p \,\mathrm{io}_{\upsilon,p} \le \sigma_x Q \le Q. \end{aligned} \tag{22.16}$$

For fixed K_s, ν and $0 < \varepsilon < 1/4$, (22.14) holds eventually.

Outline. Use the served-prime and index bounds if $m = 0$. Otherwise divide losses by Q, insert window sizes and use 22.5 and 22.6 to bound the excess over $2C\log 4$ by multiples of σ_x, with size inequalities following from growth comparisons. The complete proof is on p. 650. □

Lemma 22.8 (Splitting an integral pairing among its summands)**.** *Let O be a discrete valuation ring and $0 \neq z \in O$. If a finite family $A_j \in M_n(O)$ satisfies $\sum_j A_j = zI_n$, there are integral rectangular matrices P_j, Q_j with $P_j A_j Q_j = zI_{n_j}$ and $\sum_j n_j \ge n$.*

Proof. In Smith bases [84, §4.5] retain the n_j entries $a \mid z$, and scale their source vectors by z/a. Let W span the retained target vectors and $Z = zO^n$. The other entries belong to $\mathfrak{m}zO$, where $\mathfrak{m}$ is the maximal ideal. Hence $Z \subset W + \mathfrak{m}Z$. Nakayama's lemma applied to $(W+Z)/W$ gives $Z \subset W$ [85, Prop. 2.6]. Taking ranks gives the claim. □

22.5 Local projection estimates

Use the lattices H_Λ, R_e, R'_e and pairing $[F, G] = G(p^{A_{r'}} F)$ of 14.10. The test at rank r', depth t, denominator D, and order c_* asks for a pairing matrix $p^D I$ of rank at least $(1-\kappa)\dim V$ under the stated lattice conditions.

Lemma 22.9 (Separating the residue planes)**.** *Let $1 \le r \le k$, $F = \mathbf{F}_{p^{n'}}$, $n' \ge 1$, and $0 \le z_0 \le \cdots \le z_b = z_0 + n'$, $b \ge 1$. Put $Q_j = p^{z_{j+1}-z_j}$, $I_0 = \{1, \dots, r\}$. Let H be a rank-r row matrix of a plane. In frame j take row l to be $\sum_i T_{jli} H_{i,*}^{[p^{u_{j,l}}]}$, raised entrywise to $p^{z_j - u_{j,l}}$, with $0 \le u_{j,l} \le z_j$ and independent T_{jli}. Write $M_{j,I}$ for its minor on I and $\Delta_j = M_{j,I_0}$. The polynomial*

$$L = \Delta_0 \prod_I \left(\prod_{j<b} \Delta_j^{Q_j - 1} - \prod_{j<b} M_{j,I}^{Q_j - 1} \right)$$

over the one-column replacements I of I_0 vanishes on every competing plane and restricts on the coordinate plane to $\Delta_0 \prod_{j<b} \Delta_j^{r(k-r)(Q_j-1)} \neq 0$.

Outline. Factor minors via $d_I^{p^{z_j}}$; for $d_{I_0} \neq 0$, row-reducing to I_r kills a competing plane's replacement factor via $d^{p^{z_b} - p^{z_0}} = 1$ on $F^\times$. On the coordinate plane replacements vanish with unit diagonals, and lifting L^h gives order h. The complete proof is on p. 651. □

Lemma 22.10 (Assembling tests on incoming edges)**.** *Let $t, D \in \mathbf{N}$, $c_* \ge 4(D+1)$, and take a finite set E of incoming edges at a target Λ. For each $e \in E$ let F_{ei}, $1 \le i \le n_e$, satisfy the source conditions of 14.10 at e. Suppose for every $u \in O^{n_e}$ with a unit coordinate there is a target vector of depth t, integral in every $R'_{e'}$, whose value on $\sum_i u_i F_{ei}$ divides p^D and which belongs to $\omega^{c_*} R'_{e'}$ for $e' \in E \setminus \{e\}$. The test has retained rank $\sum_e n_e$.*

Proof. On a primitive row of length $h = \sum_e n_e$, use the test for a block with a unit coordinate. The other blocks contribute multiples of p^{D+1} since $\omega^2 = p$. Thus every primitive row takes a value dividing p^D on the image of the target lattice in O^h. Apply this to the Smith coordinate rows. The image contains $p^D O^h$, whose coordinate vectors give the required lifts. □

Lemma 22.11 (A window of integral Gram matrices)**.** *Let $A(T) = \sum_{i=0}^{J} C_i \binom{T}{i}$, $C_i \in M_d(\mathbf{Z}_p)$. Suppose $\det A \neq 0$ has degree s and leading coefficient b with $v(b) + s/(p-1) \leq dL$, where $v(p) = 1$. Let $h \in \mathbf{N}$ and $H, J+1 \leq p^h$, $H > 0$, and let β_u count Smith factors of valuation at least $E > 0$, including zero factors, on H consecutive nonnegative integers. Then*

$$H^{-1} \sum_u \beta_u \leq d\,(J/H + (L + Jh/H)/E)\,.$$

If A is the matrix of a pairing multiplied by a nonzero integral scalar, the retained Smith directions give integral pairs of valuation below E for the original pairing.

Outline. Prove integrality after substitution and use the Smith rows to bound the determinant's Gauss norm. Count integers near each root by residue classes up to height h and by single points above it, then sum and subtract the scalar valuation. The complete proof is on p. 651. □

Lemma 22.12 (Returning from the enlarged weight box)**.** *Let $|S| = N \geq 1$, and let $I_a, I_b \subseteq \mathbf{N}$ be integer intervals of sizes $w_a, w_b > 0$. Enlarge their upper endpoints by $J + c$ for $J, c \in \mathbf{N}$. Write $\mathbb{E}_+$ for the enlarged product average and $\mathbb{E}_\tau$ for the average on $(I_a + \tau)^S \times (I_b + \tau)^S$. Suppose $f, g, q \geq 0$ satisfy*

$$\sum_{j \in \{0,\dots,J\}^S} f(A - j, B - j) \prod_{s \in S} q(A_s, B_s, j_s) \leq g(A, B)$$

on the enlarged box, with natural subtraction. For $\tau = 0, c$ suppose $\sum_{i=0}^{J} q(\alpha + i, \beta + i, i) \geq \delta \geq 0$ on the translated coordinate intervals and $(w_a + J + c)^N (w_b + J + c)^N \leq 2\delta^N w_a^N w_b^N$. Then $\mathbb{E}_\tau f \leq 2\mathbb{E}_+ g$. On the original box compute the downward translation $(A - c, B - c)$ in $\mathbf{Z}^S \times \mathbf{Z}^S$. Set $f_- = f(A - c, B - c)$ when this translated pair belongs to $I_a^S \times I_b^S$, and set $f_- = 1$ otherwise. Equivalently, writing $a_- = \min I_a$ and $b_- = \min I_b$, the first case requires $A_s \geq a_- + c$ and $B_s \geq b_- + c$ for every $s \in S$. If the latter event has probability at most $1/Y$ and $0 \leq e \leq 1/Y$, then

$$\mathbb{E}_0(f_- + 2f(A + c, B + c) + e) \leq 6\mathbb{E}_+ g + 2/Y.$$

Proof. For each j translate the pairs from the τ-box by j. At each pair their total weight is at least δ^N by factorization over the coordinates. Sum and compare cardinalities. The downward shift is injective wherever it stays in the original box. Thus $\mathbb{E}_0 f_- \leq \mathbb{E}_0 f + 1/Y$, and the second claim follows. □

Definition 22.13 (Paths and their labels)**.** A signed letter of degree b denotes $\operatorname{Sym}^b V$ for a vector letter and $\operatorname{Sym}^b V^\vee$ for a covector letter. A path in rank k starts at the zero row. At a vector letter it passes from a decreasing integral row μ to a row ν with

$$\nu_1 \geq \mu_1 \geq \nu_2 \geq \mu_2 \geq \cdots \geq \nu_k \geq \mu_k, \qquad \sum_j (\nu_j - \mu_j) = b.$$

At a covector letter interchange μ, ν in these conditions. An ordinary restriction step from rank k to rank $k-1$ chooses a row η with $\mu_j \geq \eta_j \geq \mu_{j+1}$.

Fix $k = r + K$. After the signed letters take r restriction steps and denote the resulting K-row by λ. A label is a semistandard tableau of shape $\lambda - \lambda_K$ on $1, \dots, K$, with the determinant character $\det^{\lambda_K}$ understood. Equivalently, put $h(c) = \#\{a : \lambda_a \geq c\}$ for $c \in \mathbf{Z}$. Choose sets $I_c \subseteq \{1, \dots, K\}$ of size $h(c)$ for $c > 0$ and $K - h(c)$ for $c \leq 0$. Put $J_c = I_c$ for $c > 0$ and $J_c = I_c^c$ for $c \leq 0$. The tableau condition is

$$|J_{c+1} \cap [1, t]| \leq |J_c \cap [1, t]| \qquad (1 \leq t \leq K).$$

The sets I_c are empty outside a finite interval. In particular the multiplicity of the path is $\dim V_\lambda$.

Lemma 22.14 (Loss at the prefix cuts). *Take a word of n_0 signed letters followed by r restriction steps, and put $t = n_0 + r$, $k = r + K$, $q = K - 2t$. Suppose $r \geq 1$, $n_0 \geq 2r$, the first and last r signed letters are vectors, $t + r \leq K$, and $q \geq k/2$. Let $k \leq b_i \leq 2a$ be the signed-letter degrees, where $a \geq 3$ and $p \geq 13$ is prime. Degrees of successive letters of the same sign increase by a factor at least $p^2/5$.*

At a restriction step use the shorter row for computing sums of positive or negative parts. To refer to a coordinate in rank k, restore the deleted leading positions with the common value $b_* = \sum_{i \leq n_0} b_i$. Count a restriction step as a vector letter when locating the last r vector letters before a cut. Before step $i > r$, denote the oldest of these r letters by $o(i)$. It is a signed letter. Give the cut a scale $v_i > 0$ such that every earlier signed-letter degree is at most $4v_i/p$, and put $\xi_i = Lv_i/(pk)$, where $L > 0$.

Discard a path if any of the following occurs. A covector step removes more than ξ_i from the sum of positive parts. A vector step after the first r, including a restriction step, removes more than ξ_i from the sum of negative parts in absolute value. At a cut before step $i > r$, coordinate r of the restored row is less than $2b_{o(i)}/3$. The fraction discarded, counting labels, is at most

$$\exp\left(11t^2 \log(1 + 2at) - \frac{\min(L, k)}{32}\right).$$

Outline. Pieri rules and interlacing give the path count in 22.13 and propagate tail deficits. Concavity and summation by parts bound the central Weyl factors for each failed test, and bounding the other factors and counting rows gives the discarded fraction. The complete proof is on p. 651. □

Lemma 22.15 (Labels on a shared frame row). *Let Q, b, d be nonnegative integers with $Q > 0$, $d \leq 2b < Q$, and let $r \geq 2$. In every prescribed residue class for $d + \sum_{i=1}^{r} v_i$ modulo Q, there are at least*

$$Q^{r-1}(1 - 2rb/Q)$$

choices with $0 \leq v_i < Q - 2b$. For each choice, $e = d + \sum_i v_i$ satisfies $0 \leq e < rQ$.

Proof. With coordinates in $[0, Q)$ the count is Q^{r-1}, since the congruence determines the last coordinate. For a specified coordinate and a specified value in $[Q - 2b, Q)$, a different coordinate can be solved for. Thus each specified value removes Q^{r-2} choices. Sum over the $2b$ values and the r coordinates. Finally $d + \sum_i v_i \leq 2b + r(Q - 2b - 1) < rQ$. □

Lemma 22.16 (Binomial approximation to a product of digit units). *Let $p \geq 5$ be prime. For each element i of a finite set I, fix a power P_i of p and an integer $0 \leq d_i < P_i$. Put*

$$U(b) = \prod_{i \in I} \frac{(d_i + P_i b_i)!}{d_i!\, b_i!\, (P_i!)^{b_i}} \qquad (b \in \mathbf{N}^I).$$

The values of U are p-adic units. For either sign and every integer $c \geq 1$ there are coefficients $a_\nu \in \mathbf{Z}_p$, indexed by $\nu \in \mathbf{N}^I$ with $\sum_i \nu_i \leq 4c$, such that

$$U(b)^{\pm 1} \equiv \sum_{\nu} a_\nu \prod_i \binom{b_i}{\nu_i} \pmod{p^c} \qquad (b \in \mathbf{N}^I).$$

Outline. Split the factorials into blocks and expand the unit factors or their inverses in an auxiliary variable. Bound coefficient degrees by finite differences, then specialize at p, reduce modulo p^c, and use the product binomial basis. The complete proof is on p. 652. □

Lemma 22.17 (Uniform competing-edge orders of columns, bordered factors, and ties). *Let $p \geq 5$ be prime, let $n' \geq 1$, let $1 \leq r < k$ with $K = k - r$, and let O be a discrete valuation ring over $W(\mathbf{F}_{p^{n'}})$ with uniformizer ω satisfying $\omega^2 = p$. Let σ be the Frobenius automorphism of $W(\mathbf{F}_{p^{n'}})$. For $g \in \mathrm{GL}_k(W(\mathbf{F}_{p^{n'}}))$ and $a \in \mathbf{Z}$, write $g_a = \sigma^a(g)$ for its a-th Frobenius conjugate, and define the g-twisted vector row $v_a^{(g)} = X_a^{\mathrm{vec}} g_a$ and covector row $w_a^{(g)} = X_a^{\mathrm{cov}} g_a^{-\mathsf{t}}$. For any $g' \in \mathrm{GL}_k(W(\mathbf{F}_{p^{n'}}))$, let the edge substitution $s_{g'}$ send X_a^{vec} to $X_a^{\mathrm{vec}} \operatorname{diag}(I_r, \omega I_K)(g'_a)^{-1}$ and X_a^{cov} to $X_a^{\mathrm{cov}} \operatorname{diag}(\omega I_r, I_K)(g'_a)^{\mathsf{t}}$. Define $\mathrm{ord}_{g'}(F) = \sup\{m \in \mathbf{N} : s_{g'}(F) \in \omega^m O[X]\} \in \mathbf{N} \cup \{\infty\}$ (so $\mathrm{ord}_{g'}(F) \geq m$ means $s_{g'}(F) \in \omega^m O[X]$), and write $X_{a,b}^{\mathrm{vec}}$ ($r < b \leq k$) and $X_{a,b}^{\mathrm{cov}}$ ($1 \leq b \leq r$) for the normal variables.*

1. *Let L be a list of $N \geq 1$ signed letters with $N + r \leq K$ and nondecreasing integer positions $x_0 \leq \cdots \leq x_{N-1} \leq z$ such that distinct letters of the same sign have strictly increasing positions. Let P be a path of rank k on L with column heights $h_t = \#\{a : \mu_a^{(t)} \geq c\}$ for a fixed column $c \in \mathbf{Z}$, and call step i ($0 \leq i < N$) used in c if $h_{i+1} \neq h_i$. Call a used step i costly if it is a vector step from height $h_i \geq r$ or a covector step from height $h_i \leq r$, and suppose $x_i < x_j$ whenever $i < j$, i is used in c, and j is costly in c. Indexing $K^{(t)}$ by subsets $I \subseteq \{1, \ldots, k\}$ of size h_t (for $c > 0$) or $k - h_t$ (for $c \leq 0$), define $K^{(t)}$ recursively over $O[X]$ from $K_{\varnothing}^{(0)} = 1$: at the first used letter use power 1, at each subsequent used letter i raise each coordinate polynomial K_I to the polynomial power $p^{x_i - y}$ (where y is the preceding used position) and, writing $u = v_{x_i}^{(g)}$ at a vector letter and $u = w_{x_i}^{(g)}$ at a covector letter, update by right exterior multiplication $\sum_{b \in I} (-1)^{\#\{j \in I : j > b\}} u_b K_{I \setminus \{b\}}$ (for $c > 0$, vector, or $c \leq 0$, covector) or right contraction $\sum_{b \notin I} (-1)^{\#\{j \in I : j > b\}} u_b K_{I \cup \{b\}}$ (for $c > 0$, covector, or $c \leq 0$, vector), and after the last used position y_{last} raise each coordinate to $p^{z - y_{\mathrm{last}}}$ to obtain $K_c(g; J)$ for each terminal index J (with $K_c(g; \varnothing) = 1$ if no step is used in c). Let $\mathrm{oc}(c) = \sum_{i \text{ costly in } c} p^{z - x_i}$. Then:*

   (a) *$K_c(g; J) \in O[X]$ is homogeneous of multidegree $p^{z - x_i}$ at every used letter i (and 0 at unused letters);*

   (b) *$\mathrm{ord}_{g'}(K_c(g; J)) \geq \mathrm{oc}(c)$ for every $g' \in \mathrm{GL}_k(W(\mathbf{F}_{p^{n'}}))$;*

   (c) *under the chosen substitution s_g,*

$$s_g\big(K_c(g; J)\big) - \omega^{\mathrm{oc}(c)} \, \mathrm{pow}_z\big(K_c^{\mathrm{sep}}(J)\big) \in \omega^{\mathrm{oc}(c)+1} O[X],$$

   where $K_c^{\mathrm{sep}}(J)$ is the unpowered separated column polynomial built by the same recursion with power 1 at every step using only nonnormal variables at non-costly steps and normal variables at costly steps, and pow_z is the entrywise segment substitution sending each variable at letter i to its $p^{z - x_i}$-th power;

   (d) *$\mathrm{oc}(c) \leq 2p^{z - x_0}$.*

2. *Let $y_1, \ldots, y_r < x$ be r distinct vector positions strictly preceding x. Let $A_x(g)$ be the $r \times k$ matrix with entries $(v_{y_a, c}^{(g)})^{p^{x - y_a}}$, let $M_x(g)$ be its first r columns, and put $D_x(g) = \det M_x(g)$. At a vector letter at x and $r < b \leq k$, let $B_{x,b}(g)$ be the $(r+1) \times (r+1)$ minor on columns $\{1, \ldots, r, b\}$ obtained by bordering $A_x(g)$ below with $v_x^{(g)}$. At a covector letter at x and $1 \leq b \leq r$, let $P_{x,b}(g)$ be the b-th entry of $\mathrm{adj}(M_x(g)) \, A_x(g) \, (w_x^{(g)})^{\mathrm{t}}$. Then $B_{x,b}(g), P_{x,b}(g) \in O[X]$, and for every $g' \in \mathrm{GL}_k(W(\mathbf{F}_{p^{n'}}))$,*

$$\mathrm{ord}_{g'}\big(B_{x,b}(g)\big) \geq 1, \qquad \mathrm{ord}_{g'}\big(P_{x,b}(g)\big) \geq 1.$$

   Under s_g,

$$s_g\big(B_{x,b}(g)\big) - \omega s_g(D_x(g)) X_{x,b}^{\mathrm{vec}} \in \omega^2 O[X], \qquad s_g\big(P_{x,b}(g)\big) - \omega s_g(D_x(g)) X_{x,b}^{\mathrm{cov}} \in \omega^2 O[X],$$

   and every term of either difference has normal degree equal to its ω-order, at least p.

3. *At any position x carrying both a vector row and a covector row, for $E_x = \sum_{c=1}^{k} v_{x,c}^{(g)} w_{x,c}^{(g)}$ and any $P = p^e$ ($e \geq 0$), every $g' \in \mathrm{GL}_k(W(\mathbf{F}_{p^{n'}}))$ satisfies*

$$s_{g'}(E_x^P) = \omega^P \left(\sum_{c=1}^{k} X_{x,c}^{\mathrm{vec}} X_{x,c}^{\mathrm{cov}} \right)^P \in \omega^P O[X],$$

   and $s_g(E_x^P) - \omega^P \sum_{c=1}^{k} (X_{x,c}^{\mathrm{vec}})^P (X_{x,c}^{\mathrm{cov}})^P \in \omega^{P+2} O[X]$.

4. *For any $g' \in \mathrm{GL}_k(W(\mathbf{F}_{p^{n'}}))$, the order $\mathrm{ord}_{g'}$ is additive on products in $O[X]$. In particular, any product of ω^m, column factors $K_c(g; J)$ of orders a_c, q bordered or pairing factors, tie factors $E_x^{P_h}$, integral nonnormal factors, and a localizer power of order at least c_0 on competing planes has order at least $m + \sum_c a_c + q + \sum_h P_h + c_0$ under every competing $s_{g'}$.*

Outline. Put $H = (g')^{-1} g$ and expand $s_{g'}(v_a^{(g)}), s_{g'}(w_a^{(g)})$ in ω. Modulo ω^2, Frobenius power congruences and $\widetilde{W}_a^{\perp} \widetilde{W}_a = 0$ inductively prove the column orders (1) and bordered/pairing congruences (2); invariance of E_x yields (3), and Gauss valuation additivity gives (4). The complete proof is on p. 652. $\square$

Segment setup for plane projection. Use 14.10 with $p \ge 13$, $2 \le r < k$, and a marked set S of size N on an even cycle of length n'. Every cyclic window of length L_2 has at most $r-1$ marks, and every window of length G' has at most N_1. Let $u_1 = \lceil 1/\delta \rceil$ for the positive marking parameter δ. Divide the cycle into consecutive full tiles of length $(6r+1)u_1$, each partitioned into cells of length u_1 numbered from 1 to $6r+1$; in a marked tile the even cells each have one even mark and the odd cells have none, and every run of M_0+1 full tiles meets a marked tile, where $G' = (M_0+5)(6r+1)u_1$. Suppose $1 \le L_1$, $2L_1 \le L_2 < u_1$, and $3G' \le n'$. Let $a_0 \ge 1$, $Y \ge r$, $c_b, c_t, D, c_* \in \mathbf{N}$ (with a_0, Y allowed real), and let the dominant base weight have unspecified entries of absolute value at most c_b and marked extremes $(a_s, -b_s)$ with $a_0 + c_b \le a_s \le 2a_0 - c_b$, $0 \le b_s$, $b_s + 2c_b \le c_t$. Put $K = k - r$, $o = rNc_t + 2rn'c_b$, $c' = c_* + o$, $Q_* = p^{G'}$, and assume for C above an absolute threshold that

$$N_1^2 \le rN, \quad N_1 + 3r \le k, \quad CN_1^2 \log a_0 \le k, \quad 256k^2NY^2 \le p^{L_1-1},$$
$$CNk^2 \log_p a_0 \le D, \quad 8(D+1) \le c_*, \quad Ck^3r^2Np^{5G'}(c_*+o)Y \le a_0.$$

Lift the marks periodically to $\mathbf{Z}$, take the middle r even marks $f_{j,0} < \cdots < f_{j,r-1} = z_j$ in each marked tile (extended periodically by n'), and form segment j on $[f_{j-1,0}, z_j]$ with left frame $f_{j-1,0}, \dots, f_{j-1,r-1}$. At a non-left-frame letter x in segment j, put $M_x = p^{z_j - x}$, let y be its oldest of r even predecessors, write $d_s = \lambda_{s,1} - \lambda_{s,2}$ and $b_x = d_{y \bmod n'}/M_y$, and set $\alpha_j = \mathbf{1}_{j=0} + rK(p^{z_{j+1}-z_j} - 1)$ and $m_* = (N_1+1)((2Q_*+3)c' + kQ_* + c' \max_j \alpha_j + 3c')$. For a letter set I, write v_I for the least signed first-coordinate weight after a generic change of basis ($\mathrm{diag}(T, 1, \dots, 1)$ on vectors, inverse transpose on covectors). Normalizing $v(p) = 1$, put $v_{\mathrm{F}}(H) = \min_\alpha(v(H_\alpha) + v(\alpha!)/2)$, and for fixed low digits $d_{xa} \in [0, M_x)$ define the row factorial unit $u(b) = \prod_{x,a} \gamma_{M_x}(d_{xa}, b_{xa})$, where $\gamma_P(u, w) = (u + Pw)!/(u!\, w!\, (P!)^w)$.

Lemma 22.18 (The plane count and the retained rank)**.** *Under the segment setup and size hypotheses above, at every target the test passes with denominator D, order c_*, fraction $\min(1, CNr^2k/Y)$, and any integer depth at least*

$$CrN\left(CN_1^2 \log(a_0)\frac{a_0}{pk} + a_0 p^{-L_1} + c_t + kp^{G'}\right) + 2rn'c_b. \tag{22.17}$$

Outline. Counts segment symbols via 22.13, 22.14, and 22.15, lifting them with 22.9 and 22.17. Builds dual tests via isotypic projections and 22.16, 22.19, 22.20, 22.21, then sums over edges using 22.10. The complete proof is on p. 654. □

Lemma 22.19 (The plane coefficient after normal projection)**.** *Use the segment setup and size hypotheses of Lemma 22.18, together with the segment symbol space constructed in its proof with normal degree bound (A.167) and fixed determinant factor U of (A.169). In segment j, list the letters after the left frame in chronological order as $x_0, \dots, x_{l+r-1}$, where l is the number of interior letters. A normal prefix of length u consists of the first u letters of this list. Fix a simultaneous normal isotypic projection, and denote the first entry of its highest weight at this prefix by k_{ju}. Put*

$$s_u = \left\lfloor \frac{b_{x_u}}{Yk^2} \right\rfloor.$$

Let H_{ju} be the maximum of the following numbers, including the last two entries under the indicated conditions:

$$\frac{b_{x_u}}{2}, \qquad k_{j,u+1} + m_* + s_u \quad (x_u \text{ even}), \qquad k_{j,u-1} + m_* + s_u \quad (u > 0,\ x_{u-1} \text{ odd}).$$

At each physical row first select the coordinate residues modulo its non-left-frame power M_x, divide out the corresponding monomial, and replace X^{M_x} by X coordinatewise. Select the plane and normal row degrees, apply the joint projection in the normal variables, and take any normal monomial coefficient. Write F for the resulting plane polynomial. This construction applies to the flat component of every primitive combination in the segment symbol space, including after division by its scalar of exact order.

Number the shared frame slots in reverse chronological order, starting at 1. Write $P_t = p^{z_{j+1}-z_j}$ for the power at a boundary slot t of frame j. A boundary digit component of F with label ρ keeps exponents $\rho_t + P_t\beta_t$, coordinate by coordinate, and writes β_t on a new vector letter. Interior exponents are unchanged. Then the following bounds hold.

1. *For $0 \le u \le l$, on the preceding frame's quotient letters together with the first u interior letters, every boundary digit component has least signed weight at least $H_{ju} - s_u/8$.*
2. *For $1 \le m < r$, put $u = l + r - m$. On the original plane alphabet, the least signed weight of F on the interior and slots $h > m$ of frame j is at least $H_{ju} - 3s_u/8$.*

Here the least signed weight is taken after a generic change of basis on the indicated letters, as in the segment setup above. In particular, if $F = P_c$ is normalized by $v_{\mathrm{F}}(P_c) = 0$, m_0 attains this minimum, and P_{link} is the product of the projections for the second set of bounds, then

$$\langle P_c, P_{\mathrm{link}} X^{m_0} \rangle = [X^{m_0}] P_c \, m_0!, \qquad v\big(\langle P_c, P_{\mathrm{link}} X^{m_0} \rangle\big) = \tfrac{1}{2} v(m_0!).$$

Outline. Establishes the chain and link bounds on undifferentiated symbol products by separating shared boundary letters and bounding left-frame degree loss. Propagates both bounds by induction through residual operators, then uses self-adjoint link projectors. The complete proof is on p. 669. □

Lemma 22.20 (An integral preimage preserving the links)**.** *In the alphabet of Lemma 22.19, let I_j denote the interior of segment j, and let $d_{I_j} = \sum_{i \in I_j} d_i$ for the prescribed plane row degrees d_i. At each shared slot t introduce a label letter R_t, a quotient letter W_t, and a surplus letter S_t, all vectors. The recombination Φ sends their coordinates to $X_t, X_t^{P_t}, X_t$, respectively, coordinatewise, and fixes the interiors. Call a monomial allowable if its recombination has the prescribed row degrees, each coordinate exponent on R_t is less than P_t, and*

$$\deg S_{j,1} = 0, \qquad \deg S_{j,h} \le d_{I_j} \quad (1 \le h \le r).$$

An allowable polynomial is supported on allowable monomials.

Let T be an integral polynomial on the original plane alphabet of the prescribed multidegree, with least weight at least θ_{jm} on I_j and the shared slots $h > m$, for every j and $1 \le m < r$. There is an integral allowable Z with $\Phi Z = T$. Its least weight on

$$\{W_{j-1,h} : 1 \le h \le r\} \;\cup\; I_j \;\cup\; \{S_{j,h} : h > m\}$$

is at least θ_{jm} for every j, m.

Outline. Builds an integral basis of path polynomials via 22.13, 22.17, and 22.14 whose retained paths span each link cut. Reassigns interior-using boundary columns to surplus letters and splits label exponents into quotients and remainders. The complete proof is on p. 671. □

Lemma 22.21 (The detecting pairing on a boundary digit block)**.** *Use the segment setup above and the alphabets of Lemmas 22.19 and 22.20. Normalize $v(p) = 1$. For an exponent array b write $b! = \prod_i b_i!$, and use the bilinear form for which the monomials are orthogonal and X^b has square $b!$. Write $D_f X^b = f(b) X^b$ for a diagonal operator. Let F have the prescribed original row degrees and $v_{\mathrm{F}}(F) = 0$. Suppose all its boundary digit components satisfy the chain bounds θ_{ji} of Lemma 22.19(1). With the row factorial unit $u(b)$ from the segment setup above, put*

$$\Lambda(Z) = \langle F, D_u \Phi Z \rangle, \qquad n_0 = \tfrac{1}{2} v\Big(\prod_i d_i!\Big), \qquad k' = \sum_i r(1 + \log_p(d_i + r)).$$

Fix an integer $c' \ge 1$. For a chain cut i in segment j, let σ_{ji} be a slack satisfying

$$r \, d_{I_{j-1}}/P_t + 13rc' \le \sigma_{ji} \qquad (t \text{ in frame } j-1).$$

Let P_{pre} be the product of the factorial-orthogonal prefix projections on the chain cuts $\{W_{j-1,h} : 1 \le h \le r\} \cup \{x_0, \dots, x_{i-1}\}$ (with $x_0, \dots, x_{i-1}$ the first i interior letters of segment j) at thresholds $\theta_{ji} - \sigma_{ji}$, with total loss at most D_m in v_{F}. On an enlarged monomial with label exponents ρ and quotient exponents β, let α keep its interior exponents and have boundary exponents $\rho_t + P_t \beta_t$. Define

$$\Gamma = u(\alpha) \prod_{t,a} \frac{(\rho_{ta} + P_t \beta_{ta})!}{\rho_{ta}! \, \beta_{ta}! \, (P_t!)^{\beta_{ta}}}.$$

Let g_+, g_- be integral diagonal multipliers congruent on allowable monomials to Γ, Γ^{-1}, respectively, modulo $p^{c'}$. For every integral allowable Z,

$$v\big(\Lambda(D_{g_-} P_{\mathrm{pre}} D_{g_+} Z) - \Lambda(Z)\big) \geq c' + n_0 - 2k' - D_m.$$

Outline. Works blockwise, rewriting Λ via divided derivatives, approximating $D_u, D_{u^{-1}}$ by 22.16, and expanding boundary derivatives in quotient variables. Self-adjointness of P_{pre} and carry bounds from 11.7 control the error. The complete proof is on p. 672. □

Definition 22.22 (The lattice at an internal tie)**.** Work over $\mathbf{Q}_p$. On signed letters $X_i = (X_{i1}, \dots, X_{ir})$ of degrees d_i use the form

$$[X^\alpha, X^\beta]_0 = \mathbf{1}_{\alpha=\beta} \frac{\prod_{i,c} \alpha_{ic}!}{\prod_i d_i!}.$$

The action on a prefix of length j is generated by

$$E^j_{ab} = \sum_{\substack{i \leq j \\ i \text{ vector}}} X_{ia}\partial_{ib} - \sum_{\substack{i \leq j \\ i \text{ covector}}} X_{ib}\partial_{ia}.$$

A prefix with integer threshold M_j contributes the largest E^j_{ab}-stable subspace in the span of monomials of signed first-coordinate weight at most M_j. An absent threshold means $+\infty$.

Take a prefix of $m+1$ letters ending in a vector W of degree o, and put $S = \sum_i d_i$, $\rho = r$. For an integral weight w, intersect the imposed prefix spaces with the full diagonal weight space w and with the kernels of E^{m+1}_{ab} for $a < b < r$. Intersect with the integral coefficient lattice to obtain M_w. Put $x = W_\rho$,

$$M_{\geq e} = \{f \in M_w : x^e \mid f\}, \qquad G_e = \{x^e[x^e]f : f \in M_{\geq e}\}.$$

Here $[x^e]$ keeps all other variables. Thus G_e is the image lattice of the restriction map, with the power x^e restored.

Lemma 22.23 (Denominators for types and highest vectors)**.** *Let A be a finite-dimensional split integral-weight $\mathfrak{gl}_h$-module over $\mathbf{Q}_p$, with semisimple diagonal action. Let L be a full $\mathbf{Z}_p$-lattice preserved by weight extraction and by every divided root operator $E^{(j)}_{ab} = E^j_{ab}/j!$, $a \neq b$. Suppose that, for any two weights occurring in A, their entries at each fixed coordinate differ by at most l. Put*

$$d = \binom{h}{2} \lfloor \log_p(l + h - 1) \rfloor.$$

For $h = 1$ put $d = 0$. Projection onto any sum of isotypic components takes L into $p^{-d}L$. In an isotypic component of highest weight λ, let Δ_λ be the integral span of divided-root words applied to its integral highest vectors. Then

$$p^d(L \cap A_\lambda) \subseteq \Delta_\lambda.$$

Outline. Extract weight zero and use the row recursion to bound the denominator of the invariant projection on maps. Averaging rank-one maps gives the type projectors, and averaging highest-weight extraction gives the divided-root generation bound. The complete proof is on p. 674. □

Lemma 22.24 (Restriction at the last vector)**.** *Fix the first m signed letters and their degrees as in 22.22, and let the last letter be a vector W in $r \geq 2$ coordinates. For each $d \geq 0$, let Y_d be the rational polynomial space of last degree d satisfying all the prefix caps, before imposing a weight or a highest-vector condition. Write R for evaluation at $W_r = 0$, and $I = \{1, \dots, r-1\}$.*

Define X_d on the hyperplane $W_r = 0$ as follows. Keep every cap on a proper prefix. For each cap C containing W, impose the cap C on the first m letters for $\mathfrak{gl}_r$, and on all $m+1$ letters for $\mathfrak{gl}(I)$. The latter condition retains the types with first highest-weight entry at most C. Then

$$RY_d = X_d.$$

If S is the sum of the first m degrees and d, and L denotes the integral coefficient lattice in each polynomial space, then

$$p^{2b}(X_d \cap L) \subseteq R(Y_d \cap L), \qquad b = \binom{r}{2} \lfloor \log_p(4S + r - 1) \rfloor.$$

Outline. Differentiate to check the caps under restriction, then prove surjectivity by induction on degree using a highest-vector kernel calculation. Apply 22.23 twice for a larger Lie algebra action whose highest vectors lift in degree zero. The complete proof is on p. 674. □

Lemma 22.25 (The graded image and its pairing partners)**.** *Use 22.22 with $r \ge 2$, and put*

$$e_1 = \binom{r}{2}\lfloor \log_p(S+r-1)\rfloor, \qquad e_2 = \binom{r}{2}\lfloor \log_p(4S+r-1)\rfloor.$$

For $0 \le e \le o$, use the space X_{o-e} of 22.24 after subtracting e from each cap containing W. In this space take the full weight $w - e\epsilon_r$ and the highest vectors for $\mathfrak{gl}_{r-1}$. Denote the result by H. Then $\mathbf{Q}_p G_e = W_r^e H$. If $m > 0$, then

$$p^{e_1+2e_2}\big(\mathbf{Q}_p G_e \cap P_{\mathbf{Z}_p}\big) \subseteq G_e.$$

The orthogonal projection onto $\mathbf{Q}_p G_e$ for $[\ ,\ ]_0$ takes $P_{\mathbf{Z}_p}$ into $p^{-(m+1)e_1}P_{\mathbf{Z}_p}$. For $m = 0$ both inclusions hold with exponent zero. For $e > o$ the graded image is zero.

Outline. Divide by W_r^e and use 22.24 with the shifted caps to obtain the restricted image and lift. Use 22.23 to bound denominators for the highest-vector projection of the lift and for the commuting projections that give the orthogonal projector. The complete proof is on p. 676. □

Lemma 22.26 (Detecting a primitive vector of the graded lattice)**.** *In 22.22, write $V_e = \mathbf{Q}_p G_e$, and let $P_{\mathbf{Z}_p}$ be the integral coefficient lattice in the specified multidegree. Suppose*

$$p^c(V_e \cap P_{\mathbf{Z}_p}) \subseteq G_e.$$

Suppose also that, for every monomial X^α of that multidegree, there is $y_\alpha \in V_e$ with $p^D y_\alpha \in P_{\mathbf{Z}_p}$ and $[v, y_\alpha]_0 = [v, X^\alpha]_0$ for all $v \in V_e$. In any basis of G_e its Gram determinant is nonzero and has valuation at most

$$(\operatorname{rank} G_e)(D + 2c).$$

Outline. Use the lattice inclusion to bound the least coefficient valuation by c, then pair with its scaled monomial partner to get a factorial pairing of valuation at most $D + 2c$. Apply this test in Smith bases to bound the diagonal valuations and the determinant. The complete proof is on p. 677. □

Lemma 22.27 (Leading determinant at an internal tie)**.** *Use 22.22, with $r \ge 2$. Adjoin a covector Y of degree $n \ge S$. Write $\eta = Y_\rho$, $E = \sum_c W_c Y_c$, and $F = \sum_c \partial_{W_c}\partial_{Y_c}$. For an integer shift $z \ge 0$ put*

$$T_z = \sum_{l=0}^{o} \frac{(-1)^l E^l F^l}{(l!)^2 \binom{o+n+r+z-1}{l}}.$$

Define $f[n]$ by the following substitution on the prefix, followed by multiplication by η^{n-S}. On a vector V, replace V_ρ by $\sum_c V_c Y_c$ and V_c by ηV_c for $c \ne \rho$. On a covector U, replace U_ρ by ηU_ρ and U_c by $\eta U_c - U_\rho Y_c$ for $c \ne \rho$. Write $\langle\ ,\ \rangle$ for the unnormalized factorial form, with monomial diagonal $\alpha!$.

Choose a basis f_b of M_w adapted to $M_{\ge e}$. Let e_b be the order of f_b in x and $g_b = x^{e_b}[x^{e_b}]f_b$. For each e, write $d_e = \#\{b : e_b = e\}$ and $H_e = ([g_b, g_{b'}]_0)_{e_b = e_{b'} = e}$. For an integer $u \ge 0$, the matrix with values

$$A_{bb'}(n) = \frac{p^u \binom{o+n+r+z-1}{o}}{(n-S)!\,S!\prod_i d_i!}\,\langle T_z f_b[n], f_{b'}[n]\rangle$$

is polynomial in n. If the H_e are nonsingular, its determinant has degree and leading coefficient

$$s_A = \sum_b (S + o - e_b), \qquad b_A = \prod_{e : d_e > 0} \left(\frac{p^u e! \binom{z+e}{e}}{o!\,S!}\right)^{d_e} \det H_e.$$

Outline. Expand in filtration bases and get leading coefficients from the real harmonic eigenvalues of T_z. Orthogonality and Cauchy–Schwarz give strict mixed-order degree bounds, so over $\mathbb{Q}_p$ the leading determinant coefficient is the product of block determinants. The complete proof is on p. 677. □

Lemma 22.28 (Increasing the covector degree)**.** *Use the prefix and substitution of 22.27. Fix $0 \neq f \in M_w$, the degrees on the remaining letters, and all prefix thresholds. Let Z_n be the intersection of the threshold spaces with the span of products vb, where v is in the module generated by $f[n]$ on the prefix including Y, and b is a polynomial in the remaining letters of the specified multidegree. If $S \le n \le n'$, then $\dim Z_n \le \dim Z_{n'}$.*

Outline. Multiply by $Y_\rho^{n'-n}$ and project with the prefix Casimir to map the highest line onto the new highest line. The map shifts every weight by the same character and commutes with the raising operators. A maximal weight in a nonzero kernel would therefore give a highest vector. The unique highest line of the source maps nontrivially, so the map is injective. Extending it coefficientwise over the suffix monomials preserves the prefix thresholds. The complete proof is on p. 678. □

Lemma 22.29 (Integral contraction of the middle rows)**.** *Let a marked factor have $k \ge 3$ rows and put $h = k - 2$. Write c for the bound on its middle entries. Write $[P, Q] = (P, 0, \dots, 0, -Q)$ for a flat weight. Suppose λ has extremes $P + c, -Q - c$, where $P, Q \ge 0$, and put $\delta = \lambda - [P, Q]$. In a selected edge frame, there are maps*

$$i : V_{[P,Q]} \longrightarrow V_{[P+2c,Q+2c]}, \quad U_b : V_{[P,Q]} \longrightarrow V_\lambda, \quad L_a : V_\lambda \longrightarrow V_{[P+2c,Q+2c]}$$

indexed by $\dim V_{(\lambda_2,\dots,\lambda_{k-1})}$ middle directions, with $L_a U_b = \delta_{ab} p^{k_b} i$. For $k \ge 7$,

$$0 \le k_b \le \binom{h}{2} \lfloor \log_p(2c + h - 1) \rfloor.$$

Multiplication by the vector defining i gives a saturated inclusion of the selected normalized edge lattices.

Outline. Use 14.12, complementary minors and Smith reduction to pair the middle lattices, then lift the tableaux to define the contraction. Highest-tableau evaluation gives a unit leading coefficient for the multiplier of i, so adapted-frame reduction proves saturation. The complete proof is on p. 679. □

Lemma 22.30 (From prefix Smith factors to the level lattice)**.** *Let p be an odd prime. Consider a signed word of L letters in $r \ge 2$ coordinates. Mark t disjoint adjacent vector–covector pairs, in order. Impose prefix caps ending at the vectors of marked pairs or at the end of the word. Let Y be their intersection in fixed multidegree and let $M = Y \cap \mathbf{Z}_p[X]$. Use the normalized factorial form B_0 of 22.22 on the word. For pair i, use the operator T_{z_i} of 22.27 on its two letters and call it Φ_i.*

Let $X \ge 1$ bound every letter degree and let S_{tot} be their sum. Put

$$a_0 = \binom{r}{2} \lfloor \log_p(S_{\mathrm{tot}} + r - 1) \rfloor.$$

Suppose $(L + 2)a_0 \le \kappa$, $(L + 2)r^2(1 + \lfloor \log_p X \rfloor) \le \kappa$, and the degree n_i of each marked covector is at least the total degree of its preceding prefix. At each such prefix and full weight w, form the matrix of 22.27 with $u = \lceil \kappa \rceil$, using a basis of M_w. Impose the caps through the marked vector in this prefix. Write b_{iw} for the number of its Smith entries of valuation greater than $E \ge 1$, counting zero as valuation $+\infty$. For each nonzero M_w choose one nonzero $f \in M_w$, and write z_{iw} for the dimension of the space Z_{n_i} of 22.28 for this choice.

For the form $B(v, v') = B_0(\Phi_1 \cdots \Phi_t v, v')$ on M, the number of Smith entries of valuation greater than $\lfloor t(E + \kappa) + \kappa \rfloor$ is at most $\sum_{i,w} b_{iw} z_{iw}$.

Outline. Use 22.27 to identify prefix highest vectors and retain Smith entries of valuation at most E at each tie. Extend the pairing identities using 22.23 to clear denominators, and bound the number of large Smith entries by the codimension of the common fixed space. The complete proof is on p. 680. □

Lemma 22.31 (The rank-one localizing polynomial)**.** *Let s_j be the successive even starts in cyclic order modulo n', with gaps G_j and at least two starts. Write X_{ja}, $0 \le a < k$, for the vector coordinates there. In the frame in which the distinguished residue line is coordinate zero, put $h_j = p^{G_j} - 1$ and fix one start j_0. The polynomial*

$$R = X_{j_0,0} \prod_{a=1}^{k-1} \left(\prod_j X_{j0}^{h_j} - \prod_j X_{ja}^{h_j} \right)$$

has order at least one at every other residue line. At the distinguished line its reduction after the edge substitution is

$$X_{j_0,0}\prod_j X_{j0}^{(k-1)h_j}.$$

Consequently R^c supplies order c at competing lines and uses precisely $c((k-1)h_j+\mathbf{1}_{j=j_0})$ degrees at start j.

Outline. Modulo the uniformizer, edge substitution scales X_{ja} by $u_a^{p^{s_j}}$, and $u^{p^{n'}}=u$ gives $\prod_j (u^{p^{s_j}})^{h_j}=1$ for $u\neq 0$, killing R on competing lines. At $u_0=1$ this gives the monomial, and R^c gives order c and multidegrees. The complete proof is on p. 681. □

Lemma 22.32 (The depth constant for one segment)**.** *Consider a gap $G\geq 2$ with s interior positions at increasing positive odd distances $d_i<G$. Write $P_i=p^{G-d_i}$, $Q\geq p^G$, $p\geq 5$, and $k\geq 2$. Let a_1,b',L' be positive, $a_S\geq a_1$ the starting weight, $b_i\leq 2b'$ the interior weights, and $b_+\leq 2b'$ the terminal weight. Let $P_iH_i\leq b_i$, and let the capacities satisfy*

$$C_0\leq a_1/p^G,\qquad C_{i+1}=C_i+H_i-e_i,\qquad H_i,e_i\geq 0.$$

At a position with $p^{d_i}<a_1/b'$ suppose $b_i\leq a_S/p^{d_i}$ and $e_i\leq (1+4/p^2)H_i+L'C_i/k$. At the other positions suppose $b_i\geq a_1/p^{d_i}$ and $e_i\leq C_i+L'H_i/k$. Suppose also that

$$o_+\leq C_s+(L'/k+4/p^2)\sum_i 2a_1/P_i+L'b_+/k.$$

Let D_i be the sum of the nonzero-coordinate vector digits and the coordinate-zero covector digit, each digit in $[0,P_i)$. Put

$$R_*=a_1/p^2+b'/p+L'a_1/(pk)+L'b'/k+kQ.$$

The order of the block satisfies

$$o_++\sum_i(P_ie_i+D_i)\leq\sum_i\min(b_i,a_S/p^{d_i})+30(s+1)R_*.$$

Outline. Unroll the capacity recursion and bound the earlier contributions by a geometric sum using the spacing of odd positions. Bound the excess over the minimum in each range, then add the digit and terminal orders. The complete proof is on p. 682. □

Lemma 22.33 (An integral binomial expansion at an internal tie)**.** *Use the matrix of 22.27, with $m+1$ prefix letters, and let $X\geq 1$ satisfy $2S+1\leq X$. Put $\ell_X=1+\log_p X$. If*

$$u\geq\max\left\{((m+2)r+r+1)\ell_X,\ ((m+4)r+6)\lfloor\log_p(S+o+r)\rfloor\right\},$$

each entry of $A(n)$ is a linear combination over $\mathbf{Z}_p$ of $\binom{n}{j}$, $0\leq j\leq S+o$.

Outline. Use 22.27 to bound the entry degrees and 11.7 to show the scaled values at $S,\dots,3S$ are integral. Take forward differences there and translate the shifted binomial basis. The complete proof is on p. 682. □

Lemma 22.34 (Diagonal derivatives on a block with ties)**.** *Take a word of vector and covector letters in r coordinates over the fraction field of a discrete valuation ring O of characteristic zero. Impose upper bounds b_J on signed first-coordinate weights at specified prefixes J, taking the largest stable subspace at each prefix as in 22.22. Pair disjoint consecutive vector and covector letters at internal ties. Include a final vector tied to a separate covector ξ. At tie h write*

$$E_h=\sum_a W_{ha}Y_{ha},\qquad H_h=E_h+Z_hS_h',$$

where Z_h,S_h' are separate scalar variables, and fix a shift $z_h\geq 0$. For fixed full row degrees let $L(b)$ be the O-span of

$$\prod_h H_h^{j_h}Z_h^{z_h}\,y\,\xi^\alpha.$$

Here j_h ranges from zero to the lesser of the paired degrees, y has the degrees obtained by subtracting j_h from both letters of each pair, with the terminal covector recorded separately by ξ^{α}, and y is integral and satisfies all the prefix bounds.

For a multi-index ν supported on interior letter coordinates and internal scalar variables, let T_ν act on a monomial X^{μ} by multiplication by $\prod_v \binom{\mu_v}{\nu_v}$. Then

$$T_\nu L(b) \subseteq L\big((b_J + |\nu|)_J\big).$$

Furthermore, an integral polynomial in the fraction-field span of $L(b)$ belongs to $L(b)$.

Outline. Extract coefficients after scaling one variable at a time to bound the increase in prefix weights, expanding the affected tie factors. For saturation, set $Z_h = 1$ and replace S'_h by $S'_h - E_h$ to recover integral reduced coefficients. The complete proof is on p. 683. □

Lemma 22.35 (The integral rank-one column estimates)**.** *Let S have N marks on an even cycle of length n', with at least two even marks and successive even gaps at most G'. Each gap has at most N_1 interior marks. Suppose $N_1 \ge 1$, $2N_1+2 \le k-1$, $k^2Y^2 \le p < a/b'$, $Y \ge 1$, $b' > 0$, and p prime. Let $\mathcal{L} = C_0 N_1^2 \log a$, $Y_b > 0$, $\Delta \ge 0$, $c_b, D \in \mathbf{N}$, and put $c_* = 8(D+1)$, $c_4 = c_* + 2Nc_b + 2n'c_b$, $\rho_{\mathrm{loc}} = 2HN_1\mathcal{L}/k + 4\Delta/b'$. Assume*

$$\begin{aligned} &H \le Y_b, \quad 2Y_bN\mathcal{L} < k, \quad 8\log(NY) \le \mathcal{L}, \quad Y_bN\mathcal{L}b'/k \le \Delta \le b'/2, \\ &kY^2Y_bN\mathcal{L}\log a \le a/b', \quad HN\mathcal{L}^2 \le k, \quad \rho_{\mathrm{loc}} < 1/3, \quad c_b \le \mathcal{L}b'/k, \\ &HNY^2(N_1^5 + k^3)\log_p a \le D, \qquad Hk^3Np^{4G'}(c_* + (N+n')c_b)Y \le b'. \end{aligned} \tag{22.18}$$

Suppose $(a/b')[1-\rho_{\mathrm{loc}}, 1+\rho_{\mathrm{loc}}]$ contains no odd positive power of p. Take $C_0 \ge T_0$, $H \ge T_2(C_0)$, and $C = C_*(C_0)$ for suitable absolute constants $T_0, C_A, C_s > 0$ and positive functions T_2, C_*. Use the nonempty integer intervals

$$I_a = [\lceil a \rceil, \lfloor a + a/(kY) \rfloor], \qquad I_b = [\lceil b' \rceil, \lfloor b' + \Delta \rfloor].$$

Put $a_1 = \lceil a \rceil$, $L' = C_0 N_1^2 \log a_1$, $J_0 = L'b'/k$, and $J_b = \lfloor J_0 \rfloor$. For $u \ge 0$ let $P(u) = \binom{u+k-1}{k-1}$, with $P(u) = 0$ for negative u, and set

$$d(u,v) = P(u)P(v) - P(u-1)P(v-1), \qquad q(u,v,i) = d(u-i, v-i)/(P(u)P(v))$$

for $0 \le i \le \min(u,v)$, and zero otherwise. For each local model there are nonnegative f, g satisfying the pointwise inequality of 22.12 on the enlarged box with $J = J_b$, $c = c_b$, and

$$\mathbb{E}_+ g \le (C_A + C_s) N N_1 / Y. \tag{22.19}$$

For each original A, B and every dominant base weight with those extremes and other entries of absolute value at most c_b, the rank-one test passes at fraction $f_- + 2f(A + c_b, B + c_b) + 12Nkc_b/b'$, order c_*, denominator D, and the ceiling of

$$\begin{aligned} &J_{\mathrm{odd}} + CNN_1(a/p^2 + b'/p + \mathcal{L}a/(pk) + \mathcal{L}b'/k) \\ &\quad + CN\{p^{G'}(c_* + CNk\log_p a + 4Nc_b + 2n'c_b) + kp^{G'} + k\log_p a + c_b\} + 2n'c_b, \end{aligned} \tag{22.20}$$

where J_{odd} is the odd-mark sum in 14.10.

Outline. Bounds segment Gram losses via 22.11 and 22.30, lifting paired block families with 22.17, 22.32, 22.34, and 22.31. Splits into harmonic summands via 22.8, then inserts middle rows by 22.29 and 22.10. The complete proof is on p. 683. □

Lemma 22.36 (The comparison on one good orbit)**.** *Fix a family in the design class of 33.6. Write $a = xb$, $c = 4S_c$, and $L = \log x$. At a candidate prime p, put $z = \log_p(2a)$ and*

$$H_- = \frac{\mathcal{L}a}{pk} + ap^{-L_1^{\mathrm{loc}}}, \qquad H_+ = \frac{\mathcal{L}a}{pk} + \frac{a}{p^2}, \qquad L_1^{\mathrm{loc}} = \left\lceil \log_p((kx^{26})^4) \right\rceil.$$

There is a family constant $C_* \ge 3$ for which the following holds at all sufficiently large scales. Define

$$\ell_p^{\mathrm{int}} = C_*(c+\eta)z + \begin{cases} C_* r_G b' L + C_* L H_-, & p < p_0, \\ b'/x + \eta + C_* w'' L/\delta_1 + b_0 + C_* L H_+, & p_0 \le p < X', \end{cases} \qquad b_0 = \exp(x^{\varepsilon/2}).$$

Let v be a good orbit with representative t_0, let $T = T(t_0)$ be its set of marks as in 14.1, and fix $1 \le j \le m$. Write $P_{v,j} = \Pi_p(t_0, j)$ for the global price in (14.4). For $s \in T$ put

$$A_s = \sum_{\substack{h \ge 1 \\ h \text{ odd}}} [a_s/p^h - W_j(z_h(s))]_+.$$

With the depth t_v and denominator D_p of 14.9, one has

$$\sum_{s \in T} (A_s - \ell_p^{\mathrm{int}}) \le g_U(t_0, j) + \bar{\delta}(t_0) - P_{v,j} - \lceil t_v \rceil_+ - D_p - 1. \tag{22.21}$$

Outline. Split the odd-power sum into positive parts and minima and bound the change to the data of 14.4. Use 14.7 and 14.10 to bound the costs, charging odd marks above the cutoff to distinct minima at preceding even marks by 14.2. The complete proof is on p. 695. □

Theorem 22.37 (Local comparison on the design class)**.** *Given the twist selection and family duties of 33.6, the following assertion is closed on $\mathcal{D}_3(\mathcal{A})$ in the sense of 36.10. For each such family F_m, let $I(x)$ index its models at scale x. There are $C \ge 0$ and x_0 for which the following bounds hold for every $x \ge x_0$ and every $i \in I(x)$. For every word u and place w with $p(w) \ne T_u$, the containment of 21.7 holds at exponent $\lfloor e_w \tau_w \rfloor_+$, and $\ell_{u,w} \le d_M(\tau_w + \mathrm{io}_w)$, where $e_w \ell_{u,w}$ is the length of $(M_{H,u})_{(w)}/(N_u)_{(w)}$ as in 21.6, and d_M is as in 16.5. If $V_p = \varnothing$, these three quantities $\tau_w, \mathrm{io}_w, \ell_{u,w}$ vanish.*

For $v \in V_p$, write $N_v = |TT \cap v|$ and S_v for the ceiling in 14.9. Then $S_v \le 2mN_v a/p$. At a signal or inert-class prime $p \le X''$ the gain is between zero and $4n_T a/p$ per lift. At a served prime in this range, (22.2) holds with $r_p = F_p$ and $\ell_p = C_p + \mathbf{1}_{\{p \text{ inert class},\ p<X'\}}(R_p + O_p)$. For every signal or inert-class prime $p \le X''$, the same inequality holds with g_{jp} replaced by $g_{p,j}^{\mathrm{count}}$ of Definition 14.8. Use the row bound S_c of 15.12 and set $N_0 = N_m$, $t_0 = t$, $L_1(p) = 1 + \lceil \log_p(k^2 x^{78}) \rceil$ for the prime-sum loss, distinct from $L_1^{\mathrm{loc}} = \lceil \log_p((kx^{26})^4) \rceil$ in 22.36, and $b_0 = b^{x^{-\varepsilon/2}}$ in (22.4) and (22.12). For a candidate with $\Sigma_p \le x^{-\varepsilon}$ the selection V_p is the full set of good representatives. On each good candidate orbit the mean of κ_v over its local box is at most $N_v x^{-12}$, pointwise below p_0.

Outline. Extend the local tests by scalars, with loss bounds from 22.40 and 22.41. Apply 16.5 and 16.3 to bound the local length and prove containment, then sum the orbit comparisons of 22.36 to bound the gain. The complete proof is on p. 697. □

Lemma 22.38 (Numerical thresholds for the weight box)**.** *For each $0 < \varepsilon < 1/52$ there is a threshold datum $\mathcal{K}_W(\varepsilon)$ in the sense of 36.10 with the following property for every family of parameter ε and every $CH \in \mathbf{R}$ passing its size tests. Write C_0 for the middle orbit constant multiplying $N_1^2 \log a_1$ in 22.35, C_N for the last-list coefficient allowed in $N_{\mathrm{win}} \le C_N \log x$, C_{I} for the family coefficient in $C_{\mathrm{I}} N \mathcal{L}^2 \le k$, and C_ρ for the coefficient of $C_N(\log x)\mathcal{L}/k$ in the band width ρ_B of 14.8. Then $C_0 > 0$, the coefficient inequalities*

$$C_0 \ge T_0, \qquad H = T_2(C_0) \le \min(C_{\mathrm{I}}, K_{\mathrm{DI}}, C_\rho/2), \qquad 2C_*(C_0) \le K_1, \quad C_*(C_0) \le K_2 \tag{22.22}$$

hold (with T_0, T_2, C_* as in 22.35), and $N_{\mathrm{win}} \le C_N \log x$ at all sufficiently large indexed points. The tests use the numerical lists of the family and CH.

Outline. Bound N_{win} via 14.8. Use T_0 from 22.35 for the middle threshold and maximize over A_ε and the doubled T_2 and C_* values on the eligible list for the last threshold. The complete proof is on p. 699. □

Lemma 22.39 (Size bounds on good candidate orbits)**.** *Fix a family and a real CH passing the numerical tests of 22.38. Set $L_1^{\mathrm{loc}} = \lceil \log_p((kx^{26})^4) \rceil$, $N_1 = N_{\mathrm{win}}$. Eventually each good candidate orbit of length n' with N marks has even n' and*

$$2L_1^{\mathrm{loc}} \le L_2 \le 2\lceil 256 \log x \rceil < u_1, \qquad 3G' \le x^{\varepsilon/4} \le n',$$
$$N \le 40 n' \delta_1, \qquad n' \le N C_{31} \log x/\delta_1, \qquad N_1 \le C_N \log x.$$

Successive even marks give at least two cyclic segments of length at most G' with at most N_1 interior marks. Here C_N is the window coefficient of 22.38, and C_{31} is a family constant.

Outline. Bound L_2, G', N_1 from parameter bounds, and exclude odd n' since an odd order divides $o \le x^{\varepsilon/100}$, contradicting $n' \ge x^{\varepsilon/4}$. The run condition bounds even gaps by G', giving $n' \le NG'$, two gaps, and the N bound. The complete proof is on p. 699. □

Lemma 22.40 (Projection at rank r_G)**.** *Let $C \ge 0$ be the absolute constant of 22.18. Suppose the family passes the tests of 22.38 at CH and its share, depth, orbit-factor, denominator, weight-size and rank coefficients are each at least C. Then eventually every good candidate orbit below p_0 passes 14.10 at $r = r_G$, depth $\lceil t_{II} \rceil_+$, denominator D_{II}, and fraction* $\min(1, C_{\mathrm{frac}} N r^2 k / x^{26})$.

Outline. Use the orbit bounds of 22.39 and the family thresholds to check the hypotheses of 22.18 with $Y = x^{26}$. Compare its required depth and retained rank with those of the rank test. The complete proof is on p. 699. □

Lemma 22.41 (Mean projection loss at rank one)**.** *For every family and real CH passing the numerical tests of 22.38, eventually each good candidate orbit at $p \ge p_0$ has mean least passing fraction in 14.10 at most $C_m N N_1 / Y$ in its uniform integer box, with $N = N_v$, $N_1 = N_{\mathrm{win}}$, $Y = x^{13}$.*

Outline. Check the size bounds for 22.35 using 22.39 and the band exclusion of 14.8, then return to the original box by 22.12. Transport the pairings to the local models and compare depths before averaging the bound on the least passing fraction. The complete proof is on p. 700. □

22.6 Arithmetic models and common weights

For a model $i \in I(x)$ define

$$\mathrm{IO}_i = \sum_{w \in \mathrm{supp}_i} \frac{e_w \log N\mathfrak{p}_w}{[K : \mathbf{Q}]} \, \mathrm{io}_{i,w}. \tag{22.23}$$

A consistent presentation uses the model's frame, layer count, host arithmetic datum, and weighted top labels. Its w_j reads lift $j+1$ at the code of an embedding.

Theorem 22.42 (Loss bounds for a small-share model)**.** *For each inert-class prime p, let $b_p(TT)$ count the top positions whose h_p-orbit fails the window or tile condition of 8.15. Suppose that for all sufficiently large x and every model $i \in I(x)$ of F_m,*

$$\frac{\delta_1 n}{2} \le n_T \le 2\delta_1 n, \qquad \sum_{\substack{p < X' \text{ of inert class} \\ \mathrm{ord}(h_p) \ge x^{\varepsilon/4}}} \frac{\log p}{p} \, b_p(TT) \le n_T x^{-2+\varepsilon}.$$

Assume the conclusion of 22.37 with its constant written C_G and scale $b_0 = b^{\delta(x)}$ for some $\delta(x) \to 0$ (for example $\delta(x) = x^{-\varepsilon/2}$ as supplied by 22.37). There are $c_{\mathrm{Loss}} \ge 0, x_0$ such that for $x \ge x_0$, $i \in I(x)$, $R_W(i) \le x^{-9}$,

$$\mathrm{IO}_i \le c_{\mathrm{Loss}} Q_i, \qquad Q_i = |TT_i| m_i b_i. \tag{22.24}$$

For every consistent presentation with an integer cutoff satisfying

$$a_t / w_j(\rho) \le N_{\mathrm{cut}} \quad (j = 0, \dots, m_i - 1,\ t \in TT_i,\ \rho \in \Phi), \tag{22.25}$$

one has

$$\mathrm{ASK}_{N_{\mathrm{cut}}} - \mathrm{Ga}_i \le c_{\mathrm{Loss}} Q_i. \tag{22.26}$$

Outline. Check windows by 22.4, discards by 15.14 and served primes by 22.37. Use 22.5 and bad orbits for losses, 14.9 for the index, 22.1 for prime weights, 22.43 for cutoff existence and 22.7 for both loss bounds. The complete proof is on p. 701. □

Lemma 22.43 (Places and node sizes)**.** *For a model D in its own frame, the archimedean indices map bijectively to Φ. At $x \geq \max(x_1 + 1, 3)$ in an admissible family, assign the model weights at lift $j + 1$ for $0 \leq j < m$ to matched indices and assign weight one to unmatched positions. Then $w_j(\rho) \geq 1$ on Φ. At thick positions $b \leq w_j(\rho) \leq b + \eta_{\mathrm{sz}} + 1$, and elsewhere on Φ, $w_j(\rho) \leq 2b'$.*

Outline. Composing e_τ with T of 5.2 maps the archimedean indices bijectively onto Φ, since the image has size $f = 2n = |\Phi|$. Parameter inequalities of 14.2 place mark and thin weight intervals in $[1, 2b']$, while thick labels give the thick interval. The complete proof is on p. 702. □

For the top labels $\mathcal{T}(x) \subset \mathbf{N}$, an admissible integer pair for a base model satisfies

$$a \leq a_v(t) \leq a + a/(kx^{13}), \qquad b' \leq b_v(t) \leq b_{\mathrm{top}}(x) \qquad (t \in \mathcal{T}(x)). \tag{22.27}$$

For $x > x_1$, $b_{\mathrm{top}}(x) = b' + 2n\delta_1 x^{\varepsilon}\mathcal{L}b'/k$. A model realizes the pair when its top weights agree with it.

Lemma 22.44 (A single weight vector for the translations)**.** *Assume 22.42 and the following common-weight hypothesis. At each sufficiently large window point x there are a base model $i_0 \in I(x)$ and functions $a_v, b_v : \mathbf{N} \to \mathbf{Z}$ satisfying (22.27) for i_0. Some model in $I(x)$ realizes this pair, and every model i realizing it satisfies $R_W(i) \leq x^{-9}$. Then for some $c_{\mathrm{Loss}} \geq 0, x_1$ the same pair works at every translation and every realization satisfies (22.24) and (22.26) for window points $x \geq x_1$. The common-weight exponent is $e_{\mathrm{Loss}} = 0$.*

Proof. Choose a pair supplied by the common-weight hypothesis. Every realization satisfies $R_W(i) \leq x^{-9}$, so 22.42 gives both loss bounds. The covering property carries the pair to every integer translation $0 \leq u_0 < n$. □

Proposition 22.45 (Small share in the integer weight box)**.** *There is $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ there is a threshold $\mathcal{K}$ with the following property. For every family F_m of parameter ε and every $CH \in \mathbf{R}$ for which the family passes the size tests of 36.10 at threshold $\mathcal{K}$, there is $x_0(F_m, CH)$ such that for every $x \geq x_0$ and every $i \in I(x)$, one can choose integer functions A, B on the model group, zero off TT, satisfying (22.27) on TT and*

$$R_W(A, B) = \sum_{p \in \mathcal{P}_c} \frac{\log p}{p} \left(21_{\{\Sigma_p(A,B) > x^{-\varepsilon}\}} + 4\Sigma_p(A, B)\right) \leq x^{-9}.$$

Outline. Set the threshold by 22.38 and average over the integer box using 14.8, 22.41 and 22.39. Injectivity and disjointness bound $\mathbb{E}\Sigma_p$, and the excess-share and prime-sum bounds give a pair with $R_W \leq x^{-9}$. The complete proof is on p. 702. □

Lemma 22.46 (The weights of a model in the constructed family)**.** *Suppose that for all sufficiently large x, every model $i \in I(x)$ of F_m satisfies $R_W(i) \leq x^{-9}$ at its own top weights. Then F_m satisfies the common-weight hypothesis of 22.44.*

Outline. Pick $i_0 \in I(x)$ beyond the thresholds, transport its top weights to $\mathcal{T}(x)$, and extend by zero so i_0 realizes the pair and every realization satisfies the share bound by hypothesis. For the constructed family, 33.26 supplies $R_W(i) \leq x^{-9}$. The complete proof is on p. 702. □

Chapter 23

A slope bound for families of maps

23.1 Slopes and height

Let K be a totally complex number field with ring of integers $\mathcal{O}_K$. Put

$$n_K = [K : \mathbf{Q}], \qquad c_K = -\log 2 + \frac{\log|\operatorname{disc} K|}{n_K}.$$

A lattice in a finite dimensional K-space V is a finitely generated $\mathcal{O}_K$-submodule. Write $r(M) = \dim_K \operatorname{span}_K M$. Choose one embedding $K \hookrightarrow \mathbf{C}$ at each complex place v and give the scalar extension V_v a Hermitian norm. For a lattice L, its diagonal image in $\bigoplus_{v|\infty} V_v$, with squared norm $\sum_v \|x_v\|_v^2$, has real rank $\rho(L) = n_K r(L)$.

For $L \neq 0$, let

$$\mu_{\mathrm{real}}(L) = -\frac{\log \operatorname{covol}(L)}{\rho(L)}, \qquad \mu(L) = 2\mu_{\mathrm{real}}(L) + c_K,$$

where the covolume is taken in the real span of L. The Harder–Narasimhan (HN) filtration uses induced and orthogonal quotient metrics. The maximum real slope is taken over nonzero sublattices and the minimum over nonzero quotients by saturated sublattices. Use the same affine change for the extreme slopes $\mu_{\min}$ and $\mu_{\max}$. For the zero lattice the real slopes are zero by convention and the normalized slopes are c_K.

For a nonzero K-linear map $\phi : V \to W$, its height is

$$h(\phi) = \frac{1}{\#\{v \mid \infty\}} \sum_{v|\infty} \log \|\phi_v\|_{\mathrm{op}}^2.$$

Proposition 23.1 (The slope inequality for one map)**.** *Let V, W be finite dimensional K-spaces with the preceding metrics and $A \subseteq V$ a lattice. Let $\phi : V \to W$ be a K-linear map with $\phi(A) \neq 0$. Then*

$$\mu_{\min}(A) - h(\phi) \leq \mu_{\min}(\phi(A)).$$

For every lattice $B \subseteq W$ containing $\phi(A)$,

$$\mu_{\min}(A) \leq \mu_{\max}(B) + h(\phi).$$

If ϕ is injective on A, then also

$$\mu_{\max}(A) \leq \mu_{\max}(B) + h(\phi).$$

This is the totally complex integral case of the slope inequality of Bost and Chen, in doubled slopes [77, Propositions 1.2 and 2.10]. By Proposition 1.2 the maximal slope is invariant under finite extension. Duality gives the same assertion for the minimal slope, so Proposition 2.10 applies over K. Restrict the source to the span of A. The target is the span of $\phi(A)$ for the surjective inequality and the span of B for the other two. The lattice inclusions bound the finite-place norms by one. Restriction weakly decreases the archimedean norms, so twice the full height is at most $h(\phi)$.

Proposition 23.2 (Counting slopes with the HN filtration)**.** *Let L be a lattice, and let $\lambda_1(L), \dots, \lambda_{\rho(L)}(L)$ be its real HN slopes, with a factor of rank j over K contributing $n_K j$ entries. For $\theta \in \mathbf{R}$, let $L_{\geq\theta}$ be the saturated HN step containing the slopes at least θ. Then*

$$\#\{j : \lambda_j(L) < \theta\} = \rho(L) - \rho(L_{\geq\theta}).$$

For $\rho(L) > 0$ and $a \in \mathbf{R}$,

$$\sum_{j=1}^{\rho(L)} (2\lambda_j(L) + a) = \rho(L)(2\mu_{\mathrm{real}}(L) + a).$$

For $L \neq 0$, the least and greatest entries of $2\lambda_j(L) + c_K$ are $\mu_{\min}(L)$ and $\mu_{\max}(L)$. If $L_{\geq\theta} \neq 0$, its minimum real slope is at least θ. Every nonzero sublattice $M \subseteq L$ with minimum real slope at least θ lies in $L_{\geq\theta}$.

Gaudron's Proposition 5.13 gives the canonical filtration. Lemmas 5.10 and 5.11 identify its extreme slopes in the Hermitian case [87]. These are the Hermitian formulas in the normalization by real rank and covolume.

Lemma 23.3 (The share below a threshold)**.** *For each i in a finite set I, let $x_{ij} \in \mathbf{R}$ for $1 \leq j \leq m_i$, where m_i is a nonnegative integer. Put $R = \sum_i m_i$. Suppose $U, \tau \geq 0$, $s \in \mathbf{R}$, and*

$$\sum_{i,j} x_{ij} \geq Rs, \qquad x_{ij} \leq s + U.$$

Define

$$z_U(\tau) = \begin{cases} U/(U+\tau), & U > 0, \\ 0, & U = 0. \end{cases}$$

Then

$$\sum_i \#\{j : x_{ij} < s - \tau\} \leq R z_U(\tau).$$

Proof. Let q be the number of entries below $s - \tau$. For $U > 0$,

$$Rs \leq \sum_{i,j} x_{ij} \leq q(s-\tau) + (R-q)(s+U) = R(s+U) - q(U+\tau).$$

If $U = 0$, the hypotheses force every entry to equal s. □

23.2 The image of a family at one complex place

Let V_i for i in a finite set I, and Z, be finite dimensional complex Hilbert spaces. Let $\widetilde{B}_i : V_i \to Z$ be complex-linear and $c_i > 0$. Put

$$d_i = \dim_{\mathbf{C}} V_i, \qquad R = \sum_i d_i, \qquad D_Z = \dim_{\mathbf{C}} Z, \qquad P_i = c_i \widetilde{B}_i^* \widetilde{B}_i, \qquad Q_Z = \sum_i c_i \widetilde{B}_i \widetilde{B}_i^*.$$

Write $\|T\|_{\mathrm{HS}}^2 = \mathrm{tr}(T^*T)$. The two frame estimates are

$$\sum_i \|P_i - 1\|_{\mathrm{HS}}^2 \leq eR, \qquad 0 \leq e \leq \tfrac{1}{4}, \tag{23.1}$$

$$D_Z \,\mathrm{tr}(Q_Z^2) \leq (1+\eta)(\mathrm{tr}\, Q_Z)^2, \qquad \eta \geq 0. \tag{23.2}$$

Theorem 23.4 (Codimension of the images)**.** *Suppose $R > 0$ and (23.1)–(23.2) hold. Let $V_i' \subseteq V_i$ be complex subspaces with $\sum_i (d_i - \dim_{\mathbf{C}} V_i') \leq zR$. Then*

$$\mathrm{codim}_Z \sum_i \widetilde{B}_i V_i' \leq (\eta + 4z + 4\sqrt{e}) D_Z.$$

Outline. Use Cauchy–Schwarz and the source frame bound to control the trace on the orthogonal complement of the restricted images. The target frame bound gives $q - \sqrt{\eta q} \le 2(z + \sqrt{ze})$, which is estimated separately for $z \ge 1/4$ and $z < 1/4$. The complete proof is on p. 702. □

Lemma 23.5 (From complex dimension to K-rank)**.** *Let $B_i : V_i \to W$ be K-linear maps between finite dimensional K-spaces. Put $d_i = \dim_K V_i$ and $D_C = \dim_K W$. Assume $R = \sum_i d_i > 0$. Suppose that at a complex place v, after a common invertible change of target coordinates and an invertible change in each source, there are lifts*

$$\widetilde{B}_i : \mathbf{C}^{d_i} \longrightarrow Z = \mathbf{C}^{D_C} \oplus \mathbf{C}^{D_0}$$

whose projections onto the first summand represent $B_{i,v}$. Assume (23.1)–(23.2) with $D_Z = D_C + D_0$. If $\mathcal{O}_K$-submodules $N_i^\tau \subseteq V_i$ satisfy $\sum_i (d_i - r(N_i^\tau)) \le zR$, then

$$D_C - r\left(\sum_i B_i N_i^\tau\right) \le (\eta + 4z + 4\sqrt{e}) D_Z.$$

Proof. Let V_i' be the complex span of N_i^τ in the source coordinates, so $\dim_{\mathbf{C}} V_i' = r(N_i^\tau)$. The projection sends $S = \sum_i \widetilde{B}_i V_i'$ onto the complex span of $\sum_i B_i N_i^\tau$ in the target coordinates. The kernel of the projection has dimension D_0, so $D_Z - \dim_{\mathbf{C}} S \ge D_C - r(\sum_i B_i N_i^\tau)$. Apply 23.4. □

23.3 Integration of the slope count

Proposition 23.6 (A finite layer-cake inequality)**.** *Let $D \ge 1$ be an integer, $s \in \mathbf{R}$, $H \ge 0$, and $x_1, \dots, x_D \ge s - H$. Let $w : [0, H] \to \mathbf{R}$. Suppose*

$$\#\{j : x_j < s - \tau\} \le D w(\tau) \qquad (0 \le \tau \le H)$$

and $\min(1, w)$ is integrable on $[0, H]$. Then

$$s - \int_0^H \min(1, w(\tau))\, d\tau \le \frac{1}{D} \sum_{j=1}^{D} x_j.$$

The counting-measure form of Lieb and Loss [88, Theorem 1.13] gives

$$\sum_j (s - x_j)_+ = \int_0^H \#\{j : x_j < s - \tau\}\, d\tau.$$

Use $s - x_j \le (s - x_j)_+$ and bound the integrand by $D \min(1, w(\tau))$.

For $U, H \ge 0$, define

$$\varphi(U, H) = \begin{cases} U \log(1 + H/U), & U > 0, \\ 0, & U = 0, \end{cases}$$

and for $0 \le \delta < 1$ and $\eta, e \ge 0$ put

$$E_{\mathrm{fr}}(U, H; \delta, \eta, e) = \frac{4\varphi(U, H) + (\eta + 4\sqrt{e})H}{1 - \delta}. \tag{23.3}$$

Theorem 23.7 (The strong family slope inequality)**.** *Let I be finite and nonempty. Let N_i be full lattices in finite dimensional K-spaces V_i, and $C^\vee$ a full lattice in a finite dimensional K-space W. Give the spaces Hermitian norms at the complex places. Let $B_i : V_i \to W$ be K-linear. Choose $h_i \in \mathbf{R}$ for each $i \in I$ and suppose*

$$B_i N_i \subseteq C^\vee, \qquad B_i \ne 0 \implies h(B_i) \le h_i.$$

Set $d_i = \dim_K V_i$ and $R = \sum_i d_i > 0$. At one complex place suppose the maps admit lifts to a common Z as in 23.5 with the frame estimates (23.1)–(23.2). Write $D_C = \dim_K W$ and $D_Z = \dim_{\mathbf{C}} Z$, and suppose $0 \le \delta < 1$ and $(1-\delta)D_Z \le D_C$. Let $s \in \mathbf{R}$, $U, H \ge 0$, and assume

$$Rs \le \sum_i d_i(\mu(N_i) - h_i), \qquad \mu_{\max}(N_i) - h_i \le s + U \quad (i \in I), \qquad s - \mu_{\min}(C^\vee) \le H.$$

Then

$$s \le \mu(C^\vee) + E_{\mathrm{fr}}(U, H; \delta, \eta, e).$$

In particular, under the same hypotheses,

$$s \le \mu(C^\vee) + \log \#I + E_{\mathrm{fr}}(U, H; \delta, \eta, e).$$

Outline. Bound omitted source ranks by 23.2 and 23.3 and place retained images by 23.1. Convert the target deficit from 23.5 to a slope count by 23.2, integrate by 23.6, and use $\log \#I \ge 0$ for the second bound. The complete proof is on p. 703. □

23.4 The frame supplied by the period maps

Fix $C_0 \ge 1$ and $C_H > 0$. Take $x > 1$ with $\sigma = x^{-C_H} \le 1/20$. At translation u_0, let $\mathcal{W} = W(u_0)$ be the set of words. The period maps B_u of 21.32 have divided lattices N'_u and target $RR^\vee$. Put $d_u = r(N'_u)$ and $R = \sum_u d_u$. Write $M^0_{u,\mathbf{C}}$ for the unrestricted level-zero column space and $f_u = \dim_{\mathbf{C}} M^0_{u,\mathbf{C}}$. At one complex place suppose

$$\widetilde{B}_u = A_u J_u, \qquad A_u : \mathbf{C}^{f_u} \to Z, \qquad J_u : \mathbf{C}^{d_u} \to \mathbf{C}^{f_u}, \quad J_u^* J_u = 1.$$

For $c_u > 0$, put $K_{uv} = \sqrt{c_u c_v}\, A_u^* A_v$ and $\mathcal{R} = \sum_u f_u$.

The frame input consists of $D_* > 0$, $\eta_0, \eta_1 \in \mathbf{R}$, $e_0 \ge 0$ and $0 \le d_e \le 1/4$ with

$$\|K_{uv}\|^2_{\mathrm{HS}} \le (1+\eta_0)\frac{f_u f_v}{D_*} \qquad (u \ne v), \tag{23.4}$$

$$\|K_{uu} - 1\|^2_{\mathrm{HS}} \le e_0 f_u \qquad (u \in \mathcal{W}), \tag{23.5}$$

$$D_Z \le (1+\eta_1)D_*, \qquad \sum_u (f_u - d_u) \le d_e \mathcal{R}, \tag{23.6}$$

$$-1 \le \eta_0 \le \sigma/2, \qquad \eta_1 \le \sigma/2, \qquad \sqrt{e_0} \le \sigma/2. \tag{23.7}$$

Fix $e_B \ge 2e_0$. In the period model put $L = x^\varepsilon$ and $b = e^L$, with $0 < \varepsilon < 1/52$. Let m be the chain length and k the starting rank. Write f for the number of complex places of the host CM field. With $\Gamma = \exp(C_\Gamma m^5 f L)$ for $C_\Gamma > 0$, the counting inputs are

$$C_0^{-1} \le \frac{f_u \Gamma^2}{D_*} \le C_0 \quad (u \in \mathcal{W}), \qquad \#\mathcal{W} \ge C_0 \Gamma^2 x^{C_H}. \tag{23.8}$$

Theorem 23.8 (Application to the divided period family)**.** *Assume (23.4)–(23.8). Let N'_u and $RR^\vee$ be full lattices with $d_u > 0$ for each word. Suppose*

$$B_u N'_u \subseteq RR^\vee, \qquad B_u \ne 0 \implies h(B_u) \le h_u,$$

and the projections of $\widetilde{B}_u$ represent B_u in invertible coordinates at the chosen place as in 23.5. Suppose $0 \le \delta < 1$ and $(1-\delta)D_Z \le r(RR^\vee)$.

In the period model take $D_* = D^\star$ from 21.35 and $\delta = 1 - D_{\mathrm{rows}}/D_{\mathrm{all}}$, where $D_{\mathrm{all}} = D_Z$ and $D_{\mathrm{rows}} = r(RR^\vee)$ are the full and retained row counts.

Define

$$s = \frac{\sum_u d_u(\mu(N'_u) - h_u)}{R}, \qquad U = \max_u\{\mu_{\max}(N'_u) - h_u\} - s.$$

Then the frame estimates hold with

$$\eta = 5\sigma + 3\sqrt{d_e}, \qquad e = 2e_0 \le e_B, \qquad 0 \le e \le \tfrac{1}{4}.$$

For every $H > 0$ with $s - \mu_{\min}(RR^\vee) \le H$,

$$s \le \mu(RR^\vee) + \log \#\mathcal{W} + E_{\mathrm{fr}}(U, H; \delta, \eta, e).$$

Outline. Compress the source defects and combine the Gram second moment, compressed trace lower bound and dimension bound to get $\eta = 5\sigma + 3\sqrt{d_e}$. Use the mean condition of 23.7 and the slope bound from 23.2 to apply 23.7. The complete proof is on p. 704. □

Lemma 23.9 (Gram estimates in the arithmetic coordinates)**.** *Use the selected realization of 33.6 under the source and size hypotheses of (33.22) in 33.78. Use the level models of 36.5 and the row system of 37.13. Use the Bessel identification of 21.32 for the period maps. Fix the outer parameters of 33.78, so the source tuple in 33.7 is*

$$(C_0, C', C_\ell, C_{YL} + 1, \tfrac{1}{4}, 3, \tfrac{1}{2}, C_{Tu}), \qquad C_0 = 3.$$

At $A = 1$ take the signed-estimate constants C_B, C_4 of 33.99. For the joint choice in 23.12, take the common middle value $C_\Gamma = C_H = C_F$ above both $1 + \max(C_B, C_4)$ and the middle threshold of 21.33. The exponent F_0 there depends on ε and the already fixed outer constants. Use this middle value in Γ and in the column-size conditions. Then choose C', C_ℓ above the final thresholds of the spectral rows in 33.78, including the same-word thresholds of 33.18 and 33.99. Require $C_\ell \ge 40$ for 21.36. Use the absolute convergence and ties of 33.78 for this realization.

Uniformly for large x, one may take

$$\eta_0 = b^{-6}, \qquad e_0 = 2C_0/\Gamma + 2e^{-kf\log k}, \qquad \eta_1 = e^{-c_7 rf\log r}, \qquad d_e = 2f/b$$

in (23.4)–(23.6), with $r = k + m - 1$ and $c_7 = 1/16$. Under Green's parametrization of cuspidal characters by irreducible polynomials (Definition 32.6) and the local-law conditions of Theorem 32.53 and size conditions of Theorem 32.62 (at $C_s = 6$, as in Theorem 32.63), the target defect of 23.8 satisfies

$$0 \le \delta \le e^{-x^{\alpha/7}}, \qquad \alpha = \varepsilon/J \tag{23.9}$$

with $J \ge 1$ the parameter of 32.62.

Outline. Normalize the identity cell and bound the remaining Gram terms using 33.113, 33.18, and 33.99. Transfer via 21.32, bound row defects by 37.13 and 32.62, and estimate rank loss using 21.9. The complete proof is on p. 705. □

23.5 Parameters common to the family

Theorem 23.10 (Uniform frame error and the height interval)**.** *Fix an admissible family and constants $C_H > 0$, $C_0 \ge 1$. Suppose that for all sufficiently large x the full lattices and period-map data of 23.8 and the counting inputs hold uniformly, with $\sigma = x^{-C_H}$. Suppose the members admit the frame input of 23.8 with a common $d_e = d_e(x)$ and $2e_0 \le e_B = e_B(x)$, where*

$$0 \le d_e, e_B \le \tfrac{1}{4}, \qquad (3\sqrt{d_e} + 4\sqrt{e_B})x^{C_H} \le 1.$$

Then beyond a common threshold 23.8 applies to every member with $0 \le \eta \le \bar\eta$ and $0 \le e \le \bar e$, where

$$\bar\eta = 5x^{-C_H} + 3\sqrt{d_e}, \qquad \bar e = e_B, \qquad (\bar\eta + 4\sqrt{\bar e})x^{C_H} \le 6.$$

In the period model with the division data of 21.32, write RR_{u_0} for the full positive-rank lattice of translation u_0. All translations use $H = x^{C_H} b$, where $b = e^{x^\varepsilon}$. For each u_0,

$$s(u_0) \le \mu(RR^\vee_{u_0}) + \log \#W(u_0) + \frac{4\varphi(U(u_0), x^{C_H} b) + 6b}{1 - \delta(u_0)}.$$

Outline. Above the cutoffs and $20^{1/C_H}$, 23.8 gives $\eta = \bar{\eta}$, $e \le \bar{e}$, and $(\bar{\eta} + 4\sqrt{\bar{e}})x^{C_H} \le 6$. Arakelov slope duality and 21.32 give $s - \mu_{\min}(RR^\vee) \le x^{C_H} b$, so 23.8 with $H = x^{C_H} b$ yields the bound. The complete proof is on p. 709. □

Lemma 23.11 (The rank in the selected level window)**.** *Consider a family of the working class $W_1(\mathcal{A})$ of 33.6 satisfying the closed class-number condition of 37.21 and the symmetric columns from item 2 of 33.6. Use the word levels in the window of 21.35. Let f_u be the dimension of the lowest, level-zero column form space for the word u, $f = [F : \mathbf{Q}]$, and k the starting rank. Let $D^\star > 0$ be the row main term and $d_u^\star > 0$ the product of the definite mass, the pre-level index and the dimensions of the archimedean coefficient factors for u. Write Γ for the common positive factor used to define the word windows. Beyond a single family cutoff, for every point and word,*

$$f_u = d_u^\star (T_u^k - 1)^f, \qquad \frac{1}{2} \le \frac{f_u \Gamma^2}{D^\star} \le \exp(1).$$

Outline. Apply 37.21 at level zero using 8.17, 37.11 and 8.12. Evaluate the symmetric column of item 2 of 33.6 at class representatives and use 37.21 to get $f_u = d_u^\star (T_u^k - 1)^f$, then apply the selected prime window. The complete proof is on p. 709. □

Corollary 23.12 (The code and the shared precision)**.** *Under the design, selected-column and class-number hypotheses of 33.6, for every sufficiently small $\varepsilon > 0$ and every prescribed threshold record, the construction supplies a family and an exponent $C_H > 0$ such that at each constructed point and translation $\#W(u_0) = \lceil 3\Gamma^2 x^{C_H} \rceil$. Beyond a common threshold, $\#W(u_0) \le 2^{k-3}$ for every translation. This word count and the normalized rank bound of 23.11 give (23.8) with $C_0 = 3$. For the family of 23.9 the common upper bounds in 23.10 may be taken, for large x, as*

$$d_e(x) = E^2, \qquad e_B(x) = 2E^2, \qquad E = \exp(-x^{\varepsilon/8}).$$

For the constructed family, beyond one common cutoff every member satisfies $\mathcal{T} \le b^2$, with $\mathcal{T}$ as in 21.3.

Outline. Theorem 33.32 supplies the family and C_H above the thresholds of Lemma 23.9 and Lemma 21.33, and prime-sum and size bounds yield $\mathcal{T} \le b^2$. The growth of k bounds the word count by 2^{k-3}, which with Lemma 23.11 gives $C_0 = 3$. Evaluating Lemma 23.9 at $E = \exp(-x^{\varepsilon/8})$ produces the bounds d_e, e_B for Theorem 23.10. The complete proof is on p. 710. □

Chapter 24

The pointwise bound and its eight error terms

24.1 Common data and their realisation

In the numerical statements $x > 0$ and $\varepsilon \in \mathbf{R}$. In a window family $0 < \varepsilon < 1/52$. Let U_0 be a finite set of translations with a finite nonempty word set $W(u_0)$ at each u_0. Let TT be the finite set of top positions. Write $n_T = |TT|$. Let $m \geq 0$ and $f > 0$ be integers. Index the layers by $0 \leq j < m$ and the f archimedean positions by ρ. In the arithmetic model Lemma 22.43 identifies these positions with Φ from Definition 21.1. Write $w_{u_0 j\rho}$ for their weights and $a_t \geq 0$ for the top weights, which are common to all translations. Put

$$S_a = \sum_{t \in TT} a_t, \qquad Q = n_T m b, \qquad \langle z_\rho \rangle_\rho = \frac{1}{f} \sum_{\rho=1}^{f} z_\rho.$$

At a fixed translation let M_u be the zeroth-layer column lattice and N'_u the divided lattice. Write

$$d_u = \operatorname{rk} N'_u, \qquad \mu_u = \mu(N'_u), \quad \mu_u^+ = \mu_{\max}(N'_u), \qquad \mu_{M,u} = \mu(M_u), \quad \mu_{M,u}^+ = \mu_{\max}(M_u).$$

The target $C^\vee = RR^\vee$ has mean slope μ_C and minimum slope $\mu_{C,\min}$. Let $T_u > 0$ be the level, given in the arithmetic model by the word's auxiliary prime, and h_u the height allowance. The numerical statements allow arbitrary real slopes and positive real d_u. Define

$$\langle z_u \rangle_d = \frac{\sum_{u \in W(u_0)} d_u z_u}{\sum_{u \in W(u_0)} d_u}, \qquad s = \langle \mu_u - h_u \rangle_d, \qquad U = \max_{u \in W(u_0)} (\mu_u^+ - h_u) - s.$$

Use the scales $a, b, b', k, L, L_a, B_0, N_c$ of the preceding chapters, with $k > 0$. Take $X'' \in \mathbf{R}$ common to the translations. In the window model $G = E_0 \times A$ as in 5.2, with $n = |A|$. Here $X'' = a/w''$ with $w'' = b'/(xn)$. Put $\ell_g = \log(g_0 g_1)$ for the two fixed auxiliary primes. Use the net gain Ga, base term Fo, total layer loss Ee_u and local-index total IO of Definition 21.3, and the aggregated sum $\mathrm{ASK}(u_0)$ of Definition 21.1. At each translation $\mathrm{Ga}, \mathrm{Fo}, \mathrm{IO}$ are common to its words.

For real constants $c_{\mathrm{Div}}, e_{\mathrm{Div}}, c_E, c_H, c_{\mathrm{Count}}, c_{\mathrm{Slope}}, c_1, c_{\mathrm{Loss}}, e_{\mathrm{Loss}}, c_{HH}$, put

$$\begin{aligned}
\log \Omega &= \gamma + \log(\pi/v) - \mathrm{LpL}, \\
S_{\mathrm{div}} &= m \sum_{t \in TT} (a_t \log a_t - a_t - a_t \log \Omega), && (24.1) \\
E_{\mathrm{div}} &= c_{\mathrm{Div}} n_T a L_a + c_{\mathrm{Div}} m (B_0 + N_c) x^{e_{\mathrm{Div}} \varepsilon}, \\
P &= c_{\mathrm{Loss}} Q x^{e_{\mathrm{Loss}} \varepsilon}, && (24.2) \\
\mathrm{Lo}_u &= c_1 \mathrm{IO} + c_1 n_T m (a/k + b') \log T_u + c_1 X'' + 1, \qquad J = \langle \mathrm{Lo}_u \rangle_d. && (24.3)
\end{aligned}$$

Here $v > 0$ and $\gamma, \mathrm{LpL} \in \mathbf{R}$. In the arithmetic model $v = \mathrm{rd}(E)/2$ and γ is Euler's constant. The parameter LpL then equals $\mathrm{Re}\big(L'(1,\chi_d)/L(1,\chi_d)\big)$, where χ_d is the quadratic character of discriminant $-d$, as in Lemma 21.22. Set $a_t \log a_t = 0$ at $a_t = 0$.

Uniform family estimates have common constants and thresholds for all $i \in I_x$. The frame bounds may depend on x. For $e_L \geq 0$, the common-weight loss hypothesis supplies $c_{\mathrm{Loss}} \geq 0$ and, at every sufficiently large window point, a base model i_0 and functions $a_v, b_v : \mathbf{N} \to \mathbf{Z}$ admissible for i_0 as in 22.44. Some model i_1 realises both functions on the top labels. Every model realising them satisfies the losses of 22.42 with P from (24.2) and $e_{\mathrm{Loss}} = e_L$.

Use the following hypotheses at each translation. The local frame parameters are δ, η, e, with common bounds $\bar\delta, \bar\eta, \bar e$, and the local height parameter is $\mathcal{H}$. The counting threshold λ_5 of 21.35 is expressed in the common data in 24.3.

H1
$$x, b, k, v > 0, \quad a, b', c_1 \geq 0, \quad a_t \geq 0, \quad w_{u_0 j \rho} > 0, \quad d_u, T_u > 0, \quad \mu_u \leq \mu_u^{+},$$
$$\bar\delta < 1, \quad \bar\eta \geq 0, \quad 0 \leq \bar e \leq \tfrac{1}{4}.$$

H2 The division and height estimate is
$$\mu_C + h_u - \mathrm{Fo} \leq S_{\mathrm{div}} - S_a \log T_u + E_{\mathrm{div}}.$$

H3 The total layer loss satisfies
$$|\mathrm{Ee}_u - m S_a \ell_g| \leq m c_E B_0.$$

H4 For any two words,
$$|(h_u + S_a \log T_u) - (h_{u'} + S_a \log T_{u'})| \leq c_H (n_T a L_a + m B_0).$$

H5 The levels satisfy
$$\lambda_5(u_0) \leq \log T_u \leq c_{\mathrm{Count}} m L, \qquad |\log T_u - \log T_{u'}| \leq c_{\mathrm{Count}} \log k.$$

H6 The zeroth-layer column slopes satisfy
$$-c_{\mathrm{Slope}} n_T a L_a (1 + m/k) \leq \mu_{M,u}, \qquad \mu_{M,u}^{+} \leq c_{\mathrm{Slope}} n_T a L_a.$$

H7 The divided lattice satisfies
$$\mu_u^{+} \leq \mu_{M,u}^{+} + \mathrm{Ga} + \mathrm{Fo} - \mathrm{Ee}_u + c_1 m B_0,$$
$$\mu_{M,u} + \mathrm{Ga} + \mathrm{Fo} - \mathrm{Ee}_u - \mathrm{Lo}_u \leq \mu_u.$$

H8
$$\mathrm{ASK}(u_0) - \mathrm{Ga}(u_0) \leq P, \qquad \mathrm{IO}(u_0) \leq P.$$

H9 For $u > 0$ and $H > 0$ put $\vartheta(u, H) = u \log(1 + H/u)$, and set $\vartheta(0, H) = 0$. If $0 \leq \delta < 1$, $\eta \geq 0$, $0 \leq e \leq 1/4$, and $\mathcal{H} > 0$ satisfies $s - \mu_{C,\min} \leq \mathcal{H}$, then
$$s \leq \mu_C + \log |W(u_0)| + \frac{4\vartheta(U, \mathcal{H}) + (\eta + 4\sqrt{e})\mathcal{H}}{1 - \delta}.$$

H10
$$s - \mu_{C,\min} \leq \mathcal{H}, \qquad 0 < \mathcal{H} \leq x^{c_{HH}} b.$$

H11
$$0 \leq \delta \leq \bar\delta, \qquad 0 \leq \eta \leq \bar\eta, \qquad 0 \leq e \leq \bar e.$$

Theorem 24.1 (Common realisation at a window point)**.** *Let $\mathcal{F}$ be a window family as in 22.44. Fix $C_{\mathrm{abs}}, C_H, e_L \ge 0$, $C_5 \in \mathbf{R}$ and $c_6 > 0$. Assume the uniform estimates of 21.11 and 21.32–21.42, with division exponent C_{abs} and height exponent C_H as in 21.33 for the selected design. At all sufficiently large x, assume that each $i \in I_x$ has $d_u > 0$ and $\mu(N'_u) \le \mu_{\max}(N'_u)$ for every word, and $\mu_{\min}(RR^\vee) = -\mu_{\max}(RR)$.*

For all sufficiently large x assume common bounds $\bar\eta = \bar\eta(x)$ and $\bar e = \bar e(x)$ with

$$\bar\eta \ge 0, \qquad 0 \le \bar e \le \tfrac14, \qquad (\bar\eta + 4\sqrt{\bar e})x^{C_H} \le 6. \tag{24.4}$$

For each $i \in I_x$ assume that some $0 \le \eta_i \le \bar\eta$ and $0 \le e_i \le \bar e$ give H9 for every $\mathcal{H} = H > 0$ with $s_i - \mu_{C,\min,i} \le H$. Use the model's defect $\delta = \delta_i = 1 - D_{\mathrm{rows},i}/D_{\mathrm{all},i}$, as in Theorem 23.8. Assume the defect bound of 32.62 for these defects and the common-weight loss hypothesis with exponent e_L.

Under the small-ε design hypotheses of Corollary 23.12, use its d_e, e_B in Theorem 23.10 to take $\bar\eta = 5x^{-C_H} + 3\sqrt{d_e}$ and $\bar e = e_B$. For the same target, 37.13–37.14 supply the good-label injection and relative cylinder counts on the compatible row system of 32.62. The local idempotents use one common nontrivial additive character at the places above each distinguished prime. With the weight laws of 32.53 and $C_s = 6$, 32.63 gives the defect bound.

Assume that for all sufficiently large x and every $i \in I_x$, its host satisfies the thin-character and three sign-character hypotheses of Lemma 24.2 with the fixed constants c_6, C_5, and its top set satisfies

$$\tfrac12 n x^{-\varepsilon/8} \le n_T \le 2n x^{-\varepsilon/8}. \tag{24.5}$$

For the fixed c_6, assume that there exist an exponent $e_K \ge 0$ and a threshold common to the models above which the size bounds of 25.2 on $m, k, L_a, b', X'', |W(u_0)|$ and the chain-length lower bound of 25.7 hold. With $X' = \exp(x^{\varepsilon/3})$, $\theta = \exp(-c_6 x^{\varepsilon/8})$, n_2 the number of thick labels and $\eta_{\mathrm{sz}} \ge 0$ the common width, assume also

$$\begin{aligned} 2f(x+1)\theta \le 1/x, \qquad & 6f(x+1)(\eta_{\mathrm{sz}}+1) \le b/x, \\ n_2/n \ge x^{-\varepsilon}, \qquad & k \le n^2 x^{1/2-\varepsilon}. \end{aligned} \tag{24.6}$$

Then there are real constants K, c_{HH}, e_K, C_L, x_0 such that, at each window point $x \ge x_0$, there are common integer weight functions a_v, b_v and a model realising them at each translation. One can take

$$c_{HH} = C_H, \qquad C_L = C_5, \qquad e_{\mathrm{Div}} = C_{\mathrm{abs}}, \qquad e_{\mathrm{Loss}} = e_L.$$

The selected models share the frame and global scales, including m, f, q, d, and satisfy H1–H11. With $b = e^{x^\varepsilon}$, $a = xb$ and $N = \lceil 2a \rceil$, their weights satisfy

$$a \le a_t \le 2a, \qquad |a_t/b - x| \le 1, \qquad a_t/w_{u_0 j\rho} \le N \quad (t \in TT,\ 0 \le j < m,\ \rho \in \Phi).$$

They give the arithmetic data of 25.3–25.6, with ℓ from 24.2 and $b'X' = xb$. They satisfy the windows of 25.2 and the chain-length bounds of 25.7. They also satisfy (24.5) and (24.6). In particular $1 \le k \le x^{e_K}$.

If S_2 is the common set of integer thick labels and a_0 generates A_0, put $T_2 = \{sa_0 : s \in S_2\}$. The host's label-range parameter $L_w \ge 2$ gives $0 \le s \le 4L_w s < \mathrm{ord}(a_0)$ for $s \in S_2$, so $n_2 = |T_2| = |S_2|$. At the thick positions $((0,1), s_2 + u_0)$ with $s_2 \in T_2$, the weights lie in $[b, b + \eta_{\mathrm{sz}} + 1]$. They lie in $(0, 2b']$ at the other positions of Φ. For the family of 22.44 one may take $e_L = 0$.

Outline. Lemma 22.44 realises the common weights at each translation and gives $e_L = 0$. Theorem 21.11 and Theorem 21.32 through Lemma 21.42 give the word estimates, while Theorem 32.62, duality, and Lemma 22.43 give the frame and weight bounds. Lemma 5.2, the budgets, Lemma 5.3, and Lemma 24.2 supply the arithmetic data. The complete proof is on p. 711. □

Lemma 24.2 (Character constants for a host)**.** *Let F_m be an admissible family of exponent ε, and fix $c_6, C_5 \in \mathbf{R}$. For every $x \ge 0$ and model at x, take the arithmetic datum of its host in the same frame, with the q-coordinate condition of 8.3. There is a function $\ell : \widehat{G} \to \mathbf{C}$ with $\mathrm{LpL} = \mathrm{Re}\,\ell(\chi_C)$, where χ_C is the d-sign. If $\xi \ne \mathbf{1}$ and a primitive Dirichlet character χ of positive modulus satisfies $\chi(n) = \prod_{p|n} V_\xi(p^{v_p(n)})$ for every $n \ge 1$, with the local averages V_ξ of Definition 5.4, then $\ell(\xi) = (L'/L)(1, \chi)$.*

Use this ℓ in G_ξ from 5.4. Set $X' = \exp(x^{\varepsilon/3})$ and $\theta = \exp(-c_6 x^{\varepsilon/8})$. If the primitive characters of the host satisfy $|g_\chi(y)| \le y\theta$ for $y \ge X'/3$ as in 37.4, then $|G_\xi(y)| \le y\theta$ on that range for every ξ. Suppose the primitive representatives of $(-q/n)$, $(-d/n)$ and their product on units modulo $8qd$ satisfy

$$|g_\chi(x)| \le x^{1/2+\varepsilon}, \qquad \left|\frac{L'(1,\chi)}{L(1,\chi)}\right| \le C_5 \log x. \tag{24.7}$$

Then $|G_\xi(x)| \le x^{1/2+\varepsilon}$ and $|\ell(\xi)| \le C_5 \log x$ for the two sign characters and their product. The hosts constructed in Theorem 8.20 satisfy (24.7) with $C_5 = 4$ at sufficiently large window points.

Outline. Identify primitive representatives using Frobenius and quadratic reciprocity, then transfer bounds via 6.6. Select primes in stages 7.20 and 7.21, using 7.17 for window bounds and 7.16 to bound logarithmic derivatives. The complete proof is on p. 712. □

24.2 The main term

Definition 24.3 (Counting threshold and geometric main term)**.** With the convention of (24.1), put

$$r = c_{\mathrm{Count}}(L_a + m^2 L_a/k), \tag{24.8}$$

$$\lambda_5(u_0) = \sum_{j=0}^{m-1} \langle \log(w_{u_0 j\rho} v/\pi) \rangle_\rho + m\ell_g - r, \tag{24.9}$$

$$\mathrm{Main}(u_0) = \sum_{j=0}^{m-1} \sum_{t \in TT} a_t \left(\log a_t - 1 - \gamma + \mathrm{LpL} - \langle \log w_{u_0 j\rho} \rangle_\rho \right). \tag{24.10}$$

Lemma 24.4 (Exact cancellation of the geometric main terms)**.** *For $v > 0$, $f > 0$ and $w_{u_0 j\rho} > 0$,*

$$S_{\mathrm{div}} - S_a \lambda_5(u_0) + m S_a \ell_g = \mathrm{Main}(u_0) + S_a r.$$

Proof. Put $Z = \sum_j \langle \log w_{u_0 j\rho} \rangle_\rho$ and $B = \sum_t a_t(\log a_t - 1 - \gamma + \mathrm{LpL})$. The definitions give

$$S_{\mathrm{div}} = mB - mS_a \log(\pi/v), \quad \lambda_5(u_0) = Z - m\log(\pi/v) + m\ell_g - r, \quad \mathrm{Main}(u_0) = mB - S_a Z.$$

Substitution gives $S_{\mathrm{div}} - S_a\lambda_5(u_0) + mS_a\ell_g = mB - S_aZ + S_ar$. □

With the window-model parameter $\log\Gamma$ equal to $C_\Gamma m^5 fL$ as in (23.8), the earlier remainder and threshold are

$$r_{\mathrm{old}} = c_{\mathrm{Count}}(L + m^2 L/k) + \frac{2\log\Gamma}{fk}, \qquad \lambda_{\mathrm{old}}(u_0) = \lambda_5(u_0) + r - r_{\mathrm{old}}.$$

Substitution in 24.4 gives

$$S_{\mathrm{div}} - S_a \lambda_{\mathrm{old}}(u_0) + m S_a \ell_g = \mathrm{Main}(u_0) + S_a r_{\mathrm{old}}.$$

24.3 The lower bound and the slope spread

Lemma 24.5 (Lower bound for the height-adjusted mean slope)**.** *Assume H1–H3 and H5–H7. Put*

$$K_5(u_0) = c_{\mathrm{Slope}} n_T a L_a (1 + m/k) + m(S_a\ell_g + c_E B_0) + S_{\mathrm{div}} - S_a\lambda_5(u_0) + E_{\mathrm{div}}.$$

Then $\mathrm{Ga} + \mu_C - K_5(u_0) - J \le s$ at every translation.

Outline. Start from H7, bound the height-adjusted forcing term below using H2 and H5, and apply the error and slope bounds of H3 and H6. Taking the d-weighted mean gives the lower bound for s. The complete proof is on p. 713. □

Lemma 24.6 (Variation between two words)**.** *Assume* H1 *and* H3–H5*. Put*

$$V = 2mc_E B_0 + c_H(n_T a L_a + mB_0) + S_a c_{\mathrm{Count}} \log k.$$

Then $(-\mathrm{Ee}_u - h_u) - (-\mathrm{Ee}_{u'} - h_{u'}) \le V$ *for all* $u, u' \in W(u_0)$.

Proof. By H3, $\mathrm{Ee}_{u'} - \mathrm{Ee}_u \le 2mc_E B_0$. With $H_u = h_u + S_a \log T_u$,

$$h_{u'} - h_u = H_{u'} - H_u + S_a(\log T_u - \log T_{u'}).$$

The first term is at most $c_H(n_T a L_a + mB_0)$ by H4. Since $S_a \ge 0$, H5 bounds the second term by $S_a c_{\mathrm{Count}} \log k$. □

Lemma 24.7 (Maximum slope above the weighted mean)**.** *Assume* H1 *and* H3–H7*. Put*

$$K_3 = c_{\mathrm{Slope}} n_T a L_a (2 + m/k) + V + c_1 m B_0.$$

Then $0 \le U \le K_3 + J$.

Outline. Subtracting H7 at u and u' cancels common terms, and H6 with 24.6 bounds the difference by $K_3 + \mathrm{Lo}_{u'}$. Averaging over u' and maximizing over u gives $U \le K_3 + J$, while H1 gives $0 \le U$. The complete proof is on p. 713. □

24.4 The logarithmic correction and the upper bound

Lemma 24.8 (Uniform logarithmic correction)**.** *Let* $u, H, X, b, U_* \in \mathbf{R}$ *satisfy* $0 \le u \le U_*$, $0 < H \le Xb$ *and* $X, b > 0$*. With* ϑ *as in* H9*,*

$$\vartheta(u, H) \le (U_* + b)\log(1 + X).$$

Proof. Put $L_X = \log(1 + X) > 0$. The bound holds at $u = 0$. For $u > 0$, use $\log z \le z - 1$ at $z = (1 + H/u)/(1 + X)$ and multiply by u. Then

$$\vartheta(u, H) \le uL_X + \frac{H}{1 + X} - u\left(1 - \frac{1}{1 + X}\right) \le U_* L_X + \frac{Xb}{1 + X}.$$

The same inequality at $z = (1 + X)^{-1}$ gives $X/(1 + X) \le L_X$. □

Theorem 24.9 (Upper bound for the net gain)**.** *Assume* H1–H7 *and* H9–H11*. Write*

$$X = x^{c_{HH}} > 0, \qquad K_0 = S_a r + E_{\mathrm{div}} + mc_E B_0 + c_{\mathrm{Slope}} n_T a L_a (1 + m/k).$$

At every translation,

$$\mathrm{Ga}(u_0) \le \mathrm{Main}(u_0) + \log|W(u_0)| + K_0 + J + \frac{4\log(1 + X)(K_3 + J + b) + (\bar{\eta} + 4\sqrt{\bar{e}})Xb}{1 - \bar{\delta}}.$$

Outline. Combine 24.5 with H9, cancel μ_C, and rewrite $K_5(u_0)$ by 24.4. Bound the numerator using 24.7, H10, 24.8, and H11, then replace $1/(1 - \delta)$ by $1/(1 - \bar{\delta})$. The complete proof is on p. 713. □

Lemma 24.10 (Uniform bound for the mean quotient loss)**.** *Assume* H1 *and* H5*. Then*

$$J \le c_1 \mathrm{IO} + c_1 n_T m(a/k + b') c_{\mathrm{Count}} mL + c_1 X'' + 1.$$

If also H8 *holds, then* $J \le M_{\mathrm{Lo}}$*, where*

$$M_{\mathrm{Lo}} = c_1 P + c_1 n_T m(a/k + b') c_{\mathrm{Count}} mL + c_1 X'' + 1.$$

Proof. By H1, the coefficient of $\log T_u$ in (24.3) is nonnegative. Use $\log T_u \le c_{\mathrm{Count}} mL$ and take the weighted mean. Since $c_1 \ge 0$, $\mathrm{IO} \le P$ gives the second bound. □

24.5 The eight error terms

Definition 24.11 (Error at one translation)**.** Put $\bar{U} = K_3 + M_{\mathrm{Lo}}$ and $X = x^{c_{HH}}$. The error at u_0 is

$$\begin{aligned} E_8(u_0) = {} & \underbrace{S_a c_{\mathrm{Count}}(L_a + m^2 L_a/k)}_{1} \\ & + \underbrace{c_{\mathrm{Div}} n_T a L_a + c_{\mathrm{Div}} m(B_0 + N_c) x^{e_{\mathrm{Div}}\varepsilon}}_{2} \\ & + \underbrace{m c_E B_0}_{3} + \underbrace{c_{\mathrm{Slope}} n_T a L_a (1 + m/k)}_{4} \\ & + \underbrace{\log |W(u_0)|}_{5} + \underbrace{c_{\mathrm{Loss}} n_T m b x^{e_{\mathrm{Loss}}\varepsilon}}_{6} \\ & + \underbrace{M_{\mathrm{Lo}}}_{7} + \underbrace{\frac{4\log(1+X)(\bar{U}+b) + (\bar{\eta} + 4\sqrt{\bar{e}})Xb}{1-\bar{\delta}}}_{8}. \end{aligned} \tag{24.11}$$

In terms 7 and 8 the expanded quantities are

$$\begin{aligned} M_{\mathrm{Lo}} &= c_1 c_{\mathrm{Loss}} n_T m b x^{e_{\mathrm{Loss}}\varepsilon} + c_1 n_T m(a/k + b') c_{\mathrm{Count}} m L + c_1 X'' + 1, \\ \bar{U} &= c_{\mathrm{Slope}} n_T a L_a (2 + m/k) + 2 m c_E B_0 + c_H (n_T a L_a + m B_0) \\ &\quad + S_a c_{\mathrm{Count}} \log k + c_1 m B_0 + M_{\mathrm{Lo}}. \end{aligned}$$

The family identifies each word set with $W(x)$, so term 5 is constant across the selected translations.

The kernel in Theorem 23.7 has integral $\int_0^{\mathcal{H}} U/(U+\tau)\, d\tau = \vartheta(U, \mathcal{H})$, with value zero at $U = 0$. Since $W(u_0)$ is nonempty, $\log |W(u_0)| \geq 0$. Adding this quantity to the first bound of Theorem 23.7 gives its second bound and term 5 above. Use M_{Lo} to bound J in both 24.5 and 24.7.

Theorem 24.12 (The pointwise bound for ASK)**.** *Assume* H1–H11*. Then for every $u_0 \in U_0$,*

$$\boxed{\mathrm{ASK}(u_0) \leq \mathrm{Main}(u_0) + E_8(u_0)}.$$

At every sufficiently large window point this holds for the models selected in Theorem 24.1.

Proof. The coefficient of J in 24.9 is $1 + 4\log(1+X)/(1-\bar{\delta}) > 0$. Substitute $J \leq M_{\mathrm{Lo}}$ from 24.10 and apply H8. Then

$$\begin{aligned} \mathrm{ASK} - \mathrm{Main} &\leq P + \mathrm{Ga} - \mathrm{Main} \\ &\leq P + \log |W(u_0)| + K_0 + M_{\mathrm{Lo}} \\ &\quad + \frac{4\log(1+X)(K_3 + M_{\mathrm{Lo}} + b) + (\bar{\eta} + 4\sqrt{\bar{e}})Xb}{1-\bar{\delta}}. \end{aligned}$$

Expand K_0 and substitute $\bar{U} = K_3 + M_{\mathrm{Lo}}$ to obtain (24.11). □

Under the weight and constant identifications of Lemma 25.3, $\mathrm{Main}(u_0)$ is the sum of the real parts of the arithmetic main terms over top positions and layers. Under the window hypotheses of Theorem 25.2, the error satisfies

$$E_8(u_0) \leq x^{(e_{\mathrm{Div}} + e_{\mathrm{Loss}} + 3)\varepsilon} \big[n_T a(1 + m^2/k) + m(B_0 + N_c) \big].$$

Chapter 25

Four terms and the square-root bound

For the chains of Alpöge's construction [3], Theorem 24.12 bounds $\mathrm{ASK}(u_0) - \mathrm{Main}(u_0)$ by the error $E_8(u_0)$ of Definition 24.11.

Write n_T for the number of top positions, m for the chain length, k for the initial rank, and a for the common scale of the top weights. Let B_0, N_c be the parameters in the second term of Definition 24.11, and let $e_{\mathrm{Div}}, e_{\mathrm{Loss}}$ be the division and loss exponents. Set

$$\mathcal{E} = n_T a \left(1 + \frac{m^2}{k}\right) + m(B_0 + N_c), \qquad C_{\mathrm{err}} = e_{\mathrm{Div}} + e_{\mathrm{Loss}} + 3. \tag{25.1}$$

25.1 Combining the eight terms

Theorem 25.1 (A real-variable error bound)**.** *Let all the parameters below be real. Suppose*

$$\begin{gathered} y, y_D, y_L \ge 1, \qquad n_T, a \ge 1, \qquad b, m \ge 0, \qquad k > 0, \\ b \le a, \quad m \le k, \quad mb \le a, \quad 0 \le L, L_a \le 2y, \\ b' \ge 0, \quad kb' \le a, \quad 0 \le X'' \le a, \quad 0 \le S_a \le 2n_T a, \quad B_0, N_c \ge 0, \\ \ell_W \le a, \quad 0 \le \ell_k \le gy, \quad 0 \le \ell_H \le hy, \quad 0 \le d_I \le 2, \quad \tau \le Ky\, n_T a. \end{gathered}$$

Assume that each of

$$c_{\mathrm{Div}}, c_E, c_H, c_{\mathrm{Count}}, c_{\mathrm{Slope}}, c_1, c_{\mathrm{Loss}}$$

lies in $[0, K]$*, that* $h, g \ge 1$*, and that*

$$200(1+K)^2 hg \le y.$$

Define

$$\begin{aligned} P_0 &= c_{\mathrm{Loss}} n_T m b\, y_L, \\ J_0 &= c_1 P_0 + c_1 n_T m (a/k + b')\, c_{\mathrm{Count}} m L + c_1 X'' + 1, \\ V_0 &= 2mc_E B_0 + c_H(n_T a L_a + mB_0) + S_a c_{\mathrm{Count}} \ell_k, \\ \bar{U} &= c_{\mathrm{Slope}} n_T a L_a (2 + m/k) + V_0 + c_1 m B_0 + J_0. \end{aligned}$$

Let

$$\begin{aligned} R_1 &= S_a c_{\mathrm{Count}}(L_a + m^2 L_a/k), \\ R_2 &= c_{\mathrm{Div}} n_T a L_a + c_{\mathrm{Div}} m(B_0 + N_c) y_D, \\ R_3 &= mc_E B_0, \qquad R_4 = c_{\mathrm{Slope}} n_T a L_a (1 + m/k), \\ R_5 &= \ell_W, \qquad R_6 = P_0, \qquad R_7 = J_0, \\ R_8 &= d_I\{4\ell_H(\bar{U} + b) + \tau\}. \end{aligned}$$

Then

$$\sum_{i=1}^{8} R_i \le y_D y_L y^3 \left\{ n_T a(1 + m^2/k) + m(B_0 + N_c) \right\}.$$

Outline. Set $P = y_D y_L y E$, bound the first six terms by multiples of P, and use $b' \le a/k$ to bound J_0 in the estimates for $\bar{U} + b$ and R_8. The sum is at most $200(1+K)^2 hgyP \le y^2 P$ by the hypothesis on y. The complete proof is on p. 713. □

Theorem 25.2 (The error bound in a chain window)**.** *For the geometric data of Definition 24.11, take finite sets U_0, Top. For each $u_0 \in U_0$, let $W(u_0)$ be finite and nonempty. Let $m, f \in \mathbf{N}$, $n_T = \#\mathrm{Top}$, and $S_a = \sum_{t \in \mathrm{Top}} a_t$. Assume that the window data satisfy*

$$\begin{gathered} x \ge 1, \quad 0 < \varepsilon \le 1, \quad b \ge 1, \quad xb \le a, \quad n_T \ge 1, \\ m \le x^{1/2+\varepsilon/2}, \quad xm \le k, \quad e_K \ge 0, \quad 1 \le k \le x^{e_K}, \quad k \le a, \\ 0 \le L, L_a \le 2x^{\varepsilon}, \quad b' \ge 0, \quad kb' \le a, \quad 0 \le X'' \le a, \\ 0 \le a_t \le 2a \quad (t \in \mathrm{Top}), \qquad B_0, N_c \ge 0, \\ \log \#W(u_0) \le (k-3)\log 2 \quad (u_0 \in U_0), \\ \bar{\delta} \le \tfrac{1}{2}, \qquad \bar{\eta}, \bar{e} \ge 0. \end{gathered}$$

Suppose the seven coefficients of Theorem 25.1 lie in $[0, K]$, $e_{\mathrm{Div}}, e_{\mathrm{Loss}}, c_{HH} \ge 0$, and

$$(\bar{\eta} + 4\sqrt{\bar{e}}) x^{c_{HH}} b \le K x^{\varepsilon} n_T a.$$

Assume also

$$\log x \le x^{\varepsilon}, \qquad 200(1+K)^2(1+c_{HH})(1+e_K) \le x^{\varepsilon}.$$

Then, for every $u_0 \in U_0$,

$$E_8(u_0) \le x^{C_{\mathrm{err}}\varepsilon} \mathcal{E}.$$

Outline. The window hypotheses allow Theorem 25.1 with $y = x^{\varepsilon}$, $y_D = x^{e_{\mathrm{Div}}\varepsilon}$, and $y_L = x^{e_{\mathrm{Loss}}\varepsilon}$. By Definition 24.11, the sum is $E_8(u_0)$, and $y_D y_L y^3 = x^{C_{\mathrm{err}}\varepsilon}$. The complete proof is on p. 714. □

25.2 The arithmetic difference and its average

Let H be a finite abelian group, written additively, and put

$$n = \#H, \qquad \Gamma = (\mathbf{Z}/2\mathbf{Z})^2 \times H, \qquad \Phi = \{((0,e),h) : e \in \mathbf{Z}/2\mathbf{Z},\ h \in H\}.$$

Thus $f = \#\Phi = 2n$. Identify H with $\{(0,0)\} \times H$ inside Γ. Write

$$r_t(g) = ((1,1),0) + g + ((0,0),t), \quad e_1(t) = ((1,0),t), \quad \chi_C((e_0,e_1),h) = (-1)^{e_0}.$$

Characters of Γ take values in $\mathbf{C}^{\times}$, and $\mathbf{1}$ denotes the trivial character. Use the arithmetic datum D and centred functions G_ψ of Definition 5.4, with $\ell : \widehat{\Gamma} \to \mathbf{C}$ and $G_{\mathbf{1}} = G_z$. Fix $m, N \in \mathbf{N}$, top weights $a_t \in \mathbf{R}$ for $t \in H$, and local weights $w_{s,j} : \Gamma \to \mathbf{R}$ for $s \in H$ and $0 \le j < m$. Write $\mathrm{ASK}(D, N, A, t, w)$ for $\mathrm{ASK}_N(A, t, w)$ of Definition 5.4. Let γ be Euler's constant and use the real logarithm $\log r = \log|r|$ for $r \ne 0$, with $\log 0 = 0$. For $A \in \mathbf{R}$ and $w : \Gamma \to \mathbf{R}$, the one-layer main term is

$$\mathrm{MAIN}(\ell, A, w) = A\left(\log A - 1 - \gamma + \ell(\chi_C) - \frac{1}{f}\sum_{\rho \in \Phi} \log w(\rho)\right). \tag{25.2}$$

Take the geometric data of Definition 24.3 on U_0, with top set Top and f places per layer. Fix $\mathrm{tr} : U_0 \to H$ and $TT \subseteq H$. Choose bijections

$$e_T : \mathrm{Top} \longrightarrow TT, \qquad e_P : \{0, \dots, f-1\} \longrightarrow \Phi.$$

In the models of Theorem 24.1, the place bijection of Lemma 22.43 identifies the sum of log weights over Φ with that over the archimedean places at lift $j+1$ for each translation. In the construction of Theorem 24.1, choose ℓ at the base model i_0 by Lemma 24.2 and use this choice for every translation.

Write $a_\tau^{\mathrm{geom}}, w_{u_0 j}^{\mathrm{geom}}(\rho), \gamma_{\mathrm{geom}}, \mathrm{LpL}$ for the geometric top and local weights and the constants in Definition 24.3. Assume

$$a_\tau^{\mathrm{geom}} = a_{e_T(\tau)}, \quad w_{u_0 j}^{\mathrm{geom}}(\rho) = w_{\mathrm{tr}(u_0),j}(e_P \rho),$$
$$\gamma_{\mathrm{geom}} = \gamma, \qquad \mathrm{LpL} = \mathrm{Re}\,\ell(\chi_C)$$

for $\tau \in \mathrm{Top}$, $u_0 \in U_0$, $0 \le j < m$, and $0 \le \rho < f$. Write $\mathrm{ASK}_{\mathrm{geom}}$ for the geometric prime sum of Theorem 24.12, and assume

$$\mathrm{ASK}_{\mathrm{geom}}(u_0) = \sum_{t \in TT} \sum_{j=0}^{m-1} \mathrm{ASK}(D, N, a_t, t, w_{\mathrm{tr}(u_0),j}) \qquad (u_0 \in U_0).$$

Lemma 25.3 (Identification of the main term)**.** *Under these identifications, every $u_0 \in U_0$ satisfies*

$$\mathrm{Main}(u_0) = \sum_{t \in TT} \sum_{j=0}^{m-1} \mathrm{Re}\,\mathrm{MAIN}(\ell, a_t, w_{\mathrm{tr}(u_0),j}).$$

Proof. By Definition 24.3, the geometric main term is

$$\sum_{j=0}^{m-1} \sum_{\tau \in \mathrm{Top}} a_\tau^{\mathrm{geom}} \left(\log a_\tau^{\mathrm{geom}} - 1 - \gamma_{\mathrm{geom}} + \mathrm{LpL} - \frac{1}{f} \sum_{\rho=0}^{f-1} \log w_{u_0 j}^{\mathrm{geom}}(\rho) \right).$$

Reindex the logarithm sum by e_P and the top sum by e_T. The constant identities make each summand the real part of (25.2). Exchange the two outer sums. □

Let $T_2 \subseteq H$, and put $n_T = |TT|$ and $n_2 = |T_2|$. For $x > 0$, $\varepsilon > 0$, common weight $b > 0$, initial rank $k > 0$, and top-weight scale $a_E \in \mathbf{R}$, define

$$B_*(x) = 2nmbx^\varepsilon, \qquad N_*(x) = \frac{n_2 k b x^\varepsilon}{n(nx^{-\varepsilon})},$$
$$\mathcal{B}_x(a_E) = n_T a_E(1 + m^2/k) + m\{B_*(x) + N_*(x)\}.$$

For $s \in H$, set

$$\mathrm{gap}(s) = \sum_{t \in TT} \sum_{j=0}^{m-1} \{\mathrm{ASK}(D, N, a_t, t, w_{s,j}) - \mathrm{Re}\,\mathrm{MAIN}(\ell, a_t, w_{s,j})\}.$$

Theorem 25.4 (The geometric difference is the arithmetic difference)**.** *Assume the identifications stated before Lemma 25.3. Suppose the eleven hypotheses of Theorem 24.12, the window hypotheses of Theorem 25.2, and*

$$B_0 \le B_*(x), \qquad N_c \le N_*(x)$$

hold for the same geometric data. Then every $u_0 \in U_0$ satisfies

$$\mathrm{gap}(\mathrm{tr}(u_0)) = \mathrm{ASK}_{\mathrm{geom}}(u_0) - \mathrm{Main}(u_0) \le x^{C_{\mathrm{err}}\varepsilon} \mathcal{B}_x(a).$$

Proof. The assumed identity for $\mathrm{ASK}_{\mathrm{geom}}$ and Lemma 25.3 give the equality. By Theorems 24.12 and 25.2, the geometric difference is at most $E_8(u_0) \le x^{C_{\mathrm{err}}\varepsilon}\mathcal{E}$. By e_T, the top cardinalities agree. Substituting the bounds on B_0, N_c in (25.1) gives $\mathcal{E} \le \mathcal{B}_x(a)$. □

Theorem 25.5 (The one-layer Fourier identity)**.** *Let D have Frobenius elements $F_p \in \Gamma$, inertia subgroups $I_p \le \Gamma$, and distinguished integers q_1, d. Take a finite set $S \subseteq \mathbf{N}$ with $2, q_1 \in S$, and assume d is prime and $d \notin S$. Impose the local conditions of Lemma 5.3*

$$I_p = \{0\} \quad \text{for primes } p \notin S \cup \{d\},$$
$$I_p \subseteq \Phi, \quad F_p \in \Phi \quad (p \in S), \qquad I_d = \{0, ((1,0),0)\}.$$

Let $\ell : \widehat{\Gamma} \to \mathbf{C}$, $A > 0$, $t \in H$, $N \in \mathbf{N}$, and $w : \Gamma \to \mathbf{R}$. If

$$w(\rho) > 0, \qquad A/w(\rho) \le N \qquad (\rho \in \Phi),$$

then

$$\mathrm{ASK}(D, N, A, t, w) - \mathrm{MAIN}(\ell, A, w) = \frac{1}{f} \sum_{g \notin \Phi} \sum_{\psi \in \widehat{\Gamma}} \overline{\psi(g)}\, w(r_t(g)) G_\psi\left(\frac{A}{w(r_t(g))}\right)$$

as an equality in $\mathbf{C}$.

Proof. Theorem 5.7 with $a = A$ identifies the difference with $\Delta(A, t, w)$. Expand this term by Definition 5.4. □

For $A_0 \le H$, write

$$\Psi_A = \{\psi \in \widehat{\Gamma} : \psi((0,0), s) = 1 \text{ for every } s \in A_0\}.$$

A position ρ is thick at u_0 if $\rho = ((0,1), s + u_0)$ for some $s \in T_2$. Put $Q = n_T mb$ and $\delta_2 = n_2/n$.

Lemma 25.6 (Averaging over a subgroup)**.** *Let the host conditions of Theorem 25.5 hold, and let $TT, T_2 \subseteq H$, $T_2 \subseteq A_0 \le H$, $m, N \in \mathbf{N}$. Suppose $x \ge 3$, $b, b', X' > 0$, $\eta, \theta \ge 0$, and $b'X' \le xb$. Assume that for every $t \in TT$,*

$$|a_t/b - x| \le 1.$$

For every $u_0 \in A_0$ and $0 \le j < m$, assume

$$\begin{gathered} w_{u_0 j}(\rho) > 0 \quad (\rho \in \Phi), \\ a_t / w_{u_0 j}(\rho) \le N \quad ((t, \rho) \in TT \times \Phi), \\ b \le w_{u_0 j}(\rho) \le b + \eta + 1 \quad \text{at thick positions}, \\ w_{u_0 j}(\rho) \le 2b' \quad \text{at the other positions of } \Phi. \end{gathered}$$

Suppose the thin-scale estimate and the replacement costs satisfy

$$\begin{gathered} |G_\psi(y)| \le y\theta \quad (\psi \in \widehat{\Gamma},\ y \ge X'/3), \\ 2f(x+1)\theta \le 1/x, \qquad 6f(x+1)(\eta+1) \le b/x. \end{gathered}$$

Let $E, Z \in \mathbf{R}$. If

$$\mathrm{gap}(u_0) \le E \quad (u_0 \in A_0)$$

and

$$n_T Z \le \sum_{t \in TT} \mathrm{Re} \sum_{\psi \in \Psi_A} \psi(e_1(t)) G_\psi(a_t/b),$$

then

$$Q \frac{\delta_2}{2} Z \le E + \frac{2Q}{x}.$$

Proof. With $A = H$, the surviving set of Lemma 5.8 is $P = \Psi_A$, and $\delta = \delta_2$. Apply Theorem 5.10 with $\mathscr{E} = E$ and $Y_* = Z$. □

25.3 The four normalized terms

Lemma 25.7 (Both ends of the chain-length window)**.** *Let $m \in \mathbf{N}$ be the length of a chain at a scale $x > 1$, $0 < \varepsilon < 1/52$. Suppose its size condition is*

$$x^{-\varepsilon/100} \sqrt{x\, x^{-\varepsilon/8}} \le m \le x^{\varepsilon/100} \sqrt{x\, x^{-\varepsilon/8}}.$$

Then

$$x^{1/2-\varepsilon} \le m \le x^{1/2+\varepsilon/2}.$$

Proof. The two exponents $1/2 - \varepsilon/16 \pm \varepsilon/100$ lie in $[1/2 - \varepsilon, 1/2 + \varepsilon/2]$. Since $x > 1$, the bounds follow by monotonicity. □

Lemma 25.8 (The top-weight term)**.** *Let $x, X, Y, a_E, n_T, m, b \in \mathbf{R}$, with $n_T, m, b \geq 0$. If $x \leq Xm$, $X \leq Y$, and $a_E \leq 2xb$, then*

$$n_T a_E \leq 2Y(n_T m b).$$

In particular, for $x \geq 1$, $\varepsilon > 0$, and $m \geq x^{1/2-\varepsilon}$, one may take $X = x^{1/2+\varepsilon}$ and $Y = x^{1/2+13\varepsilon/8}$.

Proof. The assumptions give $x \leq Xm \leq Ym$, so

$$n_T a_E \leq 2n_T x b \leq 2n_T Y m b.$$

For the specified powers, $x = x^{1/2+\varepsilon} x^{1/2-\varepsilon} \leq Xm$ and $X \leq Y$. When $Q > 0$, the intermediate bound reads $n_T a_E / Q \leq 2x/m$. □

Lemma 25.9 (The inverse-rank term)**.** *Let $x > 0$, $Y \geq 1$, $n_T, m, b \geq 0$, $k > 0$, and $a_E \in \mathbf{R}$. If $xm \leq k$ and $a_E \leq 2xb$, then*

$$n_T a_E \frac{m^2}{k} \leq 2Y(n_T m b).$$

Proof. As $m \geq 0$ and $xm \leq k$, one has $m^2/k \leq m/x$. Therefore

$$n_T a_E \frac{m^2}{k} \leq n_T (2xb) \frac{m}{x} = 2Q \leq 2YQ.$$

When $Q > 0$, the sharper normalized estimate is $n_T a_E m^2/(kQ) \leq 2xm/k \leq 2$. □

Lemma 25.10 (The layer-position term)**.** *Let $x > 0$, $\varepsilon \in \mathbf{R}$, $m, b, n \geq 0$, and $n_T \in \mathbf{R}$. Set $Y = x^{1/2+13\varepsilon/8}$. If*

$$m \leq x^{1/2+\varepsilon/2}, \qquad \tfrac{1}{2} n x^{-\varepsilon/8} \leq n_T,$$

then

$$m(2nmbx^{\varepsilon}) \leq 4Y(n_T m b).$$

Proof. The density assumption gives $n_T \geq 0$ and $n \leq 2n_T x^{\varepsilon/8}$. Thus

$$\begin{aligned} 2nmx^{\varepsilon} &\leq 4n_T m x^{\varepsilon+\varepsilon/8} \\ &\leq 4n_T x^{1/2+\varepsilon/2+\varepsilon+\varepsilon/8} = 4n_T Y. \end{aligned}$$

Multiply by $mb \geq 0$. □

Lemma 25.11 (The thick-set rank term)**.** *Let $x \geq 1$, $\varepsilon > 0$, $m, b \geq 0$, $n > 0$, $k \geq 0$, and $n_T, n_2 \in \mathbf{R}$. Set $Y = x^{1/2+13\varepsilon/8}$. If*

$$\tfrac{1}{2} n x^{-\varepsilon/8} \leq n_T, \qquad n_2 \leq n, \qquad k \leq n^2 x^{1/2-\varepsilon},$$

then

$$m \, \frac{n_2 k b x^{\varepsilon}}{n(nx^{-\varepsilon})} \leq 2Y(n_T m b).$$

Outline. Rewrite the left side as $(mb)(n_2 k/n^2)x^{2\varepsilon}$ and use $n_2 k \leq nk$ to bound $n_2 k/n^2 \leq k/n$, together with the upper bound on k and the lower density bound on n_T, to obtain $(n_2 k/n^2)x^{2\varepsilon} \leq 2n_T Y$. Multiplying by $mb \geq 0$ yields the stated inequality. The complete proof is on p. 714. □

25.4 The square-root estimate

Theorem 25.12 (The size-window bound). *Let $C \ge 0$, $\varepsilon > 0$, $x \ge 3$, and $n, n_T, n_2, m \in \mathbf{N}$, with $n > 0$. Let $b > 0$, $a_E \ge 0$, and $k, E, G_x \in \mathbf{R}$. Suppose*

$$a_E \le 2xb, \qquad \tfrac{1}{2} n x^{-\varepsilon/8} \le n_T, \qquad x^{-\varepsilon} \le \delta_2 = \frac{n_2}{n}, \qquad n_2 \le n,$$
$$x^{1/2-\varepsilon} \le m \le x^{1/2+\varepsilon/2}, \qquad xm \le k \le n^2 x^{1/2-\varepsilon},$$
$$E \le x^{C\varepsilon} \mathcal{B}_x(a_E), \qquad 26 \le x^{3\varepsilon/8}.$$

If

$$Q \frac{\delta_2}{2} \left(G_x - 4x^{1/2+\varepsilon} \right) \le E + \frac{2Q}{x}, \qquad Q = n_T m b,$$

then

$$G_x \le x^{1/2+(C+3)\varepsilon}.$$

Outline. Summing 25.8 through 25.11 gives $\mathcal{B}_x(a_E) \le 10YQ$, hence $E/Q \le 10PY$ for $P = x^{C\varepsilon}, Y = x^{1/2+13\varepsilon/8}$. Multiplying by $2/Q$ and using $\delta_2^{-1} \le x^{\varepsilon}$ and $26 \le x^{3\varepsilon/8}$ yields the G_x bound. The complete proof is on p. 715. □

For a finite set $K_p \subseteq \Psi_A$ of retained characters and $c \in \mathbf{R}$, define

$$\omega_c(\psi) = \begin{cases} 1, & \psi \text{ is trivial on } H, \\ c, & \text{otherwise}, \end{cases} \qquad G_m^A(x) = \sum_{\psi \in K_p} \omega_c(\psi) \operatorname{Re} G_\psi(x),$$

and

$$\operatorname{Sig}_A(x) = \sum_{\psi \in \Psi_A \setminus K_p} |G_\psi(x)|.$$

In the notation of Theorem 6.5, $A = H$, $P = \Psi_A$, and the remainder is $\Sigma_A = \operatorname{Sig}_A$.

Theorem 25.13 (Bound for the retained character sum). *Fix real numbers $C, \varepsilon, C_E, K_o, c, M$, with $C \ge 0$, $\varepsilon > 0$, and $|c| \le 1$. There is $x_1 \ge 3$, depending on these constants, such that the following holds for every $x \ge x_1$.*

Let H be a finite abelian group, $A_0 \le H$, $TT \subseteq H$, and $T_2 \subseteq A_0$. Impose all the assumptions of Lemma 25.6 preceding the introduction of E and Z. Let $a_E \ge 0$, $k > 0$, and suppose

$$a_E \le 2xb, \qquad \tfrac{1}{2} n x^{-\varepsilon/8} \le n_T, \qquad x^{-\varepsilon} \le n_2/n,$$
$$x^{1/2-\varepsilon} \le m \le x^{1/2+\varepsilon/2}, \qquad xm \le k \le n^2 x^{1/2-\varepsilon}.$$

Let $K_p \subseteq \Psi_A$ contain $\mathbf{1}$. Suppose

$$\#\Psi_A \le M, \qquad \psi(((1,0),0)) = 1 \quad (\psi \in K_p),$$

and assume the character bounds

$$\left| \sum_{t \in TT} \psi((0,0),t) - c n_T \right| \le M x^{-1+\varepsilon} n_T \quad (\psi \in K_p,\ \psi|_H \ne \mathbf{1}),$$
$$|\ell(\psi)| \le K_o \qquad (\psi \in K_p \setminus \{\mathbf{1}\}),$$
$$|\ell(\psi)| \le C_E \log x \qquad (\psi \in \Psi_A \setminus K_p).$$

If every $u_0 \in A_0$ satisfies

$$\operatorname{gap}(u_0) \le x^{C\varepsilon} \mathcal{B}_x(a_E),$$

then

$$G_m^A(x) \le x^{1/2+(C+3)\varepsilon} + \operatorname{Sig}_A(x).$$

Proof. Apply Theorem 6.5 with $A = H$, $P = \Psi_A$ and $M' = M$. The host, weight and cost conditions are those of Lemma 25.6. The character and size bounds are the displayed hypotheses. Take its threshold enlarged to at least 3 and use $\Sigma_A = \operatorname{Sig}_A$. □

Assembly of Theorem 6.12

Fix a level Λ with integer parameter $o = o_\Lambda$ as in Definition 2.5, and fix real $e_D, e_L, K, c_{HH}, e_K, C_E$ and $\varepsilon > 0$. Take the threshold of Theorem 25.13 with $C = \max(e_D + e_L, 0) + 3$, $M = 4o$, $c = (3o+1)^{-1}$, and K_o as in Lemma 6.8. Enlarge the threshold to x_0 so that for all $x \geq x_0$,

$$\log x \leq x^{\varepsilon}, \qquad 200(1+K)^2(1+c_{HH})(1+e_K) \leq x^{\varepsilon}, \qquad 1 + 4o \leq x^{\varepsilon}. \tag{25.3}$$

Such a threshold exists because $x^{\varepsilon} \to \infty$ and $\log x / x^{\varepsilon} \to 0$.

Let $x \geq x_0$, take a frame ϕ and a host H_{host} as in Definition 5.1. Write H for the finite abelian group of the frame and A_0 for its distinguished subgroup. Fix $TT \subseteq H$. Suppose that H_{host} and TT admit chain data at x, ε with bounds e_D, e_L, K, c_{HH}, e_K in the sense of Definition 6.11. Suppose also that they satisfy the arithmetic requirement of Definition 8.19 for the level Λ at x, ε with constant C_E. Use the arithmetic datum of the host and choose chain data as above, with cutoff N, local weights w and compatible ℓ. The models of Theorem 24.1 take $N = \lceil 2a \rceil$.

The window hypotheses give $e_K \geq 0$. The chain's height exponent lies in $[0, c_{HH}]$, so the second inequality of (25.3) holds with the chain's exponent. The chosen chain data satisfy the remaining hypotheses of Theorem 25.4. Since $0 \leq e_{\mathrm{Div}} \leq e_D$ and $0 \leq e_{\mathrm{Loss}} \leq e_L$, we have $C = e_D + e_L + 3$ and $C_{\mathrm{err}} \leq C$. The translation map covers A_0, so for each $s \in A_0$ we may choose $u \in U_0$ with $\mathrm{tr}(u) = s$. By Theorem 25.4 and $\mathcal{B}_x(a) \geq 0$,

$$\mathrm{gap}(s) \leq x^{C_{\mathrm{err}}\varepsilon} \mathcal{B}_x(a) \leq x^{C\varepsilon} \mathcal{B}_x(a).$$

Take $K_p = K_H$, the characters whose primitive representatives on H_{host} belong to the retained family of Λ. Lemma 6.8 supplies the character conditions of Theorem 25.13 with this same ℓ. Lemma 5.3 gives the local conditions, and the chosen chain data satisfy the size, weight, cutoff, thin-scale and cost conditions. Apply Theorem 25.13 with $a_E = a$ to get

$$G_m^A(x) \leq x^{1/2 + (e_D + e_L + 6)\varepsilon} + \mathrm{Sig}_A(x).$$

For this host, Theorem 6.9 at $y = x$ gives $G_m^A(x) = G_{m,\Lambda}(x)$. With $K_p = K_H$, $\mathrm{Sig}_A(x) = \Sigma_A(x)$ term by term. Since $x \geq 3$, the arithmetic requirement of Definition 8.19 and the compatibility of the chosen ℓ give the hypotheses of Theorem 6.10. Thus $\mathrm{Sig}_A(x) \leq 4o\, x^{1/2+\varepsilon}$.

Since $e_D + e_L \geq 0$, the last inequality of (25.3) gives

$$G_{m,\Lambda}(x) \leq (1 + 4o) x^{1/2 + (e_D + e_L + 6)\varepsilon} \leq x^{1/2 + (e_D + e_L + 7)\varepsilon}.$$

Part VI

Input two: the window bound VWin

Chapter 26

The window bound and its inputs

Definition 26.1 (Local kernels and the total bound)**.** For a number field E and a prime ℓ, let $S_\ell = \{v : v \mid \ell\}$ and let G_{E,S_ℓ} be the Galois group of the maximal extension of E unramified at finite places outside S_ℓ. For $k \in \mathbf{Z}$ put

$$\mathrm{Sha}_{\ell,k}(E) = \ker\left(H^1(G_{E,S_\ell}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(-k)) \longrightarrow \prod_{v \in S_\ell} H^1(G_{E_v}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(-k)) \right).$$

Use continuous cohomology with discrete coefficients and write $s_{\ell,k}(E)$ for the $\mathbf{Z}_\ell$-length of the kernel. For $t \in \mathbf{N}$ and $B \in \mathbf{R}$, say that E satisfies the total bound up to t with bound B if the following conditions hold. For every algebraic closure Ω/E and every family of local data in this common closure, with places above S_ℓ and descended cyclotomic characters chosen for each ℓ, the lengths are finite for all ℓ and $1 \le k < t$, and

$$\sum_{\ell \in S} \log \ell \sum_{1 \le k < t} s_{\ell,k}(E) \le B \tag{26.1}$$

for every finite set S of rational primes.

Definition 26.2 (Fields, prime pairs and page functions)**.** Fix a primitive Dirichlet character ψ modulo M. Let $a \ge 0$ be an integer and let q, d be odd primes. Call E a field associated with (ψ, a, q, d) if $E \simeq_{\mathbf{Q}} \mathbf{Q}(\zeta_N)^H$ for some positive common multiple N of $2^{a+2}, M, q, d$, where ζ_N is primitive and

$$H = \{u \in (\mathbf{Z}/N\mathbf{Z})^\times : u \equiv \pm 1 \pmod{2^{a+2}},\ \psi(u) = 1,\ (u/q) = (u/d) = 1\}.$$

The last two conditions use Jacobi symbols. For $0 < \varepsilon < 1/52$ and $U > 0$, put $A = U^{\varepsilon/5}$. Call (q, d) a window pair for ε at U if

$$\begin{gathered} q \equiv 3 \pmod 8, \qquad d \equiv 7 \pmod{16}, \\ A/2 < \log q \le A, \qquad A(\log U)^2/2 < \log d \le A(\log U)^2, \\ q \mid d+1, \qquad p \mid d+1 \quad \text{for every odd prime } p \le A. \end{gathered}$$

A page function $a : \mathbf{R} \to \mathbf{N}$ satisfies $x^{2+8\varepsilon} \le 2^{a(x)} \le 2x^{2+8\varepsilon}$ for every $x \ge 1$.

Definition 26.3 (The window assertion)**.** For $C_t \in \mathbf{R}$ put $t(x) = \lfloor x^{C_t} \rfloor_{\mathbf{N}}$. The window assertion for ψ with exponent C_t requires that each $0 < \varepsilon < 1/52$ and each page function a for this ε admit reals $J \ge 1$ and X_0 with the following property. For every $X \ge X_0$ and every window pair (q, d) for ε at $X^{1/J}$, there is a $Y \ge X^{1/J}$ with $Y^2 \le X$ such that, for all $x \in [Y, Y^2]$, every field associated with $(\psi, a(x), q, d)$ satisfies the total bound of Definition 26.1 up to $t(x)$ with bound $e^{x^\varepsilon/2}$. The window bound is the assertion with $C_t = 401$ for every primitive character ψ.

Theorem 26.4 (The representation input gives the window)**.** *The window bound of Definition 26.3 follows from the four groups in Definition 26.30, the published results in Remark 26.31, and weak unitary descent in ranks two through four as in Lemma 39.51. In particular it follows from the representation assertion $\mathcal{R}$ of*

Theorem 39.2, with 39.4–39.6, using Proposition 26.32 to supply the classical inputs. The base change in that assembly is Lemma 26.22, with the character of Lemma 39.49. The trace comparisons used in attachment and descent are 26.5–26.11. The local statements used for attachment include the unlinked exponent bound of Lemma 39.55 and the weak correspondence of 40.1–40.2.

Outline. Assemble the automorphic data via 26.32 from $\mathcal{R}$, the transfer 26.22, and the published inputs of 26.31. Supplying the descent step in ranks two through four by 39.51 lets 26.39 yield 26.3 with exponent 401. The complete proof is on p. 715. □

26.1 The trace comparison

Let E/E^+ be CM and unramified at every finite place, and let U be the unitary group of the standard positive definite Hermitian form of rank $n \geq 1$. At each infinite place v of E^+, choose an embedding of E above v and a dominant integral weight λ_v of rank n. For a dominant integral weight l of rank r, write

$$H(l) = \{l_j + (r-1)/2 - j : 0 \leq j < r\}.$$

Use the rational twist $\theta_J(g) = J_r(\bar{g}^{-1})^t J_r^{-1}$ in rank r, where $(J_r)_{ij} = (-1)^i 1_{i+j=r-1}$ with indices starting at zero. The rational twisted norm is $g\theta_J(g)$. At a split place the local twist is the identity and the local norm polynomial of y is its ordinary characteristic polynomial. Fix a section Σ of the finite places of E over E^+. Give the standard integral groups and the maximal compact subgroups of the local centralizers volume one.

Let μ be a unitary idèle class character of E restricting to the quadratic character of E/E^+. Assume $\mu_w = (-1)^{\mathrm{ord}_w}$ at inert places and $\mu_v(z) = (z/|z|)^{e_v}$ at infinity, with e_v odd. For monic local polynomials Q_1, Q_2 of degrees a, b with separable product, write $O_w^{Q_1}(Q_1Q_2; f_w)$ for the sum of unitary orbital integrals with polynomial Q_1Q_2. At an inert place require both root multisets to be stable under conjugate inversion. Weight the orbit of γ there by $(-1)^{\mathrm{ord}_w \det(h|_{\ker Q_1(\gamma)})}$, where h is the Hermitian form. A change of basis multiplies the determinant by a norm of even valuation. At a split place the sign is one. Write $SO_w(Q_1, Q_2; \phi_w)$ for the sum of twisted orbital integrals with the indicated norm polynomials. Put

$$R = \mathrm{Res}(Q_1, Q_2), \qquad D = ((-1)^a Q_1(0))^b ((-1)^b Q_2(0))^a.$$

At a split place take $\Delta_w = \mu_w((-1)^{ab}D)|R^2/D|_w^{1/2}$ when $D \neq 0$, and zero when $D = 0$. At an inert place take $\Delta_w = (-1)^{\mathrm{ord}_w R}|R|_w^{1/2}$. The absolute value sends a uniformizer to the inverse residue cardinality. Matching means

$$SO_w(Q_1, Q_2; \phi_w) = \Delta_w(Q_1, Q_2) O_w^{Q_1}(Q_1Q_2; f_w). \tag{26.2}$$

The empty block has polynomial one, and a zeroth power is one.

A monic separable polynomial P over E is elliptic if its roots have modulus one at every complex embedding. Write $h_l(P)$ for the product of the conjugated weight characters evaluated on the roots at the chosen embeddings, and $r(P)$ for the number of irreducible factors. For a finite adelic centralizer C with rational subgroup Γ and maximal compact subgroup C_c, put

$$\mathrm{mass}(C) = \frac{\#(\Gamma \backslash C / C_c)}{\#(\Gamma \cap C_c)}.$$

Let $\kappa(P)$ be the rational twisted centralizer mass divided by $2^{r(P)}$, with value zero for an empty twisted class and one in rank zero. Define

$$\mathcal{S}_{a,b}(l_1, l_2; \phi) = \sum_{P_1, P_2} \kappa(P_1)\kappa(P_2) h_{l_1}(P_1) h_{l_2}(P_2) \prod_{w \in \Sigma} SO_w(P_1, P_2; \phi_w),$$

over ordered coprime elliptic polynomials of degrees a, b. Let $G_U(\lambda, f)$ sum over rational unitary classes of separable polynomial P, with summand the centralizer mass times $h_\lambda(P)$ times the product of finite orbital integrals.

Lemma 26.5 (The trace identity for two blocks)**.** *For each $a+b=n$ there is a finite collection of pairs of weights $(l_{1,I}, l_{2,I})$ and signs $s_I \in \{1,-1\}$. At each v, I_v selects a entries of $H(\lambda_v)$. The infinitesimal characters of the two weights are the selected entries shifted by $-be_v/2$ and the complementary entries shifted by $-ae_v/2$. Each of $(n,0)$ and $(0,n)$ has one term, with sign $+1$ and weight λ on the positive-rank block.*

Let $f = \bigotimes_{w\in\Sigma} f_w$ be a compactly supported Hecke test at open level $K' = \prod_w K'_w \subseteq \prod_w K_w$, where K_w is the standard integral subgroup of U. Assume $K'_w = K_w$ and $f_w = 1_{K_w}$ almost everywhere. For each $a+b=n$ take compact locally constant tests $\phi_{a,b,w}$ on the two general linear blocks, equal to the product integral unit almost everywhere. If the tests satisfy (26.2) at each $w \in \Sigma$, then

$$G_U(\lambda, f) = \sum_{a+b=n} \sum_{I} s_I \, \mathcal{S}_{a,b}(l_{1,I}, l_{2,I}; \phi_{a,b}). \tag{26.3}$$

If the nonzero set of f_{w_0} consists of elements with separable characteristic polynomial at some $w_0 \in \Sigma$, then $G_U(\lambda, f) = d_\lambda^{-1} \operatorname{tr}(f)$ on the forms of weight λ and level K', where d_λ is the weight representation's dimension.

Outline. Regroup the Hecke trace by rational conjugacy and use 26.10 for unit orbital factors. Norm reciprocity and Hermitian classification select the rational terms, common centralizers give the masses, and the resultant formula and Weyl minors give the signed block weights. The complete proof is on p. 715. □

At a split place identify U with $\mathrm{GL}_n(E_w)$. Let P be the upper block parabolic of type (a,b) with modulus δ_P. For $m = \operatorname{diag}(m_1, m_2)$ the transfer is

$$\mu_w((-1)^{ab} \det(m_1)^b \det(m_2)^a)\, \delta_P(m)^{1/2} \int_{M_{a,b}(E_w)} \int_{\mathrm{GL}_n(\mathcal{O}_w)} f_w(k^{-1} m n(X) k)\, dk\, dX, \qquad n(X) = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}. \tag{26.4}$$

The compact group and the integral matrix lattice have volume one. In the orbital integral the change from the unipotent conjugating coordinate to the off-diagonal coordinate is $Y \mapsto Y - m_1^{-1} Y m_2$. On triangularizing the two blocks its determinant is $\pm R/\det(m_1)^b$. Multiplying the absolute value of this determinant by $\delta_P(m)^{1/2}$ gives the factor $|R^2/D|_w^{1/2}$ in (26.2).

26.2 The spherical unit at an inert place

Let L/F be an unramified quadratic extension of nonarchimedean local fields of characteristic zero. Write O for the integers of L and q for the residue cardinality of F. Write $\bar{z}$ for conjugation and let v be the valuation with value one on a uniformizer. Put $\eta(z) = (-1)^{v(z)}$ for $z \neq 0$.

Let B be a finite commutative L-algebra with involution σ extending conjugation and an element g. Write P_g for the characteristic polynomial of multiplication by g. If P_g is separable, every root occurs in the minimal polynomial, so $B = L[g] \simeq L[X]/P_g$ is a product of fields. Let D_B be the product of their integer rings. Put

$$C_B = (B^\times)^\sigma / \{u\sigma(u) : u \in B^\times\}, \qquad T_B = \{u \in B^\times : u\sigma(u) = 1\}, \qquad T_B^0 = T_B \cap D_B^\times.$$

For $c \in (B^\times)^\sigma$, let $h_c(y,z) = \operatorname{Tr}_{B/L}(cy\sigma z)$ and let $X_B(c)$ consist of full O-lattices stable under g and self-dual for h_c. A full lattice is finitely generated over O and spans B over L. Define

$$N_B(c) = \sum_{[M] \in T_B \backslash X_B(c)} [T_B^0 : (T_B)_M], \qquad S(B) = \sum_{[c] \in C_B} N_B(c), \qquad (T_B)_M = \{u \in T_B : uM = M\}.$$

A lattice automorphism and its inverse are integral, so $(T_B)_M \subset T_B^0$. Multiplication by u takes lattices for $cu\sigma(u)$ to lattices for c. On $B = B_1 \times B_2$ with the componentwise involution and generator, put

$$K(B_1, B_2) = \sum_{[(c_1,c_2)] \in C_B} \eta(\det h_{c_1}) N_B(c_1, c_2).$$

The determinant uses an L-basis of B_1. A basis change multiplies it by a norm and preserves valuation parity. In dimension zero the determinant and count are one.

Lemma 26.6 (Two lattice counts for a marked vector)**.** *Suppose $\sigma(g) = g$ and P_g is separable. Let $r = \dim_L B$. For a full O-lattice M put $M^* = \{z : \mathrm{Tr}_{B/L}(zM) \subset O\}$. If $m_1, \dots, m_r$ is an O-basis of M, put*

$$s_g(M) = \eta\big(\det(\mathrm{Tr}_{B/L}(g^i m_j))_{0\le i<r,\, 1\le j\le r}\big).$$

The power basis and the trace pairing give a nonzero determinant, and its sign is independent of the integral basis. Let $G_B(u)$ sum $s_g(M)$ over full lattices satisfying $gM \subset M$, $\sigma M = M$, $u \in M$ and $1 \in M^*$. For $c \in (B^\times)^\sigma$ and $w' \in B^\times$,

$$G_B(cw'\sigma w') = \#\{M \in X_B(c) : w' \in M\}. \tag{26.5}$$

Outline. Apply Beuzart-Plessis's linearized Jacquet–Rallis theorem, checking regularity by the moment matrix. Lattice descent gives the linear trace determinant, and orthonormal bases identify the Hermitian cosets with marked self-dual stable lattices. The complete proof is on p. 716. □

Lemma 26.7 (Lattices with prescribed intersection and projection)**.** *Let $M_i \subset B_i$ be full g_i-stable lattices and suppose $R = \mathrm{Res}(P_{g_1}, P_{g_2}) \neq 0$. The number of (g_1, g_2)-stable lattices $M \subset B_1 \times B_2$ with $M \cap B_1 = M_1$ and $\mathrm{pr}_2 M = M_2$ is $q^{2v(R)}$. If $\sigma_i(g_i) = g_i$ and $\sigma_i M_i = M_i$, the number also satisfying $\sigma M = M$ is $q^{v(R)}$.*

Outline. Parametrizing lattices by $f \in V/H$ turns stability into $D(f) = g_1 f - f g_2 \in H$ with $\det D = R$, giving $q^{2v(R)}$ by elementary divisors. For self-adjoint lattices, $\tau(f) = \sigma_1 f \sigma_2$ and z of 26.6 split classes as $a + zb$, giving $q^{v(R)}$. The complete proof is on p. 717. □

Lemma 26.8 (The self-adjoint identity)**.** *Let $\sigma_i(g_i) = g_i$ and suppose $P_{g_1} P_{g_2}$ is separable. For $R = \mathrm{Res}(P_{g_1}, P_{g_2})$,*

$$K(B_1, B_2) = (-q)^{v(R)} S(B_1) S(B_2). \tag{26.6}$$

Outline. Use conductor bounds to recover weighted lattice counts by normalizing marked counts, and alternate the tests over norm classes. Away from the endpoints, the basis determinant and lifts in 26.7 give $(-q)^{v(R)}$, and normalization removes the endpoint terms. The complete proof is on p. 717. □

Lemma 26.9 (Passage from a norm-one generator to a self-adjoint generator)**.** *Suppose $g_i \sigma_i(g_i) = 1$ and $P_{g_1} P_{g_2}$ is separable. Then*

$$K(B_1, B_2) = (-q)^{v(\mathrm{Res}(P_{g_1}, P_{g_2}))} S(B_1) S(B_2). \tag{26.7}$$

Outline. Split by fixed idempotents and induct, using dual pairs and 26.7 for exchanged factors. Apply 26.8 to a self-adjoint generator of the local order over its unramified coefficient ring, then compare by trace duality and odd residue degree. The complete proof is on p. 719. □

Proposition 26.10 (Spherical matching at an inert place)**.** *Let E/E^+ be CM and unramified at finite places. At an inert place w put $L = E_w$ and $F = L^-$. Let q_0 be the residue cardinality of F. Let $a + b = n$ with $a, b \ge 0$. Take monic $Q_1, Q_2 \in L[X]$ of degrees a, b with separable product and*

$$X^{\deg Q_i} \overline{Q_i}(X^{-1}) = \overline{Q_i(0)} Q_i(X).$$

In $B_i = L[X]/Q_i$ let g_i be the image of X and extend conjugation by $\sigma_i(g_i) = g_i^{-1}$. Then

$$(-q_0)^{-v(\mathrm{Res}(Q_1, Q_2))} K(B_1, B_2) = S(B_1) S(B_2). \tag{26.8}$$

In (26.2) this matches 1_{K_w} with $1_{\mathrm{GL}_a(O)\times \mathrm{GL}_b(O)}$ on the twisted general linear groups.

Outline. Lemma 26.9 gives the unit match (26.8). For the unitary orbital integral, identify the space with a rank-one module over $B = B_1 \times B_2$ via the trace pairing and evaluate the matched transfer factor $(-q_0)^{-v(R)}$. The complete proof is on p. 720. □

26.3 Trace powers and base change

Lemma 26.11 (A bound for convolution powers on a block)**.** *Let E/E^{+} be as in 26.5 and $a, b \geq 0$. On the two general linear factors use $\theta = \theta_0$, where $\theta_0(g) = (\bar{g}^{-1})^{t}$. Fix Haar measures, a finite matrix multiplicity, compactly supported continuous end kernels with convolution operators T_1, T_2 on the rational quotient, a matrix M, and $r_0, r_1 \geq 0$. There exist $C \geq 0$ and $N \in \mathbf{N}$ with the following property.*

For $k \in \mathbf{N}$ let V be bounded and suppose $T_1 V T_2$ has a compactly supported continuous convolution kernel Φ. Put $h = \operatorname{tr}(\Phi M)$ and assume it has a common open finite level on both sides and logarithmic support size at most $r_0 + k r_1$ in each factor. Logarithmic size sums the positive logarithms of the local maximum entry norms of g and g^{-1}, with squared norms at complex places. In each factor every rational twisted class meeting the nonzero set has separable norm polynomial.

For a matrix g put $W^{\perp_g} = g(W^{\perp_{\mathrm{std}}})$, with $\perp_{\mathrm{std}}$ the standard Hermitian orthogonal complement. At every infinite place require each slice to be very cuspidal. This means invariance under twisted conjugation by the compact unitary group and

$$\int_N h(ug)\, du = 0 \quad \text{for } 0 \neq W_1 \subseteq W_2, \quad W_1^{\perp_g} = W_2, \quad W_2^{\perp_g} = W_1.$$

Fix the other variables of the slice. Let N be the unipotent radical for $W_1 \subseteq W_2$. In rank d it consists of $1 + X$ with $X(\mathbf{C}^d) \subseteq W_2$ and $XW_2 \subseteq W_1$, subject to $XW_1 = 0$. Then the regular elliptic twisted geometric sum $\mathcal{E}(h)$ satisfies

$$|\mathcal{E}(h)| \leq C(1+k)^{N} \|V\|. \tag{26.9}$$

Outline. Adelic reduction gives a compact cut containing the elliptic terms, and alternating flag weights and very cuspidal integration cancel the other terms. Cauchy–Schwarz bounds the diagonal by $\|V\|$ times a translate count whose integral grows polynomially by unfolding. The complete proof is on p. 721. □

Lemma 26.12 (The complex Lefschetz integral)**.** *In rank r over $\mathbf{C}$, use θ_0 and a dominant integral weight l. Put $\inf(l) = (H(l), -H(l))$ for the two central parameter multisets. The wave multipliers on the exterior algebra with weight coefficients give compactly supported kernels ϕ with the very cuspidal property of 26.11. If* val *is the multiplier value as a function of the two central parameters, then for $d = \operatorname{diag}(z_i)$ with $|z_i| = 1$ and distinct z_i^2,*

$$\int_{\mathrm{GL}_r(\mathbf{C})} \phi(x^{-1} d\theta_0(x))\, dx = \operatorname{val}(\inf(l))\, \overline{\operatorname{ch}_l(z_1^2, \ldots, z_r^2)}.$$

Here ch_l is the Weyl character, and the kernel uses the same Haar measure as the integral. For two blocks the formula is the product of the two integrals.

Outline. Finite propagation gives compact support, and exterior contraction gives vanishing on flag radicals. On the Hermitian cone the cochain identity reduces the supertrace to degree zero, where the multiplier at $\inf(l)$ and the flip give the conjugated Weyl character. The complete proof is on p. 723. □

Lemma 26.13 (Rational and stable geometric coefficients)**.** *For a product finite test ϕ of ranks a, b as in 26.5, assume the product of the two norm polynomials is separable wherever $\phi_{w_0} \neq 0$ for some $w_0 \in \Sigma$. Let $\mathcal{G}_{a,b}(l_1, l_2; \phi)$ be the sum over rational twisted classes with elliptic norm polynomials in each block. Each summand is weighted by the centralizer mass. The other factors are the two weight characters and the class's finite orbital integrals. With Σ_∞ the set of complex places of E,*

$$\mathcal{G}_{a,b}(l_1, l_2; \phi) = 2^{(a+b)|\Sigma_\infty|} \mathcal{S}_{a,b}(l_1, l_2; \phi).$$

For an adelic product kernel h with finite tests giving ϕ under the split and inert conventions and archimedean integrals of 26.12, write ν_v for the product of the multiplier values at the two weights at v. The same rational sum with the adelic orbital measures is

$$\mathcal{E}(h) = c_{a,b} \Big(\prod_{v \in \Sigma_\infty} \nu_v \Big) \mathcal{G}_{a,b}(l_1, l_2; \phi)$$

for a nonzero $c_{a,b}$ depending on the fixed Haar measures.

Outline. Use 26.10 and 26.5 to describe fibres by norm symbols, then count real signs by reciprocity and identify classes by the Hasse norm theorem. Combine both block counts with their masses and the archimedean multipliers of 26.12. The complete proof is on p. 725. □

Lemma 26.14 (The algebra of variable comparison tests)**.** *Use the field E, the character μ and the two-block comparison of 26.5. Write V for the space of definite unitary forms of the chosen weight and finite level. For each term s of the comparison, of ranks a, b, write H_s for the space of square-integrable functions on*

$$(\mathrm{GL}_a(E) \times \mathrm{GL}_b(E))\backslash(\mathrm{GL}_a(\mathbf{A}_E) \times \mathrm{GL}_b(\mathbf{A}_E))$$

with the finite matrix multiplicity of the test. Convolution acts on the right. For a compact open subgroup L write $e_L = \mathbf{1}_L/\operatorname{vol}(L)$.

Choose a nonempty finite set T' of inert spherical testing places containing the prescribed inert places, a finite set S_0 of split testing places containing the prescribed split places, and the remaining finite exceptional set J. Take these sets disjoint. A variable finite test may vary at a chosen split place outside $T' \cup S_0 \cup J$ where μ is unramified. The following operators are taken as tuples on V and on all the H_s.

The reference operator a_0 is the identity on V. On H_s its finite kernel is the diagonal matrix of level idempotents. Its infinite kernel is the product of the undifferentiated wave kernels of 26.12, normalized to have product of values one at the weights of s.

At a variable split place w, put $K_n = \mathrm{GL}_n(\mathcal{O}_w)$, choose a uniformizer ϖ_w, and take the characteristic function of

$$K_n \operatorname{diag}(\underbrace{\varpi_w, \dots, \varpi_w}_{j}, 1, \dots, 1) K_n, \qquad 0 \le j \le n.$$

On V use its spherical action. On H_s replace the reference idempotent at either member of the split pair, in its chosen coordinates, by the transfer (26.4) divided by $\mu_w((-1)^{ab})$. Keep the other factors of the reference test. For an infinite generator choose a block, an infinite place, one of its two rank factors and a central differential operator from either complex central action of 26.12. Apply that operator to the reference kernel in the indicated factor. On V and on the other blocks take its scalar value at the indicated weight.

Let $\mathcal{A}$ be the unital complex algebra in

$$\operatorname{End}(V) \times \prod_s \mathcal{B}(H_s)$$

generated by these tuples and their componentwise adjoints. Write ρ, R_s for the coordinate maps. Fix one separating test at each place of $T' \cup S_0$. Denote by β_s the convolution on H_s with their transferred matrix kernels, the reference level idempotents in the other finite factors, and the reference kernel at infinity. On V, let u act in each automorphic summand by the product of the prescribed m-fold local convolutions at the exceptional places, where $m \ge 1$. Then $\mathcal{A}$ is commutative, u commutes with $\rho(\mathcal{A})$, and β_s commutes with $R_s(\mathcal{A})$.

Outline. The level idempotent acts as identity, transpose gives spherical commutativity, and even time convolution commutes with central operators, with the same identities for adjoints. The variable operators act by scalars on the summands preserved by u. The complete proof is on p. 726. □

Lemma 26.15 (A spherical vector from an ordered translation character)**.** *Let L be a finite extension of $\mathbf{Q}_p$, with ring of integers O_L and uniformizer ϖ. Write $K_j = \mathrm{GL}_j(O_L)$, and let $I_j \subset K_j$ be the inverse image of the upper triangular subgroup over the residue field. Let $P \subset \mathrm{GL}_j(L)$ be the upper block parabolic with block sizes $j_1, \dots, j_s$. For smooth irreducible admissible complex representations σ_i of $\mathrm{GL}_{j_i}(L)$, form the normalized induction*

$$M = \operatorname{Ind}_P^{\mathrm{GL}_j(L)}(\sigma_1 \boxtimes \cdots \boxtimes \sigma_s).$$

Give I_j volume one. For $x_1 > \cdots > x_j$ integral, put $t_x = \operatorname{diag}(\varpi^{x_1}, \dots, \varpi^{x_j})$ and

$$A_x = \delta_B(t_x)^{1/2} \mathbf{1}_{I_j t_x I_j}, \qquad \delta_B(t) = \prod_{i<i'} |t_i/t_{i'}|_L,$$

where B is the upper triangular Borel. Suppose there is a nonzero vector in M^{I_j} on which every A_x acts by $\prod_i \xi_i^{x_i}$, with $\xi_i \in \mathbf{C}^\times$ and

$$|\xi_1| \geq \cdots \geq |\xi_j|.$$

Then each σ_i has a K_{j_i}-fixed vector. The union, with multiplicities, of their normalized Satake multisets is $\{\xi_1, \ldots, \xi_j\}$.

Outline. Read the ordered character in a subquotient's Jacquet module by Iwahori projection. Frobenius reciprocity and spherical Langlands classification give a fixed vector, which lifts by averaging, and the normalized constant term recovers the block Satake multiset. The complete proof is on p. 727. □

Lemma 26.16 (Finiteness of the discrete inducing data)**.** *Let E be a CM field. In each rank $1 \leq d \leq n$, fix a compact open finite level, finitely many maximal compact types at infinity, and a bounded set of infinitesimal parameters. Normalize unitary representations of $\mathrm{GL}_d(\mathbf{A}_E)$ to have trivial character on the positive real split center by twisting by $|\det|^{it}$. In rank d, fix one of the specified types and a bound R for the infinitesimal parameters. Suppose a continuous compactly supported right-convolution kernel acts on every automorphic form of this level, type and bound by a scalar of modulus at least δ, for some common $\delta > 0$. Fix $\epsilon > 0$. There are finitely many isomorphism classes of normalized discrete representations with these data for which every constant-term exponent c of block rank $0 < r < d$ satisfies $\mathrm{Re}\, c/[E : \mathbf{Q}] - r(d-r)/2 \leq -\epsilon r(d-r)$. Here e^{ct} is the eigencharacter on a nonzero vector in the span of translates of a constant term under left translation by $\mathrm{diag}(e^t I_r, I_{d-r})$ at the infinite places.*

For the trace comparison, the counting theorem of Müller [42, Cor. 1.3 and Thm. 1.4] also gives finiteness for the entire normalized discrete spectrum at the fixed level, types and parameter bounds. In either case the tuples of total rank at most n form finitely many isomorphism classes.

Outline. Apply Müller's adelic counting theorem modulo the split centre. Bounded infinitesimal parameters bound the Casimir eigenvalue, and finiteness of the cuspidal and residual counts gives finite lists for each type and hence for tuples. The complete proof is on p. 727. □

Lemma 26.17 (Translation tests at an inert place)**.** *Let L/L^+ be the unramified quadratic extension at an inert place in 26.5. Write $q = \#k_L$, choose a uniformiser ϖ of L^+, and use an integral Witt basis for the unitary group. Write I_U, I_r for the upper Iwahori subgroups of the unitary group and of $\mathrm{GL}_r(L)$, respectively. Give their maximal compact subgroups volume one. For an integral vector z put $t_z = \mathrm{diag}(\varpi^{z_i})$.*

Take $x_0 > \cdots > x_{n-1}$, put $i^* = n-1-i$, and set $h_i = x_i - x_{i^*}$. Assume the positive entries of h are at least two and their consecutive gaps are at least two. The unitary test is

$$u_x = \mathrm{vol}(I_U)^{-1} \mathbf{1}_{I_U t_h I_U}.$$

For $a + b = n$, partition the indices into sets A, B of sizes a, b, each invariant under $i \mapsto i^*$. Write x_A, x_B in their inherited order. Put

$$v_A(x) = \sum_{i \in A,\ j \in B} \min(h_i, h_j), \qquad d_{r,z} = \mathrm{vol}(I_r)^{-1} \mathbf{1}_{I_r t_z I_r}.$$

Then u_x matches

$$\sum_A (-\sqrt{q})^{-v_A(x)}\, d_{a,x_A} \otimes d_{b,x_B}$$

in (26.2). For a finite linear combination of the u_x, its k-th convolution power, $k \geq 1$, matches the sum over A of the k-th convolution powers of the corresponding linear combinations for that same A.

Outline. Diagonalize the translation cosets to identify the contributing orbits and compute the resultant factor of 26.5. Additivity of the exponents and resultant valuations in the chamber gives the convolution powers for each fixed assignment. The complete proof is on p. 728. □

Lemma 26.18 (The trace-power estimate)**.** *In the definite unitary comparison for an automorphic constituent π, prescribe a set S of at most one split place and a finite set T of spherical inert places. There are test levels and separating tests as in Lemma 26.14, with $S \subseteq S_0$ and $T \subseteq T'$. Use its algebra $\mathcal{A}$, with*

coordinate actions ρ and R_s. Put $D_s = \beta_s^m$, where $m \geq 1$ is the repetition exponent of the fixed tests. For each $g \in \mathcal{A}$ there are $C_s \geq 0$, $N_s \in \mathbf{N}$ and $t_s(k) \in \mathbf{C}$ such that for all $k \geq 1$,

$$\operatorname{tr}((u\rho(g))^k) = \sum_s t_s(k), \qquad |t_s(k)| \leq C_s(1+k)^{N_s}\|(D_s R_s(g))^k\|. \tag{26.10}$$

The finite level K and finite-dimensional archimedean type spaces W for the final candidates can be chosen from π, S before T is chosen.

Outline. Fix levels and types from 26.14, then separate candidates by inert interpolation and split Hecke projections. Use 26.17, 26.10, 26.12, and 26.13 in the two-block trace identity for ordered words, then apply the growth bound. The complete proof is on p. 729. □

Lemma 26.19 (A nonzero block character)**.** *Use the data of 26.18, with $uv = v$ for a nonzero vector v and $\rho(g)v = p(g)v$ for $g \in \mathcal{A}$. Put $B_s = D_s^* D_s$ and let $\mathcal{C}_s$ be the commutative unital C^*-algebra generated by $R_s(\mathcal{A})$ and B_s. For some s there is a character χ of $\mathcal{C}_s$ with $\chi \circ R_s = p$ and $\chi(B_s) > 0$.*

Outline. Under the vanishing assumption, use 26.14, a finite sum of squares and the Gelfand norm to get a test equal to one at p that contracts every block. The trace-power bound and Vandermonde inversion contradict the eigenvalue 1. The complete proof is on p. 732. □

Lemma 26.20 (Spectral realization with the reference convolution)**.** *In a block of 26.14 put $A_0 = R_s(a_0)$, $X = (\beta_s^m)^* \beta_s^m$, and let χ be a character of $\mathcal{C}_s$ with $\chi(A_0) \neq 0$. There exist characters η_j converging to χ on every variable generator and on X, with $\eta_j(A_0) \neq 0$. For each j there is a nonzero continuous vector-valued function f_j on the rational quotient of the block on which A_0, the variable generators and X act by their respective values at η_j. For the reduction height $H(x) = 1+\rho_a(x_a)+\rho_b(x_b)$ (where $\rho_d(x_d) = \inf_{\gamma \in \mathrm{GL}_d(E)} L(\gamma g)$ for a representative g of x_d and logarithmic size L as in 26.11), the functions $H^{-M} f_j$ are square-integrable for a common integer M.*

Outline. Bound weighted norms of periodized kernel rows by adelic reduction. Cyclic spectral representation and Fubini give weighted square-integrable eigenvectors near χ, and convolution by the reference kernel gives continuous representatives along a convergent sequence. The complete proof is on p. 732. □

Proposition 26.21 (Reading the block character)**.** *For a block s of ranks a, b and a character χ from 26.19, there are discrete families P_a, P_b of those total ranks, with the shifted infinitesimal characters of 26.5. In their product realization there is a nonzero vector F of the prescribed level and compact types. For each variable finite generator t of Lemma 26.14, its finite kernel acts on F by*

$$\frac{\chi(R_s(t))}{\chi(R_s(a_0))}.$$

Let b_s be the finite matrix kernel of β_s, write $*$ for convolution, and put $b_s^\dagger(x)_{ij} = \overline{b_s(x^{-1})_{ji}}$. The finite kernel $(b_s^\dagger)^{*m} * b_s^{*m}$ of $X = B_s$ acts by $\chi(X)/\chi(R_s(a_0))^{2m}$. In particular $b_s^{*m} F \neq 0$, where a kernel acts by right convolution.

Outline. Approximate the character of 26.19 by joint eigenvectors in Langlands induction fibres. The weights from 26.5 and 26.12 bound the inducing data and force the limiting twists to vanish, and the finite kernel eigenvalue equations pass to the limit. The complete proof is on p. 733. □

Lemma 26.22 (One base change for all places)**.** *Let F/F^+ be CM and unramified at finite places, and let π be automorphic on the standard definite unitary group of rank $n \geq 1$. There is a family of discrete automorphic components on $\mathrm{GL}_{n_i}(\mathbf{A}_F)$ with $\sum_i n_i = n$ and $n_i = a_i d_i$, $a_i, d_i \geq 1$. In unnormalized parabolic induction, component i embeds in the induction of*

$$\rho_i |\det|^{(a_i-1)((d_i-1)/2-j)}, \qquad 0 \leq j < d_i,$$

where ρ_i is normalized, unitary, conjugate self-dual and cuspidal on $\mathrm{GL}_{a_i}(\mathbf{A}_F)$.

At each finite place w the induced product of the local components has an irreducible subquotient P_w, spherical if π is unramified there. At an inert spherical place the Satake datum $(\alpha_i)_{1 \leq i \leq \lfloor n/2 \rfloor}$ of π gives P_w

the multiset $\{\alpha_i, \alpha_i^{-1}\}_i$, with an extra entry 1 in odd rank. At a split place P_w occurs in the local restriction of π. At each embedding $\tau : F \hookrightarrow \mathbf{C}$, the union of component infinitesimal characters is $H(\lambda_\tau)$ for the corresponding weight of π.

Outline. Choose μ via Lemma 39.49, assemble the twisted isobaric sum D_T from 26.18, 26.21, and 26.19 on each finite inert set T, and extract a conjugate self-dual algebraic base-change parameter matching the Satake parameters at every unramified place. The complete proof is on p. 735. □

Lemma 26.23 (Galois representations for definite unitary base change)**.** *Let F/F^+ be CM and unramified at every finite place, and let π be automorphic on the standard definite unitary group of rank $n \geq 1$ with dominant integral weight $\lambda = (\lambda_\tau)_\tau$. Let the discrete components on $\mathrm{GL}_{n_i}(\mathbf{A}_F)$ ($n_i = a_i d_i$, $\sum_i n_i = n$) with normalized unitary conjugate self-dual cuspidal bases ρ_i on $\mathrm{GL}_{a_i}(\mathbf{A}_F)$ be given by Lemma 26.22. Fix a prime ℓ, an isomorphism $\iota : \bar{\mathbf{Q}}_\ell \simeq \mathbf{C}$, and a prolongation of each finite place of F.*

For each component i and each $0 \leq j < d_i$, the infinitesimal weights of ρ_i at every embedding $\tau : F \hookrightarrow \mathbf{C}$ are distinct and lie in $(n - d_i)/2 + \mathbf{Z}$. Moreover, there exists a continuous semisimple representation $r_{i,j} : G_F \to \mathrm{GL}_{a_i}(\bar{\mathbf{Q}}_\ell)$ with the following properties.

1. *At every finite place $w \nmid \ell$ and every map R of 40.2, $\mathrm{WD}_\iota(r_{i,j}|_{G_{F_w}})$ satisfies the local comparison of Definition 39.5 with*
$$R\Big((\rho_{i,w} |\det|_w^{(d_i-1)/2-j}) \otimes |\det|_w^{(1-n)/2} \Big).$$
In particular, if $\rho_{i,w}$ is spherical with Satake multiset $B_w(\rho_i)$, then $r_{i,j}$ is unramified at w and inverse arithmetic Frobenius has eigenvalues $q_w^{(n-d_i)/2+j} B_w(\rho_i)$.

2. *At every finite place $w \mid \ell$, if $\rho_{i,w}$ is spherical, then $r_{i,j}|_{G_{F_w}}$ is crystalline and de Rham. If the local parameter above has zero monodromy and satisfies the toral or full tame test of Definition 39.5, then $r_{i,j}|_{G_{F_w}}$ is potentially semistable and de Rham and satisfies the local comparison of Definition 39.6. In each case its filtration jumps at an embedding $\tau : F \hookrightarrow \bar{\mathbf{Q}}_\ell$ are $(n - d_i)/2 + j - z = (n-1)/2 - (z + (d_i - 1)/2 - j)$ as z runs over the infinitesimal weights of ρ_i at $\iota \circ \tau$.*

Consequently $r_\pi = \bigoplus_i \bigoplus_{j=0}^{d_i-1} r_{i,j} : G_F \to \mathrm{GL}_n(\bar{\mathbf{Q}}_\ell)$ is a continuous semisimple representation: at every finite place w where π is unramified (with spherical transfer P_w of Lemma 26.22 and Satake multiset $B_w(P_w)$), r_π is unramified with inverse arithmetic Frobenius characteristic polynomial $P_{w,B_w(P_w)}(X) = \prod_{\alpha \in B_w(P_w)} (X - q_w^{(n-1)/2}\alpha)$ when $w \nmid \ell$, and crystalline when $w \mid \ell$, with filtration jumps $\{k - \lambda_{\iota\circ\tau,k} : 0 \leq k < n\}$ at each τ.

Outline. Extracts the weights of ρ_i from 26.22 and applies 39.2 directly in the even case or after a Hecke twist from 39.49 in the odd case. Untwisting via 26.26, Chebotarev density, and summing give r_π and 39.5–39.6. The complete proof is on p. 736. □

26.4 From representations to the window

Proposition 26.24 (Induction and cuspidal support)**.** *Let K be a nonarchimedean local field. All representations in this proposition are smooth complex representations. Write $G_n = \mathrm{GL}_n(K)$, $\nu(g) = |\det g|_K$, and $\pi_1 \times \pi_2$ for normalized parabolic induction. The cuspidal support of an irreducible representation is the multiset of supercuspidal inducing factors, up to isomorphism and permutation.*

1. *Irreducible representations of G_n are admissible. Normalized Jacquet restriction and normalized parabolic induction preserve finite length. Smooth duality commutes with normalized induction. For inducing data $V^\vee, V$, the pairing of the induced spaces is obtained by integrating the pointwise pairing over $\mathrm{GL}_n(\mathcal{O}_K)$.*

2. *Let P, Q be standard parabolics with Levi factors M, L. The normalized Q-Jacquet module of a representation induced from M has a filtration indexed by $W_L \backslash W_{G_n} / W_M$, where W denotes the Weyl group. For a minimal representative w, the corresponding quotient is obtained by Jacquet restriction in M along $M \cap w^{-1}Qw$, transport by w, and induction in L along $L \cap wPw^{-1}$. For an exterior tensor product on M, each cell multiplicity is the sum, over the constituent tensors of its Jacquet restriction, of their multiplicities times the corresponding induction multiplicities in L.*

3. *A segment is a list $\Delta = (\rho, \nu\rho, \dots, \nu^{d-1}\rho)$, $d \geq 1$, with ρ supercuspidal of positive rank. Let $Z(\Delta)$ and $L(\Delta)$ be the irreducible subrepresentation and irreducible quotient, respectively, of the product in the displayed order. The direct sum of the Grothendieck groups in all ranks, with multiplication given by induction, is the polynomial ring over $\mathbf{Z}$ on the classes $Z(\Delta)$. On the Levi with d equal blocks the Jacquet module of $Z(\Delta)$ is the tensor of the displayed factors in order.*

4. *If π_1, π_2 are irreducible and unitarizable, then $\pi_1 \times \pi_2$ is irreducible.*

5. *Let S_i be the cuspidal support of an irreducible π_i. Suppose that for every $\rho \in S_1$ and $\rho' \in S_2$ of the same rank,*
$$\rho \not\simeq \rho'\nu^j \qquad (j = -1, 0, 1).$$
Then $\pi_1 \times \pi_2$ is irreducible.

6. *Suppose $\mathrm{char}(K) = 0$. An irreducible of positive rank is essentially square-integrable if and only if it is $L(\Delta)$ for a segment. In particular every square-integrable irreducible has this form. An irreducible of positive rank is tempered if and only if it is a normalized product of square-integrable irreducibles.*

7. *Two segments are linked if their union is a segment and neither contains the other. In characteristic zero every generic irreducible is a product of the $L(\Delta)$ for pairwise unlinked segments.*

8. *In characteristic zero every irreducible has a unique Langlands multisegment $\mathfrak{m}$. Its representation $L(\mathfrak{m})$ is the irreducible cosocle of the corresponding standard module, the product of the segment factors in Langlands order. For any two multisegments, $L(\mathfrak{m} + \mathfrak{n})$ occurs as a subquotient of $L(\mathfrak{m}) \times L(\mathfrak{n})$.*

Outline. Use the geometric lemma for the Jacquet filtration, Bernstein's unitary product theorem and the segment and tempered classifications. The support condition excludes linkage across the factors, and multisegment multiplication supplies the constituent indexed by the sum. The complete proof is on p. 738. □

Proposition 26.25 (Jacquet fixed vectors, finite groups and units)**.** *Keep the notation of Proposition 26.24. Put $K_n^1 = \ker(\mathrm{GL}_n(\mathcal{O}_K) \to \mathrm{GL}_n(k_K))$, where k_K is the residue field.*

1. *Let $P = MU$ be a standard parabolic of G_n. Put $J = K_n^1(U \cap \mathrm{GL}_n(\mathcal{O}_K))$ and $J_M = M \cap K_n^1$. For an admissible V, the quotient map to the U-coinvariants gives a bijection $V^J \to (V_U)^{J_M}$.*

2. *Suppose $\mathrm{char}(K) = 0$. If every member of the cuspidal support of an irreducible V has a nonzero first-congruence-fixed vector in its own rank, then $V^{K_n^1} \neq 0$.*

3. *Suppose $\mathrm{char}(K) = 0$. Fix a nontrivial continuous unitary additive character ψ of K. Let $m \geq 1$ and let π_1, π_2 be irreducible supercuspidal representations of G_{m+1}. If for every irreducible τ of G_m their pair Whittaker functional equations admit a common scalar for the same ψ and invariant integral, then $\pi_1 \simeq \pi_2$.*

4. *For a complex representation V of a finite group G, there are subgroups H_i, linear characters λ_i of H_i, and integers a_i such that*
$$\mathrm{tr}_V - (\dim V)1_G = \sum_i a_i \,\mathrm{Ind}_{H_i}^G(\lambda_i - 1_{H_i}).$$

5. *For a prime p and $d \geq 1$, regular characters of $\mathbf{F}_{p^d}^\times$, modulo p-power conjugation, parametrize the irreducible cuspidal characters of $\mathrm{GL}_d(\mathbf{F}_p)$, with the degree and elliptic values of 32.5.*

6. *For a number field E and $m, b \geq 1$, there is an integer $a \geq 1$ prime to b such that $u \in \mathcal{O}_E^\times$ and $u - 1 \in a\mathcal{O}_E$ imply $u = v^m$ for some $v \in \mathcal{O}_E^\times$.*

Outline. Use Moy–Prasad and 26.24 for Jacquet fixed vectors, then derive the local converse hypotheses from rank m by multiplicativity and ramified twists. Apply Brauer induction, 32.31, and Chevalley's congruence theorem for the remaining assertions. The complete proof is on p. 739. □

Proposition 26.26 (Automorphic and character inputs)**.** *Use normalized Satake parameters, and let q_v denote the residue cardinality at a finite place v.*

1. *On the unitary groups of the Hermitian spaces in the host, a smooth cuspidal automorphic function of uniform moderate growth is rapidly decreasing. If h is the adelic height and $h_0(g) = \inf_{\gamma \in U(F)} h(\gamma g)$, then for each integer $N \geq 0$, $\sup_g h_0(g)^N |\phi(g)| < \infty$.*

2. *On these groups fix an open finite level, a continuous finite-dimensional compact type at infinity and the eigenvalues of the archimedean central differential operators. The corresponding automorphic forms span a finite-dimensional space. The same assertion holds for GL_a over a totally complex number field with the infinitesimal character fixed at every embedding.*

3. *On GL_a over a totally complex field the unitary cuspidal space, with the split central direction removed, is a discrete Hilbert sum of irreducibles. Its smooth compact-finite constituents are cuspidal automorphic representations. A smooth cuspidal automorphic form whose absolute value is invariant under the split centre is bounded.*

4. *Over a totally complex field, let two finite lists of cuspidal representations of positive ranks have the same union, with multiplicity, of their Satake multisets at almost every finite place. The lists agree up to isomorphism and permutation. For a unitary cuspidal representation and each spherical v, every Satake root satisfies $q_v^{-1/2} < |\alpha| < q_v^{1/2}$.*

5. *For a cuspidal representation of GL_a over a totally complex field, inverse transpose realizes the contragredient. The induced bilinear pairing of the representation with its inverse-transpose transform is nondegenerate in both variables.*

6. *For a polarized regular algebraic cuspidal input as in 39.2, the attached representation has the weights and local comparisons of 39.4–39.6.*

7. *Let E be totally complex, $a \geq 1$, and π cuspidal on $\mathrm{GL}_a(\mathbf{A}_E)$. Let m_τ be its multiset of infinitesimal weights at $\tau : E \hookrightarrow \mathbf{C}$. If $m_\tau \subset (a-1)/2 + \mathbf{Z}$ at every embedding, there is one $w \in \mathbf{Z}$ such that $m_{\bar\tau} = w - m_\tau$ for all τ. If in addition each multiset has distinct entries, π has nonzero relative Lie algebra cohomology for an algebraic coefficient system.*

8. *For each number field E, global reciprocity gives an open quotient map from the idèles onto G_E^{ab}. Its kernel is the closure of the product of $E^\times$ and the connected archimedean subgroup. Local units map onto the inertia image, a uniformizer maps to inverse arithmetic Frobenius modulo inertia, and the closure of the local image is the decomposition image. For a nonzero integral modulus $\mathfrak{f}$, the narrow ray class field is unramified away from $\mathfrak{f}$ at finite places. Its Frobenius elements away from $\mathfrak{f}$ generate the Galois group. An ideal prime to $\mathfrak{f}$ pairs trivially with all its characters precisely when it has a totally positive generator congruent to one modulo $\mathfrak{f}$.*

9. *Fix $\iota : \overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$. Let χ be an algebraic Hecke character of E with angular and norm exponents given on the connected archimedean idèles by its integral infinity type n_τ, and let μ be its continuous ℓ-adic realization with this reciprocity convention. At $v \mid \ell$ it is de Rham and potentially crystalline, crystalline when χ_v is unramified, with filtration jump $-n_{\iota\tau}$ at each induced embedding $\tau : E \hookrightarrow \overline{\mathbf{Q}}_\ell$. Its Weil–Deligne character is χ_v composed with local reciprocity in the Weil-to-multiplicative-group direction, with zero monodromy. These assertions hold for the compatible period identifications of 39.6.*

10. *For $K/\mathbf{Q}_\ell$ finite and a continuous finite-dimensional $\mathbf{Q}_\ell$-representation V of G_K, if V is de Rham, it becomes semistable over a finite extension of K.*

Outline. Use rapid decrease and finiteness, compare reciprocal Satake roots by isobaric uniqueness, and apply 39.2 and Clozel's purity and cohomology results. Reciprocity, rank-one period comparison, and Berger's monodromy theorem give the character and local Galois assertions. The complete proof is on p. 740. □

Proposition 26.27 (Finite cohomology and its local conditions)**.** *Let $r \geq 1$, let ℓ be prime, put $R = \mathbf{Z}/\ell^r$, and set $A^\dagger = \mathrm{Hom}_R(A, \mu_{\ell^r})$ for a finite discrete R-module A with continuous Galois action.*

1. *At a finite place v of a number field F, write $H_v^i(A) = H^i(F_v, A)$. These groups are finite in degrees $0, 1, 2$ and vanish in higher degrees. Cup product and the local invariant give perfect pairings of $H_v^i(A)$*

with $H_v^{2-i}(A^\dagger)$. One has

$$\frac{|H_v^1(A)|}{|H_v^0(A)|\,|H_v^2(A)|} = \begin{cases} |A|^{[F_v:\mathbf{Q}_\ell]}, & v \mid \ell, \\ 1, & v \nmid \ell. \end{cases}$$

At ℓ all three assertions also hold for the absolute Galois group of each finite subextension $K/\mathbf{Q}_\ell$ of the chosen algebraic closure, with $[K:\mathbf{Q}_\ell]$ in the formula.

2. *Suppose $v \nmid \ell$ and inertia acts trivially on A. Put $H^1_{\mathrm{ur}}(F_v,A) = \ker(H^1_v(A) \to H^1(I_v,A))$. Then $H^1_{\mathrm{ur}}(F_v,A)$ and $H^1_{\mathrm{ur}}(F_v,A^\dagger)$ are each the exact annihilator of the other.*

3. *Let S be a finite set of finite places containing every place above ℓ, and let A be a module for $G_{F,S}$, the Galois group allowing ramification at S and infinity. Write $h_i = |H^i(G_{F,S},A)|$ for $0 \le i \le 2$, and $a_\infty = \prod_{v\mid\infty} |H^0(F_v,A)|$. These three global groups are finite, and*

$$h_0 h_2 |A|^{[F:\mathbf{Q}]} = h_1 a_\infty.$$

For odd ℓ or totally complex F, with $P^i(A) = \bigoplus_{v\in S} H^i_v(A)$ and $D_R = \mathrm{Hom}_R(-,R)$, one has the exact nine-term Poitou–Tate sequence

$$\begin{aligned} 0 \to H^0(G_{F,S},A) \to P^0(A) \to D_R H^2(G_{F,S},A^\dagger) \\ \to H^1(G_{F,S},A) \to P^1(A) \to D_R H^1(G_{F,S},A^\dagger) \\ \to H^2(G_{F,S},A) \to P^2(A) \to D_R H^0(G_{F,S},A^\dagger) \to 0 \end{aligned}$$

(normalized so the cup of a uniformizer's Kummer class with the unramified character sending arithmetic Frobenius to 1 has invariant 1), perfect duality between $\mathrm{Ш}^1_S(A)$ and $\mathrm{Ш}^2_S(A^\dagger)$, and $H^i(G_{F,S},A) = 0$ for $i \ge 3$.

4. *Suppose ℓ is odd. For each $v \in S$ choose an R-submodule $L_v \subset H^1_v(A)$, and let $L_v^\perp$ be its orthogonal complement for $A^\dagger$. Let $S_L(A)$ consist of the global classes with restrictions in L_v at S, using $G_{F,S}$. Then*

$$\frac{|S_L(A)|}{|S_{L^\perp}(A^\dagger)|} = \frac{|H^0(F,A)|}{|H^0(F,A^\dagger)|\,a_\infty} \prod_{v\in S} \frac{|L_v|}{|H^0(F_v,A)|}.$$

5. *Let $K/\mathbf{Q}_\ell$ be finite and let V be a finite-dimensional continuous de Rham representation over $\mathbf{Q}_\ell$. Define $H^1_f(K,V)$ as in 36.15. Then*

$$\dim H^1_f(K,V) = \dim H^0(K,V) + \dim_{\mathbf{Q}_\ell}\big(D_{\mathrm{dR}}(V)/\mathrm{Fil}^0\big).$$

Under the Tate pairing its annihilator is $H^1_f(K,V^\vee(1))$. If $T \subset V$ is a stable full lattice, take the inverse image of the finite condition for T and the image of the rational finite condition for $V^\vee(1)/T^\vee(1)$. These conditions too are exact annihilators.

Outline. Use inflation, local duality and the local Euler characteristic of 36.14 to prove unramified orthogonality. Apply the normalized nine-term Poitou–Tate sequence and the Euler characteristics to count the Selmer groups, and use Bloch–Kato for the de Rham local condition. The complete proof is on p. 740. □

Proposition 26.28 (Abelian schemes and periods)**.**

1. *Let $R \to R_0 = R/I$ be a surjection of commutative rings with divided powers on I. Suppose there is an integer N for which every product of divided powers of elements of I of total divided-power degree at least N vanishes. For an abelian scheme A_0/R_0, evaluate its first crystalline homology on this thickening. Any lift identifies this module with its first de Rham homology. Given one lift, every direct summand reducing to the Hodge submodule is the Hodge submodule of a lift with the specified reduction. A homomorphism between special fibres lifts uniquely between two specified lifts if its crystalline map preserves the chosen filtrations.*

2. *For an abelian scheme A/R of relative dimension g, $H^1_{\mathrm{dR}}(A/R)$ is projective of rank $2g$. All its classes are primitive for the group law. Cup product identifies its exterior square with $H^2_{\mathrm{dR}}(A/R)$. The Hodge sequence is exact, with projective end terms of rank g, and commutes with arbitrary base change. In homology the Hodge submodule is isotropic for the pairing of a line bundle, and a principal polarization gives a perfect pairing.*

3. *The dual abelian scheme and its Poincaré bundle give biduality and commute with base change. On a thickening as in (1), a smooth separated quasi-compact lift of a scheme computes its crystalline cohomology by its relative de Rham complex, in particular in degrees one and two.*

4. *For a complex abelian variety with uniformization $\mathbf{C}^g/\Lambda$, integration defines $\Lambda \to H_1^{\mathrm{dR}}(A)$. The image spans, and the projection to $\mathrm{Lie}(A)$ is the lattice inclusion in the uniformizing coordinates. The map commutes with endomorphisms. If H is the Appell–Humbert form of a line bundle, the pullback of its de Rham pairing is $\epsilon\, 2\pi i \operatorname{Im} H$ on Λ, with the same $\epsilon \in \{1,-1\}$ for all such data.*

5. *For an abelian scheme over a smooth affine complex base, pull the family back to a holomorphic parameter neighborhood of 0. A given period lattice at 0 extends locally to holomorphic lattice vectors in a local trivialization of the Lie bundle. This continuation commutes with endomorphisms. A continued cycle paired with a relative de Rham class has a holomorphic period $P(z)$. If $t_1, \dots, t_d$ lie in the augmentation ideal of a divided-power thickening over $\mathbf{C}$, and every product of $s+1$ of them vanishes, evaluation on the base map given by this s-jet gives the crystalline period*

$$\sum_{|\alpha| \le s} \frac{\partial^\alpha P(0)}{\alpha!} t^\alpha.$$

6. *Let A be an abelian variety over the fraction field of a discrete valuation ring with perfect residue field. Good reduction is equivalent to trivial inertia on T_pA for some prime p invertible in the ring. Equivalently all torsion of order invertible in the ring is unramified, or this holds for infinitely many such orders. For each invertible prime p, potential good reduction is equivalent to finite inertia image on T_pA. If the residue field is finite and A has complex multiplication, it has potential good reduction.*

Outline. Use Grothendieck's deformation theorem, de Rham cohomology and the Poincaré bundle, with divided-power contraction for crystalline comparison. Obtain periods by integration and Gauss–Manin Taylor transport, and apply the Serre–Tate reduction criteria. The complete proof is on p. 740. □

Proposition 26.29 (Complex multiplication and special values)**.**

1. *Let K be imaginary quadratic, $\mathfrak{a}$ a nonzero fractional ideal, and fix an analytic uniformization by $\mathbf{C}/\mathfrak{a}$. If $\sigma \in \mathrm{Aut}(\mathbf{C}/K)$ restricts to $\mathrm{Art}_K(s)^{-1}$ on K^{ab}, with Art_K normalized as in Definition 1.15, its conjugate has uniformization by $\mathbf{C}/s_f^{-1}\mathfrak{a}$. On torsion labels the transport is multiplication by the inverse finite idèle. Here the lattice notation means the fractional ideal with the indicated finite completions.*

2. *The field $K(j(\mathbf{C}/\mathcal{O}_K))$ is the Hilbert class field. Its degree is the class number, and its conjugates of $j(\mathbf{C}/\mathcal{O}_K)$ are $j(\mathbf{C}/\mathfrak{a})$, equal for two fractional ideals exactly when their classes agree.*

3. *Let F be totally real, $\mathfrak{f} \neq 0$ an integral modulus, and b a rational function on its narrow ray classes. Use the Dirichlet series on integral ideals prime to $\mathfrak{f}$, weighted by b, and its continuation $L(s,b)$. Then $L(1-k,b) \in \mathbf{Q}$ for $k \ge 1$. If $\mathfrak{f} = (\ell^n)$, $n \ge 1$, and b takes values in $\mathbf{Z}_\ell$, then, for each integral ideal $\mathfrak{c}$ prime to ℓ, with $b_{\mathfrak{c}}(\mathfrak{a}) = b(\mathfrak{c}\mathfrak{a})$,*

$$L(1-k,b) - (\mathrm{N}\mathfrak{c})^k L(1-k, b_{\mathfrak{c}}) \in \mathbf{Z}_\ell.$$

4. *For an ordinary ideal class C of a totally real field F of degree n, put $Z_C(s) = |D_F|^{s/2}\Gamma_{\mathbf{R}}(s)^n \zeta_C(s)$, with $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$. If $CC' = [\mathfrak{d}_F]$, the class of the different, then $Z_C(s) = Z_{C'}(1-s)$ wherever the two gamma factors are finite.*

5. *Let χ be a unitary Hecke character of a totally complex field with a choice of one embedding per complex place. Suppose its angular exponents $n_\tau \in \mathbf{Z}$ include a nonzero exponent. In the completion by the local gamma factors, with the local Tate factors taken at the conductor of χ, one has*

$$\Lambda(s,\chi) = (-i)^{\sum_\tau |n_\tau|} \prod_{v\nmid\infty} \epsilon_v(s,\chi_v)\, \Lambda(1-s,\chi^{-1})$$

in the additive-character convention of 1.19.

Outline. Apply idèlic complex multiplication to lattices, torsion and j-invariants, and extend ray-class rationality and integral differences by linearity. Use the partial-zeta equation for dual lattice classes and multiply Tate's local functional equations. The complete proof is on p. 741. □

Definition 26.30 (Classical inputs for the window bound)**.** Write h_N for the conjunction of the following four groups, in the precise forms of 26.24–26.29.

1. For general linear groups over nonarchimedean local fields, use Bernstein–Zelevinsky finiteness and the geometric lemma, the Moy–Prasad isomorphism for Jacquet modules, and the classification by segments and multisegments. This includes the polynomial description of the Grothendieck ring, the classification of essentially square-integrable representations by segments, irreducibility for products of unitarizable irreducibles and for the separated supports of 26.24(5), tempered induction from square-integrable factors, and the genericity criterion for unlinked segments. Use also Langlands exhaustion and the unique irreducible quotient of a standard module, the local converse theorem and depth-zero induction. The finite-group inputs are Brauer induction for characters of degree zero and Green's parametrization of cuspidal characters by Frobenius orbits of regular characters. The unit congruence input says that for each number field and positive integers m, b, there is a positive integer a prime to b such that every unit congruent to 1 modulo a is an mth power of a unit.

2. For the proof of 26.21 also use Langlands' spectral decomposition of the full adelic L^2 space into normalized inductions from discrete Levi data, integrated over the unitary characters of the real split centre [110, Thm. 7.2]. For automorphic representations use rapid decay of cusp forms, Harish–Chandra finiteness, finiteness of spaces of general linear automorphic forms at fixed data, the cuspidal spectral decomposition, Jacquet–Shalika multiplicity one and Gelfand–Kazhdan duality. Include the definite unitary base change of Lemma 26.22, the representation assertion of Theorem 39.2, and Clozel's purity and cohomology theorems. For characters use global Artin reciprocity, ray class fields and the local comparison for the ℓ-adic realizations of algebraic Hecke characters. A de Rham representation over a finite extension of $\mathbf{Q}_\ell$ is potentially semistable.

3. For Galois cohomology with finite coefficients use local and global finiteness and Euler characteristic formulas, local Tate duality and the Poitou–Tate sequence of Theorem 27.2. At a finite place away from ℓ where the coefficient module is unramified, its unramified condition and the unramified condition for its Tate dual are exact annihilators by 26.27(2). At the places above ℓ use the local conditions of the Selmer problem and their annihilators as in 26.27(4),(5). Include the Greenberg–Wiles formula and the Bloch–Kato local conditions. Local finiteness, duality and the local Euler characteristic formula are required also over each finite extension of $\mathbf{Q}_\ell$ in the chosen algebraic closure.

4. For abelian schemes use deformation across nilpotent divided-power thickenings, the Hodge sequence and base change for de Rham cohomology, duality and crystalline comparison. For complex abelian varieties use comparison with the period lattice and its Gauss–Manin variation in families. At a discrete valuation use the good and potential good reduction criteria. The remaining assertions are elliptic CM reciprocity, generation of the Hilbert class field by j, rationality and integral translated differences for totally real ray-class values, and the two functional equations used in Lemma 1.19. Their precise forms are 26.28 and 26.29.

Remark 26.31 (Published inputs used in the proof of the window bound). In addition to the four groups in Definition 26.30, the proofs of the following numbered results use the indicated published theorems.

1. For a normalized cuspidal newform f of weight $h \geq 2$, let $V_\lambda(f)$ be its representation in Kato's normalization at a finite place λ of its coefficient field. Let $K/\mathbf{Q}$ be finite abelian, $1 \leq r \leq h-1$,

$2r \neq h$, and $T \subset V_\lambda(f)(r) = V$ a $G_{\mathbf{Q}}$-stable lattice. The subgroup of $H^1(K, V/T)$ whose localization at each place w belongs to the image of $H^1_f(K_w, V)$ is finite. For fixed K, f, r it is zero at almost all λ, for every such lattice. Lemma 30.11 uses finiteness with K quadratic, $h = 2e+1$ and $r = e \geq 1$. Thus $2r = h - 1$, including at $e = 1$ [125, §14.1 and Thm. 14.2(1)].

2. Let L be CM with maximal totally real subfield L^+, let $n \geq 2$, and let (π, χ) be a regular algebraic polarized cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_L)$. Fix $\iota : \overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$. If $r_{\pi,\iota}$ is irreducible on $G_{L(\mu_{\ell^\infty})}$, then
$$H^1_f(L^+, \operatorname{ad} r_{\pi,\iota}) = 0.$$
The coefficient space is the full matrix algebra with conjugation on G_L and the negative transpose action determined by the polarization at complex conjugation. In Theorem 30.62, the cuspidal witness supplies π, and the image of $G_{L(\mu_{\ell^\infty})}$ is exactly the subgroup in its third hypothesis [129, Thm. 6.1].

3. For every finite abelian number field L and prime ℓ, the class-group Iwasawa module of the cyclotomic $\mathbf{Z}_\ell$-extension has $\mu_\ell = 0$. The finite-layer form used in Theorem 28.6 is a bound, independent of j, on the $\mathbf{F}_\ell$-dimension of $\mathrm{Cl}(L(\mu_{\ell^j}))/\mathrm{Cl}(L(\mu_{\ell^j}))^\ell$. The bases in that theorem contain μ_ℓ for odd ℓ, and contain i for $\ell = 2$. Theorem 28.7 applies the bound to these abelian bases in the proof of 26.40 [92].

4. Let L/F be an unramified quadratic extension of nonarchimedean local fields of characteristic zero, and let η be its quadratic character. Embed GL_r in the upper left corner of GL_{r+1}. For a relatively regular semisimple $Y \in M_{r+1}(F)$, put $\Delta(Y) = \eta(\det(e^*, e^*Y, \ldots, e^*Y^r))$, using the indicated rows and the last coordinate row e^*. Then
$$\Delta(Y) \int_{\mathrm{GL}_r(F)} \mathbf{1}_{M_{r+1}(\mathcal{O}_F)}(h^{-1}Yh)\eta(\det h)\, dh = \int_{U_r(F)} \mathbf{1}_{M_{r+1}(\mathcal{O}_L)}(u^{-1}Xu)\, du$$
if Y is $\mathrm{GL}_r(L)$-conjugate to a self-adjoint X for the standard Hermitian form. The right side is zero if there is no such match. The integral subgroups on both sides have volume one. In Lemma 26.6 the nonzero determinant of the moment matrix makes the marked column and row cyclic and cocyclic, respectively. These are the relative regularity tests for the matrices used there [91, Thm. 1].

5. Fix an algebraic closure of a number field F and a finite set S of places. Let $\mathcal{L}$ be a family of cyclic extensions of prime degree such that for every finite extension D/F and every finite $v \notin S$, some $L \in \mathcal{L}$ is linearly disjoint from D and splits v completely. Suppose given continuous semisimple representations $r_L : G_L \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$, invariant up to isomorphism under $\mathrm{Gal}(L/F)$, whose restrictions from any two members to their compositum are isomorphic. There is a continuous semisimple representation over F restricting to r_L at every member. For Lemma 26.35 the members are quadratic. Infinitely many split at the prescribed place, and the Galois closure of D contains only finitely many quadratic fields. This verifies generality with $S = \varnothing$ [107, §1, Lem. 1].

6. Let O be a DVR whose fraction field has characteristic zero, let M be a finite free O-module, and let a commuting family of endomorphisms preserve M. A simultaneous system of eigenvalues on a nonzero vector of $M/\mathfrak{m}_O M$ lifts, after finite extension of the fraction field and localization at a prime above $\mathfrak{m}_O$, to the system of a characteristic-zero simultaneous eigenvector. In Lemma 30.41 the lattice is the integral Fourier lattice of cusp forms at the indicated weight, level and character. Its Hecke operators commute, the Fourier coefficient at one makes the residual eigenvector nonzero, and the congruences include the operators at the level primes [117, Lem. 6.11].

7. For $G = \mathrm{Res}_{E/\mathbf{Q}} \mathrm{GL}_d$, remove the positive real split center, fix an open compact subgroup at the finite places and an irreducible maximal compact type at infinity. For each real bound on the absolute value of the Casimir eigenvalue, the sum of the multiplicities of that level and type in the discrete spectrum under the bound is finite, including the residual spectrum. This is the finiteness consequence of the cuspidal and residual counting estimates of [42, Cor. 1.3 and Thm. 1.4, (1.17)–(1.18)]. In Lemma 26.16, E is CM, the list of compact types is finite, and the Casimir is a fixed polynomial on the bounded set of infinitesimal parameters.

8. In Langlands's spectral decomposition for GL_r over a number field, with the positive split central character fixed and unitary, the automorphic L^2 space is the Hilbert sum, over associate classes of

Levi subgroups, of the Eisenstein direct integrals formed from their discrete automorphic spectra. The inductions are normalized. Their parameters run over the imaginary dual of the real split center of the Levi modulo that of the group, with the Weyl identifications given by normalized intertwining operators. For a Levi $\prod_i \mathrm{GL}_{r_i}$, these parameters twist the factors by $|\det|^{it_i}$, $t_i \in \mathbf{R}$, subject to the fixed central character. In Proposition 26.21 the block convolutions act in the regular automorphic L^2 representation used in 26.20. The decomposition applies to that representation before extraction of generalized eigenfunctions. A nonzero spectral projection for the commuting block operators is nonzero on a set of fibers of positive measure. Thus the spectral characters used for approximation may be taken from the unitary inducing parameters [149, §7, Thm. 7.7], [134, Thms. 6.1 and 7.8].

The Galois attachment used here is Theorem 39.2. For comparison, Chenevier and Harris take a cuspidal Π over a CM field with $\Pi^c \simeq \Pi^\vee$ and nonzero relative Lie algebra cohomology with irreducible algebraic coefficients. Their Theorem 3.2.3 gives a continuous semisimple ℓ-adic representation. Away from ℓ its Frobenius-semisimplified Weil–Deligne representation has the semisimple Weil representation of the local parameter of $\Pi_v|\det|_v^{(1-n)/2}$, with monodromy bounded by that parameter in the order of part (a′). Above ℓ it is de Rham, with filtration jumps $\mu_i + n - i$, $1 \le i \le n$, for the highest weight at the corresponding embedding in the authors' coefficient convention. It is crystalline at spherical places, with the characteristic polynomial of crystalline Frobenius prescribed by the same normalized local parameter, and semistable at Iwahori-spherical places. The witness in Chapter 30 uses the cuspidal hypothesis. Its local comparisons are those of 39.4–39.6 [116, Hyp. 1.1, (1.6), Thm. 3.2.3].

Proposition 26.32 (Assembling the four groups of inputs)**.** *With the published inputs of Remark 26.31 and weak unitary descent in ranks two through four (Lemma 39.51), the representation assertion $\mathcal{R}$ of Theorem 39.2 implies the window bound of Definition 26.3.*

Outline. Combine $\mathcal{R}$ with 26.22, 26.24, 26.25, 26.26, 26.27, 26.28, and 26.29. Use 39.51 for weak descent in ranks $2, 3, 4$, then apply 26.39 to obtain 26.3. The complete proof is on p. 741. □

Lemma 26.33 (Finite multiplicity in a fixed cuspidal constituent)**.** *Let F be totally real, let π be an irreducible cuspidal constituent for GL_a/F, and let U be an open subgroup of the finite adelic group. Put $K_\infty = \prod_{v|\infty} \mathrm{O}(a)$. For a finite-dimensional complex representation ξ of K_∞, the images of all K_∞-equivariant maps $\xi \to \pi^U$ lie in a common finite-dimensional subspace of π.*

Outline. Use the Capelli operators to fix the central ideal and quotient by the common evaluation kernel to get finitely many compact types. Apply Harish-Chandra finiteness at a compact open level with those types and central ideal, then intersect with π. The complete proof is on p. 741. □

Proposition 26.34 (Coefficients, conjugation and the constituent count)**.** *Let E be totally complex, $a \ge 1$, $G = \mathrm{GL}_a$, and Ξ an irreducible algebraic coefficient representation for $\mathrm{Res}_{E/\mathbf{Q}} G$ over $\mathbf{C}$.*

1. *A cuspidal π whose infinitesimal multiset at every embedding is the negative of that of Ξ has nonzero relative Lie algebra cohomology with coefficients in Ξ.*
2. *For a regular algebraic cuspidal π and $\sigma \in \mathrm{Aut}(\mathbf{C})$, there is a cuspidal π' with finite part the scalar conjugate ${}^{\sigma}\pi_f$. Its infinitesimal multiset at $\sigma\tau$ equals that of π at τ.*
3. *For a compact open $U \subset G(\mathbf{A}_{E,f})$, put $Y = G(\mathbf{A}_{E,f})/U$ and give $\mathrm{Map}(Y, \Xi)$ the action $(\gamma f)(y) = \Xi(\gamma) f(\gamma^{-1} y)$ of $G(E)$. If S is a finite set of pairwise nonisomorphic cuspidal representations, all cohomological for this same Ξ, and each has a nonzero U-fixed finite vector, then*

$$|S| \le \sum_{j \ge 0} \dim_{\mathbf{C}} H^j\big(G(E), \mathrm{Map}(Y, \Xi)\big).$$

Outline. Use Clozel's cohomology calculation and conjugation theorem. Inject independent tensors of archimedean classes and finite fixed vectors from the cuspidal summands, and identify the target by Shapiro's lemma on the finite-level orbits. The complete proof is on p. 742. □

Lemma 26.35 (Patching at split completions)**.** *Let F be totally real, $G_F = \mathrm{Gal}(\overline{F}/F)$, and let $\mathcal{L}$ be a family of quadratic CM extensions in $\overline{F}$. Suppose that for every finite place v there are infinitely many members split at v. For $L \in \mathcal{L}$ let $r_L : G_L \to \mathrm{GL}_a(\overline{\mathbf{Q}}_\ell)$ be continuous and semisimple. Suppose r_L is invariant up to isomorphism under $\mathrm{Gal}(L/F)$, and $r_L, r_{L'}$ have isomorphic restrictions to $G_{LL'}$ for each pair. There is a continuous semisimple r of G_F with $r|_{G_L} \simeq r_L$ for all but finitely many L.*

There is also a patching assertion for a fixed cuspidal constituent π on $\mathrm{GL}_a(\mathbf{A}_F)$, an isomorphism $\iota : \overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$, and any infinite set $\mathcal{L}$ of quadratic CM extensions of F in the fixed algebraic closure. For every $L \in \mathcal{L}$ let $r_L : G_L \to \mathrm{GL}_a(\overline{\mathbf{Q}}_\ell)$ be continuous and semisimple. Suppose that at all but finitely many finite places w of L, writing v for the place below w, π_v is spherical with Satake multiset B_v, and r_L is unramified at w. Suppose its inverse arithmetic Frobenius has roots, transported by ι and counted with multiplicity,

$$\big\{(q_v^{(a-1)/2}\alpha)^{f(w/v)} : \alpha \in B_v\big\},$$

where $f(w/v)$ is the residue degree. Then there is a continuous semisimple r of G_F such that $r|_{G_L}$ is semisimple and isomorphic to r_L for all but finitely many L.

Outline. For the first assertion, avoiding the finitely many quadratic subfields in the Galois closure of D gives linear disjointness for Sorensen's prime-degree patching lemma. For the second, common normalized Satake multisets yield conjugation-invariant traces on the index-two kernels by Chebotarev and continuity; Schur's lemma rules out twisted intertwiners, and matrix patching across the index-two kernels gives r. The complete proof is on p. 742. □

Proposition 26.36 (Totally real and cyclic inputs)**.**

1. *For cuspidal representations of positive rank over a totally real field F, equality of Satake multisets almost everywhere implies isomorphism. Inverse transpose gives contragredient duality, and finite-level compact-type admissibility holds in the form of 26.33. Let f be a smooth cuspidal automorphic form of moderate growth, fixed by an open compact subgroup at the finite places. Assume that its translates by the maximal compact group at infinity span a finite-dimensional space, that $Z(U(\mathfrak{g}_{\infty,\mathbf{C}}))f$ is finite dimensional, and that the Lie algebra of the real split centre acts on f by a scalar. Then f is a finite sum of vectors in irreducible cuspidal constituents.*

2. *For a quadratic totally complex extension L/F of a totally real field and a regular algebraic cuspidal π on $\mathrm{GL}_2(\mathbf{A}_F)$, either π has the self-twist by the quadratic character of L/F, or its base change is cuspidal. The lift has the same infinitesimal multiset over each real place and copies the local representation at a split place. At common unramified places its Satake roots are the residue-degree powers of the roots downstairs.*

3. *For a normalized cuspidal newform f of weight $h \geq 2$ and level M, and $j \in \mathbf{Z}$, there is a cuspidal representation of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ of infinitesimal character $\{j + \frac{1}{2}, j + \frac{3}{2} - h\}$. At every $p \nmid M$ it is spherical, and its Satake multiset B_p satisfies*

$$\prod_{\alpha \in B_p} (X - p^{j+1/2}\alpha) = X^2 - a_p(f)X + \omega_f(p)p^{h-1},$$

   where ω_f is the nebentype.

4. *Let L/K be cyclic of prime degree with K totally complex and $a \geq 1$. A regular algebraic cuspidal representation of $\mathrm{GL}_a(\mathbf{A}_K)$ has cuspidal base change in the absence of a nontrivial self-twist by a character of $\mathrm{Gal}(L/K)$, viewed by reciprocity as a Hecke character. A regular algebraic cuspidal representation over L whose Satake multisets agree at conjugate places almost everywhere descends to a cuspidal representation over K. Infinitesimal characters pull back on embeddings and the Satake rule is the residue-degree power rule. Two cuspidal descents of the same cuspidal system differ by a character of $\mathrm{Gal}(L/K)$, including the trivial character.*

5. *If $K/\mathbf{Q}$ is imaginary quadratic, a cuspidal $\mathrm{GL}_2(\mathbf{A}_K)$ representation with distinct half-integral infinitesimal weights of integer difference at least two at each embedding, and with the cofinite Satake invariance in (4), descends with the given infinitesimal weights and with Satake matching at almost every split rational prime.*

6. *Over every totally real field of even degree there is a totally definite quaternion algebra split at every finite place. Its finite-place matrix identifications may be chosen integral almost everywhere.*

Outline. Use strong multiplicity one for duality, 26.33 for admissibility, and Harish-Chandra finiteness for finite cuspidal decomposition. Apply cyclic base change for lifting and descent, adelize and norm-twist the newform, and prescribe local quaternion invariants. The complete proof is on p. 743. □

Proposition 26.37 (The totally real rank-two attachment)**.** *Let F be totally real and π regular algebraic cuspidal on $\mathrm{GL}_2(\mathbf{A}_F)$. Suppose $\pi^\vee \simeq \pi \otimes \chi^{-1}$, where the Hecke character χ has the same sign at -1 at all real places. Fix ℓ and $\iota : \overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$. There is a continuous semisimple $r : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$ with the following properties.*

1. *At every spherical $v \nmid \ell$, r is unramified and its inverse arithmetic Frobenius roots are $q_v^{1/2} B_v$ under ι, where B_v is the Satake multiset.*
2. *If the cuspidal support at v consists of characters with no pair $\xi, \xi|\cdot|_v$, the local Weil–Deligne comparison is with the parameter of $\pi_v |\det|_v^{-1/2}$ for every correspondence of 40.2. At such a place away from ℓ the absolute values of the eigenvalues of each degree-one Weil element satisfy $|\alpha| < q_v|\beta|$ for every pair. At ℓ the representation is potentially semistable and de Rham there, and it is crystalline at a spherical place. The spherical components have the indicated unlinked character support.*
3. *At a place above ℓ satisfying the support condition in (2), the jumps at τ are the multiset $\{\frac{1}{2} - z : z \in m_{\iota\tau}\}$. There is one integer w for which each such multiset, and the multiset defined by the same formula at every embedding, is invariant under $j \mapsto w - j$.*

Outline. Take cuspidal CM lifts by 26.36(2), attach by 39.2 and patch by Chebotarev and 26.35. Use split completions for 39.5 and 39.6, the rank-two unitary classification for the root bound and 39.4 for weight symmetry. The complete proof is on p. 744. □

Write $\mathscr{H}$ for h_N, the published inputs of Remark 26.31, weak unitary descent of 39.51 in the ranks and levels of Proposition 26.32, and 26.33, 26.34, 26.36, 26.37. For rank-two descent one may supply the conclusion of the next lemma under its Galois hypotheses.

Lemma 26.38 (Descent of a specified rank-two representation)**.** *Use the cyclic and split-prime inputs of 26.36 and the attachments of 39.2 and 26.37. Let $L/\mathbf{Q}$ be abelian CM and contain the fixed field K of an odd quadratic character. Let $W : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$ be continuous with determinant factoring through a finite cyclotomic extension. Suppose $W|_{G_L}$ is irreducible and attached, via ι, to a regular algebraic cuspidal representation. Then W has a regular algebraic cuspidal attachment over $\mathbf{Q}$.*

Outline. Descend by 26.36(4), polarize using its fibre characters and attach by 39.2, then use Chebotarev and the intertwiner line to choose each twist. Over $\mathbb{Q}$, use split-prime descent by 26.36(5) and attach by 26.37 before the final twist. The complete proof is on p. 744. □

Proposition 26.39 (From automorphic inputs to counting)**.** *Assume h_N of Definition 26.30, the published inputs of Remark 26.31, and weak unitary descent in ranks two through four (39.51). Then the counting hypotheses hold. On the cuspidal spectrum they are cohomological realization, conjugation of finite Hecke data and the cohomology dimension bound. Over totally real fields they include rank-two Galois attachment, strong multiplicity one, contragredient duality, admissibility, cuspidal semisimplicity and quadratic base change. The remaining hypotheses concern newform representations, prime-degree cyclic base change over totally complex fields, rank-two descent and, over every even-degree totally real field, a totally definite quaternion algebra split at all finite places. These have the forms of 26.33, 26.34, 26.36 and 26.37; descent for a specified Galois representation is 26.38. Together these imply the window bound of Definition 26.3.*

Outline. Assemble the inputs from 26.33, 26.34, 26.36, 26.37 and 26.38. Combine the small-twist bound 28.2 with the counts of 29.15, 29.22 and 29.26 in 26.40 to obtain the window bound. The complete proof is on p. 745. □

Proposition 26.40 (Four counts suffice)**.** *Assume h_N of Definition 26.30, Remark 26.31(3), the finiteness and H^2 estimate of Theorem 28.2 for some $C_K \in \mathbf{R}$, and the four counts of 29.15, 29.22 and 29.26. Then the window bound of Definition 26.3 holds.*

Outline. Combine 28.16 at $C_t = 401$ with the plus bound 29.32 and the minus bound 27.8 via 27.10. The complete proof is on p. 745. □

Chapter 27

Three length bounds

Write $s_\ell(E,k) = s_{\ell,k}(E)$ as in 26.1 and $n = [E : \mathbf{Q}]$.

27.1 Complex conjugation and arithmetic duality

Proposition 27.1 (Splitting and addition of the bounds)**.** *For CM E and $k \in \mathbf{N}$, let $s_\ell^{\pm}(E,k)$ be the lengths of the $(-1)^k, (-1)^{k+1}$ eigenspaces $\mathrm{Sha}^{\pm}_{\ell,k}(E)$ of complex conjugation. For odd ℓ, $s_\ell = s_\ell^+ + s_\ell^-$. Fix $t \in \mathbf{N}$. For all local data in 26.1, suppose s_2 and both odd lengths are finite for $1 \le k < t$. If for every finite set T of odd primes*

$$\sum_{1\le k<t} s_2(E,k) \le B_2, \qquad \sum_{\ell\in T} \log\ell \sum_{1\le k<t} s_\ell^{\pm}(E,k) \le B_{\pm},$$

then E satisfies the total bound of Definition 26.1 up to t with bound $B_2 \log 2 + B_- + B_+$.

Proof. Conjugation c permutes the places above ℓ. At odd ℓ, $(1 \pm (-1)^k c)/2$ split the kernel. Add lengths with weights $\log \ell$. □

Theorem 27.2 (Finite-coefficient Poitou–Tate duality)**.** *Let K be a number field, ℓ prime and $r \ge 1$, with $\ell \ne 2$ or K totally complex. Put $R = \mathbf{Z}/\ell^r\mathbf{Z}$. For each primitive $\zeta \in \mu_{\ell^r}$ there exists a conjugation-compatible family of isomorphisms $\mathrm{inv}_v : H^2(K_v, \mu_{\ell^r}) \simeq R$ at finite places and their lifts to $\overline{K}$. The cup of a uniformiser's Kummer class with the unramified character sending arithmetic Frobenius to 1 has invariant 1.*

Fix this family. Let S be a finite set of finite places containing all places above ℓ, and choose a lift of each $v \in S$ to $\overline{K}$. For any finite continuous $R[G_{K,S}]$-module A, write $A^\dagger = \mathrm{Hom}_R(A, \mu_{\ell^r})$, $D_R = \mathrm{Hom}_R(-, R)$, $P^i(A) = \bigoplus_{v\in S} H^i(K_v, A)$ and $H^i_G = H^i(G_{K,S}, -)$. Then

$$\begin{aligned} 0 \to H^0_G(A) \to P^0(A) \to D_R H^2_G(A^\dagger) \\ \to H^1_G(A) \to P^1(A) \to D_R H^1_G(A^\dagger) \\ \to H^2_G(A) \to P^2(A) \to D_R H^0_G(A^\dagger) \to 0 \end{aligned}$$

is exact. The duality maps sum the local cup invariants, with the restriction of the dual class first for $i = 1$ and second for $i = 0, 2$. The localization kernels $\mathrm{Sha}^i_S(A)$ are finite for $i = 1, 2$, $\mathrm{Sha}^2_S(A) \simeq D_R\mathrm{Sha}^1_S(A^\dagger)$, and $H^i_G(A) = 0$ for $i \ge 3$.

This is the finite-coefficient form of Milne's nine-term sequence in the range where the archimedean Tate terms vanish [111, Theorem I.4.10].

27.2 The negative component

Lemma 27.3 (Bernoulli numbers at a cyclotomic level)**.** *Let θ be a Dirichlet character modulo $N \ge 1$, induced by the primitive character χ of conductor f, and let $j \ge 2$. For η modulo m, set $B_{j,\eta} =$*

$m^{j-1}\sum_{a=1}^{m}\eta(a)B_j(a/m)$, *where* $B_j(X)$ *is the Bernoulli polynomial. Put* $\mathrm{primorial}(h)=\prod_{p\le h,\ p\text{ prime}}p$. *Then* $A_{N,j}(\theta)=N\mathrm{primorial}(j+1)B_{j,\chi}$ *is an algebraic integer. For* $f\mid H$, *the character* χ_H *induced by* χ *modulo* H *satisfies*

$$B_{j,\chi_H}=B_{j,\chi}\prod_{\substack{p\mid H\\ p\nmid f}}(1-\chi(p)p^{j-1}).$$

If $\theta(-1)=(-1)^j$, *both* $B_{j,\chi}$ *and* $B_{j,\theta}$ *are nonzero. For each complex embedding* σ,

$$\max\bigl\{|\sigma A_{N,j}(\theta)|,\ N\mathrm{primorial}(j+1)|\sigma B_{j,\theta}|\bigr\}\le b(N,j):=16N^{j+1}j!. \tag{27.1}$$

Outline. Von Staudt–Clausen clears denominators, and Bernoulli distribution gives the level factors. The functional equation, Euler product and level factors give nonvanishing, while Fourier and binomial estimates give the bound. The complete proof is on p. 745. □

Theorem 27.4 (A Bernoulli bound for one character)**.** *Let* $E\subseteq\mathbf{Q}(\mu_N)$ *be CM,* ℓ *odd prime,* $\ell\nmid n$, *and* $O\subseteq\overline{\mathbf{Q}}_\ell$ *a DVR finite over* $\mathbf{Z}_\ell$ *with fraction field* K_O. *For* $k\ge1$, *let* $\theta:\mathrm{Gal}(E/\mathbf{Q})\to O^\times$ *have conjugation sign* $(-1)^{k+1}$ *and level-*N *pullback* $\widetilde{\theta}$. *With* ε_ℓ *cyclotomic, let* C_θ *be the kernel of restriction to finite inertia and decomposition at* ℓ *in* $H^1(\mathbf{Q},(K_O/O)(\theta^{-1}\varepsilon_\ell^{-k}))$. *Restriction identifies* C_θ *with the* θ*-part of* $\mathrm{Sha}_{\ell,k}(E)\otimes_{\mathbf{Z}_\ell}O$. *Normalize* v_O *by* $v_O(\pi)=1$ *for a uniformiser* π. *Then*

$$\mathrm{length}_O C_\theta\le v_O(A_{N,k+1}(\widetilde{\theta})). \tag{27.2}$$

Outline. Inflation–restriction identifies the character part. Jacobi-sum descent uses the level formula in 27.3 and cup products via 26.27(2) and 27.2, while Kummer theory gives vanishing in the residual cyclotomic case. The complete proof is on p. 746. □

Lemma 27.5 (Cyclotomic annihilation on unramified homomorphisms)**.** *Let* $N\ge1$, $K=\mathbf{Q}(\mu_N)$ *and* $G=(\mathbf{Z}/N\mathbf{Z})^\times$, *with* τ_b *acting by* b*-th powers on* μ_N. *For an odd positive integer* a *prime to* N, *set*

$$j_b=\frac{a[a^{-1}b]_N-[b]_N}{N},\qquad E_b=\frac{a-1}{2}-j_b.$$

For each ideal class c, *written additively,* $\sum_{b\in G}E_b\tau_b^{-1}c=0$. *If* $N>2$ *and* $h:G_K\to X$ *is a locally constant homomorphism to an abelian group, zero on finite inertia, then for every* $u\in G_K$ *and lifts* σ_b *of* τ_b,

$$\sum_{b\in G}E_b h(\sigma_b^{-1}u\sigma_b)=0.$$

Outline. Use the Jacobi-sum valuations in (A.226) to show the carry products are principal, then telescope and translate to obtain the required principal ideal. Inverse Artin reciprocity transfers the identity to the unramified homomorphism. The complete proof is on p. 749. □

Lemma 27.6 (Summation away from the degree)**.** *For CM* $E\subseteq\mathbf{Q}(\mu_N)$, $N,k\ge1$, $s_\ell^-(E,k)$ *is finite at odd primes* $\ell\nmid n$. *For finite sets* T *of these primes,*

$$\sum_{\ell\in T}s_\ell^-(E,k)\log\ell\le n\log b(N,k+1). \tag{27.3}$$

Proof. Take the product D of $A_{N,k+1}(\widetilde{\theta})$ over characters of $\mathrm{Gal}(E/\mathbf{Q})$ of sign $(-1)^{k+1}$, where $\widetilde{\theta}$ is the level-N pullback. By 27.3, D is a Galois-invariant nonzero algebraic integer and $\log|D|\le n\log b(N,k+1)$. Sum (27.2) in a splitting O and divide by $v_O(\ell)$ to get $s_\ell^-\le v_\ell(D)$. Summing over T proves the bound. □

Proposition 27.7 (Primes dividing the degree)**.** *Let* $E\subseteq\mathbf{Q}(\mu_N)$ *be CM,* $N,k\ge1$, *and* ℓ *odd prime. For some* $r\in\mathbf{N}$,

$$\ell^r\mathrm{Sha}^-_{\ell,k}(E)=0,\qquad r\log\ell\le30n(k+1)\log(k\ell N). \tag{27.4}$$

The length is finite and

$$s_\ell^-(E,k)\log\ell\le240\ell n^2(k+1)\log^2(k\ell N). \tag{27.5}$$

For $M = (m, H_M)$, $m > 0$, $H_M \le (\mathbf{Z}/m\mathbf{Z})^\times$, put $K_M = \mathbf{Q}(\mu_m)^{H_M}$ and $o = [K_M : \mathbf{Q}]$. Take primes $q \equiv 3 \bmod 8$, $d \equiv 7 \bmod 16$ with $\gcd(qd, m) = 1$. For $a \in \mathbf{N}$, write $E_{M,a,q,d}$ for the compositum $K_M\,\mathbf{Q}(\mu_{2^{a+2}})^+\mathbf{Q}(\sqrt{-q}, \sqrt{-d})$. If $E \simeq_{\mathbf{Q}} E_{M,a,q,d}$, every odd prime dividing n divides o.

Outline. Bernoulli distribution and 27.5 give the annihilator, and the adjugate with 27.3 and (A.230) gives (27.4). Kummer theory, (A.231), and torsion filtration give (27.5), and a Galois embedding controls odd primes. The complete proof is on p. 749. □

27.3 The negative budget and the common window

Theorem 27.8 (Uniform negative budget)**.** *Take M, o, q, d as in Proposition 27.7, integers $a, t \ge 2$, and a CM field $E \simeq_{\mathbf{Q}} E_{M,a,q,d}$. Let F be its conductor, the least level with $E \hookrightarrow \mathbf{Q}(\mu_F)$. The lengths $s_\ell^-(E,k)$ are finite for odd ℓ and $1 \le k < t$. For each finite set T of odd primes,*

$$\sum_{\ell \in T} \log \ell \sum_{1 \le k < t} s_\ell^-(E,k) \le B_-(240, o, n, F, t),$$
$$B_-(C, o, n, F, t) = 3nt^2 \log(tF) + C \sum_{\substack{\ell \mid o \\ \ell \text{ odd prime}}} \ell n^2 t^2 \log^2(t\ell F). \tag{27.6}$$

For real C and $o \in \mathbf{N}$, put $K = 3 + |C| \sum_{\ell \mid o,\ \ell \text{ odd prime}} \ell^3$ with empty sum at $o = 0$. If $F, t \ge 1$, $n \le F$, and $\log(Ft) \le 2W$, then

$$B_-(C, o, n, F, t) \le K e^{8W}. \tag{27.7}$$

Proof. Sum (27.3) at $N = F$ using $\log b(F, k+1) \le (2t+5)\log(tF)$ and $(t-1)(2t+5) \le 3t^2$. Sum (27.5) at odd $\ell \mid n$, all of which divide o by 27.7. For $u = Ft \ge 1$, $nt \le u$, so

$$3nt^2 \log u \le 3u^4, \qquad \ell n^2 t^2 \log^2(\ell u) \le \ell^3 u^4, \qquad u^4 \le e^{8W}.$$

□

Definition 27.9 (Joint window condition)**.** Fix $C_t \in \mathbf{R}$. The joint window condition at exponent C_t requires the following for every primitive Dirichlet character ψ of conductor m, every $0 < \varepsilon < 1/52$, every $A_1 \in \mathbf{R}$, and all functions $a, t : \mathbf{R} \to \mathbf{N}$ satisfying

$$a(x) \ge 2, \qquad 2^{a(x)} \le x^{A_1}, \qquad 2 \le t(x) \le x^{C_t} \qquad (x \ge 2).$$

There exist real constants $J_0 > 0$, C, P and X_1 with the following property. Let $X \ge X_1$ and let $q \equiv 3 \pmod 8$ and $d \equiv 7 \pmod{16}$ be primes with $\gcd(qd, m) = 1$. If

$$\min(q, d) > \max(2X^{C_t} + 1, P), \qquad \log(qd) \le X^{\varepsilon/(4J_0)},$$

there is a $Y \ge X^{1/J_0}$ with $Y^2 \le X$ such that both bounds below hold throughout $[Y, Y^2]$.

For each x in this interval and every field E associated with $(\psi, a(x), q, d)$ as in Definition 26.2, put $D_x = \exp(x^{4\varepsilon/5})$ and let F be the conductor of E. For every choice of local data as in Definition 26.1, the lengths $s_2(E,k)$ and $s_\ell^+(E,k)$ for odd primes ℓ are finite for $1 \le k < t(x)$. For every finite set T of odd primes,

$$\sum_{1 \le k < t(x)} s_2(E,k) \le D_x, \qquad \sum_{\ell \in T} \log \ell \sum_{1 \le k < t(x)} s_\ell^+(E,k) \le (Ft(x))^C + CD_x. \tag{27.8}$$

Theorem 27.10 (Absorption into the window bound)**.** *Let $C_t \ge 1$. The joint window condition of Definition 27.9, supplied by Theorems 28.16 and 29.32, and the negative bound of Theorem 27.8 imply the window assertion of Definition 26.3 for every primitive Dirichlet character ψ with exponent C_t.*

Outline. Under 26.3 and (A.232), use 26.2 for (A.233) and the conductor bound (A.234). Combine (27.6) and (27.7) with (27.8) via 27.1 and absorb the smaller powers of x. The complete proof is on p. 750. □

Chapter 28

The cyclotomic tower and a common window

For a number field L, a prime ℓ and $k \in \mathbf{Z}$, let S_ℓ be the places of L above ℓ. Put

$$s_\ell(L,k) = \operatorname{length}_{\mathbf{Z}_\ell} \operatorname{Sha}_{\ell,k}(L), \qquad V_\ell(L,k) = \operatorname{length}_{\mathbf{Z}_\ell} H^2(G_{L,S_\ell}, \mathbf{Z}_\ell(k+1)).$$

Both lengths may be infinite. The group $\operatorname{Sha}_{\ell,k}(L)$ is defined in 26.1. Write $D_x = \exp(x^{4\varepsilon/5})$ for $x, \varepsilon > 0$ and $j_0(\ell, R) = \lfloor \log_\ell R \rfloor + 3$, with $\lfloor \log_\ell 0 \rfloor = 0$. A cyclotomic layer L_m/L_0 of index m has degree ℓ^m. Let h_N be the data of Definition 26.30. We use local finiteness, Poitou–Tate duality and local Tate duality as in Proposition 26.27.

28.1 A bound for small twists

Call a totally complex number field L ℓ-prepared if it contains μ_ℓ for odd ℓ or i for $\ell = 2$. Recall the lifting-the-exponent identity: if $u \in \mathbf{Z}_\ell^\times$, $c = v_\ell(u-1) \geq 1$ is finite, and $c \geq 2$ for $\ell = 2$, then

$$v_\ell(u^e - 1) = c + v_\ell(e) \qquad (e \geq 1). \tag{28.1}$$

Indeed, writing $e = \ell^j e_0$ with $\ell \nmid e_0$, the linear term in $(1+(u-1))^{e_0} - 1$ has unique minimal valuation c, and iterating $(1+t)^\ell - 1 = \ell t + \cdots$ j times raises the valuation by j.

Theorem 28.1 (Class-group counting for a small twist)**.** *There is an absolute C such that for every prime ℓ, every ℓ-prepared totally complex L (containing μ_ℓ for odd ℓ or i for $\ell = 2$), and every $k \geq 1$,*

$$s_\ell(L,k) < \infty, \qquad s_\ell(L,k)\log \ell \leq \exp\big((2k[L:\mathbf{Q}])^C \log(2|D_L|)\big)\,. \tag{28.2}$$

Let F/L be cyclic over a totally complex L, with generator σ of order $\ell^{\nu-c}$. Here ℓ is prime, $1 \leq c \leq \nu$, and $c \geq 2$ if $\ell = 2$. Suppose F contains a primitive ℓ^νth root ζ satisfying $\sigma(\zeta) = \zeta^{1+\ell^c}$. Fix $k \geq 1$ and $b, B \in \mathbf{N}$. Put

$$h = \operatorname{ord}(\sigma), \qquad a = (1+\ell^c)^k, \qquad \mathcal{N}_a(\alpha) = \prod_{j=0}^{h-1} \sigma^j(\alpha)^{a^j}.$$

Let $\mathcal{E}_{\nu,k}$ be the functions φ from the finite primes of F to $\mathbf{Z}/\ell^\nu\mathbf{Z}$ satisfying

$$\sum_P \operatorname{ord}_P(\alpha)\varphi(P) = 0 \quad (\alpha \in F^\times), \qquad \varphi(\sigma^{-1}P) = a\varphi(P).$$

Suppose every finite set of $\alpha \in F^\times$ whose $\mathcal{N}_a(\alpha)^{\ell^b}$ are distinct modulo $F^{\times \ell^\nu}$ has size at most B if each α admits a divisor $\mathfrak{d}_\alpha$ with

$$\operatorname{ord}_P(\alpha) = \mathfrak{d}_\alpha(\sigma P) - a\mathfrak{d}_\alpha(P) \qquad (\text{all } P).$$

Then $\mathcal{E}_{\nu,k}$ is finite and

$$\#\mathcal{E}_{\nu,k}\,(\ell^{\nu})^{r_2(L)} \le B\,\ell^{(b+v_\ell(k))g(L)+c+v_\ell(k)}, \qquad g(L)=2[L:\mathbf{Q}]+\lfloor \log_2 h_L\rfloor+2. \tag{28.3}$$

Here h_L is the class number and $r_2(L)$ counts complex places.

Outline. Reduce to weighted norms by the snake lemma, unit determinant and Hilbert 90. Separate classes by bar cycles and Chebotarev, bound Bockstein torsion by 28.3 and 28.5 and ranks by Borel stability, then bound Sha via 28.9 and finite-layer descent. The complete proof is on p. 750. □

Theorem 28.2 (The small-twist H^2 bound)**.** *There is $C_K \ge 1$ such that whenever L is totally complex and contains μ_ℓ for odd ℓ or i for $\ell=2$, for every $k\ge 1$ and choice of local data the group $H^2(G_{L,S_\ell},\mathbf{Z}_\ell(k+1))$ is finite. With $n=[L:\mathbf{Q}]$ and $D_L=\operatorname{disc}(L)$,*

$$\log \#H^2(G_{L,S_\ell},\mathbf{Z}_\ell(k+1)) \le \exp\bigl((2kn)^{C_K}\log(2|D_L|)\bigr)\,, \tag{28.4}$$

and with $A=(\mathbf{Q}_\ell/\mathbf{Z}_\ell)(-k)$ one has the local correction identity

$$V_\ell(L,k)+\operatorname{len} H^0(L,A)=s_\ell(L,k)+\sum_{v\mid\ell}\operatorname{len} H^0(L_v,A), \qquad s_\ell(L,k)\le V_\ell(L,k)\le s_\ell(L,k)+\sum_{v\mid\ell}\operatorname{len} H^0(L_v,A). \tag{28.5}$$

Outline. Use finite coefficients, local Tate duality and 27.2 to compare localization kernels via (28.5). Bound local invariants by cyclotomic indices and (28.1), then absorb their contribution using (28.2). The complete proof is on p. 756. □

Lemma 28.3 (Integral stability over the ring of integers)**.** *Let L be totally complex and $G_r=\mathrm{GL}_r(\mathcal{O}_L)$. For $i,r,r'\in\mathbf{N}$ with $2i+4\le r\le r'$, adjoining an identity block induces an isomorphism $H_i(G_r,\mathbf{Z})\simeq H_i(G_{r'},\mathbf{Z})$. The torsion at rank r' is finite and has order $\#H_i(G_r,\mathbf{Z})_{\mathrm{tors}}$.*

Outline. Use the Chinese remainder theorem to bound the Bass stable rank by two, then compose van der Kallen's integral stability isomorphisms in positive degrees. In degree zero stabilization is the identity, and 28.5 gives torsion finiteness. The complete proof is on p. 756. □

Lemma 28.4 (An exact complex with bounded orbits)**.** *Let L be totally complex, $n=[L:\mathbf{Q}]$, $D=|D_L|$, $N\in\mathbf{N}$ and $G=\mathrm{GL}_N(\mathcal{O}_L)$. Put $d=nN$, $I=D^N d^d$, $R=N^N I^2$ and*

$$U_p = I^{1+dN}\bigl((2R+1)^d+1\bigr)^{dN(p+1)}.$$

There are G-sets X_p of ordered vertex tuples with deletion faces whose augmented complex $\mathbf{Z}[X_\bullet]\to\mathbf{Z}$ is exact. In degree $p\ge 0$ there are at most U_p orbits, and each stabilizer has order at most 3^{nN^2}. If $N\ge 1$,

$$U_p \le (2Dd)^{12(p+1)d^3N}. \tag{28.6}$$

Outline. Use vertex tuples minimizing a common functional and prove exactness on finite convex nerves by hyperplane cuts. Count orbits through common minimal vectors and active equations using Minkowski and Hadamard, and bound stabilizers by reduction modulo 3. The complete proof is on p. 756. □

Lemma 28.5 (Torsion at a fixed rank)**.** *With the notation of 28.4, let $m\ge 0$, $A=U_m$ and $s=3^{nN^2}$. The torsion subgroup of $H_m(G,\mathbf{Z})$ is finite of order $T\le (m+2)^A s^{mAs^m}$. If $k\ge 1$, $1\le N\le 4k+6$ and $m\le 2k+1$, then*

$$\log T \le \exp\bigl((2kn)^{24}\log(2D)\bigr). \tag{28.7}$$

Outline. Shift homology along the exact complex of 28.4 and bound stabilizer terms by Shapiro's lemma, transfer and bar ranks. Use Smith normal form and Hadamard for the final coinvariant torsion, then combine the bounds with (28.6). The complete proof is on p. 758. □

28.2 Generator bounds and quadratic extensions

An abelian datum M specifies $N_M \geq 1$ and a subgroup of $(\mathbf{Z}/N_M\mathbf{Z})^\times$, with fixed field K_M of degree n_M. For primes q, d with

$$q \equiv 3 \pmod 8, \qquad d \equiv 7 \pmod{16}, \qquad \gcd(qd, N_M) = 1, \tag{28.8}$$

put

$$K_{M,q,d} = K_M(i, \sqrt{-q}, \sqrt{-d}), \qquad E_{M,a,q,d} = K_M\,\mathbf{Q}(\zeta_{2^{a+2}})^+\mathbf{Q}(\sqrt{-q})\,\mathbf{Q}(\sqrt{-d}).$$

For a primitive Dirichlet character ψ modulo f, write $M_\psi = (f, \ker\psi)$ for the abelian datum whose subgroup consists of the units u modulo f with $\psi(u) = 1$. Then $E_{M_\psi,a,q,d}$ is the field associated with (ψ, a, q, d) in Definition 26.2.

For a base L_0 containing μ_ℓ for odd ℓ or i for $\ell = 2$, put $L_\infty = L_0(\mu_{\ell^\infty})$. Let $\mathcal{D}_\ell(L_0)$ be the continuous $\mathbf{Q}_\ell/\mathbf{Z}_\ell$-valued characters of G_{L_∞} unramified everywhere and split above ℓ. Its compact dual is $\mathrm{XP} = X_\ell(L_0)$. Omitting splitting gives a larger character group with dual XX, so $\mathrm{XX} \twoheadrightarrow \mathrm{XP}$.

Theorem 28.6 (Ferrero–Washington in finite layers)**.** *Let ℓ be prime and $L/\mathbf{Q}$ finite abelian, with $\mu_\ell \subset L$ for odd ℓ and $i \in L$ for $\ell = 2$. There is $r \in \mathbf{N}$ such that for every $L' = L(\mu_{\ell^n})$, $n \geq 0$,*

$$\#\big(\mathrm{Cl}(L')/\mathrm{Cl}(L')^\ell\big) \leq \ell^r.$$

Ferrero and Washington prove the vanishing of μ_ℓ for cyclotomic $\mathbf{Z}_\ell$-extensions of abelian fields [92]. For these bases, this vanishing is equivalent to a uniform bound on finite-layer ℓ-ranks.

Theorem 28.7 (Uniform numbers of tower generators)**.** *For odd prime ℓ and an abelian datum A with $\mu_\ell \subset K_A$, some $R_\ell(A) \geq 1$ bounds the $\mathbf{Z}_\ell$-generator numbers of both tower modules for every realization of A.*

For each M, some $R_0(M) \geq 1$ bounds both generator numbers at 2 for $K_{M,q,d}$ uniformly in (28.8). If g bounds the unramified mod-2 character dimension in the infinite tower of $K_M(i)$, one can take $R_0(M) = 4(g + 6n_M)$.

Outline. Bound finite-layer characters via 28.6 and class field theory, use 28.1 for descent, and apply compact Nakayama. At 2, use Frobenius indices for the tame contribution and bound each quadratic step by twice the invariant dimension. The complete proof is on p. 758. □

Lemma 28.8 (The cost of a quadratic extension)**.** *If $E \subset E'$ are number fields with $[E' : E] \mid 2$, then for every $k \in \mathbf{Z}$,*

$$s_2(E, k) \leq 1 + s_2(E', k). \tag{28.9}$$

For $a \geq 2$ and a prime pair (28.8), every realization E of $E_{M,a,q,d}$ admits a layer $L_m/K_{M,q,d}$ with $m \leq a+2$ and $s_2(E, k) \leq 1 + s_2(L_m, k)$ for every $k \geq 1$. The base satisfies

$$[K_{M,q,d} : \mathbf{Q}] \leq 8n_M, \qquad K_{M,q,d} \hookrightarrow \mathbf{Q}(\zeta_{4N_M qd}).$$

Outline. Restriction preserves Sha; for a quadratic extension, inflation-restriction yields a 2-torsion subquotient of a locally cyclic group, bounding length loss by 1. Adjoining $\mu_{2^{a+2}}$ to $K_{M,q,d}$ gives L_m with $[L_m : E] \mid 2$ and the degree and embedding. The complete proof is on p. 758. □

28.3 Twisted cokernels and their centres

For a topological generator σ of $\mathrm{Gal}(L_\infty/L_0)$, put $u = \chi_\ell(\sigma)$ and let τ be its action on $X_\ell(L_0)$. Then

$$v_\ell(u - 1) = c, \qquad c \geq 1, \qquad c \geq 2 \text{ if } \ell = 2, \tag{28.10}$$

with $\mu_{\ell^c} \subset L_0$. For an endomorphism τ of a $\mathbf{Z}_\ell$-module X, write

$$w_{\tau,u}(k, m) = \mathrm{length}_{\mathbf{Z}_\ell}\, X/((u^k\tau)^{\ell^m} - 1)X.$$

Theorem 28.9 (Descent to a twisted cokernel)**.** *Assume $\mu_\ell \subset L_0$ for odd ℓ and $i \in L_0$ for $\ell = 2$. For the tower module and generator above, every layer L_m/L_0 and every $k \geq 1$ satisfy*

$$s_\ell(L_m, k) = w_{\tau,u}(k, m). \tag{28.11}$$

Outline. Surjectivity of $u^{-k\ell^m} - 1$ on $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)(-k)$ and cohomological dimension 1 kill H^1, H^2 on Γ_m and open subgroups, so Sha is $(\mathcal{D}_\ell(L_0)(-k))^{\Gamma_m}$. Duality gives $(u^k\tau)^{\ell^m}$-coinvariants. The complete proof is on p. 759. □

Theorem 28.10 (Centres of twisted cokernels)**.** *Let X have at most R generators over $\mathbf{Z}_\ell$ and let τ be a $\mathbf{Z}_\ell$-linear endomorphism. Assume u, c satisfy (28.10). For each $j \geq 0$ there are $s_1, \dots, s_R \in \mathbf{Z}_\ell$ with*

$$w_{\tau,u}(k, j) \leq \sum_{i=1}^{R} (j + c + v_\ell(s_i - k)) \qquad (k \in \mathbf{N}), \tag{28.12}$$

where $v_\ell(0) = \infty$. If τ is unipotent modulo ℓ, then for $j_0 = j_0(\ell, R) = \lfloor \log_\ell R \rfloor + 3$ (with logarithm 0 if $R = 0$) one has the layer growth bound

$$w_{\tau,u}(k, m) \leq w_{\tau,u}(k, j_0) + R(m - j_0)_+ \qquad (k, m \in \mathbf{N}), \tag{28.13}$$

and there are $s_1, \dots, s_R \in \mathbf{Z}_\ell$ with

$$w_{\tau,u}(k, m) \leq \sum_{i=1}^{R} (\max(m, j_0) + c + v_\ell(s_i - k)) \qquad (k, m \in \mathbf{N}), \qquad j_0 = j_0(\ell, R) = \lfloor \log_\ell R \rfloor + 3, \tag{28.14}$$

where the logarithm term is taken to be 0 if $R = 0$. For these centres and $b \in \mathbf{N}$, if $|s_i - k|_\ell \geq \ell^{-b}$ for every i, then

$$w_{\tau,u}(k, m) \leq R(\max(m, j_0) + c + b). \tag{28.15}$$

Outline. Lift τ^{ℓ^j} to $\mathbf{Z}_\ell^R$ and apply Smith normal form to bound $w_{\tau,u}(k, j)$ by eigenvalue valuations. The ultrametric inequality and lifting the exponent nest the ℓ-adic balls, whose intersection gives s_i by compactness. For unipotent τ modulo ℓ, binomial expansion, Nakayama's lemma, and divisibility bound growth from j_0, yielding the uniform and far-centre bounds. The complete proof is on p. 759. □

Theorem 28.11 (The two tower bounds)**.** *Let L_0 be ℓ-prepared, and let $X_\ell(L_0)$ have at most R generators, with $R \geq 1$. Use c from (28.10) and $j_0 = j_0(\ell, R)$. There are $s_1, \dots, s_R \in \mathbf{Z}_\ell$ such that for every $k \geq 1$, $m, b \in \mathbf{N}$ and every layer L_m/L_0, if $|s_i - k|_\ell \geq \ell^{-b}$ for all i, then*

$$s_\ell(L_m, k) \leq R(\max(m, j_0) + c + b).$$

For every $k \geq 1$ and layer L_m/L_0, one also has

$$s_\ell(L_m, k) \leq V_\ell(L_{j_0}, k) + R(m - j_0)_+. \tag{28.16}$$

Proof. On $X_\ell(L_0)/\ell X_\ell(L_0)$ the continuous tower action factors through a cyclic group of order ℓ^a. Thus $(\tau - 1)^{\ell^a} = \tau^{\ell^a} - 1 = 0$ there. Apply 28.9 and 28.10 for the first bound. For the second, apply (28.13) and (28.5) at the layer of index j_0. Layers of equal index are isomorphic over L_0. □

28.4 Exceptional twists

Lemma 28.12 (At most R exceptions below each cutoff)**.** *Fix the centres of 28.11 and the corresponding c. For each $Z \in \mathbf{N}$ some $\mathcal{E}_Z \subseteq \{1, \dots, Z\}$ has size at most R. For $1 \leq k \leq Z$ outside $\mathcal{E}_Z$ and any layer L_m/L_0,*

$$s_\ell(L_m, k) \leq R\big(\max(m, j_0(\ell, R)) + c + \lfloor \log_\ell Z \rfloor\big). \tag{28.17}$$

Proof. For $Z > 0$, put $b = \lfloor \log_\ell Z \rfloor$ and exclude the $1 \leq k \leq Z$ with $|s_i - k|_\ell < \ell^{-b}$ for any i. Each centre excludes at most one integer, since the integers it excludes are congruent modulo $\ell^{b+1} > Z$. Apply 28.11. For $Z = 0$ take the empty set. □

Lemma 28.13 (A common interval for finitely many exceptions). *Let $R \in \mathbf{N}$, $0 < \eta \le C_t$, $X > 1$, and $\mathcal{E} \subset \mathbf{N}$ with $\#\mathcal{E} \le R$. Put $J = 2^{R+2}(C_t/\eta)^R$. There is Y with $X^{1/J} \le Y$ and $Y^2 \le X$ such that for every $Y \le x \le Y^2$,*

$$k \in \mathcal{E}, \quad k \le x^{C_t} \quad \Longrightarrow \quad k \le x^{\eta}. \tag{28.18}$$

Outline. Set $U = \log X$, $B = C_t/\eta$, and $T = \{\log k/C_t : k \in \mathcal{E}, k \ge 1\}$, and induct on R. Take $y = U/2$ if T misses $(U/(2B), U]$; otherwise remove a point from that interval, apply induction with endpoint $U/(2B)$, and set $Y = e^y$. The complete proof is on p. 759. □

28.5 Small exceptions and the pointwise budget

Put $P_{C,\varepsilon}(x) = C(1 + \log x) + 2\exp(Cx^{3\varepsilon/4})$ and $\eta = \varepsilon/(2C_K)$, with C_K from 28.2.

Lemma 28.14 (A pointwise bound and its sum). *Fix a prime ℓ, $R, c_1, n_0 \in \mathbf{N}$, $c_0, C_t, A_0 \ge 0$, and $J, \varepsilon > 0$. There is $C \ge 0$ depending on these and C_K for which the bound below holds. Assume the roots of unity of 28.2 and (28.16) for L_0. Suppose $\mathcal{E}$ satisfies 28.12 at $Z = \lfloor X^{C_t} \rfloor$ with $c \le c_1$, and $[L_{j_0} : \mathbf{Q}] \le n_0$. For $x \ge 1$ in $[X^{1/J}, X]$ satisfying (28.18) and $\log(2|D_{L_{j_0}}|) \le c_0 x^{\varepsilon/4}$, each $1 \le k \le x^{C_t}$ and $m \le A_0(1 + \log x)$ satisfy*

$$s_\ell(L_m, k) < \infty, \qquad s_\ell(L_m, k) + 1 \le P_{C,\varepsilon}(x). \tag{28.19}$$

For every $C_t, C \in \mathbf{R}$ and $\varepsilon > 0$, some $x_0 \ge 2$ satisfies

$$x^{C_t} P_{C,\varepsilon}(x) \le D_x \qquad (x \ge x_0). \tag{28.20}$$

Outline. Use (28.17) outside the exceptional set. Inside, combine (28.18) and (28.4) at the fixed layer, transfer via (28.16), and use the bounds on $\log X$ and m to get the logarithmic estimate. The complete proof is on p. 759. □

Theorem 28.15 (Bounds at 2 and at a fixed odd prime). *Fix $J, \varepsilon > 0$, $C_t \ge 0$ and $A_1 \in \mathbf{R}$. Call $a, t : \mathbf{R} \to \mathbf{N}$ admissible if for every $x \ge 2$,*

$$a(x) \ge 2, \qquad 2^{a(x)} \le x^{A_1}, \qquad 2 \le t(x) \le x^{C_t}. \tag{28.21}$$

Each threshold x_0 below is uniform in admissible a, t and $X \ge 2$. The estimates hold for $x \in [X^{1/J}, X]$, $x \ge x_0$, satisfying (28.18) for the corresponding exceptional set.

At 2, fix M as in 28.7. There is $x_0 \ge 2$ such that for every pair (28.8) with $\log(qd) \le X^{\varepsilon/(4J)}$, a set $\mathcal{E}_2$ of size at most $R_0(M)$ gives, for every realization E of $E_{M,a(x),q,d}$,

$$s_2(E, k) < \infty \quad (1 \le k < t(x)), \qquad \sum_{1 \le k < t(x)} s_2(E, k) \le D_x. \tag{28.22}$$

At an odd prime ℓ, fix A with $\mu_\ell \subset K_A$, put $R_\ell = R_\ell(A)$ and $j_\ell = j_0(\ell, R_\ell)$. There is $x_0 \ge 2$ such that for each a, t, X some $\mathcal{E}_\ell$ of size at most R_ℓ gives, for every realization L_0 of A and layer L_{j_ℓ}/L_0,

$$s_\ell(L_{j_\ell}, k) < \infty \quad (1 \le k < t(x)), \qquad \sum_{1 \le k < t(x)} s_\ell(L_{j_\ell}, k) \le D_x. \tag{28.23}$$

Outline. At 2, use 28.8 and the cyclotomic derivative for field bounds and choose exceptions via 28.7, 28.11, 28.12. Transfer 28.14 by quadratic comparison at 2 and layer isomorphism at odd primes, then sum via (28.20). The complete proof is on p. 760. □

28.6 Selection of the common window

Theorem 28.16 (The common Iwasawa window). *Fix $C_t \ge 1$, an abelian datum M, a finite set B' of odd primes and, for each $\ell \in B'$, a datum A_ℓ with $\mu_\ell \subset K_{A_\ell}$. Fix $0 < \varepsilon < 1/52$ and $A_1 \in \mathbf{R}$. With C_K from 28.2 and the bounds of 28.7, put*

$$R = R_0(M) + \sum_{\ell \in B'} R_\ell(A_\ell), \qquad \eta = \varepsilon/(2C_K), \qquad J = 2^{R+2}(C_t/\eta)^R.$$

There is X_0 such that for every admissible a, t of 28.15, $X \geq X_0$ and pair (28.8) with

$$\log(qd) \leq X^{\varepsilon/(4J)}, \tag{28.24}$$

there is Y with $X^{1/J} \leq Y$ and $Y^2 \leq X$. For every $x \in [Y, Y^2]$, (28.22) holds for every realization E of $E_{M,a(x),q,d}$. For each $\ell \in B'$, (28.23) holds for every realization of A_ℓ and its layers of index $j_0(\ell, R_\ell(A_\ell))$.

Proof. Let x_1 be the maximum of 2 and the thresholds in 28.15, and set $X_0 = x_1^J$. The union $\mathcal{E}_2 \cup \bigcup_{\ell \in B'} \mathcal{E}_\ell$ has size at most R. Since $0 < \eta \leq C_t$, 28.13 gives an interval $[Y, Y^2]$ satisfying (28.18) for the union. Throughout it $x \geq X^{1/J} \geq x_1$, so both estimates of 28.15 apply. □

At $C_t = 401$, Proposition 26.40 combines the window of Theorem 28.16 with the automorphic counts to give the window bound of Definition 26.3.

For a primitive Dirichlet character ψ of conductor f, take $M = M_\psi$ and $B' = \mathcal{B}_\psi$ from Definition 29.1. For each $\ell \in B'$, take $A_\ell = (\ell f, \{1\})$, so $K_{A_\ell} = \mathbf{Q}(\zeta_{\ell f})$. The dyadic estimate supplies the term at 2 in 27.10. The unweighted odd-prime sum supplies the hypothesis of 29.24. The factor $\log \ell$ enters in 29.30.

Chapter 29

Summing the plus component by classes

Write $\chi = \psi$ for the fixed primitive character of conductor f and order o. Let G_F denote the absolute Galois group of a field F. For a CM field E and a finite set S of odd primes, put

$$s^+_{\ell,k}(E) = \operatorname{length}_{\mathbf{Z}_\ell} \operatorname{Sha}^+_{\ell,k}(E), \qquad P^+(E,t;S) = \sum_{\ell\in S} \log \ell \sum_{1\le k<t} s^+_{\ell,k}(E).$$

The plus component has complex conjugation eigenvalue $(-1)^k$. For the full group over a number field L, write $s_{\ell,k}(L) = \operatorname{length}_{\mathbf{Z}_\ell} \operatorname{Sha}_{\ell,k}(L)$. The local data are those of 26.1 and 27.1.

29.1 Character components of a host

Definition 29.1 (Hosts and character classes)**.** Let $a \ge 2$ be an integer. Let q, d be primes with $q \equiv 3 \bmod 8$ and $d \equiv 7 \bmod 16$. Assume $(qd, f) = 1$. Put $N = \operatorname{lcm}(2^{a+2}, f)qd$. Let $H \le (\mathbf{Z}/N\mathbf{Z})^\times$ consist of the units u satisfying

$$u \bmod 2^{a+2} \in \{1, -1\}, \qquad \chi(u \bmod f) = 1, \qquad u \bmod q,\ u \bmod d \text{ are squares.}$$

A field of this host is a realization of $\mathbf{Q}(\mu_N)^H$. The host is large for $t \in \mathbf{N}$ if $q, d > 2t+1$ and $q, d > \max\{f, o, p_K, h_*, p_1\}$. Here p_K is the least prime at least 7 congruent to $3 \bmod 4$ and prime to f, and $h_* = \#\operatorname{Cl}(\mathbf{Q}(\sqrt{-p_K}))$. Write o_{odd} for the odd part of o. Let p_1 be the least prime congruent to $1 \bmod o_{\mathrm{odd}}$ and prime to $2fp_K$. Set

$$\mathcal{B}_\chi = \{3\} \cup \{r : r \text{ is an odd prime dividing } ofp_K h_* p_1\}.$$

For an odd prime ℓ and a DVR $O \subset \overline{\mathbf{Q}}_\ell$ finite over $\mathbf{Z}_\ell$, define

$$X_{\ell,O}(k) = \{\theta : (\mathbf{Z}/N\mathbf{Z})^\times \to O^\times : H \subseteq \ker\theta,\ \theta(-1) = (-1)^k\}.$$

Write $\theta = \theta_0\kappa$, where θ_0 has order prime to ℓ and κ has ℓ-power order. For a character η modulo N, let η_r be its Chinese remainder factor on $(\mathbf{Z}/r^{v_r(N)}\mathbf{Z})^\times$, inflated to $(\mathbf{Z}/N\mathbf{Z})^\times$ and trivial for $r \nmid N$. Put

$$\theta_P = \theta_0(\theta_0)_\ell^{-1}, \qquad \theta_1 = \theta_0(\theta_0)_2^{-1}, \qquad c_2(\theta) = v_2(\operatorname{cond}\theta_2).$$

Let f_0 be the multiplicative order of ℓ modulo $\operatorname{lcm}(4, \operatorname{ord}\theta_1)$, and put $n_0 = v_2(\ell^{f_0} - 1)$ and $R_1(\ell,\theta) = 16o(\ell f)^2\, 4^{n_0}$. For a host large for t, the triples (ℓ, k, θ) with $1 \le k < t$ and $\theta \in X_{\ell,O}(k)$ have the following classes.

I	$\ell \notin \mathcal{B}_\chi, \quad \ell - 1 \nmid 2k$
II	$k \ge 2, \quad \operatorname{ord}\theta_P$ even, $\quad (\ell \in \mathcal{B}_\chi$ or $\ell - 1 \mid 2k)$
III	$\ell \notin \mathcal{B}_\chi, \quad \ell - 1 \mid 2k, \quad \theta \ne 1, \quad \operatorname{ord}\theta$ odd
IV	$\ell \in \mathcal{B}_\chi, \quad k \ge 2, \quad \operatorname{ord}\theta_P$ odd
V	$\ell \in \mathcal{B}_\chi, \quad k = 1, \quad R_1(\ell,\theta) \le 2^{c_2(\theta)}$
Vs	$\ell \in \mathcal{B}_\chi, \quad k = 1, \quad 2^{c_2(\theta)} < R_1(\ell,\theta)$
Z	$\ell \notin \mathcal{B}_\chi, \quad \ell - 1 \mid 2k, \quad \theta = 1.$

Lemma 29.2 (Orders and conductors of host characters)**.** *The host quotient has order dividing $2^{a+2}o$. At odd r, the order of θ_r divides o if $r \mid f$ and divides 2 otherwise. In particular,*

$$\operatorname{ord}\theta \mid 2o\,2^{c_2(\theta)}, \qquad \operatorname{ord}\theta_1 \mid 2o, \qquad f_0 \le F_0 := \varphi(4o), \qquad 2^{n_0} < \ell^{F_0}. \tag{29.1}$$

If $\operatorname{ord}\theta$ is odd, then $\theta_2 = 1$. In this case $\operatorname{ord}\theta \mid o_{\mathrm{odd}}$ and $\operatorname{cond}\theta \mid f$. For a host large for t, if $\ell \in \mathcal{B}_\chi$ then $\ell < q, d$ and $\operatorname{cond}\theta \mid f$ whenever $\operatorname{ord}\theta_P$ is odd. If $\ell \notin \mathcal{B}_\chi$, then $\theta_0 = \theta$ and $\theta_\ell^2 = 1$. If also $\ell - 1 \mid 2k$ and $1 \le k < t$, then $5 \le \ell \le 2k+1 < q, d$, $k \ge 2$, $\ell \nmid N$ and $\theta_P = \theta$.

Outline. Embed the host quotient in the dyadic, character-image and square-class factors to bound its order and conductor. Largeness makes the factors at $2, q, d$ survive in θ_P and vanish at odd order, and $\ell - 1 \mid 2k$ excludes $\ell \mid N$ outside the fixed set. The complete proof is on p. 760. □

Definition 29.3 (Character lengths)**.** Let ε_ℓ be the cyclotomic character of $G_{\mathbf{Q}}$, and write D_p, I_p for a decomposition group and its inertia subgroup at p. Inflate θ via the action on μ_N and put

$$A_O = O \otimes_{\mathbf{Z}_\ell} (\mathbf{Q}_\ell/\mathbf{Z}_\ell), \qquad M_{\ell,k,\theta} = A_O(\theta^{-1}\varepsilon_\ell^{-k}).$$

Let $\mathcal{S}_{\ell,k,\theta} \subseteq H^1(G_{\mathbf{Q}}, M_{\ell,k,\theta})$ consist of the classes vanishing on finite inertia and on decomposition groups at $p \mid \ell N$. Set $L_{\ell,k,\theta} = \operatorname{length}_O \mathcal{S}_{\ell,k,\theta}$ and $e_O = \operatorname{length}_O(O/\ell O)$. At finite length put $a_{\ell,k,\theta} = L_{\ell,k,\theta}/e_O$. A character bound B means finite length and $a_{\ell,k,\theta}\log\ell \le B$.

Corollary 29.4 (Independence of the coefficient ring)**.** *If two coefficient DVRs give the same $\overline{\mathbf{Q}}_\ell$-valued character, the lengths in 29.3 are finite simultaneously and their normalized values agree when finite.*

Proof. Compare both rings with the ring of integers of a common finite extension in $\overline{\mathbf{Q}}_\ell$. Finite free extension commutes with continuous cohomology and the restriction kernels because it commutes with continuous cochains and is exact. It multiplies both lengths by the ramification index. Faithful flatness preserves the distinction between finite and infinite length. □

29.2 The automorphic counting bounds

Assume the inputs $\mathscr{H}$ defined before Lemma 26.38, and the bound of 28.2. A bar denotes reduction modulo the coefficient maximal ideal, and ω denotes the Teichmüller lift of $\bar{\varepsilon}_\ell$. Write $C_\ell = \overline{\mathbf{Q}}_\ell$ and normalize v_ℓ by $v_\ell(\ell) = 1$.

Definition 29.5 (Cuspidal attachment with witness)**.** Let K be a CM field, let $r : G_K \to \mathrm{GL}_n(C_\ell)$ be continuous and irreducible, and fix $\iota : C_\ell \simeq \mathbf{C}$. A *cuspidal attachment* of r is a regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_K)$ with the following property. At all but finitely many finite places v, both r and π are unramified. At those places ι carries the eigenvalue multiset of r at inverse arithmetic Frobenius to $q_v^{(n-1)/2}$ times the Satake multiset of π_v, where q_v is the residue cardinality.

Call a place $v \mid \ell$ *unlinked* for r if, at every prolongation of v, the decomposition group preserves a full flag whose quotient characters satisfy $\kappa_i \ne \kappa_j\varepsilon_\ell$ for all i, j. Call v *potentially unlinked* if at every prolongation one can choose a full stable flag for which $\kappa_i^m \ne (\kappa_j\varepsilon_\ell)^m$ for all i, j and all integers $m \ge 1$.

We say that r has a *cuspidal attachment with witness* if it has a cuspidal attachment and the following conditions hold. Every unlinked place above ℓ is potentially unlinked. For each cuspidal attachment π of r there is a real number t_π such that, at every finite $v \nmid \ell$ and every potentially unlinked $v \mid \ell$, the cuspidal support of π_v consists of characters of $K_v^\times$. Each such character ξ satisfies

$$q_v^{t_\pi - 1/2} < |\xi(\varpi_v)| < q_v^{t_\pi + 1/2},$$

where ϖ_v is a uniformiser.

Lemma 29.6 (Restriction in prime rank)**.** *Let G be a profinite group, H a closed normal subgroup, and r a continuous irreducible representation of G of dimension $n = 2$ or 3 over $\overline{\mathbf{Q}}_\ell$, defined over a finite extension of $\mathbf{Q}_\ell$. Suppose its residual semisimplification is a sum of characters of G. Let $D \le G$ preserve a complete flag, with quotient characters $\lambda_1, \dots, \lambda_n$. Suppose some $u \in D \cap H$ has pairwise distinct $\lambda_i(u)$, and some $s \in D$ has $\bar{\lambda}_a(s) \ne \bar{\lambda}_b(s)$. Then $r|_H$ is irreducible. If every positive power of u has distinct eigenvalues, the same holds on $H \cap U$ for every open subgroup $U \le G$.*

Outline. Translates of a simple submodule give two reducible cases, scalar action or permuted lines. Distinct eigenvalues exclude scalars, and the flag and residual separation exclude permuted lines because their reduced characters coincide on the permutation kernel. The complete proof is on p. 760. □

Lemma 29.7 (Irreducibility on the subgroups used at a marked point)**.** *Under the character and field hypotheses of Definition 30.2, let r be a characteristic-zero ordinary marked point. In Type I its restriction to every open subgroup of G_K is irreducible. In Type II let W extend the irreducible two-dimensional corner as in 30.61(3). Then $W|_{G_K \cap \ker \varepsilon_\ell}$ is irreducible.*

Outline. In Type I, distinct inertia eigenvalues exclude scalars on the normal core, and residual separation from (A.278) gives a trace contradiction for permuted lines. In Type II, a projective centralizer and the inertia trace formula exclude reducibility. The complete proof is on p. 760. □

Lemma 29.8 (The finite adjoint classes at a characteristic-zero point)**.** *At a Type I point as in 29.7, assume that $r|_{G_K}\varepsilon_\ell^{-1}$ has a regular algebraic conjugate self-dual cuspidal attachment with witness as in 30.61(2). Then the marked polarised adjoint has zero Bloch–Kato Selmer group over $\mathbf{Q}$. At a Type II point the same holds for the polarised adjoint of its corner when the extension W has determinant η_K, the newform attachment of 30.61(3), and the quadratic witness condition of Definition 30.1.*

Outline. Ordinary eigenvalues and trace identities prove irreducibility on the required subgroups. Use 29.7, the attachments of 30.61 and base change in 26.39 to identify the polarised adjoints and apply Thorne's vanishing as in 30.62. The complete proof is on p. 761. □

Lemma 29.9 (Multiplicity in the projected ordinary space)**.** *Assume $\mathscr{H}$. Use the rank-three definite unitary comparison of 30.59, at its base level. Write L/F for the CM extension, O for the coefficient DVR, Λ for the weight ring, B_0 for the universal ordinary determinant ring and M for the ordinary module of 30.23. Put $J = \operatorname{Ann}_{B_0} M$. There is a Λ-linear idempotent e on M commuting with B_0 with the following property.*

For a characteristic-zero arithmetic character x of B_0/J, put

$$M_x = \overline{\mathbf{Q}}_\ell \otimes_{\Lambda, x|_\Lambda} M, \qquad \mu_e(x) = \dim_{\overline{\mathbf{Q}}_\ell}(eM_x)[x]_{\mathrm{gen}}.$$

Here the bracket denotes the simultaneous generalized eigenspace for the action of B_0 with character x. Call x irreducible if its determinant is the determinant of an absolutely irreducible representation. If x, y are irreducible arithmetic characters of the same strictly increasing labelled algebraic weight, and the finite parts of the ordered characters of y are pairwise distinct on the units at each place above ℓ, then

$$1 \le \mu_e(x) = \mu_e(y).$$

Outline. Specialization identifies M with classical ordinary spaces by Nakayama's lemma, and nondegenerate characters of $U(k_0)$ define an idempotent e of rank $(q_0 - 1)^2$ on generic τ^{K^1} via Whittaker models. Lemma 26.22, Lemma 26.23, and strong multiplicity one fix the base change, while the two-block trace identity of Lemma 26.5 and Satake separation fix a_λ; Jacquet modules and tame types give equal positive multiplicities. The complete proof is on p. 761. □

Lemma 29.10 (Fixed-weight derivations on the Hecke quotient)**.** *In Lemma 29.9, let x be an irreducible arithmetic point of B_0/J of strictly increasing labelled algebraic weight. Every O-linear derivation at x on B_0/J which kills the values of the ordered characters on ordinary inertia is zero.*

Outline. Uses separated characters via 30.23 and 30.24 to detect B_0/J and pass to its image on eM by 29.9. Comparing multiplicities of a separating element t via 29.9 gives $x(f'(t)) \neq 0$, which kills fixed-weight derivations. The complete proof is on p. 765. □

Lemma 29.11 (Local vanishing of the Hecke ideal on the matrix space)**.** *Assume $\mathscr{H}$, and use the rank-three comparison over L in 30.59 with coefficient DVR O. Write A for the graded matrix-coordinate algebra, $B = A_0$ for its weight-zero coefficient ring, and $\phi : B_0 \to B$ for the determinant map after twisting the matrices by ε_ℓ^{-1}. The grading here is diagonal change of basis with weights $1, 0, -1$. Let $\xi : A \to \overline{\mathbf{Q}}_\ell$ be an integral O-algebra homomorphism, taking the maximal ideal of B to elements of positive valuation. Suppose*

its kernel is off the two triangular loci and the locus on which the middle coordinate splits from the outer two. For each $a \in J$ there exists $s \in A$ with

$$\xi(s) \neq 0, \qquad s\phi(a) = 0.$$

Outline. Shows by 30.59 that $A_{\mathfrak{q}}$ has no embedded primes and $\phi(J) \subset P$ at each minimal prime P. Finds a point on P via 29.10 and proves regularity there using 30.25, 29.6, and 30.65, making A_P a field. The complete proof is on p. 766. □

Lemma 29.12 (Two fixed-weight directions at a crystalline link)**.** *In the rank-three frame of 30.59, let $\xi : R \to O$ be an integral evaluation of the complete framed ordinary matrix ring, after finite coefficient extension. Write $\lambda_{v,i}$, $0 \leq i \leq 2$, for the ordered local characters. If the representation is potentially crystalline at $v \mid \ell$ and $\lambda_{v,i} = \varepsilon_\ell \lambda_{v,i+1}$ for some $i = 0, 1$, then there are two O-linearly independent O-valued derivations at ξ which kill every ordinary character value on inertia.*

Outline. Uses crystallinity of the adjacent subquotient to build an extra local cocycle via 26.27. Global duality in 26.27 and 30.55 then bounds normalized cocycles mod ℓ^m by $C|Z_m| \geq q^2$, forcing two independent O-valued derivations. The complete proof is on p. 768. □

Lemma 29.13 (No cyclotomic link at the Type I endpoint)**.** *Assume $\mathscr{H}$ and the character and field conditions of 30.5. Let r be an integral ordinary marked point as in 30.61, irreducible on every open subgroup of G_K. Suppose that at each $w \mid \ell$ its rank-three restriction to G_K has a flag of level $(N; k, 0, -k)$, with the local alternative that the flag is unlinked or the restriction is potentially crystalline. Then at each $w \mid \ell$ the level flag is unlinked. In particular this holds for $k = 1$.*

Outline. Places a linked point in the frame of 30.61 and 30.59, where 29.12 gives two independent fixed-inertia derivations. Both kill B by 29.11 and 29.10, so a combination kills a weight-1 monomial and the whole ring, a contradiction. The complete proof is on p. 769. □

Corollary 29.14 (Representation inputs for the first count)**.** *For every datum satisfying the character and field conditions of 30.5, the required Type I admissibility and adjoint rigidity follow from $\mathscr{H}$ and 28.2. The same inputs give the required Type II block rigidity; in both types the vanishing of the adjoint Bloch–Kato Selmer group at the point is Lemma 29.8.*

Outline. In Type I, gets admissibility via 29.6, 30.3, 29.13, and 30.61, and rigidity via 30.18 and 30.62. In Type II, applies 30.61, 29.6, and 30.62, killing other adjoint summands by 30.12 and 30.4. The complete proof is on p. 770. □

Theorem 29.15 (Class I, summed over primes)**.** *For every χ there are $C_{\mathrm{I}} \in \mathbf{N}$ and distinct primes r_1, r_2. Let a host be large for t with $q, d > r_1, r_2$. Fix $1 \leq k < t$ and a finite set S of odd primes with $\ell \notin \mathcal{B}_\chi$ and $\ell - 1 \nmid 2k$ for each $\ell \in S$. For every choice of coefficient rings $(O_\ell)_{\ell \in S}$ the length is finite for every $\ell \in S$ and $\theta \in X_{\ell, O_\ell}(k)$, and*

$$\sum_{\ell \in S} \log \ell \sum_{\theta \in X_{\ell, O_\ell}(k)} a_{\ell,k,\theta} \leq (Nt)^{C_{\mathrm{I}}}.$$

Outline. Chooses primes $r_1, r_2 > 30 f p_K + 1$ and verifies 30.2 and 30.5 via 29.2, 26.32, 27.2, 28.2, 29.14, and 30.7. Sums over character kernels using the two-prime estimate in 30.10 to bound the total. The complete proof is on p. 771. □

Remark 29.16 (The trivial character and $K_{4j}(\mathbf{Z})$). Fix χ, and put $B = \{2\} \cup \mathcal{B}_\chi$. For a finite abelian group D, write $D[\ell^\infty]$ for its ℓ-primary subgroup. There are constants $A_\chi, C_\chi > 0$ such that, for every integer $j \geq 1$,

$$\log \left(\prod_{\ell \notin B} |K_{4j}(\mathbf{Z})[\ell^\infty]| \right) \leq A_\chi (1 + j)^{C_\chi}.$$

The primes with $\ell - 1 \nmid 4j$ occur in 29.15. The primes outside B with $\ell - 1 \mid 4j$ have zero contribution by 29.28.

Outline. Choose host primes by 7.5 and take the trivial character of 29.1. Duality in 27.2 and Weibel's comparison identify the dual strict groups of 29.3 with the primary parts of $K_{4j}(\mathbb{Z})$, then 29.28 and 29.15 bound the sum. The complete proof is on p. 772. □

Definition 29.17 (The marked symplectic datum)**.** Use the host, $N, f, o, p_1, n_0, c_2(\theta)$ and character classes of Definition 29.1. Fix a host large for t, an odd prime ℓ, a coefficient DVR O as in that definition, $1 \le k < t$, and $\theta \in X_{\ell,O}(k)$ of class II or III. Regard Dirichlet characters as characters of $G_{\mathbf{Q}}$ by their action on roots of unity. Write ε_ℓ for the cyclotomic character and ω for the Teichmüller lift of its reduction. Put

$$\begin{aligned} n_{\mathrm{II}} &= \left\lceil \log_2 \max\{\ell N 4^{n_0},\, 2^{c_2(\theta)+6} o \ell^3 f^2\} \right\rceil, \\ n_{\mathrm{III}} &= \left\lceil \log_2 \max\{\ell f p_1 4^{n_0},\, 8\ell^2\} \right\rceil. \end{aligned} \tag{29.2}$$

In II take $(n', p', m') = (n_{\mathrm{II}}, 1, 1)$. In III take $(n', p', m') = (n_{\mathrm{III}}, p_1, \operatorname{ord}\theta)$. Choose O-valued characters ξ_2, ξ_0, where ξ_2 is primitive odd modulo $2^{n'}$ and ξ_0 has order m' modulo p'. Set

$$\xi = \xi_0\xi_2, \quad \Psi = \varepsilon_\ell^k\theta, \quad \alpha = \varepsilon_\ell\omega^{-1}\xi, \quad \nu = \alpha\Psi, \quad (\delta_0,\delta_1,\delta_2,\delta_3) = (\nu,\Psi,\alpha,1).$$

Write $r_0 = \operatorname{diag}(\delta_0,\delta_1,\delta_2,\delta_3)$ and $\bar r$ for its reduction. Let S_D be the rational primes dividing $2\ell N p'$. Choose a prolongation at each rational prime v, with decomposition group D_v and inertia I_v, and choose a complex conjugation. Set

$$T = \{\ell\} \cup \{v \in S_D \setminus \{\ell\} : \ell \mid \#\theta(I_v)\}.$$

A good datum includes these choices and satisfies the following conditions. We have $k \ge 2$, Ψ is even, and ξ is odd of order prime to ℓ and unramified at T. The characters θ, ξ are unramified outside S_D. For $v \in T$ there are arithmetic Frobenius lifts σ_v, with $\varepsilon_\ell(\sigma_\ell) = 1$, such that

$$\overline{\xi\Psi^i}(\sigma_v) \ne 1 \qquad (v \in T,\ i \in \mathbf{Z}). \tag{29.3}$$

There are markings

$$\phi = \lim_{a\to\infty} \sigma_\ell^{\ell^{a!}}, \qquad g_0 \in G_{\mathbf{Q}}, \qquad s_0 \in I_\ell \cap \ker\theta \cap \ker\omega$$

such that $\bar\alpha(g_0), \bar\Psi(g_0) \ne 1$ and $\varepsilon_\ell(s_0)$ has infinite order.

Finally, let γ be any of $\alpha, \Psi/\alpha, \nu$ or their inverses. A continuous cocycle with values in the residue field of O acted on by $\bar\gamma$ is a global coboundary if it is a coboundary on each D_v for $v \in T$ and vanishes on $I \cap \ker\bar r$ for every finite inertia group I away from T.

Definition 29.18 (Allowed lifts and their points)**.** Fix a good datum as in Definition 29.17. Use the alternating form with antidiagonal $(1,1,-1,-1)$ in the indicated coordinate order. A matrix is block diagonal here if it preserves both coordinate planes with indices $\{0,2\}$ and $\{1,3\}$. Let R be a complete Noetherian local O-algebra whose structure map is local and induces an isomorphism on residue fields. An allowed lift over R is a continuous lift $r : G_{\mathbf{Q}} \to \mathrm{GL}_4(R)$ of $\bar r$ with similitude ν and the following properties.

Away from T, r is trivial on $I \cap \ker\bar r$ for every finite inertia group I. The matrix $r(\phi)$ is block diagonal. The entries $02, 20$ of $r(g_0) + \nu(g_0)r(g_0)^{-1}$ vanish.

At each $v \in T$ require a complete Noetherian local O-domain B_v with the same residue field, a lift b_v of $\bar r|_{D_v}$ of similitude ν, and O-algebra maps $B_v \to O$, $B_v \to R$ taking b_v to $r_0|_{D_v}, r|_{D_v}$, respectively. For $v \ne \ell$ require $b_v = 1$ on $I_v \cap \ker\theta$. At ℓ put $s = s_0$, $d_i = \delta_i(s)$ and $P_i = \prod_{j\ne i}(b_\ell(s) - d_j I)$. The local equations are

$$\det(XI - b_\ell(s)) = \prod_i (X - d_i), \qquad P_j b_\ell(g) P_i = 0 \quad (i<j,\ g \in D_\ell), \qquad P_i(b_\ell(h) - \delta_i(h)I)P_i = 0 \quad (h \in I_\ell).$$

For any commutative O-algebra A, an allowed A-point is the entrywise image of an allowed lift under an O-algebra map $R \to A$. A block point has block diagonal image. A characteristic-zero point in the following statements has $A = \overline{\mathbf{Q}}_\ell$, denoted by C_ℓ. An allowed dual-number lift of such a point is an allowed point over $C_\ell[\epsilon]/(\epsilon^2)$ reducing to it.

Give entry ij weight $v_i - v_j$, where $(v_0, v_1, v_2, v_3) = ((1,1),(0,1),(1,0),(0,0))$. The weight-zero algebra of r is the O-subalgebra generated by products of entries $r(g)_{ij}$ of total weight $(0,0)$, with the group elements allowed to vary among the factors.

Lemma 29.19 (Cocycles of negative weight)**.** *Let $\eta : G_{\mathbf{Q}} \to C_\ell^\times$ be a continuous character of finite order. For $e \geq 1$, put $\lambda = \eta^{-1}\varepsilon_\ell^{-e}$. Every continuous $C_\ell(\lambda)$-valued cocycle that is a coboundary on all finite inertia groups and on D_ℓ is a global coboundary.*

Outline. Baire's theorem reduces the cocycle to finite coefficients, and 28.2 and 29.23 bound the restriction image and kernel for torsion coefficients. Scale and reduce the cocycle, then evaluate at $\lambda(g) \neq 1$ to obtain a limiting cobounding scalar. The complete proof is on p. 772. □

Lemma 29.20 (Existence of the marked datum)**.** *For every host and character of class II or III as in Definition 29.17, a finite extension of the coefficient DVR admits a good datum for this host and character, with the levels of (29.2).*

Outline. Use 29.2 to choose the odd-modulus factor and pair it with a primitive odd dyadic character to build the diagonal lift. Verify Frobenius separation and the markings, and use 30.75 to obtain residual Selmer vanishing. The complete proof is on p. 773. □

Lemma 29.21 (Construction and formal smoothness of the local domains B_ℓ and B_v)**.** *Fix a good datum $\mathcal{D}$ as in Definition 29.17, with $E = \operatorname{Frac} O$, and give entry ij of M_4 bidegree $(l_i - l_j, m_i - m_j)$, where $l = (1,0,1,0)$ and $m = (1,1,0,0)$.*

1. *For each $v \in T \setminus \{\ell\}$, the quotient $D_v/(I_v \cap \ker\theta)$ is abelian, topologically generated by σ_v and an inertia generator τ_v of finite order $m_v = \#\theta(I_v)$. Let U_v be the universal lifting ring for continuous lifts $b_v : D_v/(I_v \cap \ker\theta) \to \mathrm{GSp}_4(R)$ of $\bar r|_{D_v}$ with similitude $\nu|_{D_v}$, with base point $x_{0,v} : U_v \to O$ specialising to $r_0|_{D_v}$. Put $\zeta = \theta(\tau_v) \in O^\times$, let $J_v \subset U_v$ be the closed ideal generated by the coefficients of $\det(XI_4 - b_v(\tau_v)) - (X-\zeta)^2(X-1)^2$, and let $\mathfrak{p}_v$ be the intersection of the minimal primes of $\mathrm{sat}_\ell(\sqrt{J_v})$ contained in $\ker x_{0,v}$, where $\mathrm{sat}_\ell(\mathfrak{a}) = \{a : \exists n \geq 0,\ \ell^n a \in \mathfrak{a}\}$. Then $\mathfrak{p}_v$ is a single prime ideal invariant under Sp_4-conjugation congruent to I_4, so $B_v = U_v/\mathfrak{p}_v$ is a complete Noetherian local O-domain with a bigraded homogeneous presentation. Over E, the completion $\widehat{B_v[1/\ell]}_{\ker x_{0,v}}$ is formally smooth of dimension 10, with coordinate counts $(1,1,2)$ in first-degree-zero bidegrees $((0,-1),(0,1),(0,0))$ and 1 in each of the six cross-block bidegrees $(1,0),(-1,0),(1,1),(-1,-1),(-1,1),(1,-1)$; each cross-block tangent direction is a coboundary in $B^1(D_v, \mathfrak{sp}_4)$.*
2. *At ℓ, let U_ℓ be the universal lifting ring for continuous lifts $b_\ell : D_\ell \to \mathrm{GSp}_4(R)$ of $\bar r|_{D_\ell}$ with similitude $\nu|_{D_\ell}$, with base point $x_{0,\ell} : U_\ell \to O$. Let $J_\ell \subset U_\ell$ be the closed ideal generated by the local ordinary equations of Definition 29.18 (for $s = s_0$), and let $\mathfrak{p}_\ell$ be the intersection of the minimal primes of $\mathrm{sat}_\ell(\sqrt{J_\ell})$ contained in $\ker x_{0,\ell}$. Then $\mathfrak{p}_\ell$ is a single prime ideal invariant under Sp_4-conjugation congruent to I_4, so $B_\ell = U_\ell/\mathfrak{p}_\ell$ is a complete Noetherian local O-domain with a bigraded homogeneous presentation. Over E, the completion $\widehat{B_\ell[1/\ell]}_{\ker x_{0,\ell}}$ is formally smooth of dimension 14, with coordinate counts $(1,2,2)$ in first-degree-zero bidegrees $((0,-1),(0,1),(0,0))$, count 2 in each of the three upper cross-block bidegrees $(1,0),(-1,1),(1,1)$, and count 1 in each of the three lower cross-block bidegrees $(-1,0),(1,-1),(-1,-1)$.*
3. *In the framed presentation of 30.72 over $B = B_\ell \mathbin{\widehat{\otimes}_O} \widehat{\bigotimes}_{v\in T\setminus\{\ell\}} B_v$ (with frame at ℓ fixed to I_4), imposing the six cross-block equations $(0,1),(0,3),(2,1),(1,0),(3,0),(1,2)$ on $r(\phi) = b_\ell(\phi)$ and on each frame matrix h_v ($v \in T \setminus \{\ell\}$) eliminates $6 + 6(|T|-1)$ cross-block coordinates over E. The resulting formal chart over E has $(1,2,2)$ local coordinates from ℓ and $(1,1,2)$ from each $v \in T \setminus \{\ell\}$ in first-degree-zero bidegrees $((0,-1),(0,1),(0,0))$, together with three coordinates of nonzero first degree, namely $u : (1,0)$, $v : (-1,1)$, and $w : (1,1)$. Its first-degree-zero formal subring over E adds the two algebraically independent coordinates uv (weight 1) and wv (weight 2).*

Outline. Uses Lagrangian eigenspaces at $v \neq \ell$ and flag lifting at ℓ to show B_v, B_ℓ are formally smooth domains of dimensions $10, 14$. Eliminates cross-block coordinates in 30.72 via 27.2 and the formal implicit function theorem. The complete proof is on p. 774. □

Theorem 29.22 (Classes II and III)**.** *For each χ there are $C_{\mathrm{II}}, C_{\mathrm{III}} \in \mathbf{N}$. For every host large for t, every odd prime ℓ, every coefficient ring O and every $1 \leq k < t$, a character $\theta \in X_{\ell,O}(k)$ of class II or III has finite length and satisfies the respective bound*

$$a_{\ell,k,\theta} \log \ell \leq (\ell N k)^{C_{\mathrm{II}}}, \qquad a_{\ell,k,\theta} \log \ell \leq (\ell N k)^{C_{\mathrm{III}}}.$$

Outline. Uses 29.20, 29.21, and 30.72 to build a bigraded presentation, killing nonpositive tangents via 29.19. Proves two-sided points are irreducible (30.51) and rigid (30.19), then bounds lengths via 30.72, 30.52, and 29.4. The complete proof is on p. 775. □

29.3 Class IV and its cyclotomic layer

Lemma 29.23 (Restriction to an abelian field)**.** *Let $k \geq 1$ and $L_1/\mathbf{Q}$ be abelian of degree n with $\theta|_{G_{L_1}} = 1$. Then*

$$L_{\ell,k,\theta} \leq e_O\big(s_{\ell,k}(L_1) + n\nu_\ell(N,k)\big), \tag{6.IV}$$

where

$$\nu_\ell(N,k) = \begin{cases} \max\{v_\ell(N),1\} + v_\ell(k), & \ell \mid N \text{ or } \ell - 1 \mid k, \\ 0, & \text{otherwise.} \end{cases}$$

The inertia invariants $M_{\ell,k,\theta}^{I_\ell}$ have length at most $e_O\nu_\ell(N,k)$, and $\nu_\ell(N,k) \leq 2Nk$.

Outline. Inflation–restriction and scalar coboundaries bound the kernel length by n times the global invariant length. A cyclotomic inertia lift bounds inertia invariants, with a unit difference if $\ell \nmid N$ and $\ell - 1 \nmid k$, and restriction lands in the strict group. The complete proof is on p. 781. □

Theorem 29.24 (Transfer of the class IV sum)**.** *Fix χ, $\ell \in \mathcal{B}_\chi$ and $j \in \mathbf{N}$. Let $t \geq 2$. Assume that for every realization L_0 of $\mathbf{Q}(\mu_{\ell f})$, cyclotomic layer L_1/L_0 of degree ℓ^j and choice of local data, the lengths for $1 \leq k < t$ are finite and $\sum_{1\leq k<t} s_{\ell,k}(L_1) \leq B$. For every host large for t and every O-valued host character θ with $\operatorname{ord}\theta_P$ odd, the lengths $L_{\ell,k,\theta}$ are finite for $1 \leq k < t$ and*

$$\sum_{1\leq k<t} a_{\ell,k,\theta} \leq B + (Nt)^{C_4}, \qquad C_4 = \ell f \ell^j + 3.$$

Proof. By 29.2, $\operatorname{cond}\theta \mid f$. Take the field of 29.25, of degree $n \leq \ell f \ell^j$. After division by e_O the sum of the errors in (6.IV) is at most $2nNt^2 \leq (Nt)^{C_4}$. □

Lemma 29.25 (A field that trivializes the transferred characters)**.** *For every odd prime ℓ and $j \in \mathbf{N}$, each algebraic closure of $\mathbf{Q}$ contains abelian fields $L_0 \subseteq L_1$ with $L_0 \simeq \mathbf{Q}(\mu_{\ell f})$. The extension L_1/L_0 is a cyclotomic layer of degree ℓ^j and satisfies $[L_1 : \mathbf{Q}] \leq \ell f \ell^j$. For every integral domain A and $N \geq 1$, the Galois character associated to an A-valued Dirichlet character of level N and conductor dividing f is trivial on G_{L_1}.*

Proof. Take $L_0 = \mathbf{Q}(\mu_{\ell f})$ and $L_1 = \mathbf{Q}(\mu_{\ell^{j+1} f})$. The totient ratio is ℓ^j, and $[L_1 : \mathbf{Q}] = \ell^j\varphi(\ell f) \leq \ell^j \ell f$. The character factors through the action on $\mu_f \subseteq L_0$. □

29.4 The two classes at twist one

Theorem 29.26 (Class V)**.** *For each χ there is $C_{\mathrm{V}} \in \mathbf{N}$. For every host large for $t \geq 2$, every $\ell \in \mathcal{B}_\chi$ and $\theta \in X_{\ell,O}(1)$ with $R_1(\ell,\theta) \leq 2^{c_2(\theta)}$,*

$$L_{\ell,1,\theta} < \infty, \qquad a_{\ell,1,\theta} \log \ell \leq (\ell N)^{C_{\mathrm{V}}}.$$

Outline. Builds the Gaussian frame ring (30.54, 30.61), counts relations via 27.2, and forces Type I points via 30.75. Kills nonpositive tangents by 29.19, proves rigidity via 30.59 and 30.62, and bounds lengths by 30.72 and 30.7. The complete proof is on p. 781. □

Theorem 29.27 (Class Vs)**.** *For each χ there is $C_{\mathrm{Vs}} \in \mathbf{N}$. For every host large for $t \geq 2$, every $\ell \in \mathcal{B}_\chi$ and $\theta \in X_{\ell,O}(1)$ with $2^{c_2(\theta)} < R_1(\ell,\theta)$,*

$$L_{\ell,1,\theta} < \infty, \qquad a_{\ell,1,\theta} \log \ell \leq (\ell N)^{C_{\mathrm{Vs}}}.$$

Outline. Class Vs and the host order bounds give bounded degree after adjoining μ_ℓ. Bound the discriminant by cyclotomic inclusion and transitivity, then apply 28.2 and absorb the transfer error by restriction to obtain a power bound in ℓN. The complete proof is on p. 788. □

29.5 The trivial character

Theorem 29.28 (Vanishing in class Z). *Let $\ell \geq 5$ be prime and $k \geq 2$ even with $\ell - 1 \mid 2k$. For every O, the subgroup of $H^1(G_{\mathbf{Q}}, A_O(-k))$ vanishing on finite inertia and on a decomposition group at ℓ has length zero. In particular, $L_{\ell,k,1} = a_{\ell,k,1} = 0$ in class Z.*

Outline. For ℓ-torsion use Kronecker–Weber and decomposition at ℓ if $\ell - 1 \mid k$. Otherwise normalize on inertia and kill the restriction to $\mathbb{Q}(\sqrt{\ell})$ by Minkowski and divisor counting, using the range from 29.2 and parity. The complete proof is on p. 788. □

29.6 Summing back to the field

Lemma 29.29 (Descent to character lengths). *Let E be a CM field of the host with $n = [E : \mathbf{Q}]$. Assume O contains the $\varphi(N)$-th roots of unity. For odd ℓ and $k \geq 1$,*

$$e_O s^+_{\ell,k}(E) \leq \sum_{\theta \in X_{\ell,O}(k)} \big(L_{\ell,k,\theta} + e_O \delta_\ell(N,n,k)\big), \tag{6.R}$$

where $\omega(N)$ counts the distinct prime divisors of N and

$$\delta_\ell(N,n,k) = (n+n^2)\nu_\ell(N,k) + \omega(N)(n+1)v_\ell(n) \leq (Nk)^6. \tag{6.L}$$

If $\ell \nmid n$, one may take $\delta_\ell = 0$.

Outline. Factor the generator operators to bound the plus length by eigenspace lengths and twist to invariants. Bound restriction errors by transgression, 29.23, n-torsion and Frobenius cohomology, with zero cost for added conditions if $\ell \nmid n$. The complete proof is on p. 788. □

Theorem 29.30 (The primes in the fixed set). *Fix χ, $\ell \in \mathcal{B}_\chi$ and $j \in \mathbf{N}$. There is $C_{\ell,j} \in \mathbf{N}$ depending on these data. Let the host be large for $t \geq 2$ and let $B \geq 0$ satisfy the layer hypothesis of 29.24. For every corresponding CM field E, the lengths for $1 \leq k < t$ are finite and*

$$\log \ell \sum_{1 \leq k < t} s^+_{\ell,k}(E) \leq (Nt)^{C_{\ell,j}} + C_{\ell,j} B.$$

Outline. Choose O by 29.29; characters of odd $\operatorname{ord} \theta_P$ have conductor dividing f by 29.2 and are bounded by 29.24. The rest are bounded by 29.22, 29.26, and 29.27, and summing over $1 \leq k < t$ determines $C_{\ell,j}$. The complete proof is on p. 789. □

Theorem 29.31 (The primes outside the fixed set). *For each χ there are $C_{\mathrm{out}} \in \mathbf{N}$ and primes r_1, r_2. Let the host be large for $t \geq 2$ with $q, d > r_1, r_2$. Let E be a corresponding CM field and S a finite set of odd primes disjoint from $\mathcal{B}_\chi$. Then the lengths for $\ell \in S$ and $1 \leq k < t$ are finite and*

$$P^+(E,t;S) \leq (Nt)^{C_{\mathrm{out}}}.$$

Outline. By 29.2, 29.29 applies with zero loss and every non-class-I prime satisfies $\ell \leq 2t-1$. Summing class I via 29.15, eliminating class Z by 29.28, and bounding classes II and III by 29.22 yields a fixed power of Nt. The complete proof is on p. 789. □

Theorem 29.32 (The pointwise bound and its common window). *For fixed χ and $j : \mathbf{N} \to \mathbf{N}$ there are $C \in \mathbf{N}$ and primes r_1, r_2. Let the host be large for $t \geq 2$ with $q, d > r_1, r_2$. Suppose $B \geq 0$ satisfies the layer hypothesis of 29.24 with index $j(\ell)$ at every $\ell \in \mathcal{B}_\chi$. Then for every corresponding CM field E, odd prime ℓ and $1 \leq k < t$, the plus length is finite. For each finite set S of odd primes,*

$$P^+(E,t;S) \leq (Nt)^C + CB. \tag{6.P}$$

Fix tower data $K = (R_0, R_\ell, C_K)$ with $C_K > 0$. Use the field datum M cut out by χ and $A_\ell = \mathbf{Q}(\mu_{\ell f})$ on $\mathcal{B}_\chi$. Set $j(\ell) = \lfloor \log_\ell R_\ell(A_\ell) \rfloor + 3$ there, with the convention of 28.16, and extend j to $\mathbf{N}$. Take the corresponding C, r_1, r_2. For $C_t > 0$, $0 < \varepsilon < 1/52$ and $A_1 \in \mathbf{R}$, put

$$R = R_0(M) + \sum_{\ell \in \mathcal{B}_\chi} R_\ell(A_\ell), \qquad J = 2^{R+2}(2C_K C_t/\varepsilon)^R, \qquad D_x = \exp(x^{4\varepsilon/5}).$$

For the given a, t, q, d, X, a K-window is an interval $[Y, Y^2]$ with $X^{1/J} \le Y$ and $Y^2 \le X$ on which (28.22) holds for the corresponding fields and (28.23) holds for all realizations of A_ℓ and their degree-$\ell^{j(\ell)}$ layers at each $\ell \in \mathcal{B}_\chi$.

There is X_0 such that for every a, t admissible in (28.21), every $X \ge X_0$, and every prime pair of 29.1 satisfying

$$q, d > 2X^{C_t} + 1, \qquad q, d > \max\{f, o, p_K, h_*, p_1, r_1, r_2\}, \qquad \log(qd) \le X^{\varepsilon/(4J)},$$

the following holds on every K-window. For all x in the window, all corresponding CM fields E and all finite sets S of odd primes, the plus lengths for $1 \le k < t(x)$ are finite and

$$P^+(E, t(x); S) \le \big(\mathrm{cond}(E)t(x)\big)^C + CD_x. \tag{6.W}$$

For the tower data of 28.16 and $C_t \ge 1$, such a window exists after increasing X_0. For $C_t = 401$, the choice $t(x) = \lfloor x^{401} \rfloor$ satisfies the two bounds on t in (28.21).

Outline. Combine 29.30 and 29.31 for odd primes and use singletons for finiteness. On a common window use $B = D_x$ and $\mathrm{cond}(E) = N$ in the pointwise bound, with 28.16 supplying the tower window and bound at 2, and $t = \lfloor x^{401} \rfloor$. The complete proof is on p. 789. □

Chapter 30

Automorphic counts and deformation rigidity

Let $\mathscr{H}$ denote the inputs of 26.32 and 26.39, with duality as in 27.2. Put $G_F = \mathrm{Gal}(\overline{F}/F)$, $C_\ell = \overline{\mathbf{Q}}_\ell$, $v_\ell(\ell) = 1$, and let ε_ℓ be the cyclotomic character. Coefficient DVRs $O \subset C_\ell$ are finite over $\mathbf{Z}_\ell$. Use the Selmer group $\mathcal{S}_{\ell,k,\vartheta}$, length $L_{\ell,k,\vartheta}$ and normalised length $a_{\ell,k,\vartheta}$ of 29.3.

A representation of level $(M;\mathbf{m})$ is continuous and diagonal on inertia away from ℓ, with characters factoring through the action on μ_M. At each place above ℓ it has a flag with decomposition-group characters κ_i satisfying $\kappa_i|_I = \varepsilon_\ell^{m_i}\chi_i|_I$, where I is inertia and χ_i factors through μ_M. The flag is unlinked if $\kappa_i \neq \kappa_j\varepsilon_\ell$ for all i,j. At each such place the flag must be unlinked or the representation potentially crystalline. Separated ordinary requires an unlinked flag with the prescribed inertia characters at every place above ℓ.

Fix $\iota : C_\ell \simeq \mathbf{C}$. A place above ℓ is unlinked if each prolongation admits an unlinked flag, and potentially unlinked if each admits a flag with $\kappa_i^a \neq (\kappa_j\varepsilon_\ell)^a$ for all i,j and integers $a \geq 1$.

Definition 30.1 (Witness and automorphy conditions)**.** Let $r : G_F \to \mathrm{GL}_n(C_\ell)$ be a representation over a CM field F. The witness condition at ι requires every unlinked place above ℓ to be potentially unlinked. It also requires that for each regular algebraic cuspidal representation π attached to r via ι, there be a single $t \in \mathbf{R}$ such that the cuspidal support of π_v consists of characters at every finite $v \nmid \ell$ and every potentially unlinked $v \mid \ell$, with

$$q_v^{t-1/2} < |\xi(\varpi_v)| < q_v^{t+1/2}$$

for each support character ξ, where q_v is the residue cardinality and ϖ_v a uniformiser. For continuous irreducible $r : G_F \to \mathrm{GL}_n(C_\ell)$, this witness condition together with existence of a cuspidal attachment via ι gives exactly Definition 29.5. A representation r of $G_{\mathbf{Q}}$ satisfies the quadratic witness condition if $r|_{G_K}$ satisfies the witness condition at ι for every imaginary quadratic field K for which $r|_{G_K}$ is irreducible. The quadratic automorphy condition at ι requires that if r is continuous, then it satisfies the quadratic witness condition and each such restriction admits a regular algebraic cuspidal attachment via ι.

The attachment in 39.2 takes cuspidal input, as in [116, Theorem 3.2.3].

30.1 Trace differences and the first count

Fix an imaginary quadratic K of discriminant D_K and quadratic character η. The family $\mathcal{A}_3(K,\iota;k,N)$ consists of irreducible separated ordinary $\rho : G_K \to \mathrm{GL}_3(C_\ell)$ of level $(N;(k,0,-k))$ with conjugate self-dual cuspidal attachment and witness for $\rho\varepsilon_\ell^{-1}$. The family $\mathcal{A}_{2+1}(K,\iota;k,N|D_K|)$ consists of representations isomorphic to $W|_{G_K} \oplus 1$, with the block in positions $0,2$. Here W is irreducible over $G_{\mathbf{Q}}$, satisfies the quadratic witness condition, has level $(N|D_K|;(k,-k))$ and has determinant of cyclotomic weight zero. The twist $W\varepsilon_\ell^{-k}$ has a newform attachment of weight $2k+1$.

Definition 30.2 (The marked deformation problem and its local ring)**.** Fix an odd prime ℓ, an integer $k \geq 1$, an even integer $N \geq 1$, an imaginary quadratic field K of discriminant D_K and class number h_K, and

a Dirichlet character $\vartheta : (\mathbf{Z}/N)^\times \to O^\times$ with values in a finite DVR extension $O \subset C_\ell$ of $\mathbf{Z}_\ell$ having residue field $\mathbf{k} = O/\mathfrak{m}_O$. Assume

$$\ell \nmid \operatorname{ord} \vartheta, \quad \operatorname{comp}_\ell(\vartheta)^2 = 1, \quad \vartheta(-1) = (-1)^k, \quad \ell \nmid |D_K| h_K, \quad (N, D_K) = 1, \quad \ell - 1 \nmid 2k,$$

and, after inflating ϑ and its components to $G_{\mathbf{Q}}$ via the action on μ_N, $\ker \bar{\varepsilon}_\ell \cap \ker \vartheta_P \not\subseteq G_K$, where ϑ_P is the prime-to-ℓ order part of ϑ with its ℓ-local factor removed. Let $\eta : G_{\mathbf{Q}} \to \{\pm 1\}$ have kernel G_K, and put $\Psi = \varepsilon_\ell^k \vartheta$.

Fix a complex conjugation $c \in G_{\mathbf{Q}}$ (so $c^2 = 1$, $\eta(c) = -1$, and $\Psi(c) = 1$) and an embedding $j : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_\ell$. This embedding selects a prolongation $A_0 \mid \ell$ and an injection $\iota : G_{\mathbf{Q}_\ell} \hookrightarrow G_{\mathbf{Q}}$ with image $D_\ell = D_{A_0}$ and $\iota(I_{\mathbf{Q}_\ell}) = I_\ell = I_{A_0}$. Let $K_v \subset \overline{\mathbf{Q}}_\ell$ be the fixed field of $\iota^{-1}(G_K)$ (so $K_v = \mathbf{Q}_\ell$ if ℓ splits in K, and $K_v/\mathbf{Q}_\ell$ is the unramified quadratic extension if ℓ is inert in K, since $\ell \nmid D_K$). Identify $G_{\mathbf{Q}_\ell}$ with D_ℓ via ι, and let K_1/K_v be the extension fixed by $G_{K_v} \cap \ker \operatorname{comp}_\ell(\vartheta)$; since $\operatorname{comp}_\ell(\vartheta)^2 = 1$, ℓ is odd, and $\vartheta \operatorname{comp}_\ell(\vartheta)^{-1}$ is unramified at ℓ, K_1/K_v is totally tamely ramified of degree 1 or 2 with $I_{K_1} = \ker(\vartheta|_{I_\ell})$, and $K_1/\mathbf{Q}_\ell$ is abelian with Galois group of exponent dividing 2.

Use zero-based matrix indices $0, 1, 2$ and let $J \in \mathrm{GL}_3(\mathbf{Z})$ be the antidiagonal matrix $J_{ab} = \delta_{a,2-b}$. For any commutative ring R, define the group $\mathcal{G}_3(R) = \mathrm{GL}_3(R) \rtimes \{\pm 1\}$, where -1 acts on $\mathrm{GL}_3(R)$ by $\theta(A) = J(A^{-1})^t J^{-1}$. For $x = (A, \epsilon) \in \mathcal{G}_3(R)$, write $x_{\mathrm{mat}} = A$. Set $\mathrm{wt}_1(i) = 1 - i$ and $\mathrm{wt}(a, b) = \mathrm{wt}_1(a) - \mathrm{wt}_1(b) = b - a$. The base lift is the homomorphism

$$r_0 : G_{\mathbf{Q}} \longrightarrow \mathcal{G}_3(O), \qquad r_0(g) = \big(\mathrm{diag}(\Psi(g), 1, \Psi(g)^{-1}),\, \eta(g)\big),$$

with residual representation $\bar{r}_0 : G_{\mathbf{Q}} \to \mathcal{G}_3(\mathbf{k})$. A *marking* is a pair (c, g_1) with $g_1 \in G_K$ and $\Psi(g_1)^2 - 1 \in O^\times$.

For a complete Noetherian local O-algebra R whose structure map is local and induces an isomorphism $O/\mathfrak{m}_O \simeq R/\mathfrak{m}_R$, a *continuous lift* of $\bar{r}_0$ over R is a homomorphism $r : G_{\mathbf{Q}} \to \mathcal{G}_3(R)$ lifting $\bar{r}_0$ whose matrix entries are $\mathfrak{m}_R$-adically continuous (so its sign coordinate is η since ℓ is odd). The lift r is *normalised* (or *marked*) at (c, g_1) if

$$r(c)_{\mathrm{mat}} = I_3, \qquad \big(r(g_1)_{\mathrm{mat}} - r(cg_1^{-1}c)_{\mathrm{mat}}\big)_{01} = 0, \qquad \big(r(g_1)_{\mathrm{mat}} - r(cg_1^{-1}c)_{\mathrm{mat}}\big)_{10} = 0.$$

Let $\mathcal{K}_{\mathrm{away}} \subset G_{\mathbf{Q}}$ consist of all elements $g \in I_A \cap \ker \eta \cap \ker \vartheta$ as A ranges over all finite places of $\overline{\mathbf{Q}}$ not above ℓ.

Let U be the universal unrestricted lifting ring for continuous lifts of $\bar{r}_0|_{D_\ell}$ to $\mathcal{G}_3(R)$, with universal lift $r^{\mathrm{univ}} : D_\ell \to \mathcal{G}_3(U)$ and base point $x_{0,\ell} : U \to O$ specialising r^{univ} to $r_0|_{D_\ell}$. Choose $s_0 \in I_\ell \cap \ker(\vartheta|_{D_\ell})$ such that $\varepsilon_\ell(s_0)$ has infinite order, and put $d_i = \Psi(s_0)^{1-i} = \varepsilon_\ell(s_0)^{k(1-i)} \in O^\times$ for $i = 0, 1, 2$. These three values are pairwise distinct. Set

$$P_i = \prod_{j \neq i} \big(r^{\mathrm{univ}}(s_0)_{\mathrm{mat}} - d_j I_3\big) \in M_3(U), \qquad c_i = \prod_{j \neq i} (d_i - d_j) \in O \setminus \{0\}.$$

Let $J_{\mathrm{ord}} \subset U$ be the closed ideal generated by the coefficients of $\det(XI_3 - r^{\mathrm{univ}}(s_0)_{\mathrm{mat}}) - \prod_{i=0}^{2}(X - d_i)$, the entries of $P_j r^{\mathrm{univ}}(g)_{\mathrm{mat}} P_i$ for $0 \le i < j \le 2$ and $g \in G_{K_v}$, and the entries of $P_i(r^{\mathrm{univ}}(h)_{\mathrm{mat}} - \Psi(h)^{1-i} I_3)P_i$ for $i = 0, 1, 2$ and $h \in I_\ell$. Fix an arithmetic Frobenius lift $\varphi_v \in G_{K_v} \cap G_{K_1}$ (which exists because K_1/K_v is totally ramified), put $t_i = \mathrm{tr}(P_i r^{\mathrm{univ}}(\varphi_v)_{\mathrm{mat}}) \in U$, and define the cleared adjacent Frobenius differences

$$\delta_1 = c_1 t_0 - \varepsilon_\ell(\varphi_v) c_0 t_1 \in U, \qquad \delta_2 = c_2 t_1 - \varepsilon_\ell(\varphi_v) c_1 t_2 \in U.$$

Define the saturated ideal

$$I_{\mathrm{sat}} = \{a \in U : \exists n \ge 0,\ (\ell \delta_1 \delta_2)^n a \in \sqrt{J_{\mathrm{ord}}}\},$$

and let $I \subset U$ be the intersection of the minimal primes of I_{sat} contained in $\ker x_{0,\ell}$. The *distinguished local ring* is $B = U/I$. The *marked global deformation ring* S is the universal complete Noetherian local O-algebra with residue field $\mathbf{k}$ representing normalised continuous lifts $r : G_{\mathbf{Q}} \to \mathcal{G}_3(R)$ of $\bar{r}_0$ that vanish on $\mathcal{K}_{\mathrm{away}}$ (meaning $r(g) = 1$ for $g \in \mathcal{K}_{\mathrm{away}}$) and whose restriction $r|_{D_\ell}$ factors through $B = U/I$.

The *cohomological inputs for the marked problem* are local finiteness, local Tate duality, the local Euler characteristic formula, the local Bloch–Kato comparison, global finiteness, the global Euler characteristic formula, Poitou–Tate duality (26.32 and 27.2), and the bound of 28.2.

Lemma 30.3 (Conjugation invariance and formal smoothness of the distinguished local ring)**.** *Assume the local cohomological inputs of Definition 30.2, and keep the notation $U, I, B = U/I, x_{0,\ell}, K_v, K_1$ of that definition (for any $k \geq 1$, including $k = 1$, and whether ℓ splits or is inert in K). Put $E = \operatorname{Frac} O$.*

1. *The ideal $I \subset U$ is prime and, for every complete Noetherian local O-algebra R with residue field $\mathbf{k}$ as in Definition 30.2, is invariant under conjugation in $\mathcal{G}_3(R)$ by every $(h, 1)$ with $h \in \mathrm{GL}_3(R)$ congruent to $I_3 \bmod \mathfrak{m}_R$. Consequently $B = U/I$ is a complete Noetherian local O-domain with a homogeneous power series presentation $B = O[[X]]/(H)$ for the diagonal conjugation grading $\mathrm{wt}(a, b) = b - a$, and for every $Y \in M_3(O)$ the infinitesimal conjugation $(I_3 + \epsilon Y, 1) r_0|_{D_\ell} (I_3 + \epsilon Y, 1)^{-1}$ over $O[\epsilon]/(\epsilon^2)$ defines an O-valued derivation on B at $x_{0,\ell}$ killing H.*

2. *The completion $\widehat{B[1/\ell]}_{\ker x_{0,\ell}}$ is formally smooth over E of dimension 12 and coincides with the formal framed crystalline deformation ring $R^{\mathrm{cris}}_{r_0}$ of $r_0|_{D_\ell} : D_\ell \to \mathcal{G}_3(E)$ parameterising deformations whose restriction to G_{K_1} is crystalline. There is a graded O-algebra homomorphism $\varphi : U \to E[[t_1, \ldots, t_{12}]]$ with $\ker \varphi = I$, constant term $x_{0,\ell}$, linear terms identifying the rational cotangent space of B at $x_{0,\ell}$ with the span of the coordinates, and $1, 2, 3, 4, 2$ coordinates in weights $-2, -1, 0, 1, 2$, respectively.*

3. *Every characteristic-zero point $f : B \to C_\ell$ yields a representation $r_f : D_\ell \to \mathcal{G}_3(C_\ell)$ that is crystalline on G_{K_1} and conjugate to an upper-triangular representation on G_{K_v} with inertia characters $(\Psi, 1, \Psi^{-1})$. Every characteristic-zero tangent vector of B at f gives a cocycle in $Z^1_f(D_\ell, \operatorname{ad} r_f)$.*

Outline. Shows $R^{\mathrm{cris}}_{r_0}$ is formally smooth of dimension 12 via Bloch–Kato and filtered φ-modules. Identifies $I \cdot U^\wedge = I_{\mathrm{cris}}$ via 30.17 and slope splitting, giving primeness, conjugation invariance, and crystallinity. The complete proof is on p. 790. □

Lemma 30.4 (Marked normal form and base vanishing)**.** *In the marked problem of Definition 30.2, assume the cohomological inputs of that definition.*

1. *Every two-sided characteristic-zero point $r : G_{\mathbf{Q}} \to \mathcal{G}_3(C_\ell)$ of S (meaning one for which neither all positive-weight nor all negative-weight matrix entries vanish on G_K) is either of Type I ($r|_{G_K}$ irreducible) or of Type II ($r|_{G_K}$ block diagonal on $\{0, 2\} \oplus \{1\}$ with irreducible corner on $\{0, 2\}$). In Type II, one has $r(g)_{\mathrm{mat},11} = 1$ and $\det(\mathrm{corner}(r(g)_{\mathrm{mat}})) = 1$ for all $g \in G_K$, and defining $W(g) = \mathrm{corner}(r(g)_{\mathrm{mat}})$ for $g \in G_K$ and $W(g) = \mathrm{corner}(r(g)_{\mathrm{mat}}) \operatorname{diag}(-1, 1)$ for $g \notin G_K$ yields a continuous irreducible representation $W : G_{\mathbf{Q}} \to \mathrm{GL}_2(C_\ell)$ extending the corner with $\det W = \eta$ and $\operatorname{tr} W(c) = 0$.*

2. *Any two two-sided characteristic-zero points of S whose matrix restrictions to G_K are isomorphic lie in the same $\mathbb{G}_m$-orbit under diagonal conjugation by $\operatorname{diag}(t, 1, t^{-1})$.*

3. *At the base point r_0, let V_j be the weight-j summand of $\operatorname{ad} r_0$ over C_ℓ. Then $H^1_f(\mathbf{Q}, V_j) = 0$ for $j = -2, -1, 0$, and in particular $H^1_f(\mathbf{Q}, C_\ell(\eta)) = 0$.*

Outline. Uses the diagonal marking matrix at g_1 to classify two-sided points, applying 30.3 and class field theory in Type II and diagonal intertwiners for (2). Proves $H^1_f(\mathbf{Q}, V_j) = 0$ for $j \leq 0$ via 29.19 and class-group finiteness. The complete proof is on p. 792. □

Theorem 30.5 (A single-character trace bound)**.** *Assume the cohomological inputs for the marked problem of Definition 30.2 and 28.2, 30.12, 30.61, 30.72. Assume adjoint rigidity and admissibility for Type I, irreducible on G_K, and block adjoint rigidity for Type II, with an irreducible corner and a middle line on G_K. Let ℓ be an odd prime, $k \geq 1$, $N \geq 1$ even, and $\vartheta : (\mathbf{Z}/N)^\times \to O^\times$ satisfy*

$$\ell \nmid \operatorname{ord} \vartheta, \quad \mathrm{comp}_\ell(\vartheta)^2 = 1, \quad \vartheta(-1) = (-1)^k, \quad \ell \nmid |D_K| h_K, \quad (N, D_K) = 1, \quad \ell - 1 \nmid 2k.$$

Here h_K is the class number and comp_ℓ is the modulus component at ℓ. Let ϑ_P be the prime-to-ℓ order part with its ℓ-local component removed. Inflate the characters to $G_{\mathbf{Q}}$ via the action on μ_N and assume $\ker \bar{\varepsilon}_\ell \cap \ker \vartheta_P \not\subseteq G_K$. Put $\Psi = \varepsilon_\ell^k \vartheta$, fix $\sigma \in G_K$, and set $T = \Psi(\sigma) + 1 + \Psi(\sigma)^{-1}$.

If $\operatorname{tr} \rho(\sigma) \neq T$ throughout $\mathcal{A}_3 \cup \mathcal{A}_{2+1}$, then $L_{\ell,k,\vartheta} < \infty$, and there is a finite set $X \subset \mathcal{A}_3 \cup \mathcal{A}_{2+1}$ of pairwise nonisomorphic representations with

$$a_{\ell,k,\vartheta} \leq 2 \sum_{\rho \in X} v_\ell(\operatorname{tr} \rho(\sigma) - T).$$

Outline. Combines 30.3 and 30.72 to present S, counts obstructions by 27.2, and bounds $L_{\ell,k,\vartheta}$ via 30.6. Verifies finiteness (30.61) and tangent bounds (30.4, 30.12), then applies 30.72 and 29.3. The complete proof is on p. 793. □

Lemma 30.6 (Integral lift on the ambient algebra)**.** *In the marked problem of 30.5, let S be the normalised global ring, $x_0 : S \to O$ the base point, $q : A \to S$ the graded model map, and $x = x_0 q$. For $0 \neq b \in O$, $\mathcal{S}_{\ell,k,\vartheta}[b]$ injects additively into the $O/(b)$-valued derivations at x on A that kill F_i. Each image derivation lifts to an O-valued derivation at x supported in weight -1.*

Outline. Normalizes each b-torsion Selmer cocycle at g_1 to build a square-zero lift $(1+\epsilon tV)r_0$. Conjugation invariance in 30.3 lifts it to an O-valued weight--1 derivation on A killing $F_i \bmod b$, with $(1,0)$ entries recovering t. The complete proof is on p. 795. □

Theorem 30.7 (An integer for three-dimensional trace differences)**.** *Assume the automorphic inputs of $\mathscr{H}$. For each imaginary quadratic K there are $c, A \in \mathbf{N}$ as follows. Let $k, N \geq 1$, let $p \geq 7$ be prime with $p \nmid N|D_K|$, and take $w \mid p$ in K. Put $q = \mathrm{N}w$. Let $(z_u)_{u\in\mathcal{O}}$ be a finite indexed family of roots of unity permuted by every automorphism of $\mathbf{C}$. There is $D \in \mathbf{Z} \setminus \{0\}$, chosen before ℓ, with*

$$\log|D| \leq c\,\#\mathcal{O}\,[N(k+1)]^A \log p.$$

For every prime $\ell \neq 2, p$, every ι, every arithmetic Frobenius σ at w, and every family of finite sets $X_u \subset \mathcal{A}_3 \cup \mathcal{A}_{2+1}$ of pairwise nonisomorphic representations, the differences

$$\Delta_{u,\rho} = \operatorname{tr}\rho(\sigma^{-1}) - \iota^{-1}(q^{-k}z_u + 1 + q^k z_u^{-1})$$

are nonzero and satisfy

$$\sum_u \sum_{\rho\in X_u} v_\ell(\Delta_{u,\rho}) \leq v_\ell(D). \tag{30.1}$$

Outline. Clear denominators in the differences of 30.9, use (30.2) for nonvanishing and bound their integral Galois-invariant product by (30.3). Distinct factors and the unit denominator bound the selected valuations. The complete proof is on p. 795. □

Lemma 30.8 (From Galois level to automorphic level and infinitesimal character)**.** *Assume the automorphic inputs of $\mathscr{H}$. Let K be a CM field, $n \geq 2$, $m = (m_1, \dots, m_n) \in \mathbf{Z}^n$ strictly decreasing, $M \geq 1$, ℓ an odd prime, and $\iota : C_\ell \simeq \mathbf{C}$. Let $r : G_K \to \mathrm{GL}_n(C_\ell)$ be a continuous irreducible representation of Galois level $(M; m)$ (away from ℓ, diagonal on inertia with characters factoring through the action on μ_M; at each $w \mid \ell$, preserving a flag with inertia characters $\varepsilon_\ell^{m_i}\chi_i$ where χ_i factors through the action on μ_M). Assume that r satisfies the witness condition of Definition 30.1 at ι and that every place $w \mid \ell$ of K is unlinked or potentially unlinked for r. Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$ attached to r via ι that is polarised, i.e. $\pi^\vee \simeq \pi^c \otimes (\chi \circ N_{K/K^+} \circ \det)$ for an algebraic Hecke character χ of K^+ satisfying $\chi_v(-1) = (-1)^n$ at every real place v of K^+. Then:*

1. *At every complex embedding $\varphi : K \hookrightarrow \mathbf{C}$, π has the infinitesimal character determined by the weights m.*
2. *The finite part π_{fin} has a nonzero vector fixed by the principal congruence subgroup $K(M) = \ker(\mathrm{GL}_n(\widehat{\mathcal{O}}_K) \to \mathrm{GL}_n(\widehat{\mathcal{O}}_K/M\widehat{\mathcal{O}}_K))$.*
3. *At every finite place $w \nmid \ell$ of K where π_w is spherical with Satake multiset B, r is unramified at w and, for every arithmetic Frobenius σ_w at a place above w, ι carries the eigenvalue multiset of $r(\sigma_w^{-1})$ to $q_w^{(n-1)/2}B$.*

In particular:

(a) If π is conjugate self-dual ($\pi^\vee \simeq \pi^c$), then π is polarised by η_{K/K^+}^n and conclusions (1)–(3) hold.

(b) Let $r_0 : G_{\mathbf{Q}} \to \mathrm{GL}_n(C_\ell)$ be irreducible of Galois level $(M; m)$, satisfying the quadratic automorphy condition of Definition 30.1 at ι and $r_0(g)^t P r_0(g) = \theta(g)P$ for an invertible matrix P and a character θ of cyclotomic weight $\mathrm{wt} \in \mathbf{Z}$. Assume either that $m_i \neq m_j + 1$ for all i, j, or that $\ell \mid M$ and r_0 is

unlinked at ℓ. If K' is an imaginary quadratic field on which $r = r_0|_{G_{K'}}$ remains irreducible, then r has Galois level $(M;m)$, every place of K' above ℓ is unlinked or potentially unlinked for r, and its cuspidal attachment π over K' is polarised and satisfies (1)–(3).

Outline. Definition 30.1 makes each π_w a zero-monodromy principal series, and local-global compatibility gives (1)–(3). Parts (a)–(b) verify potential unlinkedness and polarisation using η^n_{K/K^+} and strong multiplicity one. The complete proof is on p. 796. □

Lemma 30.9 (Hecke labels and their traces)**.** *In 30.7 the two families have finite, conjugation-stable label sets S_1, S_2 with equivariant algebraic traces $t(s)$. For each ℓ, ι, σ, representations admit labels with $t(s) = \iota(\operatorname{tr}\rho(\sigma^{-1}))$, distinct for distinct isomorphism classes in each family. With constants $c_i, A_i \in \mathbf{N}$ depending on K,*

$$|t(s)| \le \sqrt{q} + 2, \qquad q^k t(s) \text{ is integral}, \tag{30.2}$$

$$\#S_1 \le c_1[N(k+1)]^{A_1}, \qquad \#S_2 \le c_2[N|D_K|(k+1)]^{A_2}. \tag{30.3}$$

In rank $n \ge 2$, fix n^2+2 imaginary quadratic fields of pairwise coprime discriminants, and consider the family of irreducible representations $W : G_{\mathbf{Q}} \to \mathrm{GL}_n(C_\ell)$ of Galois level $(M;m)$ satisfying twisted self-duality $W(g)^t P W(g) = \theta(g)P$, quadratic automorphy with witness, and one of the two alternatives of Lemma 30.8(b). If for each of these fields the irreducible restrictions have at most $P(M,H)$ conjugation-stable CM labels at level M and weight bound H, then the rational label count is at most $2^{n+1}(n^2+1)P(M,H)$.

Outline. Lemma 30.8(a) and Proposition 26.34 bound CM Hecke labels via level-$3N$ cohomology and the Weyl dimension formula. Restriction to n^2+1 quadratic fields via Lemma 30.8(b) gives the rational count, and purity yields the trace bounds. The complete proof is on p. 797. □

Theorem 30.10 (One integer for a character orbit)**.** *Assume 30.5 and 30.7. For every number field K there are $c_K, A_K \in \mathbf{N}$ as follows. Let $k, N \ge 1$, $G = (\mathbf{Z}/N)^\times$, $H \le G$, and $p \nmid 30N|D_K|$ prime. There is $D \in \mathbf{Z} \setminus \{0\}$ with*

$$\log|D| \le c_K \varphi([G:H])[N(k+1)]^{A_K} \log p.$$

For every prime $\ell \ne p$ and coefficient ring O, suppose some O-valued character of kernel H satisfies the datum conditions of 30.5. Then all characters in that O-valued orbit have finite length and

$$\sum_{\ker\vartheta = H} a_{\ell,k,\vartheta} \le 2v_\ell(D).$$

Outline. Apply 30.7 to inverse character values, using a good character for all orbit values and primitive powers to preserve the datum. Match inverse Frobenius traces with 30.5, sum by (30.1), and use two auxiliary primes. The complete proof is on p. 798. □

30.2 The two-dimensional vanishing input

Write $\overline{\varepsilon}_{\ell^r}$ for the reduction of ε_ℓ modulo ℓ^r. For a prolongation A of the ℓ-adic place, write D_A and I_A for its decomposition and inertia groups. Number the coordinates of two-dimensional column representations $0, 1$.

The normal form for (k, S, A, W) requires an odd prime ℓ, $k \ge 1$, a continuous $W : G_{\mathbf{Q}} \to \mathrm{GL}_2(C_\ell)$ with integral matrices over a finite coefficient ring, and a finite set S containing ℓ and all ramified primes. Its determinant δ is odd quadratic, and W is irreducible on $\ker\varepsilon_\ell \cap \ker\delta$. On D_A it is upper triangular with diagonal characters α_W, β. For some $h \ge 1$,

$$\alpha_W(s)^h = \varepsilon_\ell(s)^{kh} \quad (s \in I_A), \qquad \alpha_W \ne \varepsilon_\ell \quad \text{on } D_A. \tag{30.4}$$

A source for this normal form is a newform g of weight $2k+1$ and level M_g prime to ℓ, with $|\iota^{-1}(a_\ell(g))|_\ell = 1$. It is attached to $W\varepsilon_\ell^{-k}\delta^{-1}\xi$, where ξ factors through μ_ℓ and $\beta\varepsilon_\ell^k\delta^{-1}\xi = 1$ on I_A. A W-cocycle y is unramified away from ℓ if at each prime $p \ne \ell$ its restriction to inertia at a prolongation of p is a coboundary. Its second coordinate is a coboundary on D_A precisely when

$$\exists b \in C_\ell, \quad y(s)_1 = (\beta(s) - 1)b \quad (s \in D_A). \tag{30.5}$$

The spherical test at (ι, k) is imposed on every twist $W\phi$ by a character factoring through $\mathrm{Gal}(\mathbf{Q}(\mu_M)/\mathbf{Q})$ for some $M \geq 1$. If $W\phi$ has a D_A-stable flag at a prolongation $A \mid \ell$ whose successive inertia characters are $\varepsilon_\ell^{k}, \varepsilon_\ell^{-k}$, in that order, then the component at ℓ of every regular algebraic cuspidal representation attached to $W\phi$ via ι has a nonzero $\mathrm{GL}_2(\mathbf{Z}_\ell)$-fixed vector.

Let L be a number field and let U be an ℓ-adic representation of G_L. For a finite place v, choose a decomposition group G_{L_v} with inertia subgroup I_v. We use the local conditions

$$H^1_f(L_v, U) = \begin{cases} \ker\bigl(H^1(G_{L_v}, U) \longrightarrow H^1(I_v, U)\bigr), & v \nmid \ell, \\ \ker\bigl(H^1(G_{L_v}, U) \longrightarrow H^1(G_{L_v}, B_{\mathrm{cris}} \otimes_{\mathbf{Q}_\ell} U)\bigr), & v \mid \ell. \end{cases}$$

Here B_{cris} is Fontaine's crystalline period ring, with the diagonal Galois action on the tensor product. At an archimedean place the local condition is zero. Write $H^1_f(L, U)$ for the classes in continuous $H^1(G_L, U)$ whose localizations belong to these subspaces. Conjugating a decomposition group transports its local condition.

Lemma 30.11 (Kato vanishing in odd weight and quadratic descent)**.** *Let ℓ be odd, let $e \geq 1$ be an integer, and put $h = 2e + 1$. Fix $\iota : \overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$. Let W be a continuous irreducible two-dimensional representation of $G_{\mathbf{Q}}$ over $\overline{\mathbf{Q}}_\ell$. Suppose $W(-e) = W \otimes \varepsilon_\ell^{-e}$ has a weight-h newform attachment via ι, in the following sense.*

There are an integer $M \geq 1$, a Dirichlet character $\omega \bmod M$, and a cusp form $f = \sum_{n \geq 1} a_n q^n$ on $\Gamma_1(M)$ of weight h and character ω. It satisfies $a_1 = 1$ and $T_p f = a_p f$ at every prime $p \nmid M$. At a proper divisor of M there is no nonzero cusp form of weight h and any nebentype with these eigenvalues at all but finitely many primes. At all but finitely many primes $p \nmid M\ell$, $W(-e)$ is unramified and, for arithmetic Frobenius σ_p,

$$\iota\bigl(\det\bigl(X - W(-e)(\sigma_p^{-1})\bigr)\bigr) = X^2 - a_p X + \omega(p) p^{2e},$$

where ι acts on coefficients.

Let $K/\mathbf{Q}$ be imaginary quadratic and let θ be its quadratic character. Then

$$H^1_f(\mathbf{Q}, W) = H^1_f(\mathbf{Q}, W \otimes \theta) = 0.$$

Outline. Identify the primitive newform and match W with Kato's twist by Frobenius comparison. Kato's finiteness over K gives rational Selmer vanishing by divisibility and closedness of coboundaries, then restriction and corestriction give both descents. The complete proof is on p. 798. □

Theorem 30.12 (Kato vanishing for two quadratic companions)**.** *Assume $\mathscr{H}$. Let ℓ be odd, $k \geq 1$, $\ell - 1 \nmid 2k$, and K imaginary quadratic with character θ. Suppose $\ell \nmid |D_K| h_K$, $N \geq 1$ is even, and $(N, D_K) = 1$. Let $\vartheta : G_{\mathbf{Q}} \to C_\ell^\times$ factor through μ_N, have order prime to ℓ, and have square trivial on inertia at ℓ. Put $\Psi = \varepsilon_\ell^{k}\vartheta$ and assume $\Psi(c) = 1$ at complex conjugation. Let $W : G_{\mathbf{Q}} \to \mathrm{GL}_2(C_\ell)$ have level $(N|D_K|; (k, -k))$, determinant θ, and satisfy*

$$\operatorname{tr} W(s) = \Psi(s) + \Psi(s)^{-1} \quad (s \in I_A,\ A \mid \ell),$$

$$|\operatorname{tr} W(g) - \theta(g)\Psi(g) - \Psi(g)^{-1}|_\ell < 1 \quad (g \in G_{\mathbf{Q}}).$$

Assume the following two local conditions on W. There exist a prolongation $A \mid \ell$ and an integer $M \geq 1$ such that all $s \in I_A$ fixing μ_M have characteristic polynomials $(X - \varepsilon_\ell(s)^k)(X - \varepsilon_\ell(s)^{-k})$. At some $A \mid \ell$, $\theta|_{I_A} = 1$ and an ordered flag has inertia characters $\varepsilon_\ell^{k}\chi, \varepsilon_\ell^{-k}\chi$, with χ factoring through μ_ℓ. Assume $W|_{G_K \cap \ker \varepsilon_\ell}$ is irreducible. If $W\varepsilon_\ell^{-k}$ has a weight-$(2k+1)$ newform attachment via ι as in Lemma 30.11, and W satisfies the quadratic witness condition at this ι, then

$$H^1_f(\mathbf{Q}, W) = H^1_f(\mathbf{Q}, W \otimes \theta) = 0. \tag{30.6}$$

Proof. Take $e = k$ in Lemma 30.11. The level hypothesis gives continuity of W. A $G_{\mathbf{Q}}$-stable line would be stable under $G_K \cap \ker \varepsilon_\ell$, contrary to the assumed irreducibility there. The given newform is attached to $W(-k)$ via the fixed ι, and the character of $K/\mathbf{Q}$ is θ. The two conclusions of Lemma 30.11 give (30.6) directly. □

Lemma 30.13 (A source of level prime to ℓ)**.** *Assume $\mathscr{H}$, ℓ odd, and $k \geq 1$. Let V be continuous of dimension two with odd quadratic determinant δ, irreducible on $\ker \varepsilon_\ell \cap \ker \delta$. Away from ℓ let its inertia characters be diagonal of bounded cyclotomic modulus. Suppose a finite cyclotomic twist of $V\varepsilon_\ell^{-k}$ has a weight-$(2k+1)$ newform attachment. At $A \mid \ell$ suppose $\delta|_{I_A} = 1$ and an ordered flag has inertia characters $\varepsilon_\ell^{k}\chi, \varepsilon_\ell^{-k}\chi$, with χ factoring through μ_ℓ. If V passes the spherical test, it is isomorphic to a representation V' in normal form at A for some S, with a source newform g.*

Outline. Twist and use the spherical test and strong multiplicity one to get level prime to ℓ. The flag gives root valuations $0, 2k$, and the recurrence with (30.7) excludes a torsion unit root, giving (30.4) in an adapted stable lattice. The complete proof is on p. 800. □

Lemma 30.14 (Vanishing under the spherical test)**.** *Assume the cohomological inputs of $\mathscr{H}$ and take the arithmetic hypotheses and the newform attachment of 30.12. If W passes the spherical test, then $H^1_f(\mathbf{Q}, W) = H^1_f(\mathbf{Q}, W \otimes \det W) = 0$.*

Proof. The arithmetic hypotheses give $k \geq 1$, continuity of W, and irreducibility on $G_K \cap \ker \varepsilon_\ell$, hence on $G_{\mathbf{Q}}$. Apply Lemma 30.11 directly to the given attachment of $W(-k)$. Its quadratic character is $\det W$. □

Lemma 30.15 (Coefficient bounds and nonvanishing at the edge)**.** *Let g' be a cuspidal Hecke eigenform of weight $2k+1$, $k \geq 1$, and nebentype ω at level N, normalised by $a_1 = 1$ with $a_n = 0$ when $(n, N) > 1$. Then*

$$\sum_{n \leq X} |a_n|^2 \ll X^{2k+1}, \qquad |a_n| \ll n^{k+1/2}. \tag{30.7}$$

The same bounds hold before depletion. For every Dirichlet character ξ, the twisted L-value with factors at the level and modulus omitted has a nonzero right limit at each real $w \geq k+1$.

Outline. Parseval and partial summation give (A.260), and the good Euler factors give (A.261). An edge zero bounds the right side, while odd nebentype and divergence of reciprocal primes in the class $-1 \bmod M$ make the left side unbounded. The complete proof is on p. 800. □

30.3 Four-dimensional symplectic points

Throughout this section use the good data of Definition 29.17 and the lifts and points of Definition 29.18. In characteristic ℓ, write ε_ℓ for the reduction of the cyclotomic character when it takes values in the coefficient field.

Lemma 30.16 (Separation in the marked ordinary flag)**.** *At a characteristic-zero point of Definition 29.18, the local action at ℓ has an ordered flag with continuous characters λ_i. On inertia $\lambda_i = \delta_i$, and on D_ℓ their reductions agree. On D_ℓ the products $\lambda_0\lambda_3$ and $\lambda_1\lambda_2$ are both ν. If $U \leq D_\ell$ is open and contains σ_ℓ, then $\lambda_i|_U \neq (\varepsilon_\ell\lambda_j)|_U$ for $i \neq j$.*

Outline. Over B_ℓ of Definition 29.18, eigenprojectors of $b_\ell(s_0)$ give the flag, and integrality over the Henselian local ring identifies the reductions of λ_i and δ_i. The similitude equation makes the alternating form antidiagonal, yielding $\lambda_i\lambda_{3-i} = \nu$. Inertia exponents within planes and σ_ℓ of Definition 29.17 across planes rule out cyclotomic twists. The complete proof is on p. 801. □

Lemma 30.17 (Crystallinity of an ordered flag)**.** *Let K, E be finite extensions of $\mathbf{Q}_\ell$, let A be a finite-dimensional commutative local E-algebra with residue field E, and let $r : G_K \to \mathrm{GL}_n(A)$ be continuous and upper triangular. Write κ_i for its diagonal characters and $\bar\kappa_i$ for their images in $E^\times$. Suppose that $\kappa_i|_{I_K} = \varepsilon_\ell^{w_i}|_{I_K}$ in $A^\times$, where $w_1 > \cdots > w_n$ are integers. Suppose also that $\bar\kappa_i \neq \varepsilon_\ell\bar\kappa_j$ for $i < j$. Then A^n, regarded as an E-representation, is crystalline.*

Outline. Induct from crystalline twists of unramified lines. At each step, the weight order and character separation let the Bloch–Kato dimension formula, local duality and the Euler characteristic show that every extension class is crystalline. The complete proof is on p. 801. □

Lemma 30.18 (Cyclotomic restriction at a symplectic point)**.** *Let r be an irreducible characteristic-zero point of a good datum of case II or III. Take the fitted tower used for the ordinary comparison, with imaginary quadratic field K, totally real abelian field F, and $L = FK$. Suppose $r|_{G_K}$ has a cuspidal attachment with witness. Then $r|_{G_{L(\mu_{\ell^\infty})}}$ is absolutely irreducible. The same conclusion holds after the common scalar twist used to put the representation on the level group.*

Outline. Distinct residual characters exclude nontrivial self-twists trivial on G_L. The local witness and 30.16 give simple spectrum on the cyclotomic kernel, so reducibility gives, as in 29.6, a quadratic self-twist excluded by the residual marking. The complete proof is on p. 802. □

Theorem 30.19 (Infinitesimal rigidity of an automorphic symplectic point)**.** *Assume the unramified inputs from $\mathscr{H}$ and 30.62. Let D be of case II or III, with finite characters ϑ, ξ, ω factoring through μ_N for some $N \geq 1$. Fix $\iota : C_\ell \simeq \mathbf{C}$ and an irreducible characteristic-zero point r of D. Suppose that $r|_{G_K}$ has a regular algebraic cuspidal attachment with witness for every quadratic CM field K for which this restriction is irreducible. Every allowed dual-number lift r' of r has*

$$r'(g) = r(g) + \epsilon(Xr(g) - r(g)X), \qquad X \in \mathfrak{sp}_4(C_\ell).$$

Outline. Write the lift as a symplectic adjoint cocycle and verify the finite local condition by inertia averaging and 30.16, 30.17. Use 30.18 and 30.62 to obtain a fixed point over the tower, then average and project to get a symplectic coboundary. The complete proof is on p. 803. □

Lemma 30.20 (The spherical and Iwahori comparison maps)**.** *At finitely many split auxiliary places j, let the local group be GL_n and the residue cardinality Q_j. Take spherical and Iwahori fixed vectors in the same representation, with common level elsewhere. Write M, V for the resulting modules and $i : M \hookrightarrow V$ for inclusion. Use the unnormalised coset sum $T : V \to M$. Let u_{ji} denote the commuting Bernstein root operators. Suppose their elementary symmetric functions on M act by the signed coefficients of monic degree-n polynomials P_j over the spherical coefficient ring B. Let D be the ordered splitting algebra over B of those polynomials, with roots x_{ji}. Then there are maps*

$$D \otimes_B M \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} V$$

intertwining root polynomials and the common coefficient operators. Each composite is the action on its source of

$$f(x) = \prod_j \prod_{a \neq b} (x_{ja} - x_{jb})(x_{ja} - Q_j x_{jb}).$$

The assertion also holds on matching direct summands cut out by the ordinary and ordered residual idempotents.

Outline. Define the forward map by root operators and the reverse map by coset sums. Use the Bernstein relation and an alternating polynomial identity to compute both composites, then scale the reverse map to obtain the stated multiplier. The complete proof is on p. 804. □

Lemma 30.21 (Recovering the ordered flag at a separated weight)**.** *Let $K/\mathbf{Q}_\ell$ be finite, let W_K be its Weil group, and use the reciprocity map $a : W_K \to K^\times$ taking a lift of a uniformiser to an element of valuation one. Fix $\iota : \overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$. Let r be an n-dimensional potentially semistable representation of G_K over a finite extension of $\mathbf{Q}_\ell$, allowing extension of the coefficient field. Suppose that its filtration jumps at each embedding $\tau : K \hookrightarrow \overline{\mathbf{Q}}_\ell$ are*

$$h_{0,\tau} < \cdots < h_{n-1,\tau}.$$

Let $\omega_i : K^\times \to \mathbf{C}^\times$ be smooth characters whose restrictions to $\mathcal{O}_K^\times$ are pairwise distinct. Suppose the characteristic polynomial of the Weil action of r at each $s \in W_K$, after applying ι, is

$$\prod_{i=0}^{n-1} \big(X - \omega_i(a(s))\big).$$

Suppose there are continuous characters $\kappa_i : G_K \to \overline{\mathbf{Q}}_\ell^\times$ satisfying

$$\kappa_i(s) = \iota^{-1}\big(\omega_i(a(s))\big) \prod_\tau \tau(a(s))^{-h_{i,\tau}} \qquad (s \in W_K).$$

Then r has a G_K-stable complete flag whose successive characters, starting with the subline, are $\kappa_0, \dots, \kappa_{n-1}$.

Outline. Reciprocity as in 39.6 computes the rank-one parameters, and distinct inertia characters split the Weil module into lines. After twisting by the inverse of the first character, weak admissibility gives a trivial subline and induction gives the ordered flag. The complete proof is on p. 804. □

Lemma 30.22 (The ordinary projector and finite-free module)**.** *Let L/F be CM, with $F = L^+$, unramified at finite places. Let ℓ be odd and $n \geq 1$, and choose a prolongation at each finite place. Fix a distinguished place v_0 of norm an odd prime $p \neq \ell$ with $p \nmid D_L$. Let the finite set of tame type places be disjoint from the places over ℓ and omit v_0. Assume that the places over ℓ, the tame type places and v_0 split in L, and let S be their union. Fix a coefficient DVR O, residual eigenvalues at the good spherical and ordinary torus operators, and ordered residual local characters ψ_{vi}, $0 \leq i < n$, at each $v \mid \ell$.*

In the definite unitary group over F, let the away level K have principal congruence level one at v_0, spherical level outside S, and a tame type τ on a free coefficient lattice O^b given at each tame type place by the block-parahoric twisted-spherical type (constructed in the proof below) that selects precisely the irreducible smooth representations whose Weil–Deligne parameter has zero monodromy and diagonal inertia given by the Teichmüller lifts of the ordered residual characters. At each $v \mid \ell$, let the i-th compact diagonal coordinate, $0 \leq i < n$, have character $\theta = (\psi_{vi}\varepsilon_\ell^i)^{-1}$ on the reciprocity lift over the weight ring $A = \Lambda = O[[X_1, \dots, X_{n[F:\mathbf{Q}]}]]$, where each variable is a universal pro-ℓ inertia value times a fixed unit, minus one.

More generally, let A' be a complete Noetherian local O-algebra with finite residue field and let θ' be a continuous character of the compact diagonal torus at ℓ with values in $(A')^\times$. Consider $(A')^b$-valued functions invariant on the left by rational points, transforming on the right by τ^{-1} on K and $(\theta')^{-1}$ on the compact diagonal torus, invariant under the upper integral unipotents N_+ at ℓ, and fixed modulo each precision by some lower unipotent congruence subgroup. For $z = (\mathrm{diag}(\varpi_v^{n-1}, \dots, 1))_{v\mid\ell}$ put

$$U_z f(g) = \sum_{u \in N_+/(zN_+z^{-1})} f(guz),$$

and let $Y(A', \theta')$ consist of the functions fixed modulo each precision by a positive power of U_z, writing $Y = Y(A, \theta)$.

1. *Stabilizers at level K act trivially on the coefficient fibres. Factorial powers of U_z converge modulo every precision to a continuous A'-linear ordinary projector e onto $Y(A', \theta')$.*
2. *Let $i : Y(A', \theta') \to (A')^t$ evaluate at double-coset representatives at precision one, with coefficient coordinates included, and let $\sigma : (A')^t \to Y(A', \theta')$ extend values by right translation and apply e. Write V_j for the functions with all values in $\mathfrak{m}_{A'}^j (A')^b$. Then $Y(A', \theta') \cap V_j = \mathfrak{m}_{A'}^j Y(A', \theta')$ for all $j \geq 0$, σi is congruent to the identity modulo $\mathfrak{m}_{A'} Y(A', \theta')$ and is an automorphism, and $Y(A', \theta')$ is finite free over A'.*
3. *Let T_r run through the good spherical operators and the ordinary torus operators given at ℓ by putting ϖ_v on the first j diagonal entries, $1 \leq j \leq n$. For lifts a_r of the specified residual eigenvalues, the submodule*

$$M(K) = \{f \in Y : \ \forall r, a \ \exists h \geq 1, \ (T_r - a_r)^h f \in \mathfrak{m}_A^a Y\}$$

is a finite free A-module and an A-direct summand of Y.

Outline. Principal level one at v_0 trivializes fibre stabilizers, and factorial powers of U_z converge to the projector e. Precision-one evaluation and Nakayama's lemma make $Y(A', \theta')$ finite free, and lifted Hecke idempotents split off $M(K)$. The complete proof is on p. 805. □

Proposition 30.23 (The ordinary module at the three auxiliary levels)**.** *Assume $\mathscr{H}$. Let L/F be CM, with $F = L^+$, unramified at finite places. Let ℓ be odd and $n \geq 1$, and choose a prolongation at each finite place.*

Fix a distinguished place v_0 of norm an odd prime $p \neq \ell$ with $p \nmid D_L$. Let the finite set of tame type places be disjoint from the places over ℓ and omit v_0. Assume that the places over ℓ, the tame type places and v_0 split in L. Let S be their union and use the level group with $S' = S$. Fix a coefficient DVR O, pairwise distinct residual global characters with polarised product determinant, ordered residual local characters, and prescribed lifts on the ℓ-primary torsion of the local abelianisations. Suppose an ordinary determinant pair with these data exists over a complete Noetherian local coefficient algebra with residue field $k = O/\mathfrak{m}_O$.

The ordinary determinant pairs have a universal ring B. In the definite unitary level model an ordinary Hecke module is finite free over the weight ring and has B-annihilator J. The modules at the spherical and two Iwahori levels of a split auxiliary system satisfy 30.70(1). For the systems of 30.67, one may take $E = 2n!$ in the trace identity there.

Outline. Builds B from determinant and character deformation rings, acting on $M(K)$ of 30.22 via 26.22 and 30.21 at separating weights. Sheet splitting, root-algebra nilpotence, and 30.20 yield auxiliary-level freeness and trace identities. The complete proof is on p. 807. □

Lemma 30.24 (Control at an arithmetic weight)**.** *Use the definite group, tame coefficient lattice and level K of Lemma 30.22. Let A be its weight ring, θ its compact diagonal character and Y its ordinary module before localization at the residual Hecke character. Let $x : A \to \overline{\mathbf{Q}}_\ell$ be an arithmetic specialization. Write*

$$x\theta = \chi_\lambda \alpha$$

on the compact diagonal torus, where χ_λ is the lowest-weight character of the algebraic coefficient representation W_λ, and α has finite order. At each embedding, if $h_0 < \cdots < h_{n-1}$ are the ordinary weights, the exponents of χ_λ, in the order of the flag, are $h_i - i$.

At sufficiently deep Iwahori level at ℓ, the classical ordinary space of weight W_λ, tame type and level K, and right diagonal character α^{-1}, is Hecke-equivariantly isomorphic to $Y \otimes_{A,x} \overline{\mathbf{Q}}_\ell$. On the classical side the operator of a positive diagonal element a is its double-coset operator divided by $\chi_\lambda(a)$. The isomorphism restricts to the residual summand $M(K)$ of Lemma 30.22.

Outline. Identifies $O' \otimes_A Y \simeq Y'$ via 30.22 and Nakayama's lemma. Pairing with the lowest-weight functional p_0 intertwines diagonal operators, giving injectivity by U_z-divisibility and surjectivity by lifting modulo ϖ. The complete proof is on p. 809. □

Lemma 30.25 (Irreducible arithmetic points of the Hecke quotient)**.** *Assume $\mathscr{H}$. In a definite unitary ordinary model of rank $n \geq 1$ over a CM field L with $S' = S$, let J be the Hecke ideal in the universal ordinary determinant ring B. Fix ι. Let $x : B \to C_\ell$ kill J and have locally algebraic ordinary characters of strictly increasing weights at each embedding. Write D_x for the universal determinant law with its coefficients specialised by x. Let $p : G_L \to \Gamma_0$ be the quotient map to the CM subgroup of the level group. An irreducible representation r of G_L with determinant law $p^* D_x$ has a regular algebraic conjugate self-dual cuspidal attachment.*

Suppose r is continuous. Every unlinked place over ℓ is potentially unlinked. For every regular algebraic cuspidal attachment π there is $t \in \mathbf{R}$ such that at every finite place v of L away from the distinguished place of L^+, with $v \nmid \ell$ or r potentially unlinked at v, the cuspidal support of π_v is toral and its characters ξ satisfy

$$q_v^{t-1/2} < |\xi(\varpi_v)| < q_v^{t+1/2}. \tag{30.8}$$

Here q_v is the residue cardinality and ϖ_v a uniformiser.

Outline. Classical control 30.24, base change 26.22 and Galois attachment Theorem 39.2 identify the cuspidal factor by Frobenius density. Read toral characters from Jacquet eigenvalues and use 39.55 and 40.11 to test their support and ratios. The complete proof is on p. 811. □

Lemma 30.26 (Coordinates for the lifting conditions)**.** *Let F be a field, T a finite-dimensional F-space, $g, D \geq 0$, and $U \subset T^g$ a subspace of codimension at most g. Write $\iota_j : T \to T^g$ for inclusion in slot j and $W = \bigcap_j \iota_j^{-1} U$. If $\dim W + g^2 \leq D$, there are D forms X_b spanning $T^\vee$ and g^2 independent vectors $c_{ij} \in F^D$ with*

$$\sum_{j,b} c_{ij,b} X_b(a_j) = 0 \qquad (a \in U,\ 1 \leq i \leq g).$$

Proof. Pad a basis of $U^{\perp}$ with zeros to obtain g spanning forms L_i. The forms $L_i\iota_j$ have common kernel W. Put these forms, with their repetitions and zeros, in the first g^2 slots of X_b. Fill the other slots with a basis of a complement of their span and zeros. Take c_{ij} to be the unit vector of its assigned slot. The row equation is $L_i(a) = 0$. □

Lemma 30.27 (Dimension at an auxiliary-level augmentation)**.** *Let $L \to B$ be a local map of complete Noetherian local rings, with L regular of dimension d_0. Let $M \neq 0$ be a finite B-module, let $y_1, \dots, y_h \in \mathfrak{m}_B$, and write $Y = (y_1, \dots, y_h)$. Suppose M/YM is finite free over L and multiplication gives an isomorphism*

$$(M/YM)[T_1, \dots, T_h] \longrightarrow \bigoplus_{a \geq 0} Y^a M/Y^{a+1}M, \qquad T_i \longmapsto [y_i].$$

For each minimal prime Z of $\operatorname{Supp}_B M$ one has $\dim B/Z = d_0 + h$.

Outline. Lift an augmentation basis using the graded isomorphism to make M free over a power series ring over L. Each minimal support prime contracts to zero there, so integrality gives the dimension. The complete proof is on p. 813. □

Lemma 30.28 (The component with a nonzero kernel vector)**.** *Let $(P, \mathfrak{m})$ be a regular local ring. Let $g \geq 1$, let $Z = (z_{ij})$ be a g-by-g matrix and $u = (u_i)$ a column over $\mathfrak{m}$. Suppose their $g^2 + g$ entries have linearly independent images in $\mathfrak{m}/\mathfrak{m}^2$. The radical of the ideal generated by the entries of Zu and by $\det Z$ is prime. If a prime P_0 contains (Zu), omits some u_i, and satisfies $\dim P/P_0 \geq \dim P - g$, then P_0 is contained in every prime over $(Zu, \det Z)$.*

Outline. The projective incidence scheme of kernel lines containing u is regular with connected closed fibre, so its image is irreducible. The adjugate puts $\det Z$ in P_0, and the cut $Z = \operatorname{diag}(u)$ gives the dimension bound forcing equality. The complete proof is on p. 813. □

Lemma 30.29 (Comparison of two ordinary flags across a square-zero ideal)**.** *Let R be a finite local algebra over an infinite field F, with residue field F, and let $I \subset R$ satisfy $I^2 = 0$. Let D be a group. For $b = 0, 1$, let $r_b : D \to \mathrm{GL}_n(R)$ and let $\chi_{b,i} : D \to R^{\times}$, $0 \leq i < n$, be characters. Suppose the representations and their respective characters agree modulo I, and the residual characters are pairwise distinct. Suppose r_0 is upper triangular in a basis w, with ordered diagonal $\chi_{0,i}$, and*

$$\prod_{i=0}^{n-1} \big(r_1(g_i) - \chi_{1,i}(g_i)\big) = 0 \qquad (g_0, \dots, g_{n-1} \in D),$$

with the factors written in increasing order. Put

$$Z(g) = r_1(g) r_0(g)^{-1} - 1, \qquad dY(g) = r_0(g) Y r_0(g)^{-1} - Y.$$

There is $Y \in M_n(I)$ such that $Z - dY$ is upper triangular in the basis w and has i-th diagonal entry $\chi_{1,i}/\chi_{0,i} - 1$. If an F-subalgebra of $M_n(R)$ contains the representation matrices and the scalar matrices of the character values for both systems, Y can be taken in that subalgebra.

Outline. Use Lagrange projectors for a group-algebra element separating the residual characters, and use the ordered products to recover both flags. Conjugate by the sum of matching projector products and expand using $I^2 = 0$. The complete proof is on p. 813. □

Lemma 30.30 (The patched ordinary module over the auxiliary tower)**.** *Assume $\mathscr{H}$. Fix the definite ordinary unitary setting of Proposition 30.23 over a CM field L/L^+ with $d_+ = [L^+ : \mathbf{Q}]$, odd ℓ, $n \geq 1$, $S' = S$, and a coefficient DVR O containing the images of $\mathcal{O}_L$ under all ℓ-adic embeddings. Fix a residual datum, its universal ordinary determinant ring B_0, and the base ordinary module M_0 of Proposition 30.23 localised at the residual system. Let $s_1, \dots, s_q$, $q \geq 0$, lie in the CM subgroup of the base level group, with characteristic polynomials split over the residue field of O. Take a sequence of split auxiliary systems indexed by N as in Theorem 30.67, with auxiliary places $w_{N,1}, \dots, w_{N,q}$ of norms $Q_j = \mathrm{N}w_{N,j} \equiv 1 \pmod{\ell^N}$, tame generators $\tau_{N,j}$, and inverse Frobenii $\phi_{N,j}$ converging to s_j in the finite quotients of the base group.*

Divide each auxiliary level group by the kernel of the maximal pro-ℓ quotient of the inverse image of the open base subgroup, take compatible surjections $\pi_N:\Phi\to\Gamma_N$ from one finitely generated profinite group Φ, fix a nonprincipal ultrafilter on the levels N, and let $\Gamma_\infty=\Phi/K$ be the limit quotient defined by the ultrafilter, with limiting commuting elements τ_j,ϕ_j mapping to $1,s_j$ at base level. Adjoin a power series variable s, let D be the finite free ordered-root splitting algebra over $B_0[[s]]$ for the characteristic polynomials at $s_1,\dots,s_q$, and let $e_{\mathrm{ord}}\in D$ be the idempotent selecting the local factor with the prescribed ordered residual roots lifted by $\widetilde{u}_{ji}$. Put $B^{\mathrm{sp}}=e_{\mathrm{ord}}D$, a nonzero finite flat local $B_0[[s]]$-algebra, and put $d=nd_+$, $\Lambda=O[[s,w_1,\dots,w_d]]$, and $S=\Lambda[[y_{ji}]]$ in the weight parameters w_b and nq variables y_{ji}. Let R be the complete Noetherian local O-algebra for the continuous determinant on the CM subgroup of Φ, the ordered local characters with their prescribed torsion values, and the independent variables $s,U_{ji}-\widetilde{u}_{ji},y_{ji}$, with augmentation $\pi:R\to B^{\mathrm{sp}}$ sending y_{ji} to zero. Let $\mathfrak{a}\subset R$ be generated by the polarisation relations and the traces, after left multiplication by group elements, of $[k]-1$ for $k\in K$, the ordered ordinary products, the prescribed inertia differences, and

$$\prod_i([\phi_j]-U_{ji}),\qquad \prod_{b\neq i}([\phi_j]-U_{jb})([\tau_j]-1-y_{ji})$$

for all lifts to Φ. Put $Y=(y_{ji})$, $\mathfrak{l}=\mathfrak{m}_\Lambda$, $X=B^{\mathrm{sp}}\otimes_{B_0[[s]]}M_0[[s]]$ with Y acting as zero, where $M_0[[s]]=M_0\otimes_{\Lambda_0}\Lambda$ is the module of power series in s with coefficients in M_0, for the weight ring $\Lambda_0=O[[w_1,\dots,w_d]]$ of M_0, and $B_0[[s]]$ acts by multiplication of power series, and

$$f=\prod_{j=1}^{q}\prod_{i\neq b}(U_{ji}-U_{jb})^2\in R.$$

At each level N, with $a_j=v_\ell(Q_j-1)$ and $\Delta_N=\prod_{j,i}\mathbf{Z}/\ell^{a_j}$, let M_N be the ordinary residual Hecke and ordered-root summand of the deepest auxiliary module of Proposition 30.23 on which every $U_{ji}-\widetilde{u}_{ji}$ is topologically nilpotent, after scalar extension by s.

1. *Each M_N is free over $\Lambda[\Delta_N]=S/((1+y_{ji})^{\ell^{a_j}}-1:j,i)$. For fixed $b,j\geq1$, the quotient $M_N/(Y^b+\mathfrak{l}^j)M_N$ is free over $S/(Y^b+\mathfrak{l}^j)$ at all sufficiently high N.*
2. *For each $j\geq1$, on a set of levels N in the ultrafilter there are R-linear maps between $M_N/(Y+\mathfrak{l}^j)M_N$ and $X/\mathfrak{l}^jX$ whose composites in both orders are multiplication by f.*
3. *For $b\geq1$ set*

$$L_b=\prod_N M_N/\{(v_N):\ \forall j,\ v_N\in(Y^b+\mathfrak{l}^j)M_N\ \text{on a set in the ultrafilter}\},$$

$\mathcal{B}=\widehat{R[1/f]}^{\,Y}$, and $\mathcal{M}=\varprojlim_b L_b[1/f]$. Then $\mathcal{M}$ is a finite $\mathcal{B}$-module satisfying

$$\mathcal{M}/Y\mathcal{M}\simeq\mathcal{B}\otimes_R X,\qquad \mathrm{gr}_Y\,\mathcal{M}\simeq(\mathcal{M}/Y\mathcal{M})[y_{ji}],$$

and $\mathfrak{a}\mathcal{B}\subseteq\sqrt{\mathrm{Ann}_{\mathcal{B}}\,\mathcal{M}}$.

Outline. Derives freeness of M_N from 30.23 and compares $M_N/(y)M_N$ with X via 30.20. Ultrafilter quotients L_b, inversion of f, and inverse limits produce $\mathcal{M}$, its Y-graded isomorphism, and $\mathfrak{a}$-annihilation. The complete proof is on p. 814. □

Definite ordinary unitary curves and paired cohomology setup. Assume $\mathscr{H}$. Fix the definite ordinary unitary setting over L/L^+ with ℓ odd, $n\geq1$, $S'=S$, $\mu_\ell\notin L$, $d_+=[L^+:\mathbf{Q}]$, a coefficient DVR O containing the images of $\mathcal{O}_L$ under all ℓ-adic embeddings, a residual datum, its universal ordinary determinant ring B_0, the weight algebra, and ι. Let H be the CM subgroup of the level group, c its chosen complex conjugation, and $H'=H\cap\ker\varepsilon_\ell$ (using the full ℓ-adic cyclotomic character). A *curve* consists of a finite extension k of $O/\mathfrak{m}_O$, a t-adically continuous $\rho:H\to\mathrm{GL}_n(k[[t]])$, and an invertible symmetric polarising matrix P over $k((t))$ with $\rho(s)^tP\rho(csc^{-1})=\varepsilon_\ell(s)^{1-n}P$, realising the prescribed residual determinant and killing the prescribed inertia kernels, together with an ordinary flag over $k((t))$ at each selected $v\mid\ell$ with characters $\psi_{v,i}\in k[[t]]^\times$ satisfying $\psi_{v,i}\neq\psi_{v,j}$ $(i\neq j)$ and $\psi_{v,i}\neq\varepsilon_\ell\psi_{v,j}$ $(i<j)$, inducing $f:B_0\to k[[t]]$,

such that every open subset of H meeting H' contains an $s \in H'$ with separable characteristic polynomial, and $\{X \in M_n(k((t))) : \varepsilon_\ell(s)\rho(s)X = X\rho(s)\ \forall s \in D_0\} = 0$. Over $F = k((t))$ ($A = k[[t]]$, $A_m = A/(t^m)$), we consider either one absolutely irreducible block on H' (setting $g = 0$) or a direct sum of two nonzero polarised blocks of multiplier ε_ℓ^{1-n}, absolutely irreducible and inequivalent on H', in which case g is the F-dimension of ordinary odd adjoint classes, with an integral basis $\kappa_1, \dots, \kappa_g$. In the commutative slot algebra with basis $1, e_i, e_i', \eta$ ($e_ie_j' = \delta_{ij}\eta$, other positive products zero), put $\rho_1 = (1 + \sum_i e_i\kappa_i)\rho$; let $V_1(X)$ have even entries in X and odd entries in $\sum_i Xe_i$, and let $V_2(X)$ have even entries in $\sum_i Xe_i'$ and odd entries in $X\eta$ (with conjugation by ρ_1). Given q auxiliary tests $s_1, \dots, s_q \in H'$ and the auxiliary levels of 30.67 with common pro-ℓ cover $\Phi \twoheadrightarrow \Gamma_N$ and ultrafilter limit Γ_∞, set $D_* = n(d_+ + q)$; at auxiliary level N and depth m, W_m denotes the ordinary $V_1(A_m)$-cocycles (with full local condition at added places) modulo skew-transpose boundaries, and E_m denotes the degree-two odd obstruction group over A_m for V_2-lifting.

Lemma 30.31 (Uniform finite-level bounds)**.** *In the definite ordinary unitary curve setup above, write $A = k[[t]]$, $F = k((t))$, $A_m = A/t^m$, $Q = \#k$, and $d_+ = [L^+ : \mathbf{Q}]$. Write g for the dimension over F of ordinary odd adjoint classes. A single list of q auxiliary tests can be chosen for the given finite collection of curves. For each curve there are constants C_1, C_2 such that, for every $m \geq 1$, at every sufficiently high level N,*

$$\#W_m \leq Q^{mn(d_+ + q) + C_1}, \qquad \#(E_m/t^eE_m) \leq Q^{ge + C_2} \quad (1 \leq e \leq m).$$

Here the level threshold may depend on m. Both constants are independent of that threshold.

Outline. Choose trace tests detecting twisted dual classes modulo boundaries and realize them at auxiliary places via 30.67. Smith normal form, Euler characteristics and finite duality in 26.27 and 27.2 bound the ordinary classes and obstruction quotients. The complete proof is on p. 816. □

Lemma 30.32 (Ordinary classes on the limit group)**.** *In the definite ordinary unitary curve setup above, with the auxiliary levels of 30.67 and common cover $\Phi \twoheadrightarrow \Gamma_\infty$, the ordinary $V_1(F)$-classes on Γ_∞ have dimension at most $D_* = n(d_+ + q)$. Given $g + 1$ tuples of g bounded ordinary even cocycles, some nonzero F-linear combination of the tuples is the tuple of primed components of a bounded ordinary $V_2(F)$-cocycle. Inflation identifies the ordinary odd cohomology over F on the base and the limit.*

Outline. Transfer cocycles and boundaries through common finite quotients, using 30.31 and Smith normal form to bound classes and lift a nonzero combination through all depths. Odd cocycles vanish on the tame generators of 30.67 by disjoint block spectra and descend. The complete proof is on p. 819. □

Lemma 30.33 (The completed module at a curve prime)**.** *Use $R, \Lambda, \mathfrak{a}, Y, f, X, \mathcal{B}, \mathcal{M}$ of Lemma 30.30. Let p be a prime of R with $Y \subset p$, $f \notin p$, and $\Lambda/(p \cap \Lambda) = k[[t]]$ via $s \mapsto t$ for a finite extension k of the residue field of O. Put $B' = \widehat{R_p}$, and write $M' = B' \otimes_{\mathcal{B}} \mathcal{M}$. Multiplication induces*

$$(B' \otimes_R X)[T_1, \dots, T_{nq}] \ \simeq\ \mathrm{gr}_Y M'.$$

Every minimal support prime after passage to $C = B'/\mathfrak{a}B'$ has quotient dimension $1 + n(d_+ + q)$.

Outline. Uses 30.30 along the ultrafilter to prove the graded isomorphism for $\mathcal{M}$ and transfers it to M' by flat completion at p. Applying 30.27 over $\widehat{\Lambda_{p_\Lambda}}$ and passing to C gives dimension $1 + n(d_+ + q)$. The complete proof is on p. 821. □

Lemma 30.34 (Generalisations of ordinary Hecke points)**.** *In the definite ordinary unitary curve setup above, the Hecke ideal J_0 of 30.25 can be chosen to have the following properties for curves $f : B_0 \to k[[t]]$.*

(a) *If $\rho|_{H'}$ is generically absolutely irreducible and $J_0 \subseteq \ker f$, then every prime $\mathfrak{q} \subseteq \ker f$ contains J_0.*

(b) *Let $f_0, \dots, f_h$, $h \geq 0$, be curves as above. Over the fraction field of each curve require a decomposition into two nonzero representations of H, each polarised with multiplier ε_ℓ^{1-n}, absolutely irreducible and mutually nonisomorphic on H'. A prime $\mathfrak{p}$ is stable if the specialised determinant over $\overline{\kappa(\mathfrak{p})}$ is that of an irreducible representation.*

Use the universal determinant D and ordered characters $\psi_{v,i}^{\mathrm{univ}}$ in $B_0[H]$. Take the Cayley–Hamilton relations given by the coefficients of $\chi_D(z)(z)$ in the independent indeterminates T_j for every $z = \sum_j T_j z_j$, $z_j \in B_0[H]$. Adjoin the ordered products $\prod_{i=1}^n([g_i] - \psi_{v,i}^{\mathrm{univ}}(g_i))$ at the ordinary groups, the differences $[s] - 1$

on the prescribed inertia kernels, and their adjoints under $[s]^{*}=\varepsilon_{\ell}(s)^{1-n}[cs^{-1}c^{-1}]$. *Let* I *be the two-sided ideal so generated and put*

$$I^{\mathrm{cl}}=\bigcap_{b,V}\bigl(I+\mathfrak{m}_{B_0}^{b}B_0[H]+([v]-1:v\in V)\bigr),$$

where $b\geq 1$ *and* V *is open normal in* H. *A nonstable prime* $\mathfrak{p}$ *is thick if* $\overline{\kappa(\mathfrak{p})}[H]/(I^{\mathrm{cl}})$ *is nonsemisimple. Extend the ideal after taking the intersection. If consecutive curves share a thick prime below their kernels and a stable prime below* $\ker f_0$ *contains* J_0, *then every stable or thick prime below* $\ker f_h$ *contains* J_0.

Outline. Bounds auxiliary limit cohomology via 27.2 and presents each completed curve-prime ring using 30.26 and 30.29. Patching 30.23 via 30.30 and 30.27, then comparing minimal primes by 30.28, propagates J_0. The complete proof is on p. 822. □

Lemma 30.35 (A curve avoiding the excluded loci)**.** *Let* R *be a complete Noetherian local ring with finite residue field. Let* I *be an ideal,* $b\geq 1$, *and* $s\in\mathfrak{m}_R$, *with* $\dim R/I\geq b+1$. *Let* A_i,B_j *be two finite lists of ideals. Suppose* $\dim R/A_i<b$ *for every* i, *and* $\dim R/\mathfrak{q}<b$ *for every prime* $\mathfrak{q}\supseteq B_j$ *containing none of the* A_i. *There is a prime* $\mathfrak{p}\supseteq I+(s)$ *with* $\dim R/\mathfrak{p}=1$ *containing none of the listed ideals. If* $R/(s)$ *has characteristic* $\ell>0$, *there is a local map* $R\to k'[[t]]$ *with kernel* $\mathfrak{p}$, *where* k' *is a finite field.*

Outline. Enlarge the exclusion list by minimal primes and take a component after cutting by s. Repeated height-one cuts keep a product of elements from the excluded ideals nonzero, and in equal characteristic the final curve normalizes to $k'[[t]]$. The complete proof is on p. 829. □

Definition 30.36 (Four-dimensional frame cohomology)**.** The *four-dimensional frame cohomology* hypothesis is the conjunction of the following Galois cohomology inputs for finite discrete ℓ-primary modules (with ℓ odd): local finiteness, local Tate duality, the local Euler characteristic formula, unramified orthogonality, normalized Poitou–Tate duality (27.2), global finiteness, the global Euler characteristic formula, and the Greenberg–Wiles formula ([111, Ch. I, Theorems 2.1, 2.6, 2.8, 4.10, 5.1 and Corollaries 2.3, 4.15], [119, Proposition 1.6]). The hypothesis also includes the local Bloch–Kato dimension formula and rational and integral duality of 26.27(5). These inputs are contained in the standing input package $\mathscr{H}$ of 26.32 and 27.2.

Lemma 30.37 (Presentation bound for the four-dimensional framed ring)**.** *Assume Definition 30.36. Let* T *be a fitted deep tower of case II or III with totally real abelian field* F *of degree* $d=[F:\mathbf{Q}]$ *and* s *places above* ℓ, *let* $\Lambda\subset\overline{\mathbf{Q}}_{\ell}$ *be a coefficient DVR finite over* $\mathbf{Z}_{\ell}$ *with residue field* $k_{\Lambda}=\Lambda/\mathfrak{m}_{\Lambda}$, *let* A *be the matrix ring of* T *over* Λ, *and let* $x:A\to k_{\Lambda}$ *be a residual point satisfying the prescribed pairings, reduction, continuity, and either of the residual support shapes of Lemma 30.38. Then the completed framed matrix ring* R *at* x *admits a presentation over* Λ *with* t *variables and* r *relations satisfying*

$$t-r\geq 2d+2-2s.$$

Outline. Verify $\mathrm{Hom}_{\Gamma}(\mathfrak{sp}_4(k'),k'(\bar{\varepsilon}_{\ell}))=0$ and apply the cone Euler characteristic of Definition 30.36 to get $t_0-r_0\geq 2d+2$ for R_0. Fixing the local ℓ-primary flag characters adds at most $2s$ relations. The complete proof is on p. 829. □

Lemma 30.38 (Minimal components through a residual point)**.** *Assume the four-dimensional frame cohomology of Definition 30.36. Let* T *be a fitted deep tower of case II or III and* F *its totally real abelian field. Assume* $\vartheta|_{G_F}$ *has order prime to* ℓ. *Choose a coefficient DVR* $O\subseteq\Lambda\subset\overline{\mathbf{Q}}_{\ell}$, *finite over* $\mathbf{Z}_{\ell}$, *and write* $k_{\Lambda}=\Lambda/\mathfrak{m}_{\Lambda}$. *Let* $j:B\to A$ *be the coefficient and matrix rings of the tower over* Λ, *with their structure map. Let* $J\subset B$ *be a potential modularity ideal for the tower with the following conditions. For some* $e\geq 1$,

$$\mathfrak{m}_B^{e}\subseteq J+\mathfrak{m}_{\Lambda}B+(\psi_{v,i}(s)-1:v\mid\ell,\ s\in I_v,\ 1\leq i\leq 4). \tag{30.9}$$

Here $\psi_{v,i}$ *are the universal flag characters. Put* $U_v=I_v\cap\ker\varepsilon_{\ell}\cap\ker\bar{r}_D$, *with* $\bar{r}_D$ *the residual diagonal representation. For every* Λ*-map* $y:A\to k'[[t]]$, k' *finite, with* $yj(J)=0$, *constant term killing* $\mathfrak{m}_B$, *full generic matrix span and pairwise separation of its flag characters on each* U_v, *require every prime of* B *below* $\ker(yj)$ *to contain* J.

Put $d = [F : \mathbf{Q}]$ and let s be the number of local flags. Suppose s, the strict root cocycle dimensions and the global cocycle dimensions of weights $0, \pm\psi$ are at most C. For each odd root suppose the global and product-local cocycle dimensions are at most $d + C$. Assume $d \geq 30(C+1)^2$.

Take a residual point $x : A \to k_\Lambda$ satisfying the prescribed pairings, reduction and continuity. Use basis weights $(1,1),(0,1),(1,0),(0,0)$, and put $\alpha = (1,0)$, $\beta = (-1,1)$, $\psi = (0,1)$, $\theta = (1,1)$. Denote by $E(w)$ the presence of an off-diagonal representation entry of weight w. The local flag matrices n_v have ordered characters $\bar{\chi}_i|_{G_v}$. The representation support is contained in $\{\beta, -\alpha, -\theta, -\psi\}$ with flag support in $\{-\alpha, -\psi, -\theta\}$, or the representation support is contained in $\{-\alpha, -\beta, -\psi, -\theta\}$.

Let V mean support in $\{\beta, -\alpha, -\theta\}$, flag support in $\{-\alpha, -\psi, -\theta\}$ and $(n_v)_{20} \neq 0$ for every flag. Let H mean support in $\{-\beta, -\psi, -\theta\}$ and $(n_v)_{10} \neq 0$ for every flag. Assume one of

$$E(-\psi) \wedge (E(\beta) \vee E(-\alpha)), \qquad E(-\alpha) \wedge E(-\beta), \qquad V \wedge E(-\alpha),$$
$$V \wedge E(\beta) \wedge E(-\theta) \text{ with } \bar{\chi}_0/\bar{\chi}_1 \neq 1 \text{ on every } G_v, \qquad H \wedge (E(-\psi) \vee (E(-\beta) \wedge E(-\theta))).$$

If one minimal prime below $\ker x$ contains JA, every such prime does.

Outline. In the completed local ring R at x, Lemma 30.37 and connectedness give adjacent minimal primes P, Q with $\dim R/(P+Q) \geq b+1$. Bounding reducible and coincidence loci below b, apply 30.35 to build a curve propagating JR from P to Q. The complete proof is on p. 831. □

Lemma 30.39 (The reducible supports at the five residual points)**.** *Use the hypotheses, basis and root notation of 30.38. In the completed matrix ring at x, every reducible support locus has dimension at most $1 + d + 20C$, with the possible exception, in the fifth alternative, of the support contained in $\{\pm\beta, \pm\theta\}$. That exception forces equality of two flag characters on a whole local decomposition group.*

Outline. Use marked projectors to list proper supports and diagonal degeneration of the cases in 30.38 to bound their tangent spaces by cocycle counts. In the exceptional support both blocks share the first flag eigenvalue, forcing a character equality. The complete proof is on p. 831. □

Lemma 30.40 (Dimension at a coincidence of flag characters)**.** *In 30.38, let q be a prime of B containing J and the uniformiser. If $\psi_{v,i} = \psi_{v,j} \bmod q$ on U_v for some $i \neq j$, then*

$$\dim B/q \leq 2d + 1 - d_v, \qquad d_v = d/s.$$

Outside the small reducible loci of 30.39, a prime in the completed matrix ring above q has quotient dimension at most $\dim B/q + 2 + 4s$.

Outline. Use inertia generators and one cyclotomic value to count parameters after two flag characters coincide. Outside the small supports of 30.39, traces bound the matrix parameters and lower unipotent coordinates bound the flag parameters. The complete proof is on p. 833. □

Lemma 30.41 (A cuspidal Eisenstein congruence at an auxiliary prime)**.** *Fix an odd prime ℓ and $\iota : C_\ell \simeq \mathbf{C}$. Let ϕ be primitive of conductor $D > 1$ prime to ℓ. Let $w \nmid D$ be prime with $\ell \mid w - 1$ and $\phi(w) = 1$, and let θ have conductor w and order ℓ. Put $H = Dw$ and $\chi = \phi\theta$. For every sufficiently large h with $(-1)^h = \phi(-1)$ there is a normalised cuspidal eigenform of weight h, level $\Gamma_1(H)$ and character χ satisfying*

$$a_p \equiv 1 + \chi(p)p^{h-1} \pmod{\mathfrak{m}_{C_\ell}} \qquad (p \text{ prime}).$$

Congruences are transported by ι^{-1}. The eigenform can be chosen so that conjugating its Fourier coefficients gives cusp forms of the conjugate characters.

Outline. Average cusp Eisenstein series by the primitive character and use the geometric series for (A.270). Subtract an integral form to remove the constants, then lift the residual divisor-sum eigenvector by Deligne–Serre and normalise. The complete proof is on p. 834. □

Lemma 30.42 (Finiteness from valuation points)**.** *Let O be a complete DVR with finite residue field and uniformiser ϖ. Let R be a complete local O-algebra presented by finitely many coordinates $z_i \in \mathfrak{m}_R$ and homogeneous relations, with the coordinates assigned weights in a free abelian group and O of weight zero.*

Suppose that every local O-map $R \to A$ to a complete DVR with finite residue field that kills ϖ also kills every positive-degree monomial of weight zero. Then the weight-zero part of R is a finite O-module. For integer weights, the monomial condition holds if at every such point all zero-weight coordinates vanish and all positive-weight coordinates or all negative-weight coordinates vanish.

Outline. A weight-zero monomial generator outside the uniformiser radical survives in a normalised curve quotient, contrary to the DVR hypothesis. Weight-zero projection gives nilpotence modulo the uniformiser, and successive approximation gives O-finiteness. The complete proof is on p. 835. □

Lemma 30.43 (Traces at two ordinary places)**.** *Let F be real quadratic, and let the odd prime ℓ split as vv' in F. Fix an identification $\overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$ and the resulting assignment of the two real embeddings to v, v'. Fix a finite-order central character ω. At a finite place w, write O_w for the local integers and take the subgroup of $\mathrm{GL}_2(O_w)$ with bottom row congruent to $(0,1)$ modulo $\mathfrak{p}_w^{c_w}$. Assume the nonnegative integers c_w have finite support, contain the conductor of ω with its multiplicities, satisfy $c_v = c_{v'} = 1$, and satisfy $c_u > 0$ at some place u of norm two. Let $h \geq 4$, with $\omega_\tau(-1) = (-1)^h$ at both real embeddings.*

For $t \geq 0$, use the holomorphic cusp space of weight $(h + 2t, h)$, central character ω, and the indicated finite level. At w, let D_w be the sum of right translates for the double coset of $\mathrm{diag}(\varpi_w, 1)$, counting right cosets once. Define

$$T_w(t) = (\mathrm{N}w)^{h/2-1} \ell^{t\mathbf{1}_{w=v}} D_w.$$

For a word W in finite places, let $T_W(t)$ be the ordered product and let b_w be the number of occurrences of w. If $r \geq 1$, $b_v, b_{v'} \geq r$, $N \geq 0$, and $(\ell-1)\ell^N \mid t$, both traces are ℓ-integral and

$$\operatorname{tr} T_W(t) - \operatorname{tr} T_W(0) \in \ell^{\min(N,r)} \mathcal{O}_{\overline{\mathbf{Q}}_\ell}.$$

Suppose the parallel space has dimension $N > 0$ and contains a joint eigenform with integral normalized eigenvalues s_w, where $s_v, s_{v'}$ are units. Let $B \geq 1$ be an integer and put $t = (\ell-1)\ell^N B$. Then the space of weight $(h + 2t, h)$ contains a joint eigenform whose eigenvalues for the operators $T_w(t)$ are integral and reduce to s_w at every finite place w.

Outline. Jacquet–Langlands expresses both traces by the same finite coset sum on a definite quaternion algebra with ℓ-integral stabilizer denominators. Hensel's lemma separates unit roots at v, v', and the binomial theorem gives integrality and the trace congruence. Triangularization shows ordinary systems are integral, and a high ℓ-power of a separating Hecke polynomial yields a congruent eigenform. The complete proof is on p. 836. □

Lemma 30.44 (The marked integral basis)**.** *Let D be a good datum of case II or III. Start with an integral representation r reducing to the diagonal of Definition 29.17, with the prescribed similitude and inertia conditions, and an integral symplectic ordinary flag at ℓ whose quotient characters have the prescribed reductions and inertia restrictions. Use a complete discrete valuation ring with finite residue field, allowing finite extensions. These data exist for every characteristic-zero point of Definition 29.18 and for the valuation specialisations of an allowed lift used in the finiteness argument.*

Write $A = \langle e_0, e_2 \rangle$, $B = \langle e_1, e_3 \rangle$. Suppose the generic representation is absolutely irreducible, or preserves B, has full matrix span on B, and has no invariant complement to B over the algebraic closure. One can choose the order of the residual labels before choosing the initial tower. For its separating marking t there is an integral symplectic basis in which $r(t)$ is diagonal, the reduction of r is diagonal, and the flag has a lower unitriangular symplectic representative n. Every off-diagonal entry of n of weight different from $-\psi$ belongs to the maximal ideal. In this basis the generic representation has entries of both weights $\psi, -\psi$ and of at least one weight in $\{\beta, -\alpha, -\theta\}$.

Outline. Specialize the flag of 29.18 by properness and order the labels using 29.17. Lift the marking's spectral projectors and factor the flag matrix using its unit minors, then use full span or nonsplitting to find the required entries. The complete proof is on p. 838. □

Lemma 30.45 (Choice of lattice over the rational base)**.** *Use the basis and marking of Lemma 30.44. After a finite extension and diagonal conjugation by a symplectic similitude, write $R(g)$ and N for the reductions*

of the representation and its lower unitriangular flag. Write d_i for the residual characters in the chosen order. Then $R(t)$ is diagonal. The representation support lies in

$$S_P = \{\beta, -\alpha, -\theta, -\psi\},$$

the flag support lies in $S_f = \{-\alpha, -\psi, -\theta\}$, and $E(\beta)$ holds. Either $E(-\psi)$ holds, or the following conditions hold together:

$$\operatorname{supp} R \subseteq \{\beta, -\alpha, -\theta\}, \qquad N_{20} \neq 0, \qquad d_0|_{D_\ell} = d_2|_{D_\ell},$$
$$\nexists\, m \quad R_{12}(s) = m\big(d_2(s) - d_1(s)\big) \quad \text{for every } s \in D_\ell.$$

Here m ranges over the residue field, also after extending it.

Outline. Maximize the two diagonal scaling exponents in order under the valuation constraints from 30.44. The tight inequalities and the coboundary condition in 29.17, applied to the flag equations, force the stated support and exclude the local scalar. The complete proof is on p. 839. □

Lemma 30.46 (The first point on the tower)**.** *Restrict the lattice of Lemma 30.45 to the totally real field F of the initial tower, or to a fitted enlargement. Transport its ordinary flag to the chosen prolongations over ℓ. Its reduction is a residual point of the tower and satisfies one of the following alternatives of Lemma 30.38:*

$$E(-\psi) \wedge E(\beta), \qquad V \wedge E(-\alpha).$$

Outline. Transport the flag of 30.45 by triangular factorization and compute its $(2,0)$ entry. The separating marking keeps the relevant cocycles nonzero on restriction, and the excluded local coboundary forces an entry of weight $-\alpha$ in the second case. The complete proof is on p. 840. □

Lemma 30.47 (The nonsplit companion of a block point)**.** *Let an integral pair as in Lemma 30.44 be block diagonal on $A \oplus B$, with full generic matrix span on A. There is an integral pair with the same characteristic polynomials at every group element and the same ordinary inertia characters, preserving B, having full generic matrix span on B, and having no invariant complement to B.*

Outline. Finite Selmer counts and compactness give a nonzero integral cocycle in symplectic maps from A to B with the local flag condition. Twist in the square-zero subgroup and use the normalization at complex conjugation to rule out a splitting. The complete proof is on p. 841. □

Lemma 30.48 (Matrix entries in an open coset)**.** *Let G be profinite, and let r be a continuous representation over a finite extension of $k((t))$, where k is finite. Compute matrix spans and Zariski closures over an algebraic closure of the coefficient field. Suppose r is absolutely irreducible of dimension $n \in \{2,3,4\}$. Fix an eigenbasis of $r(u)$ for an element $u \in G$ whose eigenvalue ratios at distinct indices have infinite order. Suppose every open normal subgroup of G has noncommutative image.*

In each open coset there is a matrix M for which all diagonal entries of M and M^{-1} are nonzero, with one possible alternative in dimension four. In that alternative the four lines divide into two pairs, the diagonal two-by-two blocks of M vanish, and every entry of both off-diagonal blocks is nonzero.

Outline. Spectral projectors and irreducibility identify the span of the identity component as a product of full matrix algebras in equal-sized blocks. In each coset, Zariski density in a translate of the component gives simultaneous nonvanishing of the required entries. The complete proof is on p. 842. □

Lemma 30.49 (Rank-two Eisenstein lifts and imaginary quadratic twists)**.** *Assume $\mathscr{H}$. Fix an odd prime ℓ and $\iota : C_\ell \simeq \mathbf{C}$.*

1. *Let ϕ be a primitive Dirichlet character of conductor $D > 1$ prime to ℓ, let $w \nmid \ell D$ be a prime with $\ell \mid w - 1$ and $\phi(w) = 1$, and let θ be a Dirichlet character of conductor w and order ℓ. View Dirichlet characters as $C_\ell^\times$-valued characters of $G_{\mathbf{Q}}$ via ι^{-1}. For every sufficiently large integer $h \geq 2$ with $(-1)^h = \phi(-1)$, put $D_h = \varepsilon_\ell^{h-1}\phi\theta$ and $\Phi_h = \varepsilon_\ell^{h-1}\phi$. Then Lemma 30.41 and Propositions 26.36(3) and 26.37 yield a normalized cuspidal newform of weight h, level Dw and nebentype $\phi\theta$, and a continuous irreducible representation $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(C_\ell)$ of determinant D_h with residual characteristic polynomial $(X-1)(X-\bar{\Phi}_h)$ on $G_{\mathbf{Q}}$, inertia type $(1, \phi\theta)$ and unlinked principal-series or spherical cuspidal support*

away from ℓ, and an ordinary flag at ℓ of inertia weights $(h-1,0)$ and decomposition-group residual characters $(\bar{\Phi}_h, 1)$. If $H \le G_{\mathbf{Q}}$ is open normal and $\bar{\Phi}_h|_H \neq 1$, then $\rho|_H$ is irreducible; if $\bar{\Phi}_h(s)^2 \neq 1$ for some $s \in G_{\mathbf{Q}}$, then ρ is irreducible on every open subgroup of $G_{\mathbf{Q}}$.

2. *Let $K/\mathbf{Q}$ be an imaginary quadratic field of discriminant D_K prime to ℓ, with odd quadratic character η_K. There is an algebraic Hecke character of K of infinity type $(-1,0)$, unramified outside D_K and of prime-to-ℓ inertia exponent at primes dividing D_K, whose restriction to rational ideles is $\eta_K|\cdot|^{-1}$. Its ℓ-adic Galois character $\kappa_0 : G_K \to C_\ell^\times$ via ι is unramified outside ℓD_K, has prime-to-ℓ inertia order away from ℓ, satisfies $\kappa_0(g)\kappa_0(cgc^{-1}) = \varepsilon_\ell(g)^{-1}$ for complex conjugation c and $g \in G_K$, and has rational-idelic transfer $\eta_K \varepsilon_\ell^{-1}$ from $G_{\mathbf{Q}}$.*

Outline. For (1), 30.41, 26.36(3), and 26.37 produce a newform and twisted Galois representation whose local data follow from congruences and inertia eigenlines. For (2), extending an idelic character of type $(-1,0)$ gives κ_0. The complete proof is on p. 842. □

Lemma 30.50 (Twisted intertwiners and line permutations on a curve)**.** *Let A be a valuation domain with residue field k, fraction field F, and a fixed extension of its valuation to $\overline{F}$. Let $m \ge 1$.*

1. *Let D_0 be a group, $\rho : D_0 \to \mathrm{GL}_m(A)$ a representation, and $\chi : D_0 \to A^\times$ a character. Suppose there is $s_0 \in D_0$ whose residual characteristic polynomial $\det(X - \overline{\rho(s_0)})$ has a single root in $\overline{k}$ while $\bar{\chi}(s_0) \neq 1$. Then*
$$\{X \in M_m(F) : \ \chi(s)\rho(s)X = X\rho(s) \text{ for all } s \in D_0\} = 0.$$

2. *Let H be a group and $\rho : H \to \mathrm{GL}_m(A)$ a representation whose residual characteristic polynomial at every $h \in H$ is $\prod_{i=1}^m (X - \bar{\chi}_i(h))$ for characters $\bar{\chi}_1, \dots, \bar{\chi}_m : H \to \overline{k}^\times$. Suppose ρ permutes transitively a decomposition $\overline{F}^m = L_1 \oplus \cdots \oplus L_m$ into one-dimensional subspaces, and let $P \le H$ be the kernel of this permutation action. Then the eigenvalues of $\rho(p)$ on $L_1, \dots, L_m$ for $p \in P$ are valuation units whose reductions all coincide, and $\bar{\chi}_1|_P = \cdots = \bar{\chi}_m|_P$.*

Outline. For (1), distinct residual roots make the characteristic polynomials of $\chi(s_0)\rho(s_0)$ and $\rho(s_0)$ coprime, so Cayley–Hamilton forces $X = 0$. For (2), trivial residual commutators equate reduced line eigenvalues across the transitive orbit. The complete proof is on p. 843. □

Theorem 30.51 (Finiteness and automorphy of symplectic points)**.** *Assume $\mathscr{H}$, take a host h large for t as in 29.1, and let ℓ be odd, $0 < k < t$, $\vartheta \in X_\ell(k)$. For every good four-dimensional datum D of case II or III, the weight-zero algebra of each allowed lift is finite over O. At every characteristic-zero block point r,*
$$\mathrm{span}_{C_\ell}\{r(g)_{\{0,2\},\{0,2\}} : g \in G_{\mathbf{Q}}\} \neq M_2(C_\ell). \tag{30.10}$$
There is one ι for which every irreducible characteristic-zero point has an attachment with witness on each irreducible quadratic CM restriction.

Outline. Build towers and seeds in cases II (30.43) and III (30.49), linking residual points (30.44–30.47) by Selmer cocycles. Propagating 30.34 and 30.38 yields finiteness (30.42), automorphy (30.25), and (30.10). The complete proof is on p. 844. □

Theorem 30.52 (An integer for four-dimensional trace differences)**.** *Assume $\mathscr{H}$. There are $c, A \in \mathbf{N}$ as follows. Let $k \ge 2$, $M \ge 1$, and $p \ge 5$ be prime with $p \nmid M$. Let $z_i^h = 1$ for $z_1, z_2, z_3 \in \mathbf{C}$ and $h \ge 1$. There is $D \in \mathbf{Z} \setminus \{0\}$ with*
$$\log|D| \le c\varphi(h)[M(k+1)]^A \log p. \tag{30.11}$$
For each prime $\ell \neq 2, p$ dividing M, fix ι and an arithmetic Frobenius σ_p at a place above p. Let X be a finite pairwise nonisomorphic set of irreducible $G_{\mathbf{Q}}$-representations satisfying the quadratic automorphy condition at ι, of level $(M; (k+1, k, 1, 0))$ with unlinked flags at ℓ. Require for each $r \in X$ that $r(g)^t P r(g) = \varepsilon_\ell(g)^{k+1}\chi(g)P$ for some invertible P and finite cyclotomic χ. Then $\Delta_r = \mathrm{tr}\, r(\sigma_p) - \iota^{-1}(p^{k+1}z_1 + p^k z_2 + p z_3 + 1) \neq 0$ and
$$\sum_{r \in X} v_\ell(\Delta_r) \le v_\ell(D). \tag{30.12}$$

Outline. Apply 30.9 in rank 4 via Lemma 30.8(b) for a conjugation-stable label set S with bounded integral traces t_s. The Galois-invariant product $D = \prod_{s,a} F_{s,a}$ is a nonzero integer whose size and distinct factors yield both bounds. The complete proof is on p. 856. □

30.4 Three-dimensional endpoints over the Gaussian field

The three-dimensional deformation group over a coefficient ring R is $\mathrm{GL}_3(R) \rtimes \{1,-1\}$ for the polarising involution. On the sign kernel its matrix part is a representation. For a Gaussian host the sign is η_4, the quadratic character of $\mathbf{Q}(i)$. Let $\vartheta_{\ell'}$ be the prime-to-ℓ order part of ϑ, let $c_2(\vartheta)$ be its conductor exponent at 2 before taking that part, and put

$$S_b = \{v \mid N : v \neq \ell,\ \mathrm{comp}_v(\kappa_\ell(\vartheta)) \neq 1\}, \qquad T_b = \{\ell\} \cup S_b,$$

where κ_ℓ is as in 29.1. A Gaussian host has $\ell \neq 2$, $k \geq 1$, $2 \mid N$, $\vartheta(-1) = (-1)^k$, and an integer $o_1 \geq 1$ with

$$\mathrm{comp}_r(\vartheta_{\ell'})^{2o_1} = 1 \quad (r \text{ odd prime}), \qquad 8o_1 v^2 < 2^{c_2(\vartheta)} \quad (v \in T_b), \qquad c_2(\vartheta) \geq 4. \tag{30.13}$$

Write S for the places dividing ℓN, $\Gamma_{\mathbf{Q},S}$ for the Galois quotient unramified outside S, and $\Psi = \varepsilon_\ell^k \vartheta$. The frame uses the ordinary flag at ℓ. At $v \in S_b$ a basis congruent to the identity diagonalises $D_v \cap \ker \eta_4$, with prescribed diagonal on inertia. At the remaining places the frame kills inertia on $\ker \eta_4 \cap \ker \bar{\Psi}$, where a bar denotes reduction in the coefficient residue field.

Lemma 30.53 (Graded finiteness from DVR points)**.** *Let O be a complete DVR with uniformiser ϖ and finite residue field. Let R have a presentation as a quotient of $O[[X_1, \dots, X_s]]$ whose kernel is stable under projection to each weight for an assignment of integer weights to the X_i. Write z_i for their images in the maximal ideal of R and R_0 for the image of the weight-zero series. Suppose that every local O-algebra map $R \to A$ to a complete DVR with finite residue field and $\varpi A = 0$ kills the zero-weight coordinates and kills all coordinates of at least one strict weight sign. Then R_0 is finite over O.*

Outline. Use Dickson's lemma for finite monomial generation. Normalise a one-dimensional quotient and use the DVR sign condition to put each generator in the uniformiser radical, then project to weight zero and lift a residue basis to get O-finiteness. The complete proof is on p. 856. □

Theorem 30.54 (The Gaussian three-dimensional lifting assertion)**.** *Assume the marking, residue-characteristic polynomial identity, and characteristic-zero attachment of 30.61(4). For a Gaussian host and its frame F on $\Gamma_{\mathbf{Q},S}$, and any ι, there is a marking g_1 with*

$$\eta_4(g_1) = 1, \qquad \Psi(g_1)^2 - 1 \in O^\times. \tag{30.14}$$

Let u be a universal lift of the marked frame. Its weight-zero algebra R_0 consists of the elements of its coefficient ring that take equal values at any two specialisations related by conjugation by $\mathrm{diag}(t, 1, t^{-1})$. Here the specialisations take values in the same complete Noetherian local O-algebra with residue field $O/\mathfrak{m}_O$, and t is any unit of that algebra. Then R_0 is finite over O. At a C_ℓ-point of the algebra generated by the weight parts, if the matrix part of the associated r is irreducible on G_K, $K = \mathbf{Q}(i)$, then $(r|_{G_K})_{\mathrm{mat}} \varepsilon_\ell^{-1}$ is conjugate self-dual automorphic with witness.

Outline. Pull back the marking and polynomial from 30.61 and use residue triangularity and 30.53 for a characteristic-zero DVR model. A three-cycle trace Vandermonde test gives open-subgroup irreducibility, so apply 30.61 again. The complete proof is on p. 857. □

Definition 30.55 (Residual linked-frame and seed conditions)**.** Let Φ be a linked frame with group Γ over a finite field k of odd characteristic ℓ. Write η for the sign, Γ_0 for its kernel, $\bar{\Psi}$ for the residual character on Γ, and $\bar{\varepsilon}$ for the residual cyclotomic character. At an ordinary place v write I_v for inertia, $N_v = I_v \cap \ker \varepsilon_\ell$ for the kernel of the full ℓ-adic cyclotomic character on inertia, and d_v for the local degree. The local maps and cyclotomic character are continuous and the finite characters have open kernels. The marking g_0 has $\bar{\Psi}(g_0)^2 \neq 1$. The involutions c, c_w have sign -1, with $\bar{\Psi}(c) = 1$. The infinite-place indices w number $d = \sum_v d_v$. Each open subgroup of Γ has a finite space of continuous $\mathbf{F}_\ell$-characters.

Require $\bar{\varepsilon}, \bar{\varepsilon}\bar{\Psi}^{\pm 1}$ nontrivial on Γ_0, $\bar{\varepsilon}\bar{\Psi}^{\pm 2}\eta$ nontrivial on Γ, and $\bar{\varepsilon}|_{\Gamma_0}$ distinct from every integral power of $\bar{\Psi}|_{\Gamma_0}$. Each local group contains an element with $\bar{\Psi}^2 \neq 1$, and $\bar{\Psi}|_{N_v} = 1$. For each v, require d_v elements of $I_v \cap \ker \bar{\Psi}$, all but one in N_v, whose values determine the restriction to I_v of every continuous character of the local group into a finite abelian ℓ-group. Require one positive exponent killing all such characters of Γ_0 that are trivial on inertia at both lifts of every framed place.

For a finite field k' of characteristic ℓ, let V be a finite-dimensional k'-representation of Γ with open kernel. Let $b_v \subseteq V$ be stable under the local group at v. Write H^i_b for the cohomology of the cone of the restriction of continuous cochains to the local cochains with coefficients V/b_v, shifted by -1. If $\mathrm{Hom}_\Gamma(V, k'(\bar\varepsilon)) = 0$, require H^2_b finite-dimensional and

$$\dim H^2_b + \dim H^0_b + \sum_v d_v \dim b_v \le \dim H^1_b + \sum_w \dim V^{c_w=1}. \tag{30.15}$$

Take $b_v = 0$ for the strict condition. For a character χ, a strict class is a pair consisting of a locally constant cocycle y and scalars b'_v with $y(g) = (\chi(g)-1)b'_v$ on the local groups, modulo the pairs $((\chi-1)b, (b)_v)$. Require the strict groups for $\bar\Psi^{-1}\eta$ and $\bar\Psi^{-2}\eta$ to be nonzero. The local square test makes each b'_v unique for these characters. The bounds with constant C_0 are

$$\dim H^1_{\mathrm{str}}(\Gamma, k(\bar\Psi^{-2}\eta)) \le C_0, \quad \dim H^1_{\mathrm{str}}(\Gamma_0, k(\bar\Psi^{-1})) \le C_0, \quad \dim H^1(\Gamma, k) \le 1 + C_0. \tag{30.16}$$

The strict condition on Γ_0 uses both local lifts.

The seed data are an imaginary quadratic $K_0 \subset L$ for the abelian CM field $L/\mathbf{Q}$, an integer $D' > 0$ prime to ℓ, a prime $w \nmid D'$, and an integer $j \ge 0$. The $C_\ell^\times$-valued characters χ', η' of $G_{\mathbf{Q}}$ factor through $\mathbf{Q}(\mu_{D'})$ and take value one on the residue class of w. Here η' is the lift of the sign with kernel G_{K_0}. The character θ factors through $\mathbf{Q}(\mu_w)$, has order ℓ, and is trivial on G_L. Require

$$\chi'(c) = (-1)^j, \qquad \chi' \ne \eta', \qquad \ell \nmid |D_{K_0}| \operatorname{ord}(\chi'), \qquad \overline{\varepsilon_\ell^j \chi'} = \bar\Psi$$

on the absolute group for the last equality. Require $s \in G_L$ with $\overline{(\varepsilon_\ell^j \chi')(s)}^{\,2} \ne 1$.

Lemma 30.56 (The obstruction bound after imposing the inertia relations)**.** *Let F be totally real, let ℓ be odd, and let S consist of the places of F above ℓ. Let K be an open normal subgroup of G_F of index prime to ℓ. Form Γ by dividing G_F by the closed normal subgroup generated by $I_q \cap K$ at all $q \notin S$, using all prolongations. For $v \in S$ use the map $G_{F_v} \to G_F \to \Gamma$ attached to a prolongation. Let V be a representation of Γ with open kernel over a finite field k' of characteristic ℓ, of finite dimension over k'. For any locally stable $b_v \subseteq V$, form H^i_b as in Definition 30.55. If $\mathrm{Hom}_\Gamma(V, k'(\bar\varepsilon)) = 0$, these groups are finite-dimensional for $0 \le i \le 2$ and satisfy (30.15), with $d_v = [F_v : \mathbf{Q}_\ell]$.*

Outline. Use inflation to bound the degree-two kernel by the added local terms. Combine 27.2, global and local Euler characteristics and the exact sequence for the local quotients to bound the framed obstruction group. The complete proof is on p. 857. □

Lemma 30.57 (A determinantal section of a power series ring)**.** *Let O be a complete discrete valuation ring. Let a, b, D be nonnegative integers with $a = b + D$, let $g \ge 1$, and put*

$$S = O[[x_1, \dots, x_a, u_1, \dots, u_g]], \qquad u = (u_1, \dots, u_g)^t.$$

Let $f_1, \dots, f_b \in \mathfrak{m}_S$ and $Z \in M_g(\mathfrak{m}_S)$. Put

$$A = S/(f_1, \dots, f_b, (Zu)_1, \dots, (Zu)_g, \det Z).$$

Every irreducible component of $\operatorname{Spec} A$ *has dimension at least $D+1$. For any two minimal primes of A there is a chain of minimal primes from one to the other such that $\dim A/(P+Q) \ge D$ for consecutive members P, Q.*

Outline. Parametrize the universal singular matrix with a kernel vector to obtain a prime radical, then bound its height using the adjugate identity. Specialize to $Y = Z$ and $v = u$ and apply the height and connectedness theorems to the section. The complete proof is on p. 859. □

Lemma 30.58 (Numbers of even and odd relations at a block point)**.** *Consider the completed ordinary matrix deformation ring R at a residual block point of the rank-three frame of Definition 30.55. Use the coefficient DVR O' with the residue field of the point. Conjugation by the involution which acts as 1 on the outer two coordinates and as -1 on the middle coordinate gives the even and odd parts. There is a presentation over O' with a even and g odd variables, $a - (3d+1)$ even relations, and g odd relations. The odd variables generate the ideal defining the block locus. If the point is in the closure of the complement of that locus, then $g > 0$.*

Outline. Split the minimal presentation by parity and use the square-zero obstruction and marked tangent count to bound the relations via the Euler characteristic for the frame of 30.55. The odd variables generate the block ideal. The complete proof is on p. 859. □

Lemma 30.59 (Points on the linked CM frame)**.** *Assume $\mathscr{H}$. Put $\Psi = \varepsilon_\ell^e \vartheta$, $e \geq 1$, with N even, $\ell N \mid M$, and every prime ramified in η dividing M. Fix $B \mid \ell, \iota$ and a Galois link for $(\eta, \Psi, c, g_1, B, M)$ to a frame over an abelian CM field L satisfying 30.55. Assume $d > 3C_0 + 7$ and $d_v > 5$. For indices $0 \leq i < n$ put $w_{3,i} = 1 - i$ and $w_{2,i} = 1 - 2i$.*

For $n = 2, 3$, take an adically continuous framed lift over a coefficient DVR finite over O, with the prescribed diagonal reduction and inertia characters pulled back from $\Psi^{w_{n,i}}$. If its representation r on G_L is irreducible, $r\kappa_n$ is conjugate self-dual automorphic with witness. Here $\kappa_3 = \varepsilon_\ell^{-1}$ and one algebraic κ_2 works for all rank-two points. In rank three over a complete DVR A with finite residue field and $\mathfrak{m}_O A = 0$, impose the same residual condition and require the flag characters to kill the kernel of $\bar{\Psi}$ in inertia. The characteristic polynomial at g on the sign kernel is then $\prod_{i=0}^{2}(X - \bar{\Psi}(g)^{1-i})$.

Outline. Builds rank-two and adjoint rank-three seeds via 30.49 and 30.23, extracting separating curves with 30.35, 30.48, and 30.50. Propagates Hecke containment via 30.57, 30.58, and 30.34, then applies 30.25. The complete proof is on p. 860. □

Lemma 30.60 (The newform and local witness after descent)**.** *Fix $\iota : \overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$. Suppose $k \geq 1$, $L/\mathbf{Q}$ and W satisfy Lemma 26.38, and $W|_{G_L} \otimes \kappa$ has an attachment with witness in the sense of Definition 30.1 for an algebraic Hecke character with realization κ. Make the following local assumptions.*

1. *At one prolongation above ℓ, on the intersection of inertia with an open normal subgroup of $G_{\mathbf{Q}}$, W has an ordered flag with characters $\varepsilon_\ell^k, \varepsilon_\ell^{-k}$.*
2. *At $p \neq \ell$, the action of inertia factors through a finite abelian quotient on which the decomposition group acts trivially by conjugation.*

Then $W\varepsilon_\ell^{-k}$ is attached to a normalized weight-$(2k+1)$ newform, and W satisfies the quadratic witness condition.

Outline. Descend by 26.38, use 40.2, 39.6 and 39.4 to identify the weight, and obtain the newform via 26.36(3). Transfer the support bounds by local base change and use the inertia flag ratios for potential unlinking. The complete proof is on p. 868. □

Theorem 30.61 (The endpoints used in cases I and V)**.** *Assume $\mathscr{H}$. For each good imaginary quadratic datum of the first count, each ι and each $B \mid \ell$, there is one marking g_1 for assertions 1–3, with $\eta(g_1) = 1$ and $\bar{\Psi}(g_1)^2 \neq 1$. An ordinary point has sign η, is adically continuous, and reduces to the diagonal base point r_0, whose matrix part on $\ker\eta$ is $\operatorname{diag}(\Psi, 1, \Psi^{-1})$. Its matrix at complex conjugation c is 1. The $01, 10$ entries of the matrix difference $r(g_1) - r(cg_1^{-1}c)$ vanish. Away from ℓ it kills inertia with $\eta = \Psi = 1$. At B a basis congruent to the identity makes it upper triangular on $D_B \cap \ker\eta$ with diagonal $(\Psi, 1, \Psi^{-1})$ on inertia.*

1. *On one finite-index $U \leq G_{\mathbf{Q}}$, every such point over a complete DVR A with finite residue field and $\mathfrak{m}_O A = 0$ has characteristic polynomial $\prod_{i=0}^{2}(X - \bar{\Psi}(g)^{1-i})$ for $g \in U \cap \ker\eta$.*
2. *For a point over $O' \subset C_\ell$ finite over O, irreducibility on every open subgroup of $\ker\eta$ gives a conjugate self-dual cuspidal attachment with witness for its restriction to G_K twisted by ε_ℓ^{-1}, where K is the sign field.*
3. *For a Type II point over O' with trivial middle character on $\ker\eta$, every representation $W : G_{\mathbf{Q}} \to \mathrm{GL}_2(C_\ell)$ agreeing there with its irreducible corner and satisfying $\operatorname{tr} W(c) = 0$ has $W\varepsilon_\ell^{-k}$ attached to a weight-$(2k+1)$ newform, and satisfies the quadratic witness condition.*
4. *For each Gaussian host satisfying (30.13), and each ι, c, B, one marking satisfying (30.14) makes assertions 1 and 2 hold for its frame. The lifting conclusion is 30.54.*

Outline. Build a marked CM tower using Selmer classes, 28.6 and the obstruction bound of 30.56. Apply 30.59 through the link for the residual polynomials and the two automorphic ranks, then use 30.60 for the corner. The complete proof is on p. 869. □

30.5 Adjoint vanishing and ordinary derivations

A polarised ordinary frame for a CM field L gives a quotient Γ of G_{L^+}, its CM subgroup Γ_0, and a complex conjugation c. Write D_v and I_v for the local decomposition and inertia groups, and pull back through the frame maps when restricting to them. At the distinguished place the residual diagonal characters coincide on D_0, and $\bar{\varepsilon}_\ell|_{D_0} \neq 1$. Write I_0 for its inertia. For a representation ρ with polarisation P, let $R(g)X = \rho(g)X\rho(g)^{-1}$ on Γ_0, and set

$$R(c)X = -P^{-1}X^tP. \tag{30.17}$$

For any action R, write $d_RX(g) = R(g)X - X$.

An ordinary fixed-weight cocycle on Γ_0 is a continuous matrix-valued function z satisfying

$$z(gh) = z(g) + R(g)z(h), \qquad Pz(cgc^{-1}) = -z(g)^tP. \tag{30.18}$$

It vanishes on the prescribed tame inertia kernels. At each $v \mid \ell$ it is upper triangular in the ordered flag after subtraction of a local coboundary, with zero diagonal on I_v. Put $\Gamma' = \Gamma_0 \cap \ker \varepsilon_\ell$. Determinant regularity means that there are finitely many $s_j \in \Gamma'$ with separable characteristic polynomial such that a continuous cocycle u for $\varepsilon_\ell R$ with $Pu(cgc^{-1}) = u(g)^tP$ is a twisted coboundary whenever all the tests $\operatorname{tr}(u(s_j)\rho(s_j)^a)$, $a \geq 0$, vanish.

Theorem 30.62 (Bloch–Kato vanishing for the extended adjoint)**.** *Assume $\mathscr{H}$. Take a CM Hida setting at the base level S of 30.23, the union of the places of L^+ above ℓ, the tame type places and the distinguished place. Use the level frame for this set. Suppose L contains no primitive ℓ-th root of unity and O contains the images of $\mathcal{O}_L$ under all ℓ-adic embeddings. Let $2 \leq n \leq 4$, fix an admissible residual ordinary datum with determinant ring B, and take $\iota : C_\ell \simeq \mathbf{C}$, an arithmetic point x of B, and $\rho : \Gamma_0 \to \mathrm{GL}_n(C_\ell)$ whose determinant law is the specialisation D_x of the universal determinant at x. Require the following.*

1. *The representation ρ is continuous, is trivial on I_0, and its pullback to G_L is conjugate self-dual cuspidal automorphic with witness.*
2. *At every $v \mid \ell$ it has a flag whose successive characters $\kappa_{v,i}$ are the specialisations at x of the universal ordinary characters, in the prescribed order.*
3. *The restriction $\rho|_{\Gamma'}$ is irreducible.*
4. *The elements of Γ' at which ρ has separable characteristic polynomial are dense in Γ'.*
5. *At each $v \mid \ell$ there is an element of D_v mapping to Γ' on which every ratio of distinctly indexed flag characters has infinite order.*
6. *At each $v \mid \ell$ there are strictly decreasing integers m_i and an integer $h \geq 1$ with*

$$(\kappa_{v,i}(s)\kappa_{v,j}(s)^{-1}\varepsilon_\ell(s)^{m_j-m_i})^h = 1 \quad (s \in I_v,\ i,j). \tag{30.19}$$

Let $P \in \mathrm{GL}_n(C_\ell)$ satisfy $\rho(g)^tP\rho(cgc^{-1}) = \varepsilon_\ell(g)^{1-n}P$ for $g \in \Gamma_0$, and let R be the extended adjoint action (30.17). If a continuous R-cocycle z on Γ has Bloch–Kato finite pullback to G_{L^+}, then for some $X \in M_n(C_\ell)$,

$$z(g) = d_RX(g) \qquad (g \in \Gamma). \tag{30.20}$$

Outline. Identify the pullback with its cuspidal attachment by Chebotarev and Brauer–Nesbitt and match the extended adjoints via the polarisation. Apply Thorne's vanishing using cyclotomic irreducibility, then descend the coboundary by surjectivity. The complete proof is on p. 872. □

Lemma 30.63 (Cocycles with zero trace derivative)**.** *Let $\rho : G \to \mathrm{GL}_n(K)$ over a field K have full matrix span. An adjoint cocycle z with $\operatorname{tr}(z(g)\rho(g)) = 0$ for every g is a coboundary.*

Outline. Using the zero-trace condition at gh, the cocycle law, full span, and the trace pairing shows $\partial(\rho(g)) = z(g)\rho(g)$ defines a derivation on matrices. Putting $A = \sum_i \partial(E_{i1})E_{1i}$ gives $\partial(E_{jk}) = AE_{jk} - E_{jk}A$, hence $z = d_R(-A)$. The complete proof is on p. 873. □

Proposition 30.64 (An ordinary point on the prescribed unitary Hecke module)**.** *In the Hida setting of 30.65, let the level module and its annihilator J be as in 30.23. An arithmetic point x with the stated cuspidal attachment, distinguished inertia kernel and ordered unlinked ordinary characters occurs on that module at its prescribed weight and level. Its Hecke character has the determinant coefficients and ordered diamond characters of x. In particular $J \subseteq \ker x$.*

Outline. Use full span and traces to obtain unramifiedness outside the level, and local compatibility for sphericity. Descent 39.51 and split comparison 26.22 give the required eigenspace, whose determinant agrees on good Frobenii and hence everywhere. The complete proof is on p. 873. □

Theorem 30.65 (Ordinary cocycles are coboundaries)**.** *Assume $\mathscr{H}$, $2 \le n \le 4$, an admissible datum in the Hida frame at the base level S of 30.23, an arithmetic point x of weight h and an $\iota : C_\ell \simeq \mathbf{C}$. Let $\rho : \Gamma_0 \to \mathrm{GL}_n(C_\ell)$ be continuous with determinant law D_x and polarisation P as in 30.62. Write $\kappa_{v,i}$ for the ordered ordinary characters specialised at x. Assume $\rho(I_0) = 1$, conjugate self-dual cuspidal automorphy with witness on pullback to G_L, determinant regularity, and*

$$\operatorname{span}_{C_\ell} \rho(\Gamma_0) = M_n(C_\ell), \qquad \kappa_{v,i} \neq \kappa_{v,j}\varepsilon_\ell \quad (v \mid \ell,\ i,j). \tag{30.21}$$

Every ordinary fixed-weight cocycle z has $z(g) = d_R Y(g)$ on Γ_0 for some $Y \in M_n(C_\ell)$.

Outline. Realize the cocycle as an integral dual-number lift with zero inertia derivatives. Use 30.23 and 30.64 for Hecke control, 30.66 to kill trace derivatives, and 30.63 to recover a coboundary. The complete proof is on p. 874. □

Theorem 30.66 (Inertia derivatives determine trace derivatives)**.** *Assume the rank-n inputs of $\mathscr{H}$, $n \ge 1$, in the Hida frame at the base level S of 30.23. Let B be the universal ordinary determinant ring with pair $(D^{\mathrm{univ}}, \psi^{\mathrm{univ}})$, let $x : B \to O$ be an integral arithmetic point, and let $J \subseteq \ker x$. Require the level-ideal condition of 30.70(1) for the base module M, with $D = n[L^+ : \mathbf{Q}]$ weight parameters in Λ given by inertia elements. Require compatible level modules at every split auxiliary level with distinct new places outside S and the compatible map to the base. For every integral arithmetic point $x' : B \to O$, put $I_{x'} = \ker(x'|_\Lambda)$. For every t in the O-subalgebra of B generated by the coefficients of the characteristic polynomials of D^{univ} at $g \in \Gamma_0$, and every integer $k \ge 1$, require*

$$t^k M \subseteq I_{x'} M \quad \Longrightarrow \quad tM \subseteq I_{x'} M.$$

Suppose the determinant law specialised at x is that of an adically continuous representation $\rho_x : \Gamma_0 \to \mathrm{GL}_n(O)$ with an integral invertible symmetric or alternating polarisation as in 30.62. Its ordered flag characters are the specialisations $\kappa_{v,i} = x \circ \psi^{\mathrm{univ}}_{v,i}$. The representation kills the designated inertia kernels, and its matrix span over O contains $aM_n(O)$ for some $a \neq 0$. At $v \mid \ell$ assume an integral flag, a local element with distinct ordered values, and $\kappa_{v,i} \neq \kappa_{v,j}\varepsilon_\ell$ for $i < j$. At the distinguished place require

$$\big(\varepsilon_\ell(g)\rho_x(g)Z = Z\rho_x(g) \text{ for all } g \in D_0\big) \implies Z = 0 \qquad (Z \in M_n(O)). \tag{30.22}$$

Assume determinant regularity at elements $s_1, \dots, s_{q_0} \in \Gamma'$ with distinct integral roots α_{ji} at each s_j. Let $\tau^{\mathrm{univ}}(g)$ be the negative of the coefficient of X^{n-1} in the universal characteristic polynomial of g. Every O-linear x-derivation $\delta : B \to O$ killing the universal ordinary characters on inertia satisfies

$$\delta(\tau^{\mathrm{univ}}(g)) = 0 \qquad (g \in \Gamma_0). \tag{30.23}$$

Outline. Use the places of 30.67 and the bound of 30.68. Arithmetic control and 30.69 make the local weight-fibre factor a field, giving an idempotent, and the vanishing weight derivatives let 30.70 kill the trace derivatives. The complete proof is on p. 875. □

Theorem 30.67 (Taylor–Wiles auxiliary places)**.** *Assume the local and global cohomological inputs. Take the CM Hida frame at the base level S of 30.23 with residual characters $\bar\chi_i$, and suppose each open subgroup of Γ has finitely many continuous $\mathbf{F}_\ell$-characters. For an open normal subgroup H of Γ, write $N_\ell(H)$ for the kernel of its maximal pro-ℓ quotient. Put*

$$H = \Gamma_0 \cap \ker \bar\varepsilon_\ell \cap \bigcap_i \ker \bar\chi_i.$$

For any $s_1, \dots, s_{q_0} \in \Gamma'$ there are auxiliary systems of distinct $v_{r,j} \notin S$, split in L, and places $w_{r,j} \mid v_{r,j}$ with

$$\mathrm{N}w_{r,j} \equiv 1 \bmod \ell^r. \tag{30.24}$$

The enlarged level has a quotient map $h_r : \Gamma_r \twoheadrightarrow \Gamma$, inverse arithmetic Frobenii $\phi_{r,j}$, and tame elements $\tau_{r,j}$. The differences $d_{r,j} = h_r(\phi_{r,j}) s_j^{-1}$ belong to H, and for every open normal $U \supseteq N_\ell(H)$ they belong to U for all sufficiently large r. Moreover $h_r(\tau_{r,j}) = 1$ and

$$\phi_{r,j}^{-1} \tau_{r,j} \phi_{r,j} \equiv \tau_{r,j}^{\mathrm{N}w_{r,j}} \pmod{N_\ell(h_r^{-1}H)}. \tag{30.25}$$

The kernel of h_r lies in the closed normal subgroup generated by the $\tau_{r,j}$ and $N_\ell(h_r^{-1}H)$. The groups $h_r^{-1}H$ have uniformly bounded numbers of continuous $\mathbf{F}_\ell$-characters.

Outline. Apply Chebotarev in finite level quotients using cofinal pro-ℓ approximations and cyclotomic congruences. The new tame generators give the conjugation and normal generation relations, and their finitely many coset conjugates bound the character count. The complete proof is on p. 875. □

Lemma 30.68 (The ordinary Selmer trace bound)**.** *Under 30.66 use the system of 30.67. Put $A_m = O/\mathfrak{m}_O^m$ and let $\bar\rho_{r,m}$ be the pullback of ρ_x to the CM subgroup of Γ_r, reduced to A_m. Define $\mathcal{S}^{\mathrm{tr}}_{r,m}$ to be the set of functions $g \mapsto \mathrm{tr}(z(g)\bar\rho_{r,m}(g))$ for locally constant matrix cocycles over A_m satisfying (30.18) and vanishing on the prescribed tame inertia kernels. At $v \mid \ell$ require that a local coboundary makes the cocycle upper triangular in the prescribed flag, with zero diagonal on*

$$\mu_v = \{g \in D_v : \text{the image of } g \text{ in the topological abelianisation has finite } \ell\text{-power order}\}.$$

At the auxiliary places use the full local condition. Put $Q = \#(O/\mathfrak{m}_O)$, $b = \binom{n}{2}$, $d = [L^+ : \mathbf{Q}]$ and $s_\ell = \#\{v \mid \ell \text{ in } L^+\}$. There is a constant C such that for every $m \geq 1$ and all sufficiently large r,

$$\#\mathcal{S}^{\mathrm{tr}}_{r,m}\, Q^{m((n+b)s_\ell + bd)} \leq Q^{C + m((n+b)(s_\ell+d) + nq_0)}. \tag{30.26}$$

In particular $\#\mathcal{S}^{\mathrm{tr}}_{r,m} \leq Q^{m(D+nq_0)+C}$.

Outline. Use 27.2 and Euler characteristics to bound the local factors, with 30.67 identifying the auxiliary centralizers. Descend dual cocycles from auxiliary level and use determinant regularity and Smith reduction to bound their classes uniformly. The complete proof is on p. 876. □

Lemma 30.69 (Uniform trace and character comparisons)**.** *Use the integral representation, polarisation and separated flags of 30.66, with $\ell \neq 2$. On an adapted level take a compatible ordinary determinant pair over a complete local O-algebra R whose specialisation at $x_R : R \to O$ is the pullback of the determinant law of ρ_x together with its ordered characters $(\kappa_{v,i})$. At each designated tame place w, assume that the pair's inertia kernel $I_p(w)$ equals the frame's prescribed kernel $I_u(w)$. An x_R-derivation into an O-module is O-linear and satisfies $\delta(ab) = x_R(a)\delta(b) + x_R(b)\delta(a)$. There is a fixed $c_1 \in O \setminus \{0\}$, independent of level and m, such that the trace derivatives of any A_m-valued x_R-derivation multiplied by c_1 belong to $\mathcal{S}^{\mathrm{tr}}_{r,m}$ of 30.68. There is a fixed $c_2 \in O \setminus \{0\}$, independent of level and target, that kills the character derivatives of every x_R-derivation that kills traces.*

Outline. Use rank-one Cayley–Hamilton corners to recover a scaled matrix derivative and average for polarisation. Differentiate tame and flag relations at a separating element and use eigenvalue interpolation to annihilate character derivatives when trace derivatives vanish. The complete proof is on p. 877. □

Lemma 30.70 (Counting derivations in finite quotients)**.** *Let $\ell \neq 2$, $n \geq 1$, and let O be a complete coefficient DVR with finite residue field $\mathbf{k}$. Fix an adapted system as in 30.67, an ordinary determinant pair u over a complete Noetherian local O-algebra B with residue field $\mathbf{k}$, an ideal J, and an integral point x as in 30.66. Suppose the determinant coefficients and characters generate B adically, $J \subseteq \ker x$, and the following conditions hold.*

1. *The base module M is finite free over $\Lambda = O[[X_1, \dots, X_D]]$, has support $V(J)$, and each X_k acts by its weight character times a fixed unit, minus one. At auxiliary level r the ordinary pair u_r over R maps to the pullback of u via $\beta : R \to B$, with the same weight identities and the prescribed enlarged inertia kernels. The module M_1 is finite over Λ and projective over $\Lambda[\Delta]$, where $\Delta = \prod_{j,i} \mathbf{Z}/\ell^{a_j}$, $a_j \geq r$. Its standard generators act by $1 + y_{ji}$. The surjection $M_1 \to M_w$ intertwines the root operators and has kernel $\sum_{j,i} y_{ji} M_1$.*

   Write $Q_j = \mathrm{N} w_{r,j}$, let u_{ji} be the root operators, and put

$$f = \prod_j \prod_{i \neq a} (u_{ji} - u_{ja})(u_{ji} - Q_j u_{ja}).$$

   The comparison maps between M_w and M tensored with the ordered splitting algebra of the base Frobenius polynomials intertwine the coefficient action via β and the root-polynomial actions. Both composites are multiplication by f. For some fixed integer $E \geq 0$ and every j, i, the operator

$$f^E \operatorname{tr}_{u_r} \left[\left(\prod_{a \neq i} (\phi_{r,j} - u_{ja}) \right)^{E+1} (\tau_{r,j} - 1 - y_{ji}) \right] \tag{30.27}$$

   acts by zero on M_1, with the trace extended linearly in the operator coefficients.

2. *Some $t_0 \in B$ with $x(t_0) \neq 0$ satisfies $t_0 a M \subseteq \ker(x|_\Lambda) M$ for every $a \in \ker x$.*

3. *The representation ρ_x kills $N_\ell(H)$ and has distinct integral roots α_{ji} at s_j.*

4. *For some fixed nonnegative integer C, at every $m \geq 1$ and all sufficiently large r the finite-layer Selmer trace set of 30.68 satisfies*

$$\#\mathcal{S}^{\mathrm{tr}}_{r,m} \leq |\mathbf{k}|^{m(D+nq_0)+C}. \tag{30.28}$$

Then every O-linear x-derivation $\delta : B \to O$ killing the selected weight characters kills all universal traces.

Outline. Use 30.67 and the comparison operators to lift weight and diamond derivations, then adjoin a putative nonzero trace derivative. By 30.69 their combinations give Selmer traces with bounded loss of precision, exceeding the count bound. The complete proof is on p. 878. □

30.6 The algebraic length comparison

Let $A = \bigoplus_j A_j$ be a graded O-algebra over a domain and $x : A \to O$ an O-point. Put $\mathfrak{a} = \ker x$ and $S_x = \mathrm{sat}_O(\mathfrak{a}^2)$, the saturation under nonzero elements of O. Let C_j be the image of $A_j \cap \mathfrak{a}$ in A/S_x. For a family $F = (F_i)$ in $\mathfrak{a}$, put

$$C_j(F) = C_j / (C_j \cap \langle [F_i] \rangle_O).$$

A point is fixed if it kills every nonzero weight. It is two-sided if some element of positive weight and some element of negative weight have nonzero values. A point of one strict sign has a nonzero value of that sign and kills all weights of the opposite sign. The torus acts by multiplying a weight-j value by t^j.

Lemma 30.71 (Two-sided orbit slices, chart reduction and the Rees module)**.** *Let O be a complete characteristic-zero DVR with uniformiser π. Let K be an algebraically closed extension of $\operatorname{Frac} O$ with an extending valuation v normalised by $v(\pi) = 1$. Let A_0 be complete Noetherian local with residue field that of O. Assume $A = \bigoplus_j A_j$ is generated over A_0 by finitely many elements of weights $-2, -1, 1, 2$, and $x : A \to O$ is a fixed O-point. Let I be generated by $F_1, \dots, F_{r+1}$ of weight -1, $G_1, \dots, G_s$ of weight -2, $E_1, \dots, E_{r'}$ of weight 1, $E'_1, \dots, E'_{s'}$ of weight 2, and $Z_1, \dots, Z_m \in A_0 \cap \ker x$. Suppose the following five orbit conditions hold.*

1. *$r \geq 1$. There is a graded map to a formal power series ring over $\operatorname{Frac} O$ with constant term x and linear terms that identify the rational cotangent space with the span of the variables. In weights $-2, -1, 0, 1, 2$ the variable counts are $s, r, m, r' + 1, s' + 1$, with no other weights. The F_i, G_i, Z_i span the rational cotangent spaces of weights $-1, -2, 0$, respectively.*

2. *The image of A_0 in A/I is finite over O.*
3. *The fibre over the A_0-restriction of a fixed K-point of A/I distinct from x_K contains points of at most one strict sign.*
4. *Every two-sided K-point of A/I has tangent space of dimension at most one.*
5. *An element $a \in A_0 \cap \ker x$ is nonzero at every two-sided K-point of A/I.*

Then the following assertions hold.

- *For every $f \in A_{-2}$ and $g \in A_2$, the slice $f = 1,\ g \neq 0$ of $(A/I) \otimes_O K$ is finite and reduced over K, and there is a finite set T of two-sided K-points of A/I meeting every two-sided orbit.*
- *The quotient $C_{-1}(F)$ has finite length. After an invertible O-linear change among $F_1, \dots, F_{r+1}$ preserving I, there are $y_1, \dots, y_r \in A_{-1}$ and integers $b_1, \dots, b_r \geq 0$ such that*

$$F_i - \pi^{b_i} y_i \in S_x \quad (1 \leq i \leq r), \qquad F_{r+1} \in S_x, \tag{30.29}$$

and

$$\operatorname{length}_O C_{-1}(F) = \sum_{i=1}^{r} b_i. \tag{30.30}$$

- *Setting the weight-zero coordinates given by $Z_1, \dots, Z_m$ equal to zero by formal inversion defines a graded O-algebra map ϕ from A to a formal power series domain over $\operatorname{Frac} O$ in the remaining variables, and a may be replaced modulo I by an element of $A_0 \cap \ker x$ satisfying $\phi(a) \neq 0$.*
- *Inside $A[T_0, T_1, T_2]$, let R be the $\mathbf{N}^3$-graded A_0-subalgebra generated in the three unit multidegrees by $A_{-2}T_0$, A_2T_1, and $(A_0F_{r+1}^2 + A_{-2}^2A_2)T_2$. Let $J \subset R$ be the kernel of ϕ on coefficients, put $M = R/J$, and let $\mathbf{t}$ be the homogeneous list consisting of $F_i^2T_0$ $(1 \leq i \leq r)$ and G_iT_0 in the first direction, $E_i^2T_1$ and $E_i'T_1$ in the second, and $F_{r+1}^2T_2$ in the third. Let R_{++} be the product of the three ideals of R generated in unit multidegrees, call a homogeneous list admissible on a finitely generated graded R-module N if a power of $\mathfrak{m}_{A_0}R\,R_{++}$ lies in its ideal plus $\operatorname{Ann}_R N$, and for an admissible list L write $\chi_L(N)_d$ for the alternating sum of O-lengths of its Koszul homology in multidegree d. For any homogeneous section s with (s, L) admissible on N and coordinatewise sufficiently large d,*

$$\chi_{(s,L)}(N)_d = \chi_L(N/sN)_d - \chi_L(0 :_N s)_{d-\deg s}. \tag{30.31}$$

Outline. Finite generation of A_{-2}, A_2 and tangent dimension at most one make each slice $f = 1, g \neq 0$ finite and reduced, giving a cover T. Smith reduction computes $C_{-1}(F)$, formal inversion builds ϕ, and Koszul exact sequences give the length recursion. The complete proof is on p. 880. □

Lemma 30.72 (From Selmer length to trace differences)**.** Framed presentation. *Let O be complete Noetherian local with finite residue field $\mathbf{k}$ of odd characteristic ℓ. Let Γ be profinite, with finitely many continuous $\mathbf{F}_\ell$-characters on each open subgroup. Fix a block decomposition of a basis with weights w_i in a finite free lattice. An invertible symmetric or alternating form pairs the blocks by an involution and has constant total weight on its nonzero entries. Fix a residual representation $\bar{r}$ and a lift r_0. Both preserve the form up to similitude, permute the blocks and preserve the weight spaces. Allowed lifts differ from r_0 by block-diagonal isometries congruent to the identity.*

Fix finite families of continuous local maps $\alpha_v : G_v \to \Gamma$ and $\beta_q : G_q \to \Gamma$, with a distinguished frame v_0 among the v. Fix $N_q \leq G_q$ on which $\bar{r}\beta_q$ is trivial. The local conditions at q hold at r_0, are given by closed ideals, and are invariant under allowed conjugation and torus conjugation congruent to the identity. Local solutions lift across small extensions. If two allowed lifts have the same image across a small extension and the first is a solution, the second is a solution exactly when the two agree on N_q.

Let V_j be the weight-j part of the block-diagonal Lie algebra of isometries over $\mathbf{k}$, with action $\operatorname{Ad}\bar{r}$. Write $\delta b(g) = gb - b$ and $du(g,h) = u(g) + gu(h) - u(gh)$. Let $H^1_{\mathrm{fr},j}$ consist of pairs $(z, (b_v))$ with z a continuous cocycle in V_j, $z\alpha_v = \delta b_v$, $b_{v_0} = 0$, and $z\beta_q|_{N_q} = 0$. Let Ob_j be the space of continuous triples $(c, (u_v), (u_q))$ with c a 2-cocycle in V_j and $du_v = \alpha_v^* c$, $du_q = \beta_q^* c$, modulo the triples

$$\big(du, (u\alpha_v - \delta b_v)_v, (u\beta_q - e_q)_q\big).$$

Here u is a continuous global 1-cochain, the b_v are local vectors, and the e_q are continuous local cocycles vanishing on N_q. Assume each Ob_j is finite-dimensional.

Let B be a complete Noetherian local O-algebra with residue field $\mathbf{k}$ and a homogeneous power series presentation on finitely many variables. Give each G_v an allowed lift over B whose ab-entry has weight $w_a - w_b$. The universal algebra of lifts framed against these local lifts, with the v_0-frame equal to the identity, has a homogeneous presentation relative to B. The new variables are entries of $r - r_0$ or of frame matrices minus the identity. The relations can be padded with zeros to give

$$\#\mathit{variables}_j = \dim_{\mathbf{k}} H^1_{\mathrm{fr},j}, \qquad \#\mathit{relations}_j = \dim_{\mathbf{k}} \mathrm{Ob}_j\,. \tag{30.32}$$

Cotangent comparison. *Let O be a DVR with finite residue field, $x(A_j) = 0$, and $F_i \in A_j$. Suppose an O-module M injects additively into the $O/(c)$-valued x-derivations for $0 \neq c \in O$. The image derivations kill the F_i and lift to O-valued x-derivations supported in weight j. With infinite lengths allowed,*

$$\operatorname{length}_O M \leq \operatorname{length}_O C_j(F). \tag{30.33}$$

Orbit estimate. *Let O be a complete characteristic-zero DVR with uniformiser π. Let K be an algebraically closed extension of $\operatorname{Frac} O$ with an extending valuation v normalised by $v(\pi) = 1$. Let A_0 be complete Noetherian local with residue field that of O. Assume A is generated over A_0 by finitely many elements of weights $-2, -1, 1, 2$, and $x : A \to O$ is fixed. Let I be generated by $F_1, \dots, F_{r+1}$ of weight -1, $G_1, \dots, G_s$ of weight -2, $E_1, \dots, E_{r'}$ of weight 1, $E'_1, \dots, E'_{s'}$ of weight 2, and $Z_1, \dots, Z_m \in A_0 \cap \ker x$. Suppose the following conditions hold.*

1. *$r \geq 1$. There is a graded map to a formal power series ring over $\operatorname{Frac} O$ with constant term x and linear terms that identify the rational cotangent space with the span of the variables. In weights $-2, -1, 0, 1, 2$ the variable counts are $s, r, m, r' + 1, s' + 1$, with no other weights. The F_i, G_i, Z_i span the rational cotangent spaces of weights $-1, -2, 0$, respectively.*
2. *The image of A_0 in A/I is finite over O.*
3. *The fibre over the A_0-restriction of a fixed K-point of A/I distinct from x_K contains points of at most one strict sign.*
4. *Every two-sided K-point of A/I has tangent space of dimension at most one.*
5. *An element $a \in A_0 \cap \ker x$ is nonzero at every two-sided K-point of A/I.*

Then $C_{-1}(F)$ has finite length. There is a finite set T of two-sided K-points of A/I meeting every two-sided orbit. Every such T satisfies

$$\operatorname{length}_O C_{-1}(F) \leq 2 \sum_{\xi \in T} v(\xi(a)). \tag{30.34}$$

Outline. Presents the framed universal ring on pro-ℓ generators using $H^1_{\mathrm{fr},j}$ and Ob_j via small extensions. Derives the cotangent bound from x-derivations and the orbit bound from 30.71 via Koszul Euler characteristics. The complete proof is on p. 880. □

Lemma 30.73 (The lower intersection number)**.** *Use the chart, exponents, Rees algebra and admissibility notion of Lemma 30.71. If $\mathbf{t}$ is admissible on M/aM, then in coordinatewise sufficiently large multidegree d,*

$$2^{r+r'} \sum_i b_i \leq \chi_{\mathbf{t}}(M/aM)_d. \tag{30.35}$$

Outline. Pass to leading terms in the negative and positive weights and compute their Hilbert series. Removing the Smith scalars gives the lower bound through the Koszul coefficients, and 30.74 gives nonnegativity for the remaining terms. The complete proof is on p. 881. □

Lemma 30.74 (The upper intersection number)**.** *With the orbit hypotheses and construction of Lemma 30.71, the list $(\mathbf{t}, a)$ is admissible on M, and in sufficiently large multidegrees,*

$$\chi_{\mathbf{t}}(M/aM)_d \leq \chi_{(a)}(M/\mathbf{t}M)_d \leq 2^{r+1+r'} \sum_{\xi \in T} v(\xi(a)). \tag{30.36}$$

On any finite graded R-module an admissible list of nonnegative multidegrees has eventually nonnegative Koszul Euler characteristic.

Outline. Classifies primes over $J+(\mathbf{t})$ avoiding R_{++} via 30.71 to show admissibility of $(\mathbf{t},a)$ and bound horizontal multiplicities by $2^{r+1+r'}$. Prime filtrations bound $\chi_{(a)}(M/\mathbf{t}M)_d$ and give Koszul nonnegativity. The complete proof is on p. 882. □

30.7 Residual vanishing from a Bernoulli unit

For a character $\gamma : G_{\mathbf{Q}} \to \mathbf{k}^{\times}$, let $\mathbf{k}(\gamma)$ have action by γ. Consider the following inertia condition on a cocycle z in this module. For every place A, with $J_A = I_A \cap \ker\gamma$, there is a subgroup $H \le G_{\mathbf{Q}}$ such that

$$[J_A : J_A \cap H] \text{ is finite and prime to } \ell, \qquad z|_{J_A\cap H} = 0. \tag{30.37}$$

Lemma 30.75 (Residual Bernoulli-unit criterion)**.** *Let ℓ be odd, $O \subset C_\ell$ a coefficient DVR and $\mathrm{red} : O \to \mathbf{k}$ a ring map to a discrete field of characteristic ℓ. Let $\vartheta : (\mathbf{Z}/N)^{\times} \to O^{\times}$, $N \ge 1$, be nontrivial of order prime to ℓ. For a nontrivial character λ at level L, put $B_{1,\lambda} = L^{-1}\sum_{t\in(\mathbf{Z}/L)^{\times}} [t]_L \lambda(t)$, where $[t]_L$ is the least nonnegative residue. Regard $\bar\vartheta$ as the reduced Galois character. Assume*

$$\bar\vartheta \ne \bar\varepsilon_\ell, \qquad |B_{1,\vartheta^{-1}}|_\ell = 1.$$

Every continuous cocycle $z : G_{\mathbf{Q}} \to \mathbf{k}(\bar\vartheta)$ satisfying the inertia condition (30.37) has the form

$$z(g) = (\bar\vartheta(g) - 1)b \quad (g \in G_{\mathbf{Q}}), \qquad b \in \mathbf{k}. \tag{30.38}$$

Let $\phi \bmod f, \xi \bmod 2^n$ be primitive O-valued characters, with $f \ge 1$ odd, $n \ge 0$ and $d \ge 1$. Assume

$$4 \mid \ell^d - 1, \qquad \operatorname{ord}\phi \mid \ell^d - 1, \qquad \phi(-1)\xi(-1) = -1, \qquad f4^{v_2(\ell^d-1)} \le 2^n. \tag{30.39}$$

Then $|B_{1,\phi\xi}|_\ell = 1$ at level $f2^n$, and the cocycle conclusion holds for $\gamma = \overline{\phi\xi}$ and its inverse.

Outline. Use (30.37) to make the cyclotomic restriction unramified and kill it by smoothing and Stickelberger 27.5. Commutativity gives the coboundary, and 30.76 and last dyadic inertia give units and distinguish the residual characters. The complete proof is on p. 884. □

Lemma 30.76 (An effective dyadic Bernoulli unit)**.** *Let ℓ be prime, $d \ge 1$, $f \ge 1$ odd and $n \ge 0$. Let ϕ, ξ be primitive characters over C_ℓ of levels $f, 2^n$. Put $Q = \ell^d$ and $e = v_2(Q-1)$. If $4 \mid Q-1$, $\operatorname{ord}\phi \mid Q-1$, $\phi(-1)\xi(-1) = -1$, and $f4^e \le 2^n$, then $|B_{1,\phi\xi}|_\ell = 1$ at level $f2^n$.*

Outline. Group the Bernoulli sum modulo 2^n and use (A.292) for integral Frobenius-fixed coefficients. If reduction vanishes, average to restrict the residues, subtract the $1-s$ shift, and pair by the last dyadic involution to get -2 by odd parity. The complete proof is on p. 884. □

Count I uses the trace bound of 30.10 and the two vanishing statements of 30.12. It also uses the imaginary quadratic endpoints of 30.61. Counts II and III use 30.51 and the Bernoulli-unit criterion of 30.75 for the six residual characters in 29.22. Count V uses 30.54 and applies 30.75 to the product character of 29.26, whose reduction and inverse reduction are $\eta_4\bar\Psi^2$ and $\eta_4\bar\Psi^{-2}$.

Chapter 31

The exponent 401 and the downstream window

Lemma 31.1 (The rank cutoff)**.** *For a family satisfying the page normalization, $e_P = 12$, so $c_{TO} = 28e_P + 41 = 377$ and $C_t = c_{TO} + 24 = 401$.*

Let $0 < \varepsilon < 1/52$ and $x \geq 2^{100}$. At scale x, put $T_O = \lceil x^{377} \rceil$. Let R be a rank datum (from the rank-datum and frame parameters of Part VIII) with rank parameters $m, k, t \in \mathbf{N}$. Write $n = |A|$, where A is the finite abelian group of its frame. Put $f = 2n$, $r = k + m - 1$, and $L = x^{\varepsilon}$. Thus $m \geq 2$ and $m + 3 \leq k$. Define

$$n_W = 8 \left\lceil \frac{10r}{8} \left(f + \frac{f^2 (k-1)^3 L^2}{m} \right) \right\rceil .$$

Assume

$$\begin{aligned} x^{2+8\varepsilon} &\leq n \leq 2x^{2+8\varepsilon}, \\ x^{1/2-\varepsilon/16} &\leq m \leq 2x^{1/2-\varepsilon/16}, \\ x^{9/2+10\varepsilon} &\leq k \leq 2x^{9/2+10\varepsilon}, \qquad r + t = 8fr + 4 + T_O n_W . \end{aligned}$$

Then $t \leq x^{400}$.

Outline. Put $u = x^{1/100} \geq 2$ and bound f, k, r, m, L^2 by powers of u using $\varepsilon < 1/52$. Substituting these estimates into n_W, T_O, and the rank identity yields $t \leq 10708 u^{39967} \leq u^{40000} = x^{400}$ via $10708 \leq 2^{33} \leq u^{33}$. The complete proof is on p. 885. □

Theorem 31.2 (Windows at arbitrarily large scales)**.** *Fix a primitive character ψ, $0 < \varepsilon < 1/52$, and $c \geq 1$. Assume the window assertion for ψ with exponent 401 in Definition 26.3 and the prime-pair existence assertion of 7.25. Then there is $J \geq 1$ such that, for every $h \in \mathbf{R}$, there is $Z \geq h$ with the following property. For every $x \in [Z, Z^{3/2}]$ there exist $U \in \mathbf{R}$, primes q, d, and $a \in \mathbf{N}$ such that*

$$x^{1/J} \leq U \leq x, \qquad (q, d) \text{ is a good pair for } (\operatorname{cond} \psi, \varepsilon) \text{ on } [U, x], \qquad x^{2+8\varepsilon} \leq c 2^a \leq 2x^{2+8\varepsilon}.$$

Use the good-pair conditions of 7.24. Every field associated with (ψ, a, q, d) in Definition 26.2 satisfies the total bound of Definition 26.1 up to $\lfloor x^{400} \rfloor$ with budget $\exp(x^{\varepsilon}/2)$.

Proof. Apply Theorem 8.6 to $M = \operatorname{cond} \psi$, with the given ε, c, and take $y = h$. Its J, Z have the required quantifiers. At each x take $U = Y$ and $a = e$ in (8.8), keeping its primes q, d. Equation (8.9) gives the assertion for every field of Definition 26.2, since $\sqrt{\exp(x^{\varepsilon})} = \exp(x^{\varepsilon}/2)$. □

In Theorem 8.20, the rank t is a natural number, so Lemma 31.1 gives $t \leq \lfloor x^{400} \rfloor$ at the scales used there. The window estimate therefore covers every required index $1 \leq j < t$.

Part VII

Input one: Green

Chapter 32

Green's classification and its use

32.1 Green's theorem

For a prime p put $G_d = \mathrm{GL}_d(\mathbf{F}_p)$, $N_d = p^d - 1$, and $\Delta_d = \prod_{1 \le j < d}(p^j - 1)$. Write μ_N for the group of Nth roots of unity in the indicated field.

Definition 32.1 (Cuspidality)**.** A cuspidal of G_d for $d > 0$ is an irreducible character with mean zero on the standard unipotent radicals of types $(a, d-a)$, $0 < a < d$.

Definition 32.2 (Parameters)**.** Let $\mathcal{P}_d$ consist of monic irreducibles P of degree d over $\mathbf{F}_p$ with $P(0) \neq 0$. For $[K : \mathbf{F}_p] = d$, a character $\theta \in \widehat{K^\times}$ is regular if the θ^{p^i}, $0 \le i < d$, are distinct.

Definition 32.3 (Label datum)**.** A label datum L consists of a unital multiplicative map $t : \overline{\mathbf{F}}_p \to \mathbf{C}$ injective off zero and generators u_d of $\mu_{N_d}(\overline{\mathbf{F}}_p)$ with $u_{de}^{N_{de}/N_d} = u_d$ for $d, e > 0$. For $x \in \mu_{N_d}(\overline{\mathbf{F}}_p)$ set $\Theta_d(x, u_d^a) = t(x)^a$, with value zero off $\langle u_d \rangle$ in the second argument.

Definition 32.4 (Prime label)**.** An elliptic $y \in G_d$ has irreducible characteristic polynomial. A cuspidal χ has label $P \in \mathcal{P}_d$ if for every root x of P, every elliptic y, and every characteristic root z of y,

$$\chi(y) = (-1)^{d-1} \sum_{i<d} \Theta_d(x, z^{p^i}). \tag{32.1}$$

Definition 32.5 (Regular form)**.** The regular-form assertion states that for every prime p and $[K : \mathbf{F}_p] = d > 0$, there is a family τ_θ of virtual characters of G_d, indexed by $\theta \in \widehat{K^\times}$, that induces a bijection from regular Frobenius orbits to cuspidals. For regular θ, one has $\tau_\theta(1) = \Delta_d$ and

$$\tau_\theta(y) = (-1)^{d-1} \sum_{i<d} \theta(z^{p^i})$$

for elliptic y and each characteristic root $z \in K^\times$.

Definition 32.6 (Polynomial form)**.** Green asserts that for every prime p, datum L and $d > 0$, there is a family τ_P on $\mathbf{F}_p[X]$ that restricts to a bijection from $\mathcal{P}_d$ to cuspidals. For $P \in \mathcal{P}_d$, τ_P has label P and degree Δ_d.

32.1.1 Construction of the regular family

Let V have dimension n over a finite field F of size v. Put $H = \mathrm{GL}(V)$, let $c_g = \det(XI - g)$ for $g \in \mathrm{End}_F(V)$, and set $I(\phi, \psi) = \sum_{g \in H} \phi(g)\overline{\psi(g)}$ and $M = v^n - 1$.

Lemma 32.7 (Frobenius)**.** *For irreducible $P \in \mathbf{F}_p[X]$ with root x, the roots are $(x^{p^i})_{0 \le i < \deg P}$, and $x^{p^j} = x$ holds exactly when $\deg P \mid j$ for $j \ge 0$.*

Proof. This is the prime-field case of the finite-field orbit theorem [140, Ch. 2]. □

Lemma 32.8 (Lift). *There is a unital multiplicative map $t : \overline{F} \to \mathbf{C}$ injective off zero.*

Proof. Take Teichmüller lifts in the Witt ring of $\overline{F}$. Reduction distinguishes the lifts, and the nonzero lifts are roots of unity. The part of the fraction field algebraic over $\mathbf{Q}$ therefore contains the lifts and embeds in $\mathbf{C}$. □

Definition 32.9 (Root sums). For $h \geq 0$, set $s_h(P) = \sum_{P(\alpha)=0} t(\alpha)^h$, with multiplicities and $s_h(0) = 0$. The multiset of roots of a monic product gives $s_h(PQ) = s_h(P) + s_h(Q)$ for monic P, Q. Also $s_h((X-1)^j) = j$.

Definition 32.10 (Subspace inversion). For $s : F[X] \to \mathbf{C}$, set

$$\Psi_s(V, g) = s(c_g) - \sum_{\substack{W \subsetneq V \\ gW \subseteq W}} \Psi_s(W, g|_W).$$

Write $\Psi_h = \Psi_{s_h}$ and $R_h(g) = s_h(c_g)$.

Lemma 32.11 (Primitivity). *The recursion is conjugacy invariant. Call s additive if $s(PQ) = s(P) + s(Q)$ for all monic $P, Q \in F[X]$. If s is additive, then*

$$\sum_{h : V/A \to A} \Psi_s(V, g + h) = 0 \qquad (0 < A < V,\ gA \subseteq A),$$

where h is viewed as an endomorphism via quotient and inclusion. A class function on endomorphisms satisfying this identity is called primitive.

Outline. Conjugation transports invariant subspaces. Average 32.10 over the off-diagonal maps, using induction for subspaces with partial intersection with A and counting lifts for disjoint ones, so the two surviving block sums add to $s(c_g)$. The complete proof is on p. 886. □

Lemma 32.12 (Values). *If $n > 0$ and $s((X-1)^j) = j$ for all $j \geq 0$, then $\Psi_s(1) = \prod_{1 \leq j < n} (1 - v^j)$. For additive s, irreducible c_g gives $\Psi_s(g) = s(c_g)$.*

Proof. With $b_{n,a}$ the number of a-dimensional subspaces, counting maps $F^n \to F^j$ by their kernels gives, at $Y = v^j$,

$$\sum_{a=0}^{n} b_{n,a} \prod_{i=0}^{a-1} (Y - v^i) = Y^n.$$

Differentiate the polynomial identity at $Y = 1$ to obtain the recursion for the proposed values, with zero in dimension zero. For irreducible c_g, the sole proper invariant subspace is zero. □

Lemma 32.13 (Virtual characters). *For $h \geq 0$, both R_h and Ψ_h restrict to virtual characters of H.*

Outline. Over a finite extension containing the eigenvalues of H, Green's Theorem 1 applied to $\sum_i X_i^h$ shows R_h is a virtual character. Reading 32.10 as parabolic induction of Ψ_h tensored with the trivial character proves Ψ_h by induction. The complete proof is on p. 886. □

Lemma 32.14 (Mass). *Write $M_\phi(P) = \sum_{c_g = P} \phi(g)$ over endomorphisms. For additive s and monic P of degree $n > 0$, the mass for Ψ_s is $|H| s(Q)/(v^n - 1)$ if $P = Q^a$ for monic irreducible Q and $a > 0$, and zero otherwise.*

Outline. Count invariant subspaces in 32.10 and count vectors by their cyclic spans to get two convolution identities for the normalized masses. Use additivity and unique factorization to solve for the mass of Ψ_s. The complete proof is on p. 886. □

Lemma 32.15 (Torus pairing). *For primitive class ϕ and $n > 0$, put $P_\alpha = \min_F(\alpha)$ and $a_\alpha = n/\deg P_\alpha$ for $\alpha \in \mu_M(\overline{F})$. Then*

$$I(\phi, R_l) = \sum_{\alpha \in \mu_M} a_\alpha M_\phi(P_\alpha^{a_\alpha}) \overline{t(\alpha)^l}, \qquad \sum_{\alpha \in \mu_M} t(\alpha)^h \overline{t(\alpha)^l} = M \mathbf{1}_{h \equiv l \bmod M}.$$

Proof. Split a nonprimary polynomial into coprime blocks. The Sylvester map on the off-diagonal block is invertible, so all variations are conjugate and primitivity forces the value to vanish. For $c_g = P^a$ in H, the roots lie in μ_M since $\deg P \mid n$, and each contributes with multiplicity a. This gives the first identity. The second identity follows because the restriction of t bijects the two groups of Mth roots. □

Lemma 32.16 (Mixed pairing)**.** *For $n > 0$, $I(\Psi_h, R_l) = |H|\#\{i < n : hv^i \equiv l \bmod M\}$.*

Proof. Insert 32.14 and $a_\alpha s_h(P_\alpha) = \sum_{i<n} t(\alpha)^{hv^i}$ into 32.15. □

Proposition 32.17 (Expansion)**.** *For $n > 0$ and primitive class ϕ, $n|H|\phi = \sum_{0\le l<M} I(\phi, R_l)\Psi_l$ on H.*

Outline. Use 32.15 and 32.16 to show the difference has zero pairings and hence zero primary masses by Fourier inversion. On each primary fibre, descend on the kernel dimension using 32.11 and use 32.14 to show that mass determines the values. The complete proof is on p. 886. □

Lemma 32.18 (Inverted pairing)**.** *For primitive ϕ and additive s, $I(\phi, \Psi_s) = I(\phi, s \circ c)$.*

Proof. The term $W = 0$ in 32.10 vanishes. For each $0 < W < V$, fix the diagonal blocks of g stabilizing W. The restriction to W is constant while the off-diagonal block varies, so primitivity cancels the pairing. □

Lemma 32.19 (Regular exponent)**.** *If $n > 0$ and $hv^i \not\equiv h \bmod M$ for $0 < i < n$, then $I(\Psi_h, \Psi_h) = |H|$. Its restriction to H is $(-1)^{n-1}$ times an irreducible character.*

Proof. The norm follows from 32.18 and 32.16. By 32.13 the irreducible coefficients are integers, so normalized norm one leaves a single coefficient ± 1. Its sign follows from 32.12. □

Lemma 32.20 (Cuspidality)**.** *Over $\mathbf{F}_p$, the irreducible character of 32.19 is cuspidal.*

Proof. Apply 32.11 at $g = 1$, where the $1 + h$ form the unipotent radical for A. □

Theorem 32.21 (Regular family)**.** *For $[K : \mathbf{F}_p] = d > 0$, there exists a family $(\tau_\theta)_{\theta \in \widehat{K^\times}}$ with the following properties on regular parameters. Its members are cuspidal and have the values of 32.5. They agree exactly within Frobenius orbits.*

Proof. Embed K in $\overline{\mathbf{F}}_p$ and write $\theta = t^h$ on $K^\times$ using 32.8. Take $(-1)^{d-1}\Psi_h|_{G_d}$ for regular parameters by 32.19 and 32.20, and zero elsewhere. The values follow from 32.7 and 32.12. The pairings detect the orbit by 32.16 and determine the character by 32.17. □

Lemma 32.22 (Orbit injection)**.** *For $[K : \mathbf{F}_p] = d > 0$, an isomorphism $K^\times \simeq \widehat{K^\times}$ sends chosen roots of $P \in \mathcal{P}_d$ to characters $c(P)$ in distinct regular orbits.*

Proof. Both groups are cyclic of order N_d. The isomorphism commutes with pth powers, and each chosen root has an orbit of size d that determines its minimal polynomial by 32.7. □

Lemma 32.23 (Character injection)**.** *For $[K : \mathbf{F}_p] = d > 0$, with τ of Theorem 32.21 and $c(P)$ of Lemma 32.22, the map $L_d(P) = \tau_{c(P)}$ injects $\mathcal{P}_d$ into the cuspidals of G_d.*

Proof. Apply 32.21 to the orbits of 32.22. □

Let $I(C)$ be the Harish–Chandra product of a cuspidal list C of total degree n. For nontrivial additive η of $\mathbf{F}_p$, set $\xi_\eta(u) = \eta(\sum_i u_{i,i+1})$ on the upper unitriangular subgroup U_n. Write wh_η for the linearly extended ξ_η-multiplicity. On a finite group G use

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g)\psi(g^{-1}).$$

For characters of H this agrees with $I(\phi, \psi)/|H|$.

Theorem 32.24 (Generic constituent)**.** *Every $I(C)$ has an irreducible constituent of positive Whittaker multiplicity.*

Outline. The finite-field Gelfand–Graev theorem gives each cuspidal factor Whittaker multiplicity one. In Mackey restriction to U_n, Frobenius reciprocity gives one from the block-reversing double coset, so some irreducible constituent has positive Whittaker multiplicity. The complete proof is on p. 886. □

Theorem 32.25 (Harish–Chandra pairing)**.** *For equal total degrees, $\langle I(C), I(C')\rangle = 0$ unless C' permutes C. Distinct entries in C give $\langle I(C), I(C)\rangle = 1$. See [141, pp. 107–148] for the general linear groups over finite fields.*

Outline. Writing $C' = (\sigma, C'_0)$, reciprocity and Mackey restriction expand the pairing over block distributions into the Levi factors. Cuspidality kills split blocks and multi-block products, leaving $\sum_{i:C_i=\sigma}\langle I(C\setminus i), I(C'_0)\rangle$ for induction. The complete proof is on p. 887. □

Theorem 32.26 (Gelfand–Graev norm)**.** *For $n > 0$, $\Gamma_\eta = \operatorname{Ind}_{U_n}^{G_n}\xi_\eta$ satisfies $\operatorname{wh}_\eta(\Gamma_\eta) = p^n - p^{n-1}$ and $\langle \chi, \Gamma_\eta\rangle = \operatorname{wh}_\eta(\chi)$ for class functions χ.*

Proof. Over a finite field the Gelfand–Graev character is multiplicity free and has $(p-1)p^{n-1}$ constituents in the present prime-field case [141, Prop. 9.4], [142, §1, p. 76]. Its self-pairing therefore has this value. Frobenius reciprocity identifies pairing against the induced character with the ξ_η-multiplicity on restriction to U_n, and linearity gives the formula for class functions. □

Corollary 32.27 (Generic count)**.** *For $n > 0$ the number of generic irreducibles is at most $p^n - p^{n-1}$.*

Proof. Each contributes at least one to the norm in 32.26. □

Lemma 32.28 (Support count)**.** *For $n > 0$ there are at most $p^n - p^{n-1}$ cuspidal multisets of total degree n.*

Proof. The generic constituents supplied by 32.24 for distinct multisets are distinct by 32.25 and are counted by 32.27. □

Lemma 32.29 (Polynomial count)**.** *There are $(p-1)p^{n-1}$ monic degree-n polynomials with nonzero constant term for $n > 0$.*

Proof. There are $p-1$ choices for the constant coefficient and p for each of the other $n-1$ lower coefficients. □

Lemma 32.30 (Exhaustion)**.** *Injective cuspidal families L_e on $\mathcal{P}_e$ for all $e > 0$ exhaust the cuspidals.*

Proof. Replace each irreducible factor P of a polynomial counted in 32.29 by $(\deg P, L_{\deg P}(P))$. This embeds the polynomials into the multisets of 32.28 and attains the bound. A missing degree-n cuspidal would add a singleton. □

Theorem 32.31 (Green's classification)**.** *The regular form 32.5 holds. This is the prime-field cuspidal part of Green's classification [20, §§7–8, Thm. 14].*

Proof. Fix the degree and the field in 32.5, and take the family of 32.21 for that field. In every positive degree choose such a family, using the fixed one in the specified degree. Lemma 32.23 gives an injection from the prime labels into the cuspidals in each degree, and Lemma 32.30 shows that these injections are surjective. In particular the fixed regular family exhausts the cuspidals in its degree. Its values and its identification of equal characters with equal Frobenius orbits are those of Theorem 32.21. □

32.2 Polynomial labels

Fix L and $\iota : K = \mathbf{F}_{p^d} \hookrightarrow \overline{\mathbf{F}}_p$, $d > 0$.

Lemma 32.32 (Torus)**.** *The image of ι is the root set of $X^{p^d} - X$. The preimage v of u_d generates $K^\times$.*

Proof. The p^d elements of $\iota(K)$ solve the equation, and u_d has order $p^d - 1$. □

Lemma 32.33 (Duality)**.** *The map $x \mapsto \theta_x$, defined by $\theta_x(v^a) = t(x)^a$, is a bijection from $\mu_{N_d}(\overline{\mathbf{F}}_p)$ onto $\widehat{K^\times}$. It satisfies $\theta_x(k) = \Theta_d(x, \iota(k))$.*

Proof. Evaluation at v identifies the character group with $\mu_{N_d}(\mathbf{C})$, which the N_d distinct $t(x)$ exhaust. The identity follows from 32.3. □

Lemma 32.34 (Regularity). *For $x \in \mu_{N_d}$ and $a \geq 0$, $\theta_{x^a} = \theta_x^a$, and θ_x is regular exactly when the x^{p^i}, $i < d$, are distinct.*

Proof. Evaluation at v gives $\theta_{x^a} = \theta_x^a$, and the regularity criterion follows. □

Lemma 32.35 (Label roots). *Each $P \in \mathcal{P}_d$ has d distinct roots in μ_{N_d}.*

Proof. This follows from 32.7 and $P(0) \neq 0$. □

Lemma 32.36 (Root sums). *For irreducible f with root z, summing a function along $(z^{p^i})_{i<\deg f}$ sums it over the roots. For $x, w \in \mu_{N_d}$ and $a \geq 0$, $\Theta_d(x^a, w) = \Theta_d(x, w^a)$.*

Proof. The first assertion follows from 32.7 and the second from 32.3. □

Theorem 32.37 (Relabelling). *The regular form 32.5 implies 32.6.*

Outline. Choose a root giving a regular character by 32.34 and 32.35. Use 32.32 and 32.33 to evaluate at elliptic elements and 32.36 for independence of the root, then identify the regular orbits with degree-d minimal polynomials. The complete proof is on p. 887. □

The unitary-group hypotheses use 32.31. The polynomial form is used in Theorem 1.3.

32.3 Series weights

Lemma 32.38 (Proper subfields). *For $p \geq 5$, $d > 0$, the set B of elements of degree $< d$ in $\mathbf{F}_{p^d}^{\times}$ satisfies $2d|B| < N_d$.*

Proof. For $d > 1$, put $h = \lfloor d/2 \rfloor$. Every proper subfield has degree at most h, so $|B| \leq S_h = \sum_{e=1}^{h}(p^e - 1)$. Induction gives $4S_h + 4h + 5 \leq 5p^h$ and $5h \leq p^h$, hence $4hS_h < p^{2h} - 1$. This suffices for even d. For odd $d > 1$, $2dS_h < 8hS_h < 2p^{2h} < N_d$. In degree one B is empty. □

Lemma 32.39 (Power sums). *Let two lists of length $d > 0$ in $\mu_M(\mathbf{C})$, $M > 1$, have equal power sums outside $B \subset \mathbf{Z}/M$, with $2d|B| < M$. Then the first entry of the second list occurs in the first.*

Proof. The multiplicity differences c_ζ have support of size at most $2d$ and Fourier support in B. Parseval and Cauchy–Schwarz give

$$M \sum_{\zeta} |c_\zeta|^2 \leq 2d|B| \sum_{\zeta} |c_\zeta|^2.$$

Thus all differences vanish. □

Theorem 32.40 (Unique labels). *For $p \geq 5$, two prime labels of one cuspidal agree. Under Green, two cuspidals with the same prime label agree.*

Proof. Evaluate (32.1) on multiplication by u_d^a of degree d. The lifted root lists have equal power sums outside the exponents corresponding to B in 32.38. By 32.39 and injectivity of t, the two minimal polynomials share a root. Under Green, each cuspidal belongs to the family and its unique label fixes its parameter. □

Definition 32.41 (Series). An irreducible ρ has label f if it occurs in $I(C)$ for a cuspidal list whose prime labels multiply to f. Sum the corresponding central idempotents z_ρ to get E_f. For $a \in \mathbf{C}[G_n]$ put $w_a(f) = \operatorname{Re}[aE_f]_1 / \operatorname{Re}[a]_1$, where $[\,]_1$ is the identity coefficient and a zero denominator gives weight zero.

Theorem 32.42 (Squarefree series). *For $p \geq 5$ under Green, each irreducible of G_n has a unique monic degree-n label with nonzero constant term. For squarefree such f, its series consists of $\rho_f = I((\deg P, \tau_P)_{P|f})$, over monic irreducible factors.*

Proof. A minimal Levi with nonzero Harish–Chandra restriction supplies cuspidal factors by transitivity. Reciprocity gives occurrence in their product. The support is unique by 32.25, and the prime labels are unique by 32.40. For squarefree f, the factors are distinct and determined by f. Their product is invariant under permutation and has norm one by 32.25. □

Lemma 32.43 (Arithmetic relabelling)**.** *Assume $p \geq 5$ and Green. Let O label simple cuspidal representations by $\mathcal{P}_d$ in each degree $d > 0$, with equality equivalent to isomorphism. Let $E^O_{r,f}$ be the rank-r series projector for the product f of O-labels. There are inverse degreewise permutations σ, π of the prime labels. On monic f with $f(0) \neq 0$ their multiplicative extensions preserve degree and squarefreeness, with $\pi_*\sigma_* f = f$ and $E^O_{r,f} = E_{\sigma_* f}$.*

Proof. For a simple cuspidal V_P of character τ_P, set $\pi(P) = O(V_P)$. The isomorphism criterion and Green make π well-defined and injective. Finiteness gives bijectivity. Its inverse σ gives $\sigma_* f = \prod_P \sigma(P)^{v_P(f)}$. Both projectors select the same supports, and the substitution preserves the degrees and multiplicities. □

For a subgroup H and a degree-one character ξ, put $e(H, \xi) = |H|^{-1} \sum_{u \in H} \xi(u)^{-1} u$. Set $e_U = e(U_n, \xi_\eta)$. For $n = k+2s+1$, $k \geq 1$, $s \geq 0$, use zero-based indices. Let $H_J \subset U_n$ impose $u_{a,s+k} = 0$ for $s \leq a < s+k$. Set $e_J = e(H_J, \eta \circ \phi_J)$, where ϕ_J sums consecutive entries along $(0, \dots, s-1, s+k, \dots, n-1)$ and $(s, \dots, s+k-1)$. Write $w_U = w_{e_U}$ and $w_J = w_{e_J}$.

Lemma 32.44 (Idempotents)**.** *The averages e_U, e_J are idempotent with identity coefficients p^{-b}, p^{k-b}, where $b = n(n-1)/2$. For idempotent a with $\mathrm{Re}[a]_1 \geq 0$, $w_a(f) \geq 0$. The coefficient $[az_\rho]_1$ is $\rho(1)/|G_n|$ times the rank of a in the representation of ρ.*

Proof. Each average projects onto its character space. The k zero entries give $|H_J| = p^{b-k}$. Left multiplication by az_ρ has trace $|G_n|[az_\rho]_1$ on the regular representation. Only the $\rho(1)$ copies of ρ contribute, with trace equal to rank on each. This also gives nonnegativity. □

Lemma 32.45 (Multiplicity one)**.** *For a cuspidal character χ of G_d with $\chi(1) = \Delta_d$, we have $\mathrm{wh}_\eta(\chi) = 1$.*

Proof. Every complex irreducible cuspidal character occurs in the Gelfand–Graev character over a finite field, and each constituent occurs once [141, Prop. 9.4], [142, §1, p. 76]. Apply reciprocity to the nondegenerate character ξ_η of U_d. The published result allows every finite field and requires no degree hypothesis. □

Lemma 32.46 (Whittaker multiplicities under induction)**.** *For the nontrivial additive character η of $\mathbf{F}_p$ fixed above, let α and β be class functions of G_a and G_b. Write $\alpha \circ \beta$ for the Harish–Chandra induction of their external product to G_{a+b}. Then*

$$\mathrm{wh}_\eta(\alpha \circ \beta) = \mathrm{wh}_\eta(\alpha)\, \mathrm{wh}_\eta(\beta).$$

Proof. By linearity take characters. Mackey restriction to the upper unitriangular subgroup gives one summand for each Bruhat cell of the parabolic. In every cell except the block-opposite cell, a simple root subgroup lies in the intersection with the conjugate unipotent radical. The inducing representation is trivial there and the Whittaker character is nontrivial, so that summand has zero Whittaker multiplicity. In the block-opposite cell the intersection is the product of the upper unitriangular subgroups inside the two blocks, and the chosen character restricts to the product of their Whittaker characters. Frobenius reciprocity gives the product of their multiplicities. If one block has rank zero the formula is scalar multiplication by the value of the class function on the trivial group. □

Theorem 32.47 (Squarefree weight)**.** *Under 32.42, for monic squarefree f of degree n with $f(0) \neq 0$, $w_U(f) = \prod_{P|f} (p^{\deg P} - 1)^{-1}$ over monic irreducibles.*

Proof. The parabolic index times the cuspidal degrees gives $\rho_f(1) = |G_n| p^{-b} \prod_{P|f} (p^{\deg P} - 1)^{-1}$. Iterate 32.46, using 32.45 on every cuspidal factor, to obtain Whittaker multiplicity one. The rank of e_U on ρ_f is therefore one. Apply 32.44 and divide by the identity coefficient p^{-b} of e_U. □

Lemma 32.48 (Column exchange)**.** *Let e'_J be the ξ_η-average on the subgroup of U_n given by $u_{a,k} = 0$ for $a < k$. For $n = k + 2s + 1$ and every class function χ of G_n, extended linearly to the group algebra, $\chi(e_J) = \chi(e'_J)$.*

Outline. Move the inserted chain to the front one step at a time. The nondegenerate commutator pairing gives Mackey self-pairings of one, and the common intersection gives a positive cross-pairing, so induction and reciprocity preserve the trace. The complete proof is on p. 887. □

Lemma 32.49 (Mirabolic trace)**.** *For $0 \le i \le n$, let D_i^η take Harish–Chandra restriction to $(i, n-i)$, then the Whittaker multiplicity space of the second block, with the same notation for its character. Under 32.48, for every class function χ,*

$$\chi(e_J) = \sum_{i<k} (p^{k-i} - p^{k-i-1}) \operatorname{wh}_\eta(D_i^\eta \chi) + \operatorname{wh}_\eta(D_k^\eta \chi). \tag{32.2}$$

Outline. Write $\chi(e'_J)$ as the Whittaker multiplicity of a tail-averaged character on P_{k+1}, and decompose it by column-character orbits up to the first zero step i. Restricting to G_k yields a single Mackey term equal to a Harish–Chandra product with $D_i^\eta \chi$. Lemma 32.46 and Theorem 32.26 evaluate its Whittaker multiplicity, and Lemma 32.48 transfers the trace to e_J. The complete proof is on p. 888. □

Theorem 32.50 (Lower comparison)**.** *For any prime p, datum L, $n = k + 2s + 1$ with $k \ge 1$, and every f, $(1 - p^{-1}) w_U(f) \le w_J(f)$.*

Proof. For irreducible χ all terms of 32.49 are nonnegative. The $i = 0$ term is $(p^k - p^{k-1})\chi(e_U)$. Sum the coefficient formula of 32.44 over the series and divide by the identity-coefficient ratio p^k. □

Lemma 32.51 (Derivative counts)**.** *Let C have total degree n and cuspidal factors of Whittaker multiplicity one. Then $\operatorname{wh}_\eta(I(C)) = 1$, and $\operatorname{wh}_\eta(D_i^\eta I(C))$ counts subsets of positions of total degree i, for $0 \le i \le n$.*

Outline. Mackey restriction distributes inducing blocks between the Levi factors, with cuspidality killing split blocks. For each surviving distribution, 32.24 gives 1 on the opposite cell while other cells vanish on a simple root subgroup where ξ_η is nontrivial. The complete proof is on p. 888. □

Theorem 32.52 (Upper comparison)**.** *Under 32.47, if $n = k + 2s + 1$ and $k \ge 1$, then $w_J(f) \le (1 + p^{-1})^n w_U(f)$.*

Proof. For the subset counts c_i of 32.51, the trace lemma 32.49 and 32.44 give

$$\frac{w_J(f)}{w_U(f)} = p^{-k} \rho_f(e_J) \le \sum_{i \le k} c_i p^{-i} \le \prod_{P \mid f} (1 + p^{-\deg P}) \le (1 + p^{-1})^n.$$

□

The remainder of this chapter specializes Green's theory to the arithmetic point and family data constructed in Part VIII.

Theorem 32.53 (Arithmetic weights)**.** *Assume Green. Let D be an arithmetic point datum in the sense of Definition 33.6 (Part VIII, Chapter 33), with host CM field E, spectral rank $r = k + 2s + 1$ ($k \ge 1$, $s = \lfloor m/2 \rfloor - 1$), host primes $q, d > 4tr + 1$, conductor parameter Q_S divisible by 2^{e+2} ($e \ge 2$), and distinguished primes $p_0, p_1 \equiv 1 \pmod{Q_S q d}$. Fix an embedding $j_E : E \hookrightarrow \overline{\mathbf{Q}}$. At each p_j use the normalized local types of Definition 33.6 for this embedding: the Teichmüller lift and compatible generators give the label datum of 32.3, and the arithmetic projector for a polynomial f is the series projector for its cuspidal labelling as in 32.43, with value zero off the monic degree-r labels of nonzero constant term. At each type prime p_j and for nontrivial η, the arithmetic U, J weights (the local U, J weights of Section 3 pulled back along the degreewise permutation) are probabilities supported on monic degree-r labels with nonzero constant term. They satisfy 32.50 on these labels and 32.47, 32.52 on squarefree ones.*

Outline. The normalized types give $p_j \ge 5$, and Lemma 32.43 relabels the projectors via $\sigma_* f$ while preserving degrees and squarefreeness. Theorem 32.42 and Lemma 32.44 show that the resulting weights form a probability distribution. Applying Theorem 32.50, Theorem 32.47, and Theorem 32.52 at $\sigma_* f$ gives the lower bound, the squarefree formula, and the upper comparison. The complete proof is on p. 888. □

32.4 Bad rows

32.4.1 Factor degrees at one place

Let K be a finite field of cardinality u. Write $\mathcal{S}_N(K)$ for the monic squarefree polynomials of degree N with nonzero constant term. For monic f put

$$w(f) = \prod_{P^a \| f} (u^{\deg P} - 1)^{-a}, \qquad M_N(E) = \sum_{f \in \mathcal{S}_N(K) \cap E} w(f),$$

where P runs over monic irreducibles. Write $\lambda(f)$ for the multiset of factor degrees, with multiplicities, and $\lambda(\sigma)$ for the multiset of cycle lengths, including fixed points.

Lemma 32.54 (Comparison by factor degrees)**.** *For $N \geq 0$ and any collection $\mathcal{A}$ of partitions of N,*

$$M_N\{f : \lambda(f) \in \mathcal{A}\} \leq \frac{|\{\sigma \in S_N : \lambda(\sigma) \in \mathcal{A}\}|}{N!}.$$

If h is monic with $\deg h \leq N$, then $M_N\{f : h \mid f\} \leq w(h)$. For $N \geq 1$ the total mass lies between $1 - 2/u$ and 1.

Outline. Bound the number of degree-j irreducibles with nonzero constant term by $(u^j - 1)/j$ to compare each factorization type with its cycle proportion. Divide by a prescribed divisor to bound its mass, and count excluded polynomials for the total mass lower bound. The complete proof is on p. 889. □

Lemma 32.55 (A divisor of prescribed degree)**.** *For every finite field K and all integers $0 < b < N$,*

$$M_N\{f : \exists h \in K[X],\ h \mid f,\ \deg h = b\} \leq 7/8.$$

Outline. Reduce by 32.54 to uniform permutations, with X counting invariant b-sets ($\mathbb{E}X = 1$) and Y those with fixed points inside and out. Inclusion-exclusion gives $\mathbb{E}Y \geq 1/4$, and swapping gives $\mathbf{2}1_{X>0} + Y \leq 2X$, hence $7/8$. The complete proof is on p. 889. □

Lemma 32.56 (Prime degrees in the upper half)**.** *For every integer $N \geq 100$ let $\mathcal{J}_N = \{b \text{ prime} : N < 2b,\ b + 3 \leq N\}$. Then*

$$H_N = \sum_{b \in \mathcal{J}_N} \frac{1}{b} \geq \frac{1}{5 \log N}.$$

Outline. For $N < 20000$, tabulate reciprocal lower bounds using primes shared by each interval. For larger N, bound the prime factors of the central binomial coefficient to count primes in the upper half, discarding the last three possible values. The complete proof is on p. 889. □

Lemma 32.57 (Missing a long prime-degree factor)**.** *For every finite field K and integer $N \geq 100$,*

$$M_N\{f : \text{no irreducible factor of } f \text{ has degree in } \mathcal{J}_N\} \leq 1 - \frac{1}{5 \log N}.$$

Proof. For $b \in \mathcal{J}_N$ a b-cycle is unique if it occurs. Choosing and ordering its support gives $\binom{N}{b}(b-1)!(N-b)! = N!/b$ permutations. The events for distinct b are disjoint since the lengths exceed $N/2$. The complementary probability is $1 - H_N$. Apply 32.54 and 32.56. □

For prime ℓ and nonzero x of finite order, write $x_{\ell'}$ for the prime-to-ℓ component of x in $\langle x \rangle$. The projection is multiplicative and commutes with Frobenius. For $u = |K|$ and $N \geq 0$, put

$$B(u, N) = \{\operatorname{char} K\} \cup \{\ell \text{ prime} : \exists\, 1 \leq j \leq N,\ u^j < \ell^{4v_\ell(u^j - 1)}\}.$$

Lemma 32.58 (Collisions of prime-to-ℓ parts)**.** *Take roots in an algebraic closure of K. For $u = |K| \geq 4096$ and $N \geq 0$, the M_N-mass of polynomials with distinct roots x, y satisfying $x_{\ell'} = y_{\ell'}$ for some prime $\ell \notin B(u, N)$ is at most $23 \log u/\sqrt{u}$.*

Let $N \geq 1$, $c \in K^\times$, and let q, d be distinct odd primes with $\max(q, d) \geq 16$ and $16, q, d \mid u - 1$. The M_N-mass of polynomials with a root x satisfying $x_{\ell'} = c_{\ell'}$ for some prime $\ell \neq \operatorname{char} K$ is at most

$$\frac{1}{4 \min(q, d)} + \frac{1 + \log N}{u - 1}.$$

Outline. Count collisions within and across factors using Frobenius orbits and projection fibres, and sum the divisor bounds of 32.54. For a fixed target, count roots of prime-power binomials and use prime valuations to sum the bounds by degree. The complete proof is on p. 890. □

Corollary 32.59 (Exceptional factorisations in one row)**.** *Under the hypotheses of 32.58 with $u = p \geq 4096$ prime and $N = r \geq 100$, let P be a probability on the monic degree-r polynomials over $\mathbf{F}_p$ with masses $w(f)$ on $\mathcal{S}_r(\mathbf{F}_p)$. The complement has probability at most $2/p$. On $\mathcal{S}_r(\mathbf{F}_p)$ the bounds of 32.55, 32.57 and 32.58 hold for P.*

Proof. The restriction is M_r, whose total mass is at least $1 - 2/p$ by 32.54. □

32.4.2 Arithmetic rows

Write $\nu = f$ for the number of places in each type-prime tuple and m for the chain length. For arithmetic row weights $w_{A,j,v}$ put

$$P_j(S) = \sum_{t \in S} \prod_{v \mid p_j} w_{A,j,v}(t_v),$$

$$W(\nu, m, r, p, q, d) = \frac{3\nu}{p} + \frac{23\nu \log p}{\sqrt{p}} + 2\nu m \left(\frac{1}{4 \min(q, d)} + \frac{1 + \log r}{p - 1} \right) + \frac{r}{2}(7/8)^\nu + e^{-\nu/(5 \log r)}.$$

At $p = p_j$ a good tuple consists of squarefree admissible polynomials t_v. For prime $\ell \notin B(p, r)$ of 32.58, the map $x \mapsto x_{\ell'}$ is injective on the roots of each t_v. For every prime $\ell \neq p$, these parts avoid the prime-to-ℓ parts of the prescribed $\chi_{v,a}^{\pm 1} \in \mathbf{F}_p^\times$ $(1 \leq a \leq m)$ of Definition 33.6. For each $1 \leq b \leq r/2$ some t_v has no degree-b divisor. Some t_v has an irreducible factor of degree in $\mathcal{J}_r$.

Lemma 32.60 (Bad-label weight)**.** *For an arithmetic point datum of Definition 33.6 with host field E, imaginary quadratic host subfield $K_d = \mathbf{Q}(\sqrt{-d})$, and host parameters Q_S, q, d, t $(q, d > 4tr + 1)$, assume the weight laws of 32.53 and $r \geq 100$. Require both distinguished primes p_j to be at least 4096, to split completely in E and in the ray class field M_{ray} of K_d for modulus $Q_S qd$, and to satisfy $p_j \equiv 1 \pmod{Q_S qd}$. Suppose that for each $j \in \{0, 1\}$ there is a nontrivial additive character η_j such that the row idempotent at every place above p_j is either $e_U(\eta_j)$ or $e_J(\eta_j)$. Then every row weight is nonnegative. For each j and every finite set S of tuples bad at p_j,*

$$0 \leq P_j(S) \leq W(\nu, m, r, p_j, q, d).$$

Outline. Match the collision tests to 32.58 and use 32.59 with independence and a union bound for the product U-law. Apply the lower comparison of 32.53 on the complementary event to transfer the bound to the row laws. The complete proof is on p. 890. □

Definition 32.61 (Compatibility of an arithmetic row system)**.** Fix an arithmetic point with host CM field E, spectral rank r, and distinguished primes p_0, p_1. Call a row system compatible with this point if the following conditions hold.

The real places of the row system's totally real field are in bijection with the archimedean indices of the point. Its rank, totally real degree and root-discriminant parameter agree with $r, f, \mathrm{rd}(E)$. At corresponding infinite places its signature and Harish–Chandra parameter agree with those prescribed by the point. The coordinates of the parameter are strictly decreasing and belong to $(r - 1)/2 + \mathbf{Z}$. The local representation is discrete series with this parameter, and the chosen vector is a unit vector in its lowest K-type.

The level integer is p_0p_1 and its distinguished prime is p_0. If p_0 splits completely in E, it splits completely in the row system's CM field. The full row dimension is D_{all} of the point. The product of the archimedean formal degrees, the volume of the fundamental domain and the real part of the full finite-adelic kernel at the identity equals $D^\star$ of the point.

For each $j = 0, 1$ there is a nontrivial additive character η_j of $\mathbf{F}_{p_j}$ such that at every row place above p_j the local idempotent is $e_J(\eta_j)$.

Theorem 32.62 (Defect bound)**.** *Fix $0 < \varepsilon \le 1$, $c > 0$, $C_s \in \mathbf{R}$ and a family F_m of parameter ε in the sense of Definition 33.6 (Part VIII, Chapter 33) with $J \ge 1$ and $\alpha = \varepsilon/J$, whose distinguished primes p_0, p_1 split completely in E and M_{ray} and satisfy $p_j \equiv 1 \pmod{Q_S qd}$ and $q, d > 4tr + 1$. Write I_x for the indices of the members of F_m at x. For all sufficiently large x and each $i \in I_x$, suppose there is a compatible arithmetic row system A in the sense of 32.61 for which the weight laws of 32.53 hold and*

$$p_j \ge e^{8x^\alpha}, \qquad x^2 \le \nu \le x^{C_s}, \qquad m \le x^{C_s}, \qquad 100 \le r \le x^{C_s}, \qquad \min(q, d) \ge e^{x^{\alpha/6}}.$$

Assume the map from retained to all-row indices preserves labels and is injective with image exactly the indices good at both primes. For every set U of label pairs, assume that convolution with the kernel for U has image of complex dimension $\#\{a :$ the label pair of a lies in $U\}$ on the multiplicity space of the prescribed archimedean representation. Here a ranges over all row indices.

Let Ω be the set of all label pairs, e the finite-adelic kernel of the full row system, and $e_T(U)$ the kernel for $U \subseteq \Omega$. Write $C_j(S) = \{(t_0, t_1) \in \Omega : t_j \in S\}$. The kernel identities are

$$e_T(\Omega) = e, \qquad \mathrm{Re}\big(e_T(C_j(S))(1)\big) = \mathrm{Re}\, e(1) \sum_{t \in S} \prod_{v \mid p_j} w_{A,j,v}(t_v) \quad (j = 0, 1,\ S \text{ finite}). \tag{32.3}$$

For each cylinder let $N_j(S)$ count its indices. Put

$$D_j(S) = d_\infty \,\mathrm{vol}(\mathscr{F})\, \mathrm{Re}\big(e_T(C_j(S))(1)\big), \qquad D_* = d_\infty \,\mathrm{vol}(\mathscr{F})\, \mathrm{Re}\, e(1),$$

where d_∞ is the product of the archimedean formal degrees and $\mathscr{F}$ is the automorphic fundamental domain with the row system's Haar measure. Assume $D_* > 0$ and, for every cylinder,

$$|N_j(S) - D_j(S)| \le \epsilon_C D_j(S), \qquad \epsilon_C = e^{-cr\nu \log r}.$$

Eventually, uniformly in i, the relation $D_{\mathrm{rows}} = (1 - \delta_{x,i}) D_{\mathrm{all}}$ between the retained and total counts gives $0 \le \delta_{x,i} \le e^{-x^{\alpha/7}}$.

Outline. Bound discarded rows by the two bad cylinders and use relative counting and the cylinder identities to bound the defect by their weights. Use 32.61 for the idempotents in 32.60 and the size bounds for uniform decay. The complete proof is on p. 891. □

Theorem 32.63 (Closed defect bound)**.** *Assume Green. Fix $c > 0$, a family class $\mathcal{A}$ and a twist selection. Suppose the following assertion is closed on $\mathcal{D}_3(\mathcal{A})$ (Part VIII, Chapter 36). There is an x_c such that for every $x \ge x_c$ and every $i \in I_x$, one compatible row system satisfies the index, dimension and relative-count conditions of 32.62. Then the defect bound $0 \le \delta_{x,i} \le e^{-x^{\alpha/7}}$ of 32.62 is closed on this class in the sense of Definition 36.10 (Part VIII, Chapter 36).*

Outline. Keep the size thresholds of 36.10 and take a common point threshold. The selected types give 32.53 on the same rows, and the family sizes verify the numerical hypotheses of 32.62 with $C_s = 6$. The complete proof is on p. 891. □

32.5 Connected families

Theorem 32.64 (Connected families)**.** *Assume Green's polynomial classification of 32.6. Under the common hypotheses of 33.6, assume that the family bound for the weighted local lengths in 36.11 is closed on $\mathcal{D}_3(\mathcal{A})$ in the sense of 36.10. Fix $\lambda > 1$ and a request G for which hosts exist on λ-windows in the sense of 2.2. Let*

$\mathcal{E}$ consist of the families with λ-windows for which every member beyond the family's initial cutoff satisfies G. Suppose that in each family in $\mathcal{E}$, all members at sufficiently large x satisfy the finiteness and weighted length bound of 26.1 for their host field E, up to t with bound $B = \sqrt{b}$. Then there is an $\varepsilon_1 > 0$ such that for every $0 < \varepsilon < \varepsilon_1$ there is a family of exponent ε in $\mathcal{E}$ with the following property. There are constants K, c_{HH}, e_K, C_L, x_0 such that every window point $x \geq x_0$ admits the common chain realisation of 24.1 with division and loss exponents $(e_D, e_L) = (2, 0)$.

Outline. Combine the closed estimates by 36.10 and choose a family meeting their thresholds. Use the local lengths via 36.13 and 21.32 for division exponent 2 and common weights via 22.44 for loss exponent 0, then apply 24.1. The complete proof is on p. 891. □

Part VIII

Input three: the witness for HypsAlt

Chapter 33

A common witness for the 46 conditions

Let F_m be a family of parameter ε, with point indices $i \in I_x$ at $x > x_1(F_m)$. Write D, L, O for the arithmetic data, size data and objects at (x, i), respectively. Put $b = \exp(x^{\varepsilon})$, and let $\iota_D : K_D \hookrightarrow \mathbf{C}$ be the coefficient embedding. Use the two products P, F of 1.20. The join used for the test vectors is the join in the fourth part of the toric input of 1.15.

33.1 Existence and the common data

Use the family quantifiers $\mathrm{Closed}_{\mathcal{C}}$ of Definition 36.10 and the subclass $\mathcal{A}_{\mathrm{ch}}$ of Definition 17.8. Put

$$W_1(\mathcal{A}) = \{F_m \in \mathcal{A}_{\mathrm{ch}} : F_m \text{ satisfies the smoothness and twisting conditions below}\},$$
$$W_3(\mathcal{A}) = \mathcal{D}_3(\mathcal{A}) = \mathcal{D}(\mathcal{A}_{\mathrm{ch}}, 3).$$

For a frame with group A, let $n_H = |A|$ and $f = 2n_H$. The rank at layer j is $N(j) = k + j(t+1)$, where t is the rank increment parameter. Membership in $\mathcal{A}_{\mathrm{ch}}$ requires tame distinctness as in clause 4 of 33.3, the zero-free boxes and auxiliary-prime conditions of 37.4, and $C_{YL} \le 9$. In each of the latter two pointwise tests the scale Y may be chosen in $[x^{1/J}, x]$, separately for the two tests. Membership also requires that eventually, at every point and for every word u, its stabilizer denominator $D_{\Gamma,u}$ divide

$$M(2fN(m)), \qquad M(a) = \prod_{\substack{p \le a+1 \\ p \text{ prime}}} p^{\sum_{j \ge 0} \lfloor a/(p^j(p-1)) \rfloor}.$$

Smoothness means that beyond one family cutoff, at every point and for every word u, each prime divisor of $D_{\Gamma,u}$ be at most $2fN(m) + 1$. For the twisting condition use the family's own theta data and clause 7 of 33.3 at power one. Clause 7 in the duty of a member of $\mathcal{A}$ imposes the tests for every power prime to the residue characteristic. Thus $W_3(\mathcal{A})(F_m)$ holds if and only if $W_1(\mathcal{A})(F_m)$ holds, F_m satisfies the choice bounds of Definition 17.8 at $3, CH$ for some $CH \in \mathbf{R}$, and F_m satisfies the level and stabilizer size bounds of Definition 17.8.

Let h_L assert that for every number field E, algebraic closure Ω/E, finite place w and valuation subring of Ω above w, there exists a weak local Langlands correspondence over E_w in the sense of Definition 40.2. Theorem 40.1 gives this existence. Fix $c_{\mathrm{Dep}} \in \mathbf{R}$ and a condition A on the base, frame and twist at a point. Let $\mathcal{N}_{\mathrm{list}}$ be the norm-choice extension with the real expansion data and the law that rational map values give rational column maps. It also requires a character grid at $(1, c_{\mathrm{Dep}}, b)$ (Definition 34.47). The coefficient field contains the layer character values and a primitive root of order $(Nv)^{d_v}$ at each grid place of depth d_v. The signed local character lists and their theta characters agree with the grid in both directions. An origin consists of a preliminary construction, a state satisfying the same grid condition, and a lift preserving its Galois and form data and its signed norm choices. On an untwisted base its chosen places form a union of

fibres over the fixed coefficient subfield. With the extensions taken at the same point indices, put

$$\begin{aligned} \mathcal{N}_{\mathrm{orig}} &= \{n \in \mathcal{N}_{\mathrm{list}} : \text{an origin induces the given twisting set and norm choices}\},\\ \mathcal{N}_A &= \{n \in \mathcal{N}_{\mathrm{orig}} : A \text{ holds on its base, frame and twist}\},\\ p_A &= (R_{\mathrm{row}}, \mathcal{N}_A, \mathbf{1}, \top). \end{aligned} \tag{33.1}$$

Here R_{row} is the prescribed comparison of the constructed and arithmetic rows. In particular it identifies their complex row forms at the reference and twisted tuples, as in Lemma 33.33. The third entry prescribes a singleton of extra data. The last entry of the archive's member parameters is a geometric admissibility predicate on a datum, a core and a division datum. For the parameters used here it holds for every such triple, so this entry imposes no geometric condition. Theorem 33.1 uses the hypotheses P, F of Definition 1.20, the local matching and proportionality of theta integrals in Theorem 34.7, and the covolume bound of Definition 35.2. For parameters p, call a member (Definition 34.46) constructed if its columns and later stages come from one weighted construction with the prescribed row, norm, division and theta data, and its kernel entries are those of the local place matrices of that construction. Call it good if it also admits twisting data for its own theta realization satisfying the local twisting and character estimates, with disjoint auxiliary prime sets at nonreference places of different twist indices. A point is linked to a bundle at c if a weighted point with the same arithmetic datum and objects realizes that bundle: its recursion, spectral realization, rational rows, expansions, weights and selected side agree under transport, the normalization scalar is one, and the column characters agree. For p_A denote these two member conditions and this relation by Q_A, Q'_A, L_A, respectively. Put $p_0(h_L, c_{\mathrm{Dep}}) = (R_{\mathrm{row}}, \mathcal{N}_{\mathrm{list}}, \mathbf{1}, \top)$. Say that A holds eventually on bases if the following holds for all sufficiently small $0 < \varepsilon < 1/52$. Given the host constants with $C_8 \geq 1$, take a threshold record for the standard family parameters, followed in order by thresholds for c_D, c_h, x. In every good context beyond them one can prescribe finite algebraic elements in the two coefficient fields, required primes and finite sets of primes to avoid at each index. Every base over that context and every frame and twist meeting these requirements satisfy A. The first coefficient requirement may depend on the preliminary base, frame and first field, the prime requirements on both fields, and the second coefficient requirement also on the twist characters. Put

$$\mathrm{NL}(\mathcal{C}) = \mathrm{Closed}_{\mathcal{C}}[(F_m, CH) \mapsto F_m \text{ satisfies the family local-error estimate of Definition 36.11}].$$

Use the definite-side condition of Definition 36.1. In its terminal clause, quantify over all families of integral submodules of the face coordinate spaces and all compatible three-sided division data I agreeing under transport with the division datum of the selected bundle. For each word u, let e_u identify its face coordinates with the column space. The lattice in the column-chain assertion of 36.1 at that face is $e_u^{-1}(N_u)$, where N_u is the input lattice of the word. The finiteness premise of 36.11 ranges over all primes.

Theorem 33.1 (Existence of the third input)**.** *Assume the two collections of hypotheses P, F of Definition 1.20, the theta-integral comparison and local matching of Theorem 34.7, and the covolume bound formulated in Definition 35.2. For the application with O as in 39.1, Theorem 35.11 supplies this covolume bound from the premises established in Lemma 35.19; the family assembly is Proposition 35.99, with $C_0 = C_1 + C_2 + 1$. There is a common choice satisfying Definition 33.6. Its predicate on a class of families is the closure, in the sense of Definition 36.10, of the local-error estimate of Definition 36.11. Its predicate on the definite side is Definition 36.1.*

Outline. Combines P, F with 1.15 and 1.20 to obtain separated rational tests, sets $C_F = \max(C_{F0}, C_{F1})$ to align the functional and grid extensions, and projects the augmented extension with the covolume and $2, 3$ bounds. The complete proof is on p. 894. □

Definition 33.2 (Local data for the twisting tests)**.** Fix a point of the family. Let E be its CM field, let $G_E = \mathrm{Gal}(\overline{E}/E)$, and let $r = k + m - 1$. A twist index a specifies a residue prime $\ell = \ell_a$ and coefficient discrete valuation rings $R_0 \subset R \subset R_h$ for $h \geq 1$. The ring R_0 is finite over $\mathbf{Z}_\ell$ with finite residue field, and R is complete with algebraically closed residue field. The ring R_h is finite over R and contains a designated primitive ℓ^h-th root of unity ζ_h, with $\mathrm{Frac}\, R_h = (\mathrm{Frac}\, R)(\zeta_h)$. Write $K = \mathrm{Frac}\, R$. On nonzero elements of the fraction field of any of these rings A, use the valuation

$$v_A(z) = \frac{\mathrm{ord}_A(z)}{\mathrm{ord}_A(\ell)}.$$

The twisting datum specifies a finite set Σ_a of finite places of E. It contains every place above ℓ, g_0, g_1 and every place above

$$p(a) = \begin{cases} p_{\mathrm{de}}, & \ell \neq p_{\mathrm{de}}, \\ p'_{\mathrm{de}}, & \ell = p_{\mathrm{de}}. \end{cases}$$

The Hecke tests use all places $P \notin \Sigma_a$ with $P \neq \bar{P}$. For choosing the conductor there is a further exclusion. Put $d_E^* = Q_S q d$, where d is the auxiliary quadratic-field prime of Definition 5.1, and put $n_{\mathfrak{c}} = N\mathfrak{c}_K$. Let p_* be the least prime not dividing $d_E^*(p_{\mathrm{de}}-1)(p'_{\mathrm{de}}-1)n_{\mathfrak{c}}$, and let q_0 be the least prime congruent to 1 mod $p_* d_E^*$ that does not divide $n_{\mathfrak{c}}$. Let $\mathcal{V}$ be the set of places of E above

$$3,\ 5,\ g_0,\ g_1,\ p_{\mathrm{de}},\ p'_{\mathrm{de}},\ q_0, \quad \text{the word primes } T_u, \quad \text{and the prime divisors of } n_{\mathfrak{c}}.$$

For $1 \le j \le m$ and $\sigma \in \{+1, -1\}$, let V be the space of forms over K at (a, j, σ). The finite group acting on it is

$$G_{\mathrm{typ}} = \prod_{i=0}^{1} \prod_{v \text{ designated above } g_i} \mathrm{GL}_r(\mathbf{F}_{g_i}).$$

Write $d_{\mathrm{typ}} = |G_{\mathrm{typ}}|$. The point gives a subgroup $U \le G_{\mathrm{typ}}$, a character $\xi_{a,\sigma}$ of U, and the action P of the type projector on V. Set

$$V^{\circ} = P\{v \in V : uv = \xi_{a,\sigma}(u)v \text{ for every } u \in U\}.$$

The point's theta data specify a set $\mathcal{D}_{aj\sigma}$ of allowed inputs. Each z in it has a vector $\theta(z) \in V$ satisfying $u\theta(z) = \xi_{a,\sigma}(u)\theta(z)$ for every $u \in U$, and a finite set of level places $\mathcal{L}(z)$. Define the projected integral span

$$H = \sum_{z \in \mathcal{D}_{aj\sigma}} R\, P\theta(z).$$

The module for the basis test is $d_{\mathrm{typ}} H$. For a conductor set Q, level h, and Galois character $\eta : G_E \to R_h^{\times}$, each z has a set $\Theta_\eta(z)$ of twisted theta vectors in $R_h \otimes_R V$. At empty conductor and for the trivial character this set is $\{1 \otimes \theta(z)\}$. Write $P_h = 1 \otimes P$. The twisted module and its ambient module are

$$H_\eta = \sum_{z \in \mathcal{D}_{aj\sigma}} \sum_{y \in \Theta_\eta(z)} R_h P_h y, \qquad H_\eta^{\mathrm{amb}} = P_h\big(\theta_{\mathrm{amb}}(\Lambda_{\mathrm{amb}})\big),$$

where Λ_{amb} is the lattice in the auxiliary theta space at $(a, j, \sigma, h, Q, \eta)$ and θ_{amb} is its given linear map to $R_h \otimes_R V$.

For two full lattices M, N over a coefficient discrete valuation ring A in the same space, define the signed index, with the test lattice in the first argument, by

$$i_A(M, N) = \frac{\mathrm{length}_A(M/(M \cap N)) - \mathrm{length}_A(N/(M \cap N))}{\mathrm{ord}_A(\ell)}.$$

Here a full lattice is a finitely generated submodule whose fraction-field span is the given space. For a finite injective extension $R_h \to A$, write $H_{\eta,A}$ for the A-span of the image of H_η in $A \otimes_R V$, and H_A for the A-span of $1 \otimes H$.

For a finite conductor set Q, let $D_a(Q)$ be the finite abelian ℓ-group of the twisting datum, with its surjective Artin map $\alpha_Q : G_E \to D_a(Q)$. The map has open kernel, is unramified outside Q, and satisfies $\alpha_Q(cgc^{-1}) = \alpha_Q(g)^{-1}$ for complex conjugation c. For Q disjoint from the places above ℓ, every finite anticyclotomic abelian ℓ-quotient unramified outside Q factors through this map. For each abelian theta label λ the datum specifies a sign σ_λ and coefficients $b_{Q,\lambda,\delta} \in K$ for $\delta \in D_a(Q)$. For a character $\gamma : D_a(Q) \to R_h^{\times}$, put

$$A_{a,Q,\lambda}(\gamma) = \sum_{\delta \in D_a(Q)} b_{Q,\lambda,\delta}\, \gamma(\delta)^{\sigma_\lambda}, \qquad c_0(a, \lambda) = v_R\left(\sum_{\delta \in D_a(\varnothing)} b_{\varnothing,\lambda,\delta} \right).$$

(Here we use the convention $v_R(0) = 0$.) The first sum uses the coefficient embedding $K \hookrightarrow \operatorname{Frac} R_h$. On the constructed members the empty-conductor sum is nonzero by Lemma 33.45, applied at the selected index. For $P \in Q$, use the chosen place of $\overline{E}$ over P and its inertia group I_P. For $\eta = \gamma \circ \alpha_Q$ ramified at P, put

$$t_P(\eta) = \inf_{\substack{g \in I_P \\ \eta(g) \neq 1}} v_{R_h}(\eta(g) - 1).$$

The set in the infimum is nonempty and finite because the image of η is finite.

Definition 33.3 (The seven family conditions)**.** Fix F_m of parameter ε. At a point write $N_m = N(m)$, and use the scale parameters X', δ_1 of 8.15. The notation $\lceil y \rceil_+$ denotes the nonnegative ceiling. The duty of F_m consists of the following assertions.

1. The parameters $(C_p, e_P, c_{TO}, C_t, C_{\eta F}, C_c, c_w, C_{YL})$ equal $(1, 12, 377, 401, 6, 2, 9/10, 9)$. At every $x > x_1(F_m)$ and every point, the host conductor ideal $\mathfrak{c}_K \subseteq \mathcal{O}_K$ equals $\mathcal{O}_K$. The interlacing parameter vectors $v_\tau \in \mathbf{Z}^{k-1}$ agree at all embeddings τ, and $g_i \leq b^9$ for $i = 0, 1$. The designated primes $p_{\mathrm{de}}, p'_{\mathrm{de}}$ are both below X', and

$$j_0 = \lceil \log_5(n_H N_m / m) \rceil_+, \quad L_w = 10 \lceil (C_t + 4) \log x \rceil_+ + 2 \lceil 30/\varepsilon \rceil_+ + 22, \quad T_O = \lceil x^{c_{TO}} \rceil_+.$$

2. At the same points, n_H, m, k belong respectively to $[a, 2a]$ for $a = x^{2+8\varepsilon}, \sqrt{x\delta_1}, x^{9/2+10\varepsilon}$.

3. At the same points the first modulus satisfies $q_0 < X'$.

4. Beyond one cutoff, for each point, $g = g_i$, and each designated place above g, the indexed list $1, \chi_j, \chi_j^{-1}$ of tame labels has distinct entries, and each χ_j is a square in $\mathbf{F}_g^\times$. For each prime ℓ outside

$$\{g\} \cup \{\ell : \exists\, 1 \leq d \leq r,\ g^d < \ell^{4 v_\ell(g^d - 1)}\}, \qquad r = k + m - 1,$$

   the prime-to-ℓ parts of the list are distinct.

5. Beyond one cutoff every point satisfies the small-share bound of 22.46 for its own weights, as in 33.26.

6. Eventually at all $x > x_1(F_m)$ and every point, each tuple is μ_0, μ_*, μ'_*, or μ_w for a finite coefficient place w.

7. Eventually at all $x > x_1(F_m)$ and every point, at each w with $\mu_w \neq \mu_0$ the twist index, set Q, coefficient level k' and signed character agree at $w, \bar{w}$. The sign selects the character or its inverse and reverses when $w \neq \bar{w}$. Use Definition 33.2 for the point's own theta data. Choose nonnegative integers $N_0(a)$ and finite sets $S(a, j, \sigma)$ of theta inputs for all indices, layers and signs before choosing w or a character power. At each index a selected by a nonreference w, require the following for every $1 \leq j \leq m$ and each sign. The vectors $d_{\mathrm{typ}} P\theta(z)$ for all allowed z belong to V° and span it over K. The set $S(a, j, \sigma)$ is contained in $\mathcal{D}_{aj\sigma}$, and the vectors $d_{\mathrm{typ}} P\theta(z)$ indexed by this finite set form an R-basis of $d_{\mathrm{typ}} H$.

   Fix such a w. Write a, Q, k', γ_w for its index, conductor set, coefficient level and signed character of $D_a(Q)$. For every $n \in \mathbf{Z}$ with $(|n|, \ell_a) = 1$, take $\gamma = \gamma_w^n$, $\eta = \gamma \circ \alpha_Q$ and $e = \ell_a^{-2}$. Require $k' \geq 1$. Each $P \in Q$ satisfies $P \neq \bar{P}$ and has absolute residue degree and ramification index one. Require Q to be disjoint from $\Sigma_a \cup \mathcal{V}$ and from $\mathcal{L}(z)$ for every $z \in S(a, j, \sigma)$, all $1 \leq j \leq m$, and both signs. The character η has open kernel, goes to its inverse under complex conjugation, takes values in the group generated by the designated primitive $\ell_a^{k'}$-th root, is ramified exactly at Q, and is trivial on every decomposition group above ℓ_a.

   For this power there are $e_{j\sigma} \geq 0$ with $\sum_{j=1}^m \sum_{\sigma = \pm 1} e_{j\sigma} \leq e$. At (j, σ) use the character η^σ. The module H_{η^σ} lies in $H_{\eta^\sigma}^{\mathrm{amb}}$ and is a full lattice in the scalar extension of V°. For every finite injective extension of discrete valuation rings $R_{k'} \to A$ and every full A-lattice M in that scalar extension over $\operatorname{Frac} A$, require

$$i_A(M, H_{\eta^\sigma, A}) \leq i_A(M, H_A) + e_{j\sigma}.$$

   A signature s is good if its prescribed Hecke eigenvalues occur on a nonzero vector in the image of its type idempotent at its layer and sign, for the Hecke operators of orders $1, \ldots, r$ at every finite place

$P \neq \bar{P}$ outside Σ_a. Let ρ_s be its integral rank-r representation and χ_s its prescribed character before the auxiliary twist. Write O for the coefficient discrete valuation ring at level k', K for its fraction field, and $\eta_s = \eta^{\sigma_s}$, where σ_s is the sign attached to the signature s. For any such character η, define $L_s(\eta, Q)$ to be the length over O, divided by $\mathrm{ord}_O(\ell_a)$, of the following subgroup of $H^1(E, (K/O)^r)$. The action on the coefficients is $\rho_s\chi_s^{-1}\eta^{-1}$, with the original coefficients mapped to O. Above ℓ_a take the O-span of the reductions of the finite local cocycles for $\rho_s\chi_s^{-1}$ with coordinates in the original fraction field, using the prescribed local coefficient embedding. Away from ℓ_a, allow arbitrary restrictions at Q, at the places above g_0, g_1, and at the places above p_{de} if $\ell_a \neq p_{\mathrm{de}}$, or above p'_{de} otherwise. Require unramified restrictions at all remaining finite places. Put $\nu_a = \min\{v_{\ell_a}(\varepsilon_{\ell_a}(g)^{tr} - 1) : g \in G_E,\ \varepsilon_{\ell_a}(g)^{tr} \neq 1\}$, where ε_{ℓ_a} is the cyclotomic character, t is the auxiliary rank and $v_{\ell_a}(\ell_a) = 1$. The requirement is $L_s(\eta_s, Q) \leq \nu_a + e_s$ for every good s, with $e_s \geq 0$ and $\sum_{s\ \mathrm{good}} e_s \leq e$. For every abelian theta label λ, require $A_{a,Q,\lambda}(\gamma) \neq 0$ and

$$-N_0(a) \leq v_{R_{k'}}(A_{a,Q,\lambda}(\gamma)) \leq c_0(a,\lambda) + e.$$

For the same Artin character require $t\sum_{P\in Q} t_P(\eta) \leq e$. The witnesses N_0, S are common to all places and powers at this point.

Definition 33.4 (Pointwise design bounds)**.** Use the ranks and host sets of 8.17. Write n_0 for the size parameter of the second set T_2, L_w for the window length and n for the size parameter in the top-count estimate of 22.42. The thickness and top-count bounds are

$$|T_2| \geq \frac{n_0}{L_w}\exp\Bigl(-C_{T_2}\sqrt{\log n_0 \log L_w}\Bigr), \qquad \tfrac{1}{2}\delta_1 n \leq |TT| \leq 2\delta_1 n.$$

Let $\mathcal{P}$ consist of the residue-class-zero primes $p < X'$ for which the Frobenius coordinate h_p has order at least $x^{\varepsilon/4}$. Let b_p count the positions $t \in TT$ whose orbit fails the good-orbit test of 22.6. Require

$$\sum_{p\in\mathcal{P}} \frac{\log p}{p} b_p \leq 4C_b x^{-2}\delta_1 n \sum_{p\in\mathcal{P}} \frac{\log p}{p}.$$

For every primitive χ inducing, at modulus $8qd$, the prescribed q-character, the d-character or their product, require

$$|g_\chi(x)| \leq x^{1/2+\varepsilon}, \qquad |(L'/L)(1,\chi)| \leq C_5 \log x.$$

For the number M of words require

$$3\Gamma^2 x^{CH} \leq M \leq 3\Gamma^2 x^{CH} + 1, \qquad \Gamma = \exp(C_\Gamma m^5 f \log b).$$

Definition 33.5 (Good finite-type labels)**.** At either type prime g, let $f_v \in \mathbf{F}_g[X]$ be the polynomial assigned at each designated place v above g, and let $r = k + m - 1$. Each f_v must be monic and squarefree of degree r with nonzero constant term. For every prime ℓ outside the exceptional set in clause 4 of 33.3, distinct roots of f_v must have distinct prime-to-ℓ parts. For every prime $\ell \neq g$, the prime-to-ℓ part of each root must differ from those of every prescribed tame character value at v and its inverse. Here roots are taken in $\overline{\mathbf{F}}_g$, and the prime-to-ℓ part is the component of order prime to ℓ in its cyclic group.

For each integer $1 \leq a \leq r/2$, there must be a place v at which f_v has no divisor of degree a. At some place there must be an irreducible factor of prime degree d with $r < 2d$ and $d + 3 \leq r$. A tuple is good if these conditions hold at both type primes.

Definition 33.6 (The forty-six conditions on the common data)**.** Let N be a predicate on family classes. For an arithmetic datum D, objects O at that datum, scale constant c, bundle β at (D, O, c) and real numbers C, CH, let $DR(D, O, c, \beta, C, CH)$ be a predicate. A common choice for N, DR is defined by items 1–46 below. To specify the domain of the selection in item 2, write x_1 and C_{sz} for a family's cutoff and size constant. For a class $\mathcal{A}$ the selection has values

$$\sigma(F_m, h, x, h_x, i) \in \mathrm{Bd}(D_{x,i}, O_{x,i}, C_{\mathrm{sz}}), \qquad h : F_m \in \mathcal{A}, \quad h_x : x_1 < x, \quad i \in I_x. \tag{33.2}$$

Here $\mathrm{Bd}(D, O, c)$ denotes the set of bundles at (D, O, c). A bundle β comprises the recursion realization (including a compatible symmetric column system), its spectral realization, the arithmetic family of row

sides identified with the recursion, two nonnegative integers d_0, h_0 for which $\Delta = d_0$ and $\eta = mh_0$, the square-integrable columns, and their indefinite realization at the scale determined by d_0, h_0, c. The recursion selection extracts that first component with cutoff x_1 (so $x_1 \le x$ holds on (33.2)).

For closure on a class W, choose $\varepsilon_1 > 0$ and then, for each $0 < \varepsilon \le \varepsilon_1$, choose a threshold datum $\mathcal{K}$. Quantify over every family of parameter ε in W and every $CH \in \mathbf{R}$ at which it meets these thresholds. In the notation of Definition 36.10, this means $C_P \ge \mathcal{K}_{\mathrm{pre}}$, every entry of the middle list $\mathbf{m}(CH)$ is at least $\mathcal{K}_{\mathrm{mid}}(\mathbf{o})$, and every entry of the last list is at least $\mathcal{K}_{\mathrm{last}}(\mathbf{o} \,\|\, \mathbf{m}(CH))$. Here $\mathbf{o}$ is the outer list and $\|$ denotes concatenation. Point cutoffs inside the conclusion may depend on F_m, CH and are uniform in i. Reference surjectivity, the cylinder count, the real-place comparison and coefficient-lattice stability use all x beyond their point cutoffs. Rows that use the selected bundle impose $x > x_1(F_m)$.

For item 30, let h be a membership of the family in $W_3(\mathcal{A})$. Put $c = C_{\mathrm{sz}}$ and take β from (33.2) using the induced membership in $\mathcal{A}$. Write $\mathcal{B}$ for the nine indefinite-side conditions of 36.7 on the realization of β at C, CH. The paired-index and functional bounds there assume that one constant works in (36.18) eventually at every point and every lift layer. The lower weight bound assumes that at each layer a single leaf map has the prescribed values and phase of 37.24. The assertion used in item 30 is

$$\mathrm{Closed}_{W_3(\mathcal{A})}\Big[(F_m, CH) \mapsto \forall h : W_3(\mathcal{A})(F_m)\ \exists C \ge 0 \\ \text{eventually along the selection,}\ \ \mathcal{B} \wedge DR(D, O, c, \beta, C, CH)\Big]. \tag{33.3}$$

In the following list a selected point ranges over all arguments of (33.2). Each condition at that point uses the same selected bundle. Within a family, "eventually" means that one cutoff works for all later scales and every point index at those scales.

1. Choose a class $\mathcal{A}$ of families, allowing every real family parameter ε.
2. At each selected point choose one bundle as in (33.2), at $c = C_{\mathrm{sz}}$. The same choice supplies both row sides and the columns for all the requirements below.
3. Choose a function $c_T : \mathbf{R}^2 \to \mathbf{R}$.
4. Let $M(n)$ be the integer defined before Theorem 33.1. Suppose that for every $n \ge 0$ and every finite subgroup $H \le \mathrm{GL}_n(\mathbf{Q})$ one has $|H| \mid M(n)$. Fix $\lambda > 1$. Let G be any condition on a scale, an arithmetic frame, a raw host on that frame, a subset of its finite group, and an auxiliary rank. In applying G to a host, use its top set and its own auxiliary rank.

   Suppose hosts satisfying G exist on windows of exponent λ in the sense of (8.20) and the paragraph following it. In particular, for each $0 < \varepsilon < 1/52$ the host constants and the set of scales are chosen before the individual hosts. Then there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \le \varepsilon_0$ and every threshold datum $\mathcal{K}$ of Definition 36.10, there are $F_m \in \mathcal{A}$ of parameter ε and numbers $CH \ge 0$, $C_5, C_b > 0$ with the following properties. The family meets $\mathcal{K}$ at CH, has $C_F = CH$, and satisfies $C_{Tu} \ge c_T(\varepsilon, C_{YL})$. Its window contains $[Y, Y^\lambda]$ for arbitrarily large Y. At every $x > x_1$ and $i \in I_x$ its host satisfies G.

   The family belongs to $\mathcal{A}_{\mathrm{ch}}$. Its chosen signed twist satisfies the tests of clause 7 of Definition 33.3 at power one with its own theta data. Eventually at every point the second-set and top-set bounds, bad-orbit estimate with C_b, three character estimates with C_5, and both word-count bounds of Definition 33.4 hold. In particular the word-count interval is $[3\Gamma^2 x^{CH}, 3\Gamma^2 x^{CH} + 1]$.
5. Every family in $\mathcal{A}$ satisfies all seven clauses of Definition 33.3, including the tests for every power prime to the residue characteristic in clause 7.
6. At each selected point the extremal vectors exist and the single-map identities of Proposition 37.24 hold in every layer.
7. At each selected point, every place of the twisting set on the reference row side corresponds to an arithmetic place of the size support, and its tuple differs from the reference tuple.
8. At each selected point require the primitive comparison on the selected face and separation of the transferred reference types. More explicitly, the split Hecke systems on the rational rows and lifts obey

their class-function laws. The inert operators have matrices over the maximal CM coefficient subfield, and their adjoints for the class sum belong to the indexed inert operators. On the complex row and layer spaces the respective Hecke actions intertwine the maps from class functions. The finite group acts unitarily and the row and layer maps are equivariant. The selected subspaces lie in their specified row and layer blocks.

There are finite systems of complex linear tests with common kernel zero on the row space and on each layer space. On the images of the rational theta maps their values lie in the distinguished image of the coefficient field. On the reference layer the tests can also be chosen rational on the images of the coordinate basis. For each intertwining system of layer Hecke operators, each occurring good joint eigenspace on the class-function side is contained in the range of the adjoint lift, as in the decomposition of Lemma 21.20. The two faces have positive reference dimension. The layer scalars $\lambda_{j,t,p,v}$ and the common scalar κ are positive. On a type summand the layer lift times its adjoint acts by λ_{j,t,p,v_0}, and the composite row map times its adjoint acts by $\kappa \prod_{j=1}^{m} \lambda_{j,t,p,v_0}$, with the corresponding row-type indexing. Here v_0 is the distinguished embedding.

On each row side the level space is the intersection of the principal space with the prescribed character space. At principal level the tuple maps are injective with the same range as the reference map. The reference layer space is the intersection of the principal layer space with the prescribed inverse-character space. The row type summands on both sides give orthogonal direct sums. The reference layer type summands give orthogonal direct sums there, with the same dimensions as the corresponding reference row summands. The reference coefficient modules are lattices. At each coefficient embedding the rational layer coordinates preserve linear independence and span the complex space, and the transported group action is the specified action multiplied by its coefficient character. At the other infinite places the inner product of transported rational lift vectors satisfies the conjugate Gram identity of Lemma 21.20. Finally, for two distinct occurring reference types and every layer, their transfers differ at a split Hecke place or at an indexed inert operator.

9. At each selected point choose one assignment a from finite types to row indices. For every layer j and index c, the spaces of types σ with $a(\sigma) = c$ form an internal direct sum equal to the complex span of the transported rational space of index c. For a nonzero such component the arithmetic and row-side λ's agree at the reference embedding for the reference tuple and every tuple attached to a finite place. The sum of the initial row counts over $a^{-1}(c)$ is the dimension over the coefficient field of the initial row space of index c.

10. At each selected point the comparison identifies the two integral RR lattices with their respective indefinite lattices, including their metrics. At each used tuple the embedded rational lift columns span the indicated complex image. In the same coordinates the projected theta maps agree. The local ideals and their numerical corrections agree at every layer and place, with twist correction zero outside the twisting set. The column pairing after division agrees on every pair of rational inputs.

11. At each selected point, for every word, used tuple and pair of rational inputs, applying the coefficient embedding to the definite evaluation gives the arithmetic column period on the transported inputs.

12. Closed on $W_1(\mathcal{A})$, if $C_{Tu} \geq c_T(\varepsilon, C_{YL})$, then eventually, for every i and every word u,

$$b^{m/2} \leq T_u, \qquad \log T_u \leq C_{Tu} m \log b.$$

13. Closed on $W_1(\mathcal{A})$, take any $x > x_1$ and any i. Suppose g_0, g_1 split completely in the real base field and $2fk < g_a - 1$ for some $a \in \{0, 1\}$. For each word whose T_u splits completely in that field, the initial class set has cardinality $M_k I_{01}(T_u^k - 1)^f$, in the notation of Theorem 37.21.

14. There are positive constants C_1, c, C_7, C_{11} common to all arithmetic data for the zero-density bound and zero-free regions of Theorem 37.2, the explicit formula (37.1) and the unit-window zero count (37.2), in the respective ranges of Theorems 37.2 and 37.3.

15. Under the rank and gap conditions of Lemma 33.25, each real row function admits the compactly supported reproducing integral and the weighted absolute integral of that lemma. The product of the normalized real matrix coefficients is integrable.

16. On the automorphic L^2 quotient, use a finite-adelic Haar measure, a fundamental domain and the right regular action. Let e be locally constant and supported in the specified compact subgroup. At each infinite place take an irreducible unitary representation of formal degree d_v and a unit vector whose conjugate diagonal coefficient times d_v is integrable. If R_e is finite convolution by e and T_∞ is convolution by the product of these real coefficients, then $\mathrm{im}(R_e T_\infty)$ is finite-dimensional.

17. Closed on $W_3(\mathcal{A})$, the size hypotheses of (33.23) imply that inequality for every word, with the six parameters of Theorem 33.78.

18. Choose a function of finite real lists with values $(C_{\mathrm{res}}, n_0, A, C_b, C, c, c_b)$, where $n_0 \in \mathbf{N}$ and the other coordinates are real.

19. At the outer parameter list of every family, $C \geq 0$ and $A \geq 2(C_{YL} + 1)$.

20. Under the hypotheses and in the order of thresholds of Proposition 33.18, the sum of the products of the norms for any finite set of same-row pairs of the indicated two minor types satisfies the bound of that proposition.

21. For the nongeneric corank-one pairs the signed sum satisfies the first estimate of Lemma 33.99, with that lemma's point hypotheses and thresholds.

22. For the generic scalar corank-one pairs of an equal-word operator the signed sum satisfies the second estimate of Lemma 33.99, with its additional counting parameter chosen before its thresholds.

23. For every real-valued function of the outer parameter list, the two bounds (33.21) are closed on $W_3(\mathcal{A})$, using the selection in item 2 and the point hypotheses of Theorem 33.78.

24. Closed on $W_3(\mathcal{A})$, for every membership in that class, distinct words in the resulting selected spectral system satisfy (33.22) under its source hypotheses.

25. Closed on $W_3(\mathcal{A})$, the finite-level idempotent estimate of Theorem 37.13 holds with n_0, A, C, c, c_b from item 18 evaluated at the family's outer parameter list. The levels, measures and signature restriction are those in that estimate.

26. At every selected point, spaces of equal rank and admissible constant signatures have the same fundamental-domain volume for their prescribed measures. At rank $\dim Y_0$, each positive definite space has the standard genus mass. Its finite-adelic class set modulo rational isometries and the stabilizer of its base lattice is finite. For any complete set of representatives, the sum of the reciprocal stabilizer orders equals its genus mass.

27. At every selected point choose a word-to-column assignment agreeing with the division assignment, complex column coordinate isomorphisms, a linear map from complex reference coordinates into the arithmetic ambient space, and a semilinear map of row spaces. The reference coordinate map covers the good reference subspace. On rational inputs, reference evaluation in these coordinates gives the embedded definite evaluation. The row map has image the good selected row space and multiplies norms by a fixed positive scalar. It intertwines reference theta evaluation up to one nonzero scalar. For each word the column coordinate map, with the prescribed conjugation, multiplies the norm by a positive scalar that may depend on the word. In each finite spectral realization the transported row-column pairing equals the model pairing up to the prescribed conjugation and one nonzero scalar per word.

28. Closed on $W_3(\mathcal{A})$, eventually at every point the image of the reference space $V_0(\mu_0)$ under ι_{μ_0} is the entire target.

29. For every $F_m \in \mathcal{A}$, eventually at every point and every tuple μ, multiplication by $\iota_D(a)$ preserves $\widetilde{M}_\mu$ for all $a \in \mathcal{O}_{K_D}$.

30. Require (33.3). In particular the same $C \geq 0$ serves for the nine indefinite assertions and the definite-side predicate at the same selected bundle.

31. Closed on $W_3(\mathcal{A})$, eventually at $x > x_1$, reference point evaluation for every word satisfies item 1 of Theorem 33.77, including (33.17).

32. With the same family quantifiers, each used nonreference tuple has integral rational point evaluation for every word, as in item 2 of Theorem 33.77.

33. With the same family quantifiers, the common denominator hypothesis of Theorem 33.77 gives one representative of each reference period satisfying (33.18) at every finite place.

34. With the same family quantifiers and denominator hypothesis, each used nonreference period has one representative satisfying (33.18) at every place outside its excluded residue primes.

35. With the same family quantifiers, the reference scaling and the twisted column bound at T_u in item 6 of Theorem 33.77 hold for every word.

36. Closed on $W_3(\mathcal{A})$, there is one real C_1 for the two eventual estimates assumed in Lemma 17.21 and established in Proposition 17.22 at a place above $g_a \neq T_u$. The lower bound for the normalized valuation of a nonzero period representative and the upper bound for $B_{u,w} = \sum_j (F_j(w) - E_{u,j}(w))$ use the same quantity $-\sum_j \nu_{a,u,j,w} - C_1\mathcal{E}_G$, where $\nu_{a,u,j,w}$ and $\mathcal{E}_G = f(mN_m(N_m + t) + m^3\eta)$ are defined before 17.21. The period assertion includes the zero representative.

37. Closed on $W_3(\mathcal{A})$, there are $C_{\deg}, C_g, C_T, C_{\mathrm{stab}}, C_d, C_I, x_0 \in \mathbf{R}$, with $C_g \geq 0$, such that all eight conditions of Definition 21.40 hold at every point with $x \geq x_0$ and $x > x_1$.

38. The following two assertions are closed on $W_3(\mathcal{A})$. Eventually at every $x > x_1$ and every point index there are local lifts with the comparisons and the stabilizer and complementary-orbit bounds of (37.33), with initial saving one. For every membership in this class and every point above x_1 at or beyond the selection cutoff, the rational chain maps, initial stabilizers and selected kernel entries satisfy the second assertion of Proposition 37.26. Its orbit phases and signs must work for the entries of that selection.

39. Choose $c_{\mathrm{Ic}} \in \mathbf{R}$.

40. Require $c_{\mathrm{Ic}} > 0$.

41. Closed on $W_3(\mathcal{A})$, eventually at every point there is an arithmetic row system whose selected indices map injectively onto the good indices. Its real parameters, split level and finite projectors agree with those for the point. Its cylinder counts satisfy Theorem 37.13 with error parameter $\exp(-c_{\mathrm{Ic}} r f \log r)$, $r = k + m - 1$.

42. Closed on $W_3(\mathcal{A})$, for each membership in that class there is a cutoff such that at every selected point beyond it the following holds for all $0 \leq j < m$. At the reference tuple take the level subspace for the selected subgroup and its inverse character. Include its coordinate vectors into the full space of layer forms, project to the good part, and apply orthogonal projection there to the part of the selected layer subspace lying in the good part. In coordinates for that last subspace, every vector over the coefficient field is the image of a level vector under this composite.

43. Closed on $W_3(\mathcal{A})$, the real measures, formal degrees, quotient volumes, symmetric columns and level-zero characters satisfy item 5 of Theorem 33.77, with the real-place family quantifiers there.

44. Take any arithmetic datum with the following volume normalization. For every Hermitian space W of admissible constant signature there exists a Haar measure with the compact-dual normalization at infinity and finite lattice stabilizer of measure one. For each such measure the rational unitary group admits a fundamental domain, and every such domain has measure $\mathrm{vol}_D(W)$. For this datum require the unit-hyperplane assertion of Theorem 37.6, including (37.10), in each of its stated ranks and signatures.

45. For every arithmetic datum and every everywhere positive definite Hermitian space of rank two with its self-dual base lattice, the genus mass is M_2 of Proposition 37.7.

46. Require $N(W_3(\mathcal{A}))$.

Index of archive field names for the numbered items of Definition 33.6

Item	Archive field	Item	Archive field	Item	Archive field
1	$\mathcal{A}$	17	`hTF`	33	`aPer0`
2	σ	18	`c30`	34	`aPerTw`
3	`cT`	19	`hc30`	35	`aP10`
4	`hEx`	20	`sTy`	36	`aAtG`
5	`hduty`	21	`sNG`	37	`aC2`
6	`hRB`	22	`sM`	38	`hServ`
7	`hTw`	23	`sLp`	39	`cIc`
8	`hPA`	24	`sO`	40	`hcI`
9	`hLamP`	25	`h30`	41	`irrC`
10	`hDivG`	26	`hInstR`	42	`zL`
11	`hDcPtG`	27	`selJ`	43	`tie`
12	`hTu`	28	`jOn`	44	`hU`
13	`hn4`	29	`hMt`	45	`hG`
14	`hLit`	30	`rRows`	46	`nRest`
15	`hWu`	31	`aPt0`		
16	`hRF`	32	`aPtTw`		

Definition 33.7 (Size and counting conditions for the spectral estimate)**.** Use the point model of 33.78. Write $r_0 = (m-2)/2$, $n = k+m-1$, $L_b = \log b > 1$, $g = g_0 g_1$, and $v_E = \mathrm{rd}(E)/2$, where E/F is the CM extension of the model and $\mathrm{rd}(E)$ is its root discriminant. Let $\mathcal{D}$ be the spacing of the two choices of an interior word coordinate and C_Γ the size constant from the point. At a place τ of the CM type, use the $k-1$ interlacing parameters v_i, the m positive weight parameters B_j, the decreasing length-n Harish–Chandra parameter λ, and the length-k weight $u_\tau = (u_i)$ of a word u. Indices here start at one. A top place is a place over a chosen top position. There put $B_* = -u_{0,k}$ for the base word u_0.

The source condition Cnt.Hyp0 at $(C_0, C', C_\ell, C_g, c_b, C_w, c_T, C_T)$ has $C_0 \geq 1$ and the scalar inequalities

$$8C_\Gamma \leq C', \quad C' m^5 f L_b \leq k, \quad m^4 \leq f \leq k, \quad \log f \leq \sqrt{r_0},$$
$$0 < C_\ell \leq m, \quad C_\ell L_b \leq f \log k, \quad C_\ell m^2 L_b \leq k, \qquad v_E \geq 1, \quad C_\ell m \log v_E \leq k,$$
$$C_\ell m^3 L_b \log \mathrm{rd}(E) \leq k(\log k)^2, \quad \log \mathrm{rd}(E) \leq L_b / C_\ell,$$
$$(kb)^8 \leq g_i \leq b^{C_g} \ (i = 0, 1), \qquad (kfL_b)^{C_\ell} \leq \mathcal{D}, \qquad b^{c_T m} \leq T_u \leq b^{C_T m} \quad \text{for every } u.$$

At every place and all applicable indices and words require

$$C_\ell m k \mathcal{D} \leq B_j, \quad \max(k^{C_\ell}, b^{c_b}) \leq B_j \leq b^2, \qquad B_m \leq 2B_1, \quad u_1 \leq b^{C_w},$$
$$B_{j+1} - B_j \geq C_\ell n^{19} f^5 L_b, \qquad \lambda_i - \lambda_{i+1} > C_\ell k m^4 + n m^{10},$$
$$0 \leq v_i \leq 4k\mathcal{D}, \quad v_i - v_{i+1} = 4\mathcal{D}, \quad u_1 - v_1 \geq 4k\mathcal{D}.$$

For each word and $1 \leq i \leq k-2$, the choice $u_{i+1} = v_i - 1/2$ or $u_{i+1} = v_i - 1/2 - \mathcal{D}$ is constant across places. Off the top, $u_1 - v_1 \leq 8k\mathcal{D}$ and $u_k = v_{k-1} - 1/2$. On the top, $B_1 - B_* \geq 3$ and $B_1/2 \leq B_*$. The interlacing is $u_i > \lambda_i > u_{i+1}$ for $i < k$ and $u_k > \lambda_k$. Distinct words have distinct weight vectors at every place.

The distinct host primes g_0, g_1 and each level prime T_u split completely in E, with $T_u \neq g_i$ and $T_u \equiv 1 \pmod{g^2}$. Require $\mathrm{disc}\, E = (\mathrm{disc}\, F)^2$ in the model's discriminant convention. With $d_u^S, D^\star$ the column dimension and main term from 21.35, require $C_0^{-1} \leq d_u^S (\Gamma_u^{\mathrm{sp}})^2 / D^\star \leq C_0$, where Γ_u^{sp} denotes the real factor Γ_u in the spectral count of 21.35. The source hypothesis of (33.22) is this condition with $C_0 = 3$ and the constants in 33.78.

For a pair (u, u'), Cnt.Stand means $r_0 \geq 1$, that $T_u, T_{u'}$ are completely split primes at least three, and that g_0, g_1 are distinct completely split primes with $(T_u T_{u'}, g) = 1$ and $T_u \equiv T_{u'} \pmod{g}$. The condition Cnt.Hyp$(u, u'; A)$ comprises Cnt.Hyp0 and Cnt.Stand together with

$$\max(T_u/T_{u'}, T_{u'}/T_u) \leq A, \quad C_\ell m^6 L_b \leq k \log k, \quad (\log f)^2 \leq A m^2 L_b, \quad \log f \leq A r_0^2, \qquad A \in \mathbf{R}.$$

It requires also $V(u'_\tau) \leq A^k V(u_\tau)$ at each τ in both orders, where $V(w) = \prod_{i<j} (w_i - w_j)$.

Lemma 33.8 (Counting conditions from the source sizes). *Under 33.7, $C' \geq 1$ and $C_\ell > 2$,* Cnt.Hyp$(u, u; 1)$ *holds for each word u. For a pair of words and $A \geq 1$ it suffices to establish the two level-ratio bounds and the two Vandermonde bounds of 33.7. In the distinct-word row of 33.78, 21.36 gives these with $A = 3e^{7/4}$ under its size and mass thresholds.*

Outline. Derive $m \geq 4, r_0 \geq 1$, the modulo-g congruence, $T_u \neq 2$, and coprimality from the hypotheses, and chain the scalar bounds of 33.7. The two comparisons are equalities for a repeated word and follow in both orders from 21.36 for distinct words. The complete proof is on p. 894. □

Let E/F be a CM extension unramified at every finite place, let $O = \mathcal{O}_E$, and put

$$f = [F : \mathbf{Q}], \qquad \rho = |\operatorname{disc} E|^{1/(2f)} = 2v_E, \qquad \ell = \log_2(2f), \qquad c = 1 + \ell.$$

For an integral Hermitian lattice P, write $P^{\#} = \{z : h(z, P) \subset O\}$, with the dual taken in EP, and put $d(P) = |P^{\#}/P|$. All lattice indices below are cardinalities of additive quotients. Let M_t be the mass of the genus of O^t with its standard Hermitian form. Mass weights an isometry class by the reciprocal of its automorphism order.

Lemma 33.9 (Integral lattices of a given discriminant order). *A projective O-lattice of rank $t \geq 1$ has at most $i^{t+\ell}$ sublattices of index $i \geq 1$. In a fixed totally positive definite Hermitian E-space W of rank t, one has, for every integer $n \geq 1$,*

$$\sum_{\substack{[P] \text{ in } W \\ P \text{ full integral, } d(P)=n}} \frac{1}{|\operatorname{Aut} P|} \leq M_t n^{(t+\ell)/2}.$$

Outline. Count index-i inclusions via quotient ideals and residual hyperplanes. Enlarge to a maximal lattice, bound its mass by the unramified models and the Gan–Hanke–Yu formula, and count inclusions with inverse stabilizer weights using $n = bi^2$. The complete proof is on p. 895. □

Lemma 33.10 (Primitive intersections and their gluings). *Let V be totally positive definite of rank $k = j+s$, where $j, s \geq 1$. Let $\mathcal{G}$ be an isometry-invariant collection of full integral lattices in V such that $L^{\#}/L$ is cyclic of order D for every $L \in \mathcal{G}$. For $a \geq 1$ an integer put*

$$N_{j,a}(L) = \#\{U \subset V : \ U \text{ an } E\text{-subspace}, \ \dim_E U = j, \ d(L \cap U) = a\}.$$

These intersections are precisely the primitive rank-j sublattices with discriminant order a. Then

$$\sum_{[L] \in \mathcal{G}/\mathrm{U}(V)} \frac{N_{j,a}(L)}{|\operatorname{Aut} L|} \leq M_j M_s \, a^{j/2} (aD)^{(9s+8+4\ell)/2}.$$

Outline. Bound rational orbits by Landherr's classification and 33.9. Glue the two primitive intersections, bound the gluing generators by cyclicity of the ambient discriminant, and use inverse stabilizer weights and the masses from 33.9. The complete proof is on p. 895. □

Lemma 33.11 (Comparison of the two standard masses). *Suppose $v_E \geq 1$, $k \geq 3$ and $1 \leq s < k$. Put*

$$M_* = (20 f \rho^2)^{2f(6+c)}.$$

Then

$$M_s M_{k-s} \leq M_k M_* \, \rho^f \, 16^{skf}.$$

Outline. Use the trace-dual minimum and Banaszczyk's inequality to count short Gram matrices in ranks one and two, and 33.10 to count intermediate lattices. Multiply adjacent mass-ratio bounds, using the Euler factors of 37.5 in higher ranks. The complete proof is on p. 896. □

Lemma 33.12 (Primitive sublattices in the split-level genus)**.** *Suppose $v_E \geq 1$, and let L_0 be a full integral lattice in a totally positive definite Hermitian E-space of rank $k \geq 3$. Suppose T is a rational prime splitting completely in E and $L_0^{\#}/L_0 \simeq O/(T)$. Put $D = T^{2f}$, and let m_0 be its genus mass. Then*

$$m_0 = M_k \left(\frac{T^k - 1}{T - 1} \right)^f .$$

For all integers $1 \leq s < k$, $a \geq 1$, one has

$$\frac{1}{m_0} \sum_{[L] \in \mathrm{gen}(L_0)} \frac{N_{k-s,a}(L)}{|\operatorname{Aut} L|} \leq M_* \rho^f 16^{skf} (a/D)^{k/2} (aD)^{5s+5+2c}.$$

Here and below a sum over a genus takes one representative of each isometry class.

Outline. Choose one prime per conjugate pair to form self-dual enlargements. Count their residual hyperplane fibres with inverse stabilizer weights and get the mass from 33.9, then normalize the estimate of 33.10 by this mass and apply 33.11. The complete proof is on p. 897. □

Definition 33.13 (Embedded data at level g)**.** Let L be a full integral lattice in a definite space V, and let $g \geq 2$, $r \geq 0$ be integers. An embedded datum of index x consists of a full integral lattice $K \subset V$ with $d(K) = d(L)$ and an isometry $K/gK \to L/gL$ for the forms modulo g. Write $\Gamma \subset K \oplus L$ for the inverse image of its graph, and put $C = \{z : (z, z) \in \Gamma\}$. Require $[L : C] = x$ and require L/C to be generated by at most r elements. A triple consists of data $(K, \Gamma), (K', \Gamma')$ of indices x, x' and an E-linear isometry η of V taking K' to K such that $(y, z) \in \Gamma'$ if and only if $(\eta y, z) \in \Gamma$.

Lemma 33.14 (Fibre with prescribed fixed space)**.** *Use 33.13 with $L^{\#}/L$ cyclic of order D and indices $x, x' \geq 1$. Write $k = \dim_E V$, $X = xx'$, $Y = XD^2$, and $e = 3(2r + 1)$. For a subspace U of codimension s, put $a = d(L \cap U)$ and $A_s = (2/g)^{2sf} DX$, with $d(0) = 1$. The number of triples for which $\ker(1 - \eta) = U$ is zero if $a > A_s$. Otherwise it is at most*

$$(A_s/a)^{k/2} X^{2s+2c+1} Y^e (4sf)^{4sf},$$

where the last factor is one for $s = 0$. When $s = k$ one also has the bound

$$X^{k/2} Y^{c+e} \left((2/g)^{kf} (4kf)^{4f} DX \right)^k .$$

Outline. Use projection as in 33.10 to bound the fixed lattice discriminant. Count lattices via 33.9, lifts and graphs via 33.10, and isometries via 33.6, using 33.9 to count both lattices directly when the fixed space is zero. The complete proof is on p. 897. □

Lemma 33.15 (Summation over the two length shells and the two Gram shells)**.** *Let r, h be positive integers, let j, e, q be nonnegative integers with $q \geq e + 4r + 2$, let $H, E, s_* \geq 1$, and let $D, K, K' > 0$. On a measure space let $S, S' \geq 1$ and $Y, Y' > 0$ be measurable functions representing the two length maxima and the two Gram volumes. Suppose that, almost everywhere,*

$$YY'D \leq 2^j, \qquad D \leq (SS')^r, \qquad \max(Y, Y') \leq \max(S, S')^r E, \qquad s_* \leq \max\{1, (SS')^r/(2H)^{2r}\}.$$

For $i \geq 1$ the i-th length shell is $(2^i H, 2^{i+1} H]$, and the zeroth shell is $[1, 2H]$. On the pair of shells i, i' put $R = 2^{i+i'+1} H$. Let A, B be nonnegative measurable functions. For every $i, i' \geq 0$ and $y > 0$, assume that the integral of A^2 over the region where S lies in its i-th length shell, $S' \leq 2^{i'+1} H$, and $y < Y \leq 2y$ is at most $KR^e y^{2h} 2^{-2qi}$. Assume that the integral of B^2 over the region where S' lies in its i'-th length shell, $S \leq 2^{i+1} H$, and $y < Y' \leq 2y$ is at most $K' R^e y^{2h} 2^{-2qi'}$. Then

$$\int AB \leq 16 \sqrt{KK'} (2H)^{e+4r} E^2 (2^j/D)^h s_*^{-(q-e-4r-1)/r}.$$

Outline. Fix the length shells and apply Cauchy–Schwarz to the squared-integral bounds on dyadic Gram-volume shells. Use the Gram-product constraint for all but one power, bound that power by the separate Gram shells, and sum four geometric series using the length decay. The complete proof is on p. 898. □

Lemma 33.16 (Normalization and propagation of length moments)**.** *Let G be a local Cayley unitary group, let S be its Bessel subgroup, and let ν be the prescribed measure on $S\backslash G$. Let $(\sigma,\mathcal{H})$ be the discrete-series representation and let $\mathcal{H}^\infty$ denote its smooth vectors. Let η be the unitary Bessel representation of S on a finite-dimensional Hilbert space V, and let $B, L : \mathcal{H}^\infty \to V$ be continuous η-equivariant complex-linear maps. Choose a nonzero smooth vector v in the lowest compact type and put $w = v/\|v\|$ and $\Phi(g) = B(\sigma(g)v)$. Suppose there is $c \in \mathbf{C}$ such that*

$$B = cL \quad \text{on } \mathcal{H}^\infty, \qquad d > 0, \qquad d\int_{S\backslash G} \|L(\sigma(g)w)\|^2\, d\nu(g) = 1, \qquad 0 < I_B := \int_{S\backslash G} \|\Phi(g)\|^2\, d\nu(g) < \infty.$$

Then

$$c \neq 0, \qquad I_B = \frac{\|v\|^2|c|^2}{d}, \qquad d\rho(g) := \frac{\|\Phi(g)\|^2}{I_B}\, d\nu(g) = d\|L(\sigma(g)w)\|^2\, d\nu(g),$$

and ρ is a probability measure.

Let r, q be nonnegative integers. Let $l_0, \dots, l_r$ be positive measurable functions on this quotient, with $l_r \geq 1$ almost everywhere for ρ, and put $R_i = l_{i+1}/l_i$ for $0 \leq i < r$. Suppose $A, R \geq 1$ and

$$\int l_0^{2q'}\, d\rho \leq A^{q'}, \qquad \int R_i^{2q'}\, d\rho \leq R^{2q'} \quad (0 \leq q' \leq (r+1)q,\ 0 \leq i < r).$$

For every $0 \leq a \leq r$,

$$\int l_a^{2q}\, d\rho \leq (AR^{2a})^q, \qquad \int l_a^{-2q}\, d\rho \leq R^{2rq}.$$

For $b \geq 1$, the second bound is $(b^{2r})^{2q}$ when $R = b^2$.

Outline. Linearity gives $\Phi(g) = \|v\|cL(\sigma(g)w)$; squaring and integrating yield I_B, $c \neq 0$, and the probability measure ρ. Factoring l_a and l_a^{-1} through the ratios R_i and applying Hölder's inequality gives both moment bounds. The complete proof is on p. 898. □

Lemma 33.17 (One ideal clearing the leading minors)**.** *Let E be a CM field of degree $2f$, let $\mathfrak{d}$ be a nonzero conjugation-stable integral ideal, and let $T \geq 1$ be an integer. Fix $0 \leq j \leq r$, elements $p_0 = 1, p_1, \dots, p_j \in E$, and $q_l = \sum_{a=0}^{l-1}(r-a)$. Suppose that*

$$\mathfrak{a}_l = (p_l)\mathfrak{d}^{-2q_l}$$

is an integral ideal whenever $p_l \neq 0$. Put $S = \{l : 1 \leq l \leq j,\ p_l \neq 0\}$, $D_b = \prod_{l\in S} N(\mathfrak{a}_l)$, and $N_T = T^f$. Then

$$\mathfrak{c} = (T)\prod_{l\in S} \mathfrak{a}_l\overline{\mathfrak{a}_l}$$

is a nonzero conjugation-stable integral ideal contained in (T), and

$$D_b \geq 1, \qquad N(\mathfrak{c}) = (D_b N_T)^2, \qquad \frac{\mathfrak{c}\mathfrak{d}^{2q_l}}{p_l} \subset \mathcal{O}_E, \quad \frac{\mathfrak{c}\mathfrak{d}^{2q_l}}{\bar{p}_l} \subset \mathcal{O}_E \quad (0 \leq l \leq j,\ p_l \neq 0).$$

If $p_j \neq 0$, then $N(\mathfrak{a}_j) \geq 1$.

Outline. Containment and conjugation symmetry of $\mathfrak{a}_l, \overline{\mathfrak{a}_l}$ give $\bar{\mathfrak{c}} = \mathfrak{c} \subset (T)$. The identity $\mathfrak{a}_l\mathfrak{d}^{2q_l} = (p_l)$ and norm multiplicativity yield the divisibilities and norm bounds. The complete proof is on p. 899. □

Proposition 33.18 (The same-word estimate by leading minors)**.** *Fix $C_g, c_b, C_w, c_T, C_T, C_0 \in \mathbf{R}$, with $c_b, c_T > 0$. There are $C_H, C_h, C_p \geq 0$ such that, for each $C_s \geq C_p + 4$, the following holds for sufficiently large C, then sufficiently large C', C_ℓ. Use the tied models of 33.78, with* Cnt.Hyp0 *and the unramified CM condition of 37.5. Put $r = r_S = (m-2)/2$, and assume*

$$C_\ell m^6 \log b \leq k \log k, \quad (kb)^8 \leq g_i \leq b^{C_g}, \quad C_\ell \log \mathrm{rd}(E) \leq \log b, \quad \log f \leq r^2,$$
$$|\lambda_{\tau,a} - \lambda_{\tau,c}| > C_\ell k m^3\ (a \neq c), \qquad T_w \leq b^{C_T m}, \qquad (T_w, g_0 g_1) = 1 \quad \textit{for every word } w.$$

For a relevant nonidentity term of word u with $G(t) = 0$, its type (j, δ) records the rank j of the reduced pairing and its leading pairing minors, with $\delta_0 = 1$ and $\delta_l = 0$ for $l > j$. Put $q_l = \sum_{a=1}^{l}(r + 1 - a)$, $\mathfrak{d} = \mathfrak{D}_{F/\mathbb{Q}}\mathcal{O}_E$, and $H_1 = b^{C_H m}$. For $\delta_l \neq 0$, let $d_{\tau l} = |\tau(\delta_l)|\,\mathrm{rd}(E)^{-2q_l}$ and $d_l = N((\delta_l)\mathfrak{d}^{-2q_l})$, with N the absolute ideal norm. Set

$$D_\delta = d_j, \quad S_\delta = \prod_{\tau} \max\bigl(1, \max_{\substack{\delta_l \neq 0 \\ 1 \le l \le j}} d_{\tau l}/(2H_1)^{2r}\bigr), \qquad H(c) = b^{cm^2 f}.$$

The product is over the CM type. Define

$$A(j,\delta) = (C/k)^{j(k-1-j)f} D_\delta^{-(k-1-j)/2} H(C)^{Cr^4} S_\delta^{-C_s k}, \qquad M(j,\delta) = D_\delta^{k/2} H(C_h)^{C_h r^4} S_\delta^{C_p k}.$$

For every finite set U of ordered same-row pairs of terms of types $(j,\delta),(j',\delta')$ in the diagonal kernel, with row vectors $v_t, v_{t'}$,

$$\sum_{(t,t')\in U} \|v_t\|\|v_{t'}\| \le \dim(\pi_u)\,\mathrm{mass}_u(AM)(j,\delta)(AM)(j',\delta').$$

Here π_u is the archimedean coefficient representation and mass_u is the sum of inverse row-lattice stabilizer orders.

Outline. Put degenerate pairs into normal form with $j \ge 1$ and split the bound into amplitude and pair-count factors. Using a shifted Bessel family and Lemma 33.16, derive two-sided length tails, then combine Gram volumes, Lemma 33.15, and covolumes for the amplitude bound. Fibre counts, genus means via Lemma 33.12 and Lemma 33.14, and Definition 33.7 yield the pair-count bound. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 899. □

Proposition 33.19 (Weighted fibre over an embedded datum)**.** *Use the row, frame, level and unramified CM hypotheses of Proposition 33.18. Fix a row lattice, a type (j,δ), and an embedded datum of index $x \ge 1$ as in Definition 33.13. Put*

$$g = g_0 g_1, \qquad N_T = T_u^f, \qquad c = 1 + \log_2(2f), \qquad D_b = \prod_{\substack{1 \le l \le j \\ \delta_l \neq 0}} d_l.$$

The sum of relative inverse stabilizer weights of the inequivalent moving frames inducing this datum is bounded by

$$F_\delta(x) = (D_\delta/x)^{k/2} 8^{r+1} f^2 D_\delta^{3c+6} N_T^2 (D_b D_\delta N_T)^{10000(r+1)^3}. \tag{33.4}$$

Outline. Fix the block matrix and norm-one scalar, inject frame classes into normal-form orbits, and group by inner lattices and indices i_1, i_2. Unipotent reductions using Lemma 33.17 give split representatives near P_0 and bound their count. Factoring the inverse stabilizer weight across $Z \lhd R_1 \lhd R$, bounding the layers by congruence lifts and Eichler maps, and summing yields the bound. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 908. □

Proposition 33.20 (Mass of triples of embedded data)**.** *Use the definite row lattice, level $g = g_0 g_1$, and unramified CM hypotheses of Proposition 33.18. Put $N_T = T_u^f$, $c = 1 + \log_2(2f)$, $D_0 = N_T^2$, $X_* = xx'$, $Y_* = X_* D_0^2$, $\rho = 2v_E$, and $M_* = (20f(2v_E)^2)^{2f(6+c)}$. On the genus of the definite rank-k level lattice, weight each class by the reciprocal order of its isometry group. Divide the weighted count of triples of two embedded data of indices $x, x' \ge 1$ and their relative isometry by the genus mass. The result is at most*

$$X_*^{k/2} Y_*^{6r+10+4c} \left[1 + M_*(2v_E)^f \sum_{s=1}^{k-1} \bigl((32/g)^{kf}(4kf)^{4f} Y_*^7\bigr)^s + \bigl((2/g)^{kf}(4kf)^{4f} D_0 X_*\bigr)^k\right]. \tag{33.5}$$

Outline. The proof identifies $L_0^\#/L_0 \simeq O/(T)$ via 33.7 and self-duality of the flag quotient. It groups triples of 33.13 by fixed-space codimension s, multiplies the fibre bounds of 33.14 by the genus means of 33.12, and sums over s. The complete proof is on p. 916. □

Lemma 33.21 (Normalization of a nonzero Bessel average)**.** *Let $G \subset \mathrm{GL}_n(\mathbf{C})$ be a local Cayley unitary group and let $S = U \ltimes N$ be its Bessel subgroup. Fix Haar measures dg, da, dn such that multiplication carries $da\,dn$ to a two-sided Haar measure on S. Fix a measure ν on $S\backslash G$ and $c_S > 0$ such that*

$$\int_{S\backslash G}\int_{U\times N} f(anx)\,da\,dn\,d\nu(x) = c_S\int_G f(g)\,dg$$

for every nonnegative measurable f. Let $(\sigma,\mathcal{H})$ be a continuous unitary representation of G, let (π,V) be a continuous finite-dimensional unitary representation of U, and let ψ be the unitary Bessel character of N. Fix a unit vector $w \in \mathcal{H}$ and $x_0 \in G$, and define

$$m(y) = \langle \sigma(x_0)w, \sigma(y)w\rangle, \qquad a(x) = \int_{U\times N} |m(anx)|\,da\,dn,$$

$$B(x) = \int_{U\times N} m(anx)\overline{\psi(n)}\pi(a)^{-1}\,da\,dn.$$

Suppose a is S-invariant and

$$a(x) \le C\|x\|_{\mathrm{op}}^M, \qquad \int_G |m(y)|\|y\|_{\mathrm{op}}^M\,dy < \infty$$

for some $C > 0$, $M \ge 0$. Suppose B is continuous,

$$B(anx) = \psi(n)\pi(a)B(x),$$

and $B(x_1) \ne 0$ for some $x_1 \in G$. There is $\xi \in V$ such that $A(x) = B(x)\xi$ satisfies

$$0 < \mathcal{N} := \int_{S\backslash G} \|A(x)\|^2\,d\nu(x) \le c_S C\|\xi\|^2\int_G |m(y)|\|y\|_{\mathrm{op}}^M\,dy < \infty.$$

The function $W^0 = \mathcal{N}^{-1/2}A$ has quotient square norm one and the same equivariance. Every absolutely convergent identity

$$\int_G e(g)A(xg)\,dg = A(x)$$

for a scalar function e also holds with A replaced by W^0.

Outline. Pick ξ with $A(x_1) \ne 0$; unitarity, Tonelli's theorem, and quotient integration give $\mathcal{N} < \infty$. Disintegrating a neighborhood of x_1 where $\|A\| > \epsilon$ proves $\mathcal{N} > 0$, and scaling by $\mathcal{N}^{-1/2}$ gives W^0. The complete proof is on p. 917. □

Lemma 33.22 (A nonzero reproducing coefficient and the period normalization)**.** *Let G be a local Cayley unitary group and let $S = U \ltimes N$ be its Bessel subgroup. Fix Haar measures dg, da, dn such that multiplication carries $da\,dn$ to a two-sided Haar measure on S. Let ν be the quotient measure determined by*

$$\int_G f(g)\,dg = \int_{S\backslash G}\int_{U\times N} f(anx)\,da\,dn\,d\nu(x) \qquad (f \ge 0 \text{ measurable}).$$

Let $(\sigma,\mathcal{H})$ be a unitary discrete-series representation with formal degree $d > 0$, let (π,V) be a finite-dimensional unitary representation of U, and let ψ be the unitary Bessel character of N. Fix a smooth unit vector $w \in \mathcal{H}$, and put

$$\mathcal{D} = \mathrm{span}_{\mathbf{C}}\{\sigma(x)w : x \in G\}.$$

For $u, v \in \mathcal{D}$, define the operator-valued period

$$\mathcal{P}(u,v) = \int_{U\times N} \langle v, \sigma(an)u\rangle\overline{\psi(n)}\pi(a)^{-1}\,da\,dn.$$

Suppose these integrals converge absolutely. Let $\ell : \mathcal{H}^\infty \to V$ be a continuous complex-linear Bessel map, meaning that $\ell(\sigma(an)u) = \psi(n)\pi(a)\ell(u)$. Suppose that

$$\mathcal{P}(u,v)\eta = \langle \ell(v), \eta\rangle \ell(u) \quad (u, v \in \mathcal{D},\ \eta \in V)$$

and that, for every $v \in \mathcal{D}$,

$$\int_{S\backslash G} \|\ell(v)\|^2 \|\ell(\sigma(x)w)\|^2 \, d\nu(x) = \frac{\|\ell(v)\|^2}{d}.$$

Let $W : G \to V$ be continuous and satisfy

$$W(anx) = \psi(n)\pi(a)W(x), \qquad W(x) = d\int_G \overline{\langle w, \sigma(g)w\rangle} W(xg)\, dg,$$

with an absolutely convergent integral for every x. Then $W(x) = \kappa\ell(\sigma(x)w)$ for some $\kappa \in \mathbf{C}$. If W is nonzero, there is $x_0 \in G$ such that $\xi = \ell(\sigma(x_0)w) \neq 0$. For this choice,

$$A(x) := \mathcal{P}(\sigma(x)w, \sigma(x_0)w)\xi = \|\xi\|^2 \ell(\sigma(x)w), \qquad A(x_0) \neq 0,$$

$$\int_{S\backslash G} \|\ell(\sigma(x)w)\|^2 \, d\nu(x) = d^{-1}, \qquad \|A\|^2_{L^2(\nu)} = \frac{\|\xi\|^4}{d}.$$

Consequently

$$W^0(x) := \frac{A(x)}{\|A\|_{L^2(\nu)}} = \sqrt{d}\, \ell(\sigma(x)w)$$

has quotient square norm one, and $W = (\kappa/\sqrt{d})W^0$.

Outline. Left translation, Fubini's theorem, quotient integration, and the rank-one period identity yield $W = \kappa\Phi$. When $W \neq 0$, taking $\xi = \Phi(x_0) \neq 0$ in the period and norm identities gives the formulas for A, $\int \|\Phi\|^2 \, d\nu$, and W^0. The complete proof is on p. 918. □

Lemma 33.23 (Bessel uniqueness under an opposite-sign gap)**.** *Let $k \geq 1$, $r \geq 0$, and $n = 2r + k + 1$. Let $H_{k,r}$ be the $n \times n$ Cayley Hermitian matrix of signature $(k + r, r + 1)$ with ones on the antidiagonal pairing positions $1, \dots, r$ with $n, \dots, n + 1 - r$, diagonal entries $+1$ on positions $r + 1, \dots, r + k$ and -1 at position $r + k + 1$, and zeros elsewhere, let $G = \mathrm{U}(H_{k,r})$ be its unitary group, let $W_b \subset \mathbf{C}^n$ be its central coordinate subspace on positions $r + 1, \dots, r + k + 1$, and choose $z \in W_b$ with $-\operatorname{Re} h_{H_{k,r}}(z, z) > 0$. Write $S_z = U_z \ltimes N$ for its Bessel subgroup. Fix a generic unitary Bessel character ψ_c of N, with nonzero simple-root parameters $c_a \in \mathbf{C}$, $1 \leq a \leq r$, and an irreducible unitary representation (π, V) of U_z with compact Harish–Chandra parameter u.*

Let $(\sigma, \mathcal{H})$ be a discrete series representation of G with signature assignment $\varepsilon \in \{+, -\}^n$ (having $k + r$ positive and $r + 1$ negative entries) and Harish–Chandra parameter λ, where

$$\lambda_1 > \cdots > \lambda_n, \qquad \lambda_i - \frac{n-1}{2} \in \mathbf{Z}.$$

Suppose that, for a real number h,

$$h \geq 120n^3, \qquad |\lambda_i - \lambda_j| \geq h \quad \text{whenever } \varepsilon_i \neq \varepsilon_j.$$

Let $\mathcal{H}^\infty$ be the space of smooth vectors with its smooth-vector Fréchet topology. If $B_1, B_2 : \mathcal{H}^\infty \to V$ are continuous complex-linear maps satisfying

$$B_j(\sigma(an)v) = \psi_c(n)\pi(a)B_j(v) \quad (a \in U_z,\ n \in N,\ v \in \mathcal{H}^\infty,\ j = 1, 2),$$

then there are $a, b \in \mathbf{C}$, not both zero, such that

$$aB_1(v) + bB_2(v) = 0 \qquad (v \in \mathcal{H}^\infty).$$

Outline. Orbit boundedness and coefficient decay from the opposite-sign gap give a smooth unit vector w_0, a character-normalizing $\tau \in G$, a map Φ, and scalars c_1, c_2 with $B_j(\sigma(\tau)\sigma(g)w_0) = c_j\Phi(g)$. Smooth cyclicity makes the span of these translates dense in $\mathcal{H}^\infty$. Taking $(a, b) \neq (0, 0)$ with $ac_1+bc_2 = 0$ forces $aB_1+bB_2 = 0$ on $\mathcal{H}^\infty$ by continuity. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 918. □

Proposition 33.24 (Real Bessel integrals and local proportionality)**.** *Put $n = k + m - 1 = 2r_S + k + 1$, and suppose $n \geq 1801$ and $b_{\tau,j+1} - b_{\tau,j} \geq 16n^{19}$ at every CM embedding τ and every $0 \leq j < m - 1$. In a spectral realization X as in Theorem 33.78, fix a word u and an infinite place v of the totally real field F with corresponding CM embedding τ_v. Let $G_v \simeq \mathrm{U}(p_v, q_v)$ ($p_v + q_v = n$, with negative signature index q_v) be the local unitary group with Haar measure dg_v, and let $(\sigma_v, \mathcal{H}_v)$ be the local discrete series representation of G_v in X with Harish–Chandra parameter λ_v, formal degree $d_v > 0$, and unit lowest-K_v-type vector $w_v \in \mathcal{H}_v$. Pass to the local Cayley signature coordinates on G_v, and denote the Cayley group, Haar measure and Bessel subgroup of word u by G_v^0, dg_v^0 and $S_{u,v}^0 = U_{u,v}^0 \ltimes N_{u,v}^0 \leq G_v^0$. Here $N_{u,v}^0$ is the unipotent radical stabilizing the local isotropic flag of length r_S and orthogonal negative-norm vector $z_{u,v}$, equipped with its generic unitary Bessel character $\psi_{u,v}^0 : N_{u,v}^0 \to \mathbf{C}^\times$, and $U_{u,v}^0 \simeq \mathrm{U}(k)$ is the compact definite stabilizer acting unitarily on the finite-dimensional Hilbert space $V_{u,v}$ by the irreducible representation $\pi_{u,v}$ with compact Harish–Chandra parameter $u_{\tau_v} = (a_1, \ldots, a_k)$. The local numerical data of X have even $k \geq 2$. Both u_{τ_v} and $\lambda_v = (\lambda_{v,1}, \ldots, \lambda_{v,n})$ are strictly decreasing, with*

$$a_i - \frac{k-1}{2} \in \mathbf{Z}, \qquad \lambda_{v,s} - \frac{n-1}{2} \in \mathbf{Z}, \qquad a_i - \lambda_{v,s} \in \mathbf{Z} + \frac{1}{2} \quad (1 \leq i \leq k,\ 1 \leq s \leq n).$$

Their interlacing is

$$a_i > \lambda_{v,i} > a_{i+1} \quad (1 \leq i < k), \qquad a_k > \lambda_{v,k}, \qquad \lambda_{v,k+j} = -b_{\tau_v,j} \quad (0 \leq j < m).$$

The simple-root character parameters of $\psi_{u,v}^0$ are nonzero. Fix Haar measures da on $U_{u,v}^0$ and dn on $N_{u,v}^0$, let $\nu_{u,v}^0$ be a quotient measure on $S_{u,v}^0 \backslash G_v^0$ disintegrating dg_v^0 with respect to $da\,dn$, and set $N_v = 4n^2\sqrt{2q_v}$.

1. *For every $x_0 \in G_v^0$, the nonnegative integral*

$$a_v(y; x_0) = \int_{U_{u,v}^0 \times N_{u,v}^0} |\langle \sigma_v(x_0)w_v, \sigma_v(any)w_v \rangle|\, da\, dn$$

is finite for all $y \in G_v^0$ and satisfies $a_v(y; x_0) \leq C_v(x_0)\|y\|_{\mathrm{op}}^{N_v}$ for a constant $C_v(x_0) > 0$ independent of y. For all $x, x_0 \in G_v^0$, the operator-valued Bessel integral

$$B_v(x, x_0) = \int_{U_{u,v}^0 \times N_{u,v}^0} \langle \sigma_v(x_0)w_v, \sigma_v(anx)w_v \rangle\, \overline{\psi_{u,v}^0(n)}\, \pi_{u,v}(a)^{-1}\, da\, dn \tag{33.6}$$

converges absolutely in $\mathrm{End}_{\mathbf{C}}(V_{u,v})$, depends continuously on $x \in G_v^0$, and satisfies $B_v(a_0n_0x, x_0) = \psi_{u,v}^0(n_0)\pi_{u,v}(a_0)B_v(x, x_0)$ for every $(a_0, n_0) \in U_{u,v}^0 \times N_{u,v}^0$.

2. *There exist $x_1, x_0 \in G_v^0$ and $\xi \in V_{u,v}$ such that $B_v(x_1, x_0)\xi \neq 0$.*

3. *With x_0, ξ as in (2), put $A_v(x) = B_v(x, x_0)\xi$. Then $0 < \int_{S_{u,v}^0 \backslash G_v^0} \|A_v(x)\|^2\, d\nu_{u,v}^0(x) < \infty$, and if the global row function W_u of word u in X does not vanish identically on G_∞, the local factor $W_{u,v} : G_v \to V_{u,v}$ of W_u transported to G_v^0 is a scalar multiple of $W_v^0 = A_v/\|A_v\|_{L^2(\nu_{u,v}^0)}$.*

Outline. Weighted coefficient decay, a nilpotent chart on $N_{u,v}^0$, and dominated convergence prove (1). Model nonvanishing gives (2), and Lemma 33.21 with Schur orthogonality yields the nonzero L^2 norm and reproduction of W_v^0. Realizing W_v and W_v^0 by smooth Bessel intertwiners via Lemma 33.22 and applying Lemma 33.23 proves proportionality. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 920. □

Lemma 33.25 (Reproduction and integrability of the real row function)**.** *Put $n = k+m-1 = 2r_S+k+1$. Suppose $n \geq 1801$ and $b_{\tau,j+1} - b_{\tau,j} \geq 16n^{19}$ at every embedding and $0 \leq j < m - 1$. In a spectral realization*

X as in 33.78, write $G_\infty = \prod_v G_v$, let S_u be the real Bessel stabilizer for word u, and let ν_u be its quotient measure. With negative signature indices q_v, put

$$N = 4n^2\sqrt{2\max_v q_v}, \qquad L_u(x) = \inf_{s\in S_u}\prod_v \|(sx)_v\|_{\mathrm{op}}.$$

For each real row function W_u there is a bounded measurable a_u supported on a compact set containing the identity such that

$$W_u(x) = \int a_u(z)W_u(xz)\,dz, \qquad \int \|W_u(x)\|L_u(x)^N\,d\nu_u(x) < \infty.$$

The convolution is absolutely convergent at each x. The product ϕ of the conjugate lowest-type coefficients times their formal degrees belongs to $L^1(G_\infty)$.

Outline. At each infinite place, Cartan integration and Lemma 37.16 give weighted L^1 integrability of the lowest-type coefficient and $\phi \in L^1(G_\infty)$. Factoring W_u across places and unfolding the Bessel model of Proposition 33.24 yields the weighted L^1 bound. Averaging Schur orthogonality near 1 on each factor, transporting via Proposition 33.24, and multiplying across places builds a_u. The complete proof is on p. 925. □

Proposition 33.26 (The fifth construction duty)**.** *For the class $\mathcal{A}$ of 33.32, every $F_m \in \mathcal{A}$, $x > x_1(F_m)$, and $i \in I_x$ has $R_W(i) \le x^{-9}$ for the model's own weights, where R_W is the prime sum of 22.46. Thus $x_W = x_1(F_m) + 1$ works in that assertion.*

Outline. Choose integer weights with prime sum at most x^{-9} by 22.45. Use 33.32 to prescribe them at all translations and impose a common cutoff, and 33.27 to identify the boxes, extended weights, and prime sum. The complete proof is on p. 927. □

Lemma 33.27 (A trial pair on a host and at a translated point)**.** *Fix a scale $s = (x, \varepsilon, J)$, ranks R, a host H, and the orbit constants K of 14.8. Write A_H for the enumerated frame group. For exponents c_D, c_h put $\Delta = \lceil x^{c_D}\rceil$, $h_0 = 2\lceil x^{c_h}\rceil$, $\mathrm{Sc} = 4k\Delta$, and $\eta = mh_0$. For integer functions α, β on A_H, define $R_H(\alpha, \beta)$ by (22.9) at $\theta = x^{-\varepsilon}$, with the host's prime classes, Frobenius coordinates and $TT = TT(H)$. Use the least frame label in each candidate orbit and the two shares of 14.8, with $c = 4\mathrm{Sc}$ in 14.10.*

Let z be a point constructed at a translation of H with prescribed top weights. Its scale, ranks, auxiliary primes q, d, top labels, prime classes and Frobenius labels agree with those of H. Suppose its size data give Sc, η as above and have orbit factor $\mathcal{L} = K_\Lambda N_{\mathrm{win}}^2 \log a_*$ as in 14.8. Use the same representative convention at z and the local projection fraction of 14.10. Under the frame identification the integer boxes agree, and for every trial pair

$$R_z(\alpha, \beta) = R_H(\alpha, \beta).$$

If the prescribed top weights are given by α, β and the two functions vanish off TT, the point's extended weights are exactly α, β.

Outline. Common arithmetic data and matching frames give the same candidate-prime tests by 14.8, small-prime shares directly, and large-prime shares by 14.10. The box parameters and top labels identify the endpoints and zero-extended weights. The complete proof is on p. 927. □

33.2 From member parameters to the common data

Lemma 33.28 (Finite dimension of the projected row space)**.** *Let $G = U(V)$ be the unitary group of a nondegenerate Hermitian space over the CM extension E/F of 33.6, equipped with a full self-dual $\mathcal{O}_E$-lattice. Give $G(\mathbb{A}_F)$ product Haar measure. Write G_∞ and G_f for its infinite and finite factors, and put $Y = G(F)\backslash G(\mathbb{A}_F)$. For an integrable kernel q, write $R_q h(x) = \int q(z)h(xz)\,dz$, using the factor on which q is defined.*

At each real place v, let σ_v be an irreducible unitary representation with formal degree d_v for the chosen measure. Take a unit vector w_v for which $\phi_v(g) = d_v\overline{\langle w_v, \sigma_v(g)w_v\rangle}$ is integrable. Put $\phi = \prod_v \phi_v$. For every locally constant kernel e on G_f supported in the compact lattice stabilizer, the image of $R_e R_\phi$ on $L^2(Y)$ is finite dimensional.

Outline. Use formal-degree orthogonality to project by convolution, then use height twists and Fourier inversion to show that unipotent averages vanish. Control the cusps by this vanishing and use equicontinuity on the compact part to bound dimension by finitely many evaluations. The complete proof is on p. 927. □

Theorem 33.29 (Assembly on an augmented extension)**.** *Assume P and the separating rational row tests of 34.41. Fix weak local correspondences h_L as in Theorem 40.1, $c_{\mathrm{Dep}} \in \mathbf{R}$, and a condition A holding eventually on bases in the sense of* (33.1)*. Assume the linked $2,3$-bound of 36.16 at c_{Dep}. Along p_A of* (33.1)*, assume the covolume and functional rows closed on $W_3(\mathcal{A})$. For each eligible F_m, CH, the covolume row gives $C_0 \geq 0$ such that for each $C \geq C_0$, the third-layer covolume bound of 35.101 at C, CH holds eventually at selected points. Under the family division estimate* (36.18)*, the functional row gives the bound of 34.69 with the same quantifiers. The point cutoffs may depend on C. Then the common data of Definition 33.6 exist with norm condition* NL *and definite-side condition 36.1.*

Outline. Transfers member data via 33.55, 36.106, and 33.74, and derives all member rows from 33.34 using 34.17, 37.13, 33.28, and 33.75. Invokes 33.30 with 37.1, 37.5, 37.6, and 37.7. The complete proof is on p. 929. □

Theorem 33.30 (Filling the common record from all rows)**.** *Let KP be a condition on a base and frame, Q, Q' member conditions, and L_k a bundle linkage. Fix N, DR as in 33.6. Assume the following.*

1. *The adjacent-rank mass formula of 37.5.*
2. *The uniform analytic estimates of 37.1, the reproduction and weighted integral of 33.25, finite-dimensional row spaces as in 33.6, the minor-label estimate of 33.18, and both signed upper bounds of 33.99.*
3. *37.6 for each datum satisfying the quotient-volume hypothesis and 37.7 for every datum.*
4. *Assume Minkowski division implies the following four requirements. First, beyond the construction thresholds one can fix a preliminary base and frame such that any two prescribed finite sets of algebraic elements can be accommodated in the respective coefficient fields of a KP-base over that frame. Frame-preserving enlargement preserves KP. Second, an eligible untwisted KP-base with the required primes admits a Q-member. Third, every such member supplies the norm-choice and theta data used in the next enlargement. Fourth, such a member admits an enlargement with the same frame and prime set to meet new avoidance and coefficient conditions, with a member satisfying Q, Q' and twisting compatibility.*
5. *For KP, Q, Q', L_k, assume the bundle supply defined before 33.32 and all 26 member conditions of 33.34, using N for the norm condition and DR for the definite side.*

Then the common data of Definition 33.6 exist for N, DR, with

$$c_T(\varepsilon, y) = \max(11, 16 + 3\max(5, y+1)). \tag{33.7}$$

Outline. Fix the counting constants in 33.34 and intersect the member rows with 37.10. Use 33.32 to select a common family, choose joins pointwise with the closed thresholds, and fill the record of 33.6 using 33.32 and 33.26. The complete proof is on p. 930. □

33.3 Choosing the family after imposing the rows

Let L_k link points to bundles $\mathrm{Bd}(D, O, c)$. Write $Q \Rightarrow_{KP} R$ if, beyond the construction thresholds, every member satisfying Q on an eligible base and frame satisfying KP also satisfies R, for every choice of the subsequent column, row, division and core data. Define the construction class $\mathcal{C}$ as follows. Require the fixed family constants and all the pointwise equalities and rank bounds in clauses 1 and 2 of Definition 33.3. At every $x > x_1$ and every $i \in I_x$, impose the following conditions on the point.

- The prime q_0 of Definition 33.2 belongs to the host level set, satisfies $q_0 \equiv 1 \pmod{p_* d_E^*}$, $\log q_0 \leq x^{\varepsilon}$, $q_0 < X'$ and $q_0 \leq l_J$, and differs from $p_{\mathrm{de}}, p'_{\mathrm{de}}, g_0, g_1$ and every T_u. Here $l_J = mbx^{\varepsilon}/[k(m+t)\log(3(b+\mathrm{Sc}+\eta+t+1))]$, with the spacing and height parameters of Lemma 33.27. In particular this same prime supplies a host level prime congruent to one modulo d_E^* below X'.

- The tame labels pass clause 4 of Definition 33.3. The zero-free boxes and auxiliary-prime tests of Proposition 37.4 hold, with a scale in $[x^{1/J}, x]$ for each test. For every positive integer M divisible by the orders of all finite subgroups of $\mathrm{GL}_{2fN(m)}(\mathbf{Q})$, and every word u, require $D_{\Gamma,u} \mid M$.
- The point's own two weight functions belong to the boxes of Proposition 22.45 and satisfy its small-share bound $R_W(i) \le x^{-9}$ as in Proposition 33.26.
- Each tuple is used as in clause 6 of Definition 33.3. The point's own theta data pass the tests in clause 7 at every nonreference place for all the stated powers. These include the power-one test. For every tuple μ, the coefficient lattice $\widetilde{M}_\mu$ is stable under multiplication by $\iota_D(\mathcal{O}_{K_D})$.

For a condition $U(D,O)$, let $\mathcal{C}(U)$ consist of those families in $\mathcal{C}$ for which $U(D_{x,i},O_{x,i})$ holds at every $x > x_1$ and $i \in I_x$. A row $\mathrm{Row}(Q,L_k,H)$ gives $E(\kappa,D,O,c,\beta)$ on linked bundles of Q-points beyond the cutoffs. Here $\kappa = (\varepsilon, C, c_D, c_h, C_5, c_6, C_b)$ is linked to F_m if $C_5, c_6, C_b > 0$ and ε, C match F_m modulo entry 13 (C_{Tu}); if every $F_m \in \mathcal{A} \subseteq \mathcal{C}$ has linked κ and each selected bundle comes from a linked Q-point satisfying E, then $H(\mathcal{A},\sigma)$. Put

$$H \text{ holds after strengthening} \iff \exists R\,[(Q \Rightarrow_{KP} R) \wedge \mathrm{Row}(Q' \wedge R, L_k, H)]. \tag{33.8}$$

For a member condition U, write $\mathsf{B}(U,L_k)$ if beyond the point cutoffs every U-point admits an L_k-linked bundle for all $c \in \mathbf{R}$. Bundle supply for KP,Q,Q',L_k means $\exists R$ with $Q \Rightarrow_{KP} R$ and $\mathsf{B}(Q' \wedge R, L_k)$.

Lemma 33.31 (The scale before the word levels are chosen)**.** *Fix the construction constants and the exponents c_D, c_h of 33.27, with $c_D \ge 0$ and $c_h \ge 402$. Use a good host of 8.17, weights anywhere in the two prescribed boxes, and any prescribed code words. Write $L = x^{\varepsilon}$, $r = k+m-1$, and M_j for the standard definite genus mass in rank j. For $g = g_0, g_1$, let $I_c(g)$ be the index of the upper unitriangular group in $\mathrm{GL}_k(\mathbf{F}_g)$. Let $I_r(g)$ be the index in $\mathrm{GL}_r(\mathbf{F}_g)$ of the upper unitriangular group with the k entries above the diagonal in the last column of the middle block of size $k+1$ set to zero. The middle block begins after $(m-2)/2$ coordinates.*

For a decreasing parameter z of length j, put

$$W_j(z) = \frac{\prod_{a<b}(z_a - z_b)}{\prod_{a=0}^{j-1} a!}.$$

At the f embeddings of the CM type use the preliminary column parameter $U_{v,\rho}$ of word v, and the common row parameter $\lambda_\rho = (v_{0,\rho}, \dots, v_{k-2,\rho}, -b_{0,\rho}, \dots, -b_{m-1,\rho})$ in the coordinates of 15.9. The two pre-level factors are

$$d_v^\star = M_k \prod_{g=g_0,g_1} I_c(g)^f \prod_\rho W_k(U_{v,\rho}),$$
$$D^\star = M_r \prod_{g=g_0,g_1} I_r(g)^f \prod_\rho W_r(\lambda_\rho).$$

With $\Gamma = \exp(C_\Gamma m^5 f L)$, one has, for all sufficiently large x,

$$\frac{1}{fk} \log \frac{D^\star}{d_v^\star \Gamma^2} \;\ge\; mL$$

uniformly in the prescribed weights and words.

Outline. Combine adjacent-rank mass bounds from 37.5 with the two finite index ratios and the Vandermonde gap bounds from 15.9. The mass factorials cancel the Weyl denominators, and the prime lower bounds in 8.12 absorb the losses. The complete proof is on p. 930. □

Theorem 33.32 (A common family and selection)**.** *Assume the mass formula of 37.5 and the volume ratio of 37.6 for every D satisfying its quotient-volume hypothesis. Fix $KP,Q,Q',\mathrm{Bd},L_k,H$, and $c_T : \mathbf{R}^2 \to \mathbf{R}$. Assume item 4 of 33.30, bundle supply for KP,Q,Q',L_k, and (33.8) for H. There exist $\mathcal{A},\sigma$ with the following properties.*

1. *Minkowski division implies the design supply of Definition 33.6 for $\mathcal{A}, c_T$.*

2. *Every $F_m \in \mathcal{A}$ satisfies the seven duties of 33.3, with the following stronger clauses, and eventually all its coefficient lattices $\widetilde{M}_\mu$ are stable under $\iota_D(\mathcal{O}_{K_D})$. In clause 3, at each point above $x_1 = x_1(F_m)$, the prime q_0 is in the host level set and is 1 modulo the specified auxiliary-prime factor times the host conductor. It satisfies $\log q_0 \le L = x^{\varepsilon}$, $q_0 < X'$, $q_0 \le l_J$, and differs from $p_{\mathrm{de}}, p'_{\mathrm{de}}, g_0, g_1$ and every T_u. Here X' is the scale bound of 14.8 and l_J is the tame-generator limit. In clause 5 the model's own weights satisfy the box and share conditions of 33.26 at every $x > x_1$ and every point.*

3. *Every $F_m \in \mathcal{A}$ belongs to the construction class $\mathcal{C}$ with a linked parameter κ. At every $x > x_1(F_m)$, $i \in I_x$, the selected bundle comes from a linked Q'-point with that κ.*

4. *$H(\mathcal{A}, \sigma)$ holds.*

Outline. Intersect bundle and row conditions via 33.30 to define $\mathcal{A}, \sigma$ and verify items 2–4 and 33.3. Under Minkowski division, Theorem 8.17 hosts, 8.7, and 22.45 weights assemble the family of Definition 33.6. The complete proof is on p. 931. □

33.4 The member rows

For a property r of a member condition put

$$\mathrm{Lg1}(KP, Q, Q'; r) \iff \exists R\,[(Q \Rightarrow_{KP} R) \wedge r(Q' \wedge R)]. \tag{33.9}$$

Along p, take Q, Q' to be constructed and good membership and L_k the point-to-bundle relation defined after (33.1). The base condition $KP_x(B, f)$ has two clauses. For some field data δ, the designated subfield of $\mathbf{C}$ attached to B is exactly the compositum of the host character's value field, the fields M' and the ray field of δ, the prescribed roots-of-unity field, the value field of the frame characters and the fields $K_{\Psi,j}$ of δ for $1 \le j \le m$. The standard embedding to level m must also send each starting frame vector to its counterpart.

Lemma 33.33 (Row forms in the arithmetic coordinates)**.** *Fix a constructed member with its row comparison. For a tuple μ, let S_μ be the specified finite generating set of $\widetilde{M}_\mu$ in its complex ambient space. Write Λ_μ for the integral lattice of rational row forms before transport. For every word u there is a complex-linear identification $e_{u,\mu}$ of the period row space with the ambient space. It is the prescribed reference identification at μ_0 and the twisted identification at every other tuple. If $r_{u,\mu}(z)$ is the row form of a rational input z, every element of S_μ is $e_{u,\mu} r_{u,\mu}(z)$ for some z, and*

$$z \in \Lambda_\mu \quad \Longleftrightarrow \quad e_{u,\mu} r_{u,\mu}(z) \in \sum_{v \in S_\mu} \iota_D(\mathcal{O}_{K_D}) v.$$

Outline. Lift each designated generator in the rational row coordinates via the ι_D-semilinear coordinate embedding; the coordinate lattice condition preserves integrality and ambient evaluations across all integral linear combinations. The complete proof is on p. 935. □

Theorem 33.34 (The twenty-six rows)**.** *Assume global Artin reciprocity, the second convergent unitary Siegel–Weil formula of 34.17, discrete-series cover and classification, Harish–Chandra modules, cusp decay, the Casselman–Wallach input, surjectivity onto the point space, and the lower-weight bound. Fix weak local correspondences h_L as in Theorem 40.1, a norm-choice extension $\mathcal{N}$, an extra extension S, and a geometry condition Γ. For $p = (R_{\mathrm{row}}, \mathcal{N}, S, \Gamma)$, with the row comparison of (33.1), assume the open conditions of 33.74, the covolume and functional implications on this extension, with the family and point quantifiers of 35.101 and 34.69, respectively, and the linked row $\mathcal{A} \mapsto NN(W_3(\mathcal{A}))$ for a predicate NN on family classes. Then all 26 member conditions hold for KP_x, Q, Q', L_k, with NN as the norm condition and Definition 36.1 on the definite side. Each has the form (33.9), with closure on W_1 or W_3 where specified. The join type is nonempty, and the counting row gives both estimates with their constants.*

Outline. Take rows from 33.77, 33.78, 37.21, 37.6 and link periods for 17.21, 17.22. Check 21.40 via 15.12, 20.7, intersect with 33.74, 37.24, 37.26, 37.13, 34.17, and apply 36.7. The complete proof is on p. 935. □

Lemma 33.35 (An initial state with zero local contributions)**.** *Assume P and the rational test assertion of Theorem 34.41. Work beyond the starting-column thresholds of item 4 of Theorem 33.30. Fix an eligible starting column w, including its base, frame and twist. There is a choice of row data over w and a compatible later state with the following properties. Its set of local Galois indices is empty, and its coordinate $\eta_{j,v}$ is zero for every $j \in \mathbf{N}$ and every finite coefficient place v. In the untwisted case the selected set of places is a union of fibres of restriction to the fixed coefficient subfield.*

Outline. Pull back the untwisted reference support and use the rational tests of 34.41 with the multiplicity identity to lift the rows. Assemble the integral and spectral data with zero length coordinates and an empty local Galois index set. The complete proof is on p. 939. □

Lemma 33.36 (One cyclotomic specialization for the Selmer and theta polynomials)**.** *Let $O_0 \hookrightarrow O$ be an injective local homomorphism of discrete valuation rings of mixed characteristic $(0, \ell)$, with O_0 of finite residue field. For either ring A, write $d_A = \operatorname{ord}_A(\ell)$, where a uniformizer has order one. On any discrete valuation ring extending O by an injective local map, use the valuation v_ℓ with $v_\ell(\ell) = 1$.*

Fix nonnegative integers R, D, b. Write $Z = (X_1, \dots, X_R, Y_1, \dots, Y_R)$. Suppose that for every sufficiently large integer M we have the following two finite families, with index sets and bounds independent of M.

1. *For each s, the polynomial $F_{M,s} \in O_0[Z]$ has a coefficient of total degree at most D and order at most μ_s.*
2. *For each y, there are $f_{M,y}, A_{M,y}, g_{M,y} \in O[Z]$. Suppressing the subscripts, suppose that*

$$f - Ag \in \big(Y_i, (1+X_i)^{\ell^M} - 1 : 1 \le i \le R\big), \qquad \operatorname{ord}_O g(0) \le v_y.$$

In the residue polynomial ring, suppose that

$$\overline{A} \equiv u \prod_{i=1}^{R} X_i^{n_i} \pmod{(Z_j^{\ell^M} : 1 \le j \le 2R)}, \qquad u(0) \neq 0, \quad 0 \le n_i \le b.$$

The polynomials and the exponents n_i may depend on M, y.

Here μ_s, v_y are nonnegative integers. Given $\delta > 0$, there exist M, $1 \le k \le M$, and integers $0 \le e_j < k$, with $\ell^{e_j}/\varphi(\ell^k) \le \delta$, having the following property. For every primitive ℓ^k-th root ζ in a coefficient ring as above, set $z_j = \zeta^{\ell^{e_j}} - 1$. All the following values are nonzero and satisfy

$$v_\ell(F_{M,s}(z)) \le \mu_s/d_{O_0} + \delta, \qquad v_\ell(f_{M,y}(z)) \le v_y/d_O + \delta.$$

Outline. Use a diagonal subsequence and a weighted substitution to find coefficients of bounded degree and valuation, using a weighted valuation for the theta products. Choose a cyclotomic level avoiding ties for the least term valuation and evaluate at $\zeta - 1$. The complete proof is on p. 940. □

Lemma 33.37 (The Selmer length at the common character)**.** *Fix a twist index a, write $\ell = \ell_a$, and let $\mathcal{S}$ be its finite set of good signatures. Write $L_s(\eta, Q)$ for the normalized global Selmer length of Definition 33.3, with the character acting on the coefficient module by η^{-1}. Fix a finite set of forbidden places of the host field E. Suppose the following data are given for every sufficiently large M, with R independent of M.*

There are $2R$ split primes P_j of E whose conjugate pairs are disjoint. Their union Q avoids the forbidden places and the places above ℓ; its places have residue degree and ramification index one over $\mathbf{Q}$. Write $G_E = \operatorname{Gal}(\overline{E}/E)$. For each j there is a continuous anticyclotomic character $d_j : G_E \to \mathbf{Z}/\ell^M$, trivial at the decomposition groups above ℓ, unramified outside the pair of P_j, and surjective on inertia at both places of that pair.

Put $\Delta = (\mathbf{Z}/\ell^M)^{2R}$ and let b_j be its coordinate generators. For each $s \in \mathcal{S}$, suppose there is a square matrix Φ_s over $O_0[\Delta]$ presenting the Selmer condition in the following sense. For each designated coefficient ring A at level k and roots $u_j \in A$ of order dividing ℓ^M, put $\eta(g) = \prod_j u_j^{d_j(g)}$. The Selmer group for s, η, Q is identified over A with the kernel of the matrix specialized by $[b_j] \mapsto u_j$ on copies of $\operatorname{Frac}(A)/A$. Suppose $\det \Phi_s$ has a polynomial representative $F_{M,s}$ under $Z_j \mapsto [b_j] - 1$ satisfying the first hypothesis of Lemma 33.36 with $\mu_s = d_{O_0}\nu_a$. Include in that lemma any finite family of theta polynomials satisfying its second hypothesis on these same primes.

Then for any $e > 0$ there is one ray character ξ of ℓ-power order such that, for every $h \in \mathbf{Z}$ prime to ℓ and every choice of signs $\sigma_s \in \{1, -1\}$,

$$L_s(\xi^{h\sigma_s}, Q) \le \nu_a + \delta, \qquad \delta = \frac{e}{1 + |\mathcal{S}| + 2m + 4Rt}.$$

Here m is the chain length and t the auxiliary rank. At the same character the theta polynomials have the bounds of Lemma 33.36, with error δ, for either sign of h. The character is ramified exactly at Q and is trivial above ℓ. The sum of the signature errors is at most e, as is t times the sum of the tame valuations.

Outline. Use 33.36 to build one ray character and bound its tame valuations. For signed powers prime to ℓ, elementary divisors give Selmer lengths as normalized determinant valuations, and the same specialization bounds the theta values. The complete proof is on p. 941. □

Definition 33.38 (Twisting data of an untwisted member)**.** Let E be the CM host field. Call a twist index a eligible if restriction to the fixed coefficient subfield gives the place of a at some coefficient place w with $\mu_w \ne \mu_0$. Write $\ell = \ell_a$, let c act on $G_E = \mathrm{Gal}(\overline{E}/E)$ by complex conjugation, and write ε_ℓ for the cyclotomic character. For a good signature s put $T_s = \rho_s \chi_s^{-1}$, with the conventions of Definition 33.3. A parenthesized integer on a representation denotes a Tate twist.

The conditions on an eligible index are the following.

1. For each good s there is an integral isomorphism
$$(T_s^\vee(1))^c \simeq T_s(t).$$
At every place above ℓ, for each embedding of the local field, every jump of the de Rham filtration of T_s lies outside $[-1, t]$.
2. For each abelian theta label λ, the sum of its untwisted ray coefficients at empty auxiliary conductor is nonzero. Its prescribed doubled integer parameter is $2u$ with
$$1 \le u \le k + m(t + b_r + 1) + 1, \qquad b_r = \text{the odd member of } \{r + 6, r + 7\}.$$
3. The ranks and type primes satisfy $100 \le r$ and $2r \le g_i$ for $i = 0, 1$.

The twisting data exported by the member consist of nonnegative integers $N_0(a)$ and finite sets $S(a, j, \sigma)$ of its allowed theta data, with the following properties. At every w with $\mu_w \ne \mu_0$, the chosen index satisfies the preceding conditions. For each $1 \le j \le m$ and sign σ, the projected allowed theta vectors span the prescribed space over the fraction field. The projections indexed by $S(a, j, \sigma)$ form a basis of their integral span taken over *all* allowed data.

There is also a uniform assertion at every eligible index satisfying the three conditions. Suppose the projected vectors have the indicated full rank. Fix $e > 0$ and any finite sets of allowed data giving integral bases as above. For every finite set A of places of E, there is a triple (Q, k', ξ) for which all powers ξ^h, $h \in \mathbf{Z}$ prime to ℓ, pass the character, projected-lattice, Selmer, abelian theta and tame tests of Definition 33.3(7), with error e. The conductor set avoids A, the host exclusions and the levels of the chosen basis vectors. The sign at a signature or a layer is applied to the tested power. The integers $N_0(a)$ serve for every choice of e, bases and A.

Lemma 33.39 (Polarization at the Selmer normalization)**.** *Let $T = T_s$ be the representation of an attached good signature, over its base coefficient discrete valuation ring O_0. Suppose its residual image spans the full matrix algebra. In the unramified correspondence for T, the attached conjugate self-dual cuspidal representation has shift $(1 - t)/2$. For any finite-order anticyclotomic character η over an extension of O_0, put $V = T\eta^{-1}$. Then*

$$(V^\vee(1))^c \simeq V(t).$$

Over the fraction field V and $(V^\vee(1))^c$ are inequivalent. If their reductions are equivalent over an extension of the residue field, then $\nu_a > 0$.

Outline. Compare split Frobenius eigenvalues and use 36.24 to obtain equal traces. Full matrix span and the trace pairing give integral conjugacy, which survives anticyclotomic twisting, and determinants give the cyclotomic obstruction. The complete proof is on p. 942. □

Lemma 33.40 (Ray coordinates with prescribed Frobenius values)**.** *Fix the host, ℓ, and $M \geq 1$, and put $n = \ell^M$. Let T_0 be the places above p_{de} if $\ell \neq p_{\mathrm{de}}$, and above p'_{de} otherwise. There are an open normal subgroup $H \subset G_E$, containing the inertia groups at T_0, and a finite exceptional set S with the following property. At every split prime $P \notin S$ whose decomposition group lies in H, there is an additive character $d_P : G_E \to \mathbf{Z}/n$ with the local properties of Lemma 33.37.*

Let $U \triangleleft G_E$ be open, let z belong to inertia at a place of T_0, and let $h_1, \dots, h_R \in H$ fix μ_{2n}. Outside any additional finite set one can choose R primes P_i followed by R primes P'_i, all with disjoint conjugate pairs, with local degree one over $\mathbf{Q}$ and norms $1 \bmod 2n$. At the first primes the Frobenius conjugacy class modulo U is that of z. At P'_i it is that of h_i, and

$$d_{P_j}(\mathrm{Frob}_{P'_i}) = \delta_{ij}.$$

All these primes have the indicated ray coordinates.

Outline. Principalize ideals over the S-integers and use Artin reciprocity in 1.20 to form anticyclotomic ray coordinates. Apply 36.24 to choose the first primes, then use their inertia to correct the second Frobenius targets before applying it again. The complete proof is on p. 942. □

Lemma 33.41 (The square matrix for the Selmer condition)**.** *Use a finite family of ray coordinates from Lemma 33.40 whose conductor pairs avoid Σ_a. Let Q be the union of their conductor pairs, Δ the product of their cyclic targets, $\Lambda = O_0[\Delta]$, and $\kappa : G_E \to \Lambda^\times$ the inverse of their product in $\Delta \subset \Lambda^\times$. Fix a good signature and write $T = T_s$. For a discrete ℓ-primary Λ-module B, use $T \otimes_{O_0} B$ with the additional action of κ. Denote its Selmer group by $S(B)$. Use the local conditions of Definition 33.3, taking above ℓ the B-linear combinations of integral finite cocycles, that is, finite sums of cocycles $g \mapsto z(g)b$ with $b \in B$.*

Assume $r \geq 100$ and the integral polarization and filtration conditions of Definition 33.38(1), with $T = \rho_s \chi_s^{-1}$. Require ρ_s to be unramified outside the places above ℓ, g_0, g_1. At each $v \mid \ell$, require both ρ_s and T to be de Rham and potentially crystalline, and crystalline when $\ell \notin \{g_0, g_1\}$. At every finite place v, require

$$H^0(E_v, T \otimes_{O_0} \mathrm{Frac}(O_0)) = H^0(E_v, T^\vee(1) \otimes_{O_0} \mathrm{Frac}(O_0)) = 0.$$

Require the sums of the dimensions of $D_{\mathrm{dR}}/\mathrm{Fil}^0$ at conjugate embeddings to equal r. The residual image of ρ_s must span the full matrix algebra. At each place above a type prime $g_i \neq \ell$, require wild inertia to act trivially on $\bar{\rho}_s$, and require an inertia element γ with

$$\det\bigl(\bar{\rho}_s(\gamma) - \bar{\chi}_s(\gamma) I_r\bigr) \neq 0.$$

Use the local Bloch–Kato dimension formula and rational and integral duality of 26.27(5) over every finite extension of $\mathbf{Q}_p$, for every prime p, and the finite-coefficient cohomology hypotheses in Definition 1.20. Suppose T and its Tate dual are unramified outside the indicated exceptional places. There is a square matrix Φ over Λ, with entries in its maximal ideal, for which

$$S(B) \simeq \ker(\Phi : B^a \longrightarrow B^a)$$

naturally in B.

Outline. The local cocycle counts, Euler characteristics from 1.20, and duality in 27.2 bound the obstruction to lifting Selmer classes. Additive duality gives a representing module, and that bound gives at most as many relations as generators. The complete proof is on p. 943. □

Lemma 33.42 (The coefficient detected by first-order variation)**.** *Let P_i, P'_i, $1 \leq i \leq R$, be primes as in Lemma 33.40. Use the universal matrix of Lemma 33.41 for a fixed signature. Suppose the residual local cohomology in degree one vanishes at the first pairs, and the residual action of T and the cyclotomic character are trivial on the decomposition groups at the second pairs. Put $k_0 = O_0/\mathfrak{m}_0$. Let S_0, S'_0 be the residual Selmer spaces for T and its Tate dual at empty auxiliary conductor, with orthogonal local conditions. Use the dot product for the Tate pairing after choosing a generator of the residual Tate line.*

Choose an arithmetic Frobenius ϕ_i at each P'_i. On cocycle representatives $x \in S_0$, $y \in S'_0$, define

$$B_i(x, y) = x(\phi_i) \cdot y(\phi_i) - x(c\phi_i c^{-1}) \cdot y(c\phi_i c^{-1}).$$

Suppose a linear combination $\sum_i a_i B_i$ has left kernel of dimension at most b, where $b = 0$ or 1. If $4rR+2 < \ell^M$, the following hold. For $b = 0$, every polynomial representative of $\det \Phi$ has a unit coefficient of total degree at most $4rR$. For $b = 1$ and a nonempty matrix, there is an entry of $\operatorname{adj} \Phi$ for which the same conclusion holds for every representative.

Outline. Use reciprocity in 27.2 and the ray coordinates of 33.40 to bound the dimension of the kernel that lifts to first order. Apply elementary divisors to the matrix of 33.41 to detect a low-degree coefficient of the determinant or a cofactor. The complete proof is on p. 944. □

Lemma 33.43 (The cyclotomic factor in the integral cocycle calculation)**.** *Let A be a commutative local O_0-algebra. On A^r let U be the representation obtained from T_s by an anticyclotomic character, and put $U' = U^\vee(1)$. Suppose the matrices of U span $M_r(A)$, and write*

$$U'(cgc^{-1}) = J\,\varepsilon_\ell(g)^t U(g) J^{-1}$$

with $J \in \mathrm{GL}_r(A)$. Let $L \triangleleft G_E$ be stable under c, with U and U' trivial on L. Suppose there is z with $\varepsilon_\ell(z) = 1$, $U(z) = d\,1_r$, and $d^2 - 1 \in A^\times$. For cocycles x, y of U, U', respectively, write $\bar h = chc^{-1}$. If

$$x(h) \cdot y(h) = x(\bar h) \cdot y(\bar h) \qquad (h \in L),$$

then, for all $u, v \in L$ and all coordinates i, j,

$$\ell^{\nu_a} x(u)_i y(v)_j = 0 \quad \text{in } A.$$

Outline. Polarize the cocycle identity and use the scalar at z to isolate the cross terms. Modulo their annihilator, the matrix-unit argument of 33.39 gives determinant equality, so the cyclotomic factor from 33.3 kills every coordinate product. The complete proof is on p. 945. □

Lemma 33.44 (A finite coefficient ring detecting a cofactor)**.** *For O_0 a discrete valuation ring of mixed characteristic $(0, \ell)$ with finite residue field as in Lemma 33.36, write $d = \operatorname{ord}_{O_0}(\ell^{\nu_a})$, and let ϖ be a uniformizer. Suppose a polynomial $G \in O_0[X_1, \dots, X_N]$ has a unit coefficient of total degree at most w on a monomial involving none of the variables in a specified subset J. Put $L = (d+1)w$. There are a finite local O_0-algebra A, coefficients $a_i \in A$ zero for $i \in J$ and units otherwise, and $u \in A$ such that*

$$\varpi^{d+1} = 0, \qquad u^{2L+2} = 0, \qquad \ell^{\nu_a} u^{2L+1} \neq 0, \qquad G((a_i u)_i) \mid u^L.$$

Outline. Set the specified variables to zero and choose nonzero residue-field coordinates detecting a homogeneous part. Lift them to units in a finite free local algebra and adjoin nilpotent u, using the residue order to get divisibility and the free basis for nonvanishing. The complete proof is on p. 946. □

Lemma 33.45 (Local conditions for the representations of a member)**.** *Assume the hypotheses of Theorem 33.55. Take its untwisted constructed point, with its chosen column, and an eligible index a of Definition 33.38. Write O_0 for its base coefficient ring, k_0 for the residue field, $\ell = \ell_a$, and $T = T_s = \rho_s \chi_s^{-1}$ for a good signature. Put $T' = T^\vee(1)$. Both representations are unramified outside the places over ℓ and the exceptional places in Definition 33.3(7). The local hypotheses of Lemma 33.41 and the three index conditions of Definition 33.38 hold.*

Let D_0 consist of the places over p_{de} if $\ell \neq p_{\mathrm{de}}$, and over p'_{de} otherwise. There are a place $v \in D_0$ and an inertia element z at v, both independent of the signature, such that for every open normal subgroup $U \subset G_E$ some power z^n with $\gcd(n, \ell\,\mathrm{N}v) = 1$ lies in U, and for every good signature s,

$$\varepsilon_\ell(z) = 1, \qquad T_s(z) = d_s 1_r, \qquad d_s^2 - 1 \in O_0^\times.$$

At the exceptional places away from ℓ, both residual representations have zero local cohomology in degrees zero and one.

Outline. Use 32.60 and 36.25 for residual irreducibility and tame inertia for local vanishing. The attachments in 39.6 and 39.8 give the filtration, 33.39 gives polarization, and 34.1 gives the toric nonvanishing. The complete proof is on p. 946. □

Lemma 33.46 (Choosing the residual Frobenius tests)**.** *Fix a and a finite set of good signatures as in Lemma 33.45. For each $M \geq 1$ and the subgroup H of Lemma 33.40, there is an open normal c-stable subgroup $L \subset H$ containing every commutator $[g, z]$, fixing $\mu_{2\ell^M}$, and acting trivially on the residual cyclotomic character and on $T_s \bmod (\ell^{\nu_a})\mathfrak{m}_0$ for every signature s in the set. There is also a nonnegative integer R_0, independent of M and L, with the following property. Given L, there are $h_1, \dots, h_{R_0} \in L$ and an open normal subgroup $U \subset L$ such that for any $R \geq R_0$, any $h_{R_0+1}, \dots, h_R \in L$, any open normal subgroup $H' \triangleleft G_E$, and any finite exclusion set, applying Lemma 33.40 to $U \cap H'$ and $h_1, \dots, h_R$ yields R first primes whose inertia groups lie in L and whose Frobenius classes modulo $L \cap H'$ agree with z, together with R second primes whose decomposition groups lie in L, such that on the first R_0 pairs the residual local conditions of Lemma 33.42 hold for every signature in the set, and the corresponding forms have a linear combination of left corank at most b, with $b = 0$ if $\nu_a = 0$ and $b = 1$ otherwise. The determinant and adjugate conclusions of that lemma require $4rR_0 + 2 < \ell^M$.*

Outline. Definition 36.15, Theorem 27.2, Definition 1.20, Lemma 33.41, and Lemma 33.45 give $\dim S = \dim S'$. Intersecting kernels inside H gives L, and Lemma 33.39 with duality shows the pairings B_u span all forms or all alternating forms, giving left corank at most one. Choosing $h_1, \dots, h_{R_0} \in L$ and $U \subset L$ lets Lemma 33.40 yield primes meeting Lemma 33.42. The complete proof is on p. 948. □

Lemma 33.47 (Preserving the tests when a pair is added)**.** *Suppose $\nu = \nu_a > 0$. Put $I = (\ell^\nu)\mathfrak{m}_0$. Fix R first primes as in Lemma 33.46. At a stage with $n \leq R$ second primes use variables $X_0, \dots, X_{R-1}$ for the first primes and $Y_0, \dots, Y_{n-1}$ for the second primes. The first ray coordinates at a second Frobenius are the corresponding coordinate vector. The second decomposition groups lie in the group L of Lemma 33.46. Use the union of all conductor pairs for the relaxed Selmer condition.*

For a signature and a nonnegative integer c_0, consider the following tests on every *square presentation of this condition. The determinant test requires every polynomial representative of the determinant to have a coefficient outside I in degree $< c_0$. The cofactor test permits a matrix of size zero. For any positive size it requires an entry of the adjugate such that every representative has a unit coefficient in degree $< c_0$ on a monomial avoiding $X_n, \dots, X_{R-1}$.*

Suppose $n < R$, $M \geq \nu + 2$, and $c_0 + 2r \leq \ell^{M-\nu-1}$. Adjoin any second prime satisfying the same conditions for index n, with a new conductor pair. Each test already satisfied at c_0 holds at $c_0 + 2r$ for the enlarged family.

Outline. Recover the old Selmer module by killing $2r$ inertia generators annihilated by p. The resulting Fitting ideal inclusions multiply old minors by p^{2r}, whose leading term preserves detection within the enlarged degree cutoff. The complete proof is on p. 950. □

Lemma 33.48 (A dual class detected by an old cofactor)**.** *Use the stage $n < R$ of Lemma 33.47. Let $d = \operatorname{ord}_{O_0}(\ell^\nu)$, $w \geq 0$, $l = (d+1)w$, and suppose*

$$M \geq \nu + 2, \qquad 2l + 2 + 2r \leq \ell^{M-\nu-1}.$$

Suppose a signature passes the cofactor test at $w+1$ and fails the determinant test at $2l+2$. There are a finite local specialization A of the coefficients, an old presentation Ψ specialized to A, indices i, j, and $u \in A$, with

$$IA = 0, \quad u^{2l+2} = 0, \quad \ell^\nu u^{2l+1} \neq 0, \quad \det \Psi = 0, \quad (\operatorname{adj} \Psi)_{ij} \mid u^l.$$

Let x be the old Selmer class corresponding to column j of $\operatorname{adj} \Psi$. There is a cocycle y for the specialized Tate dual with coefficients in A, orthogonal to the old Selmer condition with additively dual coefficients, for which $\ell^\nu u^l[y] \neq 0$. On the subgroup L_1 obtained from L by killing all old ray coordinates except the unused first coordinates $n, \dots, R-1$, some h satisfies $B_h(x, y) \neq 0$, with the formula of Lemma 33.42 read in A.

Outline. Use 33.44 and 33.47 to kill the determinant while detecting a cofactor. The order formula 27.2 forces a surviving dual class, and 33.43 with the commutator argument of 33.46 forces a nonzero pairing. The complete proof is on p. 951. □

Lemma 33.49 (The relation determinant at a new prime)**.** *Under the numerical hypotheses of Lemma 33.48, a signature passing the cofactor test at $w+1$ admits an extension by one second prime for which it passes the determinant test at $2(d+1)w + 2 + 2r$. The first primes retain their prescribed Frobenius classes. All conductor pairs remain disjoint and avoid the prescribed finite set.*

Outline. Carry forward a passing determinant by 33.47. Otherwise use 33.48 and 36.24 to choose a prime with nonzero pairing, then use tame relations and reciprocity 27.2 to make that pairing a coefficient of a relation determinant. The complete proof is on p. 952. □

Proposition 33.50 (Uniform degree on a common set of primes)**.** *For the actual good signatures, the Selmer polynomial hypothesis of Lemma 33.37 holds with* $\mu_s = \operatorname{ord}_{O_0}(\ell^{\nu_a})$*. The numbers of primes and the degree bound are independent of* M*, of the prescribed first-prime congruence, and of the finite exclusions.*

Outline. Start with the primes of 33.46 and tests of 33.42. For positive ν_a, add one prime per signature by 33.49 and preserve earlier tests by 33.47, then take polynomial representatives of the matrices from 33.41. The complete proof is on p. 953. □

Lemma 33.51 (The integral class form giving a specialized theta vector)**.** *Fix an eligible index of Definition 33.38, a layer and a sign. Let* O *be the coefficient discrete valuation ring at this index. For an admissible twist* t*, let* O_t *be its designated coefficient ring. Let* M_t *be the lattice of definite class forms whose value at each finite adelic point belongs to the coefficient lattice at that point. Write* A_t *for the theta map on class forms, with the reciprocal stabilizer orders in its class sum. Every allowed specialized theta generator belongs to* $A_t(M_t)$*. This includes the trivial character at empty auxiliary conductor.*

Outline. Form an integral class form by summing the transported polynomials over the contributing rational elements. The dual kernel law makes their theta values equal, canceling the reciprocal stabilizer order, and a determinant twist unit gives the required generator. The complete proof is on p. 954. □

Lemma 33.52 (Removing the frame scalar from a lattice bound)**.** *Let* O *be a discrete valuation ring with fraction field* $K \subset \mathbf{C}$*, and suppose the rational prime* ℓ *lies in its maximal ideal. Let* V *be a finite-dimensional* K*-space,* $B \subset V$ *an* O*-lattice, and* $r : V \to W$ *a* K*-linear reading into a complex vector space. Suppose the readings of a* K*-basis of* V *are linearly independent over* $\mathbf{C}$*. Let* $x_\alpha \in V$ *be any family. If a fixed* $\lambda \in \mathbf{C}^\times$ *satisfies*

$$\lambda r(x_\alpha) \in r(B) \qquad \text{for every } \alpha,$$

there is $N \geq 0$ *such that* $\ell^N x_\alpha \in B$ *for every* α*. The same* N *works after adding coefficients with the same image under a map of finite ray groups.*

Outline. For $x_\alpha \neq 0$, $\lambda r(x_\alpha) = r(y)$ $(y \in B)$ and $\mathbb{C}$-linear independence of basis readings give $\lambda \in K^\times$, so $\lambda x_\beta \in B$. Taking N with $\ell^N/\lambda \in O$ gives $\ell^N x_\beta \in B$, preserved by fibre sums. The complete proof is on p. 955. □

Proposition 33.53 (Twisting data of the constructed point)**.** *Assume* P *and the separating rational row tests of Theorem 34.41. Fix weak local correspondences* h_L *as in Theorem 40.1,* $c_{\mathrm{Dep}} \in \mathbf{R}$*, and a condition* A *holding eventually on bases. Each untwisted constructed point for the parameters of* (33.1) *has the twisting data of Definition 33.38 for its own norm choices and theta series.*

Outline. Bounds toric denominators by 1.19 and builds full-rank projected spans via 33.51 and 34.41. Computes auxiliary-pair factors on ray polynomials via 36.106, 33.52, and 33.45, then applies 33.36, 33.50, and 33.37. The complete proof is on p. 955. □

Lemma 33.54 (The local indices and lattices of a twisted layer)**.** *Fix the rational row realization at the finite level of the member construction. Let* K_0 *be its coefficient field,* m *its number of layers, and* w *a finite place of* K_0 *of residue characteristic* ℓ*. Write* $\mu(w)$ *for the tuple selected at* w *and* μ_0 *for the reference tuple. Suppose* $1 \leq j \leq m$ *and* $\mu(w) \neq \mu_0$*. Let* W *be the rational space of the twisted layer at* (j, w)*.*

Use the split Hecke operators on W *of orders* $1, \dots, n_V$*, where* n_V *is the rank of the group acting on the layer. The places used are outside the conductor of the twist and away from* ℓ, g_0, g_1 *and* p_{de}*, with* p'_{de} *in place of* p_{de} *when* $\ell = p_{\mathrm{de}}$*. Let* K' *be the local coefficient field chosen for these operators at* w*, let* O *be its valuation ring, and put* $V = K' \otimes_{K_0} W$*. Let* $\mathbb{T} \subseteq \operatorname{End}_{K'}(V)$ *be the* O*-algebra generated by the indicated operators.*

The indices over (j, w) *in the localized realization are precisely*

$$(j, w, \mathfrak{m}), \qquad \mathfrak{m} \in \operatorname{Max} \mathbb{T}.$$

For such an index i, the coefficient ring O_i is O. Let $V_{\mathfrak{m}}$ be the sum of the joint generalized eigenspaces whose characters reduce with kernel $\mathfrak{m}$. Write $p_{\mathfrak{m}}$ for projection onto that sum along the other summands. If $S \subset W$ is the inverse image of the specified finite set of integral layer generators under the rational row embedding, then the local module and its comparison with the row are

$$M_i = \sum_{s \in S} O\, p_{\mathfrak{m}}(1 \otimes s), \qquad K' \otimes_O M_i \xrightarrow{\sim} V_{\mathfrak{m}}, \qquad a \otimes z \longmapsto az.$$

In these coordinates the map from W to the local module after inverting ℓ is $s \mapsto p_{\mathfrak{m}}(1 \otimes s)$. In particular,

$$\sum_{i \in I_{j,w}} \operatorname{rank}_{O_i} M_i = \dim_{K_0} W.$$

The length coordinate of the localized member is the sum in Definition 36.11 formed with these M_i and their Hecke actions. At a reference place the localized index set is empty.

Outline. Split the commuting integral Hecke action and group the joint eigenspaces by residual maximal ideal. Chinese remainder projectors decompose the layer lattice, and the quotient in 36.11 gives the sum of the normalized local lengths. The complete proof is on p. 960. □

Theorem 33.55 (Member and bundle supply on the same extension)**.** *Assume P and the separating rational row tests of 34.41. For every choice h_L of weak local correspondences as in Theorem 40.1, $c_{\mathrm{Dep}} \in \mathbf{R}$, and every A holding eventually on bases, the following hold on the parameters of (33.1):*

the mass ratio (37.9), *the member conditions for KP_x, Q_A, Q'_A,* *bundle supply for KP_x, Q_A, Q'_A, L_A.* (33.10)

The member assertion consists of the four conditions in item 4 of 33.30.

Outline. Apply 37.5 for the mass ratio and construct members using 33.35 and 33.54. Use 33.53 to choose twists before coefficient enlargement, then match the kernel and spectral data in the common frames to form linked bundles. The complete proof is on p. 962. □

Lemma 33.56 (The relative scalar at a finite coefficient place)**.** *In a linked realization on the list extension, assume P and work beyond the cutoffs for the row comparison. Fix a word u. For a tuple t, let Q_t be its chosen set of twisting places of the host CM field, and let S_t be the set of rational primes below Q_t. Write $t_0 = \mu_0$ for the reference tuple. The unnormalized comparison scalars $c_t \neq 0$ can be chosen so that*

$$(c_t/c_{t_0})^N = r, \qquad N \in \mathbf{N}_{>0}, \quad r \in \mathbf{Q}^{\times}, \quad v_p(r) = 0 \quad (p \notin S_t).$$

For every finite coefficient place w, its residue characteristic is outside S_{μ_w}.

Outline. Cancel level indices against the averaging factors and rescale the kernel charts to remove the intermediate phases. The remaining scalar ratio is a root of unity, and the selected twist is unramified above the residue characteristic of the coefficient place. The complete proof is on p. 965. □

Lemma 33.57 (The range of the arithmetic row embedding)**.** *At a linked point of (33.1), take the spectral realization at the reference row level and its prescribed normalization. Let H be the image of the reference finite-level and lowest-type projectors on the automorphic L^2 space. Let G be the sum of the good finite-type idempotents on H, and E_{ar} the complex arithmetic reference row space with its selected orthonormal basis. The conjugate of the model row embedding, after the prescribed identification of ambient spaces, maps E_{ar} onto GH. For each good label t, its selected basis vectors span the whole single-label space.*

Outline. Use formal-degree orthogonality to identify level intertwiners by evaluation at a lowest-type vector, with cuspidality from 33.25 and 33.28. Transport the good projector through the arithmetic coordinates and apply 32.44 label by label. The complete proof is on p. 967. □

Lemma 33.58 (First occurrence on the row chain)**.** *Put $r = k+m-1$. The row spaces at consecutive stages $j, j+1$ have ranks $r+jt, r+(j+1)t$. Use the real theta correspondence in the convention of the polynomial model, taking the contragredient parameter on passing from the row representation to its dual. At a fixed real place subtract the half-exponent of the target splitting character from the Harish–Chandra parameter. Suppose the resulting two lists for the smaller member are*

$$(A, B\,;\, C, D),$$

where A, C have positive entries, B, D have negative entries, all entries are distinct integers, and all have absolute value at least $(t+3)/2$. Here t is positive and odd. On the larger member the centered lists are

$$(A, I_t, D\,;\, C, B), \qquad I_t = ((t-1)/2, (t-3)/2, \dots, -(t-1)/2).$$

For the smaller member as source, its local theta lift vanishes on removing any positive number of hyperbolic planes from the indicated target. The same holds with the larger member as source and the smaller member as target. Consequently a global lift with these data at a real place and with cuspidal source is cuspidal whenever the tower constant-term formula of Theorem 9.5 applies.

Outline. Apply Atobe's discrete series criterion to identify first occurrence in both directions. Vanishing of the lower lifts gives cuspidality by the constant-term formula of 9.5. The complete proof is on p. 968. □

Lemma 33.59 (The value of the chain numerator)**.** *Let S contain the exceptional finite places of the CM field E. At $w \notin S$ let B_w be the multiset of bottom parameters, of cardinality at most r, with $|\alpha| \le (Nw)^{r/2}$ for $\alpha \in B_w$. Let $\mu_1, \dots, \mu_m$ be the unitary splitting characters of the row chain, unramified off S, with the prescribed distinct archimedean exponents at one CM embedding. For $0 \le j < m$, twist the bottom parameters by μ_{j+1}^{-1} and adjoin, for $1 \le i \le j$, the local roots*

$$(\mu_i/\mu_{j+1})(\varpi_w)(Nw)^{\,l-(t-1)/2}, \qquad 0 \le l < t.$$

Here ϖ_w is a uniformizer and $t \ge r+5$. The partial Euler product of this multiset, evaluated in the variable $s+1/2$, has a holomorphic continuation to a neighborhood of $s = t/2$ with nonzero value there.

Outline. Factor the chain Euler product, using absolute convergence for the bottom factor and 34.1 for the character ratios. The complete proof is on p. 969. □

Lemma 33.60 (The column dimension at a linked member)**.** *At every good linked member in the range of the size conditions of Theorem 33.74, let u be any word, $T = T_u$ its level prime, and d the dimension of its unrestricted column space. Use the pre-level factor $d_u^\star$ and the row factor $D^\star$ of Lemma 33.31. Put*

$$T_* = \left(\frac{D^\star}{d_u^\star \Gamma^2}\right)^{1/(fk)}.$$

Then

$$d = d_u^\star (T^k - 1)^f, \qquad \frac{1}{2} \le \frac{d\Gamma^2}{D^\star} \le \exp(1) < 3.$$

The unrestricted column space is equivariantly isomorphic to the space of complex functions on a finite free $(\mathbf{F}_T^\times)^f$-set. In these coordinates the ramified space is the common kernel of the sums over the individual factors.

Outline. Uses 37.21 away from T to get $d_u^\star$ from 33.31, then adjoins $(T^k-1)^f$ free $(\mathbf{F}_T^\times)^f$-orbit choices at T to model the space and ramified kernel. The level window for T_* yields the ratio bounds. The complete proof is on p. 969. □

Lemma 33.61 (The collision sum from finite fibres)**.** *Fix one unrestricted word space at a member of Lemma 33.60. Write*

$$r = (m-2)/2, \quad g = g_0 g_1, \quad v = \mathrm{rd}(E)/2, \quad L = \log b, \quad N = T^f, \quad R = d/D^\star, \quad B = kf \log k.$$

Use the size conditions of Definition 33.7 with width 3 *and* $C_\ell \ge 40$. *For terms* t,t' *in the same row use the orbital vectors* $v_t, v_{t'}$ *and the finite-type factor* Γ_1 *of* (33.19) *before Theorem 33.78. Let* $\mathcal{B}$ *consist of the ordered pairs for which both Gram invariants are nonzero and the corank is at least two.*

The following finite-fibre test suffices for

$$\Gamma_1 \sum_{(t,t')\in\mathcal{B}} |\langle v_t, v_{t'}\rangle| \le g^{-kf/2}\frac{d^2}{D^\star} + e^{-B/2}d.$$

For the test, index the rows and columns of a reduced Gram matrix Z *from* 0 *to* r. *Write* $p_l(Z)$ *for its leading minor of order* l, $\rho = 2v$, *and*

$$z_{l,\tau} = \rho^{-2\sum_{j=1}^{l}(r+1-j)}|\tau(p_l(Z))|, \qquad S_a(Z) = \prod_{\tau}\prod_{l=1}^{r+1}\max\{1, z_{l,\tau}b^{-am^2}\}.$$

The product over τ *uses one embedding of each conjugate pair. Put*

$$G_Z = g^{-mf}\prod_{\tau}(z_{r+1,\tau}^2 - T^2 z_{r,\tau}^2), \qquad n_Z = \left(\frac{\exp(1)\pi}{kv}\right)^{kf}(G_Z/N)^k.$$

Use labels on represented support with $G_Z > 0$. *For constants* $c \ge \max(1, C_T)$, $C_h, C_2, C_a \ge 0$, *write*

$$h_Z = b^{c^2m^3f}S_c(Z)^4, \quad S_Z = S_{C_h}(Z), \quad e_s = c(s+m)m + c(1+\log_2(2f)).$$

For each entry position a,b', *suppose the vectors* $(\rho^{a+b'-2r}\tau(Z_{ab'}))_\tau$ *belong to an additive lattice on which each nonzero vector has product of absolute values at least one. Suppose the product of the maxima of one and the absolute values of these coordinates is at most* h_Z.

For $2 \le s \le k$, *let* $\Xi_s(h)$ *be* M_sM_{k-s}/M_k *for* $s \le k-2$, *and* $(ck)^{kf}(hN)^{ck}$ *for* $s = k-1, k$, *with* M_j *the standard genus masses of Lemma 33.11. For a same-row pair* $(t,t') \in \mathcal{B}$ *of corank* s *with fixed flag* $F = F_u$, *moving flags* $F' = F'_t, F'' = F'_{t'}$, *relative lattice isometry* β, *complements* $H' = H(F,F') = W_t^{\perp}$ *and* $H'' = H(F,F'') = W_{t'}^{\perp}$, *and* $\Theta_{\mathrm{pairs}} = \{(x',x'') \in H'\times H'' : x'-x'' \in X_{\mathrm{sp}}(F),\ \beta x' - x'' \in X_{\mathrm{sp}}(F'')\}$ *having projections* $\Theta \subseteq H'$, $\Theta'' \subseteq H''$ *(*$s = k - \dim_E\Theta$*), define the sesquilinear pairing* $b_\beta(x',x'') = h(x',x'') - h(\beta x', x'')$ *on* $H'\times H''$. *Place* (t,t') *in the nondegenerate class* $\mathcal{P}_{\mathrm{nd},s}$ *(*$i=1$*) when the left and right radicals of* b_β *on* $H'\times H''$ *lie in* Θ *and* Θ'', *respectively (equivalently,* b_β *is nondegenerate on* $(H'/\Theta)\times(H''/\Theta'')$*), and in the degenerate class* $\mathcal{P}_{\deg,s}$ *(*$i=2$*) otherwise. Set* $b_1(Z) = 1$. *Set* $b_2(Z) = T^{-f/2}K(Z)$, *where* $K(Z) = 1$ *if* $p_r(Z) = 0$, *and otherwise*

$$K(Z) = \prod_{\tau}\frac{1+c_\tau}{\sqrt{c_\tau^2-1}}, \qquad c_\tau = \frac{|\tau(p_{r+1}(Z)/p_r(Z))|}{T}.$$

Use this expression on its represented support $c_\tau > 1$.

Suppose every finite fibre in class i *with labels* Z, Z' *has normalized absolute contribution at most* $d\,W_iA_i(Z)A_i(Z')$, *where the contribution includes* Γ_1, *and* W_i, A_i *are nonnegative. Require*

$$W_i \le \Xi_s(h_Zh_{Z'})(h_Zh_{Z'})^{e_s}(csf)^{6sf}\left((2/g)^{(s-i)f}\frac{\sqrt{G_ZG_{Z'}}}{N}b_i(Z)b_i(Z')\right)^k,$$

$$n_ZA_i(Z)^2b_i(Z)^{2k} \le h_Z^{2cr^2}\Gamma S_Z^{a_i-2q/(r+1)}F_i,$$

where

$$a_1 = C_2m^2, \quad a_2 = C_2(m^2+2m^4), \quad F_1 = 2R, \quad F_2 = \exp(2C_2m^6fL)(C_a/k)^{kf}.$$

It suffices to have the amplitude inequality for the single integer

$$q = \left\lceil (r+1)\left(4ck + 4e_k + 4cr^2 + \tfrac{1}{2}a_2 + 12(r+1)^2\right)\right\rceil.$$

For these hypotheses a sufficient size threshold is

$$\begin{aligned} A &= c^2 + 2\max(C_h - c, 0),\\ C &= \max\{c,\, 4 + A(6c+2),\, 2C_2,\, C_a,\, 408\exp(1)\pi\},\\ C' &\geq \max\{8C_\Gamma,\, 3C_2,\, 16(\exp(1)C)^{32}\}. \end{aligned}$$

In particular $q \leq (9c+3)km^2$.

Outline. Bounds the label sum $\sum_Z S_Z^{-12(r+1)^2}$ by disjoint-disk volume comparison on the entry lattices, then combines the fibre and amplitude inequalities via 33.100. Estimates Ξ_s using 37.5 and 33.11 and sums over coranks $2 \leq s \leq k$. The complete proof is on p. 969. □

Lemma 33.62 (The nonzero-Gram part from three pair sums)**.** *Fix a word u and write its Gram operator as a sum $K = \sum_t \mathrm{op}(t)$ that converges absolutely in Hilbert–Schmidt norm. Let the orbital vectors satisfy $\sum_t \|v_t\| < \infty$. Each term t has a row $\mathrm{row}(t)$, a finite-type index $\iota(\mathrm{row}(t))$ and a Gram invariant $G(t)$. Each ordered pair (t,t') of terms in the same row has a corank $s(t,t')$. Let E_{gen} and E_{ng} be the generic and the nongeneric terms with $G(t) \neq 0$. For a set E_1 of terms put $\mathrm{part}(E_1) = \sum_{t\in E_1} \mathrm{op}(t)$ and $w_i(E_1) = \sum_{t\in E_1:\, \iota(\mathrm{row}(t))=i} v_t$. For a set U of ordered pairs of terms put*

$$P(U) = \Gamma_1 \sum_{(t,t')\in U} \mathrm{Re}\langle v_t, v_{t'}\rangle, \qquad |P|(U) = \Gamma_1 \sum_{(t,t')\in U} |\langle v_t, v_{t'}\rangle|,$$

where $\Gamma_1 > 0$ is the finite-type factor. Assume the following.

(i) The terms of the identity cell have $G(t) = 0$ and sum to the identity. Thus $K - I = X + Z$, with $X = \mathrm{part}(E_{\mathrm{gen}}) + \mathrm{part}(E_{\mathrm{ng}})$ and Z the sum over the remaining terms.

(ii) For $E_1 = E_{\mathrm{gen}}$ and $E_1 = E_{\mathrm{ng}}$, the inequality of Lemma 33.112 (proved independently below) holds with some $P_g \geq 0$: $\|\mathrm{part}(E_1)\|_{\mathrm{HS}}^2 \leq P_g\Gamma_1 \sum_i \|w_i(E_1)\|^2$.

(iii) $\langle v_t, v_{t'}\rangle = 0$ whenever $\mathrm{row}(t) \neq \mathrm{row}(t')$ and $\iota(\mathrm{row}(t)) = \iota(\mathrm{row}(t'))$.

(iv) $s(t,t') \geq 1$ for terms t, t' in the same row with $G(t) \neq 0$ and $G(t') \neq 0$.

Let $\mathcal{B}$ be the set of same-row ordered pairs with both Gram invariants nonzero and corank at least two, as in Lemma 33.61. Let $\mathcal{U}_{\mathrm{gen}}$ and $\mathcal{U}_{\mathrm{ng}}$ be the same-row ordered pairs of corank one in E_{gen} and in E_{ng}, respectively. If

$$|P|(\mathcal{B}) \leq c_{\mathrm{col}}, \qquad P(\mathcal{U}_{\mathrm{gen}}) \leq c_{\mathrm{gen}}, \qquad P(\mathcal{U}_{\mathrm{ng}}) \leq c_{\mathrm{ng}},$$

then

$$\|X\|_{\mathrm{HS}}^2 \leq 2P_g(c_{\mathrm{col}} + c_{\mathrm{gen}} + c_{\mathrm{ng}}).$$

Suppose further that $P_g = (1+3/g_0)^f(1+3/g_1)^f$, with $(kb)^8 \leq g_0, g_1$, $b \geq 2$ and $1 \leq f \leq k$. Let $d, D^\star, Y, B$ be as in Lemma 33.71 (proved independently below), with $Y \geq 1$, and let $C_{\mathrm{gen}}, C_{\mathrm{ng}} \geq 0$. Suppose the three bounds above hold with

$$c_{\mathrm{col}} = (g_0g_1)^{-kf/2}\frac{d^2}{D^\star} + e^{-B/2}d, \qquad c_{\mathrm{gen}} = e^{C_{\mathrm{gen}}Y}\frac{d^2}{D^\star}, \qquad c_{\mathrm{ng}} = e^{C_{\mathrm{ng}}Y}\frac{d^2}{D^\star}.$$

Put $C_1 = \max(C_{\mathrm{gen}}, C_{\mathrm{ng}}) + 1$ and $a = C_1Y$. Then $C_1 \geq 1$, and the first equal-word estimate of Lemma 33.71 holds:

$$\|X\|_{\mathrm{HS}}^2 \leq \frac{12}{5}\left((1+e^a)\frac{d^2}{D^\star} + e^{-B/2}d\right).$$

Outline. Expands $\sum_i \|w_i(E_1)\|^2$ into same-row pairs by (iii), splits by corank via (iv), and applies (ii) with $\|x+y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$ and disjointness of $E_{\mathrm{gen}}, E_{\mathrm{ng}}$. Uses $1 + x \leq e^x$ and $2 \leq e^Y$ for the numerical bound. The complete proof is on p. 971. □

Lemma 33.63 (Orthogonality of orbital rows)**.** *Fix a tied equal-word model and a word u. Let $\mathcal{R}_u$ be the set of ordered pairs of row lattices. The orbital Hilbert space is the Hilbert direct sum of the spaces of normalized orbital operators indexed by $\mathcal{R}_u$. For a term t, its row is $\mathrm{row}(t) = (\Lambda_t, \Lambda'_t, \iota_t)$ with finite-type index $\iota(\mathrm{row}(t)) = \iota_t$ and lattice-pair coordinate $\rho_t = (\Lambda_t, \Lambda'_t) \in \mathcal{R}_u$. Write A_t for the Hilbert–Schmidt image of its normalized orbital amplitude and $w_t \in \mathbf{R}$ for its row weight. Its orbital vector is*

$$v_t(\rho) = \begin{cases} w_t A_t, & \rho = \rho_t, \\ 0, & \rho \neq \rho_t. \end{cases}$$

For terms with distinct lattice-pair coordinates,

$$\rho_t \neq \rho_{t'} \quad \Longrightarrow \quad \langle v_t, v_{t'} \rangle = 0.$$

Consequently hypothesis (iii) *of Lemma 33.62 holds for this model.*

Outline. The inner product is the sum of the coordinate inner products, and the supports of v_t and $v_{t'}$ are $\{\rho_t\}$ and $\{\rho_{t'}\}$. The complete proof is on p. 971. □

Lemma 33.64 (Full corank-one classes and their amplitudes)**.** *Use the term vectors, row map, corank function and signed pair sum of Lemma 33.62, and assume $\sum_t \|v_t\| < \infty$ and $\Gamma_1 > 0$. Let E_{gen} be the set of generic terms with nonzero Gram invariant. Suppose that*

$$t \sim t' \quad \Longleftrightarrow \quad \mathrm{row}(t) = \mathrm{row}(t') \ \ \textit{and} \ \ s(t,t') = 1$$

is an equivalence relation on the terms with nonzero Gram invariant. Let $\mathcal{C}$ be the set of equivalence classes of its restriction to E_{gen}, and let $\mathcal{U}_{\mathrm{gen}}$ consist of all same-row ordered pairs in E_{gen} of corank one.

Let $\mathcal{P}$ and $\mathcal{N}$ be sets of point labels and normal labels, with maps $\gamma : E_{\mathrm{gen}} \to \mathcal{P}$ and $\nu : E_{\mathrm{gen}} \to \mathcal{N}$. For $c \in \mathcal{C}$, $\gamma_0 \in \mathcal{P}$ and $\nu_0 \in \mathcal{N}$ put

$$V(c, \gamma_0, \nu_0) = \sum_{\substack{t \in c \\ \gamma(t) = \gamma_0,\ \nu(t) = \nu_0}} v_t.$$

Let $J_2 : \mathcal{P} \to [0, \infty)$ satisfy $J_2(\gamma(t)) = 0 \Rightarrow v_t = 0$ for $t \in E_{\mathrm{gen}}$. Define the normalized squared amplitude by

$$a_2(c, \gamma_0, \nu_0) = \begin{cases} \|V(c, \gamma_0, \nu_0)\|^2 / J_2(\gamma_0), & J_2(\gamma_0) > 0, \\ 0, & J_2(\gamma_0) = 0. \end{cases}$$

Then $a_2 \geq 0$, and

$$P(\mathcal{U}_{\mathrm{gen}}) = \Gamma_1 \sum_{c \in \mathcal{C}} \Big\| \sum_{\gamma_0 \in \mathcal{P},\ \nu_0 \in \mathcal{N}} V(c, \gamma_0, \nu_0) \Big\|^2, \qquad \|V(c, \gamma_0, \nu_0)\|^2 = a_2(c, \gamma_0, \nu_0)\, J_2(\gamma_0).$$

The represented point set $\mathcal{P}_* = \{\gamma(t) : t \in E_{\mathrm{gen}},\ v_t \neq 0\}$ consists precisely of the point labels represented by a nonzero term in one of these class and normal fibres.

For the finite amplitude estimate, fix $\gamma_0 \in \mathcal{P}$ and a finite set $S \subset \mathcal{C} \times \mathcal{N}$. Let r, c_1, c_0 be nonnegative integers and put $m = 2r + 2$. Let $\mathcal{V}$ be a finite index set and $\Omega \subset \mathcal{V}$. For each $p \in \mathcal{V}$ let D_p be a finite set, let $w_p : D_p \to [0, \infty)$ and $a_p : D_p \to \mathbf{C}$, and let $H_p, n_p, n_p^{\mathrm{gen}}$ be real numbers with $H_p \geq 1$, $n_p, n_p^{\mathrm{gen}} \geq 0$, and $H_p \geq 2$ for $p \in \Omega$. Assume

$$\sum_{d \in D_p} w_p(d) |a_p(d)|^2 \leq H_p^{c_1 (r+1)^2} n_p, \qquad \begin{cases} n_p \leq 8 H_p^{c_0 (r+1)^2 + 2m} n_p^{\mathrm{gen}}, & p \in \Omega, \\ n_p = n_p^{\mathrm{gen}}, & p \notin \Omega. \end{cases}$$

Suppose there are nonnegative weights ρ_c, a number $V_0 \geq 0$, a subset $S_0 \subset S$ and a map $\mathbf{d} : S_0 \to \prod_{p \in \mathcal{V}} D_p$ such that, for $(c, \nu_0) \in S$,

$$a_2(c, \gamma_0, \nu_0) = \begin{cases} \rho_c \Big| \prod_{p \in \mathcal{V}} a_p\big(\mathbf{d}(c, \nu_0)_p\big) \Big|^2, & (c, \nu_0) \in S_0, \\ 0, & (c, \nu_0) \notin S_0, \end{cases}$$

and, for every tuple $\mathbf{e} \in \prod_{p\in\mathcal{V}} D_p$,

$$\sum_{\substack{(c,\nu_0)\in S_0\\ \mathbf{d}(c,\nu_0)=\mathbf{e}}} \rho_c \le V_0 \prod_{p\in\mathcal{V}} w_p(e_p).$$

Put $H_{\mathrm{tot}} = \prod_{p\in\mathcal{V}} H_p$, $M_{\mathrm{gen}} = V_0 \prod_{p\in\mathcal{V}} n_p^{\mathrm{gen}}$ *and* $e_7 = (c_1+c_0)(r+1)^2 + 2m + 5$. *Then*

$$\sum_{(c,\nu_0)\in S} a_2(c,\gamma_0,\nu_0) \le H_{\mathrm{tot}}^{e_7} M_{\mathrm{gen}}.$$

Outline. Partitions $\mathcal{U}_{\mathrm{gen}}$ into class squares, regrouping the pair sum by Cauchy–Schwarz to express $P(\mathcal{U}_{\mathrm{gen}})$ via fibre norms. Groups S_0 by tuples $\mathbf{e}$ and combines the fibre and local weighted bounds to bound $\sum a_2$. The complete proof is on p. 972. □

Lemma 33.65 (The unitary dimension and mass bound)**.** *Fix a tied equal-word model of Theorem 33.78 and a word u. Let E/F be its CM extension, let g_0, g_1 be its two auxiliary primes, assumed to split completely in E, put $g = g_0 g_1$, assume $(T_u, g) = 1$, and let $\mathcal{V}_{\mathrm{typ}}$ be the finite set of places of F dividing g, with residue fields $\mathbf{F}_{q_v}$ for $v \in \mathcal{V}_{\mathrm{typ}}$. Let k be the initial rank, put*

$$\mathcal{I} = \prod_{v\in\mathcal{V}_{\mathrm{typ}}} \mathrm{GL}_k(\mathbf{F}_{q_v}),$$

and let $\mathcal{U} \le \mathcal{I}$ be the subgroup of tuples that are upper unitriangular at every $v \in \mathcal{V}_{\mathrm{typ}}$. Let Γ_1 be the finite-type factor of Lemma 33.112, namely the product over $v \in \mathcal{V}_{\mathrm{typ}}$ of the values at 1 of the normalized induced Whittaker characters on $\mathrm{GL}_k(\mathbf{F}_{q_v})$. Let C_u be the complex column space of u, let π_u be its archimedean coefficient representation, and put

$$d = \dim_{\mathbf{C}} C_u, \qquad n_\pi = \dim_{\mathbf{C}} \pi_u.$$

Let (V, h_V) be the Hermitian E-space of the model (with h_V, and its adelic extension h_A below, conjugate-linear in the first variable and linear in the second), let F_u be the fixed flag of u with top isotropic subspace $X_u \subset V$ of dimension r and rational flag stabilizer $\mathrm{Stab}(F_u)$ (fixing the flag basis of X_u and the final vector modulo X_u), and let $\mathcal{R}_0(u)$ be the model's finite set of representatives for the $\mathrm{Stab}(F_u)$-orbits of row lattices $\Lambda \subset V$. For $\Lambda \in \mathcal{R}_0(u)$, let $\mathrm{Aut}_{F_u}(\Lambda)$ be the group of E-linear automorphisms of $X_u^{\perp}/X_u$ induced by elements of $\mathrm{Stab}(F_u)$ preserving Λ, set $w(\Lambda) = |\mathrm{Aut}_{F_u}(\Lambda)|^{-1}$ when $\mathrm{Aut}_{F_u}(\Lambda)$ is finite and $w(\Lambda) = 0$ otherwise, and put

$$\mu_u = \mathrm{mass}_u = \sum_{\Lambda\in\mathcal{R}_0(u)} w(\Lambda).$$

Then $\Gamma_1 = |\mathcal{I}/\mathcal{U}| \ge 1$; in particular $\Gamma_1 > 0$ and $\Gamma_1 \ge 0$, and

$$\Gamma_1 \mu_u n_\pi \le d.$$

Outline. Evaluates 33.112 at 1 for $\Gamma_1 = |\mathcal{I}/\mathcal{U}|$. Lifts $\mathcal{I}/\mathcal{U}$ into $H_{\mathrm{int}}(u)$ via split-place frames, bounds each double-coset fibre by $|\mathrm{Aut}_{F_u}(\Lambda)|$, and sums over $\Lambda \in \mathcal{R}_0(u)$. The complete proof is on p. 972. □

Lemma 33.66 (Vanishing of orbital vectors on nonrelevant terms)**.** *Fix a tied equal-word model of Theorem 33.78 and a word u. Let E/F be its CM extension. For a term t of word u, let F_u be the fixed flag of u and F_t' the transported flag of t, let $\mathfrak{n}(F_u)$ and $\mathfrak{n}(F_t')$ be their rational unipotent radical Lie algebras with root-sum functionals ℓ_{F_u} and $\ell_{F_t'}$ into E, and call t relevant when $\mathrm{Tr}_{E/F}\circ\ell_{F_u}$ and $\mathrm{Tr}_{E/F}\circ\ell_{F_t'}$ agree on $\mathfrak{n}(F_u) \cap \mathfrak{n}(F_t')$. Let v_t be the orbital vector of t as in Lemma 33.63. If t is not relevant, then $v_t = 0$. In particular, if $\mathcal{T}_0$ is the set of nonidentity terms of word u with zero Gram invariant and $\mathcal{T}_{\mathrm{rel}} \subseteq \mathcal{T}_0$ is its subset of relevant terms, then*

$$v_t = 0 \qquad (t \in \mathcal{T}_0 \setminus \mathcal{T}_{\mathrm{rel}}).$$

Outline. Writes v_t via 33.63 as an orbital integral and shows $\Gamma_{\mathrm{orb}}(t)\backslash R_t$ is compact with χ_t trivial on $\Gamma_{\mathrm{orb}}(t)$. For nonrelevant t, exponentiation gives $\chi_t(r_0) = -1$, so averaging kills v_t. The complete proof is on p. 974. □

Proposition 33.67 (The norm expansion on the three nonidentity cells)**.** *Fix a tied equal-word model of Theorem 33.78 (with g_0, g_1 split completely in E and $(T_u, g) = 1$) and a word u. Let C_u be the complex column space of u, and let the orbital Hilbert space and inner product $\langle \cdot, \cdot \rangle$ (conjugate-linear in the first variable and linear in the second) be those of Lemma 33.63. Let $\mathcal{I} = \prod_{v \in \mathcal{V}_{\mathrm{typ}}} \mathrm{GL}_k(\mathbf{F}_{q_v})$ and $\mathcal{U} \le \mathcal{I}$ be as in Lemma 33.65. At each $v \in \mathcal{V}_{\mathrm{typ}}$ with selected additive character ψ_v of $\mathbf{F}_{q_v}$, let $\psi_{\mathrm{nat},v} : \mathrm{GL}_k(\mathbf{F}_{q_v}) \to \mathbf{C}$ equal $\psi_v\big(\sum_{a=1}^{k-1} x_{a,a+1}\big)$ for upper unitriangular x and 0 otherwise, put $\psi_{\mathcal{I}}(x) = \prod_{v \in \mathcal{V}_{\mathrm{typ}}} \psi_{\mathrm{nat},v}(x_v)$ for $x \in \mathcal{I}$, and define the finite-type Gram matrix $\mathcal{G} = (\mathcal{G}_{ij})_{i,j \in \mathcal{I}}$ by*

$$\mathcal{G}_{ij} = \overline{\gamma_{\mathrm{fin}}(j^{-1}i)}, \qquad \gamma_{\mathrm{fin}}(x) = |\mathcal{U}|^{-1} \sum_{y \in \mathcal{I}} \psi_{\mathcal{I}}(yxy^{-1}).$$

For each term t of word u, let O_t be its Hilbert–Schmidt operator on C_u, let v_t be its orbital vector, and let $\iota_t \in \mathcal{I}$ be its finite-type index. Assume

$$\sum_t \|O_t\|_{\mathrm{HS}} < \infty, \qquad \sum_t \|v_t\| < \infty.$$

Let E_{gen} and E_{ng} be the sets of generic and nongeneric terms with nonzero Gram invariant, and let $\mathcal{T}_0$ be the set of nonidentity terms with zero Gram invariant. For $S \in \{E_{\mathrm{gen}}, E_{\mathrm{ng}}, \mathcal{T}_0\}$, put

$$\mathrm{part}(S) = \sum_{t \in S} O_t, \qquad w_i(S) = \sum_{\substack{t \in S \\ \iota_t = i}} v_t \quad (i \in \mathcal{I}).$$

Then

$$\|\mathrm{part}(S)\|_{\mathrm{HS}}^2 = \mathrm{Re} \sum_{i,j \in \mathcal{I}} \mathcal{G}_{ij} \big\langle w_i(S), w_j(S) \big\rangle.$$

In particular, for $Z = \mathrm{part}(\mathcal{T}_0)$ and $w_i = w_i(\mathcal{T}_0)$,

$$\|Z\|_{\mathrm{HS}}^2 = \mathrm{Re} \sum_{i,j \in \mathcal{I}} \mathcal{G}_{ij} \langle w_i, w_j \rangle.$$

Outline. Uses block Parseval and bijections Φ_Λ built via 33.65 to lift $\|\mathrm{part}(S)\|_{\mathrm{HS}}^2$ to $\mathcal{I}^2$ by stabilizer invariance. Summing the Whittaker phases over $\mathcal{I}^2$ yields the Gram matrix $\mathcal{G}_{ij}$. The complete proof is on p. 975. □

Proposition 33.68 (Summation of the zero-Gram pair amplitudes)**.** *Fix $C_H, C_h, C_p \ge 0$ and $C \ge 2(C_H+1)$, and put $C_s = C_p + 4$. There is a constant $C_{\ell,\mathrm{pair}} \ge 16$, depending only on these constants, with the following property.*

Let a tied equal-word model satisfy the source conditions of Definition 33.7, the unramified CM hypothesis, and the parameter conditions of Proposition 33.18, with $C_\ell \ge C_{\ell,\mathrm{pair}}$. Assume the fixed-type estimate of Proposition 33.18 for the chosen constants. Write

$$r = (m-2)/2, \qquad L = \log b, \qquad \rho = \mathrm{rd}(E), \qquad H(c) = b^{cm^2 f}.$$

In particular, $m = 2r+2$, $k \ge m+3$, $L > 1$, $1 \le f \le k$, and

$$C_\ell m^6 L \le k \log k, \qquad C_\ell \log \rho \le L, \qquad (kb)^8 \le g_0, g_1.$$

Fix a word u. Let C_u be its complex column space, let π_u be its archimedean coefficient representation, and put

$$d = \dim_{\mathbf{C}} C_u, \qquad n_\pi = \dim_{\mathbf{C}} \pi_u, \qquad \mu_u = \mathrm{mass}_u\,.$$

Here μ_u is the sum of the inverse row-lattice stabilizer orders, with the normalization of Proposition 33.18. Let Γ_1 be the finite-type factor of Lemma 33.62, and put

$$P_g = (1 + 3/g_0)^f (1 + 3/g_1)^f.$$

Let $\mathcal{T}_0$ be the set of terms with zero Gram invariant outside the identity cell, and let $\mathcal{T}_{\mathrm{rel}}$ be its relevant terms. For a term t, write

$$\mathrm{row}(t) = (\Lambda_t, \Lambda'_t, \iota_t)$$

for its complete row label, including its finite-type index ι_t. Let v_t be its orbital vector, and assume

$$\Gamma_1 \geq 0, \qquad \Gamma_1 \mu_u n_\pi \leq d, \qquad v_t = 0 \quad (t \in \mathcal{T}_0 \setminus \mathcal{T}_{\mathrm{rel}}). \tag{33.11}$$

Define the set of ordered pairs

$$\mathcal{Q}_0 = \{(t, t') \in \mathcal{T}_0^2 : \mathrm{row}(t) = \mathrm{row}(t')\}.$$

Then the family $(\|v_t\|\|v_{t'}\|)_{(t,t')\in\mathcal{Q}_0}$ is summable, and

$$P_g \Gamma_1 \sum_{(t,t')\in\mathcal{Q}_0} \|v_t\|\|v_{t'}\| \leq d \exp(-kf \log k). \tag{33.12}$$

Outline. Sums S_δ^{-3r} over rank-j labels via minor integrality (33.18) and dyadic disk packing in $\mathcal{O}_E$. Combines 33.18, 33.7, $C_{\ell,\mathrm{pair}}$, and $P_g \leq 2$ to bound the pair sum by $d\exp(-kf\log k)$. The complete proof is on p. 977. □

Lemma 33.69 (From pair amplitudes to the zero-Gram operator)**.** *Fix a tied equal-word model satisfying the selected local hypotheses of Theorem 33.78 (as used in Lemma 33.112), and fix a word u. Use the notation $k, f, \mathcal{T}_0, \mathcal{Q}_0, v_t, \mathrm{row}(t), \iota_t, d, P_g, \Gamma_1$ of Proposition 33.68. Let O_t be the Hilbert–Schmidt operator of a term t, and assume*

$$\sum_t \|O_t\|_{\mathrm{HS}} < \infty, \qquad \sum_t \|v_t\| < \infty, \qquad \Gamma_1 \geq 0.$$

Let $\mathcal{I}$ be the finite set of finite-type indices, and put

$$Z = \sum_{t\in\mathcal{T}_0} O_t, \qquad w_i = \sum_{\substack{t\in\mathcal{T}_0 \\ \iota_t = i}} v_t \quad (i \in \mathcal{I}).$$

Let $\mathcal{G} = (\mathcal{G}_{ij})_{i,j\in\mathcal{I}}$ be the finite-type Gram matrix. Assume the norm expansion

$$\|Z\|_{\mathrm{HS}}^2 = \mathrm{Re} \sum_{i,j\in\mathcal{I}} \mathcal{G}_{ij} \langle w_i, w_j \rangle. \tag{33.13}$$

If (33.12) holds, then

$$\sum_{t\in\mathcal{T}_0} \|O_t\|_{\mathrm{HS}} < \infty, \qquad \|Z\|_{\mathrm{HS}}^2 \leq d e^{-kf\log k}.$$

Outline. Applies $\mathcal{G} \preceq P_g\Gamma_1 I$ from 33.112 to bound $\|Z\|_{\mathrm{HS}}^2 \leq P_g\Gamma_1 \sum_i \|w_i\|^2$. Expands $\|w_i\|^2$ via row orthogonality from 33.63 and Cauchy–Schwarz to reduce to the pair-amplitude bound. The complete proof is on p. 980. □

Theorem 33.70 (Uniform thresholds for the zero-Gram estimate)**.** *Fix $C_g, c_b, C_w, c_T, C_T, C_0 \in \mathbf{R}$, with $c_b, c_T > 0$. There exist thresholds $C'_{\mathrm{req}}, C_{\ell,\mathrm{req}}$ with the following property. For every $C' \geq C'_{\mathrm{req}}$ and $C_\ell \geq C_{\ell,\mathrm{req}}$, use a tied model of Proposition 33.18 satisfying the source conditions of Definition 33.7 and the unramified CM hypothesis. Fix any word u, and use the term operators, vectors, and column space of Lemma 33.69. Suppose that the absolute-convergence hypotheses of that lemma hold. Then, with*

$$Z = \sum_{\substack{t \text{ outside the identity cell} \\ G(t)=0}} O_t, \qquad d = \dim_{\mathbf{C}} C_u,$$

one has

$$\sum_{\substack{t \text{ outside the identity cell} \\ G(t)=0}} \|O_t\|_{\mathrm{HS}} < \infty, \qquad \|Z\|_{\mathrm{HS}}^2 \leq d e^{-kf\log k}. \tag{33.14}$$

The thresholds depend on the fixed constants and are uniform in the datum, the tied model, and the word.

Outline. Takes the maximum of thresholds from 33.18 and 33.68 and checks 33.18 from 33.7. Combines 33.65, 33.66, 33.68, 33.67, and 33.112, then concludes via 33.69. The complete proof is on p. 980. □

Lemma 33.71 (The numerical test on an unrestricted word space)**.** *Let C be the unrestricted complex column space for one word, $d = \dim C$, and $K = \alpha^{-1}P^*P$ its Gram operator, where P is the period map and $\alpha > 0$ the unfolding constant. Let $k \geq 5$ be the initial rank, $m \geq 2$ the chain length, $2 \leq f \leq k$ the degree of the real field, and $L \geq 1$ the logarithmic scale. Put*

$$Y = m^5 f L, \qquad a = C_1 Y, \qquad B = kf \log k,$$

where $C_1 \geq 1$. Suppose $K - I = X + Z$, with X the sum over the nonzero Gram invariant after removing the identity cell, and Z the remaining nonidentity sum. Suppose the two equal-word estimates, in the normalization with main term $D^\star > 0$, are

$$\|X\|_{\mathrm{HS}}^2 \leq \frac{12}{5}\left((1+e^a)\frac{d^2}{D^\star} + e^{-B/2}d\right), \qquad \|Z\|_{\mathrm{HS}}^2 \leq e^{-B}d.$$

Let $d^\star > 0$ be the pre-level dimension factor, let $\Gamma = \exp(C_\Gamma Y)$, $C_\Gamma \geq C_1 + 1$, and set $T_* = (D^\star/(d^\star\Gamma^2))^{1/(fk)}$. Assume the level prime T satisfies

$$d = d^\star(T^k - 1)^f, \qquad T_* \leq T \leq (1 + 1/(fk))T_*, \qquad T \equiv 1 \pmod{(g_0 g_1)^2}, \qquad g_0 \geq k^8,$$

where g_0, g_1 are the two type primes. Suppose the level model identifies C with functions on a finite free $(\mathbf{F}_T^\times)^f$-set, with each diamond operator acting by translation in its coordinate. Let U be the common kernel of the f coordinate sums. Then

$$\|K - I\|_{\mathrm{HS}}^2 \leq ed, \qquad \dim U \geq (1-\delta)d, \qquad 2e < 1 - \delta, \qquad e = 8e^{-a} + 6e^{-B/2}, \quad \delta = \frac{f}{T-1}.$$

Outline. Use the level window to bound $d/D^\star$ and combine the two Hilbert–Schmidt estimates. Count characters nontrivial in every coordinate on each free orbit, then use the level congruence and size bounds to show $2e < 1 - \delta$. The complete proof is on p. 981. □

Lemma 33.72 (A nonzero bottom pairing)**.** *Let C_u be the complex column space of a word u, $d_0 = \dim C_u > 0$, $P_u : C_u \to H$ its period map into the spectral Hilbert space, and $\alpha_u > 0$ its unfolding constant. Put $K_u = \alpha_u^{-1}P_u^*P_u$. Let $U_u^{\mathrm{ram}} \subset C_u$ be the ramified subspace in the column coordinates. Suppose, for real e, δ,*

$$\|K_u - 1\|_{\mathrm{HS}}^2 \leq ed_0, \qquad \dim U_u^{\mathrm{ram}} \geq (1-\delta)d_0, \qquad e < 1 - \delta.$$

Then some $\phi \in U_u^{\mathrm{ram}}$ satisfies $P_u\phi \neq 0$. At a linked realization with the full reference level of Lemma 33.57, suppose the column identification intertwines the unrestricted period maps. Then a full reference multiplicity map has nonzero translated pairing with a model column representing ϕ.

Outline. Bound the period kernel dimension by $\|K_u - 1\|_{\mathrm{HS}}^2$ to find a ramified column with nonzero period. Surjective full-level evaluation from 33.57 and invertible central translation realize that period by a reference row. The complete proof is on p. 981. □

Lemma 33.73 (Comparison with a full reference row)**.** *Use the row and column comparisons at a linked realization, with the common finite translate and level measures of Lemma 33.56. Fix a word u. Let T_0 be a full level-zero reference multiplicity map, with a preimage under descent on the reference chain. Suppose that its translated model pairing with a permitted column is $b_0 \neq 0$. For a tuple t, a column ϕ in the complex realization of the permitted space and a rational row F, let $A_t(\phi, F)$ be the arithmetic top period in the complex model. Use the column change from the chain and the bottom column isometry to form the translated model pairing $B_t(\phi, F)$ with the descended row. Before dividing the column change by its norm multiplier, there is a $p \neq 0$, independent of t, ϕ, F, such that*

$$c_t A_t(\phi, F) b_0 = p B_t(\phi, F).$$

Here c_t is the unnormalized scalar of Lemma 33.56.

Outline. Unfold each double theta step to descend relations among weighted periods, using cusp decay as in 34.23. Transfer the bottom relation to translated pairings and use the nonzero reference pairing to obtain the comparison scalar. The complete proof is on p. 982. □

Theorem 33.74 (The twelve open conditions)**.** *Assume P. For every choice h_L of weak local correspondences as in Theorem 40.1 and $c_{\mathrm{Dep}} \in \mathbf{R}$, the parameters $p_0(h_L, c_{\mathrm{Dep}})$ satisfy the following linked rows.*

Primitive rows	*Primitive rows and separated types.*
Finite types	*Finite types assigned to each row index and layer. The isotypic spaces form a direct sum equal to the transported row space and have prescribed λ on nonzero components at used tuples. The assigned counts sum to the base row dimension.*
Divisor comparison	*Both RR lattices agree with their indefinite realizations, and rational lifts span at used tuples. The theta maps, numerical data and column pairings agree.*
Definite evaluation	*At used tuples the embedded definite evaluation equals the arithmetic column period on transported inputs.*
Join maps	*The join maps and identities of 33.6 at every selected point.*
Recursion	*Compatible indefinite recursion at every point.*
Label realization	*Choose a Hida realization X, a good operator and an isometry from spectral to model space. The automorphic datum has the prescribed rank, degree, root discriminant, signatures, Harish–Chandra parameters, auxiliary levels and dimension. Its archimedean representations have the specified discrete series and lowest unit vectors, and its formal-degree and volume expression equals $D^\star$. There is a complex-linear isometry from the arithmetic reference row space onto the complex conjugate of the full good subspace cut out by the operator. After the ambient identification, conjugating this isometry gives the spectral row embedding. At each type prime the assigned local polynomials are valid labels, each row label tuple is good in the sense of Definition 33.5, and the realized row belongs to its labelled type space.*
Good labels	*Eventually at every selected point, every finite-type row satisfies Definition 33.5 at both type primes, for the prescribed tame characters and their inverses.*
Geometry	*Point summands with class-sum pinning, recursion ties and an evaluation grid.*
Multiplicity	*Eventually, the multiplicity identity (33.16) with the unit conditions below.*
Infinite-place factors	*Eventually, for each $e : K_D \hookrightarrow \mathbf{C}$ at its infinite place v_e, the factor of the division datum at e equals the selected row-side factor $\kappa(v_e)$.*
Definite side	*Eventual definite-side condition of 36.1 for each $C \geq C_0$.*

(33.15)

For the multiplicity identity, use the definite face and complex realization of the selected bundle β. Write K for the coefficient field of this face and $\iota : K \hookrightarrow \mathbf{C}$ for its embedding. Let u range over its column indices and t over its tuples. Let $\phi \in K^{a_u}$ be a column coordinate vector and $F \in K^{n_t}$ a row coordinate vector, with a_u, n_t their respective dimensions. Write e_t for the rational row realization, j_t for its map to the complex realization, Θ_t for the theta map, s_{EE} for the isometry on its output space, and P_u for the spectral column map. There are a complex-linear map V, column matrices U_u, and scalars $a'_u \neq 0$, $\zeta_{u,t} \neq 0$ such that for every u, t, ϕ, F,

$$\zeta_{u,t}\, \iota\big(\mathrm{Pt}_{u,t}(\phi, F)\big) = a'_u \left\langle s_{EE}\big(V \Theta_t j_t e_t F\big),\, P_u\big(U_u \iota(\phi)\big)\right\rangle . \tag{33.16}$$

Here $\mathrm{Pt}_{u,t}$ is the definite evaluation pairing and $\iota(\phi)$ is taken coordinatewise. The map V preserves the norm on the good theta subspace and sends it into itself. Each U_u preserves the prescribed column norm, and $\zeta_{u,\mu_0} = 1$. At each finite coefficient place w, any $z \in K$ with $\iota(z) = \zeta_{u,\mu_w}$ has $v_w(z) = 0$.

The good-label, multiplicity, infinite-place and definite-side conditions are closed on W_3. For the definite-side condition, choose $C_0 \geq 0$ per eligible F_m, CH, then a cutoff per $C \geq C_0$. For extensions with the same h_L, S, Γ, a proof, for every choice in N', that a choice in N exists at the same point indices transfers each linked row with conclusion $H(\mathcal{A}, \sigma)$ from N to N'.

Outline. Combine the reference pairing of 33.72 with descent in 33.73 and scalar units from 33.56. Use 33.57 for the full good subspace and 32.37 and 36.5 for the finite-type and definite-side rows. The complete proof is on p. 983. □

Lemma 33.75 (The lower weight before the choice of a family). *Assume P of Definition 1.20. On the class $\mathcal{F}_{\mathrm{cov}}$ of Definition 35.2, the following assertion is closed in the sense of Definition 36.10. For each eligible F_m, C_H there is x_0 such that, whenever $x > x_1$ and $x \geq x_0$, the following holds for every $i \in I_x$ and every point in $\mathcal{P}$ of Definition 36.6 over (x, i). For every $C \in \mathbf{R}$, every layer $1 \leq j \leq m$, every primitive block a and every infinite coefficient place v,*

$$L_{6,j} \leq \log \lambda^{(1)}_{j,\mu_0,a,v}.$$

Here the superscript denotes level one, and $L_{6,j}$ is the sixth numerical entry at C, C_H in the notation of 21.12.

Outline. Check the size and prime conditions of 21.16 at one family cutoff using the level-one realization of 33.74. The lift comparison 21.19 and scalar identities 21.21 give the bound for every C, whose coefficient in $L_{6,j}$ is zero. The complete proof is on p. 986. □

Lemma 33.76 (A common cutoff for the reference index bounds). *Use the level-zero realization at the points of $\mathcal{P} \cap \mathcal{R}$ in Definition 36.6. Use $D_R, i_{jw}, e_w, \ell_w, c_w, F_{0,j,w}, J_{0,j,w}$ as in 36.8, and set $U = L(B_0 + N_c)$. Here $D_R > 0$, $m \geq 1$, and $U \geq 0$.*

Suppose that on a family satisfying the division estimate (36.18) there are $a \geq 0$ and x_0 with the following property. For $x > x_1$, $x \geq x_0$, every index $i \in I_x$ and every such point over (x, i) admit numbers $u_{jw} \geq 0$ satisfying

$$\frac{1}{D_R} \sum_j i_{jw} \leq \sum_j (2e_w F_{0,j,w} + u_{jw}) + c_w \quad (w \in \mathrm{Tw}), \qquad \sum_{w \in \mathrm{Tw}} \ell_w (u_{jw} + e_w J_{0,j,w}) \leq aU \quad (1 \leq j \leq m).$$

Suppose $J_{0,j,w} \geq 0$ on these points. Then the three reference inequalities (36.12)–(36.14) hold at every one of the same points for every $C \geq 2a$. In particular, if the hypotheses hold with the family quantifiers of 36.10 on $\mathcal{F}_{\mathrm{cov}}$, so does the reference index implication of 33.34.

Outline. Take the positive part of the excess index at each place and divide it equally among the layers. Averaging the weighted errors and adding the bound for the nonnegative reference terms gives all three inequalities for every $C \geq 2a$ at the original cutoff. The complete proof is on p. 986. □

33.5 Evaluation, periods, and the selected spectrum

Theorem 33.77 (Six arithmetic rows). *Fix construction conditions $\mathcal{S}$ and a point-to-bundle relation L_k. Take $KP = KP_x$ and let Q, Q' mean constructed and good membership for $\mathcal{S}$, respectively, in the sense defined after (33.1). The following* Lg1*-rows are closed on W_3 and eventual at selected points.*

1. *For each word u, reference evaluation is injective on the specified labels and bijective on their stabilizers. Its complex map Pt_u takes $F_i(m)$ into ptC_u. An integral rational realization P_u satisfies, for each infinite v and rational level-m vector z,*

$$q_{u,\mathrm{pt},v}(P_u z) \leq \mathrm{ptB}(D, u) q_{u,m,v}(z), \qquad \mathrm{ptB}(D, u) = \prod_\tau T_u^{-\deg_\tau} \mathrm{beOn}_\tau^2. \tag{33.17}$$

   Here q_v is the real majorant diagonal, and $\deg_\tau, \mathrm{beOn}_\tau$ are the chain degree and norm factor. The same constant bounds $\|\mathrm{Pt}_u \phi\|^2 / \|\phi\|^2$ for nonzero $\phi \in F_i(m)$.

2. *Each used nonreference tuple has a rational integral twisted evaluation for every word.*

3. *Assume the common denominator condition of item 6 of Definition 15.12, namely that every twisted chain has denominator D_{Γ,u_0} and every class stabilizer order at every layer $0 \leq j \leq m$ of every twisted chain divides this integer. For each word u, reference point-lattice vector, and $H \in \widetilde{M}_{\mu_0} \cap V_0(\mu_0)$, one $y \in K_D$ represents the period through ι_D. At every finite w, with residue characteristic p and ramification index e_w,*

$$y = 0 \quad \text{or} \quad v_w(y) \geq -e_w v_p(D_{\Gamma,u}). \tag{33.18}$$

4. *Under the same common denominator condition, for each word u, used $\mu \neq \mu_0$, point-lattice vector, and $H \in \widetilde{M}_\mu \cap V_0(\mu)$, one representative of the twisted period satisfies (33.18) at every finite w with $p \notin Q'(\mu)$.*

5. *At each point, reindexing a signature basis identifies the scaled Haar measure with the prescribed compact-dual normalization. At every infinite place the prescribed discrete-series formal degree is attained and every discrete series with the specified parameters has that degree for this measure. The adelic quotient measures give the prescribed fundamental-domain volumes for all admissible signatures. The point also has a compatible symmetric column system. For each word the level-zero character has finite exponent and values in $\iota_D(K_D)$.*

6. *For each word u and finite place w above $p = T_u$, put $E^T_{u,w} = T^*_{u,w}$, with $T^*_{u,w}$ as in 17.12. For every input in $N_{\lambda,u,w}$ and $c \in K_D^\times$ with $v_w(c) \geq e_w m E^T_{u,w}$, scaling the reference column by $\iota_D(c)$ puts it in the image of $S_m(w)$. For each used $\mu \neq \mu_0$ with $p \notin Q'(\mu)$, the twisted column is represented by az under the terminal embedding, with z in the local twisted level-m lattice and $a = 0$ or $v_w(a) \geq -e_w m E^T_{u,w}$.*

Outline. Extracts the column system, lowering maps, and Petersson decomposition from good members, with local Weyl-dual row lifts from 33.33. Applies 17.3 for items 1 to 4, the real comparison for item 5, and 17.20 for item 6. The complete proof is on p. 987. □

For the term vectors v_t of the spectral rows and a set S of term pairs, put

$$\mathrm{PairAbs}_{u,u'}(S) = \Gamma_1 \sum_{(t,t') \in S} |\langle v_t, v_{t'} \rangle|, \tag{33.19}$$

where $\Gamma_1 > 0$ is the finite-type normalization. Among same-row pairs with both Gram values $G \neq 0$, let S_{big} have corank at least two and S_{nsc} consist of nonscalar generic pairs of corank one. Write $d^C_w, D^\star_C$ for the model column dimensions and main term, and $d^S_w, D^\star$ for their arithmetic counterparts. Set

$$\epsilon_L(D) = e^{-kf \log k/3}, \qquad \Delta_{u,u'} = \frac{d^C_u d^C_{u'}}{D^\star_C}. \tag{33.20}$$

Theorem 33.78 (Three estimates for the selected spectral rows)**.** *Assume the finite-volume and nonsingular Gram-density assertions of the second formula of 34.17. For every KP, Q, Q', L_k, the following estimates give* Lg1*-rows closed on W_3. Put*

$$C_0 = 3, \qquad C' = C_k, \qquad C_\ell = \min\{C_f, C_w, C_\Delta, C_{\mathrm{count}}, C_r, C_{\mathrm{gap}}\}, \qquad p = (C_{YL} + 1, \tfrac{1}{4}, 3, \tfrac{1}{2}, C_{Tu}).$$

Use the family constants of Definition 14.11. With $B = \log b$, the rank bounds there are $C_k m^5 f B^2 \leq k$ and $C_f B \leq f \log k$. The exponent C_w occurs in $k^{C_w} \leq w''$. The spacing bound uses C_Δ and the term $(NfB)^{C_{\mathrm{count}}}$. The height bound uses $C_r r^{19} f^5 B \leq H_0$, and C_{gap} occurs in the consecutive-gap bound. Here $(C_0, C', C_\ell, p) = (3, C', C_\ell, C_{YL} + 1, \frac{1}{4}, 3, \frac{1}{2}, C_{Tu})$ matches the eight-parameter signature $(C_0, C', C_\ell, C_g, c_b, C_w, c_T, C_T)$ of Definition 33.7 (with $C_0 = 3$ first and weight upper-bound parameter $C_w = 3$ in the third slot of p). Fix any real-valued function a_F of the outer list. For each eligible F_m, CH above the closed thresholds there is a common cutoff x_0. Each assertion holds under its point hypotheses for all $x \geq x_0$ with $x > x_1(F_m)$ at or above the selection cutoff, and every $i \in I_x$.

For (33.21) assume Cnt.Hyp$(D, L, u, u'; C_0, C', C_\ell, p, a_F(\mathbf{o}(F_m)))$ *of 33.7, absolute summability of $\|\mathrm{op}(t)\|$ and $\|v_t\|$, and the following compatibility of the Hida model X with its full data Y. It identifies $d^C_w = d^S_w$, $D^\star_C = D^\star$. It includes level-subgroup normalization and neatness for every word, the row-function conditions and integrable coefficient projector of 33.25, and the standard adelic additive character with compatible measures, local models and local arithmetic data. Require also $T_w\big(\pi/(v_E g_0 g_1)\big)^{2r_S} \geq 1$ for every word w, with $v_E = \mathrm{rd}(E)/2$ and $r_S = m/2 - 1$. Then*

$$\mathrm{PairAbs}_{u,u'}(S_{\mathrm{big}}) \leq \epsilon_L(D)\Delta_{u,u'}, \qquad \mathrm{PairAbs}_{u,u'}(S_{\mathrm{nsc}}) \leq \epsilon_L(D)\Delta_{u,u'}. \tag{33.21}$$

For a set of terms E_1 write $\mathrm{part}(E_1) = \sum_{t \in E_1} \mathrm{op}(t)$. Let K_0 be the sum of these operators for the identity cell and the generic and nongeneric cells with $G \neq 0$. Under the source condition Cnt.Hyp0$(D, L; C_0, C', C_\ell, p)$

of 33.7 (at $(3, C', C_\ell, C_{YL}+1, \frac{1}{4}, 3, \frac{1}{2}, C_{Tu})$), distinct words satisfy

$$\|K_0(u,u')\|_{\mathrm{HS}}^2 \le (1+b^{-6})\frac{d_u^S d_{u'}^S}{D^\star}. \tag{33.22}$$

Under Cnt.Hyp0$(D, L; C_0, C', C_\ell, p)$, *every level prime satisfies*

$$1 \le T_w \left(\frac{\pi}{v_E g_0 g_1}\right)^{2r_S}, \qquad r_S = m/2 - 1. \tag{33.23}$$

Outline. Use the volume ratio of 37.6 and the critical-level bounds to force the level inequality. Apply 33.106 for the loop estimates and use the ratios from 21.36 with 33.113 and 33.99 for distinct words. The complete proof is on p. 987. □

Lemma 33.79 (The terminal-flag Gram inequality)**.** *Use the selected local models and row representatives under the source and size hypotheses of (33.22). The levels T, T' are congruent modulo $(g_0 g_1)^2$. The scalar-partner coefficients $c_{\rho p}$ and the terminal kernel K_{top} use the same local characters and measures. Let $\alpha > 0$ be the product of local normalizers and $s_d = \sup \mathrm{Dens}/\alpha$ over represented matrices. For finite sets Y of terminal flags and S of scalar-partner rows and for $z \in \mathbf{C}^Y$,*

$$\sum_{\rho \in S} \left| \sum_{p \in Y} \overline{z_p} c_{\rho p} \right|^2 \le s_d \sum_{p,q \in Y} \overline{z_p} K_{\mathrm{top}}(p,q) z_q.$$

Outline. Expand the kernel into positive rank-one summands and use rational approximation and gluing to identify partner fibres. Shifted determinants fix the frame normalizers on each support, so the row multipliers bounded by s_d give the inequality termwise. The complete proof is on p. 988. □

Lemma 33.80 (The interpolation grid at a real place)**.** *Put $n = k + 2r + 1$, $K = 3$, $N = 3(r+1)$, with $k \ge 2$. Let $T_p, T_q > 0$ be the normalized final norms. Use two decreasing weights u, u' with coordinates in $u_1 + \mathbf{Z}$ and a decreasing spectral parameter λ, with all coordinates bounded by $b \ge \max(3, k)$ in absolute value. Require the common interlacing inequalities*

$$\max(u_{i+1}, u'_{i+1}) + \tfrac{1}{2} \le \min(u_i, u'_i) - \tfrac{1}{2} \qquad (1 \le i < k).$$

For a cut at coordinate $1 \le l \le k$ use

$$Y = \{2 - \max(u_l, u'_l) + 5j : 0 \le j < N(2r+1)\}.$$

Assume $|u_l - u'_l| \ge 30(r+1)^2 + 6$ and $c > 0$, $c + 2 \le b$. The adjacent gaps of the first $k-1$ coordinates of λ are at least $1 + c$. For each of the two weights $a = u, u'$, assume half-unit interlacing with these coordinates, or the following one-index variant. At $x = -y$, for each $1 \le i < k$ outside one fixed exceptional index, require $a_i \ge \lambda_i + 1/2$ if $\lambda_i > x$ and $a_{i+1} \le \lambda_i - 1/2$ otherwise. These conditions hold at each $y \in Y$. Put

$$P_*(y) = \prod_{a=1}^{n} (|y + \lambda_a| + c), \qquad \Lambda^2 = \max\{1, (4/3)^N b^{8r+20}/(T_p T_q)\},$$

$$q(y) = -T_p T_q \prod_{i=1}^{k} (y + 1/2 + u_i)(y + 1/2 + u'_i).$$

A positive letter based at $y \in Y$ has evaluation point y, and a negative letter based there has evaluation point $y + 1$. Take lists w of such letters based in Y, with distinct evaluation points separated by more than two. Define $q_w(y)$ by multiplying $q(y)$ by $(y - z + 1)/(y - z)$ for each positive letter with point z and by $(z - y)/(z - y - 1)$ for each negative one.

If $|w| + 3 \le N$, there are three disjoint rows y_{aj} of $d+1$ nodes in Y avoiding the bases of w. Here $d = 2i$ for a wave at level $0 \le i \le r$, or $d = 2(i - s) + s$ for a lift of length $1 \le s \le i \le r$. For $L_{aj} = \prod_{h \ne j} (y_{aj} - y_{ah})^{-1}$ they satisfy

$$\sum_{j_1, j_2, j_3} \prod_{a=1}^{3} |L_{aj_a}| \left(\frac{P_*(y_{aj_a}) P_*(y_{aj_a} + 1)}{q_w(y_{aj_a})} \right)^{1/2} \le \Lambda^3. \tag{33.24}$$

Outline. Use displacement and interlacing to bound the denominators and show $q(y) > 0$. Pair spectral factors with neighboring weight factors and bound the ratios by the gaps, then use three disjoint rows of unused nodes and the Lagrange coefficient sums. The complete proof is on p. 989. □

Lemma 33.81 (Normalization in a determinant expansion)**.** *Let $s = r+1,\ m = 2s,\ n \ge s,\ b \ge \max(2,n)$, and $C_h \ge 0$. Write $\mathcal{J}_\mu$ for finitely many vector-valued functions on a measured coordinate chart and I_μ for their finite squared L^2 norms, with the common point weight included in the measure. Let D be a scalar function on the chart. Start at $\mu = 0$. At each index reached in fewer than a steps suppose*

$$D\mathcal{J}_\mu = \sum_{\substack{I\subset\{1,\dots,n\}\\ |I|=s}} \tau_{\mu I}\mathcal{J}_{\mu+\mathbf{1}_I}, \qquad |\tau_{\mu I}| \le H^s, \qquad H \le b^{(C_h+10)m}.$$

Assume the identities hold at the represented points under consideration, that zero-energy functions vanish there, and that $I_{\mu+\mathbf{1}_I} \le (\exp(1)n)^s I_\mu$ for $I_\mu > 0$. There is an expansion of $D^a\mathcal{J}_0$ at those points in functions of energy at most I_0, with sum of absolute coefficients at most $b^{(C_h/2+6)m^2a}$.

Outline. Rescale each positive-energy function to the initial energy and sum the absolute coefficients of its children using the energy ratio. Iterate the one-step bound over paths and combine terms at the same index. The complete proof is on p. 990. □

Definition 33.82 (Partial orbital pairings and Capelli-weighted energies)**.** Fix integers $k \ge 2$, $r \ge 0$, $m = 2r+2$, and $n = 2r+k+1$. A local Bessel datum of rank (k,r) consists of the Cayley Hermitian matrix $H_{k,r}$ on $\mathbf{C}^n$ of signature $(k+r, r+1)$, its unitary group $G = \mathrm{U}(H_{k,r})$, a discrete series representation $(\sigma, \mathcal{H}_\sigma)$ of G with decreasing Harish–Chandra parameter $\lambda = (\lambda_1,\dots,\lambda_n)$, two Bessel frame vectors $z, z' \in \mathbf{C}^n$ in the central coordinate subspace on positions $r+1,\dots,r+k+1$ with positive final norms $T_p = -\operatorname{Re} h_{H_{k,r}}(z,z) > 0$ and $T_q = -\operatorname{Re} h_{H_{k,r}}(z',z') > 0$, compact definite stabilizers $U_z, U_{z'} \simeq \mathrm{U}(k)$ equipped with probability Haar measures da, da' and irreducible finite-dimensional unitary representations $(\pi, W_p), (\pi', W_q)$ of decreasing compact Harish–Chandra parameters $u, u' \in \mathbf{R}^k$, unipotent radicals $N_z, N_{z'}$ carrying unipotent Haar measures dn, dn' of finite positive scale κ_N, and Bessel characters $\psi_z, \psi_{z'}$ of nonzero parameter $c_\psi \in \mathbf{C}^\times$. Standardized coordinates mean $c_\psi = -2i$ and standard unipotent Haar measure $(\kappa_N = 1)$.

For $0 \le j \le r+1$, a level-j block point $\gamma = ((t_1,s_1),\dots,(t_d,s_d))$ is a list of pairs $(t_i,s_i) \in \mathbf{C}^\times \times \mathbf{N}_{\ge 1}$ with $\sum_{i=1}^d s_i = j$. Its block matrix $M(\gamma)$ is the $j \times j$ block-diagonal matrix with i-th block $t_i J_{s_i}$, where J_s has ones on its antidiagonal, and its order-l leading minor is $D_l(\gamma) = \det M(\gamma)_{1\le a,b\le l}$ $(1 \le l \le j,\ D_0(\gamma) = 1)$. At top level $j = r+1$, define

$$G_{\mathrm{loc}}(\gamma) = |D_{r+1}(\gamma)|^2 - T_pT_q|D_r(\gamma)|^2.$$

The level-j point weight $p_{k,r}(T_p,T_q;\gamma)$ equals 1 at $j = 0$, and for $j \ge 1$ equals the product over the blocks (t_i,s_i), starting after $a_i = \sum_{h<i} s_h$ positions, of $|t_i|^{2s_i(2(r+1-a_i-s_i)+k+s_i-2)}$, with the last factor replaced by $(|t_d|^2 - T_pT_q)^{k-1}$ when $j = r+1$ and $s_d = 1$. On any top-level block point represented by an element $g \in G$ whose frame-pairing matrix is $M(\gamma)$, one has $G_{\mathrm{loc}}(\gamma) \ge 0$ and $p_{k,r}(T_p,T_q;\gamma) \ge 0$.

For a smooth operator-valued coefficient $H : G \to \mathrm{Hom}_{\mathbf{C}}(W_q, W_p)$ and a top-level block point γ, whenever $g \in G$ represents $M(\gamma)$, ψ_z and g-conjugated $\psi_{z'}$ agree on the common unipotent stabilizer $R_g = N_z \cap gN_{z'}g^{-1}$, μ_R is the root-coordinate measure on R_g, and ν disintegrates $(da\,dn) \times (da'\,dn')$ along μ_R, define the orbital integral $J(H;g,\nu) \in \mathrm{Hom}_{\mathbf{C}}(W_q,W_p)$ by integrating $\psi_z(n)\overline{\psi_{z'}(n')}\,\pi(a)H(a^{-1}n^{-1}gn'a')\pi'(a')^{-1}$ against $d\nu$. Let $\|J(H;\gamma)\|_{\mathrm{HS}}^2 \in [0,\infty]$ be the supremum of $\|J(H;g,\nu)\|_{\mathrm{HS}}^2$ over all admissible (g,μ_R,ν), with value 0 when none exists.

In standardized coordinates, for $0 \le j \le r+1$ and level-j block points γ, γ', the partial orbital pairing of two smooth rapidly decaying coefficients H_1, H_2 integrates the first j flag positions against $\psi_z, \overline{\psi_{z'}}$ and π, π', multiplies the two arguments by $p_{k,r}(T_p,T_q;\gamma)^{1/2}$ and $p_{k,r}(T_p,T_q;\gamma')^{1/2}$, and takes the inner product of the resulting functions on the residual corank-$(r+1-j)$ Bessel slice. At $j = 0$ $(\gamma = \gamma' = [])$ this is the sesquilinear Bessel pairing $\beta(H_1,H_2)$ with empty-list weight 1; at $j = r+1$ and $\gamma = \gamma'$ with $G_{\mathrm{loc}}(\gamma) > 0$, its diagonal value $E(H;\gamma)$ equals $p_{k,r}(T_p,T_q;\gamma)\|J(H;\gamma)\|_{\mathrm{HS}}^2$.

For the word forms fix a cut $1 \le l \le k$ and put $\sigma_l = (-1)^{k-l}$. A letter is a triple $a = (\epsilon_a, s_a, y_a)$, consisting of a sign, a position in a finite set S, and an actual evaluation point $y_a \in \mathbf{C}$. A positive grid letter

with base y has evaluation point y; its negative companion has evaluation point $y+1$. The shift is included in the letter and is applied only once. Put $I_S = \{0,1\}^S$. For an unpositioned signed coefficient word w, assign the distinct positions $0,\dots,|w|-1$ in list order, and denote the resulting positioned word by $\mathrm{pos}(w)$. Write $I_w = \{0,1\}^{\{0,\dots,|w|-1\}}$.

We now give the radial operator datum used to define the forms. Let G_r be the subgroup of G fixing the first r isotropic coordinate vectors and their duals, let $\Omega = (\mathbf{C}^\times)^r$, let $\zeta(g)$ be the terminal radial coordinate supplied by the actual data, and put $\mathcal{G}(g) = |\zeta(g)|^2 - T_pT_q$. Let V be the complex space of functions

$$\phi : \mathbf{C}^r \times \mathrm{Mat}_n(\mathbf{C}) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(W_q, W_p)$$

that vanish outside $\mathcal{D} = \{(c,g) : c \in \Omega,\ g \in G_r,\ \mathcal{G}(g) > 0\}$, are locally restrictions of smooth ambient functions on $\mathcal{D}$, and satisfy

$$\phi(c, hgh') = \pi(h)\phi(c,g)\pi'(h') \quad ((c,g) \in \mathcal{D},\ h \in U_z,\ h' \in U_{z'}).$$

Fix Haar measure μ on G_r. Operators are recorded in the paired algebra $\mathscr{A} = \mathrm{End}_{\mathbf{C}}(V) \times \mathrm{End}_{\mathbf{C}}(V)^{\mathrm{op}}$: the first component acts on V, the second is its designated adjoint component, and a complex scalar λ acts by $\lambda(A,B) = (\lambda A, \bar\lambda B)$, so that the involution is conjugate-linear. The involution interchanges these components, and the adjoint of an operator matrix also transposes its array indices. An integral adjoint identity is required on the relevant test domain whenever it is used; the operator record alone is not such an identity.

The compatible actual data consist of the real-place parameter and compact-type conditions, the actual quantum-minor operators, a level datum L in $\mathscr{A}$, and start data $(Q_{\mathrm{st}}, q, x, \xi, e, \mathcal{I})$. The real-place parameter and compact-type conditions require $k \ge 1$, $T_p, T_q > 0$, $u_i, u_i' \in u_1 + \mathbf{Z}$ for $1 \le i \le k$, and $\max(u_{i+1}, u_{i+1}') + \frac{1}{2} \le \min(u_i, u_i') - \frac{1}{2}$ for $1 \le i < k$, with π, π' realized in highest-weight models of $U_z, U_{z'}$ of compact Harish–Chandra parameters u, u'. The level datum contains the coordinate and inverse multiplication operators, their coordinate differential operators, and the terminal differentiation operators, with their specified commutation and adjoint relations. Write $X_{j,a}$, $0 \le j < r$, $0 \le a < 6$, for the actual coordinate operators and $\nabla, \nabla^\star$ for the terminal differentiation operators. Its coordinate pairs are

$$p_j = (X_{j,0}, -X_{j,3}), \quad \bar p_j = (X_{j,3}, -X_{j,0}), \quad c_j = (X_{j,1}, X_{j,4}), \quad c_j^{-1} = (X_{j,2}, X_{j,5}),$$

with the conjugate coordinate pairs reversed. Its terminal multiplication pairs are multiplication by $\zeta, \zeta^{-1}, \overline{\zeta}, \overline{\zeta}^{-1}, \mathcal{G}, \mathcal{G}^{-1}$, paired with the conjugate functions, and its scalar parameter is T_pT_q. The actual-operator compatibility requires the first components to act on the radial domain as the corresponding quantum-minor, coordinate differential, and multiplication operators. Write $Q_{\mathrm{act}}(Y)_{ab}$ $(a,b \in \{0,1\})$ for the pair in $\mathscr{A}$ of the actual terminal quantum-minor operators at (Y,a,b) and $(\overline{Y},b,a)$, in that order. The full start and level conditions are retained. Specifically, the starting-matrix relation for (Q_{st}, q) requires, with $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and the transposition operator P on $\mathbf{C}^2 \otimes \mathbf{C}^2$,

$$\begin{gathered} Q_{\mathrm{st}}(Y)^\dagger = Q_{\mathrm{st}}(Y) \quad (Y \in \mathbf{R}), \qquad q(Y)^* = q(Y) \quad (Y \in \mathbf{R}), \\ \big((Y'-Y)I - P\big)Q_{\mathrm{st},1}(Y')Q_{\mathrm{st},2}(Y) = Q_{\mathrm{st},2}(Y)Q_{\mathrm{st},1}(Y')\big((Y'-Y)I - P\big), \\ Q_{\mathrm{st}}(Y+1)^{\mathsf{T}} J Q_{\mathrm{st}}(Y) = Q_{\mathrm{st}}(Y) J Q_{\mathrm{st}}(Y+1)^{\mathsf{T}} = q(Y) J, \end{gathered}$$

together with letter regularity $Y - y_a \ne 1$ when $\epsilon_a = +$ and $y_a - Y \ne 1$ when $\epsilon_a = -$, and

$$Q_{\mathrm{st}}(Y) = \sigma_l Q_{\mathrm{act}}(Y), \qquad \zeta^* = \overline{\zeta}, \quad \nabla^* = \nabla^\star, \quad (T_pT_q)^* = T_pT_q.$$

Let $\mathcal{Z}_L$ denote the specified family of level operators in the level datum, and write $[A,B] = AB - BA$. Its start conditions include

$$\begin{gathered} [Q_{\mathrm{st}}(Y)_{10}, Q_{\mathrm{st}}(Z)_{10}] = 0, \qquad x(Y)\xi(Y) = \xi(Y)x(Y) = 1 \quad (Y \in \mathcal{I}), \\ [x(Y), x(Z)] = 0, \quad [x(Y), f] = [e(Y), f] = 0 \quad (f \in \mathcal{Z}_L), \quad [x(Y), \nabla] = 0, \\ x(Y)^* = x(Y) \quad (Y \in \mathbb{R}), \qquad Q_{\mathrm{st}}(Y)_{11} = \mathcal{G}\, x(Y), \\ Q_{\mathrm{st}}(Y)_{10} = \mathcal{G}\, x(Y)\big(\ell_0(Y) + \overline{\zeta}\,\zeta^{-1}\mathcal{G}^{-1}e(Y)\big) \quad (Y \in \mathcal{I}), \\ \ell_0(Y) = -\nabla - (Y + \tfrac{1}{2})\overline{\zeta}\,\mathcal{G}^{-1}. \end{gathered}$$

These are conditions on the actual operator realization; the starting-matrix relation and the specified family of level operators are carried with that datum.

The scalar data entering admissibility are explicit. For $1 \le i < k$, put

$$a_i = \max(u_{i+1}, u'_{i+1}) + \tfrac{1}{2}, \quad b_i = \min(u_i, u'_i) - \tfrac{1}{2}, \quad d_i = \max(0, \lfloor b_i - a_i \rfloor),$$

and let $\mathcal{T} = \prod_{i=1}^{k-1} \{0, \dots, d_i\}$. For $\rho \in \mathcal{T}$, set

$$x_\rho(Y) = \prod_{i=1}^{k-1} (Y + a_i + \rho_i), \quad \mathcal{I} = \{Y : x_\rho(Y) \neq 0 \text{ for every } \rho \in \mathcal{T}\}, \quad q(Y) = -T_p T_q \prod_{i=1}^{k} (Y + \tfrac{1}{2} + u_i)(Y + \tfrac{1}{2} + u'_i).$$

A word is admitted from the start when its positions are distinct, all its evaluation points are real, each letter point Y satisfies

$$Y - 1, Y, Y + 1 \in \mathcal{I}, \qquad q(Y)q(Y-1) \neq 0,$$

and the difference of the evaluation points at any two different positions is not an integer of absolute value at most two. It is admissible for the cut when, in addition, every letter point Y satisfies

$$u_l + 2 \le -Y \le u'_l - 2 \quad \text{or} \quad u'_l + 2 \le -Y \le u_l - 2,$$

and the evaluation points at any two different positions have distance greater than two. These conditions apply to the actual evaluation points of both signs.

The level matrices and word operators are defined as follows. For $0 \le j < r$, put

$$T_j(Y) = \begin{pmatrix} Y + p_j & -c_j \\ c_j^{-1} & 0 \end{pmatrix}, \qquad t_r(Y) = 1, \quad t_i(Y) = t_{i+1}(Y) T_i(Y), \qquad Q_i(Y) = t_i(\overline{Y})^* Q_{\mathrm{st}}(Y) t_i(Y).$$

Let e_a^{uv} be the matrix unit on the a-th binary position, acting as the identity on the other positions. If M is a two-by-two matrix whose entries are operator matrices on I_S, write $\mathrm{ins}_a M = \sum_{u,v=0}^{1} e_a^{uv} M_{uv}$, and write $\mathrm{lift}\, Q$ for the entrywise diagonal embedding of Q. Put $P_a = (e_a^{vu})_{u,v=0}^{1}$ and $Q_a = (e_a^{uv})_{u,v=0}^{1}$. The exchange matrices are

$$\mathcal{R}_a(d) = (d-1)^{-1}(dI - P_a), \qquad \mathcal{R}_a^\vee(d) = (d-1)^{-1}(dI - Q_a).$$

Scalar inverses use the convention $0^{-1} = 0$. The dressed letter matrices are defined by

$$\mathsf{T}_i[\varnothing; Y] = \mathrm{lift}\, Q_i(Y),$$
$$\mathsf{T}_i[a :: w; Y] = \begin{cases} \mathsf{T}_i[w; Y] \mathcal{R}_{s_a}(Y - y_a), & \epsilon_a = +, \\ \mathcal{R}_{s_a}^\vee(y_a - Y) \mathsf{T}_i[w; Y], & \epsilon_a = -. \end{cases}$$

Thus a letter inserted at a new position a acts on V^{I_S} by

$$L^+_{i,a,Y;w} = \mathrm{act}\big(\mathrm{ins}_a\, \mathsf{T}_i[w; Y]\big), \qquad L^-_{i,a,Y;w} = \mathrm{act}\big(\mathrm{ins}_a\, \mathsf{T}_i[w; Y]^{\mathsf{T}}\big).$$

Both domain and target are V^{I_S}. Their prescribed adjoints are obtained by taking the adjoint of the displayed operator matrices. Define

$$R_i(\varnothing) = 1, \qquad R_i(a :: w) = R_i(w) \,\mathrm{ins}_{s_a} \begin{cases} \mathsf{T}_i[w; y_a], & \epsilon_a = +, \\ \mathsf{T}_i[w; y_a]^{\mathsf{T}}, & \epsilon_a = -. \end{cases}$$

For an operator matrix M, its action is $(\mathrm{act}\, M\, x)_a = \sum_b (M_{ab})_1 x_b$, where the subscript denotes the first component of the paired operator. The one-step slice operator $B_i(w)$ is the action of the ordered product, in word order, of the slot insertions of $T_i(y_a)$ for a positive letter and its entrywise adjoint for a negative letter.

All pairings below are linear in their first argument. For $c \in \Omega$, replace its coordinates $i, \dots, r-1$ by $t_i, \dots, t_{r-1}$ to obtain $c \#_i t$; for a single $t \in \mathbf{C}$, $c \#_i t$ replaces only the coordinate i by t. With planar Lebesgue measure in these coordinates, define

$$b_c(\phi, \psi) = \int_{G_r} \langle \phi(c,g), \psi(c,g) \rangle_{\mathrm{HS}} \, d\mu(g),$$

$$\Phi_i(x, y; c) = \int_{\mathbf{C}^{r-i}} \left(\prod_{j=i}^{r-1} |t_j|^{-2} \right) \sum_{a \in I_S} b_{c \#_i t}(x_a, y_a) \, dt,$$

$$\Xi_{i,w}(x, y; c) = \Phi_i(\mathrm{act}\, R_i(w) x, y; c).$$

In these integrals $|0|^{-2} = 0$, the value of the total scalar inverse used in the definitions; the relevant integrability assertions are those of Lemma 33.87. The products and coordinate integrals at $i = r$ are empty. The positive subspaces $\mathcal{K}_i \subset V$ are supplied by Lemma 33.87; its sesquilinearity and nonnegativity assertions apply on $\mathcal{K}_i^{I_S}$, meaning that every component belongs to $\mathcal{K}_i$. A transfer to a partial orbital kernel requires an admissible positioned word, a coordinate in Ω, an array in this product subspace for each point, and equality of the kernel with the displayed form, including its measure and point-weight factors. These identification and membership requirements are retained whenever the form is used on coefficient slices.

The coefficient insertions used before slicing have a separate domain and target. For a matrix size d, write $\mathcal{F}_d$ for the space of functions from $\mathrm{Mat}_d(\mathbf{C})$ to $\mathrm{Hom}_{\mathbf{C}}(W_q, W_p)$, on which differential operators act on the open sets where the functions are smooth, and let $\mathcal{A}_t(u), \mathcal{A}_t^{\vee}(u) : \mathcal{F}_n \to \mathcal{F}_n$ be the two quantum-minor differential operators in the actual coefficient data. Their action and formal adjoints are carried by those data and their radial intertwining identities. For $0 \le j \le r$, the positive and negative insertions $\mathcal{V}_{j,Y}^{\pm} : \mathcal{F}_n \to \mathcal{F}_n \oplus \mathcal{F}_n$ have components

$$(\mathcal{V}_{j,Y}^{+} F)_e(g) = D_j(g)^{-1} \mathcal{A}_{j-e}\big(Y + (n-1)/2\big) F(g), \qquad (\mathcal{V}_{j,Y}^{-} F)_e(g) = \overline{D_j(g)}^{-1} \mathcal{A}_{j-e}^{\vee}\big(Y - (n-1)/2\big) F(g)$$

Here $D_j(g)$ is the order-j frame minor, with $D_0 = 1$, and scalar inverses are set to zero at zero. Write $G_{\mathrm{loc}}(g) = |D_{r+1}(g)|^2 - T_p T_q |D_r(g)|^2$ for the frame Gram polynomial on the group. The formulas hold for $e \in \{0, 1\}$, except that the component $e = 1, j = 0$ is defined to be zero. In particular both insertions at $j = 0$ are $F \mapsto (F, 0)$. Their iterated components are obtained by applying the first letter to the array for the remaining word and then applying the partial orbital slice. The two signs use their respective prescribed differential operators; no assertion equating one to the formal adjoint of the other is imposed. The matrix adjoints and the integral adjoint identities needed for the word forms are the ones in the actual radial realization above, with its specified measure and test domain.

For $0 \le j \le r$, level-j coefficient arrays have components in $\mathcal{F}_{n-2j}$, the operator-valued functions on the residual matrix space. Let $R_j^{\mathrm{coef}}(w)$ be the word construction on these arrays for the signed two-by-two quantum-minor operator matrix of the fixed normalized level-j datum, and let β_j be its Bessel pairing. For an array of coefficient operators define $(\mathrm{act}_{\mathrm{coef}}\, M\, x)_a = \sum_b M_{ab}(x_b)$. The corresponding level form is

$$\Xi_{j,w}^{\mathrm{coef}}(x, y) = \sum_a \beta_j\big((\mathrm{act}_{\mathrm{coef}}\, R_j^{\mathrm{coef}}(w) x)_a, y_a\big),$$

The input arrays are obtained by applying the iterated coefficient insertions $\mathcal{V}_{j,Y}^{\pm}$ on the original matrix space and then taking the partial orbital slice. The operator $R_j^{\mathrm{coef}}(w)$ in the displayed form acts on these residual arrays after slicing. The residual datum has rank $(k, r-j)$. Its radial realization uses a new radial space and its level-zero subspace $\mathcal{K}_0^{(j)}$; the coefficient index j and the radial level index i have these distinct meanings. Lemma 33.88 gives the comparison with the radial word form for $\mathrm{pos}(w)$. At the terminal level $j = r+1$, the actual tower carries instead a terminal orbital vector $v_H(\gamma)$, terminal lifts $\Theta_{w,a}$ ($a \in I_w = \{0,1\}^{|w|}$) and orthogonal compact-type projections P_ρ ($\rho \in \mathcal{T}$) on W_p, and for a nonempty word w it sets

$$\tau_w(H; \gamma, \gamma') = \sum_{a \in I_w} \sum_{\rho \in \mathcal{T}} d_{w,a,\rho}(\gamma)\, d_{w,a,\rho}(\gamma') \, \langle P_\rho v_{\Theta_{w,a} H}(\gamma), P_\rho v_{\Theta_{w,a} H}(\gamma') \rangle_{\mathrm{HS}},$$

where $d_{w,a,\rho}(\gamma)$ is the square root of $\max\big(0, (|D_{r+1}(\gamma)|^2 / G_{\mathrm{loc}}(\gamma))^{|w|} \operatorname{Re} f_{w,a}(\rho)\big)$ and $f_{w,a}(\rho)$ is the a-th diagonal entry of c_w in the scalar word recursion of Lemma 33.87 with x replaced by $x_{\rho,l}$. The terminal

vector uses the prescribed representing group element, root measure and frame compatibility in the terminal datum; it is zero unless the block point is valid, has positive Gram value, and has such a representative. Its terminal unweighted pairing is

$$\mathscr{R}_{r+1,w}(H;\gamma,\gamma') = \begin{cases} \langle v_H(\gamma), v_H(\gamma')\rangle_{\mathrm{HS}}, & w = [], \\ \tau_w(H;\gamma,\gamma'), & w \neq []. \end{cases}$$

It is zero when either argument is not a level-$(r+1)$ block point. For $j \le r$, write $x_{j,w,\gamma}[H]$ for the inserted coefficient array after the partial orbital slice, and put

$$\mathscr{R}_{j,w}(H;\gamma,\gamma') = \Xi^{\mathrm{coef}}_{j,w}\big(x_{j,w,\gamma}[H], x_{j,w,\gamma'}[H]\big)$$

on valid level-j points, extending it by zero otherwise. At every level define the actual tower energy

$$\widetilde{E}_j(w;H;\gamma) = p_{k,r}(T_p,T_q;\gamma)\,\mathrm{Re}\,\mathscr{R}_{j,w}(H;\gamma,\gamma).$$

In particular, $\widetilde{E}_{r+1}([];H;\gamma) = p_{k,r}(T_p,T_q;\gamma)\|v_H(\gamma)\|^2_{\mathrm{HS}}$. We retain $E(H;\gamma)$ for the physical orbital energy defined above, with its root-coordinate measure. In the level-zero endpoint formula, E_0 abbreviates $\widetilde{E}_0$. The cut coefficient is $H_\varepsilon = \chi(G_{\mathrm{loc}}/\varepsilon)H$. For the endpoint let $\mathcal{C}(Y)$ denote the $(0,0)$-entry of the unsigned two-by-two coefficient quantum-minor operator matrix at level zero. This is the full quantum-determinant differential operator on coefficients; its eigenvalue action is a separate hypothesis. Write $\mathrm{Reg}(a,Y)$ for the letter-regularity predicate carried by the exchange datum. Suppose H is smooth on the group (here β and the Bessel averages in it are Bochner integrals, with value zero for a nonintegrable integrand), the positions of w are distinct, its evaluation points are real, and $\mathrm{Reg}(b,y_a)$ holds whenever a precedes b in w. Define

$$\mathcal{D}_{\varnothing}H = H, \qquad \mathcal{D}_{a::w}H = \sigma_l\mathcal{C}(y_a)\mathcal{D}_wH.$$

Then

$$E_0(w;H;[]) = \mathrm{Re}\,\beta(\mathcal{D}_wH, H).$$

This is the level-zero endpoint of the word construction. Its specialization to a finite sum of coefficient eigenfunctions retains their individual eigenvalue equations and pairwise orthogonality for β.

For a displacement bound $c > 0$, retain the positive normalization

$$P_*(z) = \prod_{a=1}^{n}(|z+\lambda_a| + c), \qquad n_c(w) = \prod_{a\in w} P_*(y_a).$$

For K new grid nodes avoiding the old positions and evaluation points, prepend positive letters at y_{aj_a}, or negative letters at $y_{aj_a}+1$, in the same order. If Ξ_w is the unextended form and x_η, y_η are the old-index arrays obtained by fixing the new binary indices to $\eta \in \{0,1\}^K$, their alternating contraction is

$$\mathcal{J}_{\Xi_w}(x,y) = \sum_{\eta\in\{0,1\}^K} (-1)^{K-\sum_a \eta_a}\Xi_w(x_\eta, y_{\mathbf{1}-\eta}).$$

One inserted pair contributes $\Xi_w(x_1,y_0) - \Xi_w(x_0,y_1)$.

Lemma 33.83 (Polynomial contractions between compact types)**.** *Let $k, r \in \mathbf{N}$, $k \ge 2$, and let $W_b \subset \mathbf{C}^{2r+k+1}$ be the central coordinate subspace on positions $r+1,\dots,r+k+1$ for the Cayley Hermitian form h. Let $z \in W_b$ satisfy $-\mathrm{Re}\,h(z,z) > 0$, and put $E_z = W_b \cap z^{\perp}$. Let (π,V) be a nonzero finite-dimensional continuous irreducible unitary representation of U_z, with compact parameter $u = (u_1,\dots,u_k)$ in the Capelli normalization, whose shifted coordinates are $u_j - k/2 + 1/2$. Suppose that every $u_j - u_{j+1}$ is a positive integer. Fix $1 \le l < k$, $d \in \mathbf{N}$, and $b \ge 1$ such that*

$$|u_j| \le b \quad (1 \le j \le k), \qquad d \le b, \qquad 2d+1 \le u_l - u_{l+1}.$$

There are compact types (π_+,V_+), (π_-,V_-) with parameters $u + d\mathbf{1}_{l+1}$, $u - d\mathbf{1}_l$, and maps

$$A_\pm : \mathbf{C}^{2r+k+1} \longrightarrow \mathrm{Hom}_{\mathbf{C}}(V_\pm, V)$$

with the following properties. Each map is a finite sum of fixed operators multiplied by polynomials of total degree at most d in ω and $\overline{\omega}$. For $a \in U_z$, $t \in \mathbf{R}$, and $\omega \in \mathbf{C}^{2r+k+1}$,

$$A_{\pm}(a\omega)\pi_{\pm}(a) = \pi(a)A_{\pm}(\omega), \qquad A_{\pm}(t\omega) = t^{d}A_{\pm}(\omega).$$

For $\omega \in E_z$,

$$\|A_{\pm}(\omega)\|^{2} \le (\operatorname{Re} h(\omega,\omega))^{d},$$

and for every $v \in V$,

$$(\operatorname{Re} h(\omega,\omega))^{d}\|v\|^{2} \le (3b)^{2d}\big(\|A_{+}(\omega)^{*}v\|^{2} + \|A_{-}(\omega)^{*}v\|^{2}\big).$$

Outline. Decompose V into interlacing Capelli eigenspaces V_M for the stabilizer of a unit vector $e \in E_z$, and compose one-step contractions into degree-d chains $A_{\pm}$. Using the gap $2d+1 \le u_l - u_{l+1}$, bound one chain eigenvalue below by $(3b)^{-2d}$ on each V_M. The spectral theorem yields the lower bound at e, and equivariance with homogeneity extends it to E_z. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 990. □

Lemma 33.84 (Boundary kernels and lattice summation)**.** *Use planar Lebesgue measure and inner products linear in the second argument.*

(a) Let $s \in \mathbf{N}$, $s \ge 2$, $\omega \in \mathbf{C}$ with $|\omega| = 1$, and let $\chi : \mathbf{C} \to \{z \in \mathbf{C} : |z| = 1\}$ be a nontrivial continuous additive character. Let Y be a finite set. For $p \in Y$, let $B_p : \mathbf{C} \to \mathcal{H}$ be strongly measurable, where $\mathcal{H}$ is a complex Hilbert space, and put

$$b_p(z) = s|z|^{s-1}B_p(z), \qquad A_{p,x}(D) = \sum_{z^s = \omega D} \chi(szx)B_p(z).$$

Suppose b_p is square integrable on every compact subset of $\mathbf{C}^{\times}$. Suppose also that $U_{p,x} \in L^2(\mathbf{C},\mathcal{H})$ for $x \ne 0$, that

$$\sup_{x \ne 0} \|U_{p,x}\|_2^2 \le M_p < \infty,$$

and that, for every compact $K \subset \mathbf{C}^{\times}$, there is $C_{p,K}$ such that

$$\|U_{p,x}(D) - A_{p,x}(D)\| \le C_{p,K}|x|^{-1} \quad \text{for almost every } D \in K, \quad x \ne 0.$$

Define

$$(T_x V)(c) = |x|^{-(s-1)}V(c/x^{s-1}), \quad u_{p,x} = T_x U_{p,x}, \quad a_{p,x} = T_x A_{p,x},$$

and let U, Q, P_b be the kernels obtained by pairing, respectively, (u,u), (u,a), and (a,a). Then $b_p \in L^2$, $\|b_p\|_2^2 \le M_p$, the block kernel

$$\begin{pmatrix} U & Q \\ Q^{*} & P_b \end{pmatrix}$$

is positive semidefinite on every finite set of nonzero inverse coordinates, and $P_b(p,x;p,x) \le sM_p$. For every $p, q \in Y$ and $\sigma \in \mathbf{C}$,

$$\lim_{|x|\to\infty} P_b(p,x;q,x+\sigma) = \lim_{|x|\to\infty} Q(p,x;q,x+\sigma) = \lim_{|x|\to\infty} Q^{*}(p,x;q,x+\sigma) = \int_{\mathbf{C}} \chi(\sigma z)\langle b_p(z), b_q(z)\rangle\, dz. \tag{33.25}$$

(b) Let $f \in \mathbf{N}$, let $L \subset \mathbf{C}^f$ be a full lattice, and define its dual by

$$L^{\vee} = \left\{ \sigma : 2\operatorname{Re}\sum_{v} \sigma_v \ell_v \in \mathbf{Z} \text{ for every } \ell \in L \right\}.$$

For a nonnegative, lower semicontinuous, integrable function $D : \mathbf{C}^f \to \mathbf{R}$, put

$$\widehat{D}(\sigma) = \int_{\mathbf{C}^f} D(c) \exp\left(2\pi i\, 2\operatorname{Re}\sum_{v} \sigma_v c_v\right) dc.$$

Let $A : L^\vee \to \mathbf{C}$ satisfy

$$\operatorname{Re} \sum_{\sigma,\tau\in T} A(\sigma-\tau)c_\sigma\overline{c_\tau} \geq 0$$

for every finite $T \subset L^\vee$ and every family (c_σ). If $J > 0$, $F : L^\vee \to [0,\infty)$ is summable, and

$$|\widehat{D}(\sigma) + A(\sigma)| \leq JF(\sigma),$$

then every finite $S \subset c_0 + L$ satisfies

$$\operatorname{covol}(L)\sum_{c\in S} D(c) \leq J\sum_{\sigma\in L^\vee} F(\sigma). \tag{33.26}$$

(c) Let $g, R > 0$, $z_v \geq 0$, and $K \in \{3,5\}$. Put

$$m_K(z,t) = \begin{cases} 1, & t = 0,\\ \min\{1,(z/|t|)^K\}, & t \neq 0,\end{cases} \qquad M_z(t) = \prod_{v=1}^{f} m_K(z_v,t_v), \qquad L_R = (R(\mathbf{Z}+i\mathbf{Z}))^f.$$

There is an absolute $C \geq 1$, valid for both values of K, such that the following series converges and

$$R^{-2f}\sum_{\sigma\in L_R^\vee} M_z(g\sigma) \leq R^{-2f}\prod_{v=1}^{f} C\{1+(Rz_v/g)^2\}. \tag{33.27}$$

Consequently the limit superior of the left side as $R\to\infty$ is at most $C^f\prod_v (z_v/g)^2$.

Outline. Part (a) expands $A_{p,x}$ via $D = \omega^{-1}z^s$ and uses Fourier decay for the boundary limits. Part (b) bounds Fejér means of periodized D using positivity of A, and (c) sums $M_z(g\sigma)$ over dyadic annuli in $L_R^\vee$. The complete proof is on p. 992. □

Lemma 33.85 (Successive extraction of leading coefficients)**.** *Let V be a complex vector space, let $K, d \in \mathbf{N}$, and let $\sigma_0,\dots,\sigma_{K-1} \in \mathbf{C}$. For $0 \leq a \leq K$, let $W_a : \mathbf{R}^K \to V$. For $y \in \mathbf{R}^K$, write $y[a \leftarrow t]$ for the vector obtained by replacing coordinate a by t. Suppose that for each $a < K$ and each y there are $c_0,\dots,c_d \in V$ such that*

$$W_{a+1}(y[a\leftarrow t]) = \sum_{q=0}^{d} t^q c_q \quad (t\in\mathbf{R}), \qquad c_d = \sigma_a W_a(y).$$

For each $a < K$, choose distinct real numbers $y_{a,0},\dots,y_{a,d}$, and put

$$L_{a,j} = \prod_{h\neq j}(y_{a,j}-y_{a,h})^{-1}.$$

Then

$$\left(\prod_{a=0}^{K-1}\sigma_a\right) W_0(0) = \sum_{j_0,\dots,j_{K-1}=0}^{d}\left(\prod_{a=0}^{K-1} L_{a,j_a}\right) W_K(y_{0,j_0},\dots,y_{K-1,j_{K-1}}).$$

For $K = 0$, the sum is over the unique empty tuple and the products equal one.

Outline. Apply the Lagrange formula for the leading coefficient of a polynomial to one coordinate at a time. The complete proof is on p. 993. □

Lemma 33.86 (Two-component contraction at level zero)**.** *Let V be a complex vector space and let $\Xi : V\times V \to \mathbf{C}$ be linear in its first argument and conjugate-linear in its second. Fix $x, y \in V$, $\omega \in \mathbf{C}$, and a complex-linear map $W_\omega : V \to V$. Define*

$$I^+, I^- : V \longrightarrow V\oplus V, \qquad I^+(v) = I^-(v) = (v,0),$$

and define

$$\widetilde{W}_\omega : V\oplus V \longrightarrow V\oplus V, \qquad \widetilde{W}_\omega(v_0,v_1) = (W_\omega v_0, W_\omega v_1 + \omega W_\omega v_0).$$

For $A = (A_0, A_1)$ and $B = (B_0, B_1)$, put

$$\mathscr{J}_{\Xi}(A, B) = \Xi(A_1, B_0) - \Xi(A_0, B_1).$$

Then

$$\mathscr{J}_{\Xi}\big(\widetilde{W}_{\omega} I^{+}(x), I^{-}(y)\big) = \omega\, \Xi(W_{\omega} x, y).$$

More generally, let $K \geq 0$ be an integer. For $\epsilon, \eta \in \{0,1\}^K$, put

$$A_{\epsilon} = \omega^{|\epsilon|} W_{\omega} x, \qquad B_{\eta} = \begin{cases} y, & \eta = (0, \ldots, 0), \\ 0, & \textit{otherwise}, \end{cases} \qquad |\epsilon| = \sum_{a=1}^{K} \epsilon_a.$$

Write $\mathbf{1} - \epsilon = (1 - \epsilon_1, \ldots, 1 - \epsilon_K)$. The contraction obtained by applying the one-slot rule successively is

$$\mathscr{J}_{\Xi, K}(A, B) = \sum_{\epsilon \in \{0,1\}^K} (-1)^{K - |\epsilon|} \Xi(A_{\epsilon}, B_{\mathbf{1} - \epsilon}) = \omega^{K} \Xi(W_{\omega} x, y).$$

The map W_{ω} is applied once to the original vector. Each added positive slot applies the extension $v \mapsto (v, \omega v)$. For $K \geq 1$, the contraction is zero at $\omega = 0$, without requiring $\Xi(W_0 x, y) = 0$.

Outline. Compute the one-slot contraction directly. In the iterated signed sum only the term with $\epsilon = (1, \ldots, 1)$ survives. The complete proof is on p. 993. □

Lemma 33.87 (Theorem P, compact form)**.** *Use the compatible actual data and the forms of Definition 33.82, with finite-dimensional complex Hilbert spaces W_p, W_q, a finite position set S, a cut $1 \leq l \leq k$, and the same vectors and representations in the real-place datum D. Put $n = 2r + k + 1$ and $N = |S|$. For a real number A and an integer $M \geq 0$ satisfying*

$$A \geq 127(r+1)^3 (N n^2 + n^3), \qquad M \geq 5 + 3(r+1)(N+2)n + Nr,$$

there are complex subspaces $\mathcal{K}_i \subset V$, $0 \leq i \leq r$, with the following properties. For every admissible word w and $c \in \Omega$, the form $\Xi_{i,w}(\ ,\ ; c)$ is sesquilinear and positive semidefinite on $\mathcal{K}_i^{\{0,1\}^S}$. Each one-step slice matrix $B_i(w)$ maps $\mathcal{K}_i^{\{0,1\}^S}$ into $\mathcal{K}_{i+1}^{\{0,1\}^S}$. The level-zero composite integral (A.333) (formed in the proof below from the word-frame top density δ_w) is absolutely convergent and equals $\Xi_{0,w}$. Finally, if a coefficient F on the matrix space has compact support, satisfies every weighted differentiability bound, and satisfies $F(g) \neq 0 \Rightarrow G_{\mathrm{loc}}(g) \geq \varepsilon > 0$ on the group, its averaged coefficient has a radial realization in $\mathcal{K}_0$.

Outline. For any compatible actual datum, uniform positive and negative spectral lifting operators are built from the right group action and its adjoint while preserving equivariance and test classes. Combining them with the coordinate and adjoint identities yields the radial spectral identity and decay. The one-letter density estimate and compact construction then give $\mathcal{K}_i$ with slice closure, positivity, and the composite integral. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 993. □

Lemma 33.88 (Compact coefficient forms on residual slices)**.** *Let W_p, W_q be finite-dimensional complex Hilbert spaces. Use the actual place data and a level tower of Definition 33.82, with compact measures of mass one and the prescribed unipotent measure at each residual level. Suppose its base frame is (ν', m_0), where ν' lies in the central residual subspace, $h(\nu', z') = 0$, $h(\nu', \nu') = 1$, $m_0 \in G_r$, and $m_0 z' = \sqrt{T_q / T_p}\, z$. Use its compact-type projections and actual terminal lifts. Require that, for each compact coefficient F, cut word w (an unpositioned signed word whose evaluation points lie on the cut of Definition 33.82 with pairwise distances greater than two), and radial representative ϕ_F, the representative of $\Theta_{w,a} F$ is the a-component obtained by applying the full ordered slice product and then the terminal word intertwiner to the array with ϕ_F in its all-zero component and zero elsewhere. A coefficient at level $j \leq r$ is called compact if it is smooth on the residual group, its nonzero locus there is bounded in matrix norm, and its residual Gram polynomial is bounded below by a positive number on that locus. Write $\mathrm{pos}(w)$ for the positioning of a coefficient word in its list order. The following statements hold.*

(a) *For a compact coefficient F on the original group, a valid level-j block point γ, and every word w, let $x_{\gamma, w}$ be the array obtained by the coefficient insertions and the partial orbital slice of Definition 33.82.*

Every component is a compact coefficient for the residual datum of rank $(k, r-j)$. For compact arrays x, y and a word on the fixed cut whose evaluation points at distinct positions have distances greater than two, there are radial representatives $\phi_{x,a}, \phi_{y,a} \in \mathcal{K}_0^{(j)}$ for this residual datum and a constant $c_j > 0$, fixed by its Haar normalization, such that

$$\Xi^{\mathrm{coef}}_{j,w}(x,y) = c_j \Xi^{(j)}_{0,\mathrm{pos}(w)}(\phi_x, \phi_y; \mathbf{1}).$$

Here the superscript (j) denotes the residual datum, and $\mathcal{K}_0^{(j)}$ is the level-zero subspace of Lemma 33.87 for that datum. Consequently the coefficient form is sesquilinear and positive semidefinite on compact arrays.

(b) *Put $f_s(t) = \exp(st - \overline{s}\,\overline{t})$. In every letter position apply the triangular substitution $x_{1,\eta} \mapsto x_{1,\eta} + s_\ell x_{0,\eta}$, where $s_\ell = s$ for a positive letter and $s_\ell = \overline{s}$ for a negative letter, and denote their product by $U_w(s)$. For a cut word w whose evaluation points are pairwise more than two apart, define*

$$\mathscr{W}_{j,w}(s)x(g) = f_s(D_1(g))U_w(s)x(g)$$

using the first residual frame minor. Both this array and every finite linear combination of compact arrays are compact, and

$$\Xi^{\mathrm{coef}}_{j,w}(\mathscr{W}_{j,w}(s)x, \mathscr{W}_{j,w}(s)y) = \Xi^{\mathrm{coef}}_{j,w}(x,y).$$

Let w^+ and w^- prepend positive letters at $y_1, \dots, y_K$ and negative letters at $y_1+1, \dots, y_K+1$, respectively. Suppose both are cut words and $|y_a - y_b| > 2$ for $a \neq b$. Starting with $q_\varnothing(Y) = q(Y)$, define

$$q_{(+,t)::w}(Y) = q_w(Y)\frac{Y-t+1}{Y-t}, \qquad q_{(-,t)::w}(Y) = q_w(Y)\frac{t-Y}{t-Y-1}.$$

Then $q_w(y_a) > 0$, and for compact extended arrays

$$\left|\mathcal{J}_{\Xi^{\mathrm{coef}}_{j,w}}(x,y)\right|^2 \prod_{a=1}^{K} q_w(y_a) \le \mathrm{Re}\,\Xi^{\mathrm{coef}}_{j,w^+}(x,x)\ \mathrm{Re}\,\Xi^{\mathrm{coef}}_{j,w^-}(y,y).$$

(c) *At level $r+1$, using the Hilbert–Schmidt pairing linear in its first argument, the actual unweighted terminal kernel has the finite Gram representation*

$$\mathscr{R}_{r+1,w}(F;\gamma,\gamma') = \sum_{\alpha} \langle Z_{w,\alpha}(\gamma), Z_{w,\alpha}(\gamma')\rangle_{\mathrm{HS}}.$$

For the empty word the sole vector is $v_F(\gamma)$. For a nonempty word the index is $\alpha = (a,\rho)$, with $a \in I_w$ and ρ a compact type, and the vector is

$$Z_{w,a,\rho}(\gamma) = d_{w,a,\rho}(\gamma)\, P_\rho v_{\Theta_{w,a}F}(\gamma),$$

where $\Theta_{w,a}$, P_ρ, and the real multiplier $d_{w,a,\rho}$ are the terminal lift, type projection, and word multiplier of Definition 33.82. All vectors are zero at invalid points. Multiplying each vector by the square root of its point weight and by its real coordinate factor gives the corresponding weighted Gram representation.

Outline. On a valid slice, local differential insertions preserve bounded support and the slice Gram identity transfers the positive lower bound to the residual group. Applying Lemma 33.87 to the residual datum of rank $(k, r-j)$ yields representatives in $\mathcal{K}_0^{(j)}$ and the form comparison in (a). Stability under residual matrix entries gives the mixed contraction, intertwining gives wave isometry, and expanding the terminal branch proves (c). A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 998. □

Proposition 33.89 (Standardized minor-moment bound on the wave grid)**.** *Fix $C_h, C_a \in \mathbf{R}$ with $C_a \ge 0$. There exist $C \ge 0$ and $C_{g,1} \ge 1000$ such that for every $C_g \ge C_{g,1}$ the following holds at any standardized local Bessel datum of Definition 33.82 whose discrete series has the prescribed signature $\varepsilon_{k,r}$, positive on the indices $1, \dots, k-1$ and alternating $+, -, +, \dots$ on the indices $k, \dots, n$, with $k \ge 2$, $m = 2r+2$, $n = 2r+k+1$, scale $b \ge 2$, final norms $1 \le T_p, T_q \le b^{2|C_h|m}$, K-finite discrete-series coefficient F, and a big-cell section. In coordinates $(c_0, \dots, c_{r-1}, d)$, the section uses a central block depending smoothly on d for $|d|^2 > T_p T_q$,*

preserving the central Hermitian form and pairing the image of the central part of z' with the central part of z to give d. Let I_{std} be the Lebesgue integral on this region of the squared Hilbert–Schmidt norm of the two-sided Bessel average of F at the section point, with density

$$(|d|^2 - T_pT_q)^{k-1} \prod_{a=0}^{r-1} |c_a|^{2(2(r-a)+k-1)}.$$

Assume that λ, u, u' have coordinates bounded by b in absolute value and satisfy the strict half-integral interlacing ($a_i - \frac{k-1}{2} \in \mathbf{Z}$, $\lambda_s - \frac{n-1}{2} \in \mathbf{Z}$, $a_i > \lambda_i > a_{i+1}$ for $1 \le i < k$, and $a_k > \lambda_k$, for $a = u, u'$), adjacent gaps $\lambda_s - \lambda_{s'} \ge C_g m^4$ ($s < s'$), opposite-sign gap at least $C_g m^{9/2}(3(r+1)n^2+n^3)$, a splitting scale $b_v \ge C_g n^7$ satisfying

$$0 \le \lambda_a \le b_v/8 \quad (1 \le a < k), \qquad -2b_v \le \lambda_a \le -b_v \quad (k \le a \le n),$$

and the following gap alternative. If $u = u'$, there is an index $1 \le l_0 < k$ with $u_{l_0} - u_{l_0+1} \ge 2C_g(r+1)^2 + 1$, and there are indices $1 \le a, b < k$ with $\lambda_a \ge u_{l_0}$ and $\lambda_b \le u_{l_0+1}$ (which under strict interlacing forces $k \ge 3$ and $2 \le l_0 \le k-2$). If $u \ne u'$, there are $1 \le l_0 \le k$ and $x \le y$ such that

$$y - x \ge C_g(r+1)^2, \qquad \min(-u_{l_0}, -u'_{l_0}) + 2 \le x \le y \le \max(-u_{l_0}, -u'_{l_0}) - 2.$$

Then for every $1 \le l \le r+1$, every integer $0 \le e \le C_a m^3$, and every top-level block point γ with $G_{\mathrm{loc}}(\gamma) \ne 0$,

$$|D_l(\gamma)|^{2e}\, p_{k,r}(T_p, T_q; \gamma)\, \|J(F;\gamma)\|_{\mathrm{HS}}^2 \le b^{Cm^5} \max\bigl(1, |G_{\mathrm{loc}}(\gamma)|^{-1}\bigr)^{40(r+1)^2} I_{\mathrm{std}}. \tag{33.28}$$

Outline. Tensor decomposition gives a finite orthogonal eigenfunction expansion of $G_{\mathrm{loc}}^{p_0}F$ for the signed endpoint. Double cutoffs put every slice array into Lemma 33.88; Lemmas 33.80, 33.84, and 33.85 shorten the tower energy before removing both cutoffs, using Lemma 33.83 when $u = u'$. A shift family, Pieri expansion via Lemma 33.81, and the π-factor energy conversion yield the bound. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 999. □

Lemma 33.90 (Torus rescaling of local Bessel integrals)**.** *Under the hypotheses of Proposition 33.89 except that $1 \le T_p, T_q \le b^{|C_h|m}$, the Bessel character has parameter $-2i\kappa_0$ with $\kappa_0 > 0$, $\kappa_0^{\pm 1} \le b^{|C_h|}$, $\kappa_0^{2r}T_p \ge 1$, $\kappa_0^{2r}T_q \ge 1$, and the unipotent Haar measure has arbitrary finite positive scale κ_N, diagonal torus conjugation with parameters $\kappa = \kappa_0^{-1}$ and $\tau = \kappa_0^r$ carries the datum to character parameter $c_\psi = -2i$, with a finite positive unipotent Haar scale, and final norms $\kappa_0^{2r}T_p, \kappa_0^{2r}T_q$. Put $g_*(\kappa_0) = \kappa^{2r(r+1)}\tau^{4(r+1)} = \kappa_0^{2r(r+1)}$ and $F_*(\kappa_0) = \max(\kappa_0, \kappa_0^{-1})^{2m^2} \max(\kappa_0^r, \kappa_0^{-r})^{2m}$. Then $F_*(\kappa_0) \le b^{4|C_h|m^2}$, and every $1 \le l \le r+1$, integer $0 \le e \le C_a m^3$, and top-level block point γ with $G_{\mathrm{loc}}(\gamma) \ne 0$ satisfy*

$$|D_l(\gamma)|^{2e}\, p_{k,r}(T_p, T_q; \gamma)\, \|J(F;\gamma)\|_{\mathrm{HS}}^2 \le b^{2|C_h|m^2 e}\, F_*(\kappa_0)\, b^{Cm^5} \max\bigl(1, |g_*(\kappa_0) G_{\mathrm{loc}}(\gamma)|^{-1}\bigr)^{40(r+1)^2} I,$$

where I is the squared big-cell section integral in the original coordinates and C is the constant of Proposition 33.89.

Outline. Find a compact-finite F_1 whose torus-translated operators scale those of F by one nonzero factor, and show that Haar rescaling preserves the inequality of Definition 33.82. Diagonal conjugation with $\kappa = \kappa_0^{-1}$ and $\tau = \kappa_0^r$ moves to parameter $-2i$ with a common factor $K_1 > 0$. Verify the transformed size and minor bounds, apply Proposition 33.89, and cancel K_1. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1009. □

Lemma 33.91 (The weighted sum of real orbital integrals)**.** *Fix $A_w \ge 0$ and $C_h \in \mathbf{R}$. There are $C_b, C_{g,0}$ such that the following holds for $C_g \ge C_{g,0}$. Put $r = r_S$, $m = 2r+2$, $n = k+2r+1$, $\xi = \pi/(v_E g_0 g_1)$. Use the same $T, T' \in \mathbf{N}$ and b at every real place, with*

$$\xi > 0, \quad \xi^{\pm 1} \le b^{C_h}, \quad 1 \le T, T' \le b^{C_h m}, \qquad T\xi^{2r},\ T'\xi^{2r} \ge 1.$$

At each place take the prescribed signature, standard character, positive finite section normalization, two Bessel frames of final norms T, T', a K-finite discrete-series coefficient, and a big-cell section. Use the

Harish–Chandra parameter λ prescribed by the point, with coordinate gaps at least $C_g m^4$. For each length-k weight u, u' require $u_i > \lambda_i > u_{i+1}$ for $1 \le i < k$ and $u_k > \lambda_k$, with the corresponding inequalities for u'. The parameter and type coordinates have the prescribed integral or half-integral normalization, so the strict interlacing separates them by at least $1/2$. All their coordinates are bounded by b in absolute value. Opposite signature signs have gap at least

$$C_g m^{9/2}\big(3(r+1)n^2 + n^3\big).$$

There is $b_v \ge C_g n^7$ with $\lambda_a \in [0, b_v/8]$ when $\min_i u_i < \lambda_a < \max_i u_i$, and $\lambda_a \in [-2b_v, -b_v]$ otherwise. If $u = u'$, some consecutive gap $u_l - u_{l+1} \ge 2C_g(r+1)^2 + 1$ has a coordinate of the first group at or above u_l and one at or below u_{l+1}. If $u \ne u'$, some interval of length at least $C_g(r+1)^2$ lies between $-u_l$ and $-u'_l$, at distance at least two from both endpoints.

Sum over rational block points γ with an integral representative and nonzero integral $G = D_{r+1}\overline{D_{r+1}} - TT' D_r \overline{D_r}$. Write D_j for its leading minors and use the block ends in products over j. At an infinite place v, use the complex absolute value of the chosen embedding. Put

$$W_v = |G|_v \prod_j |D_j|_v^2, \qquad w_v = \prod_j (1 + |D_j|_v^2)^2, \qquad W = \prod_v W_v, \quad w = \prod_v w_v.$$

A block of length s with entry t has matrix tJ_s, where J_s has ones on its antidiagonal. For a block starting after a positions its local entry is $z = \mathrm{rd}(E)^{-(2r+3-2(a+1)-s)} v(t)$. The rotated entry is $(-1)^r(-i)^{2a+s-1} z$. The local point weight p_v is the product of $|z|^{2s(2(r+1-a-s)+k+s-2)}$ over the blocks. For a represented final singleton, use the factor $(|z|^2 - TT')^{k-1}$. Put $p(\gamma) = \prod_v p_v(\gamma)$.

Write the coefficient as $F(h) = P\sigma(h)Q$, with the images of Q and P^* compact-finite, and let π, π' be the two definite types. For each unitary representative h of the rotated matrix of frame pairings, require the two unipotent characters to agree on the common stabilizer. In compact and unipotent coordinates $(a, n), (a', n')$ the integrand of the orbital operator is

$$\psi_z(n)\overline{\psi_{z'}(n')}\,\pi(a) F(a^{-1}n^{-1}hn'a')\pi'(a')^{-1}.$$

Here $\psi_z, \psi_{z'}$ are the Bessel characters of the two frames with parameter $-2i\xi$ in the arithmetic chart. Use each quotient measure that disintegrates the product Haar measure along the common stabilizer with the prescribed root-coordinate measure. Write $\|J_v(\gamma)\|_{\mathrm{HS}}^2$ for the supremum of the squared norms over these choices, with value zero on empty support. Let I_v be the squared section integral in diagonal coordinates $(c_0, \dots, c_{r-1}, d)$, with Lebesgue density

$$\mathbf{1}_{|d|^2 \ge TT'}(|d|^2 - TT')^{k-1} \prod_{a=0}^{r-1} |c_a|^{2(2(r-a)+k-1)}.$$

Then

$$\sum_\gamma w(\gamma) p(\gamma) W(\gamma)^{A_w (r+1)^2} \prod_v \|J_v(\gamma)\|_{\mathrm{HS}}^2 \le b^{C_b m^5 f} \prod_v I_v.$$

Outline. Apply Proposition 33.89 to fix the thresholds, and dominate each local weight by one maximal minor power. Apply Lemma 33.90 at $\kappa_0 = \xi$, then multiply over infinite places using $0 < |G(\gamma)|_v \le |D_{r+1}(\gamma)|_v^2$ and $\prod_v |G(\gamma)|_v \ge 1$. Bounding $\sum_\gamma w(\gamma)^{-1}$ by a dilated polydisk count and summing over γ completes the bound. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1011. □

Lemma 33.92 (A character sum on the norm congruence)**.** *Let F be a nonarchimedean local field of characteristic zero, with valuation ring O, uniformizer ϖ, and residue cardinality q. Let O_E be the integers of the unramified quadratic extension or $O \times O$ with exchange involution. Write h for an integral Hermitian form, linear in the first variable, on an O_E-lattice U. Suppose $W \subseteq U$ contains $\rho \ge 4$ orthonormal vectors. Let $a \ge 0$, $N \ge a + 1$, and use probability measure on $U/\varpi^N U$. Write Z for the congruence $h(u, u) \in \varpi^a O$. For every additive character χ of this quotient,*

$$|q^a \mathbb{E}(\mathbf{1}_Z \chi) - \mathbf{1}_{\chi=1}| \le \frac{8}{7} q^{1-\rho}.$$

The probability that Z holds and $h(u, W)$ is a proper ideal of O_E is at most $q^{-a}(16/7)q^{1-\rho}$.

Outline. Expand the norm congruence in additive characters on the orthonormal coordinates. Square each coordinate sum and use trace orthogonality, multiply the bounds and sum by conductor, then restrict the coordinates to each maximal ideal to bound the proper-pairing event. The complete proof is on p. 1012. □

Lemma 33.93 (Counting the two normal coefficients)**.** *Let E be a CM field of degree $2f$, write $\bar{x}$ for complex conjugation, and write N for the absolute ideal norm. For a nonzero fractional ideal J and $a \in E^{\times}$,*

$$\#\{x \in J : x\bar{x} = a\}\,(NJ)^2 \le 8f^2\,|N_{E/\mathbf{Q}}a|.$$

In particular let $T, T' \ge 3$, $P \ge 1$, and suppose a set of labels injects into pairs (x, x') with $x\bar{x} = x'\bar{x}' = 1$, $x \in J$, $x' \in J'$. If

$$(NJ)^{-2} \le T^{2f}P^2, \qquad (NJ')^{-2} \le T'^{2f}P^2,$$

the number of labels is at most $(TT')^{6f}P^6$.

Outline. Count the ideals $(x)J^{-1}$ among the divisors of $(a)(J\bar{J})^{-1}$ and bound each fibre by the number of roots of unity using their cyclotomic degrees. Apply the estimate to both coordinates and use injectivity of the labels. The complete proof is on p. 1012. □

Definition 33.94 (Corank-one partner classes and normal data)**.** Fix words u, v in a tied Hida model (X, Y) on the Hermitian E-space (V, H_E) of dimension $n = k + 2r + 1$, where $r = r_S \ge 1$, $m = 2r + 2$, $f = [F : \mathbf{Q}]$, and $g = g_0g_1$. For a double-coset term t of (u, v) with nonzero Gram invariant $G(t) \ne 0$, let $\mathrm{row}(t) = (\Lambda_t, \Lambda_t', \iota_t)$ be its complete row label, F_u the fixed flag of u, and F_t' the moving flag of t. Their $2r + 2$ vectors span a nondegenerate subspace $W_t = \mathrm{span}_E(F_u, F_t') \subset V$ whose orthogonal complement $W_t^{\perp}$ is totally positive definite of rank $k - 1$.

At each finite place p of F of residue cardinality q_p, let $\theta_p \in E_p^{\times}$ be the local frame-difference scale and $\Lambda_{t,p}$ the p-adic row lattice of t. The local saturation index (defect) $\beta_p(t) \ge 1$ is the index inside $\Lambda_{t,p} \cap W_{t,p}$ of the submodule generated by the first frame and the θ_p^{-1}-scaled differences of the two frames. The term t is generic ($t \in E_{\mathrm{gen}}$) if $\beta_p(t) = 1$ at every finite place p, and nongeneric ($t \in E_{\mathrm{ng}}$) if $\beta_p(t) > 1$ at some finite place p.

Two terms t, t' with $G(t) \ne 0$ and $G(t') \ne 0$ in the same complete row have corank $s(t, t') = 1$ when the first projection Θ of Θ_{pairs} in Lemma 33.61 has dimension $k - 1$, using $F = F_u$, $F' = F_t'$, $F'' = F_{t'}'$, and the relative rational isometry β of the pair. As in Lemma 33.64, this equivalence relation partitions E_{ng} into nongeneric corank-one partner classes c; for $u = v$, the generic scalar-partner classes c inside E_{gen} are the classes of Lemma 33.79. Each $t \in c$ has a rational block point $\gamma = \gamma(t)$ of level $r+1$ over E with leading minors $D_l(\gamma)$, nonzero integral Gram invariant $G(\gamma) = D_{r+1}(\gamma)\overline{D_{r+1}(\gamma)} - T_uT_vD_r(\gamma)\overline{D_r(\gamma)}$, Gram norm factor $N_G(\gamma) = \prod_\tau |\tau(G(\gamma))| \ge 1$ over the CM type, block-end minor norm factor $N_D(\gamma) = \prod_\tau \prod_j |\tau(D_j(\gamma))|^2 \ge 1$ so $W(\gamma) = N_G(\gamma)N_D(\gamma)$ in Lemma 33.91, and finite-place heights $H_p(\gamma) \ge 1$ whose product is $H(\gamma) = \prod_p H_p(\gamma) = g^{-(2r+1)f}d_0^{-e_\gamma}N_G(\gamma)N_D(\gamma)$, with $d_0 = \mathrm{rd}(E)$ and $e_\gamma = 2f\big(r(r+1) + \sum_j j(2r+1-j)\big)$. Within (c, γ), a term is indexed by its normal datum $\nu = \nu(t)$, whose two norm-one coordinates $x \in J, x' \in J'$ ($x\bar{x} = x'\bar{x}' = 1$) lie in fractional ideals of E attached to (c, γ), with vector subtotal

$$V(c, \gamma, \nu) = \sum_{\substack{t \in c \\ \gamma(t) = \gamma,\ \nu(t) = \nu}} v_t.$$

At each finite place p, write $d \in D_p$ for the local normal data at γ with orbit weights $w_d \ge 0$, unsigned local reference mass $n_p \ge 0$, generic local mass $n_p^{\mathrm{gen}} \ge 0$, global generic mass $n_{\mathrm{gen}}(\gamma) = V_0 \prod_p n_p^{\mathrm{gen}}$ as in Lemma 33.64, and local orbital character sums $a_N(d)$ (over presentations with $\beta_p > 1$) and $a_Z(d)$ (over presentations with $\beta_p = 1$). Let Ω be the finite set of exceptional places where $\beta_p > 1$ or $n_p \ne n_p^{\mathrm{gen}}$ can occur, so $H_p(\gamma) \ge q_p \ge 2$ for $p \in \Omega$.

Lemma 33.95 (Changing the normalization of Gram densities)**.** *Let w range over the archimedean product place ∞ and the finite places of a number field. At each w, let V_w be the local tuple space, $\mathcal{H}_w$ its space of Hermitian Gram matrices, $q_w : V_w \to \mathcal{H}_w$ the Gram map, and $\mathcal{H}_w^{\mathrm{nd}}$ the open subset of nonsingular matrices.*

Fix measures μ_w^0, ν_w^0 on $V_w, \mathcal{H}_w$, a test function $\Phi_w : V_w \to \mathbf{C}$, and a continuous function $f_w : \mathcal{H}_w^{\mathrm{nd}} \to \mathbf{C}$ such that, for every compact $K \subset \mathcal{H}_w^{\mathrm{nd}}$,

$$\int_{q_w^{-1}(K)} \Phi_w \, d\mu_w^0 = \int_K f_w(S) \, d\nu_w^0(S).$$

Let $c_w, e_w > 0$, and put

$$\mu_w = c_w \mu_w^0, \qquad \nu_w = e_w \nu_w^0, \qquad a_w = c_w/e_w, \qquad g_w = a_w f_w.$$

Then g_w is continuous on $\mathcal{H}_w^{\mathrm{nd}}$, and

$$\int_{q_w^{-1}(K)} \Phi_w \, d\mu_w = \int_K g_w(S) \, d\nu_w(S).$$

Fix a nonsingular rational Gram matrix T, and write T_w for its image at w. Suppose the finite-place products $\prod_v a_v$ and $\prod_v f_v(T_v)$ converge as nets over finite sets of places, and

$$a_\infty \prod_v a_v = 1.$$

Then the finite-place product of the $g_v(T_v)$ converges and

$$g_\infty(T_\infty) \prod_v g_v(T_v) = f_\infty(T_\infty) \prod_v f_v(T_v).$$

Outline. Rescales each local integral by $a_w = c_w/e_w$, preserving continuity. Factoring finite partial products of $g_v(T_v)$ into a_v and $f_v(T_v)$ products, taking the net limit, and applying $a_\infty \prod_v a_v = 1$ yields the global identity. The complete proof is on p. 1013. □

Lemma 33.96 (Siegel–Weil generic mass comparison)**.** *Under the hypotheses of Lemma 33.99 (fixing the six standing constants before the thresholds for C', C_ℓ), above the resulting thresholds for C', C_ℓ the tied model admits positive mass-normalization data $(A_0, c_0, c_0', V_1, V_{ZZ})$ and real integrals $I_C, I_N, I_{\mathrm{rat}}, I_{\mathrm{re}}$ with $d_0 = \mathrm{rd}(E)$ and archimedean mass-to-section normalization $A_{\mathrm{re}} = d_0^{2r(r+1)(k-1)f} A_0 > 0$ satisfying $I_N = (V_{ZZ}/V_1) I_C$, $\Gamma_1 A_0 (c_0 c_0')^{-1} I_N \le \Delta_{u,v}$, $V_1/V_{ZZ} = d_0^{f(r+1)(8r^2+10r+3)/3} g^{f(r+1)(2r+3)}$, $(c_0 c_0')^{-1} I_C = 2^{f(r+1)} I_{\mathrm{rat}}$, and $I_{\mathrm{rat}} = d_0^{2r(r+1)(k-1)f + fr(r+1)(8r+10)/3} I_{\mathrm{re}}$, whence*

$$\Gamma_1 A_{\mathrm{re}} I_{\mathrm{re}} \le (d_0/2)^{f(r+1)} g^{f(r+1)(2r+3)} \Delta_{u,v}.$$

Moreover, with $C_0' = 9$, at each represented block point γ one has $0 < n_p^{\mathrm{gen}} < \infty$, $n_p \le 8 H_p(\gamma)^{C_0'(r+1)^2+2m} n_p^{\mathrm{gen}}$ for $p \in \Omega$, and $n_p = n_p^{\mathrm{gen}}$ for $p \notin \Omega$; the discriminant-normalized Gram norm $\widehat{N}_G(\gamma) = d_0^{-2r(r+1)f} N_G(\gamma)$ and the real point weight $p(\gamma)$ of Lemma 33.91 satisfy $\widehat{N}_G(\gamma)^{k-1} \le H(\gamma)^{6m} p(\gamma)$; and with $e_1 = 2$ and $e_N = 10(r+1)^2 + 6m + 1$ the global generic mass $n_{\mathrm{gen}}(\gamma)$ satisfies

$$0 < n_{\mathrm{gen}}(\gamma) \le A_{\mathrm{re}} e_1 H(\gamma)^{3(r+1)^2} \widehat{N}_G(\gamma)^{k-1} \le A_{\mathrm{re}} e_1 H(\gamma)^{e_N} p(\gamma). \tag{33.29}$$

Outline. The proof uses Proposition 33.24 for local uniqueness and archimedean convergence, and combines the measure form of the convergent Siegel–Weil formula with Lemma 33.95 to obtain the normalized theta mean. The mass data of the tied model then satisfy the admissible normalization, and the local and global generic mass bounds follow. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1013. □

Lemma 33.97 (Normalization of the rotated orbital factors)**.** *Use the rank parameters and scale $b > 1$ of Lemma 33.91, and fix $C_1 \in \mathbf{R}$. Let τ run over a CM type of cardinality $f > 0$. At each τ, fix a representative of a rotated block point, a Haar measure $\mu_{G,\tau}$ on the full local unitary group containing the two frames, a*

root-coordinate measure $\mu_{R,\tau}$ on its common stabilizer, and a quotient measure $\nu_{\tau,1}$ which disintegrates $\mu_{G,\tau}$ along $\mu_{R,\tau}$. Thus, for every nonnegative measurable function F,

$$\int F(x)\,d\mu_{G,\tau}(x) = \int\!\!\int F(sx)\,d\mu_{R,\tau}(s)\,d\nu_{\tau,1}(x).$$

Write

$$\mathcal{A}_\tau(\nu) = \int \Phi_\tau(x)\,d\nu(x)$$

for the local row-function orbital operator, with its fixed normalization included in the kernel Φ_τ, which takes values in linear maps between finite-dimensional unitary spaces. Let $Q(\gamma) > 0$ be the squared global orbital factor at the represented row point γ, for a fixed pinned family of quotient measures. At every element of the archimedean group the quotient measure disintegrates the fixed archimedean Haar measure along a stabilizer Haar measure. At every regular element, where the leading minors of orders $1,\dots,r$ and the Gram invariant are nonzero at every infinite place, the quotient measure equals the fixed archimedean Haar measure. At the designated representative of each represented block point the stabilizer Haar measure in this disintegration can be chosen to have the fixed root-coordinate measure as its image in root coordinates. Suppose

$$P(\gamma) = \prod_\tau \|\mathcal{A}_\tau(\nu_{\tau,1})\|_{\mathrm{HS}}^2 > 0, \qquad \vartheta = \left(\frac{P(\gamma)}{Q(\gamma)}\right)^{1/(2f)}, \qquad b^{-C_1 m^2} \le \vartheta \le b^{C_1 m^2}.$$

Then $\nu_{\tau,\vartheta} = \vartheta^{-1}\nu_{\tau,1}$ is a quotient measure for the stabilizer measure $\vartheta\mu_{R,\tau}$, with the prescribed normalization window, and

$$Q(\gamma) = \prod_\tau \|\mathcal{A}_\tau(\nu_{\tau,\vartheta})\|_{\mathrm{HS}}^2. \tag{33.30}$$

For a subtotal $V(c,\gamma,\nu)$, define $a_2(c,\gamma,\nu) = \|V(c,\gamma,\nu)\|^2/Q(\gamma)$ using the same pinned quotient measure. Then

$$\|V(c,\gamma,\nu)\|^2 = a_2(c,\gamma,\nu)\prod_\tau \|\mathcal{A}_\tau(\nu_{\tau,\vartheta})\|_{\mathrm{HS}}^2.$$

Outline. Canceling $\vartheta, \vartheta^{-1} > 0$ in disintegration verifies $\nu_{\tau,\vartheta}$. Linearity scales each squared norm by ϑ^{-2}, so multiplying over τ via $\vartheta^{2f} = P(\gamma)/Q(\gamma)$ and by a_2 yields both identities. The complete proof is on p. 1016. □

Proposition 33.98 (Rotated archimedean reduction for corank-one subtotals)**.** *Fix the six standing constants of Proposition 33.18 and any real weight parameter $A_w \ge 0$. For both pinned nongeneric corank-one classes of (u,v) under the base and pair standing conditions of Definition 33.7, and equal-word generic scalar corank-one classes of (u,u) under the same base and pair standing conditions, there are constants $C_a, C_n \in \mathbf{R}$, $C_b \ge 0$, and thresholds for C', C_ℓ with the following normalized estimates. On the support where the global squared orbital factor $Q(\gamma)$ is positive, use the same pinned quotient measure for Q and $a_2(c,\gamma,\nu) = \|V(c,\gamma,\nu)\|^2/Q(\gamma)$. For the selected local row-function orbital operators of Lemma 33.97,*

$$Q(\gamma) = \prod_\tau \|\mathcal{A}_\tau(\nu_{\tau,\vartheta})\|_{\mathrm{HS}}^2, \qquad \|V(c,\gamma,\nu)\|^2 \le a_2(c,\gamma,\nu) C_J \prod_\tau \|J_\tau(\gamma)\|_{\mathrm{HS}}^2, \qquad C_J = \prod_\tau \big(K_\tau b^{C_a m^2}\big),$$

where $K_\tau > 0$ are the local Bessel-comparison constants and $\|J_\tau(\gamma)\|_{\mathrm{HS}}^2$ denotes the supremum of Lemma 33.91. At every represented point, including $Q(\gamma) = 0$, the analytic comparison is

$$Q(\gamma) \le C_J \prod_\tau \|J_\tau(\gamma)\|_{\mathrm{HS}}^2.$$

If M_τ is the local row-function section integral, obtained by using the squared row-function orbital operator and ambient Haar measure with the diagonal-coordinate density of Lemma 33.91, the section comparison is

$$\left(\prod_\tau K_\tau\right)\prod_\tau I_\tau \le \prod_\tau M_\tau \le b^{C_n m^2 f} I_{\mathrm{re}},$$

where $I_{\mathrm{re}} \ge 0$ is the normalized row integral of an admissible tuple formed from the mass data that the proof of Lemma 33.96 constructs from any choice of its auxiliary data; for these data I_C is the constructed integral multiplied by the fixed unipotent scale factor κ, and the two frame identities of that lemma determine I_{rat} and I_{re} from I_C. Lemma 33.91 applies at exponent A_w, with these normalization losses absorbed into C_b.

Outline. Uniform restriction and Proposition 33.24 supply the bound on $\prod_\tau M_\tau$ and the scale window for ϑ when $Q(\gamma) > 0$. Local Bessel comparison selects $K_\tau > 0$ giving $K_\tau I_\tau \le M_\tau$ and local orbital bounds, which combine with Lemma 33.97 to bound Q and $\|V\|^2$. Cancelling K_τ and applying Lemma 33.91 absorb the remaining factors into C_b. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1016. □

Lemma 33.99 (The two signed corank-one sums)**.** *For a set U of term pairs put $P(U) = \Gamma_1 \sum_{(t,t')\in U} \mathrm{Re}\langle v_t, v_{t'}\rangle$. Assume the second formula of 34.17 and fix the six standing constants of 33.18. There is $C_4 \ge 0$, followed by thresholds for C', C_ℓ, such that, under the size and split-prime hypotheses of 33.7 for u, v, the Hida-model compatibility specified before (33.21), and absolute convergence,*

$$P(U_{\mathrm{ng}}) \le 2^{-(k-2r_S-5)} e^{C_4 m^5 f \log b} \Delta_{u,v}.$$

Here U_{ng} consists of same-row nongeneric corank-one pairs. The pair standing conditions of 33.7 include $r_S = r_0 = (m-2)/2 \ge 1$, complete splitting of $T_u, T_v \ge 3$ and distinct g_0, g_1, $(T_u T_v, g_0 g_1) = 1$, and $T_u \equiv T_v \pmod{g_0 g_1}$.

For every fixed additional real counting parameter A, there is $C_B \ge 0$, followed by thresholds for C', C_ℓ, such that $\mathrm{Cnt.Hyp}(u, u; A)$ and the same ties and convergence give

$$P(U_{\mathrm{sc}}) \le e^{C_B m^5 f \log b} \Delta_{u,u}$$

for same-row generic scalar corank-one pairs of the equal-word operator.

Outline. Rewrite $P(U_{\mathrm{ng}})$ and $P(U_{\mathrm{sc}})$ as class-square sums via Definition 33.94 and Lemma 33.64. Bound local defect sums using a symmetric isometry kernel and Lemma 33.92, then compare total and generic masses using Lemma 33.91. Count normal labels with Lemma 33.93, apply Cauchy–Schwarz, and combine with Proposition 33.98 for both global bounds. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1018. □

Lemma 33.100 (Cancellation of the two represented-frame determinants)**.** *Let $k, f \ge 1$, $s \ge 2$, and $g, v, N > 0$. Let $G_1, G_2 > 0$ be the normalized Gram products of two represented frames. Put*

$$n_i = \left(\frac{\exp(1)\pi}{kv}\right)^{kf} (G_i/N)^k.$$

Let $M \ge 0$ be the weight of a fibre of pairs, $A_i \ge 0$ bounds for the two amplitudes, and $W \ge 0$ the remaining counting factor. For $d \in \{1, 2\}$ and $B_i > 0$, suppose

$$M \le W \left((2/g)^{(s-d)f} \frac{\sqrt{G_1 G_2}}{N} B_1 B_2\right)^k, \qquad n_i A_i^2 B_i^{2k} \le L_i \quad (i = 1, 2).$$

Then

$$M A_1 A_2 \le W (2/g)^{(s-d)kf} \left(\frac{kv}{\exp(1)\pi}\right)^{kf} \sqrt{L_1 L_2}.$$

Proof. Multiply the amplitude inequalities and take the nonnegative square root. The factor on $A_1 A_2$ is

$$(B_1 B_2)^k \sqrt{n_1 n_2} = (B_1 B_2)^k \left(\frac{\exp(1)\pi}{kv}\right)^{kf} \left(\frac{\sqrt{G_1 G_2}}{N}\right)^k.$$

Division by its positive value and substitution into the fibre bound give the assertion. □

Lemma 33.101 (The saving on the small side of an ordered pair)**.** *Let $T, T', g > 0$, and let v range over f places. At each place let $s_{iv} > 0$ and $c_{iv} = \sqrt{1 + s_{iv}^2}$ for $i = 1, 2$. Suppose*

$$\left(\frac{\sqrt{TT'}}{g}\right)^f \le \prod_v \frac{c_{1v} + c_{2v}}{s_{1v} s_{2v}}.$$

Then for at least one i, putting $\nu_v = T' s_{iv}^2$, one has

$$\prod_v \min(\nu_v, T') \le (4g\sqrt{T'/T})^f.$$

The same conclusion for the single available side holds if its $s_v > 0$ satisfy $(\sqrt{TT'}/g)^f \le \prod_v s_v^{-1}$.

For any side satisfying this conclusion let $k \ge 1$, $r \ge 0$, $m = 2r + 2$ be integers. Let $1 \le C \le k^{1/3}$, $0 \le \eta \le 1/4$, and let $T_{\mathrm{crit},v} > 0$ satisfy

$$g^{-m} T_{\mathrm{crit},v} \ge k^{3r}, \qquad \prod_v (T_{\mathrm{crit},v}/T')^k \le 2C_0\Gamma^2$$

for $C_0, \Gamma > 0$. Put $a_v = T_{\mathrm{crit},v}/T'$. The local minimum factors satisfy

$$\left(\prod_v \min\{1, \nu_v a_v g^{-m} (C/k)^{2r(1-\eta)}\}\right)^k \le 2C_0\Gamma^2 \left(4\sqrt{T'/T}\, g^{1-m} k^{-r}\right)^{kf}.$$

Outline. Bound the product of the two clipped local sizes and take the smaller side, using $\min(1, s)^2 \le s$ for a single side. Insert this bound into the product of local minima using the two bounds on the critical scales. The complete proof is on p. 1021. □

Definition 33.102 (Reduced Gram labels and partner classes)**.** In a tied Hida model (X, Y) under the hypotheses of (33.21), fix words u, u' with level primes $T_u, T_{u'}$ in the Hermitian E-space (V, H_E) of dimension $n = k + 2r + 1$, where $r = r_S = (m-2)/2 \ge 1$, $m = 2r + 2$, $f = [F : \mathbf{Q}]$, $v = \mathrm{rd}(E)/2$, $\rho = 2v = \mathrm{rd}(E)$, $g = g_0 g_1$, $L_b = \log b$, and $N = T_u^f$. Each double-coset term t for (u, u') with nonzero Gram invariant $G(t) \ne 0$ has complete row $\mathrm{row}(t) = (\Lambda_t, \Lambda'_t, \iota_t)$, row weight $\mathrm{wt}(t) = \big(w(\Lambda_t) w(\Lambda'_t)\big)^{1/2}$, orbital vector v_t, fixed flag F_u of rank $r+1$, moving flag F'_t of rank $r+1$, and $(r+1) \times (r+1)$ reduced Gram matrix $Z = Z(t) = (Z_{ab'})_{0 \le a, b' \le r}$ over E with leading minors $p_l(Z) = \det(Z_{ab'})_{0 \le a, b' < l}$ $(1 \le l \le r+1$, $p_0(Z) = 1)$. At each place τ of the CM type, set $d_{l,\tau}(Z) = \rho^{-2q_l} |\tau(p_l(Z))|$ with $q_l = \sum_{i=1}^{l} (r + 1 - i)$. For $a \ge 0$, define

$$S_a(Z) = \prod_\tau \prod_{l=1}^{r+1} \max\{1, d_{l,\tau}(Z) b^{-am^2}\}, \qquad G_Z = g^{-mf} \prod_\tau \big(d_{r+1,\tau}(Z)^2 - T_u T_{u'} d_{r,\tau}(Z)^2\big),$$

so $G_Z > 0$ on represented support. For corank $s \ge 2$, define the represented-frame normalizer $n_Z^{(2)} = \big(\exp(1)\pi/(kv)\big)^{kf} (G_Z/N)^k$, and put $K(Z) = 1$ if $p_r(Z) = 0$ and $K(Z) = \prod_\tau (1 + c_\tau)/\sqrt{c_\tau^2 - 1}$ with $c_\tau = |\tau(p_{r+1}(Z)/p_r(Z))|/\sqrt{T_u T_{u'}} > 1$ if $p_r(Z) \ne 0$. For any real $c \ge \max(1, C_{Tu})$, define the switch height $h_Z = b^{c^2 m^3 f} S_c(Z)^4$.

For a same-row pair (t, t') with $G(t) \ne 0$ and $G(t') \ne 0$, put $W_t = \mathrm{span}_E(F_u, F'_t)$, $H' = W_t^\perp$, and $H'' = W_{t'}^\perp$. These complements are totally positive definite of dimension $k-1$. For a frame F, write $X_{\mathrm{sp}}(F)$ for the totally isotropic E-span of its first r vectors. For the relative rational isometry β, let $\Theta \subset H'$ be the first projection of

$$\{(x', x'') \in H' \times H'' : x' - x'' \in X_{\mathrm{sp}}(F_u),\ \beta x' - x'' \in X_{\mathrm{sp}}(F'_{t'})\}.$$

Define $s(t, t') = k - \dim_E \Theta$, as in Lemma 33.61. For $2 \le s \le k$, partition the corank-s pairs of S_{big} into the nondegenerate class $\mathcal{P}_{\mathrm{nd},s}$ and the degenerate class $\mathcal{P}_{\mathrm{deg},s}$ as in Lemma 33.61. When $s(t, t') = 1$, the first projection Θ equals H'. For a fixed first term, its integral partner data are the equivalence classes of generic moving frames that represent admissible integral terms in the same row lattice and admit a compatible pair isometry of corank one. The frames obey the prescribed integral frame convention, the intersection of their isotropic subspace with the row lattice is the prescribed flag lattice, and their Gram matrix is equivalent to the specified label under integral lower and upper unitriangular changes. The equivalence is generated by the integral fixed-frame unipotent stabilizer and integral upper-unitriangular changes of the moving frame.

Proposition 33.103 (Higher-corank loop pair sums)**.** *Under the hypotheses and thresholds of (33.21), use the notation of Definition 33.102, put $f_w = d_w^S$, $R' = f_{u'}/D^\star$, and $\Gamma = e^{C_\Gamma m^5 f L_b}$, and choose $C \geq \max(1, 4\sqrt{a_F(\mathbf{o}(F_m))}, C_{Tu})$ above the count and amplitude thresholds. Set $H = b^{Cm^2 f}$, let $\Xi_s = M_s M_{k-s}/M_k$ for $2 \leq s \leq k-2$ with the standard definite genus masses M_j of Lemma 33.11, and at the endpoint ranks put*

$$\Xi_s = (Ck)^{kf}(H^{Cm}N)^{Ck}, \qquad s = k-1, k. \tag{33.31}$$

Define $F_s = (kv)^{kf}\Xi_s C^{kf} H^{C(s+m)m^2}(Csf)^{6sf}\Gamma$, $U_s = (2/g)^{(s-1)kf}F_s$, and $V_s = (2/g)^{(s-2)kf}F_s e^{Cm^6 f L_b}(C/k)^{kf}$. Then for every $2 \leq s \leq k$ and every finite subset U,

$$\begin{aligned} \Gamma_1 \sum_{(t,t')\in U} |\langle v_t, v_{t'}\rangle| &\leq f_u R' U_s \quad (U \subset \mathcal{P}_{\mathrm{nd},s}), \\ \Gamma_1 \sum_{(t,t')\in U} |\langle v_t, v_{t'}\rangle| &\leq f_u V_s \qquad (U \subset \mathcal{P}_{\deg,s}). \end{aligned} \tag{33.32}$$

Outline. Group pairs by reduced Gram labels (Z, Z') and rank-s lattice, combining weighted fibre counts with orbital amplitude majorants in both cases. Cancel $\sqrt{n_Z^{(2)} n_{Z'}^{(2)}}$ via Lemma 33.100 and choose one permitted moment $q \leq (9c+3)km^2$ to produce summable decay $(\sigma_Z \sigma_{Z'})^{-\zeta}$. Summing over labels, comparing constants, and applying Lemma 33.65 completes both bounds. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1022. □

Lemma 33.104 (The single-side product estimate)**.** *Let $T, T', g > 0$, and let $s_\tau > 0$ be indexed by a set of f places. If*

$$\left(\frac{\sqrt{TT'}}{g}\right)^f \prod_\tau s_\tau \leq 1,$$

then

$$\prod_\tau \min(T' s_\tau^2, T') \leq \left(g\sqrt{T'/T}\right)^f \leq \left(4g\sqrt{T'/T}\right)^f.$$

Proof. For $0 < s \leq 1$, one has $T's^2 \leq T's$; for $s \geq 1$, one has $T' \leq T's$. Thus $\min(T's^2, T') \leq T's$. Put $q = \sqrt{TT'}/g$, so $T' = g\sqrt{T'/T}\, q$. Multiplication gives

$$\prod_\tau \min(T' s_\tau^2, T') \leq (T')^f \prod_\tau s_\tau = \left(g\sqrt{T'/T}\right)^f q^f \prod_\tau s_\tau \leq \left(g\sqrt{T'/T}\right)^f.$$

The last inequality in the statement follows from positivity. □

Proposition 33.105 (Represented-frame genus mean and switch partner bounds)**.** *Under the hypotheses and thresholds of Proposition 33.103, there exist a constant $c \geq \max(1, C_{Tu})$, determined before the family and point thresholds from C_{Tu} and the rank-independent constants of the switch and partner estimates, and positive local represented-frame factors $\nu_\tau(T_u, T_{u'}, Z) > 0$ at the places τ of the CM type, with product $n_Z = \prod_\tau \nu_\tau(T_u, T_{u'}, Z) > 0$ and switch height $h_Z = b^{c^2 m^3 f} S_c(Z)^4$, with the following properties.*

1. *On a row family of total mass M, for every represented reduced Gram label Z, the inverse-stabilizer-weighted sum of inequivalent first frames of label Z (the represented-frame genus mean) is at most $M h_Z^{cr^2} n_Z$.*
2. *Suppose that for each of Z, Z', multiplying column b by $(-1)^{r-b}$ for $0 \leq b \leq r$ gives a matrix $LM(\gamma)U$ for a block point γ over E and lower and upper unitriangular matrices L, U over E, respectively. For a fixed generic first term t of label Z, sending a generic corank-one partner t' of label Z' in the same row to the integral partner class of its moving frame, as defined in Definition 33.102, is injective. The number of these classes is at most $c^f (h_Z h_{Z'})^{cr^2}$. Consequently, every finite fibre of same-row generic corank-one pairs with these labels has weighted cardinality bounded in the two orders by*

$$M h_Z^{cr^2} n_Z\, c^f (h_Z h_{Z'})^{cr^2}, \qquad M h_{Z'}^{cr^2} n_{Z'}\, c^f (h_{Z'} h_Z)^{cr^2}, \tag{33.33}$$

and hence by their geometric mean $M c^f (h_Z h_{Z'})^{3cr^2/2}\sqrt{n_Z n_{Z'}}$.

3. *Each corank-one term of label Z has squared orbital amplitude per unit row weight and coefficient dimension carrying the inverse factor n_Z^{-1}, and on every pair $(t,t') \in S_{\mathrm{nsc}}$ whose two terms are relevant, at least one side satisfies the small-side saving of Lemma 33.101. Enlarging C above the threshold determined by c and setting $J = C^f H^{Cm^3} \Gamma e^{Cm^6 f L_b} (Cg^{-(2r+1)}k^{-r})^{kf/2}$, every finite subset $U \subset S_{\mathrm{nsc}}$ satisfies*

$$\Gamma_1 \sum_{(t,t')\in U} |\langle v_t, v_{t'}\rangle| \le \max(1, C_0) f_u R' J. \tag{33.34}$$

Outline. Enlarge the counting constants to a common c, glue rank-$(k-1)$ complements via Lemma 33.79, and use frame equivalence and corank-one isometry uniqueness to inject partners and take the geometric mean. Split U via Lemma 33.101 to cancel $n_Z, n_{Z'}$ and gain the small-side saving. Choosing a permitted moment with Lemma 33.61 and summing over labels and rows proves (3). A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1024. □

Lemma 33.106 (Summing the loop pairs)**.** *Under the hypotheses and thresholds of (33.21),* $\mathrm{PairAbs}_{u,u'}(S_{\mathrm{big}} \cup S_{\mathrm{nsc}}) \le e^{-P/3}\Delta_{u,u'}$, *where* $P = kf \log k$.

Outline. Partition $S_{\mathrm{big}} \cup S_{\mathrm{nsc}}$ and apply Proposition 33.103, Proposition 33.105, and Definition 33.7 to majorize the sum via U_s, $\Gamma^2 V_s$, and J. Cap Γ to γ at cost ρ^3 with $6\log(3A) \le P$. Bounding Ξ_s by Lemma 33.11 and Definition 33.7 and summing the decaying bounds over $2 \le s \le k$ gives $e^{-P/3}\Delta_{u,u'}$. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1027. □

Lemma 33.107 (The two flag lattices and their nonzero frequencies)**.** *Let E be the CM field, $f = [E : \mathbf{Q}]/2$, $g = g_0 g_1$, and $\mathfrak{d}$ the absolute different of E. For a block of length s starting at b_0 put $\nu = 2r+3-2b_0-s$. For fractional ideals A, B write $[A,B]_s = \{t \in A : t - (-1)^s \bar{t} \in B\}$. The period lattice, inverse-difference lattice and character dual of the period lattice are respectively*

$$L_{b_0 s} = [g\mathfrak{d}^\nu, g^2\mathfrak{d}^\nu]_s, \quad L'_{b_0 s} = [g^{-1}\mathfrak{d}^{-\nu}, \mathfrak{d}^{-\nu}]_s, \quad L^*_{b_0 s} = \{l \in E : \mathrm{Tr}_{E/\mathbf{Q}}(lt/g) \in \mathbf{Z}\ (t \in L_{b_0 s})\}.$$

Suppose J is a nonzero fractional ideal, L' an additive subgroup of J, and $z_v \ge 0$ with $\prod_v z_v \le \sqrt{N(J)}(a/\Theta)^f$, $a > 0$, $\Theta > 2a$. Here $N(J)$ is the absolute ideal norm and v runs through one embedding of each conjugate pair. With an absolute constant C, every finite $S \subset L' \setminus \{0\}$ satisfies

$$\sum_{l\in S} \prod_v \min\{1, (z_v/|v(l)|)^5\} \le [C(a/\Theta)^{5/2}]^f. \tag{33.35}$$

Outline. Use ideal norms to separate points in dyadic boxes, subdivide into cells with at most one point, and sum the fifth-power bounds. Identify the dual by the trace character and bound its ideal using the containment of $g^2\mathfrak{d}^\nu$ in the period lattice. The complete proof is on p. 1028. □

Lemma 33.108 (The oscillatory integral at a merging pair of blocks)**.** *Let $s = n+1 \ge 2$, and let J_j be the matrix with ones on the reverse diagonal in size j. For $\beta \in \mathbf{C}^\times$ put*

$$\psi_\beta(z) = \exp(i\,\mathrm{Re}(\beta z)), \qquad d\mu_\beta(z) = |\beta|(2\pi)^{-1}\, d\,\mathrm{Re}\, z\, d\,\mathrm{Im}\, z.$$

Use this measure on each independent complex entry. For a strictly upper triangular matrix Y put $\ell(Y) = \sum_i Y_{i,i+1}$. Let Φ be a smooth compactly supported function on the complex s-by-s matrices with values in a complex Banach space. Define

$$L_\Phi(z) = \int \psi_\beta(\ell(Y))\Phi\big(zJ_s(1+Y)\big)\, dY,$$
$$B_\Phi(t,c) = \int \psi_\beta(a_n + \ell(V) + b_n)\Phi\left(\begin{pmatrix}1 & 0\\ a & 1\end{pmatrix}\begin{pmatrix}tJ_n & 0\\ 0 & c\end{pmatrix}\begin{pmatrix}1+V & b\\ 0 & 1\end{pmatrix}\right) da\, dV\, db.$$

Here a is a row of length n, b a column of length n, V is strictly upper triangular of size n, and Y is strictly upper triangular of size s. Entries are indexed from one, so $(aJ_n)_1 = a_n$.

For $D = ct^n$ in a fixed compact subset of $\mathbf{C}^\times$, uniformly as $t \to 0$,

$$|t|^{n^2} B_\Phi(t,c) = \frac{1}{s} \sum_{z^s=(-1)^n D} |z|^{n^2} \psi_\beta(sz/t) L_\Phi(z) + O_{\Phi,D}(|t|).$$

The error constant can be chosen for the entire compact set.

Outline. Use the reversed matrix determinant to eliminate the first diagonal entry and locate critical points at $z^s = (-1)^n D$. Integrate by parts away from the roots and use stationary phase with the determinant derivative and triangular change of variables to get $L_\Phi(z)$. The complete proof is on p. 1029. □

Lemma 33.109 (Boundary pairings in the inverse entry)**.** *Let $s \geq 2$, $|\omega| = 1$, and let χ be a nontrivial continuous additive character of $\mathbf{C}$. Use planar Lebesgue measure. For each of finitely many prefixes p, let $B_p(z)$ be a strongly measurable Hilbert-space vector such that $b_p(z) = s|z|^{s-1} B_p(z)$ is square integrable on compact subsets of $\mathbf{C}^\times$. Set*

$$A_{p,x}(D) = \sum_{z^s=\omega D} \chi(szx) B_p(z).$$

Suppose $\sup_{x\neq 0} \|U_{p,x}\|_2^2 \leq M_p < \infty$ and for each prefix p and compact annulus $K \subset \mathbf{C}^\times$, there is $C_{p,K} \geq 0$ such that, for every $x \neq 0$, $\|U_{p,x}(D) - A_{p,x}(D)\| \leq C_{p,K}|x|^{-1}$ for almost every $D \in K$. To compare fibres at different entries use the unitary change of variable

$$(T_x V)(c) = |x|^{-(s-1)} V(c/x^{s-1}).$$

Then $\|b_p\|_2^2 \leq M_p$. For $x' = x + \sigma$, each of the three pairings

$$\langle T_x A_{p,x}, T_{x'} A_{p',x'} \rangle, \quad \langle T_x U_{p,x}, T_{x'} A_{p',x'} \rangle, \quad \langle T_x A_{p,x}, T_{x'} U_{p',x'} \rangle$$

tends to

$$\int_{\mathbf{C}} \chi(\sigma z) \langle b_p(z), b_{p'}(z) \rangle \, dz.$$

The inner products are linear in the second argument. The original and boundary fibres together have a positive semidefinite block Gram matrix.

Outline. Apply Fourier decay to distinct-root terms in squared annular norms to get global square integrability. Use domination and continuity of dilations to pass to boundary fibres, keep matching-root frequencies, and realize both families in one Hilbert space for positivity. The complete proof is on p. 1029. □

Lemma 33.110 (Regularity of the truncated group coefficient)**.** *Let $G = \mathrm{U}(p,q)$ in coordinates with Hermitian matrix $\mathrm{diag}(1_p, -1_q)$. Put $\|g\|_F^2 = \sum_{ij} |g_{ij}|^2$. Let θ be a smooth real function equal to zero on $(-\infty, 0]$ and to one on $[1, \infty)$. For a smooth coefficient F with values in a finite-dimensional normed space put*

$$F_n(g) = \theta\left(\frac{4 - \|g\|_F^2/(n+1)^2}{3}\right) F(g) \qquad (n \in \mathbf{N}).$$

Then $F_n = F$ for $\|g\|_F \leq n+1$, and F_n has compact support in $\|g\|_F \leq 2(n+1)$. Fix n, an integer $d \geq 0$ and a real A. The quantities $\|g\|_F^A \|L_{\mathbf{u}} R_{\mathbf{v}} F_n(g)\|$ are uniformly bounded over $g \in G$ and all words of left and right invariant derivatives of total length at most d whose directions have norm at most one.

Outline. Checking the cutoff argument gives $F_n = F$ on the inner ball and support in the intersection of G with the outer closed ball, which is compact and avoids 0. Continuity in g, multilinearity in the directions, and compactness yield the uniform bound. The complete proof is on p. 1030. □

Lemma 33.111 (The distinct-word scalar contribution)**.** *Fix the standing parameters of Theorem 33.78 and Proposition 33.18. There exist thresholds $k_0, C_n, C_0', C_{\ell,0}$ such that for every compatible tied model of Theorem 33.78 satisfying the source and size hypotheses of (33.22) together with (33.23), $C' \geq C_0'$, $C_\ell \geq C_{\ell,0}$, $b \geq 16$, $k \geq k_0$, and $8C_n m^5 f \log b \leq k$, the signed generic scalar corank-one sum for distinct words u, v is at most $A\Delta_{u,v}$, where*

$$A = (1+\delta_s)e_1, \qquad 0 \leq \delta_s \leq b^{-7}, \qquad 1 \leq e_1 \leq \exp(12 f\, 2^{-(k-2r_S-2)}).$$

Outline. Choose a differing interior bit and build the threshold scales and actual tower realization. Norm cutoffs put all slice arrays into Lemma 33.88, giving positive kernels, five-node wave and lift blocks, and boundary lattice bounds via Lemmas 33.84 and 33.107; entrywise limits remove the cutoff. Contracting arithmetic majorants via Lemma 33.79, taking the signed endpoint limit, and formal-degree conversion yield the bound. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1030. □

Lemma 33.112 (The finite-type matrix)**.** *Under the selected local hypotheses of 33.78, the finite-type Gram matrix is at most $P_g\Gamma_1 I$ in positive semidefinite order, where $P_g = \prod_{j=0}^{1}(1+3/g_j)^f$. For a cell E_1 with absolute summability and the norm expansion, put $w_i(E_1) = \sum_{t\in E_1:\,\iota(\mathrm{row}(t))=i} v_t$, where ι gives the finite-type index of a row. Then*

$$\|\operatorname{part}(E_1)\|_{\mathrm{HS}}^2 \le P_g\Gamma_1 \sum_i \|w_i(E_1)\|^2.$$

Outline. Diagonalize the local Whittaker convolution using 32.23, 32.24, 32.25, 32.29 and multiplicity one from 32.26. Bound eigenvalues by last-column restrictions, tensor the matrix bounds, and pass to the limit by absolute summability. The complete proof is on p. 1040. □

Lemma 33.113 (The scalar row and the full kernel)**.** *Assume the second formula of Theorem 34.17 and fix the standing parameters of Theorem 33.78 and Proposition 33.18. There exist thresholds $k_0, C_n, C_0', C_{\ell,0}$ such that for every compatible tied model of Theorem 33.78 satisfying the source and size hypotheses of (33.22) together with (33.23), (33.21), $C' \ge \max(C_0', 29)$, $C_\ell \ge \max(C_{\ell,0}, 40)$, $b \ge 16$, $k \ge k_0$, and $8C_n m^5 f \log b \le k$, distinct words u, v satisfy the bound $(1+b^{-6})\Delta_{u,v}$ for the squared Hilbert–Schmidt norms of K_0 and the full normalized Gram operator.*

Outline. Unfold absolutely by 33.25 and discard the identity and zero-Gram terms by local pairings, irrelevance, and Schur's lemma. Use 33.112 and row orthogonality, then combine bounds from 33.111, 33.106, and 33.99 at the size thresholds. The complete proof is on p. 1041. □

Chapter 34

Theta correspondence, test vectors and evaluation

Let E/F be a CM extension with conjugation σ and adele rings $\mathbb{A}_E, \mathbb{A}_F$. For a group H over F write $[H] = H(F)\backslash H(\mathbb{A}_F)$ and let $\mathrm{avg}_{[H]}$ be integration divided by the quotient volume. Write $\mathscr{P}$ for the product F of local theta hypotheses in Definition 1.20, and $\mathscr{J}$ for the conjunction of the five toric assertions in Definition 1.15. At scale $x > 1$ put $L = x^{\varepsilon} = \log b$, with $0 < \varepsilon < 1/52$.

34.1 Matching theta integrals

Let V, V' be Hermitian and W skew Hermitian, of dimensions $a, a', a + a'$ and respective signatures $(a - p, p), (q, a' - q), (a + a' - p - q, p + q)$ at every infinite place. Require

$$a' < a, \qquad a + a' \equiv 0 \pmod 2, \qquad 2(p+q) + 1 < \frac{a - a'}{2}, \qquad 2(p+q) + 1 < a'. \tag{34.1}$$

Take the prescribed Cartan involutions and self-dual lattices with split frames. Assume $E/\mathbf{Q}$ is Galois and E/F is unramified at finite places. The additive character ψ of $\mathbb{A}_F/F$ has infinite part $y \mapsto \exp(-2\pi i \operatorname{Tr}_{F_\infty/\mathbf{R}} y)$. Put $T = S_0 \sqcup Q \subseteq P$, where P consists of the prescribed split places with chosen embeddings. At inert places the residue characteristic is odd and $Nv > 3$, with Nv the residue cardinality of F_v. At inert places and on P, ψ_v has conductor $\mathcal{O}_v$. Fix $v_1 \notin T$ above a rational prime split completely in E.

The unitary Hecke characters α, β restrict to $\eta^{a'}, \eta^a$, where η is quadratic for E/F. Their angular exponents satisfy $m_{\alpha,w} - m_{\beta,w} > a + a'$ at every infinite place w of F. Both characters are unramified off T, and $\nu = \alpha/\beta$ is ramified at both split places of E above every $v \in T$ (so its local conductor exponent satisfies $c_v \geq 1$ for every $v \in T$, and in particular for every $v \in S_0 \subseteq T$). Thus $\nu|_{\mathbb{A}_F^\times} = 1$ and $\nu(\bar{t}) = \nu(t)^{-1}$. The Weil models ω, ω' on $V \otimes_E W, V' \otimes_E W$ use the splitting pairs $(\beta, \beta/\alpha), (\alpha, \alpha/\beta)$.

For a split embedding $t : E \hookrightarrow F_v$, let U_t be the set of primed Schwartz vectors whose image in some allowed barred model for t vanishes on matrices of less than full row rank. For the chosen embedding t_1 at v_1, let $\mathcal{S}'_{t_1}$ be the span of restricted pure tensors whose v_1-factor belongs to U_{t_1}. Put

$$I(\Phi, h) = \int_{[\mathrm{U}(V)]} \theta_\omega(g, h; \Phi)\, dg$$

and define I' with V', ω'. The theta kernels sum the transformed tests over rational points of the Schrödinger spaces. Real polynomial vectors are the span of Heisenberg derivatives of Gaussians.

Matching maps p_∞, p'_∞ on real polynomial vectors and p_v, p'_v on finite Schwartz vectors are surjections onto common nonzero irreducible quotients Π_∞, Π_v. They intertwine the target action, at infinity through its Lie algebra and maximal compact. They kill the finite source differences and the real source Lie action, and they are invariant under the real source maximal compact. Off T the two vacua have the same nonzero

image. At $v \in T$ use split and barred model maps ι, ι' sending vacua to integral-matrix indicators. Put $D = a + a'$, $X_0 = (I_a\ 0)$, $Y_0 = (0\ I_{a'})$, and $w(g) = \nu_v(\det g)|\det g|_v^{D/2}$ at the chosen embedding. Normalize $K_j = \mathrm{GL}_j(\mathcal{O}_v)$ to volume one. In these coordinates the matching images agree when $P(\iota\phi) = P'(\iota'\phi')$, where the two operators are given by

$$\begin{aligned} P(\phi)(h) &= \beta_v(\det h)|\det h|_v^{-a/2} \sum_{\mathbf{n}}^{\mathrm{fin}} \int_{K_a \varpi^{\mathbf{n}} K_a} \phi(gX_0h^{-t})w(g)\,dg, \\ P'(\phi')(h) &= \alpha_v(\det h)|\det h|_v^{a'/2} \int_{\mathrm{GL}_{a'}(F_v)} \phi'(gY_0h)w(g)\,dg. \end{aligned} \tag{34.2}$$

Here $h \in \mathrm{GL}_D(F_v)$, ϖ is a uniformizer, and the first operator uses the finite-support sum over Cartan double cosets indexed by weakly decreasing integer tuples.

A polynomial match has equal images under the tensors of the finite and real maps. The finite tensor includes T and every non-vacuum place. General matches are limits in the real Schwartz topology of polynomial matches with fixed finite factors and primed vectors in $\mathcal{S}'_{t_1}$.

Lemma 34.1 (Hecke nonvanishing at the required integers)**.** *Let χ be a unitary Hecke character of a CM field E. At each infinite place v let m_v be its angular exponent, with local factor $\Gamma_{\mathbf{C}}(s+|m_v|/2)$, and suppose that some $m_v \neq 0$. Then $L(k,\chi)$ is finite and nonzero at every integer $k \geq 1$, and at every integer $k \leq 0$ for which $k + |m_v|/2 > 0$ at all infinite places. At $k \geq 1$ one may omit any finite set of finite Euler factors.*

Outline. The Euler product gives $L(k,\chi) \neq 0$ for $k > 1$, and Proposition 26.29(5) makes $L(s,\chi^{\pm 1})$ entire. If $L(1,\chi) = 0$, the nonnegative Taylor series of $\zeta_E(s)^2 L(s,\chi)\overline{L(\bar{s},\chi)}$ at $s = 2$ contradicts harmonic divergence. Proposition 26.29(5) transfers nonvanishing to allowed $k \leq 0$, and omitting Euler factors at $k \geq 1$ multiplies by nonzero factors. The complete proof is on p. 1043. □

Fix a real place w. Put

$$P = a - p, \quad Q = p, \quad P' = q, \quad Q' = a' - q, \qquad R = P + Q', \quad S = Q + P', \quad D = R + S.$$

Thus the target has signature (R, S). Write

$$c = \frac{m_{\alpha,w} - m_{\beta,w}}{2}, \qquad n = c - \frac{R-S}{2}, \qquad m = c + \frac{R-S}{2}.$$

Evaluating the restrictions of α, β at the idele which is -1 at w and 1 elsewhere gives $m_{\alpha,w} \equiv a'$ and $m_{\beta,w} \equiv a \pmod 2$. Since D is even, c is integral. The inequality $2c > D$ gives $n > S$ and $m > R$.

For a source of signature (b, e), use polynomial coordinates z_{ij} with $b+e$ rows and D columns, in signature order. Denote this polynomial space by $\mathcal{F}_{b,e}$. For a coordinate $l = (i, j)$, define two operators

$$(A_l, B_l) = \begin{cases} (z_l, \partial_l), & \text{if the row and column have equal signs,} \\ (-\partial_l, z_l), & \text{if their signs differ.} \end{cases} \qquad H_{lt} = A_l B_t + \tfrac{1}{2}\delta_{lt}.$$

Here a variable in an operator formula means multiplication by that variable. The source and target matrix units act, before the character twists, by

$$D_1(E_{ih}) = \sum_j H_{(i,j),(h,j)}, \qquad D_2(E_{jk}) = \sum_i H_{(i,j),(i,k)}.$$

These actions commute, as follows from $[B_l, A_t] = \delta_{lt}$. Write $\Theta_{b,e}(h)$ for the quotient of $\mathcal{F}_{b,e}$ by the span of $(D_1(Z) - h\,\mathrm{tr}\,Z)f$, for all source matrices Z and polynomials f. Write $t^{[j]}$ for a string of j coordinates equal to t.

Lemma 34.2 (Generators of the two real theta modules)**.** *Give $T = \Theta_{P,Q}(c)$ the target action D_2, and give $T' = \Theta_{P',Q'}(-c)$ the target action $D_2 + c\,\mathrm{tr}$. Both modules are nonzero. Each is generated by an irreducible compact type of highest weight*

$$\lambda = \big((d+n)^{[P]}, d^{[Q']}\,;\,(m-d)^{[Q]}, (-d)^{[P']}\big), \qquad d = (P-Q)/2.$$

The compact Lie algebra here is $\mathfrak{k} = \mathfrak{gl}_R(\mathbf{C}) \oplus \mathfrak{gl}_S(\mathbf{C})$, with upper triangular positive roots in both blocks.

Outline. Use a Fischer pairing to show that the determinant products survive the source relations. Joint harmonics give the compact types, 1.22 gives generation, and the vacuum shifts and trace twist identify the highest weights. The complete proof is on p. 1044. □

Lemma 34.3 (The first noncompact operators)**.** *In either module of Lemma 34.2, let L denote the generating compact type and let $\mathfrak{p}$ be the two off-diagonal blocks of $\mathfrak{gl}_D(\mathbf{C})$ relative to R, S. Let $C_K = \sum E_{ij}E_{ji}$, summed over ordered pairs in the same block. If C_K acts on L by b_0, every eigenvalue on $\mathfrak{p}L = \operatorname{span}\{Yv : Y \in \mathfrak{p},\ v \in L\}$ is strictly greater than $b_0 + D$.*

Outline. Use the minimum degree from 34.2 to represent $\mathfrak{p}L$ by joint harmonics of degree d_0+2. The possible weight changes increase the compact Casimir eigenvalue by $D + 2(\lambda_i - \lambda_j) > D$. The complete proof is on p. 1045. □

Lemma 34.4 (A common irreducible real quotient)**.** *For the actions of Lemma 34.2, the modules T, T' admit surjections onto a common nonzero irreducible $\mathfrak{gl}_D(\mathbf{C})$-module, intertwining every target matrix unit.*

Outline. By 34.2, the module generated by the graph of the compact-type identification projects onto both modules. Filter by off-block degree and use 34.3 to show that the graph is a full Casimir eigenspace, then quotient by the sum of proper submodules. The complete proof is on p. 1046. □

Lemma 34.5 (Nonzero definite theta integrals)**.** *Assume Theorem 34.16, and keep the triple and Weil models of (34.1) with $p = q = 0$. There exist a real-polynomial Schwartz vector Φ on the unprimed side with $I(\Phi, \cdot) \neq 0$ and a real-polynomial Schwartz vector $\Phi' \in \mathcal{S}'_{t_1}$ on the primed side with $I'(\Phi', \cdot) \neq 0$.*

Outline. Averages a real polynomial from 34.2 with congruence indicators at T and doubles against the constant function to relate the squared theta norm to primitive Hecke L-values. Nonvanishing via 34.1 yields a witness on each side. The complete proof is on p. 1046. □

Lemma 34.6 (Constancy of the reverse lift)**.** *Keep the triple and Weil models of (34.1), let (S, ω_S) be (V, ω) or (V', ω'), and let P be a Schwartz vector on $(S \otimes_E W)(\mathbb{A}_F)$. Let u be a continuous automorphic function on $[\mathrm{U}(W)]$ such that $h \mapsto u(h)\overline{\theta_{\omega_S}(x, h; P)}$ is integrable on $[\mathrm{U}(W)]$ for every $x \in \mathrm{U}(S)(\mathbb{A}_F)$ and*

$$T_P u(x) = \int_{[\mathrm{U}(W)]} u(h)\overline{\theta_{\omega_S}(x, h; P)}\, dh$$

is continuous on $[\mathrm{U}(S)]$. Suppose there is a finite set E_{exc} of finite places of F containing $T \cup \{v_1\}$ such that at every $v \notin E_{\mathrm{exc}}$ above a rational prime split completely in E, $u = \sum_j U_j(\xi_j)$ for finitely many $\xi_j \in \Pi_v$ and $\mathrm{U}(W)(F_v)$-equivariant linear maps $U_j : \Pi_v \to R_v$ from the unramified split matching quotient $\Pi_v = i_{P_{a,a'}}^{\mathrm{GL}_{a+a'}}(\alpha_v \boxtimes \beta_v)$ (with full-rank realization via Lemma 34.9) into a subspace R_v of continuous functions on $[\mathrm{U}(W)]$ integrable against $\overline{\theta_{\omega_S}(x, \cdot; Q)}$ for every Schwartz vector Q on $(S \otimes_E W)(\mathbb{A}_F)$ and every $x \in \mathrm{U}(S)(\mathbb{A}_F)$. Then $T_P u$ is constant on $\mathrm{U}(S)(\mathbb{A}_F)$.

Outline. At good split places $v \notin E_{\mathrm{exc}}$, covariant matrix distributions vanish on singular strata by stabilizer-twist mismatch, giving $\mathrm{U}(S)(F_v)$-invariance of $T_P u$. Strong approximation and Hecke L-function holomorphy at 1 force constancy. The complete proof is on p. 1047. □

Theorem 34.7 (Matching and proportional theta integrals)**.** *Assume Theorem 34.16. Matching maps exist for every triple above. Fix such maps. The integral I converges on Schwartz vectors. For $g \in \mathrm{U}(V')(\mathbb{A}_F)$, let $H_{\mathrm{quot}}(g) = \inf_{\gamma \in \mathrm{U}(V')(F)} \mathrm{ht}(\gamma g)$, using adelic height. For every $\Phi' \in \mathcal{S}'_{t_1}$ and $N \in \mathbf{N}$ there are $C_N \in \mathbf{R}$, $k_N \in \mathbf{N}$ such that*

$$|\theta_{\omega'}(g, h; \Phi')| H_{\mathrm{quot}}(g)^N \leq C_N\, \mathrm{ht}(h)^{k_N} \qquad (g, h \text{ arbitrary}). \tag{34.3}$$

The integral I' converges with moderate growth on this space and is nonzero for some test there. There is a unique $c \in \mathbf{C}^\times$ such that

$$I(\Phi, h) = c I'(\Phi', h) \tag{34.4}$$

for every match and every h.

At $v \notin T$ above a prime split completely in E, the kernels are the local source differences and $p'_v(U_t) = \Pi_v$ for each $t : E \hookrightarrow F_v$. A Schwartz functional killing the primed relations factors through p'_v. The primed Weil action preserves $\mathcal{S}'_{t_1}$. The sign triples in the step formulas below belong to $\{\pm 1\}^3$ and are uniform in the base fields.

For fractional ideals $\mathfrak{b}, \mathfrak{c}$, signs e_1, e_2, s and $z \in \mathbf{Z}$, put

$$\mathfrak{f}(e_1, e_2, s; \mathfrak{b}, \mathfrak{c}; z) = \nu(\mathfrak{b})^{e_1}\nu(\mathfrak{c})^{e_2} \prod_{v \in T} \nu_v(s)\epsilon_v(\psi, \nu_v, z). \tag{34.5}$$

Use the prescribed ideal character and ν_v at the chosen embedding. For uniformizer ϖ_v and conductor exponent c_v, set

$$\epsilon_v(\psi, \nu_v, z) = (Nv)^{-c_v z}\nu_v(\varpi_v)^{c_v} \sum_{u \in (\mathcal{O}_v/\varpi_v^{c_v})^\times} \nu_v(u)^{-1}\psi_v(u/\varpi_v^{c_v}).$$

In the β-step write $a' = b_0 + 2,\ \ell = (a - b_0 - 2)/2$. Keep V and remove adapted planes from V', W at frame positions $(0, b_0 + 1), (a, a + b_0 + 1)$. The lower indices are $(p, q - 1)$. For lower comparison constant c_1,

$$cL^{S_0}(\nu, \ell + 1) = \frac{c_1}{C\,\mathfrak{f}(e_{\beta,1}, e_{\beta,2}, s_\beta; \mathfrak{b}, \mathfrak{c}; -\ell)}. \tag{34.6}$$

In the α-step write $q = 0,\ a = b_0 + 2,\ \ell = (b_0 + 2 - a')/2$. Keep V' and remove planes from V, W at $(0, b_0 + 1)$ on each side. The lower indices are $(p - 1, 0)$, and

$$c = C\,\mathfrak{f}(e_{\alpha,1}, e_{\alpha,2}, s_\alpha; \mathfrak{b}, \mathfrak{c}; 1 - \ell)L^{S_0}(\nu, \ell)c_1. \tag{34.7}$$

For the plane vectors x, x' in Lemma 34.14 use the fractional ideals $\mathfrak{b} = \{s \in E : sx \in \Lambda_{\mathrm{source}}\}$ and $\mathfrak{c} = \{s \in E : sx' \in \Lambda_W\}$. The L-function $L^{S_0}(\nu, \cdot)$ omits places above S_0; because ν is ramified above every $v \in T \supseteq S_0$, every primitive Euler factor above S_0 equals 1, so $L^{S_0}(\nu, s) = L_E(\nu, s)$ identically. The fixed models, measures, domains, planes and real maps determine $C \neq 0$ before the characters, Weil actions and finite matching maps are chosen.

Outline. Builds local matching maps via 34.2, 34.4, and 1.21. Inducts on $p + q$, proving $p = q = 0$ via 34.5, 34.6, and Petersson proportionality, and positive $p + q$ via Levi reductions and two-step contraction. The complete proof is on p. 1047. □

Lemma 34.8 (The Petersson norm of a single theta integral)**.** *Keep the triple preceding Theorem 34.7. For any Haar measure and rational fundamental domain on the target, $I(\Phi, \cdot)$ is square integrable for polynomial tests, and $I'(\Phi', \cdot)$ is square integrable for polynomial tests in $\mathcal{S}'_{t_1}$. Both integrals converge absolutely and are continuous. At fixed allowed finite factors they depend continuously on the real Schwartz factor, locally uniformly in the target. The primed kernel satisfies (34.3) and the primed integral has moderate growth.*

Outline. Apply Poisson summation and count on the Siegel sets of 1.22. Pair tests with their reflections so full-rank minors give chamber decay, then integrate against chamber densities for convergence and square integrability. The complete proof is on p. 1051. □

Lemma 34.9 (Source differences on split matrix spaces)**.** *In either barred split realization put $k = F_v,\ d = a + a'$, and let n be the source dimension. Let $\mathcal{S} = C_c^\infty(\mathrm{Mat}_{n,d}(k))$ and let $\mathcal{S}^\circ$ be supported on rank n. Write $A(U)$ for the span of $\omega(g, 1)\phi - \phi$ with $\phi \in U,\ g \in \mathrm{GL}_n(k)$. If the character ζ in the action*

$$\phi(Y) \longmapsto \zeta(\det g)|\det g|^{d/2}\phi({}^t gY)$$

is nontrivial, then the natural maps are isomorphisms

$$\mathcal{S}^\circ/A(\mathcal{S}^\circ) \longrightarrow \mathcal{S}/A(\mathcal{S}) \longrightarrow \Pi_v.$$

The second map is induced by the matching map. At a completely split good place the character is nontrivial for either embedding. At the selected primed place $\mathcal{S}^\circ$ corresponds to $\operatorname{span} U_{t_1}$.

Outline. Row averaging and scaling give full-rank representatives. Weighted fibre integration realizes the coinvariants as an induction irreducible by Proposition 26.24, and the Hecke law at split primes gives non-triviality of the good-place character. The complete proof is on p. 1053. □

Lemma 34.10 (Descent of the two integrals)**.** *Fix any matching maps for the triple preceding Theorem 34.7. An unprimed polynomial test matching zero has zero integral. A primed polynomial test matching zero has zero integral if it belongs to $\mathcal{S}'_{t_1}$. On these domains each integral is determined by its image in the common restricted tensor product.*

Outline. Use the Petersson radical from 34.8, split coinvariants from 34.9, and compact averaging with 34.20. At infinity extend orthogonality from the generating type of 1.22 by Lie adjointness, then descend factor by factor. The complete proof is on p. 1054. □

Lemma 34.11 (Real vanishing after zero evaluation)**.** *Use compatible models over the same good base, with source and target signatures $(r,s),(r',s')$ and target dimension $d_1 = r' + s'$. Let m_w be the source exponent at w. If $|m_w| > d_1$ everywhere at infinity and $\min(r',s') < \min(r,s)$, the source integral vanishes on full-rank tests. If also $d_1 + 2\min(r,s) < 2(r+s)$, it vanishes on all Schwartz tests.*

Outline. Apply the continuous invariant Schwartz functional $\mathcal{L}$ from Lemma 34.8 to the differentiated Fock action $d\omega_{\mathrm{pol}}(Y)P$; nonpositive derivative weights force annihilation on rank-one compact elements, and polarization completes the vanishing. The complete proof is on p. 1055. □

Lemma 34.12 (Doubling a matrix coefficient in the definite case)**.** *Keep the matched models of Theorem 34.7 with $p = q = 0$. Write $G = \mathrm{U}(W), H' = \mathrm{U}(V')$, and use the Petersson product linear in its first argument. Give G any Haar measure and rational fundamental domain. Let Y be the polynomial matches $y = (\Phi, \Phi')$ with $\Phi' \in \mathcal{S}'_{t_1}$, and put $J_y = I'(\Phi', \cdot)$, $F_y = sI(\Phi, \cdot) + tJ_y$ for fixed $s, t \in \mathbf{C}$. If*

$$\langle F_y, F_z \rangle_{[G]} = \lambda \langle J_y, J_z \rangle_{[G]} \qquad (y, z \in Y),$$

then for each polynomial primed test P and $y \in Y$,

$$\|T_P F_y\|^2_{[H']} = \lambda \|T_P J_y\|^2_{[H']}, \qquad T_P f(x) = \int_{[G]} f(h) \overline{\theta_{\omega'}(x, h; P)}\, dh.$$

Outline. Unfold the doubled pairing into matrix coefficients and use Petersson proportionality to compare them. Continue to the definite Siegel–Weil point by 34.16, where Fubini gives the squared reverse-lift norms with a common measure factor. The complete proof is on p. 1055. □

Lemma 34.13 (Removing a hyperbolic plane)**.** *For $q > 0$ the β-planes can be made adapted by choosing Cartan involutions and split frames. Their complements have indices $(p, q-1)$. For $q = 0 < p$ the α-step gives $(p-1, 0)$. Removal on the V-side is possible for every $p > 0$ with q fixed.*

Outline. Use Hasse–Minkowski to find rational isotropic lines and lattice duality with a trace-one correction to split off isotropic planes. Choose compatible frames and Cartan involutions and check the resulting indices, sizes and character restrictions. The complete proof is on p. 1056. □

Lemma 34.14 (The two Levi identities)**.** *Fix an adapted step and the upper and lower real maps. Assume polynomial comparison and nonzero polynomial witnesses on both sides at smaller index sums, with the primed witness in $\mathcal{S}'_{t_1}$. Let c_1 be the lower comparison scalar. Write $A + B, A' + B'$ for the constant terms along the removed target line, in order of the characters $\alpha|\cdot|_E^{a'/2}, \beta|\cdot|_E^{a/2}$ on $\mathbb{A}_E^\times$. Put $z = \ell + 1$ for β and $z = \ell$ for α, and set $S = \mathfrak{f}(e_1, e_2, s; \mathfrak{b}, \mathfrak{c}; 1-z) L^{S_0}(\nu, z)$ with uniform signs. There is $C \neq 0$ fixed before the characters, Weil actions and finite maps such that at each Levi lift m of $(1, h_1)$ from the lower group,*

$$\begin{aligned} \beta: &\quad CSB(m) = c_1 B'(m) \quad \text{for polynomial matches with } \Phi' \in \mathcal{S}'_{t_1}, \\ \alpha: &\quad B' = 0, \quad A(m) = CSc_1 A'(m) \quad \text{for every polynomial match.} \end{aligned}$$

Some unprimed polynomial test has $B(m) \neq 0$ in the β-step. Some polynomial match has $A'(m) \neq 0$ in the α-step.

Outline. Unfolds the line constant term into zero evaluation and a mixed-polarization cone integral, keeping full-rank support at v_1 via 34.9. Local factorization computes Euler and Gauss factors on standard tests; restriction uniqueness and lower comparison extend them. The complete proof is on p. 1056. □

Lemma 34.15 (A line test for theta-integral differences)**.** *Set*

$$\Sigma = \{I(\Phi) - I'(\Phi') : \Phi, \Phi' \text{ polynomial}, \ \Phi' \in \mathcal{S}'_{t_1}\}.$$

For nonzero totally isotropic rational $U \subset W$, let C_U be the average over the radical of its stabilizer. If $f \in \Sigma$ has zero constant term on the line of one rational hyperbolic plane, then $C_U f = 0$ for every such U. If also $p + q > 0$, then $f = 0$.

Outline. Inducts on $\dim_E U$, killing Whittaker coefficients via shear stabilizers ($j \geq 3$) and smoothed 34.8 bounds ($j = 2$). For $p + q > 0$, shows f is orthogonal to both integrals via 34.9, 34.10, 34.6, and 34.5. The complete proof is on p. 1059. □

34.2 The integral formulas used in the construction

Theorem 34.16 (Doubling continuation and the definite Siegel–Weil value)**.** *Let Y be Hermitian or skew Hermitian of dimension $n \geq 1$ over the good base, with common signature and the prescribed Cartan data, self-dual lattice and split frames. Fix a doubling frame, with Iwasawa height a and standard maximal compact K. Let χ be a continuous Hecke character of $\mathbb{A}_E^\times$, trivial on $E^\times$, with $\chi|_{\mathbb{A}_F^\times} = \eta^c$, $c \in \mathbf{N}$. For continuous right K-finite f satisfying $f(px) = \chi(\det t)|\det t|_E^{s_1+n/2} f(x)$ on the Siegel parabolic with Levi coordinate t, the series*

$$E(s, x) = \sum_{\gamma} a(\gamma x)^{s-s_1} f(\gamma x)$$

over rational Siegel-parabolic cosets is holomorphic in a right half-plane. Near each s_0 some $k \in \mathbf{N}$ gives an extension of $(s - s_0)^k E(s, \cdot)$ to continuous functions of moderate growth, locally uniformly near s_0 (uniform N, C on a closed disc $\overline{D(s_0, r)}$). Evaluation by a linear functional bounded on each class $|h(x)| \leq \mathrm{ht}(x)^N$ is holomorphic there.

Let A, Y have opposite Hermitian signs and dimensions $m, n \geq 1$ with the stated model conditions. Suppose A has common definite signature. Take $\chi|_{\mathbb{A}_F^\times} = \eta^m$ and compatible doubling data with functional l. For K-finite Schwartz Φ, the section $x \mapsto l(\omega(x)\Phi)$ at $s_0 = (m - n)/2$ has holomorphic Eisenstein value

$$E(s_0, x; \Phi) = \begin{cases} \operatorname{avg}_{[\mathrm{U}(A)]} \theta(x, g; \Phi), & n < m, \\ 2 \operatorname{avg}_{[\mathrm{U}(A)]} \theta(x, g; \Phi), & n \geq m \end{cases} \tag{34.8}$$

for any Haar measure and fundamental domain.

Outline. Unitarity of χ and Langlands's continuation give moderate growth on a disc, so Cauchy's formula gives holomorphy in X_N. The second assertion is the anisotropic Siegel–Weil formula. The complete proof is on p. 1061. □

Theorem 34.17 (Convergent Siegel–Weil and Rallis formulas)**.** *For the first and third assertions let A, B be Hermitian of dimensions a, b over the CM extension E/F of the same arithmetic point, and let r_B be the rational Witt index of B. For the first assume*

$$a < b \leq 2a, \qquad a + r_B < b. \tag{34.9}$$

With compatible Iwasawa, Cartan, doubled Weil $(1, \chi_B)$, and covolume-one data, any Haar measure on $\mathrm{U}(B)(\mathbb{A}_F)$, and any allowed oscillator test Φ annihilated by every sufficiently long word of archimedean annihilation operators for the chosen Cartan data, the theta kernel $h \mapsto \theta(g, h; \Phi)$ is integrable at every g, with quotient average $E(g, f_\Phi, s_0)$ at $s_0 = (b - a)/2$.

Over a CM extension take nonsingular Hermitian J, T of sizes M, n with $2n < M$ (so $[\mathrm{U}(J)]$ has finite volume) and $\Phi = \Phi_\infty \prod_{v<\infty} \Phi_v$ (Φ_∞ smooth compactly supported, Φ_v Schwartz–Bruhat, almost everywhere

the integral-tuple indicator in a fixed F-basis of E). For every Haar measure and measurable fundamental domain (the normalization cancelling in the quotient average) and self-dual tuple and matrix measures for a nontrivial character of $\mathbb{A}_F/F$, the Gram pushforwards have continuous nonsingular densities g_∞, g_v with

$$\operatorname{avg}_{[\mathrm{U}(J)]} \sum_{\operatorname{Gram}(X)=T} \Phi(g^{-1}X) = g_\infty(T) \prod_{v<\infty} g_v(T) \tag{34.10}$$

and convergent product.

For the inner product require $a < b \le 2a$ and $r_B = 0$ or $a + r_B < b$, with compatible Weil models ω, ω_d, Iwasawa, Cartan, covolume-one, and source Haar data. Let π be irreducible cuspidal on $\mathrm{U}(A)$ whose lifts to all signatures removing $i \ge 1$ hyperbolic planes from B vanish on the specified smooth forms. For allowed tests Φ_1, Φ_2 (Lemma 34.21, Definition 34.34), require

$$\delta(\Phi_1, \Phi_2)(x) = \Phi_1(\operatorname{res}_L x)\overline{\Phi_2(-\operatorname{res}_R x)}$$

to be K-finite, with the two restrictions indicated. For f_1, f_2 in the specified smooth cuspidal realization, the integrals $\theta_{\Phi_i} f_i(h) = \int_{[\mathrm{U}(A)]} \theta_\omega(g, h; \Phi_i)\overline{f_i(g)}\, dg$ converge in $L^2([\mathrm{U}(B)])$, and

$$\mathrm{CT}_{s=s_0}\, Z_{\chi_B^{-1}}(s, \delta(\Phi_1, \Phi_2), f_1, f_2) = \operatorname{vol}[\mathrm{U}(B)]^{-1} \langle \theta_{\Phi_1} f_1, \theta_{\Phi_2} f_2 \rangle_{[\mathrm{U}(B)]}. \tag{34.11}$$

Here CT *is the constant Laurent coefficient and Z_μ uses $\mu(\det g_2)^{-1}$ at the second group argument.*

Outline. A global rational polarization converts oscillator tests into Schrödinger Schwartz–Bruhat tests; specialize Yamana's convergent Siegel–Weil theorem, Weil's nonsingular orbital density formula, and the convergent Rallis identity via K-finite approximation and the Siegel-cone bound. The complete proof is on p. 1061. □

34.3 Vacuum vectors and separation

Lemma 34.18 (Unramified vacuum generation)**.** *Let L_{loc} be a complete discretely valued field of characteristic zero with integers $\mathcal{O}$, uniformizer ϖ and residue cardinality q^2 for odd $q \ge 2$. Its involution σ preserves valuation and fixes ϖ, and some $u \in \mathcal{O}$ has $\sigma u - u \in \mathcal{O}^\times$. Take Hermitian $J_S \in \mathrm{GL}_n(\mathcal{O}), J_T \in \mathrm{GL}_{n'}(\mathcal{O})$, and $\delta \in \mathcal{O}^\times$ with $\sigma\delta = -\delta$. Let the additive character $\psi : L_{\mathrm{loc}} \to \mathbf{C}^\times$ be trivial on $\mathcal{O}^\sigma$ and nontrivial on $\varpi^{-1}\mathcal{O}^\sigma$.*

On a complex space M take operators $\rho(x)$ for $x \in \mathrm{Mat}_{n'\times n}(L_{\mathrm{loc}})$ and a representation ω of $\mathrm{U}(J_S)\times\mathrm{U}(J_T)$. Write $x^* = {}^t\sigma(x)$ and $H(x, y) = \delta \operatorname{tr}(J_S^{-1} y^* J_T x)$. Assume

$$\rho(0) = 1, \quad \rho(x)\rho(y) = \psi((H(x, y) + \sigma H(x, y))/2)\rho(x + y), \quad \omega(g, h)\rho(x) = \rho(hxg^{-1})\omega(g, h).$$

The integral groups K_S, K_T fix the vectors fixed by all integral translations. Suppose one such vector $s \ne 0$ has translates spanning M. Then

$$M^{K_S} \subseteq \operatorname{span}_{\mathbf{C}}\{\omega(1, h)s : h \in \mathrm{U}(J_T)\}, \qquad M^{K_T} \subseteq \operatorname{span}_{\mathbf{C}}\{\omega(g, 1)s : g \in \mathrm{U}(J_S)\}.$$

Outline. Decompose into lattice character lines and use stabilizer phases to find integral Gram representatives. Isolate the required congruences by averaging vacuum translates, use Witt extension and induction on lattice index for spanning, and apply the adjoint model. The complete proof is on p. 1062. □

Definition 34.19 (Pointwise size inequalities)**.** Use the scale and host of Proposition 14.7. Write $n = f/2$, $r = k + m - 1$, $M = N(m) = k + m(t + 1)$, and put

$$\theta = \varepsilon/100, \quad u = x^\theta, \quad \varepsilon_z = \varepsilon/J, \quad \delta_1 = x^{-\varepsilon/8}, \quad \delta_2 = |T_2|/n, \quad A = 2n\delta_1.$$

Here $J \ge 1$ is the scale parameter, T_2 is the thick set, and TT is the set of top positions. Put $X' = \exp(x^{\varepsilon/3})$, $b' = xb/X'$, $w'' = b'/(xn)$, $X'' = xb/w''$. Write Λ_G for the price parameter, η for the weight-window width,

Δ for the node separation parameter, and $S_c = 4k\Delta$. The constant C_Γ normalizes the member modulus by $\log \Gamma_P = C_\Gamma m^5 f L$. Set

$$A_1 = \frac{kL}{m\delta_2}, \qquad A_* = \frac{mb}{nt^2(M+t)L^2 + kn^2(S_c+t+L)}.$$

The pointwise size hypothesis in Proposition 14.7 consists of the following inequalities. In the first display $y \in [z/u, uz]$ means both endpoint inequalities.

$$n \in [x^{2+8\varepsilon}/u, ux^{2+8\varepsilon}], \qquad m \in [\sqrt{x\delta_1}/u, u\sqrt{x\delta_1}], \qquad k \in [x^{9/2+10\varepsilon}/u, ux^{9/2+10\varepsilon}],$$
$$\max(um^5fL^2, 8C_\Gamma m^5 fL, m^2L, m\sqrt{x}\log x) \le k, \qquad k \le \sqrt{x}\, n^2 x^{-\varepsilon}, \qquad \sqrt{k/m} \le nx^{-\varepsilon},$$
$$um^6L \le k\log k, \qquad um^3 L \log \mathrm{rd}(E) \le k(\log k)^2, \qquad \log f \le \sqrt{m/2-1}, \qquad (\log f)^2 \le m^2 L,$$
$$u\log\log b \le rf\log r, \qquad uL \le f\log k, \qquad f \le \min(m^2 n, r^2), \qquad 1 \le t \le x^{C_t+\theta},$$
$$(Q_S qd)^{36} t^{150} \le \sqrt{b}.$$

For the host primes, the top weights b_s, and the quadratic character χ_d of $\mathbf{Q}(\sqrt{-d})$, require

$$\log q \ge x^{\varepsilon_z/6}, \qquad x^{\varepsilon_z/5-\varepsilon_z/100} \le \min(\log q, \log d), \qquad \max(\log q, \log d) \le x^{\varepsilon/5+\theta},$$
$$\left|\mathrm{Re}\,\frac{L'(1,\chi_d)}{L(1,\chi_d)}\right| \le L, \qquad 1 \le \mathrm{rd}(E) \le Q_S\sqrt{qd}/2, \qquad n\delta_1/u \le |TT| \le A, \qquad L \le \Lambda_G \le uL,$$
$$(4AL + u\log x)\Lambda_G/k \le x^{-2}, \qquad b_s \le b' + AL\Lambda_G b'/k \quad (s \in TT), \qquad x^{-\varepsilon} \le \delta_2, \qquad L_w \le (\log x)^2.$$

Here L_w is the host word-length parameter. The width and spacing inequalities are

$$t \le \eta \le b/(t^3 f k^3), \qquad \log\eta \le x^{\varepsilon/4}, \qquad t\eta \le w''/x, \qquad m(t+1)\eta \le w'', \qquad m(S_c+\eta) \le w''L,$$
$$M\eta \le \delta_2 m w'', \qquad \eta^2 rL \le \delta_2 m (w'')^2, \qquad r(b'+S_c)(\log x)A_1 \le \delta_2 mb,$$
$$\max\{12b/A_* + 2t + 8,\ ktL,\ ur^{19}f^5L,\ 4M^3+2t+6,\ ukm^{9/2},\ 2(r+mt)^3+t+3,\ Mm^{10}\} \le \eta/m,$$
$$\max\{b'+S_c+\eta+b/A_*,\ 2\eta+t+8b/A_*+7\} < A_*,$$
$$\max(Mm^{10}, umM) \le \Delta, \qquad \log\Delta \le x^{\varepsilon/4}/2, \qquad \max(kS_c, umk\Delta, \sqrt{b}) \le w'',$$
$$4kS_c \le \Lambda_G b', \qquad 4uk^2\Delta \le \Lambda_G b', \qquad um\log(\mathrm{rd}(E)/2) \le k.$$

At every infinite position τ require $b_{\tau,m-1} \le 2b_{\tau,0}$; for every two layer weights $w_{j,\tau}, w_{j',\tau'}$, $1 \le j, j' \le m$, require $w_{j,\tau} \le X'' w_{j',\tau'}$. For $i, i' \in \{0,1\}$ require $(kb)^8 \le g_i \le 2g_{i'}$ and $\log g_i \le uL$. Both $\log p_{\mathrm{de}}$ and $\log p'_{\mathrm{de}}$ are at most L. There are at most 2^{k-3} words, and the host cutoff up to which every odd prime splits in $\mathbf{Q}(\sqrt{-d})$ is at least 11.

Corollary 34.20 (Two averages of an inert test)**.** *At an inert place of a triple preceding Theorem 34.7, let H, G be the source and target, K_H, K_G their integral groups, $\mathcal{S}$ the local model and s its vacuum. Put $A = \mathrm{span}_{\mathbf{C}}\{hx - x : h \in H,\ x \in \mathcal{S}\}$. Then $\mathcal{S} = A + \mathrm{span}_{\mathbf{C}}\, Gs$. For each $x \in \mathcal{S}$ there are $k_1, \dots, k_b \in K_G$, $b \ge 1$, $c \in \mathbf{C}$ with $\sum_i k_i x - cs \in A$, and $k_i s = s$.*

Outline. Verify the unimodular frame, unramified character, and vacuum properties at the inert place, then apply 34.18 to both finite averages. The source average keeps x mod A in $\mathrm{span}_{\mathbb{C}}\, Gs$; the target average is a scalar multiple of s mod A. The complete proof is on p. 1063. □

For a real realization R, let $\mathcal{V}$ be the tensor product of the smooth archimedean representation spaces of the bottom row. Let $\mathcal{E}_0$ be the subspace of its automorphic multiplicity maps at the trivial level index satisfying the prescribed level character and fixed by the type idempotent. Write $t(w)$ for evaluation under tensor linearization. Let H be the top definite group, embedded by ι in the last row group. Let N, ψ_N be the Bessel unipotent and character. For real test ρ and grid test Φ, let $F_{\rho,\Phi,u}$ be the successive lift of u. For coefficient action ρ_{coef}, dual ξ and column Ψ, put $\phi_{\xi,\Psi}(h) = \xi(\rho_{\mathrm{coef}}(h^{-1})\Psi(h_f))$. The period is

$$\mathcal{B}_{\rho,\xi,\Psi,\Phi}(u) = \int_{[H]} \int_{[N]} \phi_{\xi,\Psi}(h) F_{\rho,\Phi,u}(n\iota(h)) \overline{\psi_N(n)}\, dn\, dh.$$

The row size conditions comprise the pointwise size inequalities of Definition 34.19, the existence of compatible parameters with weight-gap constant 10, and the following additional bounds. Write $r = k + m - 1$ for the bottom row rank, $f = [F : \mathbf{Q}]$, $N(m) = k + m(t+1)$ for the last chain rank, S_c for the center bound and g_{sep} for the prescribed grid spacing at gap parameter one. The upper bound for ranks across the whole grid is $N_* = k + m(t + r + 7) = k + m(t + k + m + 6)$, where t is the auxiliary rank. The numerical bounds are

$$m^4 \le f, \quad 44 \le m, \quad x^2 m^3 L \log \mathrm{rd}(E) \le k, \quad fmN(m)^2 d \le b, \tag{34.12}$$

$$fm \le (r+1)^3, \quad \log g_i \le 9L + 1, \quad \log q_0 \le L, \tag{34.13}$$

$$\log \mathrm{rd}(E) \le \log \mathrm{cond}(E) \le L/100, \quad S_c + (m+1)g_{\mathrm{sep}} + \tfrac{1}{2} \le b. \tag{34.14}$$

The conductor ideal condition is $\mathfrak{c}_K = \mathcal{O}_{K_d}$ for $K_d = \mathbf{Q}(\sqrt{-d})$. At each infinite place and weight position l, $b_{\tau,l} > 0$, $e_{\tau,l} \le 4b$.

Lemma 34.21 (Reconstruction at a fixed Heisenberg level)**.** *Let S, T be two spaces in a grid Weil pair over E/F. Put $X = \mathrm{Hom}_E(S,T)$, and write $X_{\mathbb{A}}$ for its adelic points. Let $[x,u]$ be the alternating form of the pair and ψ_0 its standard additive character of sign -1. Write*

$$H(u)f(x) = \psi_0([x,u]/2) f(x+u).$$

A theta test in this model is a continuous function on $X_{\mathbb{A}}$, smooth at infinity, fixed by $H(U)$ for some open finite adelic lattice U, and satisfying

$$f(x+\gamma) = \psi_0(-[\gamma,x]/2) f(x) \qquad (\gamma \in X(F)).$$

Here and below $X(F)$ denotes the rational points of the underlying F-space. Use Haar measure for which $X(F)$ has covolume one. The pairing (f,g) is the integral of $f\overline{g}$ on $X(F)\backslash X_{\mathbb{A}}$.

For a finite adelic lattice $\Gamma \subset X_{\mathbb{A}}$, let $D(\Gamma)$ consist of the theta tests fixed by $H(2v)$ for every $v \in \Gamma$, and by $H(v)$ whenever $v \in \Gamma$ is supported on one factor above a split finite place of F. Let Γ° consist of the $u \in X_{\mathbb{A}}$ such that $\psi_0([u,2v]) = 1$ for the first set of v and $\psi_0([u,v]) = 1$ for the second.

Suppose that L is linear on theta tests, that $u \mapsto L(H(u)f)$ is continuous and bounded for each test f, and that

$$L\left(\int a(u) H(u) f \, d\nu(u)\right) = \int a(u) L(H(u)f)\, d\nu(u)$$

whenever ν is a countable sum of finite measures, $a \in L^1(\nu)$, and the pointwise integral is a theta test. For every nonzero $v_0 \in D(\Gamma)$ and every $f \in D(\Gamma)$,

$$(v_0, v_0) L(f) = \int_{\Gamma^\circ} (f, H(u)v_0) L(H(u)v_0)\, du.$$

Outline. Use the Heisenberg commutator for support and differentiation for decay. Fourier expansion and unfolding recover f up to the vacuum norm squared from its vacuum matrix coefficients, and superposition gives the formula for L. The complete proof is on p. 1063. □

Lemma 34.22 (The prescribed tests detect a depth-fixed period)**.** *Fix a row arrow i, $0 \le i < m$. Write $S = Y_{i+1}^{\mathrm{col}}$, $T = Y_i^{\mathrm{col}}$ for the column-0 spaces at grid nodes $(i+1,0),(i,0)$ ($Y_0^{\mathrm{col}} = Y_0$) and Λ_S, Λ_T for their lattices. Put $M = \mathrm{Hom}_{\mathcal{O}_E}(\Lambda_S, \Lambda_T)$. Let e select the marked factor at each of the split depth places, and let $\mathfrak{m}$ be the product of the selected prime powers with their grid depths. The integral oscillator lattice Λ_X is the finite completion of the lattice generated by the maps*

$$s' \longmapsto a\, t\, h_S(s,s'), \qquad s \in \Lambda_S, \quad t \in \Lambda_T, \quad a \in (\mathfrak{c}_K \mathcal{O}_E)\mathfrak{b}^{-1},$$

where $h_S(s,s') = \bar{s}^{\,t} J_S s'$ in the grid frame, $\mathfrak{c}_K$ is the host conductor ideal and $\mathfrak{b}$ is the chosen factor of the relative different. The row depth lattice is

$$\Gamma_i = \{u \in \Lambda_X : eu \in \widehat{\mathfrak{m}M}\}.$$

For $\lambda \in M/\mathfrak{m}M$, let v_λ be the coset vacuum in the oscillator model for $(-S,T)$. Its selected split factor is the indicator of $e\widehat{\mathfrak{m}M} + e\lambda$ on $eX_{\mathbb{A}}$; elsewhere one uses the grid vacuum. For a complex polynomial P in the real-place creation operators a^+, put

$$E_i(\lambda, P) = \overline{P(a^+)v_\lambda}.$$

Conjugation identifies $(-S,T)$-tests with (S,T)-tests. Under the host and grid coprimality of the depth primes $q_0, g_0, g_1, p_{\mathrm{de}}, p'_{\mathrm{de}}$ to $Q_{SQ}\,\mathrm{N}\mathfrak{c}_K$ (namely $\gcd(p_{\mathrm{de}}, Q_{SQ}) = \gcd(p'_{\mathrm{de}}, Q_{SQ}) = 1$ together with the host and grid coprimalities of Definitions 34.46(1) and 34.47(1)), every $E_i(\lambda, P)$ belongs to $D(\Gamma_i)$. For every L as in Lemma 34.21,

$$\big(\forall \lambda, P \quad L(E_i(\lambda, P)) = 0\big) \quad \Longrightarrow \quad \big(\forall f \in D(\Gamma_i) \quad L(f) = 0\big).$$

Outline. Finite translations permute the coset vacua up to scalars. At infinity the holomorphic vacuum translates and reproducing integral extend vanishing to all real translates, so 34.21 gives vanishing on the depth space. The complete proof is on p. 1064. □

Lemma 34.23 (Exchange of periods on a column chain)**.** *Fix a grid and the coprimalities of Lemma 34.22. Use the row measures and any core Haar measures and fundamental domains. Write Y_i^{col}, V_i for the column-0 space at $(i,0)$ ($Y_0^{\mathrm{col}} = Y_0$) and core at height i. Let $F_0, \dots, F_m$ be a chain as in Lemma 34.33 below, with its depth condition on every downward test. Let u be a bottom row form whose partial row lifts with the prescribed tests are rapidly decreasing, including u. Write $j : \mathrm{U}(Y_0^{\mathrm{col}}) \to \mathrm{U}(V_0)$ for the stabilizer embedding. If*

$$\int_{[\mathrm{U}(Y_0^{\mathrm{col}})]} F_0(j(h))u(h)\,dh \neq 0,$$

then there are polynomials P_i and cosets λ_i at all $0 \le i < m$ such that the top Bessel period of F_m against the row lift with these tests is nonzero.

Outline. Use 34.21 to recover a nonzero bottom Fourier–Jacobi period and propagate nonvanishing by unfolding and Fourier uniqueness. Use 34.22 to select the prescribed tests at each step, then conjugate at the top to obtain the Bessel period. The complete proof is on p. 1065. □

Definition 34.24 (Finite-group series at the two theta primes)**.** For $g = g_0, g_1$, a labeling consists of a multiplicative map from $\overline{\mathbf{F}}_g$ to $\mathbf{C}$, injective on nonzero elements, and generators of $\mathbf{F}_{g^a}^\times$ for $a \ge 1$ compatible under norm. Use the finite general linear group series of Definition 1.20. The series condition requires, for each g, a labeling with the prescribed Teichmüller character such that, for every polynomial $P \in \mathbf{F}_g[X]$, the given series element in $\mathbf{C}[\mathrm{GL}_r(\mathbf{F}_g)]$ equals the sum of the central character idempotents of the irreducibles labeled by P.

Theorem 34.25 (Real functionals separate rows)**.** *Assume $\mathscr{P}$ together with the absolutely convergent unitary Siegel–Weil formula for $b > 2a$ [48], the unramified local doubling zeta-integral formula for both split and inert unitary groups [150, §4.2.1][83, p. 334, Remark 3], Harish–Chandra finiteness [98], Lemma 34.18, Lemma 34.31 (via [156]), and the first formula of Theorem 34.17. For every $\varepsilon \in \mathbf{R}$, $C_F \ge 6510$ there is x_0 such that the following holds for data of exponent ε at $x \ge x_0$ satisfying the row size conditions above and the series condition of Definition 34.24. For a real realization R, require the prescribed level characters at the base index. Identify its finite group with the product of the residue-field matrix groups at the chosen places above g_0, g_1, so that reduction agrees componentwise, each place lies over the prime selected by the frame, and the identification carries the embedded coefficients of the level idempotent to those of the type idempotent. Every such realization with a grid at $(1, C_F, b)$ (Definition 34.47), its Weil system and compatible row measures satisfies*

$$\forall w \in \mathcal{V} \setminus \{0\} \quad \exists \rho, \xi \quad \forall t \in \mathcal{E}_0 \setminus \{0\} \quad \exists \Psi, \Phi \quad \mathcal{B}_{\rho,\xi,\Psi,\Phi}(t(w)) \neq 0$$

with convergence of the lifts and integrations.

Outline. Use 34.30 and 34.33 to choose a nonzero column chain, with cuspidality and decay from 9.5 and 1.22. Apply 34.23 and use real Bessel multiplicity one to choose a real pair that works for every nonzero row map. The complete proof is on p. 1069. □

Lemma 34.26 (The two factors in the lowering projection)**.** *Let $G = \mathrm{U}(p,q)$, let z be a positive coordinate unit vector in a diagonal signature frame, and let $H = \mathrm{U}(p-1,q)$ be its stabilizer. Put $n = p+q$, and suppose $p-1 \geq q \geq 1$ and $n \geq 4$. Let π, σ be discrete series of G, H with decreasing Harish–Chandra parameters λ, μ. Suppose the entries of both parameters have absolute value at most B, consecutive entries in each list differ by at least two, and entries of opposite signs in either signed list differ in absolute value by at least $1000n^3$. Put $X = 8nB$. Write $e_l u$ for the tuple with u in coordinate l and zero elsewhere.*

On the smooth vectors of $(\mathbf{C}^n)^\vee \otimes \pi$, let p_s denote the projection to the discrete series π_s of parameter $\lambda - e_s$, where e_s in a parameter denotes the s-th standard basis vector. These projections sum to the identity. There are component maps R_s to the unitary models of π_s and nonzero real constants c_s with the following properties. For every lowest compact type unit vector v of π, every l, and every bounded H-intertwiner $P : \pi \to \sigma$,

$$|c_s|\,\|R_s(e_l v)\|^2 \leq X^{4n}, \qquad P\big((p_s(e_z u))_z\big) = \gamma_s P u \quad (u \in \pi^\infty), \qquad \gamma_s = \frac{\prod_{a=1}^{n-1}(\mu_a - \lambda_s + \frac{1}{2})}{\prod_{j \neq s}(\lambda_j - \lambda_s)}.$$

Here $e_z u$ means insertion in the coordinate of z. In particular $|\gamma_s| \leq X^n$. For the same Haar measure on G, the formal degrees satisfy $d_\pi / d_{\pi_s} \leq X^n$.

For later use, if $D(A)$ is the complexified derived action of a matrix A and v is as above, then

$$\|D(A_1)\cdots D(A_t)v\| \leq \big(8\sqrt{n}(B+S)\big)^t \quad \text{if } t \leq S, \quad \|A_j\|_{\mathrm{HS}} \leq 1.$$

Outline. Decompose the tensor product by the coherent character identity and globalize by 36.9. The Capelli comatrix gives the projections and their stabilizer scalars, while compact Casimir bounds and 1.22 control the norms and formal degrees. The complete proof is on p. 1070. □

Lemma 34.27 (Haar volumes for a bounded column)**.** *Use the real groups and parameters of Lemma 34.26. Give G, H Haar measures and give $H\backslash G$ the corresponding quotient measure. Write $B_H(1) = \{h : \|h-1\|_{\mathrm{HS}} < 1\}$ and*

$$D = \{Hg : \|g^{-1}z\|^2 \leq 2X^{8n}\}.$$

For a lowest H-type unit vector w of σ, its diagonal coefficient is integrable, and

$$\frac{1}{d_\sigma} \leq X^{67n^2}\,\mathrm{vol}(B_H(1)), \qquad d_\pi\,\mathrm{vol}(D)\,\mathrm{vol}(B_H(1)) \leq 4X^{24n^3}.$$

Every coset in D has a representative of Hilbert–Schmidt norm at most X^{10n}.

Outline. Combine Cartan decay from 37.16 with ball coverings and Schur orthogonality from 1.22. Hyperbolic rotations give bounded coset representatives, and 34.26 bounds the coefficient variation on a small ball for the quotient volume estimate. The complete proof is on p. 1071. □

Lemma 34.28 (A column moment for the branching coefficient)**.** *In Lemma 34.26, suppose the two signed parameters satisfy the discrete branching rule that in their merged decreasing list, adjacent signs agree exactly when the entries belong to different lists. Let $i : \sigma \hookrightarrow \pi|_H$ be an isometric intertwiner. For lowest-type unit vectors w of σ and v_0 of π,*

$$\int_G \|g^{-1}z\|^2\,|\langle iw, \pi(g)v_0\rangle|^2\,dg \leq \frac{X^{6n+3}}{d_\pi} \leq \frac{X^{8n}}{d_\pi}.$$

Outline. Use 34.27 and Schur orthogonality to project onto the branching line, using multiplicity one. Apply the tensor projections and bounds of 34.26 to the column-weighted coefficient and combine the squared-integral estimates by Cauchy–Schwarz. The complete proof is on p. 1072. □

Lemma 34.29 (A coefficient for the bottom period)**.** *Let Y be the bottom row of rank r, $V = Y \perp Ez_0$ the bottom core of rank $n = r + 1$, and $j : \mathrm{U}(Y) \to \mathrm{U}(V)$ the stabilizer embedding of the prescribed norm-one z_0. Let $\mathfrak{m}$ be the grid modulus. Under the size and level assumptions of Theorem 34.25, there is an absolute $C \geq 0$ such that eventually for each ε, C_F, uniformly in the grid and discrete-series models, the following holds. Put $L_{\mathrm{loss}} = C f n^4 \log b$. There is a nonzero pure smooth row tensor w_* and a core coefficient k, the product of the real coefficients and the finite core congruence indicator. For every nonzero row map t, put $u = t(w_*)$. For the row Haar measure there is $c \neq 0$ with*

$$\int_{\mathrm{U}(Y)(\mathbb{A}_F)} k(j(y)^{-1} j(h)) u(h)\, dh = c\, u(y). \tag{34.15}$$

For $y \in \mathrm{U}(Y)(\mathbb{A}_F)$, put $N_\tau(z) = \|j(y)_\tau^{-1} z_\tau\|$ in the adapted Euclidean frames. For rational $\gamma \in \mathrm{U}(V)(F)$ and $z = \gamma z_0$,

$$\int_{\mathrm{U}(Y)(\mathbb{A}_F)} |k(j(y)^{-1}\gamma j(h))|\, dh \prod_\tau N_\tau(z)^{2n+5} \leq e^{L_{\mathrm{loss}}} |c|. \tag{34.16}$$

The integrand vanishes unless $z - z_0 \in \mathfrak{m}\Lambda_{j(y)}$, where the lattice has finite completion $j(y)_f \widehat{\Lambda}_V$. For X in the row Lie algebra in one adapted real frame,

$$|u(y \exp(sX)) - u(y)| \leq e^{L_{\mathrm{loss}}} \sup |u|\, |s|\, \|X\|_{\mathrm{HS}}. \tag{34.17}$$

Outline. Use discrete branching, the column moment 34.28 and the volumes 34.27 to find a bounded translate with large row projection. A compact operator gives a reproducing eigenvector, with decay from 37.16 and support from the finite congruence indicator. The complete proof is on p. 1072. □

Lemma 34.30 (A bottom period at the prescribed vector)**.** *For each $\varepsilon \in \mathbf{R}$ and $C_F \geq 6510$ there is a threshold for grids satisfying the row size conditions of Theorem 34.25. For any bottom row and core models of the prescribed parameters, any nonzero row multiplicity map t at row level and any nonzero w in its smooth tensor space, there are a core multiplicity map y at core level and an a in the span of pure local K-finite tensors with*

$$\mathrm{Per}_0(y(a), t(w)) = \int_{[\mathrm{U}(Y)]} y(a)(j(h)) t(w)(h)\, dh \neq 0.$$

Outline. Use 34.29 and cuspidality to rule out short lattice vectors where the row form has large modulus. Unfold the translated Poincaré series so congruence depth makes the identity term dominate, then use cyclicity and compact-type approximation. The complete proof is on p. 1074. □

Lemma 34.31 (The inert spherical Hecke identity in even ranks)**.** *Let L/K be an unramified quadratic extension of nonarchimedean local fields of characteristic zero, with Q elements in the residue field of K, Q odd. Take the split skew-Hermitian space of dimension $2a$ and the split Hermitian space of dimension $2b$ over L, with $1 \leq a \leq b$. Use self-dual lattices, an additive character of K of conductor $\mathcal{O}_K$, and trivial splitting characters. Normalize Haar measure on each group so that the lattice stabilizer has volume one. Let ω be the Weil representation and ϕ_0 its lattice vacuum.*

Write $\mathcal{H}_a, \mathcal{H}_b$ for the spherical Hecke algebras and $\mathcal{S}_a, \mathcal{S}_b$ for their normalized Satake transforms. Use coordinates in which the base-change roots in dimension $2a$ are $t_1, t_1^{-1}, \ldots, t_a, t_a^{-1}$. For $h' \in \mathcal{H}_b$ there is an $h \in \mathcal{H}_a$ with

$$\omega(1, h')\phi_0 = \omega(h, 1)\phi_0.$$

For every such h, with $c = b - a$,

$$(\mathcal{S}_a h)(t_1, \ldots, t_a) = (\mathcal{S}_b h')(Q, Q^3, \ldots, Q^{2c-1}, t_1, \ldots, t_a).$$

The list preceding the t_j's is empty for $c = 0$.

Outline. Use 34.18 and compact averaging to turn the target action on the vacuum into source convolution. Liu's spherical-module theorem gives the Satake substitution adjoining $Q, Q^3, \ldots, Q^{2(b-a)-1}$. The complete proof is on p. 1076. □

Lemma 34.32 (Spherical parameters in the even core chain)**.** *Consider a gap-one grid and admissible level chain with Weil system and row measures, core ranks $n = r+1,\ d = r+5,\ n_i = n+id$ $(0 \le i \le m,\ r$ odd), and exceptional set S consisting of the base place above q_0 and the places of F above $g_0 g_1 p_{\mathrm{de}} p'_{\mathrm{de}}$. The two splitting characters in step i are $\widetilde{\chi}_i = \chi_{m-i}^{-1}$, with χ_{m-i} the grid character. At $w \mid v \notin S$, the base-change Satake multiset $B(w)$ of a nonzero conjugate bottom spherical cuspidal eigenfunction satisfies*

$$|B(w)| \le n, \qquad |\beta| \le (Nw)^{(n-1)/2} \quad (\beta \in B(w)). \tag{34.18}$$

If the conjugate of a cuspidal value at stage i has multiset $B_i(w)$, the conjugate of its nonzero upward vacuum lift has multiset

$$B_{i+1}(w) = B_i(w) \sqcup \{\widetilde{\chi}_i(w)(Nw)^{(d-1)/2-l} : 0 \le l < d\}.$$

Outline. Cuspidal decay and the L^1-norm bound for spherical convolution give the trivial Satake bound (34.18) at split and inert places. For the upward vacuum lift, moving Hecke convolutions across the Weil pairing computes $B_{i+1}(w)$ by Gaussian-binomial lattice counting at split places and Lemma 34.31 at inert places. The complete proof is on p. 1076. □

Lemma 34.33 (Core values spanned by column chains)**.** *Assume the local inputs $\mathscr{P}$ of Theorem 34.25 together with the absolutely convergent unitary Siegel–Weil formula for $b > 2a$ [48], unramified local doubling for both split and inert unitary groups [150, §4.2.1][83, p. 334, Remark 3], Harish–Chandra finiteness [98], Lemma 34.18, Lemma 34.31, and the first formula of Theorem 34.17. Require $p_{\mathrm{de}}, p'_{\mathrm{de}}$ coprime to $Q_S q$, $r > 1800$, and $b_{\tau,j+1} - b_{\tau,j} > 2N_*^3$ at every infinite place τ for $j+1 < m$. Use a gap-one grid, an admissible level chain, its Weil system and row measures. Choose Haar measures and fundamental domains for the core groups. For every bottom core model of the prescribed parameters and Cartan data, core multiplicity map y at core level and K-finite tensor a, the value $y(a)$ belongs to the span of the following F_0. There are rapidly decreasing F_i on the core quotients with*

$$F_m = \phi_{\xi,\Psi}, \quad \Psi \in \mathcal{C}_{\mathrm{top}}, \qquad F_i(g) = \int_{[\mathrm{U}(V_{i+1})]} \theta_i(h, g; P_i) F_{i+1}(h)\, dh \quad (0 \le i < m).$$

Here $\mathcal{C}_{\mathrm{top}}$ is the space of automorphic forms on the definite top group with the prescribed coefficient representation, top level and trivial level character. Each test P_i is fixed by the depth-lattice Heisenberg translations, with selected components in $\mathfrak{m}\widehat{\mathrm{Hom}}(\Lambda_S, \Lambda_T)$ for the pair's source and target lattices.

Outline. Decomposes the bottom multiplicity space into Hecke eigenvectors and shows each upward lift nonzero via doubling, 34.17, Siegel–Weil, and 34.1. Distinct Satake multisets give downward surjectivity, and Lie stability spans K-finite vectors. The complete proof is on p. 1077. □

34.4 Twisted square models

Fix an arithmetic point with CM extension E/F, and choose its realization θ at the infinite places. Let W, W_1, X have ranks $u, v, d = u+v$, with d even and signatures $(u-p, p), (v-q, q), (d-p-q, p+q)$. Require $v + 2(2(p+q)+1) < u$ and $2(p+q)+1 < v$. The characters α, β restrict to η^v, η^u and have rational exponents a_τ, b_τ with $2(a_\tau - b_\tau) < -d$. At each finite component both characters are unramified or their quotient is ramified at a split component prime to the different of $F/\mathbf{Q}$. Write $\mathcal{P}(W, X)$ for polynomial Fock vectors. Algebraic Fock functions are finite sums of $\mathrm{Fock}(f, P)$ with permitted finite factors f and real polynomials P. Let $\mathcal{R}_{\mathrm{fin}}$ be the finite source differences and $\theta\,\mathrm{Int}_\omega$ the source theta integral.

Definition 34.34 (Twisted-square triples, seesaw transfer and adjoint model identifications)**.** Write $x \mapsto \bar{x}$ for conjugation on E/F and $\xi_0 \in E^\times$ $(\bar{\xi}_0 = -\xi_0)$ for the host imaginary scalar, a unit at every inert place of E/F satisfying $\mathrm{Im}\,\tau(\xi_0) > 0$ for every $\tau \in \theta$. Assume W, W_1, X carry Cartan involutions and self-dual $\mathcal{O}_E$-lattices. A tuple $(W, W_1, X, \alpha, \beta)$ satisfying the rank, signature, exponent and ramification conditions above (where the unitary Hecke characters α, β act on the unit circle at each $\tau \in \theta$ with integer angular exponents $2a_\tau, 2b_\tau$) is called a *twisted-square triple*.

Fix frames (A, A_1, B) for (W, W_1, X) with Hermitian Gram matrices J_W, J_{W_1}, J_X, and write $H_X(h)$ for the adelic matrix height of $h \in \mathrm{U}(X)(\mathbb{A}_F)$ in frame B. For Hermitian spaces M, X over E/F with

frames (A_M, B) (in particular for $M \in \{W, W_1\}$), an oscillator Weil representation ω of sign $\epsilon \in \{+,-\}$ and splitting pair (χ_M, χ_X) in the model of Lemma 34.21 carries a continuous additive character ψ of $\mathbb{A}_F/F$ of sign -1 (infinite part $y \mapsto \exp(-2\pi i \operatorname{Tr}_{F_\infty/\mathbf{R}} y)$), acts on functions on $\operatorname{Hom}_E(M,X)(\mathbb{A}_F)$ through J_M for $\epsilon = +$ and $-J_M$ for $\epsilon = -$, and has theta kernel $\theta_\omega(g,h;\Phi) = (\omega(g,h)\Phi)(0)$ and source theta integral $\theta \operatorname{Int}_\omega(\Phi)(h) = \int_{\mathcal{F}} \theta_\omega(g,h;\Phi)\, d\nu(g)$ for a Haar measure ν and $\mathrm{U}(M)(F)$-fundamental domain $\mathcal{F}$ on $\mathrm{U}(M)(\mathbb{A}_F)$. Permitted tests Φ are theta tests in the model of Lemma 34.21, and permitted finite functions f are smooth oscillator theta functions annihilated by the archimedean Fock annihilation operators for $(\epsilon M, X)$ in frames (A_M, B). For $P \in \mathcal{P}(W,X)$ and $P' \in \mathcal{P}(W_1,X)$, write $\mathrm{Fock}(f,P)$ and $\mathrm{Fock}'(f',P')$ for the Fock functions of signs $+$ and $-$; their complex spans are the unprimed and primed algebraic Fock spaces, on which $\mathcal{R}_{\mathrm{fin}}$ denotes the complex span of finite-adelic source differences $\omega(u_f,1)\Psi - \Psi$. *Matching data* $(\omega, \omega', \Pi, p, p')$ for ψ consist of oscillator Weil representations ω, ω' for $(W,X), (W_1,X)$ of signs $+,-$ and splitting pairs $(\beta, \beta/\alpha), (\alpha, \alpha/\beta)$, a nonzero complex vector space Π jointly irreducible under the commuting actions of $\mathrm{U}(X)(\mathbb{A}_{F,f})$ and the complexified target Lie algebras at all infinite places, and target-equivariant linear maps p, p' mapping the unprimed and primed algebraic Fock spaces onto Π and vanishing on $\mathcal{R}_{\mathrm{fin}}$ and on the archimedean source Lie actions.

Given Haar measures and fundamental domains $(\nu, \mathcal{F}), (\nu_1, \mathcal{F}_1)$ on $\mathrm{U}(W)(\mathbb{A}_F), \mathrm{U}(W_1)(\mathbb{A}_F)$, a *seesaw transfer* consists of a bijection $e : \{1,\dots,d\} \simeq \{1,\dots,u+v\}$, Hermitian spaces V, V' of ranks u, v and a skew-Hermitian space W^{skew} of rank $u+v$ with global frames (together with local split frames of their self-dual lattices) and Gram matrices

$$J_V = J_W, \qquad J_{V'} = -J_{W_1}, \qquad J_{W^{\mathrm{skew}}} = \xi_0\,(J_X^t)^e \quad \big(((J_X^t)^e)_{ij} = (J_X^t)_{e^{-1}(i),e^{-1}(j)}\big),$$

infinite embeddings of E conjugate to those of θ, good base data preceding Theorem 34.7 with the same additive character ψ, Schrödinger Weil models $(\tilde\omega, \tilde\omega')$ on $(V \otimes_E W^{\mathrm{skew}})(\mathbb{A}_F)$ and $(V' \otimes_E W^{\mathrm{skew}})(\mathbb{A}_F)$ forming a triple preceding Theorem 34.7 with characters (α, β) and source measures and domains pushed forward from $(\nu, \mathcal{F}), (\nu_1, \mathcal{F}_1)$ along the frame matrix identifications $\kappa_W : \mathrm{U}(W) \xrightarrow{\sim} \mathrm{U}(V)$ and $\kappa_{W_1} : \mathrm{U}(W_1) \xrightarrow{\sim} \mathrm{U}(V')$, and the target isomorphism $\kappa_X : \mathrm{U}(X) \xrightarrow{\sim} \mathrm{U}(W^{\mathrm{skew}})$ sending a matrix h in frame B to $(\bar h)^e$ (entrywise conjugation followed by reindexing by e). Given $\mathbb{A}_F$-linear topological symplectic isomorphisms

$$c_A : (V \otimes_E W^{\mathrm{skew}})(\mathbb{A}_F) \xrightarrow{\sim} \operatorname{Hom}_E(W,X)(\mathbb{A}_F), \qquad c_B : (V' \otimes_E W^{\mathrm{skew}})(\mathbb{A}_F) \xrightarrow{\sim} \operatorname{Hom}_E(W_1,X)(\mathbb{A}_F)$$

preserving the rational subspaces, the *adjoint model identifications* $(\mathcal{W}_A, \mathcal{W}_B)$ send Schrödinger tests Φ_D on $\mathbb{A}_F^{u\times(u+v)}$ and Φ'_D on $\mathbb{A}_F^{v\times(u+v)}$ to $\Phi = \mathcal{W}_A(\Phi_D)$ and $\Phi' = \mathcal{W}_B(\Phi'_D)$ defined by

$$\Phi(c_A(w)) = \sum_{\xi \in F^{u\times(u+v)}} (\rho_\psi(w)\Phi_D)(\xi), \qquad \Phi'(c_B(w')) = \sum_{\xi' \in F^{v\times(u+v)}} (\rho'_\psi(w')\Phi'_D)(\xi'),$$

where ρ_ψ, ρ'_ψ denote the Darboux Heisenberg actions for ψ on the two Schrödinger spaces.

Lemma 34.35 (Seesaw transfer of a twisted-square triple)**.** *Let $(W, W_1, X, \alpha, \beta)$ be a twisted-square triple with frames (A, A_1, B), Haar measures and fundamental domains $(\nu, \mathcal{F})$ on $\mathrm{U}(W)(\mathbb{A}_F)$ and $(\nu_1, \mathcal{F}_1)$ on $\mathrm{U}(W_1)(\mathbb{A}_F)$, and oscillator Weil representations ω, ω' on $\operatorname{Hom}_E(W,X)(\mathbb{A}_F)$ and $\operatorname{Hom}_E(W_1,X)(\mathbb{A}_F)$ of signs $+,-$ and splitting pairs $(\beta, \beta/\alpha), (\alpha, \alpha/\beta)$ for an additive character ψ of $\mathbb{A}_F/F$. Then there exist a seesaw transfer $(e, V, V', W^{\mathrm{skew}}, \tilde\omega, \tilde\omega', \kappa_W, \kappa_{W_1}, \kappa_X)$ and symplectic isomorphisms (c_A, c_B) as in Definition 34.34 with the following properties. The spaces V, V', W^{skew} have signatures $(u-p, p), (q, v-q), (d-p-q, p+q)$ satisfying (34.1), at every infinite place w of F with chosen embedding $\tau \in \theta$ the angular exponents under $\bar\tau$ satisfy $m_{\alpha,w} - m_{\beta,w} = 2(b_\tau - a_\tau) > d$, and there exist $C' > 0$ and $N' \in \mathbf{N}$ with $\mathrm{ht}(\kappa_X(h)) \le C' H_X(h)^{N'}$ for all $h \in \mathrm{U}(X)(\mathbb{A}_F)$. The adjoint model identifications $(\mathcal{W}_A, \mathcal{W}_B)$ map the full polynomial Schwartz spaces of $(\tilde\omega, \tilde\omega')$ $\mathbf{C}$-linearly and bijectively onto the unprimed and primed algebraic Fock spaces of (ω, ω'), intertwine the finite-adelic and archimedean Lie actions of source and target via (κ_W, κ_X) and (κ_{W_1}, κ_X), and satisfy*

$$\theta_\omega(g,h;\Phi) = \theta_{\tilde\omega}(\kappa_W(g), \kappa_X(h); \Phi_D), \qquad \theta_{\omega'}(g_1,h;\Phi_0) = \theta_{\tilde\omega'}(\kappa_{W_1}(g_1), \kappa_X(h); \Phi_{0D})$$

for all $g \in \mathrm{U}(W)(\mathbb{A}_F)$, $g_1 \in \mathrm{U}(W_1)(\mathbb{A}_F)$, $h \in \mathrm{U}(X)(\mathbb{A}_F)$, and corresponding algebraic pairs (Φ_D, Φ) and (Φ_{0D}, Φ_0). Under the unitary Siegel-set estimate of Proposition 1.22, for every $h \in \mathrm{U}(X)(\mathbb{A}_F)$, every corresponding unprimed algebraic pair (Φ_D, Φ), and every corresponding primed algebraic pair (Φ_{0D}, Φ_0) with $\Phi_{0D} \in \mathcal{S}'_{t_1}$, both integrals below converge absolutely and satisfy

$$\theta \operatorname{Int}_\omega(\Phi)(h) = I(\Phi_D, \kappa_X(h)), \qquad \theta \operatorname{Int}_{\omega'}(\Phi_0)(h) = I'(\Phi_{0D}, \kappa_X(h)).$$

Outline. Transfers (W, W_1, X) to $(V, V', W^{\mathrm{skew}})$ matching 34.34 and 34.7. Adjoint maps c_A, c_B induce bijections $(\mathcal{W}_A, \mathcal{W}_B)$, giving the kernel identities and, via 1.22 and 34.8, the integral identities. The complete proof is on p. 1080. □

Lemma 34.36 (The seesaw comparison scalar)**.** *Assume $\mathscr{P}$. Keep the twisted-square triple $(W, W_1, X, \alpha, \beta)$ with frames (A, A_1, B), matching data $(\omega, \omega', \Pi, p, p')$ for ψ, and seesaw transfer $(e, V, V', W^{\mathrm{skew}}, \tilde{\omega}, \tilde{\omega}', \kappa_W, \kappa_{W_1}, \kappa_X)$ with adjoint model identifications $(\mathcal{W}_A, \mathcal{W}_B)$ from Lemma 34.35. Fix matching maps $(q_{\mathrm{ss}}, q'_{\mathrm{ss}})$ onto the jointly irreducible target quotient Π_{ss} preceding Theorem 34.7 for $(V, V', W^{\mathrm{skew}}, \tilde{\omega}, \tilde{\omega}')$. Then there exists a unique nonzero scalar $s \in \mathbf{C}^\times$, called the* seesaw comparison scalar*, such that for every corresponding unprimed algebraic pair (Φ_D, Φ) and every corresponding primed algebraic pair (Φ_{0D}, Φ_0) with $p(\Phi) = p'(\Phi_0)$ in Π (and in particular whenever $\Phi_{0D} \in \mathcal{S}'_{t_1}$, on which subclass s is already uniquely determined), the pair $(\Phi_D, s\Phi_{0D})$ is a polynomial match for $(q_{\mathrm{ss}}, q'_{\mathrm{ss}})$.*

Outline. Transports p, p' via 34.35 and factors Π, Π_{ss} through one simple restricted tensor product using 1.22, 34.9, 34.20, 34.10, and 34.7. Countable-dimensional Schur's lemma yields the unique scalar $s \in \mathbf{C}^\times$. The complete proof is on p. 1081. □

Theorem 34.37 (Convergence and matching for twisted squares)**.** *Assume $\mathscr{P}$, Theorem 34.7 and the unitary Siegel-set estimate of Proposition 1.22. For either Hermitian sign and compatible data for M, X, suppose at some infinite place $\mathrm{sig}(M) = (\mathrm{rank}\, M - p, p)$ and $\mathrm{rank}\, X < 2(\mathrm{rank}\, M - p)$. For each permitted Φ, Haar measure and fundamental domain there are $C \in \mathbf{R}$, $N \in \mathbf{N}$ with*

$$\int_{[\mathrm{U}(M)]} |\theta_\omega(g, h; \Phi)|\, dg \le C H_X(h)^N \qquad (h \text{ arbitrary}) \tag{34.19}$$

for the prescribed target height.

Fix the standard additive character of sign -1, the frames and the exponents a_τ, b_τ. There are surjections q, q' from $\mathcal{P}(W, X), \mathcal{P}(W_1, X)$ onto a common P_{real}. For every twisted-square triple with these data, matching data exist with signs $+, -$ and splitting pairs $(\beta, \beta/\alpha), (\alpha, \alpha/\beta)$. The common quotient $\Pi \neq 0$ is jointly irreducible for the finite adelic and real Lie actions. The maps p, p' on the stable algebraic Fock spaces intertwine these actions and kill the finite source differences and source Lie action. For each match there are an isomorphism $e : P_{\mathrm{real}} \otimes_{\mathbf{C}} P_{\mathrm{fin}} \to \Pi$ and maps r, r' onto P_{fin} on permitted finite functions with

$$p(\mathrm{Fock}(f, P)) = e(q(P) \otimes r(f)), \qquad p'(\mathrm{Fock}'(f', P')) = e(q'(P') \otimes r'(f')).$$

For any additive character with a twisted-square triple, matching data and compatible measures and domains, there is $c \neq 0$ such that every primed algebraic Φ' has an algebraic representative $\Phi_0 \bmod \mathcal{R}_{\mathrm{fin}}$ satisfying (34.19). For every unprimed algebraic Φ,

$$p(\Phi) = p'(\Phi_0) \quad \Longrightarrow \quad \theta\, \mathrm{Int}_\omega(\Phi) = c\, \theta\, \mathrm{Int}_{\omega'}(\Phi_0).$$

Outline. Proves moderate growth on a Siegel set of 1.22 via 34.8. Builds matching data and $P_{\mathrm{real}} \otimes_{\mathbf{C}} P_{\mathrm{fin}} \simeq \Pi$ via 1.20, 34.35, and 34.7, then transfers representatives and proportionality via 34.36. The complete proof is on p. 1082. □

34.5 Fields of definition and finite tests

A complete choice fixes a grid with Weil system, row measures, a nonzero smooth tensor with real tests, square constants and a test lattice. For preliminary datum P with frame f, write $\iota_X : K_X \hookrightarrow \mathbf{C}$ for the coefficient embedding of a completion X. The coefficient-field condition requires in $\iota_X(K_X)$ the finite local component values of the row characters $\mu_{0,j}$, the level twists ξ_t and the layer characters χ_i for $1 \le i \le m$. It requires the finite local values of η with the half-weight normalization and of $\chi_\Psi, \delta, \delta'$ with weight zero. It contains the eighth roots of unity. At index t the finite test places are the chosen lifts of the exceptional places of F, together with the twist places Q_t. At each such w it contains the roots of unity of order $(Nw)^{d_w}$, where d_w is the prescribed base depth at the chosen base place, two at the other chosen lifts, and one elsewhere. It contains $\sqrt{Nw}$ at every finite test place and at each additional place chosen by the grid to represent the

support of the auxiliary ideal and its conjugate. Each phase and jet value on a support term belongs to the field.

For each t, i, j and integer $1 \le s \le N_* + 1$, the ratio $L_t(s)/L_{\mu_0}(s)$ for the corresponding layer characters belongs to the field whenever $L_{\mu_0}(s) \neq 0$. For each (i, j) for which a bottom triple exists, require a choice of such a triple and two nonzero periods Ω, Ω'. For every t, finite target point and monomial test for which the corresponding integral exists absolutely, its value divided by its own side's period belongs to the field.

Lemma 34.38 (A common period field for the two bottom integrals)**.** *Assume the local theta inputs $\mathscr{P}$. Fix a grid with its Weil system, a finite set T of level indices satisfying the prescribed character and ramification conditions, and $0 \le i, j < m$. For a standard bottom triple at (i, j), write A, A' for its two source spaces and W for its target. For $t \in T$, $h \in \mathrm{U}(W)(\mathbb{A}_{F,f})$ and a Fock monomial X^k on the respective source, write $I_t(h, k)$ and $I'_t(h, k)$ for the two theta integrals. There are $\Omega, \Omega' \in \mathbf{C}^\times$ and a number field $M \subset \mathbf{C}$ such that*

$$I_t(h, k)/\Omega \in M, \qquad I'_t(h, k)/\Omega' \in M$$

for every such test on its respective side.

Outline. Reduce translates and monomials to finite sets by the class-number theorem and central rotation. Definite class sums give algebraic projections, and the CM Taylor formula gives normalized values whose compositum works for both families. The complete proof is on p. 1084. □

Theorem 34.39 (A finite field requirement for a finite character set)**.** *Assume $\mathscr{P}$, $\mathscr{J}$, the Hecke–Tate functional equation and Theorems 34.7, 34.25, 34.37. There is $C_* \in \mathbf{R}$ such that for each $\varepsilon \in \mathbf{R}$, $C_F \ge C_*$ there is x_0 with the following property. Given a scale s of exponent ε with $x(s) \ge x_0$, P, f and a finite set Ξ of Hecke characters of E, there is a finite $S \subset \overline{\mathbf{Q}} \cap \mathbf{C}$. For each completion X with $S \subseteq \iota_X(K_X)$, size datum over X at s, and compatible Hida realization R of frame f with twisting places away from q_0 and characters in Ξ, there is a complete choice at $(1, C_F, b(s))$ satisfying the coefficient-field condition above.*

Outline. For each level key, use 34.7, 34.37, 34.25 and 34.38 to obtain data and fields. Adjoin character, phase and jet values and toric ratios from 1.15 proved in 1.19, then take the union of field bases and transport by level agreement. The complete proof is on p. 1086. □

Definition 34.40 (Data for the finite rational tests)**.** The coefficient-field condition on the base and frame requires a choice of field data for which the embedded base coefficient field is exactly the compositum of the embedded host E, the auxiliary field M' of the choice, its ray field, the frame character field and the fields of the layer characters for $1 \le j \le m$, together with the prescribed cyclotomic fields. Their orders are the order of the finite group for the remaining rows, $g_i(g_i - 1)$ for $i = 0, 1$, and $3^4, 5^4$. The terminal inclusion $X' \hookrightarrow X_m$ sends each frame vector of X' to the frame vector of X_m with the same index.

At g_i, fix the labeling used for the frame and let $\omega_i \in \mathbf{F}_{g_i}^\times$ be the label of its character. Each type place must lie above the frame's specified prime in the distinguished imaginary quadratic field. The Teichmüller character and every polynomial series idempotent are those of this fixed labeling, and at every indexed type place the layer label numbered $j = 0, \dots, m-1$ is $\omega_i^{2(j+1)}$. The indexed type places, taken over both primes, must be in bijection with the selected group places, preserving their underlying places of E and their specified primes in the quadratic field. The element defining the trivial type is a central idempotent of the complex group algebra of the product of the residue-field matrix groups.

The theta-map condition requires a nonzero archimedean theta intertwiner on each successive row arrow and on the terminal arrow, common to all level indices on that arrow. Its matrix coefficients satisfy the smooth tensor norm bound, and it intertwines the target action and kills the diagonal source relations for the compact groups and Lie algebras. At each level index there must be multiplicity maps on these arrows given by theta integration with the prescribed measures and domains. Their composite, followed by the bottom projection, must be realized by a map into the row Hilbert space. On each auxiliary layer there must also be a compact-equivariant polynomial test map taking some unit lowest-type tensor and unit coefficient vector to the distinguished polynomial. At each index its theta integral must take values in the prescribed automorphic multiplicity space and its projection must admit a lift to the corresponding Hilbert model.

For geometric evaluation, at every layer and compatible embedding require a semilinear map from the rational cuspidal forms to the Hilbert space whose evaluation at the chosen vector gives the geometric form.

This condition uses the prescribed vector at the good embeddings and a unit vector giving the conjugate form at the other embeddings. At good embeddings evaluation must be injective. At every embedding the resulting realization must carry each finite linearly independent family to a complex linearly independent family and its image must span. Require a rational action of the finite group on the forms which preserves integrality at every valuation and under realization gives the geometric action with the prescribed character factor. The forms integral at all the specified valuations must be a full lattice, and the dimension of the geometric cuspidal space must equal the multiplicity dimension in the real model at each layer.

Theorem 34.41 (Finite rational test functionals)**.** *Assume the inputs of Theorem 34.39, the unitary Siegel-set estimate of Proposition 1.22 and the finite-group cuspidal character input. In the eventual large-field construction take a base and frame with the coefficient field and frame normalization of Definition 34.40. Require the set of twist primes to contain g_0, g_1, d and every rational prime at most $X''(n)$. Fix a pinned choice of columns and any completion by the remaining row, division, local, type and geometric data. For any admissible fields and set of places, take a stage z satisfying the finite-group, theta-map and geometric evaluation conditions of Definition 34.40. Let EE be the complex row space attached to the realization and face of z. At each level index t, let FR_t be the source over the coefficient field K, let j_t realize it in the complex source, and let Θ_t be the theta map from that source to EE. For the stage embedding $v_0 : K \hookrightarrow \mathbf{C}$ there are $N \in \mathbf{N}$, $\psi_i \in EE^\vee$ with*

$$\bigcap_{0 \le i < N} \ker \psi_i = 0, \qquad \psi_i(\Theta_t(j_t x)) \in v_0(K) \quad (t \text{ arbitrary},\ 0 \le i < N,\ x \in FR_t). \tag{34.20}$$

Outline. Use separation from 34.25 and spanning columns to get finitely many functionals with zero common kernel. Project to good summands and compare models by 34.7 and 34.37, then normalize and transfer signs to obtain $v_0(K)$-valued evaluations. The complete proof is on p. 1087. □

34.6 Preserving a prescribed grid

Lemma 34.42 (Finite field requirements for full grids)**.** *For a grid G, let v_0 be its base place, $w(v_0)$ the chosen place above it, and χ_{a+1} its layer characters. Put $\gamma(G) = (v_0, w(v_0), (\chi_{a+1})_{0 \le a < m})$. A full grid over R (Definition 34.49) admits a compatible complete choice with the evaluation and square-field conditions. The bottom clause concerns the base index μ_0. Suppose there is a common pair (δ, t_0) admitting bottom pairs for every $i, j < m$. Then one can choose a common pair and bottom witnesses for all i, j so that both norms of each witness lie in $\iota(K)$. Put $h_{ij} = \lfloor (r_{i+1,j+1} + r_{i,j})/2 \rfloor$. For $i, j < m$, $1 \le s' \le h_{ij}$ and either twist, the corresponding untwisted and twisted toric values x, y have $y/x \in \iota(K)$.*

Fix the scale s, the constant C_F, a completion X, a frame f and its allowed set Ξ of twist characters. Among the realizations of Theorem 34.39 take those with square-recursive size data (Definition 34.49), the row size conditions of Theorem 34.25 and the prescribed host with $F = E^+$. For the quadratic character χ_d of $\mathbf{Q}(\sqrt{-d})$, require $\left|\mathrm{Re}(L'(1,\chi_d)/L(1,\chi_d))\right| \le x^{\varepsilon}$. Say that prescribed grids extend at this completion if, for each such realization, every grid G_0 at $(1, C_F, b(s))$ admits a full grid G at the same parameters with $\gamma(G) = \gamma(G_0)$. The grid extension condition at C_F requires this property at every completion of the construction. The finite field requirement for prescribed grids is the assertion of Theorem 34.39 with the conclusion that prescribed grids extend at every completion whose coefficient field contains the requested finite set.

Suppose at the requisite thresholds that each permitted grid admits a Weil system and initial choice extending to a complete choice over a number field $N_1 \subset \mathbf{C}$ with the same grid choice and Weil system. Suppose also that the square-field condition holds over a number field $N_2 \subset \mathbf{C}$ for each square-recursive datum, admissible level character system with base index and permitted grid at $(1, C_F, b)$. Assume level agreement transports the grid, Weil system, evaluations and field conditions. If level keys for finite character sets and grid keys at fixed gap are finite, then the finite field requirement for prescribed grids holds.

Proof. Take the union of bases of $N_1 N_2$ for representatives of the realized pairs of keys at gap one. In a coefficient field containing the union, transport the full representative to the realization with its pair of keys. □

Lemma 34.43 (Algebraic ratios on a prescribed grid)**.** *Under the inputs of Theorem 34.39, for each permitted grid and square-recursive datum there is a number field $N_2 \subset \mathbf{C}$ satisfying the square-field condition of Lemma 34.42.*

Outline. Form the character $\chi = \mu_{0,j}\xi_{\mu_0}/\chi_{m-i}$ with grid twist δ_c, compare the floored grid midpoints $c_{\tau,a}$ against $e^{\mathrm{gr}}_{\tau,j+1}$, and deduce algebraicity of the resulting norm ratios over the prescribed field. The complete proof is on p. 1088. □

Theorem 34.44 (Eventual existence of evaluation grids)**.** *Under the inputs of Theorem 34.39 there is $C_{F1} \in \mathbf{R}$ such that for every $C_F \geq C_{F1}$ the grid extension condition of Lemma 34.42 eventually holds in the large-field sense of Theorem 33.55.*

Outline. Use 34.39 and 34.43 to obtain N_1, N_2 and transport the conditions by level agreement. Ray class finiteness and the finite base-place choices give finitely many grid keys, so 34.42 gives the fourth-stage field request. The complete proof is on p. 1088. □

Lemma 34.45 (The finite height of the higher toric ratios)**.** *Use a fitted grid (Definition 34.49) under the size hypotheses of Theorem 34.68, with coefficient field K and $f = [E^+ : \mathbf{Q}]$. At finite w of K let v_w be the uniformizer-one valuation and $\ell_w = \log N\mathfrak{p}_w/[K : \mathbf{Q}]$. For $i, j < m$ put $h = h_{ij}$ as in Lemma 34.42. For $1 \leq s \leq h$ use u, u_c of Lemma 34.43, with $z_c = u_c/u \in K$ through the specified embedding. Put $z_T = 1$, $z_{T,c} = z_c$, $e^{\mathrm{gr}}_{\tau,j+1} = e_{\tau,j}$ (Definition 34.46), and in Definition 1.15*

$$\kappa_\tau = \max(0, e^{\mathrm{gr}}_{\tau,j+1} + c_{\tau,m-i} - s), \quad M = 2sf + 2\sum_\tau \kappa_\tau, \quad C_0 = |D_F|^{(2s-1)/2} \prod_\tau \Gamma(2s+\kappa_\tau)(2\pi/\sqrt{d})^{\sum_\tau \kappa_\tau}.$$

For $v = u, u_c$, let P_v be the monic minimal polynomial over K of $\beta(v) = (C_0 v)^{12}/\operatorname{disc}(\Lambda)^M$, with $\Lambda = \mathcal{O}_{K_d}$. Put $y_w(v) = v_w(P_v(0))/(12 \deg P_v)$ and $g_T(w) = \operatorname{trunc}_0 y_w(u)$, truncating toward zero. Outside $1 \leq s \leq h$ use both ratios one and $g_T = 0$.

Beyond the joint grid and point thresholds, for every sufficiently large C_1 the in-range ratios are nonzero, and for every integer s and $z = z_T, z_{T,c}$ the sum has finite support and satisfies

$$\sum_w \ell_w |v_w(z) + g_T(w)| \leq B_0, \qquad B_0 = C_1 f b(\log b + \log x). \tag{34.21}$$

Outline. Use 34.43 and 1.18 and take denominator bounds from 1.15 proved in 1.19. Use Hecke estimates, CM transport, Chowla–Selberg and the product formula to bound valuations, then scale by z_c^{12} with a common shift. The complete proof is on p. 1089. □

34.7 The functional bound

Let RR be the glued row lattice of positive rank with infinite frames $X_{E,v}$ and normalizations $\kappa_v > 0$. The squared-norm condition is

$$\mu_{\max,RR} \leq x^{C_H} b - \langle \log \kappa \rangle_\infty. \tag{34.22}$$

Here C_H is the family height exponent and $\langle \log \kappa \rangle_\infty$ is the arithmetic mean of $\log \kappa_v$ over the infinite places of the coefficient field.

Definition 34.46 (Members and member points)**.** Fix a frame as in Definition 5.1, with finite abelian group A, half-degree $n = |A|$, and target character order $o \geq 1$. Fix the real degree $f = 2n$, even starting rank $k \geq 2$, even chain length $m \geq 2$, and odd auxiliary rank $t \geq 1$, with bottom row rank $r = k+m-1$, auxiliary rank $n_V = r + t$, and column-layer ranks $N(j) = k + j(t+1)$ $(j \in \mathbf{N})$, satisfying

$$2 \leq m, \quad m+3 \leq k, \quad r+5 \leq t, \quad 3m < t, \quad m \leq f \leq k, \quad m^2 \leq k + m(t+1).$$

A Harish–Chandra parameter tuple at these ranks is a quadruple $H = (\lambda^{\mathrm{HC}}, v, b, e)$ with $\lambda^{\mathrm{HC}} = (u_0, \ldots, u_{k-1}) \in (\mathbf{Z} + \frac{1}{2})^k$, $v = (v_0, \ldots, v_{k-2}) \in \mathbf{Z}^{k-1} \subset \mathbf{Q}^{k-1}$, $b = (b_0, \ldots, b_{m-1}) \in \mathbf{Z}^m$, and $e = (e_0, \ldots, e_{m-1}) \in \mathbf{Z}^m$ satisfying $u_{a+1} < v_a < u_a$ $(0 \leq a < k-1)$, $-b_0 < u_{k-1}$, $e_j - b_j \geq (t+3)/2$ $(0 \leq j < m)$, and $b_{j+1} - e_j \geq (t+3)/2$ $(0 \leq j < m-1)$.

An *arithmetic datum* D at these ranks consists of the following host, Hermitian-space, row-and-division, member, local-Galois, and theta-jet data:

1. A *host datum* over an abelian CM extension $E/\mathbf{Q}$ (unramified over its maximal totally real subfield $F = E^+$ at finite places, with $[F:\mathbf{Q}] = f$, CM conjugation $x \mapsto \bar{x}$, CM type Ψ of cardinality f, and distinguished embedding $\varphi_0 \in \Psi$, $\varphi_0 : E \hookrightarrow \mathbf{C}$) and imaginary quadratic subfield $K_d = \mathbf{Q}(\sqrt{-d}) \subset E$ (for a prime $d > 4$ with $d \equiv 7 \pmod{16}$): an auxiliary prime $q \neq 3$ with $q \equiv 3 \pmod 8$, $\left(\frac{-d}{q}\right) = 1$ and $\sqrt{-q} \in E$ and $4tr+1 < q, d$; a positive integer factor $Q_S \leq n^2$ and split bound $B_{\mathrm{sp}} \in \mathbf{N}$ such that every odd prime $p \leq B_{\mathrm{sp}}$ satisfies $\left(\frac{-d}{p}\right) = 1$ and every odd prime $l \mid Q_S$ satisfies $4l \leq B_{\mathrm{sp}}$; an ideal $\mathfrak{b} \subseteq \mathcal{O}_E$ preserved by Galois automorphisms projecting to Ψ and factoring the relative different as $\mathfrak{D}_{E/K_d} = \mathfrak{b}\bar{\mathfrak{b}}$ with $\mathfrak{b} + \bar{\mathfrak{b}} = \mathcal{O}_E$; a finite set $\mathcal{P}_{\mathrm{lev}}$ of primes coprime to Q_S containing distinct theta primes $g_0 \neq g_1$ and distinct prime-twist primes $p_{\mathrm{de}} \neq p'_{\mathrm{de}}$ with $p_{\mathrm{de}} \equiv p'_{\mathrm{de}} \equiv 1 \pmod{15}$ and $\{p_{\mathrm{de}}, p'_{\mathrm{de}}\} \cap \{g_0, g_1\} = \varnothing$; a nonzero primitive ideal $\mathfrak{c}_K \subseteq \mathcal{O}_{K_d}$ coprime in norm to $30\,q\,d\,Q_S$ and $\mathcal{P}_{\mathrm{lev}}$ whose prime factors in $\mathcal{O}_{K_d}$ have prime norms splitting in K_d; finite top-position and thick sets TT and $T_2 \subset \mathbf{N}$, word-length integer $L_w \geq 2$, and constant $C_{T_2} \in \mathbf{R}$.

2. *Hermitian spaces* over E/F (each a finite-dimensional E-vector space V with nondegenerate conjugate-symmetric E-sesquilinear form h_V, E-basis, signature map, unimodular $\mathcal{O}_E$-lattice $L_0(V)$, and volume $\mathrm{vol}(V) > 0$): definite column spaces X_j^{col} $(0 \leq j \leq m)$ of rank $N(j)$, a subspace $X' \hookrightarrow X_m^{\mathrm{col}}$ of rank $r + mt$ isometric onto the orthogonal complement of an anisotropic vector $z \in X_m^{\mathrm{col}}$, reference row spaces Y_j $(0 \leq j \leq m)$ of rank $r + jt$ and signature $(r + jt - \lfloor (m-j)/2 \rfloor, \lfloor (m-j)/2 \rfloor)$ at every embedding $E \hookrightarrow \mathbf{C}$ with isometric embeddings $X_j^{\mathrm{col}} \hookrightarrow Y_j$ $(0 \leq j < m)$, an isometry $X' \xrightarrow{\sim} Y_m$, and a vector $z_m \in Y_0$ orthogonal to the image of X_0^{col} and of norm $-T_{u_0}$ (with T_{u_0} the base level prime of item (4)(b)) and $r_S = (m-2)/2$ pairwise orthogonal hyperbolic planes in Y_0 orthogonal to X_0^{col} and z_m, row-layer spaces W_j $(1 \leq j \leq m)$ of rank r finite-adelically isometric to Y_0, and auxiliary spaces V_j^{sp} $(1 \leq j \leq m$, written V_j^{sp} to distinguish them from the core spaces V_i of Section 3) of rank $n_V = r + t$ and sign $\epsilon_V(j) \in \{\pm 1\}$ making $L_0(V_j^{\mathrm{sp}})$ orthonormal for $\epsilon_V(j) h_{V_j^{\mathrm{sp}}}$.

3. *Coefficient field, row data, and division lattices*: a totally complex Galois number field $K/\mathbf{Q}$ with involution $x \mapsto x^*$, distinguished embedding $\iota : K \hookrightarrow \mathbf{C}$ $(\iota(x^*) = \overline{\iota(x)})$, integral subring $O_{\mathrm{row}} \to K$ containing $\mathcal{O}_E$, base subfield $K_0 \subseteq K$, embedding $E \hookrightarrow K_0 \subseteq K$ pulling ι back to φ_0 and carrying $\bar{x}$ to x^*, and finite-place set $\Sigma_f(K) \simeq \mathrm{Spec}_{\neq 0}(\mathcal{O}_K)$ with valuations v_w, ramification indices e_w, and involution $\bar{w}$; a finite CM-embedding index set $\mathcal{T}$ $(\mathcal{T} \xrightarrow{\sim} \Psi$, $|\mathcal{T}| = f)$, finite word set $\mathcal{W}$ with base word $u_0 \in \mathcal{W}$, and Harish–Chandra tuples $H_{u,\tau} = (\lambda_{u,\tau}^{\mathrm{HC}}, v_\tau, b_\tau, e_\tau)$ $(u \in \mathcal{W},\ \tau \in \mathcal{T})$ agreeing with $H_\tau = H_{u_0,\tau}$ on v_τ, b_τ, e_τ and on the first and last entries $u_{u,\tau,0}, u_{u,\tau,k-1}$ of $\lambda_{u,\tau}^{\mathrm{HC}}$; a tuple set $\mathcal{T}_{\mathrm{tup}}$ with base index μ_0, prime-twist indices μ_*, μ'_*, and conjugation-invariant place maps $\mu_{\mathrm{at}}, \mu_{\mathrm{tw}} : \Sigma_f(K) \to \mathcal{T}_{\mathrm{tup}}$ with $\mu_{\mathrm{at}}(w) \in \{\mu_0, \mu_{\mathrm{tw}}(w)\}$; finite-dimensional complex inner-product spaces $\mathcal{E}$ (dimension $D_{\mathrm{rows}} = (1 - \delta_p) D_{\mathrm{all}}$, $\delta_p \leq 1$), $\mathcal{E}_0$, ambient spaces $\mathcal{A}_\mu$ with symmetric idempotents and O_{row}-lattices, multiplicity spaces $\mathcal{M}_{\mu,j}$ $(\mathcal{M}_{\mu,0} \simeq \mathcal{E}$, $\mathcal{M}_{\mu,m} \simeq \mathcal{A}_\mu$, $\mathcal{M}_{\mu,j+1} \to \mathcal{M}_{\mu,j})$, word column spaces $\mathcal{M}_u$ with pairings $\mathcal{M}_u \times \mathcal{A}_\mu \to \mathbf{C}$, $\mathbf{C}$-bilinear pairings $B_u : \mathcal{M}_u \times \mathcal{E} \to \mathbf{C}$, units $\zeta_{u,\mu} \in K^\times$ $(\zeta_{u,\mu_0} = 1)$, and layer spaces M_j and $M_{t,\mu,j}$ (with K-forms, O_{row}-lattices, symmetric idempotents, eigenvalues, integer local transition indices, and rational invariants $\eta_j(w), L_\sharp(l,e) \in \mathbf{Q}_{\geq 0}$); and Hermitian $\mathcal{O}_K$-lattices RR and $\mathcal{N}_u$ $(u \in \mathcal{W}$, each a K-vector space V with K-basis $\mathbf{b}$, invertible infinite-place frame matrices $X_v \in \mathrm{GL}_d(\mathbf{C})$, and $\mathcal{O}_K$-submodule $\mathcal{L} \subset V)$ with K-linear period maps $V(\mathcal{N}_u) \to V(RR)^\vee$, heights $h_u \in \mathbf{R}$, parameter $\eta_F \in \mathbf{R}$, and ι-semilinear basis-spanning realizations $j_{\mathcal{E}} : V(RR) \to \mathcal{E}$ (isometric at ι) and $j_{N,u} : V(\mathcal{N}_u) \to \mathcal{M}_u$.

4. At each word $u \in \mathcal{W}$, a word-level *member* (C_u, A_u, P_u) together with *member forms*:

   (a) a *column chain* C_u assigning to each $i \geq 0$ a finite class set $\mathcal{C}_{u,i}$ (with $\mathcal{O}_E$-lattices $\Lambda_{u,i}(c) \subset X_i^{\mathrm{col}}$ in D for $0 \leq i \leq m$), finite stabilizer groups $\Gamma_{u,i}(c)$ $(c \in \mathcal{C}_{u,i})$, an integer $D_{\Gamma,u} > 0$ divisible by $|\Gamma_{u,i}(c)|$ for all $0 \leq i \leq m$ and $c \in \mathcal{C}_{u,i}$, fibre sets $Y_{u,i}$ with factors $d_{u,i}(y,\tau) \in \mathbf{C}^\times$, unit-norm determinant factors $\rho_{u,i}(c), \rho'_{u,i}(c')$ for $i < m$ given in D through an abelian determinant group D_t with characters $\tilde{\mu}_{i+1}, \chi_D : D_t \to \mathbf{C}^\times$ and class map $\delta_{u,i} : \mathcal{C}_{u,i} \to D_t$ by

   $$\rho_{u,i}(c) = \tilde{\mu}_{i+1}(\delta_{u,i}(c))^{-1} \chi_D(\delta_{u,i}(c))^{-N(i+1)/2}, \qquad \rho'_{u,i}(c') = \tilde{\mu}_{i+1}(\delta_{u,i+1}(c')) \chi_D(\delta_{u,i+1}(c'))^{N(i)/2},$$

   unfolding maps $\mathrm{unf}_{u,i}(c,c') : \mathbf{C}^{Y_{u,i}} \to \mathbf{C}^{Y_{u,i+1}}$, and shift exponents $s_{u,i}^\sharp(\tau) = e_{\tau,i} - N(i+1)/2 \in \mathbf{Z}$ and $s_{u,i}^{\mathrm{div}}(\tau) = e_{\tau,i} - N(i)/2 \in \mathbf{Z}$ $(0 \leq i < m)$;

(b) a *top-level column datum* A_u specifying a prime $T_u \in \mathcal{P}_{\mathrm{lev}} \setminus \{p_{\mathrm{de}}, p'_{\mathrm{de}}\}$ (with $T_u = \pi_u \bar{\pi}_u$ for $\pi_u \in \mathcal{O}_E \cap K_d$ satisfying $\pi_u \equiv 1 \pmod{Q_S\, q\, d\, (g_0 g_1)^2 \mathcal{O}_E}$, $c_u \bar{c}_u T_u = T_{u_0}$ for $c_u \in E^\times$, and $T_u^\star \le T_u \le (1+(fk)^{-1}) T_u^\star$), a finite place set $\mathcal{V}_{T,u}$ of cardinality f with finite fields $\mathbf{F}_v$ of cardinality T_u, nontrivial additive characters $\psi_{T,v} : \mathbf{F}_v \to \mathbf{C}^\times$, actions $\rho_{T,v} : \mathbf{F}_v^\times \to \mathrm{GL}_{\mathbf{C}}(\mathbf{C}^{\mathcal{C}_{u,0} \times Y_{u,0}})$, a finite-dimensional $\rho_{T,v}(\mathbf{F}_v^\times)$-stable subspace $M_{u,0} \subset \mathbf{C}^{\mathcal{C}_{u,0} \times Y_{u,0}}$ (paired with its ramified subspace $M_{\mathrm{ram},u} \simeq \mathcal{M}_u$), an integer $d_u^\star > 0$, a common real factor $D^\star > 0$, and a common price parameter $\Gamma_P > 1$;

(c) a *period map* P_u on C_u, consisting of complex vector spaces $M_{\mathrm{t},u} \simeq \mathcal{A}_{\mu_0}$, $W_{z,u}$, and $W_{\mathrm{row},u}$, $\mathbf{C}$-linear maps $\mathrm{ev}_{z,u} : \mathbf{C}^{\mathcal{C}_{u,m} \times Y_{u,m}} \to W_{z,u}$ and $\mathrm{low}_u : W_{z,u} \to W_{\mathrm{row},u}$, and a $\mathbf{C}$-bilinear pairing $\mathrm{per}_u : W_{\mathrm{row},u} \times M_{\mathrm{t},u} \to \mathbf{C}$, accompanied in D at every $\mu \in \mathcal{T}_{\mathrm{tup}}$ by a twisted chain $C_{u,\mu}$ ($C_{u,\mu_0} = C_u$), a base-layer isomorphism $e_{0,u,\mu} : \mathbf{C}^{\mathcal{C}_{u,0} \times Y_{u,0}} \xrightarrow{\sim} \mathbf{C}^{\mathcal{C}_{u,\mu,0} \times Y_{u,\mu,0}}$, and a period map $P_{u,\mu}$ with $M_{\mathrm{t},u,\mu} \simeq \mathcal{A}_\mu$ realizing the μ-th column-period pairing on $\mathcal{M}_u \times \mathcal{A}_\mu$; and

(d) *member forms* on (C_u, A_u, P_u): for each $i \ge 0$, $v \in \mathcal{V}_{T,u}$, and $c \in \mathcal{C}_{u,i}$, rank-$N(i)$ $\mathbf{F}_v$-spaces $U_{u,i,v,c}, K_{u,i,v,c}$ with a nonzero covector on $U_{u,i,v,c}$, a nonzero vector in $K_{u,i,v,c}$, and raising/lowering maps from $U_{u,i,v,c}$ and $K^\vee_{u,i,v,c}$ to $\mathcal{C}_{u,i}$ and $\mathrm{GL}_{\mathbf{C}}(\mathbf{C}^{Y_{u,i}})$; $\mathbf{F}_v^\times$-dilations on $\mathcal{C}_{u,i}$ and $\mathbf{C}^{Y_{u,i}}$ preserving $|\Gamma_{u,i}(c)|$ and inducing $\rho_{T,v}$ at $i = 0$; a family of bijections $\mathcal{V}_{T,u} \xrightarrow{\sim} \mathcal{T}$ indexed by $w \in \Sigma_f(K)$; antitone integer dominant weights $\lambda_{u,j}(\tau) \in \mathbf{Z}^{N(j)}$ given for $0 \le j \le m$ and $0 \le a < N(j)$ by $\lambda_{u,j}(\tau)_a = u_{u,\tau,a} - \rho_{k,a}$ ($a < k$, with rank-k half-sum vector $\rho_{k,a} = (k-1)/2 - a$) and $\lambda_{u,j}(\tau)_a = -e_{\tau,q} + \frac{k}{2} + \frac{(2q+1)(t+1)}{2}$ ($k \le a < N(j)$, $q = \lfloor (a-k)/(t+1) \rfloor$), plus a weight in $\mathbf{Z}^{N(m)-1}$; fibre subspaces $F_{u,i} \subseteq \mathbf{C}^{\mathcal{C}_{u,i} \times Y_{u,i}}$ ($F_{u,0} = M_{u,0}$) with sesquilinear fibre forms $\langle \cdot, \cdot \rangle_{V,u,i,c}$ and stabilizer-weighted inner product $\langle \varphi, \psi \rangle_{C,u,i} = \sum_{c \in \mathcal{C}_{u,i}} |\Gamma_{u,i}(c)|^{-1} \langle \varphi(c), \psi(c) \rangle_{V,u,i,c}$; a finite point-class set mapping to $\mathcal{C}_{u,m}$ with injective stabilizer maps, a point subspace $W_{\mathrm{pt},u} \subseteq W_{\mathrm{row},u}$ with sesquilinear form $\langle \cdot, \cdot \rangle_{P,u}$, and good Hermitian $\mathcal{O}_K$-lattices $L_{u,i}$ and $L_{\mathrm{pt},u}$ whose ι-semilinear realizations carry their K-bases to $\mathbf{C}$-bases of $F_{u,i}$ and $W_{\mathrm{pt},u}$ and match at ι the real parts of $\langle \cdot, \cdot \rangle_{C,u,i}$ and $\langle \cdot, \cdot \rangle_{P,u}$, with $V(\mathcal{N}_u)$ embedding K-linearly and norm-isometrically onto the preimage of $M_{\mathrm{ram},u}$ in $V(L_{u,0})$.

5. *Split-group, p-adic Galois, Haar, and theta-jet packages*: a split-group package with positive rational spacing $\Delta \in \mathbf{Q}_{>0}$ satisfying $v_{\tau,l} - v_{\tau,l+1} = 4\Delta$ ($0 \le l < k-2$), an injective word coding $\ell : \mathcal{W} \hookrightarrow \{0,1\}^{k-2}$ setting $u_{u,\tau,l+1} = v_{\tau,l} - \frac{1}{2} - \Delta \mathbf{1}_{\ell_u(l)=1}$, and a Hilbert space H_S ($\dim_{\mathbf{C}} H_S = D_{\mathrm{all}}$, containing an anti-unitarily isometric copy of $\mathcal{E}$) receiving $M_{u,0}$; p-adic period-ring $(A_{\mathrm{inf}}, B_{\mathrm{cris}}, B_{\mathrm{dR}})$, Galois-bridge, local Hecke, base-change, twisted-module, and finite-group type packages with $G_{0,g} \simeq \prod_{k=0,1} \mathrm{GL}_r(\mathbf{Z}/g_k\mathbf{Z})^f$ and central series idempotents at g_0, g_1; compatible Haar measures on $\mathrm{U}(P,Q)$ (probability on $\mathrm{U}(N,0)$); global theta-jet situations at signal and inert primes between ranks $N(j-1)$ and $N(j)$ ($1 \le j \le m$); and local level modules of rank $\dim_{\mathbf{C}} M_{u,0}$ at primes $p \ge 5$ in the host's inert class, outside $\mathcal{P}_{\mathrm{lev}}$ and not dividing T_u, whose arithmetic Frobenius component in A has even order.

A *size datum* L_{sz} over D fixes a scale $s = (x, \varepsilon, J)$ ($x > 1$, $0 < \varepsilon < 1/52$, $J \ge 1$, $L = x^\varepsilon$, $b = e^L$, $a = xb$, $X' = \exp(x^{\varepsilon/3})$), integer top weights $a_t, b_t \in \mathbf{Z}$ ($t \in TT$) in $[a, a + a/(kx^{13})]$ and $[b', \infty)$ (with $b' = xb/X'$ and $X''(n) = xnX'$ as in Definition 34.19) fixing $u_{u_0,\tau,0} = a_t + (k-1)/2$ and $u_{u_0,\tau,k-1} = -b_t - (k-1)/2$ at top embeddings $\tau = \tau_1(t)$, where also $b_{\tau,0} - b_t > (k-1)/2 + 3$, a finite support $S_{\mathrm{supp}} \subset \Sigma_f(K)$ for the local division-gain, finite-factor, error, and twist-indicator functions $G_{D,j}, F_j, E_{u,j}, \tau_{\mathrm{loc}}$ on $\Sigma_f(K)$ and local transition indices (with $G_{D,j} = 0$ above g_0, g_1, $F_j(w) = A_j(w) = 0$ for $1 \le j \le m$ when $\mu_{\mathrm{at}}(w) = \mu_0$, $F_j(w) = 0$ for every j when $\mathrm{char}(w) > X''(n)$ outside $\{g_0, g_1, d\}$, and $F_j(\bar{w}) = F_j(w)$, where $A_j(w) = I_j(\mu_{\mathrm{at}}(w), w) + I_j(\mu_{\mathrm{at}}(w), \bar{w}) + J_j(\mu_{\mathrm{at}}(w), w)$ uses the row transition indices I_j and J_j), $b_{\tau,j} \le b^2$, a drop set avoiding all T_u, depth $j_0 \ge \log_5(n\, N(m)/m)$ with $10j_0 + 20 \le L_w$, $\mathrm{Re}(L'(1,\chi_d)/L(1,\chi_d))$, input $\mathcal{O}_K$-submodules $\mathrm{NuLat}_u \subseteq V(\mathcal{N}_u)$, thick density $\delta_2 = |T_2|/n \in (0,1]$, width $\eta > 0$, center bound $S_c = 4k\Delta > 0$ bounding $0 \le v_{\tau,a}, u_{u,\tau,a'} \le S_c$ ($0 < a' < k-1$) and windows for $w_j(\tau) = e_{\tau,j-1}$ ($1 \le j \le m$) given by $[b, b+\eta+1]$ at thick places, $[b_t + b'/x, b_t + b'/x + \eta]$ at top positions $t \in TT$, and $[w'', w'' + \eta]$ at thin places, with $|b_{\tau,i} - e_{\tau,j}| \le \eta$ for every τ and $0 \le i, j < m$, subsets $TT_w \subseteq TT$ with $TT_w = TT$ when $\mathrm{char}(w)$ is in the host's signal class or $\mathrm{char}(w) \ge X'$, $1 \le t \le x^{C_t}$, $\log \Gamma_P = C_\Gamma m^5 f L$ ($C_\Gamma > 0$), partition weights ≥ 1, and local length ratios in $\mathbf{Q}$.

A *construction context* $\mathcal{K}$ fixes a scale s, family constant record, gap/spacing exponents $c_D, c_h \in \mathbf{R}$, preliminary host-word-parameter datum P_{pre} with $\mathfrak{c}_K = \mathcal{O}_{K_d}$, auxiliary prime $q_1 \in \mathcal{P}_{\mathrm{lev}} \setminus \{g_0, g_1, p_{\mathrm{de}}, p'_{\mathrm{de}}\}$,

integer box weights a, b, injective word coding $c : \mathcal{W} \hookrightarrow \{0,1\}^{k-2}$ meeting the host-and-window checks, and level-prime choices $T_u = \pi_u \bar{\pi}_u$ with $d_u^\star, D^\star$ given by the standard unitary mass-index products and $\Gamma_P = \exp(C_\Gamma m^5 f L)$. Over $\mathcal{K}$, a *built point* β consists of: a seven-stage tuple (base field K with $E \hookrightarrow K_s \subseteq K_0 \subseteq K$ containing the group/prime cyclotomic roots and capitulation generators, word columns and member forms, remaining row spaces, division stage, split-group stage, local Galois-bridge stage, and Haar/theta-jet stage) assembling an arithmetic datum D; a character frame over P_{pre} and finite-place twisting data (a finite twist-prime set $\mathcal{P}_{\mathrm{tw}}$, split prime-ideal sets $Q(w) \subset \mathrm{Spec}_{\neq 0}(\mathcal{O}_E)$ coprime to $\mathrm{char}(w)$, exponents $k(w) \in \mathbf{N}$, and conjugation-invariant tame Hecke characters ψ_w of E of order dividing $\mathrm{char}(w)^{k(w)}$, trivial off $\mathcal{P}_{\mathrm{tw}}$); a core specialization fixing tuple twists ξ_μ ($\xi_{\mu_0} = 1$, $\xi_{\mu_*} = \delta$, $\xi_{\mu'_*} = \delta'$), active twist places W_{tw} (forcing $\{g_0, g_1, d\}$ and all primes $\le X''(n)$ into $\mathcal{P}_{\mathrm{tw}}$ and vanishing of lift base terms off W_{tw}), Δ, $\eta_F = 6x^{-C_F}$, and Teichmüller labels $\omega^{2(j+1)}$ at g_0, g_1; an object layer equipping D with glued-lattice characterization on RR, Shimura/divisor models, automorphic/theta actions, local-to-global chain kernels, base-level row side, ambient lattices, and unitary genus-mass Haar gauges; and a ledger specialization equipping D with a size datum L_{sz} at (s, a, b), NuLat_u comparison with $M_{\mathrm{ram},u}$ away from T_u, and local error/valuation/recursion ties. For a parameter predicate κ, point predicate Q, and bundle linkage Lk at $c \in \mathbf{R}$, a *built member* $(D, O, \mathfrak{b})$ is realized by $\mathcal{K}$ satisfying κ, a built point β over $\mathcal{K}$ satisfying Q with datum D and object package O, and a bundle $\mathfrak{b}$ linked to β at c via Lk, supplying the notions of member and built member in Section 1 of Chapter 33.

Along a family F_m at $x > x_1$ and $i \in I_x$, a *member point* p over (D, L_{sz}) packages: an indefinite row-side full Hecke datum over a finite group $G_p \simeq G_{0,g}$, tuple set T with base index $\mu_0 \in T$, subgroups $P_0, P_1 \le G_p$ with characters $\psi_0 : P_0 \to K^\times$ and $\psi_1 : P_1 \to K^\times$, definite subspaces, two-level indefinite data, definite columns, Hilbert-space periods, and three-sided division ties; integers $\Delta, h_0 \in \mathbf{N}$ matching $\Delta_D = \Delta$ and $\eta = m h_0$, stabilizer divisibility into D_{Γ,u_0}, $F = E^+$, and host parameter-interlacing and $100r^4/256r^4$ gap bounds; a projective-lattice core over $K_s \subseteq K$ and level-3 real realization R weakly characterized on the induced face, with standard sign--1 additive character ψ, sign--1 residue-trace compatibility of ψ_0 at P_g, coprimality of Q_t to $Q_S\, q\, N\mathfrak{c}_K$, unit phases, isometric top projection, positive definite original Hermitian forms on X' and V_{i+1}^{sp} ($0 \le i < m$), with their prescribed frames orthonormal for these forms and their $\mathcal{O}_E$-lattices equal to the spans of these frames, and F-diagonal Gram matrices on W_{i+1} and V_{i+1}^{sp}; weight tuples κ_i and extremal vectors $v_0(i, \sigma)$ ($\sigma : K \hookrightarrow \mathbf{C}$) with geometric cusp-form evaluation, conjugate, finite-group action, equivariance, and phase pins; residue-field matrix-group and series-idempotent pins at g_0, g_1; and a tuple map $\mathrm{tup}_T : \mathcal{T}_{\mathrm{tup}} \to T$, a left inverse of the tuple inclusion $T \to \mathcal{T}_{\mathrm{tup}}$ and inverse to it on μ_0 and on every $\mu_{\mathrm{at}}(w)$, identifying μ_0 with the base tuple and sending $\mu_{\mathrm{at}}(w)$ to a gluing index $\mu(w) \in T$ at each finite place w of K. Fixing a representative complex embedding $v : K \hookrightarrow \mathbf{C}$ at each infinite place $v \mid \infty$ of K, an invertible frame matrix $X_v \in \mathrm{GL}_d(\mathbf{C})$ on K^d defines $\|y\|_v = \|X_v^{-1} v(y)\|_2$ for the Euclidean norm $\|\cdot\|_2$ on $\mathbf{C}^d$, with positive definite metric $H_v = (X_v^{-1})^\dagger X_v^{-1}$ ($\|y\|_v^2 = v(y)^\dagger H_v v(y)$) and canonical frame $X_v = H_v^{-1/2}$. For $p \in \mathcal{P}$ of Definition 35.1, the level-zero data furnish at each $t \in T$ a source evaluation $\mathcal{O}_K$-lattice $M_t \subset K^{n_V(t)}$ with Hermitian matrix $H_V(t) \in M_{n_V(t)}(K)$ having positive definite embeddings $H_V(t)_v = v(H_V(t))$ and frames $X_V(t)_v = H_V(t)_v^{-1/2}$ ($v \mid \infty$), a surjective quotient matrix $P_t \in M_{d_Q(t) \times n_V(t)}(K)$ ($P_{t,v} = v(P_t)$) equipping $K^{d_Q(t)}$ with the positive definite quotient Hermitian metric

$$H_Q(t)_v = \left(P_{t,v} H_V(t)_v^{-1} P_{t,v}^\dagger\right)^{-1} \in M_{d_Q(t)}(\mathbf{C})$$

(inducing $\|q\|_v = \inf_{P_{t,v} x = q} \|x\|_v$) and the *direct quotient Hermitian frame*

$$X_Q(t)_v = H_Q(t)_v^{-1/2} = \left(P_{t,v} H_V(t)_v^{-1} P_{t,v}^\dagger\right)^{1/2} \in \mathrm{GL}_{d_Q(t)}(\mathbf{C})$$

satisfying $(X_Q(t)_v^{-1})^\dagger X_Q(t)_v^{-1} = H_Q(t)_v$, and a row-realization matrix $\iota_t : K^{d_Q(t)} \to RR \otimes_{\mathcal{O}_K} K$ such that ι_{μ_0} is injective, $\iota_{\mu_0} P_{\mu_0}$ is onto $RR \otimes_{\mathcal{O}_K} K$, $\mu(w) = \mu_0$ outside finitely many twist places, and $RR_w = \iota_{\mu(w)} P_{\mu(w)} M_{\mu(w),w}$ at every finite place w of K.

Definition 34.47 (Character grids and grids)**.** Fix an arithmetic datum D over E/F (Definition 34.46). A *character frame* over D specifies prime ideals $\bar{\pi}_k \subset \mathcal{O}_{K_d}$ above g_k ($k = 0, 1$), injective characters $\omega_k : (\mathbf{Z}/g_k\mathbf{Z})^\times \hookrightarrow \mathbf{C}^\times$, and unitary Hecke characters of E: base row characters $\mu_{0,j}$ ($0 \le j < m$) trivial on $\mathbb{A}_F^\times$, of angular infinity exponent $-2e_{\tau,j}$ across Ψ, tamely ramified everywhere, unramified outside primes of E

above g_0g_1, and sending any integer a coprime to g_k at any prime $w \mid \bar{\pi}_k$ of E to $\omega_k(a \bmod g_k)^{-2(j+1)}$; the everywhere unramified splitting character η restricting to the quadratic character of E/F on $\mathbb{A}_F^\times$ with angular infinity exponent 1 across Ψ; the everywhere unramified central character χ_Ψ trivial on $\mathbb{A}_F^\times$ with angular infinity exponent 2 across Ψ; and finite-order prime-twist characters δ, δ' at $p = p_{\mathrm{de}}, p'_{\mathrm{de}}$ trivial on $\mathbb{A}_F^\times$, tamely ramified everywhere, unramified outside primes of E above p, and restricting on $\mathcal{O}_{E,v}^\times$ at every prime $v \mid p$ of E to a character of exact order 15. A *level-character datum* $\mathcal{L}$ over D indexed by T adjoins to a character frame level-twist Hecke characters ξ_t of E ($t \in T$), compatible unitary characters on $\mathbb{A}_E^\times = (\mathbb{A}_F \otimes_F E)^\times$ for $\mu_{0,j}, \eta, \xi_t$, a finite set P_g of finite places of E above g_0, g_1, finite ramification-place sets Q_t ($t \in T$) of finite places of E, families of residue-field characters at each finite place of E, one indexed by $\mathbf{N}$ for the base-row data and one indexed by T for the level twists, and reference $\mathcal{O}_E$-lattices $\Lambda_{Y,j}, \Lambda_{X'}, \Lambda_{V,i}, \Lambda_{W,i}$ with finite-adelic frames in $\mathrm{GL}(\cdot, E \otimes_F \mathbb{A}_{F,f})$ on the reference row spaces $Y_{\min(j,m)}$ ($j \in \mathbf{N}$, equal for $0 \le j \le m$ to Y_j of rank $r + jt$ and signature $(r + jt - \lfloor (m-j)/2 \rfloor, \lfloor (m-j)/2 \rfloor)$), the definite space X' (rank $r + mt$, signature $(r + mt, 0)$), the auxiliary spaces V_{i+1}^{sp} ($0 \le i < m$, rank $n_V = r + t$), and the row-layer spaces W_{i+1} ($0 \le i < m$, rank r). Fix a real parameter triple $(c_{\mathrm{gap}}, c_{\mathrm{Dep}}, b)$.

A *character grid* over $\mathcal{L}$ at $(c_{\mathrm{gap}}, c_{\mathrm{Dep}}, b)$ consists of a grid place datum and a grid character datum at gap c_{gap}:

1. The *grid place datum* specifies a finite base place v_0 of F dividing the split auxiliary prime q_0 determined by D in Definition 33.2 (with $q_0 \notin \{g_0, g_1, p_{\mathrm{de}}, p'_{\mathrm{de}}\}$, so $v_0 \notin S_1$ below) and a lift $v \mapsto w(v)$ from finite places of F to finite places of E. Let S_1 be the finite set of prime ideals of $\mathcal{O}_F$ dividing $g_0g_1p_{\mathrm{de}}p'_{\mathrm{de}}$, put $S_0 = \{v_0\} \cup S_1$, and put $W_0 = \{w(v) : v \in S_0\}$. For every $v \in S_0$, $w(v)$ lies over v and admits a *split local idempotent* $e \in \mathbb{A}_E = \mathbb{A}_F \otimes_F E$, defined by $e^2 = e$, $e\bar{e} = 0$, $e + \bar{e} = \varepsilon_v$ (for the place idempotent $\varepsilon_v \in \mathbb{A}_F$ equal to 1 at v and 0 at ∞ and at all finite places $v' \ne v$), and the condition that a nonzero $x \in \mathcal{O}_E$ lies in $\mathfrak{P}_{w(v)}$ if and only if the finite part of ex^{-1} lies outside $\widehat{\mathcal{O}}_E$; for $v \mid g_k$ ($k = 0, 1$), $w(v) \in P_g$; for each $p \in \{p_{\mathrm{de}}, p'_{\mathrm{de}}\}$, a single prime ideal $\mathfrak{p} \subset \mathcal{O}_{K_d}$ lies below $w(v)$ for all $v \mid p$; and for every $t \in T$ and $\mathfrak{P} \in Q_t$ over v of F, one has $v \ne v_0$, while $v \in S_1$ implies $\mathfrak{P} = w(v)$. Each $q \in S_0$ has prime ideal $\mathfrak{q} \subset \mathcal{O}_F$ of prime-power absolute norm $N\mathfrak{q}$ (also written q inside logarithms and w-adic valuations), selected prime ideal $\mathfrak{P}_{w(q)} \subset \mathcal{O}_E$, grid conductor exponent $c_{v_0} = 1$ and $c_q = 2$ for $q \in S_0 \setminus \{v_0\}$, and prescribed positive integer grid depth $d_{\mathrm{pl}}(q)$ at $(c_{\mathrm{gap}}, c_{\mathrm{Dep}}, b)$ defined by $d_{\mathrm{pl}}(v_0) = e_{v_0}(c_{\mathrm{Dep}}, b) = \lfloor P_{\mathrm{pf}}(c_{\mathrm{Dep}}, b) / \log N\mathfrak{q}_{v_0} \rfloor_+ + 1$ and $d_{\mathrm{pl}}(q) = 2$ for $q \in S_1$, where P_{pf} and $\lfloor \cdot \rfloor_+$ are as in Definition 34.50. Put
$$Q_{\mathrm{lev}} = \prod_{q \in S_0} (N\mathfrak{q})^{c_q}, \quad c_S = \log Q_{\mathrm{lev}} = \sum_{q \in S_0} c_q \log N\mathfrak{q}, \quad \mathfrak{f}_F = \prod_{q \in S_0} \mathfrak{q}^{c_q} \subset \mathcal{O}_F, \quad \mathfrak{m}_E = \prod_{q \in S_0} \mathfrak{P}_{w(q)}^{c_q} \subset \mathcal{O}_E.$$
2. The *grid character datum* specifies Hecke characters $\chi_1, \dots, \chi_m$ of E paired with continuous unitary characters on $\mathbb{A}_E^\times$ trivial on the diagonally embedded $E^\times$ and agreeing with the given local characters on $E^\times$ embedded into each finite local factor such that for each $1 \le i \le m$: χ_i is trivial on $\mathbb{A}_F^\times$ (that is, on $F_v^\times$ at every finite and infinite place v of F); its archimedean component satisfies $\chi_{i,\infty}(k) = \prod_{\tau \in \mathcal{T}} (\sigma_\tau(k)/|\sigma_\tau(k)|)^{2\,\mathrm{cen}(\tau,i)}$ ($k \in E^\times$), where $\mathrm{cen}(\tau, i) = c_{\tau,i} \in \mathbf{Z}$ is the floor of the i-th adjacent-weight midpoint of the signed Harish–Chandra grid parameter attached to H_τ, $d_s = r + 6$, and gap c_{gap} (as in Lemma 34.43); for every finite place $\mathfrak{P}$ of E not above S_0, $\chi_{i,\mathfrak{P}}(\mathcal{O}_{E,\mathfrak{P}}^\times) = \{1\}$; for $\mathfrak{P}$ above S_1, $\chi_{i,\mathfrak{P}}(1 + \mathfrak{P}^2\mathcal{O}_{E,\mathfrak{P}}) = \{1\}$ and $\chi_{i,\mathfrak{P}}(1 + \mathfrak{P}\mathcal{O}_{E,\mathfrak{P}}) \ne \{1\}$; for $\mathfrak{P} \mid p_{\mathrm{de}}p'_{\mathrm{de}}$, $\chi_{i,\mathfrak{P}}(x) = 1$ whenever $x \in E_{\mathfrak{P}}^\times$ satisfies $x^{N\mathfrak{P}-1} = 1$; and for $\mathfrak{P}$ above v_0, $\chi_{i,\mathfrak{P}}(1 + \mathfrak{P}\mathcal{O}_{E,\mathfrak{P}}) = \{1\}$ and $\chi_{i,\mathfrak{P}}(\mathcal{O}_{E,\mathfrak{P}}^\times) \ne \{1\}$.

For Hermitian spaces A, B over E/F with Gram matrices J_A, J_B in their frames, an integer $c \ge 0$, and $\epsilon \in \{\pm 1\} \subset E$, a *Hermitian orthogonal-extension decomposition* from A to B of codimension c and sign ϵ consists of an isometric E-linear embedding $\iota_{A,B} : A \hookrightarrow B$ (with frame matrix $M_{\iota_{A,B}} \in M_{\mathrm{rank}\,B \times \mathrm{rank}\,A}(E)$ and $\mathrm{rank}\,A + c = \mathrm{rank}\,B$) and a matrix $a \in M_{\mathrm{rank}\,B \times c}(E)$ satisfying $M_{\iota_{A,B}}^\dagger J_B a = 0$ and $a^\dagger J_B a = G_{c,\epsilon}$, where $(\cdot)^\dagger$ is CM-conjugate transpose and the $c \times c$ signed anti-diagonal Gram matrix $G_{c,\epsilon}$ has (p,q)-entry ($0 \le p, q < c$) equal to ϵ if $p + q + 1 = c$ and $p = q$, 1 if $p + q + 1 = c$ and $p \ne q$, and 0 if $p + q + 1 \ne c$; it assigns to any $\mathcal{O}_E$-lattice Λ_A on A and c-tuple $(\mathfrak{I}_0, \dots, \mathfrak{I}_{c-1})$ of nonzero fractional ideals of $\mathcal{O}_E$ the extension $\mathcal{O}_E$-lattice $\iota_{A,B}(\Lambda_A) \oplus \bigoplus_{p=0}^{c-1} \mathfrak{I}_p a_p$ on B (with a_p the p-th column of a). For nonzero fractional ideals $\mathfrak{b}_1, \mathfrak{b}_2$ of $\mathcal{O}_E$ in E and $c \ge 0$, define the *staircase fractional-ideal tuple* of length c by
$$\mathrm{stair}(\mathfrak{b}_1, \mathfrak{b}_2; c)_p = \mathfrak{b}_1^{c-p}\mathfrak{b}_2^{-(p+1)} \qquad (0 \le p < c).$$

Recall from Lemma 34.22 the host oscillator fractional ideal $\mathfrak{a}_D = (\mathfrak{c}_K\mathcal{O}_E)\mathfrak{b}^{-1}$ of E and its CM conjugate $\bar{\mathfrak{a}}_D$, and let $\mathfrak{a}_{\mathrm{st}}, \bar{\mathfrak{a}}_{\mathrm{st}}$ be the conjugate pair of first-column fractional ideals of $\mathcal{O}_E$ attached to (v_0, w) at $(c_{\mathrm{gap}}, c_{\mathrm{Dep}}, b)$ (as in Lemma 34.23). Put $n_{\mathrm{rk}} = r+1 = k+m$, $d_s = r+6 = k+m+5$, $r_{i,j} = r + jt + id_s$ $(0 \le i, j \le m)$,

$$n_i = n_{\mathrm{rk}} + i(d_s - 1) = (r+1) + i(r+5) \quad (0 \le i \le m), \qquad n'_j = n_m + j(t+1) \quad (0 \le j \le m).$$

A *grid* G over $\mathcal{L}$ at $(c_{\mathrm{gap}}, c_{\mathrm{Dep}}, b)$ adjoins to a character grid at $(c_{\mathrm{gap}}, c_{\mathrm{Dep}}, b)$ a vertex system of Hermitian spaces with $\mathcal{O}_E$-lattices indexed by: grid nodes (i,j) $(1 \le i \le m,\ 0 \le j \le m)$ of rank $r_{i,j}$ and negative signature $\lfloor (m-j)/2 \rfloor$ (with row $i = 0$ given by the reference spaces Y_j $(0 \le j \le m)$ of $\mathcal{L}$ and lattice $\Lambda_{Y,j}$); core nodes P_i $(0 \le i \le m)$ of rank n_i and negative signature $\lfloor (m-i+1)/2 \rfloor$; and positive definite top nodes X_j $(1 \le j \le m)$ of rank n'_j and negative signature 0 (with initial top node $X_0 = P_m$ of rank $n'_0 = n_m$), linked by four families of Hermitian orthogonal-extension decompositions identifying each target $\mathcal{O}_E$-lattice with the corresponding extension lattice of its source: from Y_0 to P_0 of codimension 1 and sign $+1$ with ideal $\mathcal{O}_E$ on $\Lambda_{Y,0}$; from P_i to $(i,0)$ $(1 \le i \le m)$ of codimension $i-1$ and sign -1 with tuple $\mathrm{stair}(\mathfrak{a}_{\mathrm{st}}, \bar{\mathfrak{a}}_{\mathrm{st}}; i-1)$; from X_j to (m,j) $(1 \le j < m)$ of codimension $m-1-j$ and sign -1 with tuple $\mathrm{stair}(\mathfrak{a}_D, \bar{\mathfrak{a}}_D; m-1-j)$ (and at $j = 0$ the $i = m$ decomposition from $X_0 = P_m$ to $(m,0)$); and from (m,m) to X_m of codimension 1 and sign $+1$ with ideal $\mathcal{O}_E$. At a member point p with scale weight b and $C_F \in \mathbf{R}$, a grid on p at C_F is a grid over the level-character datum of p's real realization at $(1, C_F, b)$.

Definition 34.48 (Definite CM theta pairs and coordinate integrality allowances)**.** Let D be an arithmetic datum with imaginary quadratic subfield $K_d = \mathbf{Q}(\sqrt{-d}) \subset E$, real degree $f = [F : \mathbf{Q}]$, CM-embedding index set $\mathcal{T}$, and distinguished embedding $\varphi_0 : E \hookrightarrow \mathbf{C}$.

1. An ideal $\mathfrak{c} \subseteq \mathcal{O}_{K_d}$ is a *curve ideal* for D if $\mathfrak{c} \neq 0$ and $\gcd(N\mathfrak{c}, 2d) = 1$ (in particular $\mathfrak{c} = \mathcal{O}_{K_d}$ is a curve ideal). For $\mathfrak{c} \subseteq \mathcal{O}_{K_d}$ and $\delta \in \mathbf{C}$, set

$$\Lambda(\mathfrak{c}, \delta) = \mathrm{span}_{\mathbf{Z}}\{\delta\,\varphi_0(c) : c \in \mathfrak{c}\} \subset \mathbf{C}, \qquad \mathrm{area}(\Lambda) = \inf\big\{|\operatorname{Im}(\bar{w}_1 w_2)| > 0 : w_1, w_2 \in \Lambda\big\},$$

   with Weierstrass invariants $g_2(\Lambda), g_3(\Lambda) \in \mathbf{C}$, modular discriminant $\mathrm{disc}(\Lambda)$, and Weierstrass function $\wp_\Lambda$. A *curve datum* $(\mathfrak{c}, \delta)$ over D consists of a curve ideal $\mathfrak{c} \subseteq \mathcal{O}_{K_d}$ and a *discriminant twelfth-root period* $\delta \in \mathbf{C}$ defined by $\delta^{12} = \mathrm{disc}(\Lambda(\mathfrak{c}, 1))$. A point $t_0 \in \mathbf{C}$ is an *unkilled two-torsion point* for $\Lambda = \Lambda(\mathfrak{c}, \delta)$ if $2t_0 \in \Lambda$ and $\frac{1 \pm i\sqrt{d}}{2} t_0 \notin \Lambda$; write $\theta_0(\Lambda, t_0)$ for the associated normalized genus-one CM theta series and $\nu(\Lambda, t_0)$ for its Gaussian weighted mean square, the average over a fundamental domain $\mathcal{F}_\Lambda$ of Λ in $\mathbf{C}$ of $|\theta_0(\Lambda, t_0)(z)|^2 \exp(-\pi|z|^2/\mathrm{area}(\Lambda))$. An integer $Q \ge 1$ is an *admissible level modulus* for D if every rational prime $p \mid Q$ satisfies $p = 2$ or $\left(\frac{-d}{p}\right) = 1$; for a rational prime p, a level $Q' \in \mathbf{N}$ is *p-compatible with Q* if its p-adic valuation in $\mathbf{N}$ (with $v_p(0) = 0$) satisfies $v_p(Q') \le v_p(Q)$.

2. For $d_{\mathrm{rk}} \in \mathbf{N}$, let $G_{\mathbb{A}}(d_{\mathrm{rk}}) \le \mathrm{GL}_{d_{\mathrm{rk}}}(\mathbb{A}_{E,f})$ $(\mathbb{A}_{E,f} = E \otimes_{\mathbf{Q}} \mathbb{A}_{\mathbf{Q},f})$ be the finite-adelic unitary group preserving the identity Hermitian form $I_{d_{\mathrm{rk}}}$ over $\mathbb{A}_{\mathbf{Q},f}$ under E's CM conjugation, with rational subgroup $G_{\mathrm{rat}}(d_{\mathrm{rk}}) \le G_{\mathbb{A}}(d_{\mathrm{rk}})$ generated by F-rational unitary matrices and integral subgroup $G_{\mathrm{int}}(d_{\mathrm{rk}}) \le G_{\mathbb{A}}(d_{\mathrm{rk}})$ generated by $\widehat{\mathcal{O}}_E$-integral unitary matrices; for a rational prime p, an element $h \in G_{\mathbb{A}}(d_{\mathrm{rk}})$ is *integral over p* if hk is trivial above p for some $k \in G_{\mathrm{int}}(d_{\mathrm{rk}})$. Given a curve datum $(\mathfrak{c}, \delta)$ over D, an unkilled two-torsion point $t_0 \in \mathbf{C}$ for $\Lambda = \Lambda(\mathfrak{c}, \delta)$, and $Q \in \mathbf{N}$, a *definite CM theta pair* P at $(\mathfrak{c}, \delta, t_0, Q)$ consists of:

   (a) source and target ranks $a_P, d_P \in \mathbf{N}$ with $a_P \le d_P$ and d_P even, and complex configuration space $V_{\mathbf{C}}(a_P, d_P) = \prod_{\tau \in \mathcal{T}} M_{d_P \times a_P}(\mathbf{C})$;

   (b) an *odd-rank CM vacuum* $e_0 : V_{\mathbf{C}}(a_P, d_P) \to \mathbf{C}$ at $(\mathfrak{c}, \delta, t_0)$, defined as an element of the CM theta space of the standard lattices $\mathcal{O}_E^{a_P}, \mathcal{O}_E^{d_P}$ on the CM torus of $(\mathfrak{c}, \delta)$ having algebraic Taylor coefficients and squared norm $\nu(\Lambda, t_0)^{f\,a_P d_P}$ on $\Lambda_0 = \Lambda_0(a_P, d_P)$. For a K_d-basis $b = (b_j)$ of $M_{d_P \times a_P}(E)$, orthonormal for $(x,y) \mapsto \mathrm{tr}_{E/K_d}\mathrm{tr}(x\bar{y}^t)$, put $b_j^{\mathcal{T}} = (\varphi_\tau(b_j))_\tau$, $\Lambda_b = \sum_j \Lambda b_j^{\mathcal{T}}$, and $\theta_b(z) = \prod_j \theta_0(\Lambda, t_0)(\sum_\tau \mathrm{tr}(z_\tau \varphi_\tau(b_j)^\dagger))$. Require $e_0 = \kappa I_{\Lambda_b, \Lambda_0}\theta_b$ for some such b, where I is the normalized lattice average of Lemma 17.14 and $\kappa \in \overline{\mathbf{Q}}$ has absolute value one at every complex embedding. Call b adapted at p if it is also a basis of $\mathfrak{b}^{-1}M_{d_P \times a_P}(\mathcal{O}_E)$ over $\mathcal{O}_{K_d}$ after inverting the integers prime to p, with $\mathfrak{b}$ as in Lemma 34.22. For every prime $p \nmid N\mathfrak{c}$ and every embedding $\lambda : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$, require a presentation $e_0 = \varepsilon I_{\Lambda_b, \Lambda_0}\theta_b$ with b adapted at p, $p \nmid [\Lambda_0 : \Lambda_0 \cap \Lambda_b]$, and $\varepsilon \in \overline{\mathbf{Q}}$ satisfying $|\lambda(\varepsilon)|_p = 1$.

(c) a *compatible Weil representation* $W = (\chi_V, \chi_V', \epsilon_W, \epsilon_W', \Omega_W)$ for $e_0 \neq 0$, consisting of unitary Hecke data χ_V, χ_V' (with archimedean exponent tuple $(m_{W,\tau})_{\tau\in\mathcal{T}} \in \mathbf{Z}^{\mathcal{T}}$ for χ_V'), each of finite conductor, with exponents congruent modulo 2 to its rank (a_P or d_P), whose finite character is trivial on $x\bar{x}$ for every finite idèle x and on integral idèles supported at inert primes, signs $\epsilon_W, \epsilon_W' \in \{\pm 1\}$, and a $\mathbf{C}$-linear group action Ω_W of $G_{\mathbb{A}}(a_P) \times G_{\mathbb{A}}(d_P)$ preserving the CM theta space on the torus of $(\mathfrak{c}, \delta)$, intertwining Heisenberg–Weyl translations in the following sense: for every (g, g'), $z \in M_{d_P\times a_P}(E)$, and CM theta function F, there is an integer $m \geq 1$ such that

$$\Omega_W(g,g')\big(\mathcal{W}_{\mathfrak{c},\delta}(\pi_\delta(z))F\big) = \mathcal{W}_{\mathfrak{c},\delta}(\pi_\delta(z'))\big(\Omega_W(g,g')F\big)$$

for every $z' \in M_{d_P\times a_P}(E)$ with $z' - g'zg^{-1}$ entrywise in $m\widehat{\mathfrak{c}\mathcal{O}_E\mathfrak{b}^{-1}}$, fixing e_0 under integral elements supported at inert primes, and acting on e_0 at elements (g, g') supported at split primes (moving $\mathcal{O}_E^{a_P}, \mathcal{O}_E^{d_P}$ to L_1, L_1' with split-adjusted determinants $x, x' \in \mathbb{A}_E^\times$ trivial mod inert primes) by $\chi_V(x')^{\epsilon_W}\chi_V'(x)^{\epsilon_W'}$ times the lattice-change operator from $\Lambda_0(a_P, d_P)$ to $\Lambda(L_1, L_1')$;

(d) a degree tuple $n_P = (n_{P,\tau})_{\tau\in\mathcal{T}} \in \mathbf{N}^{\mathcal{T}}$ satisfying $2n_{P,\tau} = \epsilon_W' m_{W,\tau} - d_P$ for all $\tau \in \mathcal{T}$;

(e) a test function $F_P : V_{\mathbf{C}}(a_P, d_P) \to \mathbf{C}$ invariant under $\Omega_W(k, 1)$ for all $k \in G_{\mathrm{int}}(a_P)$ and *preserved at level Q relative to e_0*, meaning that there exist $k_{\mathrm{pr}} \in \mathbf{N}$, $r_1, \dots, r_{k_{\mathrm{pr}}} \in \overline{\mathbf{Q}}$, and $z_1, \dots, z_{k_{\mathrm{pr}}} \in M_{d_P\times a_P}(E)$ with $Q\cdot\pi_\delta(z_s) \in \Lambda_0(a_P, d_P)$, $Q^{a_P d_P} r_s$ integral over $\mathbf{Z}$ ($1 \leq s \leq k_{\mathrm{pr}}$), $\big(\sum_{s=1}^{k_{\mathrm{pr}}} |\rho(r_s)|\big)^2 \leq Q^{a_P d_P}$ for every ring homomorphism $\rho : \overline{\mathbf{Q}} \to \mathbf{C}$, and $F_P(y) = \sum_{s=1}^{k_{\mathrm{pr}}} r_s\big(\mathcal{W}_{\mathfrak{c},\delta}(\pi_\delta(z_s))e_0\big)(y)$; and

(f) a subgroup $J_P \leq G_{\mathrm{int}}(d_P)$ of finite index $1 \leq [G_{\mathrm{int}}(d_P) : J_P] \leq Q^{d_P^2+4}$ acting on F_P under $\Omega_W(1, j)$ ($j \in J_P$) by scalars of modulus 1.

For $N \in \mathbf{N}$ and $B \in \mathbf{R}$, P is (N, B)-*small* if $d_P \leq 2N$ and $n_{P,\tau} \leq 2B$ for all $\tau \in \mathcal{T}$.

3. For $u \in \mathbf{N}$, a subgroup $J \leq G_{\mathbb{A}}(u)$ and $g \in G_{\mathbb{A}}(u)$, write $\mathrm{Cls}_D(u, J) = G_{\mathrm{rat}}(u)\backslash G_{\mathbb{A}}(u)/J$ for the double-coset class set and $|\Gamma_D(J, g)|$ for the number of $\gamma \in G_{\mathrm{rat}}(u)$ with $g^{-1}\gamma g \in J$. For a definite CM theta pair P, the class sets $\mathrm{Cls}_D(a_P, G_{\mathrm{int}}(a_P))$ and $\mathrm{Cls}_D(d_P, J_P)$ are finite and the stabilizer orders $|\Gamma_D(G_{\mathrm{int}}(a_P), x)|$ and $|\Gamma_D(J_P, h)|$ ($x \in G_{\mathbb{A}}(a_P)$, $h \in G_{\mathbb{A}}(d_P)$) are positive integers. Let $\mathrm{Tup}_D(a_P, d_P, n_P)$ be the finite set of *minor tuples* $T = (T_{\tau,j})_{\tau\in\mathcal{T},\, 1\leq j\leq n_{P,\tau}}$, ordered in j and with repetitions allowed, where each $T_{\tau,j}$ is an a_P-element subset of $\{1, \dots, d_P\}$. Write $z_{\tau,r,i}$ for the (r, i) entry of the τ-component of $z \in V_{\mathbf{C}}(a_P, d_P)$, and let

$$\partial_T = \prod_{\tau\in\mathcal{T}} \prod_{j=1}^{n_{P,\tau}} \det\Big(\frac{\partial}{\partial z_{\tau,r,i}}\Big)_{r\in T_{\tau,j},\, 1\leq i\leq a_P}$$

be the constant-coefficient holomorphic differential operator of the product of these minors, the rows of each determinant taken in increasing order (no factorial normalization). Fix a representative x_c of each $c \in \mathrm{Cls}_D(a_P, G_{\mathrm{int}}(a_P))$ and a representative h_q of each $q \in \mathrm{Cls}_D(d_P, J_P)$. The *class-tuple coordinate* $c_P(h, T) \in \mathbf{C}$ of $h \in G_{\mathbb{A}}(d_P)$, its pointwise squared sum, and the *norm* of P are

$$c_P(h,T) = \sum_{c\in\mathrm{Cls}_D(a_P, G_{\mathrm{int}}(a_P))} |\Gamma_D(G_{\mathrm{int}}(a_P), x_c)|^{-1}\big(\partial_T\, \Omega_W(x_c, h)F_P\big)(0),$$

$$\mathrm{sq}_P(h) = \sum_{T\in\mathrm{Tup}_D(a_P,d_P,n_P)} |c_P(h,T)|^2,$$

$$\mathrm{norm}(P) = \sum_{q\in\mathrm{Cls}_D(d_P,J_P)} |\Gamma_D(J_P, h_q)|^{-1}\, \mathrm{sq}_P(h_q) \geq 0.$$

The summands of $c_P(h, T)$ are unchanged under $x_c \mapsto \gamma x_c k$ ($\gamma \in G_{\mathrm{rat}}(a_P)$, $k \in G_{\mathrm{int}}(a_P)$), and sq_P and $|\Gamma_D(J_P, \cdot)|$ are unchanged under $h \mapsto \gamma h j$ ($\gamma \in G_{\mathrm{rat}}(d_P)$, $j \in J_P$), so $c_P(h, T)$ and $\mathrm{norm}(P)$ do not depend on the chosen representatives. On a grid G over $\mathcal{L}$ at $(c_{\mathrm{gap}}, c_{\mathrm{Dep}}, b)$, for $t \in T$, $0 \leq i, j < m$, an ideal $\mathfrak{m} \subseteq \mathcal{O}_E$, and (δ, t_0, Q), define the larger and smaller bottom source ranks and splitting characters by

$$a_+(i,j) = r_{i+1,j+1} - 2\big\lfloor \tfrac{m-j-1}{2} \big\rfloor, \qquad a_-(i,j) = r_{i,j} - 2\big\lfloor \tfrac{m-j}{2} \big\rfloor,$$

$$\alpha_{t,i,j} = \mu_{0,j}\xi_t\eta^{r_{i,j} \bmod 2}, \qquad \beta_{i,j} = \chi_{m-i}\eta^{r_{i,j} \bmod 2}, \qquad \nu_{t,i,j} = \mu_{0,j}\xi_t\chi_{m-i}^{-1}.$$

A pair (P,P') of definite CM theta pairs at $(\mathcal{O}_{K_d},\delta,t_0,Q)$ is a *standard bottom pair* on G at $(t,i,j,\mathfrak{m})$ if: $a_P=a_+(i,j)$, $a_{P'}=a_-(i,j)$, and $d_P=d_{P'}=a_+(i,j)+a_-(i,j)$; on all finite idèles, $(\chi_{V,P}^{\epsilon_{W,P}},\chi'_{V,P}{}^{\epsilon'_{W,P}})=(\beta_{i,j}^{-1},\nu_{t,i,j}^{-1})$ and $(\chi_{V,P'}^{\epsilon_{W,P'}},\chi'_{V,P'}{}^{\epsilon'_{W,P'}})=(\alpha_{t,i,j},\nu_{t,i,j}^{-1})$; $F_P=\sum_{r\in R}\Omega_{W,P}(r,1)\phi_{\cos}(P;\mathfrak{m},0)$ and $F_{P'}=\sum_{r'\in R'}\Omega_{W,P'}(r',1)\phi_{\cos}(P';\bar{\mathfrak{m}},a_+(i,j))$ for finite coset-representative sets $R\subseteq G_{\mathrm{int}}(a_P)$ modulo $\mathfrak{m}$ and $R'\subseteq G_{\mathrm{int}}(a_{P'})$ modulo $\bar{\mathfrak{m}}$ of the coset vacua at target-row offsets 0 and $a_+(i,j)$; and both J_P and $J_{P'}$ equal $\{h\in G_{\mathrm{int}}(d_P):h_{u,v}\in\widehat{\mathfrak{m}}\text{ for all }0\le v<a_+(i,j)\le u<d_P\}$.

4. For $C\in\mathbf{R}$, integers $f',N,Q\ge1$, and $B,L\in\mathbf{R}$, set

$$I(C;f',N,B,L,Q)=2Cf'N\big(\log(2f'N)+BL+2NB\log Q\big).$$

Let S' be a finite set of rational primes and $\delta_{\mathrm{vp}}:\mathbf{N}\to\mathbf{R}_{\ge0}$ a function vanishing outside S' with $\sum_{p\in S'}\delta_{\mathrm{vp}}(p)\log p\le b_{\mathrm{int}}$. Say that δ_{vp} satisfies the *p-adic coordinate integrality bound* for (D,N,B,Q) if for every discriminant twelfth-root period $\delta\in\mathbf{C}$ of $\mathcal{O}_{K_d}$, unkilled two-torsion point $t_0\in\mathbf{C}$ for $\Lambda(\mathcal{O}_{K_d},\delta)$, rational prime p ($|p|_p=p^{-1}$ on $\mathbf{C}_p$), p-compatible level $Q'\in\mathbf{N}$, and (N,B)-small definite CM theta pair P at $(\mathcal{O}_{K_d},\delta,t_0,Q')$: every class $q\in\mathrm{Cls}_D(d_P,J_P)$ admits a representative $h\in q$ integral over p; every $h\in G_{\mathbb{A}}(d_P)$ satisfies $|\Gamma_D(J_P,h)|\ne0$ and $v_p(|\Gamma_D(J_P,h)|)\le\delta_{\mathrm{vp}}(p)$; and for every $h\in G_{\mathbb{A}}(d_P)$ integral over p and $T\in\mathrm{Tup}_D(a_P,d_P,n_P)$, one has $c_P(h,T)\in\overline{\mathbf{Q}}$ and

$$|\lambda(c_P(h,T))|_p\le p^{\delta_{\mathrm{vp}}(p)}$$

for every ring homomorphism $\lambda:\overline{\mathbf{Q}}\to\mathbf{C}_p$. Say that $(S',\delta_{\mathrm{vp}})$ satisfies the *norm valuation bound* for (D,N,B,Q) with budget b_{int} if for every discriminant twelfth-root period $\delta\in\mathbf{C}$ of $\mathcal{O}_{K_d}$, unkilled two-torsion point $t_0\in\mathbf{C}$ for $\Lambda(\mathcal{O}_{K_d},\delta)$, totally complex number field K with $\sigma:K\hookrightarrow\mathbf{C}$, index set $\mathcal{I}$, families $(P_\alpha)_{\alpha\in\mathcal{I}},(P'_\alpha)_{\alpha\in\mathcal{I}}$ of (N,B)-small definite CM theta pairs at $(\mathcal{O}_{K_d},\delta,t_0,Q)$ each having at least one nonzero coordinate, and $z_{A,\alpha},z'_{A,\alpha}\in K$ with $\sigma(z_{A,\alpha})=\mathrm{norm}(P_\alpha)$ and $\sigma(z'_{A,\alpha})=\mathrm{norm}(P'_\alpha)$, the local allowance $d_\ell(w)=\sum_{p\in S'}\delta_{\mathrm{vp}}(p)v_w(p)\ge0$ satisfies $\sum_{w\in U}\ell_w d_\ell(w)\le b_{\mathrm{int}}$ for every finite set U of finite places of K, together with $z_{A,\alpha}\ne0$, $z'_{A,\alpha}\ne0$, $v_w(z_{A,\alpha})\ge-3d_\ell(w)$, and $v_w(z'_{A,\alpha})\ge-3d_\ell(w)$ for all $\alpha\in\mathcal{I}$ and finite w. Finally, for a grid G at $(c_{\mathrm{gap}},c_{\mathrm{Dep}},b)$, define the two integer bottom grid weights at (i,j,τ) $(0\le i,j<m,\ \tau\in\mathcal{T})$ by

$$w_A(c_{\mathrm{gap}};i,j,\tau)=\Big\lfloor\frac{2\,e^{\mathrm{gr}}_{\tau,j+1}-(r_{i,j}\bmod2)-a_-(i,j)}{2}\Big\rfloor\in\mathbf{Z},$$
$$w_{A'}(c_{\mathrm{gap}};i,j,\tau)=\Big\lfloor\frac{-2\,\mathrm{cen}(c_{\mathrm{gap}},\tau,m-i)-(r_{i,j}\bmod2)+a_+(i,j)}{2}\Big\rfloor\in\mathbf{Z},$$

where $e^{\mathrm{gr}}_{\tau,j+1}\in\mathbf{Z}$ is the $(j+1)$-st upper grid weight of τ at gap c_{gap} (as in Lemma 34.43), and put

$$B_w(c_{\mathrm{gap}})=\max\Big(1,\max_{\substack{0\le i,j\le m\\ \tau\in\mathcal{T}}}\max\big(|w_A(c_{\mathrm{gap}};i,j,\tau)|,|w_{A'}(c_{\mathrm{gap}};i,j,\tau)|\big)\Big)\in\mathbf{N}_{>0}\subset\mathbf{R}_{\ge1},$$

abbreviated $B_w=B_w(1)$ at gap 1; a function d_ℓ from finite places of K to $\mathbf{R}$ is a *grid integrality allowance* on G at constant $C\in\mathbf{R}$ if some $(S',\delta_{\mathrm{vp}})$ satisfies both the p-adic coordinate integrality bound and the norm valuation bound for $(D,N_*,B_w(c_{\mathrm{gap}}),Q_{\mathrm{lev}})$ with budget $I(C;f,N_*,B_w(c_{\mathrm{gap}}),\log b,Q_{\mathrm{lev}})$ and $d_\ell(w)=\sum_{p\in S'}\delta_{\mathrm{vp}}(p)v_w(p)$ for every finite place w of K.

Definition 34.49 (Square-recursive size data, full grids, and fitted grids)**.** Let D be an arithmetic datum and L_{sz} a size datum over D at scale s of exponent ε, with $L=x^\varepsilon$ and $b=\exp(L)$.

1. The size datum L_{sz} is *square-recursive* if: L_{sz} satisfies the row size conditions, comprising (34.12)–(34.14), the pointwise size inequalities of Definition 34.19, the existence of compatible window/tower parameters with weight-gap constant 10, $\mathfrak{c}_K=\mathcal{O}_{K_d}$, and $b_{\tau,l}>0$, $e_{\tau,l}\le4b$ for all $\tau\in\mathcal{T}$ and $0\le l<m$; D satisfies $F=E^+$ together with the host parameter-interlacing and gap conditions ($(t+3)/2$ bounds on $e_{\tau,j}-b_{\tau,j}$ and $b_{\tau,j+1}-e_{\tau,j}$, consecutive row-parameter and cloud-boundary gaps at least $100r^4$, and node gaps at least $\max(4N(m)^3+2t+6,256r^4)$); and the quadratic character χ_d of K_d satisfies $\big|\mathrm{Re}(L'(1,\chi_d)/L(1,\chi_d))\big|\le x^\varepsilon$. On an indefinite row-side datum over a finite group G_p with subgroups $P_0,P_1\le G_p$ and characters $\psi_0:P_0\to K^\times$, $\psi_1:P_1\to K^\times$, the *level-2 face* F_{fc} retains

(P_0,ψ_0,P_1,ψ_1) while setting the definite-column index set to $\varnothing$, the period subspace to $0\subseteq\mathbf{C}$, and all eleven auxiliary face scalars to 0. Given a real realization R over G_p, tuple index set T with base index μ_0, and subfield $K_s\subseteq K$, the *standard sign--1 normalization at* (P_0,ψ_0) requires R's additive character $\psi:\mathbb{A}_F/F\to\mathbf{C}^\times$ to have archimedean part $x_\infty\mapsto\exp\bigl(-2\pi i\sum_\rho\theta_\rho(x_\infty\otimes 1)\bigr)$ and, for every family of split local idempotents $(e_{\mathfrak{P}})_{\mathfrak{P}\in P_g}$ and every $u\in P_0$,

$$\iota(\psi_0(u))=\prod_{\mathfrak{P}\in P_g}\psi_{e_{\mathfrak{P}}}\bigl((N\mathfrak{P})^{-1}\,S_{\mathrm{tr}}(e_G(u)_{\mathfrak{P}})\bigr),$$

where $S_{\mathrm{tr}}(g_{\mathfrak{P}})\in\mathcal{O}_E$ lifts the superdiagonal residue sum of $g_{\mathfrak{P}}\in\mathrm{GL}_r(\mathcal{O}_E/\mathfrak{P})$. The triple $(L_{\mathrm{sz}},R,F_{\mathrm{fc}})$ satisfies the *Hida row-setting conditions* if L_{sz} satisfies the row size conditions, D satisfies the host conditions with $F=E^+$, R satisfies the Hida level and residue-group type conditions of Theorem 34.39 and Definition 34.40 (including $Q_t\subseteq\mathrm{Ram}(\xi_t)\subseteq Q_t\cup\overline{Q}_t$ disjoint from q_0, and realized along $\mathcal{P}$ from the series condition of Definition 34.24), and F_{fc} carries the standard sign--1 normalization at (P_0,ψ_0). At the base level $\ell_0=\ell(P_0,\psi_0)$ of F_{fc}, write $M_{t,\mathrm{lv}}(F_{\mathrm{fc}})\subset K^{n_V(t)}$ for the source evaluation $\mathcal{O}_K$-lattice at $t\in T$ (equal at $p\in\mathcal{P}$ to M_t of Definition 34.46), and for $x\in K^{n_V(t)}$ define the row-source form $\mathrm{form}_{F_{\mathrm{fc}}}(t,x)=\sum_{\alpha=1}^{d_{\mathrm{amb}}(t)}(v_{E,t}(\ell_0)x)_\alpha b_{t,\alpha}\in FR_t$ via the level-inclusion matrix $v_{E,t}(\ell_0)\in M_{d_{\mathrm{amb}}(t)\times n_V(t)}(K)$ into the K-basis $(b_{t,\alpha})$ of FR_t in Theorem 34.41.

2. Let G be a grid at $(1,C_F,b)$ with $W_0=\{w(v):v\in S_0\}$. A *Weil system* W on G extending R assigns to each grid arrow (S,T_{sp}) of G with sign $\epsilon\in\{\pm1\}$ and unitary Hecke splitting characters $(\chi_S,\chi_{T_{\mathrm{sp}}})$ restricting on $\mathbb{A}_F^\times$ to $(\eta^{\mathrm{rank}\,S},\eta^{\mathrm{rank}\,T_{\mathrm{sp}}})$ a continuous unitary oscillator Weil representation $\omega_{S,T_{\mathrm{sp}}}$ of $\mathrm{U}(S)(\mathbb{A}_F)\times\mathrm{U}(T_{\mathrm{sp}})(\mathbb{A}_F)$ on theta tests on $\mathrm{Hom}_E(S,T_{\mathrm{sp}})(\mathbb{A}_F)$ for R's sign--1 character ψ (in the model of Lemma 34.21 and Definition 34.34, agreeing with R on row arrows) that intertwines Heisenberg shifts by hw_0g^{-1}, fixes evaluation at 0 under $\mathrm{U}(S)(F)\times\mathrm{U}(T_{\mathrm{sp}})(F)$, preserves L^2 norm on the rational additive quotient, and acts at every split local idempotent $e\in\mathbb{A}_E$ on e-invariant functionals by a positive real multiple of the local determinant factor of $(\chi_S,\chi_{T_{\mathrm{sp}}})$. For $t\in T$ and any finite place w' of E over v of F, set $\mathcal{P}(G,t)=W_0\cup Q_t$, $e_{\mathrm{dep}}(G,w')=d_{\mathrm{pl}}(v)$ if $w'\in W_0$ and 0 otherwise, and $d_G(w')=\max(e_{\mathrm{dep}}(G,w'),1)$. A *finite-adelic lattice frame* of an $\mathcal{O}_E$-lattice $\Lambda\subset E^d$ is an element $\beta\in\mathrm{GL}_d(E\otimes_F\mathbb{A}_{F,f})$ satisfying $\widehat{\Lambda}=\beta(\widehat{\mathcal{O}}_E^d)$. A *grid choice* S_0^{gr} on $U\subseteq T$ consists of: finite-adelic lattice frames $\beta_{i,j}$ on the grid nodes (i,j) $(0\le i,j\le m$, equal to $\mathcal{L}$'s frames at $i=0)$ and $\beta_{P,j}$ on the top nodes X_j $(0\le j\le m)$ such that at every $t\in U$ and $w'\in\mathcal{P}(G,t)$ with uniformizer $\varpi_{w'}\in E_{w'}$, for each $0\le j<m$ (with d_j the codimension-c_j decomposition from X_j to (m,j), $c_j=m-1-j$), $\beta_{m,j,w'}=\bigl(M_{\iota_{d_j}}\beta_{P,j,w'}\mid a_{d_j}\,\mathrm{diag}(u_{j,w',0},\dots,u_{j,w',c_j-1})\bigr)$ where $u_{j,w',p}=\epsilon_{\mathrm{fr}}(c_j,p)\varpi_{w'}^{\mathbf{1}_{j=0}\,e_{\mathrm{dep}}(G,w')(p+1)}$ $(\epsilon_{\mathrm{fr}}(c,p)=-1$ for $2p+1>c$ and $+1$ for $2p+1\le c)$, while at $j=m$ (for d_{top} from (m,m) to X_m) $\beta_{P,m,w'}=\bigl(M_{\iota_{d_{\mathrm{top}}}}\beta_{m,m,w'}\mid a_{d_{\mathrm{top}}}\bigr)$; *standard normalizations* $(N_{\mathrm{std}},Z_{\mathrm{norm}})$ on U, comprising a finite rational prime set S_{bad} containing all primes below $S_0,\bigcup_{t\in U}Q_t$ and the prime divisors of $Q_S\,q\,d\,g_0g_1\,N\mathfrak{c}_K$, global E-frames on all nodes of G orthonormal on definite nodes and compatible with d_{top}, finite-adelic unitary elements supported at split places off S_{bad} carrying those frames' $\mathcal{O}_E$-lattices to $\widehat{\Lambda}_X$ on definite nodes, with the elements at X_m and (m,m) intertwining the top embedding and the element at X_m fixing the last extension column, and a curve datum $(\mathfrak{c}_K,\delta_Z)$ with unkilled two-torsion point $t_{0,Z}$ (Definition 34.48(1)), odd-rank CM vacua $e_{0,\wp}$, oscillator vacua $\phi_{0,\wp}$, and scalars $c_{0,\wp}\in\mathbf{C}^\times$ across definite grid pairs $\wp$ (independent of t on top pairs) equating evaluation at 0 of Fock monomials on $\phi_{0,\wp}$ with $c_{0,\wp}$ times the δ_Z-period product and the $t_{0,Z}$-jet of $e_{0,\wp}$ at 0; and a finite set P_A of places of E in the support of $\mathfrak{a}_D$ or $\bar{\mathfrak{a}}_D$ containing a unique place above each place of F below the support of $\mathfrak{a}_D$. Writing $\mathcal{W}_L(\lambda)=\bigotimes_{\tau\in\mathcal{T}}\nabla_L(\lambda(\tau))$ for the tensor product of Schur–Weyl highest-weight polynomial modules over a field L, a *sum choice* S on $U\subseteq T$ extends S_0^{gr} by a $\mathbf{C}$-linear map $\Phi_{\mathrm{co}}:W_{c_X}\to\mathcal{W}_{\mathbf{C}}(\lambda_{\mathrm{list}}(0,m))$ from R's top-column coefficient space to the listed-weight polynomial module at $(0,m)$ that acts, under a ι-semilinear bridge family tie $c_{0,R}:\mathcal{W}_K(\lambda_R)\to W_{c_X}$ (and $c_{0,L,i}$ on V_{i+1}^{sp}) intertwining the unitary actions, pairings, and integral lattices off $S_b\cup Q_t$, by applying ι to polynomial coefficients and reindexing $\{1,\dots,\mathrm{rank}\,X'\}\simeq\{1,\dots,r_{0,m}\}$. For $U=T$, S satisfies the *evaluation-field condition* at μ_0 under $\iota:K\hookrightarrow\mathbf{C}$ (that is, the coefficient-field condition preceding Theorem 34.39) if $\iota(K)$ contains the normalized local values of $\mu_{0,j}$ $(0\le j<m)$, ξ_t $(t\in T)$, and χ_i $(1\le i\le m)$ at weight 0, η at weight $1/2$, and χ_Ψ,δ,δ' at weight 0, all 8-th roots of unity,

all $(Nw')^{d_G(w')}$-th roots of unity for $w' \in W_0 \cup \bigcup_{t\in T} Q_t$, $\sqrt{Nw'}$ for $w' \in P_A \cup \bigcup_{t\in T}(W_0 \cup Q_t)$, the local Heisenberg phases on nonvanishing moved test terms at $w' \in \mathcal{P}(G,t)$ and CM jet coefficients on supported coset-expansion classes, the Hecke L-value ratios $L(s,\nu_{t,i,j})/L(s,\nu_{\mu_0,i,j})$ $(0 \le i,j < m,\ t \in T,\ 1 \le s \le N_* + 1)$ whenever $L(s,\nu_{\mu_0,i,j}) \ne 0$, and the period-normalized bottom class sums of Lemma 34.38. On $X_0 = P_m$ of rank $n_p = n_m = (r+1)+m(r+5)$, let $\mathrm{Cl}_+(G)$ be its finite congruence double-coset class set in $\mathrm{U}(X_0)(\mathbb{A}_{F,f})$, let $\mathcal{D}_{\mathrm{test}}(G)$ be the finite index set of prescribed row-coset grid test data, put $\mathcal{I}_{\mathrm{test}}(G) = \mathcal{D}_{\mathrm{test}}(G) \times \mathrm{Cl}_+(G)$, let $\lambda_{P,G} = (\lambda_{P,G}(\tau))_{\tau\in\mathcal{T}}$ be the antitone integer dominant weight tuple of X_0 at gap 1 with shifted nonnegative tuple $\nu_{P,G}(\tau)_a = \lambda_{P,G}(\tau)_a - \lambda_{P,G}(\tau)_{n_p-1} \in \mathbf{N}$, and put $\mathcal{I}_{\mathrm{coef}}(G) = \prod_{\tau\in\mathcal{T}} \mathcal{T}_{\mathrm{ss}}(n_p,n_p;\nu_{P,G}(\tau))$, where $\mathcal{T}_{\mathrm{ss}}(n_p,n_p;\nu)$ is the finite set of interlaced $n_p \times n_p$ nonnegative integer matrices $(m_{a,q})$ with column sums $\sum_{a=0}^{n_p-1} m_{a,q} = \nu_q$ indexing the Weyl basis of $\nabla(\lambda_{P,G}(\tau))$. A *complete choice* a of ambient dimension $n \in \mathbf{N}$ adjoins to a sum choice S on T: Haar measures and fundamental domains on $\mathrm{U}(V_X)(\mathbb{A}_F)$ for all nodes X of G and on the top Bessel unipotent flag of $(m,0)$; a real test vector ρ and linear functional ξ on the dual highest-weight module of $\lambda_{P,G}$ on X_0; downward chain vectors (Fock polynomials on (Y_{l+1},Y_l) for $0 \le l < m-1$ and on (X',Y_{m-1}), together with a smooth tensor in the bottom-row archimedean space M_X); real matching data on W (arrow vacua matching W's row vacua at $j=0$ and proper submodules of polynomial-route pairs at each (i,j) surjecting onto both routes); class representatives $x_c \in \mathrm{U}(X_0)(\mathbb{A}_{F,f})$ $(c \in \mathrm{Cl}_+(G))$; a bijection $e_a : \{1,\dots,n\} \xrightarrow{\sim} \mathcal{I}_{\mathrm{test}}(G) \times \mathcal{I}_{\mathrm{coef}}(G)$; and seesaw constants $c_{ij,t} \in \mathbf{C}$ $(0 \le i,j < m,\ t \in T)$, meeting the separation, rationality, and series conditions of Theorem 34.41 and Definition 34.24; for $t \in T$, $x' \in FR_t$, and $1 \le i \le n$, write $v_{\mathrm{row}}(a,t,x')_i \in \mathbf{C}$ for the evaluated row-pairing sum on the realization of x' at t, using the grid test datum, class representative and coefficient basis vector selected by $e_a(i)$. A *full grid* over R at $(1,C_F,b)$ is a grid G with compatible Weil system W over R admitting a complete choice satisfying both the evaluation-field condition and the square-field condition of Lemma 34.42.

3. At a member point p and constant C_F, a grid G on p at C_F *fits* p if: $\iota(K_0)$ contains the unramified normalized values of p's character frame $(\mu_{0,j},\chi_\Psi,\delta,\delta'$ at weight 0 and η at weight $1/2)$, the unramified weight-0 values of $\chi_1,\dots,\chi_m$, and a primitive $(N\mathfrak{q})^{d_{\mathrm{pl}}(q)}$-th root of unity for every $q \in S_0$ (as in Section 1 of Chapter 33); at every twisted finite place w of K $(\mu_{\mathrm{at}}(w) \ne \mu_0)$, its twist index $a_{\mathrm{tw}} = \mathrm{tw}(w)$ has residue prime $\ell_{a_{\mathrm{tw}}} = \mathrm{char}(w)$, its underlying place lies under w or $\bar{w}$, and $\mathrm{tw}(w)$ agrees across paired twisted places whenever $Q(w) \ne \varnothing$; for every such w, every $\mathfrak{P} \in Q(w)$ splits in E/F, $Q(w)$ is closed under CM conjugation, and $Q(w)$ is disjoint from the excluded finite set $S_0(a_{\mathrm{tw}})$ of places of E at a_{tw} (which contains the places above $\ell_{a_{\mathrm{tw}}}, g_0, g_1$ and the selected prime-twist prime in $\{p_{\mathrm{de}},p'_{\mathrm{de}}\} \setminus \{\ell_{a_{\mathrm{tw}}}\}$); for every active twist index a_{tw} there exists $N_{\mathrm{tw}}(a_{\mathrm{tw}}) \in \mathbf{N}$ such that for every twisted place w with $\mathrm{tw}(w) = a_{\mathrm{tw}}$, integer m' coprime to $\ell_{a_{\mathrm{tw}}}$, and abelian layer index λ at a_{tw}, the evaluated twisted toric element for $\gamma_w^{m'}$ is nonzero with normalized $\ell_{a_{\mathrm{tw}}}$-adic valuation in $[-N_{\mathrm{tw}}(a_{\mathrm{tw}}),\ c_0(a_{\mathrm{tw}},\lambda)+\ell_{a_{\mathrm{tw}}}^{-2}]$ (clause 7 of Definition 33.3, in the notation of Definition 33.2); at every twisted place w (with $a_{\mathrm{tw}} = \mathrm{tw}(w)$) and Frobenius element at a finite place $v \notin Q(w)$ of E off $\ell_{a_{\mathrm{tw}}}$, preimages in K of $\psi_w(v)^{-1}$ match the local Artin character values; and at every active a_{tw}, $S_0(a_{\mathrm{tw}})$ contains all places above the base conductor at $w(v_0)$ and the abelian layer characters and toric elements at a_{tw} match those of G through a single embedding of $\mathbf{C}$ into the local algebraic closure extending the prescribed embedding of K_0, using this embedding or its conjugate as prescribed by the layer's side. Every abelian layer matches some $0 \le j,a < m$ and $1 \le s \le N_* + 1$; conversely, for every $0 \le j,a < m$, $1 \le s \le N_* + 1$, and either side, a matching layer on that side exists. A *fitted grid* at (p,C_F) consists of a grid G fitting p, a Weil system W on G over p's real realization, and a complete choice on (p,G,W) satisfying the evaluation-field condition and the square-field condition of Lemma 34.42. Put $T_{\mathrm{use}}(p) = \{\mu_0\} \cup \{\mu(w) : w \text{ a finite place of } K\}$. A *fitted evaluation datum* d at (p,C_F) packages (G,W) with $n \in \mathbf{N}$, matrices $X'_v \in M_n(\mathbf{C})$ $(v \mid \infty)$, an $\mathcal{O}_K$-submodule $M \subset K^n$, matrices $V_t^0 \in M_{n\times nV(t)}(K)$ $(t \in T)$, elements $r_{ij,t} \in K$ $(t \in T,\ 0 \le i,j < m)$, a function d_ℓ on finite places of K, finite sets $S_d, S_e \subset \mathbf{N}$, functions $N_0(i,j,s,l) \in \mathbf{R}$ $(0 \le i,j < m,\ s \in \mathbf{Z},\ l \in \mathbf{N})$, bottom elements $z_A(i,j), z'_A(i,j) \in K$, higher toric elements $z_T(i,j,s), z_{T,0}(i,j,s), z_{T,1}(i,j,s) \in K$ with shifts $g_T(i,j,s,w) \in \mathbf{Z}$, selector $c(w) \in \{0,1\}$, and $B'_w \in \mathbf{R}$, for which some complete choice a of ambient dimension n on (p,G,W) satisfies:

   (a) a satisfies the separation, rationality, series, evaluation-field, and square-field conditions;

(b) $(M,(X'_v)_{v|\infty})$ is the full-rank direct-sum Hermitian test $\mathcal{O}_K$-lattice in K^n ($M\otimes_{\mathcal{O}_K}K=K^n$) defined from the form lattice $L_{\mathrm{form}}(G,x_c,e_a)\subset K^n$ and Hermitian form Gram matrix $G_{\mathrm{form}}(G,e_a)\in \mathrm{GL}_n(K)$ assembled along $e_a:\{1,\dots,n\}\xrightarrow{\sim}\mathcal{I}_{\mathrm{test}}(G)\times\mathcal{I}_{\mathrm{coef}}(G)$ from the class-representative lattices at x_c on X_0 and the Weyl module of $\lambda_{P,G}$ by taking the dual lattice $M=L_{\mathrm{form}}(G,x_c,e_a)^\vee$ and frames satisfying $(X'^{-1}_v)^\dagger X'^{-1}_v=\big(v(G_{\mathrm{form}}(G,e_a))^{-1}\big)^t$ at every $v\mid\infty$;

(c) for every $t\in T_{\mathrm{use}}(p)$, $x\in K^{n_V(t)}$, and $1\le i\le n$, $\iota((V^0_t x)_i)=v_{\mathrm{row}}(a,t,e_{V,0}(p,t,x))_i$, where $e_{V,0}(p,t,x)=\sum_{\alpha=1}^{d_{\mathrm{amb}}(t)}(v_{E,t}(\ell_0)x)_\alpha b_{t,\alpha}\in FR_t$ is the level-zero row-source image of x at $\ell_0=\ell(P_0,\psi_0)$; and

(d) for every $0\le i,j<m$, there exist seesaw proportionality constants $c_{ij}(t)\in\mathbf{C}$ ($t\in T$, obtained via the seesaw comparison scalar of Definition 34.34, Lemma 34.36, and Theorem 34.37) with $c_{ij}(\mu_0)\neq 0$ such that at every $t\in T_{\mathrm{use}}(p)$, on the (i,j) grid square between $(i,j+1)$ and $(i+1,j)$ through $(i+1,j+1)$ and (i,j), every t-admissible grid volume package (defined by the following normalizations: at each node X, the finite level subgroup $J_X(t)$ has volume one, and the infinite Haar factors are probability measures on definite groups and the row Haar measures of exponential-coordinate density $2^{-n_X(n_X-1)/2}$ relative to the prescribed Lie measure on indefinite groups of rank n_X; here $J_{(i,0)}(t)$, $J_{\mathrm{core}(i)}(t)$, and $J_{\mathrm{top}(0)}(t)$ are the grid congruence subgroups, $J_{(0,j+1)}(t)$ is the prescribed column level, and the remaining positive-column grid and top nodes use the maximal lattice stabilizer; each domain is a rational fundamental domain, and the Bessel unipotent quotient has volume one), matched kernel pair (Θ,Θ') with big and small theta kernels $K_{\mathrm{big}}(W,\Theta,g,g_1;\cdot)$ and $K_{\mathrm{small}}(W,\Theta',g,g_1)$, and convergent Hecke translation datum (z,g,g_1) of formal degree $\deg(z)$ and translation operator T_z satisfy

$$\deg(z)\int_{\mathcal{F}_{\mathrm{big}}}K_{\mathrm{big}}(W,\Theta,g,g_1;u)\,d\nu_{\mathrm{big}}(u)=c_{ij}(t)\int_{\mathcal{F}_{\mathrm{small}}}\big(T_zK_{\mathrm{small}}(W,\Theta',g,g_1)\big)(u)\,d\nu_{\mathrm{small}}(u)$$

with the right-hand integral nonzero for at least one such choice, and $\iota(r_{ij,t})c_{ij}(\mu_0)=c_{ij}(t)$ (so $r_{ij,\mu_0}=1$). Moreover, there exist a common discriminant twelfth-root period $\delta_{\mathrm{per}}\in\mathbf{C}$ of $\mathcal{O}_{K_d}$, an unkilled two-torsion point $t_0\in\mathbf{C}$ for $\Lambda(\mathcal{O}_{K_d},\delta_{\mathrm{per}})$, and for all $0\le i,j<m$ standard bottom pairs (P_{ij},P'_{ij}) on G at $(\mu_0,i,j,\mathfrak{m}_E)$ over $(\mathcal{O}_{K_d},\delta_{\mathrm{per}},t_0,Q_{\mathrm{lev}})$ (Definition 34.48(3)) with $\iota(z_A(i,j))=\mathrm{norm}(P_{ij})$ and $\iota(z'_A(i,j))=\mathrm{norm}(P'_{ij})$. For every $0\le i,j<m$, $z_T(i,j,s)=1$ for all $s\in\mathbf{Z}$, while for $1\le s\le h_{ij}$ ($h_{ij}=\lfloor(r_{i+1,j+1}+r_{i,j})/2\rfloor$), $(z_{T,0}(i,j,s),z_{T,1}(i,j,s))$ and $g_T(i,j,s,\cdot)$ satisfy the toric-ratio valuation relation of Lemma 34.45 at μ_0 (for $\nu_{\mu_0,i,j}$ and prime-twist characters $\delta_0=\delta$ at p_{de} and $\delta_1=\delta'$ at p'_{de}, agreeing with $z_T,z_{T,c},g_T(w)$ of Lemma 34.45), and for rational ideal-class partial zeta values $z(C)=\zeta_F(1-2s,C)\in\mathbf{Q}$ ($C\in\mathrm{Cl}(\mathcal{O}_F)$) of the totally real field F at $1-2s$ (as in Definition 1.15 and Lemma 34.45, with weight $\kappa_{i,j,s}(\tau)=\max(0,e^{\mathrm{gr}}_{\tau,j+1}+c_{\tau,m-i}-s)$ and $M_{i,j,s}=2sf+2\sum_{\tau\in\mathcal{T}}\kappa_{i,j,s}(\tau)$), $N_0(i,j,s,l)$ is given at every prime l by

$$N_0(i,j,s,l)=\left\lceil\frac{\beta_{\mathrm{vp}}(d,s,\kappa_{i,j,s},\mathfrak{f}_F,z,l)}{12}\right\rceil_+=\left\lceil c_{E,l}(z)+n_l(d,s,\kappa_{i,j,s},\mathfrak{f}_F)+M_{i,j,s}\,a_0(\mathfrak{f}_F,l)\right\rceil_+\in\mathbf{N}$$

(from the partial-zeta, ramification/conductor, and $\mathfrak{f}_F$-valuation exponents of Definition 1.15), while for $s\notin[1,h_{ij}]$ one has $z_{T,0}(i,j,s)=z_{T,1}(i,j,s)=1$, $g_T(i,j,s,\cdot)=0$, and $N_0(i,j,s,\cdot)=0$. Globally, at each finite place w of K, $c(w)=1$ if $p_{\mathrm{de}}\in\mathfrak{p}_w$ and $c(w)=0$ otherwise; $S_d=\{l\in\mathbf{N}:$ l is prime and $\exists\, i,j,s,\ N_0(i,j,s,l)\neq 0\}$ and $S_e=\{l\in\mathbf{N}:l$ is prime and $\exists\, w,\ \mu(w)\neq\mu_0\wedge l\in\mathfrak{p}_w\}$; d_ℓ is a grid integrality allowance on G at constant C_F (Definition 34.48(4)); and $B'_w=B_w$ is the gap-1 bottom weight bound of Definition 34.48(4) (henceforth denoted B_w).

Definition 34.50 (Grid modulus allowances and evaluation budgets)**.** For a grid G over F, recall from Definition 34.47 the finite set $S_0=\{v_0\}\cup S_1$ of grid places of F, the prime-power absolute norms $N\mathfrak{q}>1$ ($q\in S_0$, also written q inside logarithms and w-adic valuations), the grid conductor exponents $c_{v_0}=1$ and $c_q=2$ ($q\in S_0\setminus\{v_0\}$), $Q_{\mathrm{lev}}=\prod_{q\in S_0}(N\mathfrak{q})^{c_q}$, and $c_S=\log Q_{\mathrm{lev}}$. Put $R_*=k+m-1=r$, $n_{\mathrm{rk}}=k+m=R_*+1$, $d_s=k+m+5=R_*+6$, and $N_*=k+m(t+d_s+1)=k+m(t+k+m+6)$. For an estimate constant $C_1\in\mathbf{R}$ and scale weight $b=\exp(L)>0$, define the prime-power modulus budget

$$P_{\mathrm{pf}}=P_{\mathrm{pf}}(C_1,b)=C_1f^2(k+m)^6\log\big(b(k+m)g_0g_1p_{\mathrm{de}}p'_{\mathrm{de}}|D_E|\big),$$

and define the *grid modulus allowances* $e_q = e_q(C_1, b) \in \mathbf{N}$ $(q \in S_0)$ by

$$e_{v_0}(C_1, b) = \big\lfloor P_{\mathrm{pf}}(C_1, b)/\log N\mathfrak{q}_{v_0} \big\rfloor_+ + 1, \qquad e_q(C_1, b) = 2 \quad (q \in S_0 \setminus \{v_0\}),$$

where $\lfloor y \rfloor_+ = \max(0, \lfloor y \rfloor) \in \mathbf{N}$; thus $c_q \le e_q(C_1, b)$ for all $q \in S_0$, and put $\Sigma_0 = \Sigma_0(C_1, b) = \sum_{q \in S_0} e_q(C_1, b) \log N\mathfrak{q}$.

Enumerate $\mathcal{T} \simeq \{0, \dots, f-1\}$ and define the integer cloud coordinates $\mathrm{cl}(\tau, i) = \lfloor v_{\tau, i-2} \rfloor \in \mathbf{Z}$ for $0 \le \tau < f$ and $2 \le i \le k$ (from the Harish–Chandra tuple $v_\tau \in \mathbf{Q}^{k-1}$ of Definition 34.46) and $\mathrm{cl}(\tau, i) = 0$ otherwise, with integer cloud spread

$$L_c = \max_{0 \le \tau < f} \max\big(0,\ \mathrm{cl}(\tau, 2) - \mathrm{cl}(\tau, k)\big) \in \mathbf{N}.$$

Writing $\lceil y \rceil_+ = \max(0, \lceil y \rceil) \in \mathbf{N}$, define the initial top-node rank $n_p \in \mathbf{N}_{>0}$ (equal to $n_m = n_0' = \mathrm{rank}(X_0)$ of Definition 34.47), the integer gap bound $\gamma_{\mathrm{gap}}(C_1) \in \mathbf{N}$, and the dominant-weight spread factor $R_p = R_p(C_1) \ge 1$ by

$$\begin{aligned} n_p &= (R_* + 1) + m(R_* + 5) = k + m + m(k + m + 4), \\ \gamma_{\mathrm{gap}}(C_1) &= \max\Big(\big\lceil C_1 N_*^3 + C_1 n_{\mathrm{rk}}^7 + C_1 n_{\mathrm{rk}}^3 \sqrt{L_c} \big\rceil_+,\ 2N_*^3 + 2d_s + 3,\ \big\lceil 48 n_{\mathrm{rk}}^2 \sqrt{m} \big\rceil_+ \Big), \\ R_p(C_1) &= (m+1)\big(\gamma_{\mathrm{gap}}(C_1) + 1\big) + L_c + n_p. \end{aligned}$$

Recall the place weights $\ell_w = \log N\mathfrak{p}_w / [K : \mathbf{Q}]$ and toric budget $B_0 = C_1 f b(\log b + \log x)$ of Lemma 34.45 and the gap-1 bottom weight bound $B_w \ge 1$ of Definition 34.48. Define

$$E_M = 4 f n_p R_p \log(n_p R_p), \quad H_{I,0} = C_1 f m N_* b L + C_1 m^2 N_*^2 \Sigma_0, \quad B_F = C_1 f b m^2 (N_* + N_*^2)(L + \Sigma_0).$$

Lemma 34.51 (Combination of the eight finite losses)**.** *Let K be the coefficient field with embedding $\iota : K \hookrightarrow \mathbf{C}$, let $m \ge 1$, and let $L_{\mathrm{form}} \subset K^n$ be the full form lattice of Definition 34.49(3), with dual $M = L_{\mathrm{form}}^\vee$. For a finite place w, say that a complex scalar z has loss at most a if $z = \iota(z_K)$ for some $z_K \in K$ such that $z_K = 0$ or $v_w(z_K) \ge -a$. For a coefficient lattice $\mathcal{L}$ in a K-vector space, say that a complex coefficient vector has loss at most a in $\mathcal{L}$ if it is the image of a vector u for which some $s \in K^\times$ satisfies $v_w(s) \le a$ and $su \in \mathcal{L}$.*

Choose an $\mathcal{O}_K$-generating set of L_{form}. Suppose the pairings of $y \in K^n$ with its generators, after applying ι, have finite expansions of the form

$$s_{\mathrm{nor}} \sum_\beta c_\beta \sum_\gamma \gamma_{\beta,\gamma} \sum_\eta d_{\beta,\eta} \left\langle T_{m-1} \cdots T_0 \Psi_0, R_{m-1} \cdots R_0 F_0 \right\rangle.$$

Here β runs over the supported finite local terms, γ over their finite class sets, and η over a finite coset expansion. The factors c_β, $\gamma_{\beta,\gamma}$ and $d_{\beta,\eta}$ are respectively the product of local coefficients, the inverse stabilizer factor, and the product of local datum values. The operators T_j and R_j, their coefficient lattices and their initial vectors depend on the term indices. They are respectively the successive top and row kernel maps in that expansion. The final pairing is the contraction between the two terminal coefficient spaces.

Let $\mathcal{B} \ge 0$, and suppose there are eight nonnegative functions $\delta_S, \delta_C, \delta_D, \delta_G, \delta_T, \delta_R, \delta_F, \delta_P$, each supported on finitely many finite places, with the following properties. Their weighted sums satisfy

loss function	$\sum_w \ell_w \delta(w)$
$\delta_S, \delta_C, \delta_D, \delta_G$	$m\mathcal{B}$
$\delta_T, \delta_R, \delta_F, \delta_P$	$\mathcal{B}$

where the right column is an upper bound for each function in its row. At every place under consideration, uniformly in all the term indices, $s_{\mathrm{nor}}, c_\beta, d_{\beta,\eta}, \gamma_{\beta,\gamma}$ have losses at most $\delta_S(w), \delta_C(w), \delta_D(w), \delta_G(w)$, respectively. The initial top vector Ψ_0 has loss zero, and F_0 has loss at most $\delta_F(w)$. Each T_j sends a vector of loss a to one of loss at most $a + \delta_T(w)$ in the next coefficient lattice, and each R_j has the analogous property with $\delta_R(w)$. If terminal vectors have losses a, a', their contraction has loss at most $a + a' + \delta_P(w)$.

Put

$$\delta(w) = \delta_S(w) + \delta_C(w) + \delta_D(w) + \delta_G(w) + m\delta_T(w) + m\delta_R(w) + \delta_F(w) + \delta_P(w).$$

Then δ is nonnegative and finitely supported,

$$\sum_w \ell_w \delta(w) \le (6m+2)\mathcal{B} \le 8m\mathcal{B},$$

and for every such w there is $s_w \in K^\times$ with

$$v_w(s_w) \le \delta(w), \qquad s_w y \in M.$$

Outline. Track losses over m top and row steps, add the contraction and scalar losses, and apply the nonarchimedean triangle inequality to pairings with L_{form}. Local duality with a uniformizer gives s_w, and summing the eight bounds gives the estimate. The complete proof is on p. 1090. □

Lemma 34.52 (The finite layer-sum denominator estimate)**.** *Assume $\mathscr{P}$. There is $C_{F0} \in \mathbf{R}$ such that for every $C_F \ge C_{F0}$ there is $C_0 \in \mathbf{R}$, and for every $\varepsilon \in \mathbf{R}$ and $C_1 \ge C_0$ there is $x_0 \in \mathbf{R}$ with the following property. Fix an arithmetic datum D and size datum L_{sz} at scale s of exponent ε with $x(s) \ge x_0$, a real realization R, and a level-2 face F_{fc} such that $(L_{\mathrm{sz}}, R, F_{\mathrm{fc}})$ satisfies the Hida row-setting conditions of Definition 34.49(1), together with a grid G at $(1, C_F, b)$, a Weil system W on G, and a complete choice a of ambient dimension n on (G, W) over R satisfying the evaluation-field condition of Definition 34.49(2). Put $\sigma(G) = \sum_{q \in S_0} d_{\mathrm{pl}}(q) \log N\mathfrak{q}$, where $d_{\mathrm{pl}}(q)$ is the prescribed grid depth of Definition 34.47. Then there exists a nonnegative real-valued denominator-loss function δ_{sum} on the finite places of K, vanishing outside a finite set T_{sum} of exceptional finite places of K and satisfying*

$$2 \sum_{w \in T_{\mathrm{sum}}} \ell_w \delta_{\mathrm{sum}}(w) \le C_1 f b m^2 N_*^2 \big(L + \sigma(G)\big),$$

such that for every Hermitian test $\mathcal{O}_K$-lattice $(M, (X'_v)_{v|\infty})$ in K^n built from a as in Definition 34.49(3), every $t \in T$ and finite place w of K such that every $\mathfrak{P} \in Q_t$ either lies over a place in S_0 or has rational residue prime different from that of w, every source element $x \in M_{t,\mathrm{lv}}(F_{\mathrm{fc}})$, and every vector $y = (y_1, \dots, y_n) \in K^n$ satisfying $\iota(y_i) = v_{\mathrm{row}}(a, t, \mathrm{form}_{F_{\mathrm{fc}}}(t, x))_i$ for all $1 \le i \le n$, there exists a nonzero scalar $s_0 \in K^\times$ with

$$v_w(s_0) \le \delta_{\mathrm{sum}}(w) \qquad \text{and} \qquad s_0 y \in M.$$

In particular, at every finite place $w \notin T_{\mathrm{sum}}$ one has $\delta_{\mathrm{sum}}(w) = 0$, so $v_w(s_0) \le 0$ and $y \in M_w$.

Outline. The proof expands each class pairing into pure coset data with modulus valuation bounded by the grid depth of Definition 34.47. It bounds the eight local factor losses uniformly and applies Lemma 34.51 to assemble δ_{sum} and s_0. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1090. □

Lemma 34.53 (The finite sum estimates)**.** *Assume $\mathscr{P}$. There is $C_{F0} \in \mathbf{R}$ such that for each $C_F \ge C_{F0}$, in the closed sense of Definition 36.10 on the family class of Definition 35.2 under the division bound of Lemma 36.13, there is $C_0 \in \mathbf{R}$ with the following property. For every $C_1 \ge C_0$, eventually at every member point $p \in \mathcal{P}$ and every fitted evaluation datum d at (p, C_F) with unnormalized evaluation matrices V_t^0, there exist a finite set T_F of finite places of K and integers $d_F(w)$ vanishing outside T_F such that*

$$V^0_{\mu(w)}(M_{\mu(w),w}) \subseteq \mathfrak{p}_w^{-d_F(w)} M_w \quad (\text{all finite } w), \qquad 2 \sum_{w \in T_F} \ell_w d_F(w) \le B_F.$$

Moreover, for any family of roots of unity $\zeta_t \in K$ $(t \in T)$, replacing V_t^0 by $\zeta_t^{-1} V_t^0$ preserves both properties with the same T_F and d_F.

Outline. Combine the closed cutoffs for place avoidance, the finite-sum estimate, and the Hida row setting so Lemma 34.52 applies at $t = \mu(w)$. Set $T_F = \mathrm{supp}(\delta_{\mathrm{sum}})$ and $d_F = \lfloor \delta_{\mathrm{sum}} \rfloor$, and use Definition 34.49(1) with localization for the inclusion. Monotonicity of Σ_0 gives B_F, and integrality of ζ_t^{-1} preserves both. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1093. □

Lemma 34.54 (Height through a direct quotient metric)**.** *Let K be a totally complex number field, put $n_\infty = [K:\mathbf{Q}]/2$, and write A_v for the image of a matrix A under the embedding at an infinite place v. Let $H \in M_r(K)$ have positive-definite Hermitian images H_v, and let $P \in M_{q\times r}(K)$ have rank q. Suppose that the invertible source frames X_v and the direct quotient frames $X_{Q,v}$ satisfy*

$$(X_v^{-1})^\dagger X_v^{-1} = H_v, \qquad X_{Q,v} = (P_v H_v^{-1} P_v^\dagger)^{1/2}.$$

For invertible target frames X'_v and a matrix $B \in M_{n\times q}(K)$, define

$$h_{X_Q,X'}(B) = \frac{1}{n_\infty}\sum_{v|\infty} \log \|(X'_v)^{-1} B_v X_{Q,v}\|^2,$$

and define $h_{X,X'}(BP)$ by replacing $B_v X_{Q,v}$ with $B_v P_v X_v$. Here the norm is the Euclidean operator norm, $\dagger$ denotes conjugate transpose, and $\log 0 = 0$ in the height definition. Then

$$h_{X_Q,X'}(B) = h_{X,X'}(BP).$$

Proof. The source-frame identity gives $H_v^{-1} = X_v X_v^\dagger$. Set

$$U_v = X_{Q,v}^{-1} P_v X_v.$$

Then $U_v U_v^\dagger = 1$. Hence U_v is a coisometry and $U_v^\dagger$ is an isometry. For a matrix A with q columns, the inequalities

$$\|AU_v\| \le \|A\|, \qquad \|A\| = \|AU_vU_v^\dagger\| \le \|AU_v\|$$

give equality. Apply this to $A = (X'_v)^{-1}B_vX_{Q,v}$, square the norms, take logarithms, and average over the infinite places. □

Lemma 34.55 (The base evaluation estimate)**.** *Assume $\mathscr{P}$. There is $C_{F0} \in \mathbf{R}$ such that for each $C_F \ge C_{F0}$, in the closed sense of Definition 36.10 on the family class of Definition 35.2 under the division bound of Lemma 36.13, there is $C_0 \in \mathbf{R}$ with the following property. For every $C_1 \ge C_0$, eventually at every member point $p \in \mathcal{P}$ and every fitted evaluation datum d at (p, C_F) with base matrix $V^0_{\mu_0}$, every matrix $V_b \in M_{n\times d_Q(\mu_0)}(K)$ satisfying $V_b P_{\mu_0} = V^0_{\mu_0}$ obeys*

$$h(V_b) \le H_{I,0},$$

where $n_\infty(K) = [K:\mathbf{Q}]/2$ and $h(V_b) = \frac{1}{n_\infty(K)}\sum_{v|\infty}\log\|V_b\|_v^2$ (with $\log 0 = 0$ in this height definition) is the operator-norm height from $K^{d_Q(\mu_0)}$ equipped with the direct quotient Hermitian frames $(X_Q(\mu_0)_v)_{v|\infty}$ of Definition 34.46 to $(K^n, (X'_v)_{v|\infty})$.

Outline. Under Definition 36.10 and the setting of Lemma 34.52, the proof bounds the operator-norm height $h_{X,X'}(V)$ of compatible evaluation matrices by $H_{I,0}$. Taking $X_v = H_V(\mu_0)_v^{-1/2}$ and applying Lemma 34.54 transfers the bound to V_b. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1094. □

Lemma 34.56 (The frame estimate for the evaluation lattice)**.** *For every $C_F \ge 0$, in the closed sense of Definition 36.10 on the family class of Definition 35.2 under the division bound of Lemma 36.13, for every $C_1 \ge 1$, eventually at every member point $p \in \mathcal{P}$ and every fitted evaluation datum d at (p, C_F), the $\mathcal{O}_K$-submodule $M \subset K^n$ is finitely generated and nonzero, X'_v is invertible ($\det X'_v \ne 0$) at every infinite place v of K, and with respect to the standard basis of K^n and the frames $(X'_v)_{v|\infty}$,*

$$\mu_{\max}(M) \le E_M = 4fn_pR_p\log(n_pR_p).$$

Outline. Use the test-lattice tie of Definition 34.49(3) once the Hida row setting holds at the member point. Bound the dominant-weight spread on the rank-n_p top space by $R_p(C_1) \ge 1$ using monotonicity of γ_{gap} for $C_1 \ge 1$. The slope bound $\mu_{\max} \le 4|\mathcal{T}|\,\mathrm{rank}(X_0)R\log(\mathrm{rank}(X_0)R)$ for the dual of a direct sum of Weyl lattices of weight spread at most R then yields finite generation of M, invertibility of every X'_v, and $\mu_{\max}(M) \le E_M$. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1095. □

Lemma 34.57 (The valuation budget of a square tower)**.** *Let K be a number field with involution, let w be a finite place of K, and write $v = v_w$, with the convention $v(0) = 0$. Let e, p, q, e_0, h_0 be nonnegative integers such that*

$$e_0 + p = e + q, \qquad e_0 \le h_0, \qquad h = h_0 + p + q.$$

Let $R, \zeta, \gamma, A, A' \in K$, let $t_0, \dots, t_{p+q-1} \in K$, and let $\rho_s \in K$ for $1 - e_0 \le s \le h$. Suppose that $A, \gamma, \zeta, t_k, \rho_s$ are nonzero, that $v(\zeta) = 0$, and that

$$R \prod_{k=0}^{q-1} \rho_{e+k+1} = \zeta \left(\prod_{k=0}^{p+q-1} t_k \right) \gamma \prod_{k=0}^{p-1} \rho_{e+q-k}.$$

Let $a_s, n_s \ge 0$ for $1 \le s \le h$, let $\eta \ge 0$, and let $d, B \in \mathbf{R}$. Assume

$$\begin{aligned} v(\rho_s) &\le a_s + \eta, \qquad -v(\rho_s) \le a_s + n_s \qquad (1 \le s \le h), \\ v(\rho_s) &= v(\rho_{1-s}) \qquad (1 - e_0 \le s \le 0), \qquad |v(t_k)| \le B, \\ v(\gamma) &\le v\left(A \prod_{s=1-e_0}^{h_0} \rho_s \right) + v(A') + 6d. \end{aligned}$$

Then $R \neq 0$ and

$$v(R) \le |v(A)| + |v(A')| + 4 \sum_{s=1}^{h} a_s + 5 \sum_{s=1}^{h} n_s + 2h\eta + 6d + (p+q)B.$$

Outline. Check that all tower indices lie in $[1, h]$, deduce $R \neq 0$, take valuations, and substitute the $v(\gamma)$ bound. Reflection and the $v(\rho_s)$ bounds give $4A_1 + 5N_1 + 2h\eta$, and replacing $v(A), v(A')$ by their absolute values finishes. The complete proof is on p. 1096. □

Lemma 34.58 (The square valuation estimate)**.** *Assume $\mathscr{P}$, $\mathscr{J}$, the Hecke–Tate functional equation, and Theorem 34.7. There is $C_{F0} \in \mathbf{R}$ such that for each $C_F \ge C_{F0}$, in the closed sense of Definition 36.10 on the family class of Definition 35.2 under the division bound of Lemma 36.13, there is $C_0 \in \mathbf{R}$ with the following property. For every $C_1 \ge C_0$, eventually at every member point $p \in \mathcal{P}$ and every fitted evaluation datum d at (p, C_F), every element of $S_d \cup S_e$ is prime, $c_q \le e_q$ for all $q \in S_0$, and for all $0 \le i, j < m$ and every finite place w of K, putting $y_{ij} = m - 1 - j$ (so $0 \le y_{ij}$ and $y_{ij} + 1 \le m$), one has $h_{ij} \le N_*$, $r_{ij,\mu_0} = 1$, $r_{ij,\mu(w)} \neq 0$,*

$$\begin{aligned} v_w(r_{ij,\mu(w)}) \le{}& |v_w(z_A(i,j))| + |v_w(z'_A(i,j))| \\ &+ 4 \sum_{s=1}^{h_{ij}} \big(|v_w(z_T(i,j,s)) + g_T(i,j,s,w)| + |v_w(z_{T,c(w)}(i,j,s)) + g_T(i,j,s,w)| \big) \\ &+ 5 \sum_{s=1}^{h_{ij}} \sum_{l \in S_d} N_0(i,j,s,l) v_w(l) + 2h_{ij} \sum_{l \in S_e} \frac{v_w(l)}{l^2} + 6d_\ell(w) + 2y_{ij} \sum_{q \in S_0} c_q v_w(q), \end{aligned}$$

and, for every integer $1 \le s \le h_{ij}$, $N_0(i,j,s,l) \ge 0$ for every rational prime l (vanishing at primes outside S_d) with $\sum_{l \in S_d} N_0(i,j,s,l) \log l \le B_0$.

Outline. Combine the closed cutoffs with the fitted allowance of Definition 34.48 and Definition 34.49; the case $\mu(w) = \mu_0$ is immediate. For $\mu(w) \neq \mu_0$, form Hecke ratios ρ_s by toric bounds and reflection, compare bottom constants, and descend the seesaw tower. Lemma 34.57 yields nonvanishing and the valuation bound, and Definition 34.50 gives $c_q \le e_q$. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1097. □

Lemma 34.59 (An explicit prime-denominator allowance)**.** *Let $f, N, Q \ge 1$ and $d \ge 0$ be integers, let $B, L \ge 1$, and let $S \subset \mathbf{N}$ be finite. Use $\log 0 = 0$ throughout this lemma. Suppose that $a_p \ge 0$ for every $p \in \mathbf{N}$, that $a_p = 0$ outside S, and that*

$$\sum_{p \in S} a_p \log p \le L, \qquad \log d \le 2L.$$

Put

$$M = 8^{4fN}(8fN)!, \qquad S' = \big(\{p : p \mid M\} \cup S \cup \{d\} \cup \{p : p \mid Q\}\big) \cap \{p : p \text{ is prime}\}.$$

For prime p, write v_p for the exponent of p in a positive integer and set

$$e_p = v_p(M) + 4fNB\left(a_p + \frac{\mathbf{1}_{p=d}}{4} + v_p(Q)\right) + 4N^2 v_p(Q).$$

Set $e_p = 0$ when p is not prime. Then $e_p \geq 0$, $e_p = 0$ outside S', and

$$\begin{aligned}\sum_{p \in S'} e_p \log p &\leq 36fN\log(2fN) + 6fNBL + 4fNB\log Q + 4N^2\log Q \\ &\leq I(18; f, N, B, L, Q) \\ &= 36fN\log(2fN) + 36fNBL + 72fN^2B\log Q.\end{aligned}$$

Outline. Bound $\log M \leq 18r\log r$ for $r = 2fN$ via $(4r)! \leq (4r)^{4r}$, rewrite $\log M$ and $\log Q$ by prime factorization, and insert the bounds on a_p and $\log d$. Elementary bounds using $fB, N \geq 1$ then give $I(18; f, N, B, L, Q)$. The complete proof is on p. 1100. □

Lemma 34.60 (The coordinate denominator bound)**.** *There is an absolute constant $C_{\mathrm{int}} > 0$ (one may take $C_{\mathrm{int}} = 18$) with the following property. Let D be an arithmetic datum with $f = [F : \mathbf{Q}]$, let $Q, N \geq 1$ be integers such that Q is an admissible level modulus for D in the sense of Definition 34.48(1), and let $B, L \geq 1$ be real numbers with $\log \mathrm{rd}(E) \leq L$. Then there exist a finite set S' of rational primes and a real-valued exponent function $\delta_{\mathrm{vp}} : \mathbf{N} \to \mathbf{R}_{\geq 0}$ vanishing outside S' and satisfying $\sum_{p \in S'} \delta_{\mathrm{vp}}(p)\log p \leq I(C_{\mathrm{int}}; f, N, B, L, Q)$ such that:*

1. *δ_{vp} satisfies the p-adic coordinate integrality bound of Definition 34.48(4) for (D, N, B, Q), namely that for every discriminant twelfth-root period $\delta \in \mathbf{C}$ of $\mathcal{O}_{K_d}$ with CM lattice $\Lambda = \Lambda(\mathcal{O}_{K_d}, \delta)$, unkilled two-torsion point $t_0 \in \mathbf{C}$ for Λ, rational prime p, level $Q' \in \mathbf{N}$ that is p-compatible with Q ($v_p(Q') \leq v_p(Q)$), and (N, B)-small definite CM theta pair P at $(\mathcal{O}_{K_d}, \delta, t_0, Q')$, every class in $\mathrm{Cls}_D(d_P, J_P)$ has a representative h integral over p, every $h \in G_{\mathbb{A}}(d_P)$ satisfies $|\Gamma_D(J_P, h)| \neq 0$ and $v_p(|\Gamma_D(J_P, h)|) \leq \delta_{\mathrm{vp}}(p)$, and for every h integral over p and $T \in \mathrm{Tup}_D(a_P, d_P, n_P)$ one has $c_P(h, T) \in \overline{\mathbf{Q}}$ and $|\lambda(c_P(h, T))|_p \leq p^{\delta_{\mathrm{vp}}(p)}$ for every ring homomorphism $\lambda : \overline{\mathbf{Q}} \to \mathbf{C}_p$; and*
2. *$(S', \delta_{\mathrm{vp}})$ satisfies the norm valuation bound of Definition 34.48(4) for (D, N, B, Q) with budget $I(C_{\mathrm{int}}; f, N, B, L, Q)$, namely that for every discriminant twelfth-root period $\delta \in \mathbf{C}$ of $\mathcal{O}_{K_d}$ with $\Lambda = \Lambda(\mathcal{O}_{K_d}, \delta)$, unkilled two-torsion point $t_0 \in \mathbf{C}$ for Λ, totally complex number field K with $\sigma : K \hookrightarrow \mathbf{C}$, index set $\mathcal{I}$, families of (N, B)-small definite CM theta pairs P_α, P'_α ($\alpha \in \mathcal{I}$) at $(\mathcal{O}_{K_d}, \delta, t_0, Q)$ each possessing at least one nonzero coordinate, and elements $z_{A,\alpha}, z'_{A,\alpha} \in K$ with $\sigma(z_{A,\alpha}) = \mathrm{norm}(P_\alpha)$ and $\sigma(z'_{A,\alpha}) = \mathrm{norm}(P'_\alpha)$, setting $d_\ell(w) = \sum_{p \in S'} \delta_{\mathrm{vp}}(p) v_w(p) \geq 0$ gives $z_{A,\alpha} \neq 0$, $z'_{A,\alpha} \neq 0$, $v_w(z_{A,\alpha}) \geq -3d_\ell(w)$, and $v_w(z'_{A,\alpha}) \geq -3d_\ell(w)$ at every finite place w of K, and $\sum_{w \in U} \ell_w d_\ell(w) \leq I(C_{\mathrm{int}}; f, N, B, L, Q)$ on every finite set U of finite places of K.*

In particular, for any grid G at $(c_{\mathrm{gap}}, c_{\mathrm{Dep}}, b)$ whose level Q_{lev} is admissible for D with $1 \leq \log b$, $\log \mathrm{rd}(E) \leq \log b$, and $C \geq C_{\mathrm{int}}$, there exists a grid integrality allowance d_ℓ on G at constant C in the sense of Definition 34.48(4).

Outline. Set $a_p = v_p(|\operatorname{disc}(E)|)/(2f)$ and use a trace-dual basis of $\mathcal{O}_E$ to obtain the coordinate and stabiliser bounds of Definition 34.48. Lemma 34.59 then gives $(S', \delta_{\mathrm{vp}})$ within budget $I(18; f, N, B, L, Q)$. Expanding $\mathrm{norm}(P)$ with the ultrametric inequality yields $v_w(z) \geq -3d_\ell(w)$ and the grid allowance for $C \geq 18$. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1100. □

Lemma 34.61 (The lattice norm bound)**.** *Assume $\mathscr{P}$, the Hecke–Tate functional equation, and Theorem 34.7. There is $C_{F0} \in \mathbf{R}$ such that for each $C_F \geq C_{F0}$, in the closed sense of Definition 36.10 on the family class of Definition 35.2 under the division bound of Lemma 36.13, there is $C_0 \in \mathbf{R}$ with the following property. For every $C_1 \geq C_0$, eventually at every member point $p \in \mathcal{P}$ and every fitted evaluation datum d*

at (p, C_F), one has $d_\ell(w) \geq 0$ at every finite place w of K, and for every finite set U of finite places of K, all $0 \leq i, j < m$, and each $z = z_A(i,j), z'_A(i,j)$,

$$\sum_{w \in U} \ell_w d_\ell(w) \leq D_0, \qquad z \neq 0, \qquad v_w(z) \geq -3d_\ell(w) \quad (\textit{all finite } w),$$

where $D_0 = 2C_1 f N_* \big(\log(2fN_*) + B_w L + 2N_* B_w c_S\big)$.

Outline. Under $\mathscr{P}$, the logarithmic-derivative bound yields the square-recursive regime with (N_*, B_w)-small bottom pairs (P_{ij}, P'_{ij}), while the Hecke–Tate functional equation and Theorem 34.7 give each a nonzero coordinate. Definition 34.49(3) and Definition 34.48(4) then give the valuation bounds and, using monotonicity for $C_1 \geq C_F \geq 1$, the budget D_0. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1102. □

Lemma 34.62 (The two Galois means)**.** *There is an absolute constant $C_A > 0$ such that every complex lattice of discriminant one has area at most C_A. Fix such a constant and put*

$$C_{\mathrm{gal}} = 18 + \frac{3}{2}\big(\log^+(2\pi^2) + \log^+ C_A + 2\big).$$

Let D be an arithmetic datum, let $(\mathfrak{c}, \delta)$ be a curve datum with lattice Λ, and let t_0 be an unkilled two-torsion point. Suppose $L \geq 1$, $\log \mathrm{rd}(E) \leq L$, and

$$\left|\mathrm{Re}\,\frac{L'(1,\chi_d)}{L(1,\chi_d)}\right| \leq L.$$

Let $M \subset \mathbf{C}$ be any number field containing $g_2(\Lambda), g_3(\Lambda), \wp_\Lambda(t_0)$. For each $\rho : M \hookrightarrow \mathbf{C}$, let Λ_ρ be the conjugate lattice and choose an unkilled two-torsion point t_ρ on it. Write $A_\rho = \mathrm{area}(\Lambda_\rho)$ and let $\nu_\rho = \nu(\Lambda_\rho, t_\rho)$ be the Gaussian weighted mean square of Definition 34.48(1). Then

$$\frac{1}{[M:\mathbf{Q}]}\sum_\rho \log^+(A_\rho^{-1}) \leq (\log^+ C_A + 2)L \leq C_{\mathrm{gal}} L, \qquad \frac{1}{[M:\mathbf{Q}]}\sum_\rho \log^+ \nu_\rho \leq C_{\mathrm{gal}} L.$$

Outline. The proof bounds discriminant-one areas via $y^6|\Delta(\tau)|$ and averages $\log^+(A_\rho^{-1})$ by the period identity. It applies a pointwise estimate for $\log^+ \nu$ of Definition 34.48(1) to conjugate lattices to bound the mean of $\log^+ \nu_\rho$. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1103. □

Lemma 34.63 (The archimedean norm estimate)**.**

1. *There is an absolute constant $C_{\mathrm{arch}} > 0$ with the following property. Let D be an arithmetic datum with $f = [F:\mathbf{Q}]$, let $(\mathfrak{c}, \delta)$ be a curve datum over D with CM lattice $\Lambda = \Lambda(\mathfrak{c}, \delta)$, let $t_0 \in \mathbf{C}$ be an unkilled two-torsion point for Λ (Definition 34.48(1)), and let $Q, N \geq 1$ be integers and $B, L \geq 1$ real numbers satisfying $\log \mathrm{rd}(E) \leq L$ and $\big|\mathrm{Re}(L'(1,\chi_d)/L(1,\chi_d))\big| \leq L$. For any totally complex number field K with $\sigma : K \hookrightarrow \mathbf{C}$, any families of (N,B)-small definite CM theta pairs P_α, P'_α $(\alpha \in \mathcal{I})$ at $(\mathfrak{c}, \delta, t_0, Q)$, and any $z_{A,\alpha}, z'_{A,\alpha} \in K$ satisfying $\sigma(z_{A,\alpha}) = \mathrm{norm}(P_\alpha)$ and $\sigma(z'_{A,\alpha}) = \mathrm{norm}(P'_\alpha)$, each $z = z_{A,\alpha}, z'_{A,\alpha}$ satisfies*

$$\mathrm{mean}_v \log^+ |z|_v \leq C_{\mathrm{arch}} f(2N) B\big(L + \log(2NB)\big) + C_{\mathrm{arch}} f N^2 \big(L + \log(2fNB)\big) + C_{\mathrm{arch}} N^2 \log Q,$$

where $\mathrm{mean}_v\, \varphi(v) = \frac{2}{[K:\mathbf{Q}]}\sum_{v|\infty} \varphi(v)$ is the mean over infinite places of K and $|z|_v$ denotes the standard complex absolute value at v.

2. *Assume $\mathscr{P}$. There is $C_{F0} \in \mathbf{R}$ such that for each $C_F \geq C_{F0}$, in the closed sense of Definition 36.10 on the family class of Definition 35.2 under the division bound of Lemma 36.13, there is $C_0 \in \mathbf{R}$ such that for every $C_1 \geq C_0$, eventually at every member point $p \in \mathcal{P}$ and every fitted evaluation datum d at (p, C_F), every $0 \leq i, j < m$ and each $z = z_A(i,j), z'_A(i,j)$ satisfies $\mathrm{mean}_v \log^+ |z|_v \leq A_0$, where*

$$A_0 = C_1 f N_* \big[2B_w\big(L + \log(2N_* B_w)\big) + N_*\big(L + \log(2fN_* B_w)\big)\big] + C_1 N_*^2 c_S.$$

Outline. In (1), the proof passes to a Galois field $M/\mathbf{Q}$, bounds $\mathrm{mean}_v \log^+ |z|_v$ via Cauchy–Schwarz and conjugate coordinate estimates, and averages using Lemma 34.62. Part (2) specializes (1) under Definition 36.10 via the L'/L bound. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1105. □

Lemma 34.64 (Existence of a fitted evaluation datum)**.** *Assume $\mathscr{P}$, the Hecke–Tate functional equation, Theorem 34.7, and the ordinary lattice-norm transport and partial-zeta denominator clauses of $\mathscr{J}$. There is $C_{F0} \in \mathbf{R}$ such that for each $C_F \geq C_{F0}$, in the closed sense of Definition 36.10 on the family class of Definition 35.2 under the division bound of Lemma 36.13, eventually at every member point $p \in \mathcal{P}$, every grid G fitting p at C_F and Weil system W on G over p's real realization admitting a complete choice that satisfies both the evaluation-field condition and the square-field condition of Lemma 34.42 extends to a fitted evaluation datum d at (p, C_F) with underlying grid G, realizing the higher toric data of Lemma 34.45 and the bottom weight bound B_w of Definition 34.48.*

Outline. The proof uses Definition 36.10 and Theorem 34.7 for algebraic row sums, and a tower identity with Lemma 34.1 for $c_{ij,t}/c_{ij,\mu_0} \in \iota(K)$. It builds the datum of Definition 34.49(3) via Lemmas 34.42, 34.45, and 34.60. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1108. □

Lemma 34.65 (Comparison map and base kernel on a fitted grid)**.** *Assume $\mathscr{P}$ and Theorems 34.7 and 34.25. There is $C_{F0} \in \mathbf{R}$ such that for each $C_F \geq C_{F0}$, in the closed sense of Definition 36.10 on the family class of Definition 35.2 under the division bound of Lemma 36.13, eventually at every member point $p \in \mathcal{P}$ and every fitted evaluation datum d at (p, C_F) with unnormalized matrices V_t^0 and seesaw products $r_t = \prod_{i,j} r_{ij,t}$, one has $r_{\mu_0} = 1$ and $r_{\mu(w)} \neq 0$ for every finite place w of K, $RR_w = \iota_{\mu(w)} P_{\mu(w)} M_{\mu(w),w}$ at every finite place w of K, $\iota_{\mu_0} P_{\mu_0}$ maps $K^{n_V(\mu_0)}$ onto $RR \otimes_{\mathcal{O}_K} K$, $\ker V_{\mu_0}^0 = \ker P_{\mu_0}$, and there exist roots of unity $\zeta_t \in K$ $(t \in T)$ with $\zeta_{\mu_0} = 1$ and a K-linear map $\Psi : RR \otimes_{\mathcal{O}_K} K \to M \otimes_{\mathcal{O}_K} K$ satisfying*

$$\Psi \iota_t P_t = r_t^{-1} (\zeta_t^{-1} V_t^0) \qquad (t = \mu_0 \text{ or } t = \mu(w)).$$

Outline. The proof uses Definition 34.49(3) and Theorem 34.7 under Definition 36.10 for RR_w and row functionals ℓ_α. Separation via Theorem 34.25 produces an injective K-linear Ψ, yielding $\ker V_{\mu_0}^0 = \ker P_{\mu_0}$. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1110. □

Lemma 34.66 (Fitted-grid size, weight, and modulus bounds)**.** *For every $C_F \geq 0$, in the closed sense of Definition 36.10 on the family class of Definition 35.2 under the division bound of Lemma 36.13, for every $C_1 \geq 42$, eventually at every member point $p \in \mathcal{P}$ and every fitted evaluation datum d at (p, C_F),*

$$1 \leq x, \quad 1 \leq L, \quad L + \log x \leq 3L, \quad 2 \leq m, \quad m + 3 \leq k, \quad m^4 \leq f \leq k,$$
$$mfN_*^2 \leq b, \quad 80R_*^4 \leq N_*, \quad 1 \leq n_p \leq 2mR_*, \quad 1 \leq R_p \leq C_1 m N_*^3 b, \quad \log(n_p R_p) \leq 4L,$$
$$\Sigma_0 \leq 2P_{\mathrm{pf}}, \quad c_S \leq 45fL, \quad 1 \leq B_w \leq C_1 b, \quad \log B_w \leq C_1 L,$$

and

$$100000\, C_1^3 f m^2 N_*^2 (R_* N_* L + P_{\mathrm{pf}}) \leq x^{C_H}.$$

Outline. Derives rank, gap, and weight bounds from the family windows and 34.50, using $b = \exp(x^\varepsilon)$ to control n_p, R_p, and $\log(n_p R_p)$. Sums prime logarithms over S_0 for c_S, Σ_0 and bounds the polynomial prefactor by x^{C_H}. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1112. □

In Proposition 34.67, use the member-point maps, bottom weight bound, and evaluation budgets of Definitions 34.46, 34.48, and 34.50.

Proposition 34.67 (Evaluation estimates on a fitted grid)**.** *Assume $\mathscr{P}$, $\mathscr{J}$, the Hecke–Tate functional equation, and Theorems 34.7 and 34.25. There is $C'_{F0} \in \mathbf{R}$ such that for each $C_F \geq C'_{F0}$ there is $\varepsilon_1 > 0$, and for each $0 < \varepsilon \leq \varepsilon_1$ there are size thresholds $\mathcal{K}$ with the following property. Let F_m be an eligible family of Definition 35.2 meeting $\mathcal{K}$ at C_H in the sense of Definition 36.10 and satisfying the division bound of Lemma 36.13. Then there is a threshold $C_0 \geq 42$ such that for every $C_1 \geq C_0$ there is a point cutoff $x_0 > x_1(F_m)$ after which every member point $p \in \mathcal{P}$ at $x \geq x_0$ carrying a fitted grid at C_F with*

a compatible Weil system, complete choice satisfying the evaluation-field and square-field conditions, and finite-group series of Definition 34.24 admits a Hermitian evaluation lattice M, evaluation matrices V_t, a map $\Psi : RR\otimes K \to M\otimes K$, and integers $d_F(w)$ supported on a finite set T_F with the following simultaneous properties. Put $r_t = \prod_{i,j} r_{ij,t}$ for the product of the layer-pair factors, with $r_{\mu_0} = 1,\ r_{\mu(w)} \neq 0$. After a unit normalization,

$$\Psi\iota_t P_t = r_t^{-1} V_t \quad (t = \mu_0 \text{ or } t = \mu(w)), \qquad \ker V_{\mu_0} = \ker P_{\mu_0}. \tag{34.23}$$

Also $RR_w = \iota_{\mu(w)} P_{\mu(w)} M_{\mu(w),w}$ and $\iota_{\mu_0} P_{\mu_0}$ is onto $RR\otimes K$. With direct quotient norms,

$$\begin{gathered} V_{\mu(w)}(M_{\mu(w),w}) \subseteq \mathfrak{p}_w^{-d_F(w)} M_w, \qquad 2\sum_w \ell_w d_F(w) \le B_F, \\ h(\Psi\iota_{\mu_0}) \le H_{I,0}, \qquad \mu_{\max}(M) \le E_M, \end{gathered} \tag{34.24}$$

where h is the mean logarithm of the squared operator norm at infinity.

Use the half-ranks h_{ij} of Lemma 34.42. Write z_A, z'_A for the bottom numbers, $z_T, z_{T,c}, g_T$ for the higher data of Lemma 34.45, and $c(w)$ for the selected twist. Then

$$\begin{aligned} v_w(r_{ij,\mu(w)}) \le{}& |v_w(z_A)| + |v_w(z'_A)| \\ &+ 4\sum_{s=1}^{h_{ij}} \big(|v_w(z_T(i,j,s)) + g_T(i,j,s,w)| + |v_w(z_{T,c(w)}(i,j,s)) + g_T(i,j,s,w)| \big) \\ &+ 5\sum_{s=1}^{h_{ij}} \sum_{l\in S_d} N_0(i,j,s,l) v_w(l) + 2h_{ij} \sum_{l\in S_e} \frac{v_w(l)}{l^2} \\ &+ 6d_\ell(w) + 2y_{ij} \sum_{q\in S_0} c_q v_w(q). \end{aligned} \tag{34.25}$$

Here S_d, S_e are the finite rational prime sets for the count and degree terms, $d_\ell \ge 0$ is the local denominator allowance, and $S_0, c_q \le e_q$ are the grid primes and modulus exponents. The indices satisfy $y_{ij} + 1 \le m$ and $h_{ij} \le N_*$. The prime count satisfies

$$\sum_{l\in S_d} N_0(i,j,s,l) \log l \le B_0 \qquad (1 \le s \le h_{ij}), \qquad N_0(i,j,s,l) \ge 0 \quad (l \in S_d). \tag{34.26}$$

For bottom weight bound B_w and logarithmic level allowance c_S put $l_1 = \log(2fN_*)$, $l_2 = \log(2N_*B_w)$, $l_3 = \log(2fN_*B_w)$, and

$$\begin{aligned} D_0 &= 2C_1 fN_*(l_1 + B_w L + 2N_* B_w c_S), \\ A_0 &= C_1 fN_*[2B_w(L + l_2) + N_*(L + l_3)] + C_1 N_*^2 c_S. \end{aligned}$$

For every finite set U of finite places and each $z = z_A, z'_A$,

$$\sum_{w\in U} \ell_w d_\ell(w) \le D_0, \qquad z \neq 0, \quad v_w(z) \ge -3d_\ell(w), \qquad \mathrm{mean}_v \log^+ |z|_v \le A_0, \tag{34.27}$$

where the last mean is over infinite embeddings.

Outline. Take C'_{F0} above the grid thresholds of Lemmas 34.45 and 34.53–34.66, intersect their closed ε-ranges, and combine the fitted evaluation, polynomial bound, and grid separation estimates uniformly over all admissible member points. The complete proof is on p. 1113. □

Theorem 34.68 (The constant-function Fourier construction satisfies the functional bound)**.** *Assume $\mathscr{P}, \mathscr{J}$, the Hecke–Tate functional equation and Theorems 34.7 and 34.25. Assume the following closed assertion, in the sense of Definition 36.10, on the family class of Definition 35.2. Eventually at every member point in $\mathcal{P}$ of Definition 35.1, for all $C \in \mathbf{R}$, every layer j, block p and infinite place v, the level-one base eigenvalue satisfies $\log \lambda^{(1)}_{j,\mu_0,p,v} \ge L_{6,j}$, where $L_{6,j} = \sum_\rho \log \mathrm{Hook}(a^+_{j\rho}) - C_{\mathrm{sz}} f n_V^2 L_p$ is the hook-length lower bound of Theorem 21.12 with family constant $C_{\mathrm{sz}} = C_{\mathrm{sz}}(F_m)$ and $L_p = \log(b\ \mathrm{rd}(E) g_0 g_1 n_V)$.*

There is $C'_{F0} \in \mathbf{R}$ such that for each $C_F \geq C'_{F0}$ there is $\varepsilon_1 > 0$, and for each $0 < \varepsilon \leq \varepsilon_1$ there are size thresholds $\mathcal{K}$. Let F_m belong to the family class of Definition 35.2, let its constants at C_H meet the thresholds $\mathcal{K}$ of Definition 36.10, and assume the division bound of Lemma 36.13, namely that for some A, x_1, every point at $x \geq x_1$ satisfies $\eta_j \leq AD_0L(B_0 + N_c)$ in every lift layer j. There is x_0 after which every member point in $\mathcal{P}$ carrying a fitted grid at C_F as specified below satisfies, at level zero,

$$\forall C \in \mathbf{R}, \qquad \mu_{\max,RR} \leq x^{C_H} b - \langle \log \kappa \rangle_\infty .$$

The fitted grid must carry a Weil system realizing the given theta action and a complete choice satisfying the coefficient-field condition of Theorem 34.39 and the square-field condition of Lemma 34.42. Its finite-group series must satisfy Definition 34.24.

Outline. Glues inclusions from 34.67 for injective Ψ and bounds $h(\Psi)$ via 21.12. Sums finite losses via 34.45, then applies 34.66 to bound $\mu_{\max,RR} + \langle \log \kappa \rangle_\infty$ by $x^{C_H} b$. The complete proof is on p. 1114. □

34.8 The functional row on the chosen members

Theorem 34.69 (The functional condition along members)**.** *Assume discrete-series coverage, Harish–Chandra realization, rapid decay of cusp forms, Casselman–Wallach globalization, the ray-class-field input and Theorem 34.68. There is $C_{F0} \in \mathbf{R}$ with the following property. For $C_F \geq C_{F0}$, take an allowed local Langlands system h_L with auxiliary extension S. Fix a geometry condition Γ. For every norm-choice extension N, assume each datum satisfying N admits an extension with list at C_F. Adjoin to N the grid extension condition at C_F of Lemma 34.42 and assume the open conditions of Theorem 33.74.*

A condition R eventually holds for every constructed member over the permitted base frames. On good members satisfying R, a row condition E eventually holds for every linked bundle. For a compatible selection on $\mathcal{A}$ for this row, assume the closed hypotheses on $W_3(\mathcal{A})$ and the division bound of Lemma 36.13. There is $C_0 \geq 0$ such that for each $C \geq C_0$ the selected bundles eventually satisfy the slope inequality (34.22) at parameters (C, C_H), with point threshold allowed to depend on C. The class $W_3(\mathcal{A})$ requires $W_1(\mathcal{A})$, a level-3 height choice and eventual bounds on $T_u, D_{\Gamma,u}$. For $D_{\Gamma,u}$ some C', x_0 give $\log D_{\Gamma,u} \leq C' f N(m) \log(f N(m))$ on every member at $x \geq x_0$ and every word u.

Outline. Transport and intersect the divisor and geometry data to select fitted grids tied to the original bundles. Realization and the family conditions of 33.3 let 34.68 apply at its threshold, and the ties carry the inequality back with $C_0 = 0$. The complete proof is on p. 1116. □

Use Theorem 34.41 in the member supply of Theorem 33.55. Apply Theorem 33.29 at $c_{\mathrm{Dep}} = C_F = \max(C_{F0}, C_{F1})$ with the grid and functional rows, the linked $2,3$-bound of Theorem 36.16 and the covolume row of Theorem 35.101.

Chapter 35

Covolume bounds

Let Λ be a rank-r lattice over a totally complex number field K with $s_K = [K : \mathbf{Q}]/2$ infinite places. In the archimedean frame X, write $\mathrm{ht}(X, \Lambda)$ for its $\mathbf{Z}$-covolume multiplied by $2^{s_K r} |\operatorname{disc} K|^{-r/2}$. Put

$$\mathrm{lcv}(X, \Lambda) = \frac{2}{[K : \mathbf{Q}]} \log \mathrm{ht}(X, \Lambda).$$

For a nonzero map φ, let $h(\varphi)$ be the mean over the infinite places of $\log \|\varphi\|_v^2$.

$$\|\varphi\|_v \le e^a \quad (v \mid \infty) \quad \Longrightarrow \quad h(\varphi) \le 2a. \tag{35.1}$$

At a point p over x, let E be the host field and put $f = [E : \mathbf{Q}]/2$, $L_x = x^{\varepsilon}$, $b = e^{L_x}$ and $E_p = fmbL_x \log\log x$. At layer $0 \le i < m$, write $j = i + 1$. For the layer weights $w_{j,\rho}$, set

$$M_{p,i} = (k + j - 1 - (m - j)) \log\Omega \sum_{\rho} w_{j,\rho}, \qquad \ell_{3,i}(C, C_H) = M_{p,i} + CE_p. \tag{8.24a}$$

At fixed p, i, C the right side is independent of C_H. For the host field E and the Kronecker character χ_d of $\mathbf{Q}(\sqrt{-d})$, the period of 21.12 satisfies

$$\log\Omega = \gamma + \log(2\pi/\operatorname{rd}(E)) - L_E, \qquad L_E = \operatorname{Re} \frac{L'(1, \chi_d)}{L(1, \chi_d)}. \tag{35.2}$$

Here $\operatorname{rd}(E) = |\operatorname{disc} E|^{1/[E:\mathbf{Q}]}$.

35.1 The estimate and its geometric inputs

Let $\mathcal{U}$ be the class of all families. For a family F_m, write x_1 for its lower threshold and I_x for its index set at x. Let $\mathcal{F}_{\mathrm{cov}}$ consist of the families in $W_3(\mathcal{U})$ satisfying clauses 1–3 of Definition 33.3 and the following wall bound. Beyond one family cutoff, at every index the prime sum of 22.46, evaluated at the point's own weights, is at most x^{-9}. We use admissibility in the sense of Definition 35.1.

Definition 35.1 (Admissible points for the covolume bound)**.** Let p be a point of the family, K its realisation field and $r = k + m - 1$. The next three paragraphs define the class $\mathcal{P}$ of member points used for the lower-weight estimate. An admissible point for the covolume bound is a point of $\mathcal{P}$ satisfying also the two arithmetic requirements below.

At each step of the theta chain, including the last step, the intertwiners and the maps defined by integration against the theta kernel exist. For every tuple the resulting theta maps agree under evaluation, both on the base and on each layer. At the embeddings extending the prescribed base embedding, evaluation at the chosen vector is injective. At the other embeddings the conjugate evaluation is realised by a unit vector whenever the conjugate weight space is identified with the layer space. On each core cusp space the finite level group acts over the core field, preserves integral sections, and under evaluation acts by the realised action

multiplied by the prescribed character. In every layer the extremal vector pairs to the prescribed product of minors, and the one-map comparison holds, including the comparison over an extension containing the image of K under the prime restrictions of 37.24.

The primitive and divisor rows satisfy the identities of 33.74. Distinct finite indices in the reference row have distinct transported types at every layer. The level-one row and its lifts have the prescribed Hecke decompositions: the inert operators commute, split over the specified field and commute with the finite group action, with distinct indexed inert places. The frames at the two auxiliary primes have the prescribed bases. Both global row lattices agree with their indefinite realisations, the rational lifts span the indicated spaces, and the theta maps and column pairings agree under these identifications. The two level characters come from the prescribed residue-field additive characters under the group identification and satisfy the Weil character relation. At the reference tuple and base character, the selected row subspace has dimension D_0. The rational layer maps into the exported row spaces induce the prescribed maps on coordinates and identify the layer lattices and global row lattice. Every finite row type satisfies the local type conditions of 33.74. Every place passing the single-index test for the reference numerical data at $(0,0)$ belongs to the twisting set.

The point comes from compatible spectral and indefinite realisations with the prescribed normalisation and recursion. The core data, base pairing, tame types and series agree under transport. The reference subgroup corresponds to the prescribed level subgroup, and its inverse character gives the two local level characters after scalar extension. At each twist index and layer, on either side, every theta datum transforming by the level character belongs to the allowed set. At each place with nonreference tuple the twist character takes values in the prescribed coefficient field. The embedded realisation field satisfies the ray-level condition at $g_0 g_1 p_{\mathrm{de}} p'_{\mathrm{de}}$, contains the $g_a(g_a-1)$-th roots of unity for $a=0,1$, and contains the ray fields at the tuple levels. The point satisfies the twist and basis conditions of clause 7 of 33.3 at its own twist data.

For the first arithmetic requirement let $F \subset \mathbf{C}$ be the field generated over $\mathbf{Q}$ by the image of the coefficient ring in the prescribed complex embedding. Put $H = 19 f r^2 L_x + 198 f^2 L_x$. Require

$$\log[F:\mathbf{Q}] \le H, \qquad \log \mathrm{rd}(F) \le H.$$

For the second, at each $g = g_0, g_1$ there must be a label datum of 32.3 with the prescribed Teichmüller map. For every $P \in \mathbf{F}_g[X]$, the specified series element in $\mathbf{C}[\mathrm{GL}_r(\mathbf{F}_g)]$ must equal the sum of the central primitive idempotents of the irreducible characters with label P for that same datum.

Definition 35.2 (The family covolume bound)**.** For each eligible F_m, C_H in the closed range of 36.10 on $\mathcal{F}_{\mathrm{cov}}$, there exist $C_0 \ge 0$ and $x_0 \in \mathbf{R}$ such that for every $x > x_1$ with $x \ge x_0$, every $\iota \in I_x$, every admissible point over (x,ι), every $C \ge C_0$ and every $0 \le i < m$,

$$\frac{\mathrm{lcv}(X_{W,i}, \Lambda_i)}{D_{\mathrm{norm}}} \le M_{p,i} + C E_p. \tag{8.24b}$$

Here $X_{W,i}$ and Λ_i are the frame and lattice of layer i in the reference indefinite realisation at p. The normalising dimension is $D_{\mathrm{norm}} = \dim X_E$, with the convention $a/0 = 0$.

Write $r = k + m - 1$ and $n_V = r + t$, where the chain ranks are $N(j) = k + j(t+1)$. The precision parameter at the point is $P_* = C_P (f r L_x)^{e_P}$, where $C_P \ge 1$ and $e_P \ge 12$ are the chosen precision constants, and throughout this chapter eligible families are taken in the closed range of 36.10 with stage-three spacing exponent $C_{\mathrm{count}} \ge 4e_P + 11$. The layer signature is $(P,Q) = (k + j - 1, m - j)$. Use one CM embedding τ per conjugate pair. Call a deep cover Y of the layer good with parameters (e,o), for $e \in \mathbf{N}$ and $o \in \mathbf{R}$, under the following conditions. Its level extends the base level by the coprime factor 15^e, with auxiliary level parameter one and complementary conjugate ideal factors. The level sections respect the covering map, every geometric frame lifts, and the weight-zero comparison section is nowhere zero. The finite adelic subgroup is the principal congruence subgroup of level $M' = g_0 g_1 15^e$ in the lattice stabiliser. The cover's coefficient field F' is relatively algebraically closed in the component function fields and satisfies $\log[F':\mathbf{Q}] + \log \mathrm{rd}(F') \le o$. The intermediate coefficient extension used to form the cover is normal over the core coefficient field. An embedding of the coefficient field is called good when its restriction to the imaginary quadratic host $K = \mathbf{Q}(\sqrt{-d})$ is the distinguished embedding. Every complex-conjugate pair contains exactly one good embedding. At each good embedding, the components have their analytic uniformisations, the uniformised

frames come by base change from test rings, and the characters of dominant weights are trivial on the component groups acting on the domains. The adelic frame parametrisation agrees with uniformisation. Its distinguished minors have absolute value one. In rank r the component Gram matrix is the negative transpose of the layer Gram matrix and its polarising scalar is the negative of the host scalar, of prescribed absolute value at every embedding. One embedding of the layer lattice identifies it with the uniformising lattice and gives compatible orthogonal frames on all components at every such embedding. At a nonterminal layer, a good cover without displayed parameters means one with $e = \lfloor 3L_x/\log 3 \rfloor_{\mathbf{N}}$ and $o = 391(frL_x)^2$.

Let η_H be the Hodge weight (0 in the first P columns, -1 in the last Q per τ), with zero-based anchor $a_\tau = \lambda_{\tau,k-2}$, $u_W = (\lambda_{\tau,i} - a_\tau)_{\tau,i}$, and $u_M = u_W - c_0\eta_H$ (scalar exponent 0), where $c_0 = c_0(D,j)$ is the least multiple of 8 at least

$$\max\left(r + \binom{fPQ+1}{2} + 2,\ 8fr + 4\right).$$

Write $G_s(t) = (\lambda + t\eta_H, n_{\mathrm{col}}/2)_{\mathrm{row}}$ for shifted row weights (distinguishing row pairs $(u,m)_{\mathrm{row}}$, $m = n_{\mathrm{col}}/2$, from theta pairs $(a,N)_\theta$, $N = 2m$), $\mu_Y = 2\widehat{\deg}/\operatorname{rk}$ for squared-norm slope, and *algebra of rules* for the graded ring of modular sections on frames (Definition 35.53).

Definition 35.3 (Section lattices and flat degree)**.** Let $r_Y(G)$ be the rank over the coefficient field of the integral cusp lattice of weight G on Y. A weight has a regular section lattice if its Petersson norms admit Gram frames and its integral cusp sections form a finitely generated module over the coefficient integers spanning the cusp space. On a component this condition uses the sections supported on that component and their component norms.

Deep goodness means that the row weights at shifts $-c_0, 0, c_0$ and the anchored weight u_M have regular section lattices, with $r_Y(G_s(-c_0)) > 0$. The component set is finite and nonempty, and the weight-u_M lattice is regular on each component.

The flat-degree law requires that on every component, for every CM embedding τ, the flat line at τ have an invertible section with constant norm on the frames at each infinite place and constant valuation on the frames at each finite place after subtraction of the local different defect. The corrected finite valuations have finite support. Every such section has degree

$$\frac{P-Q}{2}\log\Omega + Q\big(\log d - \log \mathrm{rd}(E)\big).$$

Lemma 35.4 (The flat line in the row metric)**.** *Under the flat-degree law on a nonempty signature-(P,Q) component system, the first Chern value at τ is*

$$c_{1,\tau} = -\tfrac{P-Q}{2}\log\Omega - Q(\log d - \log \mathrm{rd}(E)). \tag{35.3}$$

For a good cover ($d \equiv 7 \pmod{16}$), assume $\mathcal{H}_P$ of Definition 35.5, the Tamagawa formula for special unitary groups, and the existence of a smooth toroidal compactification with simple normal-crossings boundary, logarithmic Kodaira–Spencer isomorphism and canonical extension of de Rham cohomology. Assume compatible complex uniformisation and level goodness at every rational prime, deep goodness, the existence of flat sections, the host and weight size conditions, the anchor bounds $|M_{p,i} - (P-Q)\log\Omega\sum_\tau a_\tau| \le 100E_p$, $QL_x\sum_\tau |a_\tau| \le 100E_p$, and $frn_V L_x \le b$, and

$$|c_{1,\tau} + (P-Q)\log\Omega/2| \le 3QL_x \qquad \text{for every } \tau.$$

Assume also the unramified-point hypotheses of 35.10 and $\sum_{\ell \mid 2dD_E M'} \log\ell \le 40L_x$ as in 35.21. Then

$$\mu_Y(G_s(-c_0)) - \mu_Y(u_M) \ge -M_{p,i} - 1183E_p.$$

Outline. The CM eigenlattice comparison of 21.22 and the local different give the first Chern value. The integral twist preserves rank, and comparison with $M_{p,i}$ and bounds on the different, twist and bad primes give the slope loss $1183E_p$. The complete proof is on p. 1116. □

Definition 35.5 (The assertions in the Hodge-bundle hypothesis)**.** Write $\mathcal{H}_P$ for the conjunction below (each with its own universally quantified data); on an integral projective arithmetic variety of generic dimension D, write $I(\overline{L})$ for the order-$(D+1)$ arithmetic self-intersection of $\overline{L}$.

1. For a unitary datum of positive rank and level M', there is a smooth projective model over its coefficient field with SNC boundary and a logarithmic Kodaira–Spencer isomorphism on the interior. The smooth integral PEL toroidal models at good primes $\ell \nmid 2dD_E M'$, with relative normal-crossings boundary and multiplicity-one special fibres, are used under the separate integral-model hypotheses of Theorem 35.11. The split-fibre condition in Theorem 35.11 requires that at each point the semiabelian fibre be an extension of an abelian variety by a split torus after a finite separable extension of the residue field.

2. If $\overline{L}$ is nef on fibres and its metric is semipositive, then $I(\overline{L}) \le \widehat{\mathrm{vol}}(\overline{L})$, with the normalisation of Theorem 35.33.

3. If $\overline{L}$ is arithmetically nef, then $\widehat{\mathrm{vol}}(\overline{L}) = I(\overline{L})$.

4. Arithmetic heights and the geometric and arithmetic intersection functionals exist for finite families of Hermitian lines. The arithmetic functional obeys the rational-section recursion, is invariant under isometry, and is multilinear in tensor powers. Dominant generically finite pullback multiplies it by the generic degree. Adding a smooth conjugation-invariant function u to one metric weight adds one half of the integral of u against the product of the remaining Chern forms, summed over the embeddings. In the constant log-norm shift convention this changes $I(\overline{L})$ by $(D+1)t\deg(L_{\mathbf{Q}}^{D})$ for a shift of size t.

5. For an abelian scheme of relative dimension g over an affine base, H^1_{dR} is finite locally free of rank $2g$. Its classes are primitive for the group law, cup product identifies its second exterior power with H^2_{dR}, and the Hodge submodule is a direct summand of rank g. The Hodge sequence commutes with base change. A lift over a nilpotent divided-power thickening identifies the crystal with the de Rham module of the lift. The polarisation pairing is zero on the Hodge submodule and is perfect for a principal polarisation.

6. For a nonzero integral ideal $\mathfrak{m}$ of a number field, use the ray modulus consisting of $\mathfrak{m}$ and all real places. There is a finite abelian extension unramified at every finite prime not dividing $\mathfrak{m}$. The Frobenius elements at those finite primes generate the Galois group. On coprime ideals the common kernel of the characters is the principal ray subgroup, with totally positive generators congruent to one at the modulus.

7. A principally polarised abelian variety over a finite extension of $\mathbf{Q}_p$ has a semiabelian model after a finite extension.

8. Let G be the special unitary group of a nondegenerate Hermitian space of rank at least two over a CM extension E/F. For a finite set S of places such that $\prod_{v\in S} G(F_v)$ is noncompact, $G(F)$ is dense in the adeles of G away from S. Here G is simply connected and absolutely almost simple, as required for strong approximation [19, Th. 7.12].

9. For g prime and $N > 0$, Green's full classification [20, Th. 14] assigns distinct generic irreducible characters of $\mathrm{GL}_N(\mathbf{F}_g)$ to monic squarefree degree-N parameters with nonzero constant term. The degree belonging to a parameter times the order of its torus is $\prod_{i=1}^{N}(g^i-1)$.

10. For a bundle with a good metric on the open of a smooth compactification, the Chern integral is the intersection number of the extended classes. On its full flag bundle let Δ be the root Vandermonde and let v be the fibre point class. If the alternation of a root polynomial Q is ΔR, with R symmetric in each factor, then Q and Rv have the same top intersection on the flag bundle.

11. Nonsingular Hermitian forms of equal rank over a CM field are isometric if their signatures agree at every archimedean place and the quotient of their determinants is a norm from the CM field to its real subfield.

12. For $\delta, x \in F^{\times}$, where F is a number field, $u^2 - \delta v^2 = x$ has a solution in F if it has a solution in each real and finite completion.

13. For semiabelian schemes over a normal Noetherian base, a homomorphism over a dense open extends uniquely over the base [167, Prop. 3.3.1.5].

14. Over a field of characteristic $p > 0$, the induced module for a dominant integral GL_N-weight λ is nonzero and simple if $\lambda_i - \lambda_j + j - i \le p$ for every $i < j$.

15. A line inducing a principal polarisation on an abelian variety over an algebraically closed field has a one-dimensional space of sections.

16. If A is an abelian variety of dimension g over an algebraically closed field and n is invertible in that field, then $\#A[n] = n^{2g}$.

17. Over a nilpotent divided-power thickening on which p is nilpotent, the de Rham module of an abelian lift is identified with the evaluation of the crystal. Given such a lift, an admissible lift of the Hodge filtration is realised by an abelian lift. A homomorphism on the closed fibre lifts uniquely when its action on the crystal preserves the two lifted filtrations.

18. Over the p-adically complete valuation ring of the algebraically closed characteristic-zero field in Lemma 35.19, with p a nonunit, the contravariant prismatic module of a p-divisible group is free of rank its height. With ξ the generator of the kernel of Fontaine's map determined by the chosen roots of unity, the Frobenius image contains $\varphi(\xi)M$, and reduction modulo ξ gives the functorial Hodge sequence. On the constant and multiplicative height-one groups the Frobenii are respectively φ and $\varphi(\xi)\varphi$. Frobenius-compatible linear maps from the multiplicative module to the constant module correspond to homomorphisms from the constant group to the multiplicative group.

19. For a smooth integral projective variety over an algebraically closed field of characteristic zero and a nef and big line N, $H^i(X, K_X \otimes N) = 0$ for $i > 0$.

20. On a projective D-dimensional scheme over an algebraically closed field, a top product of nef lines is nonnegative. On an integral such scheme a nef line L is big exactly when $L^D > 0$. The section count for a nef line satisfies $h^0(X, L^m) = (L^D)m^D/D! + O(m^{D-1})$.

21. For a compact complex n-fold X ($n \geq 1$) admitting a positive line, sections $s_1, \ldots, s_n$ of a Hermitian line L (weight ϕ_L) with $\sum_j |s_j|_L^2 \leq 1$ and transverse finite common zero set Z, $\psi = n \log \sum_j |s_j|_L^2$, and a Hermitian line E (weight ϕ_E) with $\phi_E + \alpha\psi$ semipositive off Z for $1 \leq \alpha \leq 1 + \delta$ ($\delta > 0$), prescribed values $f(P)$ in local frames of $K_X \otimes E$ extend to $F \in H^0(X, K_X \otimes E)$ satisfying

$$2^n \int_X \frac{|F|^2 e^{-\psi}}{1 + \delta^2 \psi^2}\, d\mu \leq \frac{34}{\delta} \frac{2^{n+1}\pi^n}{(n-1)!} \sum_{P \in Z} \left| \frac{f(P)}{\operatorname{Jac} s(P)} \right|^2 e^{-\phi_E(P) + n\phi_L(P)}. \tag{35.4}$$

22. For an arithmetically nef Hermitian line L of top self-intersection a on an integral arithmetic variety of generic dimension D and a fixed Hermitian twist N, the unit ball in $H^0(L^j \otimes N)$ is eventually finite and has logarithmic cardinality

$$\frac{a j^{D+1}}{(D+1)!} + o(j^{D+1}). \tag{35.5}$$

23. A geometrically integral projective D-fold in characteristic zero has a Chow form over its field of definition, unique up to a nonzero scalar. It vanishes on the $(D+1)$-tuples of hyperplanes meeting the variety in a common point over the algebraic closure. Its degree in each block is the degree of the variety.

24. Arithmetic cycle heights satisfy the rational-section recursion. If an integral arithmetic variety maps generically finitely onto its image of the same dimension, the self-intersection of the pulled-back line is the generic degree times the height of the image cycle.

25. On a smooth projective complex variety analytification identifies the algebraic and holomorphic sections of a line bundle.

26. Let (X, Δ) be a normal integral projective complex pair with Δ effective and $K_X + \Delta$ $\mathbf{Q}$-Cartier. If M is a line numerically equivalent to $K_X + \Delta + A$ with A nef and big, the multiplier ideal $\mathcal{J}$ has cosupport the non-klt locus and $H^i(X, M \otimes \mathcal{J}) = 0$ for $i > 0$. For the second assertion suppose the pair is klt, fix a closed point x, and put $n = \dim X \geq 1$. If $A^j \cdot Z > c_j^j$ for every irreducible j-fold Z through x, with $c_j > 0$ and $\sum_{j=1}^n j/c_j \leq 1$, there is an effective rational Cartier divisor $B \equiv A$ such that $(X, \Delta + B)$ is log canonical near x and has x as its isolated non-klt locus there. In both assertions A is a positive rational multiple of a nef and big line.

27. On a smooth integral projective variety in characteristic zero, a nonzero coherent ideal becomes effective Cartier with simple normal crossings support after a sequence of smooth blow-ups which is an isomorphism off its support.

28. Let R be Noetherian and complete for an ideal I. Compatible morphisms modulo all I^{j+1} from a proper R-scheme to a separated finite-type R-scheme algebraize uniquely. A compatible system of proper schemes over R/I^{j+1} with compatible invertible sheaves whose closed-fibre member is ample algebraizes to a proper R-scheme with a relatively ample invertible sheaf.

In what follows, write $\mathcal{H}_X$ for the package consisting of the prescribed-fan model, semiabelian Lie-bundle seminegativity, the separation conclusion under (A.430), the cuspidal Green parametrisation of 32.31 and the Fubini–Study formula for Chow height; write $\mathcal{H}_H$ for the package consisting of the Siegel model, the prescribed-fan boundary charts, their compatibility with the Siegel charts and the arithmetic degree formula.

Lemma 35.6 (A generator on each component)**.** *Let Z be a component of a good cover over its coefficient field F'. On the moduli open of Z, let A be the universal abelian scheme. For an embedding τ in the CM type, write $V_\tau = (\operatorname{Lie} A)_\tau$ and put*

$$\mathcal{L}_\tau = \det V_\tau \otimes (\det V_{\bar\tau})^{-1}.$$

There is an F'-rational section of $\mathcal{L}_\tau$ which generates the line everywhere on the open. At each infinite place its invariant norm is constant on the frames of Z.

Outline. The polarized Hodge sequence identifies the line with a de Rham determinant, and the level congruence gives trivial determinant monodromy. Deligne's regular-singular comparison algebraizes the horizontal generator, which descends to F' and has constant Hodge norm. The complete proof is on p. 1119. □

Lemma 35.7 (The projected Hodge lattice and the two integral Lie frames)**.** *Let R be a discrete valuation ring of mixed characteristic $(0,\ell)$, with fraction field K containing the images of the embeddings of E. Write v for its additive valuation. Suppose that the integral Hodge sequence, in homology, is*

$$0 \longrightarrow L^\vee \longrightarrow H \longrightarrow L \longrightarrow 0,$$

where L is the Lie lattice, the action on its dual is conjugated, and H is locally free of rank r over $R \otimes_{\mathbf{Z}} \mathcal{O}_E$. For an embedding $\sigma : E \to K$, put

$$L[\sigma] = L \cap (L \otimes_R K)_\sigma, \qquad H(\sigma) = \operatorname{pr}_\sigma H, \qquad h_\sigma = \dim_K (L \otimes_R K)_\sigma.$$

The Hodge sequence identifies $\det H(\tau)_K$ with $\det L[\tau]_K \otimes \det L[\bar\tau]_K^{-1}$.

Complete R. In the factor of $\mathcal{O}_E \otimes R$ containing τ, let S be the embeddings with the same residual embedding as τ. Choose a uniformiser π of the corresponding completion of E over its maximal unramified subfield, and put $a_\sigma = \sigma(\pi)$ for $\sigma \in S$. In the above identification the determinant of a pair of integral Lie frames generates the sublattice of $\det H(\tau)$ of valuation

$$\sum_{\sigma \in S \setminus \{\tau\}} h_\sigma \, v(a_\sigma - a_\tau).$$

In particular, a section whose coefficient in a generator of $\det H(\tau)$ is b has valuation $v(b)$ minus this sum in the Lie frames.

Outline. Splits the unramified part over completed R so $H(\tau) \simeq H/TH$ ($T = t - a_\tau$). Lifting a basis of $L[\tau]$ and adjoining $\operatorname{pr}_\tau F \simeq L[\bar\tau]^\vee$ gives $\operatorname{coker}(T | L/L[\tau])$, whose length is the stated sum. The complete proof is on p. 1120. □

Lemma 35.8 (Formal biextension germs and the integral Hodge bundle on a normal model)**.** *Let Z be a component of signature (P,Q), with $r = P + Q \geq 1$, of a good cover over its coefficient field F', and let $O \subset F'$ be a discrete valuation ring of mixed characteristic $(0,\ell)$ with fraction field F' and uniformiser $\varpi \mid \ell$. Let $\pi_X : X \to \operatorname{Spec} O$ be an integral, normal, flat, projective scheme over O with function field Ω and generic point $\xi : \operatorname{Spec} \Omega \to X$, whose generic fibre $X_{F'}$ contains the moduli open U of Z as a dense open subscheme. Suppose X carries a semiabelian scheme $G \to X$ of relative dimension $d = \frac{1}{2}[E : \mathbf{Q}]r$ with identity section*

$e : X \to G$, an action of $\mathcal{O}_E$ on G, and a symmetric cubical line bundle L_{PP} on G rigidified along e, whose Mumford biextension $\Lambda(L_{\mathrm{PP}}) = m^*L_{\mathrm{PP}} \otimes p_1^*L_{\mathrm{PP}}^{-1} \otimes p_2^*L_{\mathrm{PP}}^{-1}$ on $G \times_X G$ (where $m, p_1, p_2 : G \times_X G \to G$ are multiplication and the two projections, rigidified along $e \times \mathrm{id}_G$ and $\mathrm{id}_G \times e$) satisfies the following conditions:

- *(a) for every $a \in \mathcal{O}_E$, there is a bi-rigidified isomorphism $\mathrm{ad}_a : (a \times \mathrm{id}_G)^*\Lambda(L_{\mathrm{PP}}) \xrightarrow{\sim} (\mathrm{id}_G \times \bar{a})^*\Lambda(L_{\mathrm{PP}})$ on $G \times_X G$;*
- *(b) L_{PP} is isomorphic as a rigidified line bundle to $(\mathrm{id}_G, w_0)^*\Lambda(L_{\mathrm{PP}})$ for an element $w_0 \in \mathcal{O}_E$ with $w_0 + \bar{w}_0 = 1$;*
- *(c) U equals the abelian locus of $G_{F'}$ in $X_{F'}$, and $(G_U, \mathcal{O}_E\text{-action}, \Lambda(L_{\mathrm{PP}})_U)$ agrees with the universal principally polarised abelian scheme on U with its $\mathcal{O}_E$-action and Poincaré biextension;*
- *(d) at every geometric point $\bar{x} : \operatorname{Spec} \bar{k} \to X$ of residue characteristic ℓ, with Chevalley torus extension $0 \to T \to G_{\bar{x}} \xrightarrow{q_{\bar{x}}} B \to 0$ of an abelian variety B by a torus T, there is a rigidified line bundle M on B inducing a principal polarisation $\lambda_B : B \xrightarrow{\sim} B^\vee$ such that $L_{\mathrm{PP}}|_{G_{\bar{x}}} \simeq q_{\bar{x}}^*M$.*

Write $\mathcal{L} = \mathrm{Lie}(G/X)$ and $V = \mathcal{L}_\xi$. Assume also that the semiabelian fibres become extensions by split tori after finite separable extensions of their residue fields. For every complete Noetherian normal local domain R of characteristic zero and every morphism $\operatorname{Spec} R \to X$ meeting the abelian locus and having split closed fibre, require a formal PEL uniformisation of the pulled-back semiabelian scheme, compatible with its action and polarisation, whose cubical period sections are units on the pulled-back abelian locus and whose induced torsion maps are isomorphisms. Then the following statements hold:

1. *For every integer $n \geq 0$ and every nonempty open subscheme $W \subseteq X$, pullback along the structure morphism $\pi_n : G_W^{\times_W n} \to W$ is a bijection $\Gamma(W, \mathcal{O}_W) \xrightarrow{\sim} \Gamma(G_W^{\times_W n}, \mathcal{O}_{G_W^{\times_W n}})$.*
2. *Write $\mathbf{x} = (x_1, \ldots, x_d)$ and $\mathbf{y} = (y_1, \ldots, y_d)$ (and $\mathbf{x}', \mathbf{x}'', \mathbf{y}', \mathbf{y}''$) for d-tuples of formal variables. Over a commutative ring R, a d-dimensional formal biextension germ B consists of a commutative formal group law $\mu_B(\mathbf{x}, \mathbf{y}) \in R[[\mathbf{x}, \mathbf{y}]]^d$ with formal $\mathcal{O}_E$-action $a \mapsto [a]_B(\mathbf{x}) \in R[[\mathbf{x}]]^d$ of linear part $\rho_B(a) = D([a]_B)(0) \in \mathrm{Mat}_d(R)$, together with bi-rigidified partial-multiplication, symmetry and Rosati-adjointness series $\varphi_1(\mathbf{x}', \mathbf{x}'', \mathbf{y})$, $\varphi_2(\mathbf{x}, \mathbf{y}', \mathbf{y}'')$, $\mathrm{sy}(\mathbf{x}, \mathbf{y})$ and $\mathrm{ad}_B(a)(\mathbf{x}, \mathbf{y})$ ($a \in \mathcal{O}_E$) satisfying the symmetric commutative biextension cocycle laws and*

$$\begin{aligned}\varphi_1([a]_B\mathbf{x}', [a]_B\mathbf{x}'', \mathbf{y})\,\mathrm{ad}_B(a)(\mu_B(\mathbf{x}', \mathbf{x}''), \mathbf{y}) &= \mathrm{ad}_B(a)(\mathbf{x}', \mathbf{y})\,\mathrm{ad}_B(a)(\mathbf{x}'', \mathbf{y})\,\varphi_1(\mathbf{x}', \mathbf{x}'', [\bar{a}]_B\mathbf{y}),\\ \varphi_2([a]_B\mathbf{x}, \mathbf{y}', \mathbf{y}'')\,\mathrm{ad}_B(a)(\mathbf{x}, \mu_B(\mathbf{y}', \mathbf{y}'')) &= \mathrm{ad}_B(a)(\mathbf{x}, \mathbf{y}')\,\mathrm{ad}_B(a)(\mathbf{x}, \mathbf{y}'')\,\varphi_2(\mathbf{x}, [\bar{a}]_B\mathbf{y}', [\bar{a}]_B\mathbf{y}'').\end{aligned}$$

   Require also the Rosati identities

$$\begin{aligned}\mathrm{ad}_B(1) &= 1,\\ \mathrm{ad}_B(ab)(\mathbf{x}, \mathbf{y}) &= \mathrm{ad}_B(a)([b]_B\mathbf{x}, \mathbf{y})\,\mathrm{ad}_B(b)(\mathbf{x}, [\bar{a}]_B\mathbf{y}),\\ \varphi_1([a]_B\mathbf{x}, [b]_B\mathbf{x}, \mathbf{y})\,\mathrm{ad}_B(a+b)(\mathbf{x}, \mathbf{y}) &= \mathrm{ad}_B(a)(\mathbf{x}, \mathbf{y})\,\mathrm{ad}_B(b)(\mathbf{x}, \mathbf{y})\,\varphi_2(\mathbf{x}, [\bar{a}]_B\mathbf{y}, [\bar{b}]_B\mathbf{y}),\\ \mathrm{ad}_B(a)(\mathbf{x}, \mathbf{y})\,\mathrm{ad}_B(\bar{a})(\mathbf{y}, \mathbf{x})\,\mathrm{sy}([a]_B\mathbf{x}, \mathbf{y}) &= \mathrm{sy}(\mathbf{x}, [\bar{a}]_B\mathbf{y}).\end{aligned}$$

   There exists a system S of d-dimensional formal biextension germs B_q over $\mathcal{O}_{X,q}$ for all points $q \in X$, obtained from local relative parameters of G along e and bi-rigidified local trivialisations of $\Lambda(L_{\mathrm{PP}})$, equipped with compatible formal biextension isomorphisms (φ, ψ, λ) (where $\psi = \varphi^{-1}$) along every specialisation $q' \rightsquigarrow q$ in X.
3. *For any bi-rigidified series $f(\mathbf{x}, \mathbf{y}) \in 1 + (\mathbf{x})(\mathbf{y})R[[\mathbf{x}, \mathbf{y}]]$, write $\mathrm{bil}(f)_{1,1} \in \mathrm{Mat}_d(R)$ for its bilinear $(1,1)$-coefficient matrix in $f(\mathbf{x}, \mathbf{y}) \equiv 1 + \mathbf{x}^{\mathsf{t}}\,\mathrm{bil}(f)_{1,1}\mathbf{y} \pmod{(\mathbf{x})^2 + (\mathbf{y})^2}$. For any formal biextension germ B over R coming by scalar extension from S, setting*

$$\rho_H(B)(a)(v, \eta) = \big(\rho_B(a)v,\ \rho_B(\bar{a})^{\mathsf{t}}\eta - \mathrm{bil}(\mathrm{ad}_B(a))_{1,1}^{\mathsf{t}}v\big) \qquad (a \in \mathcal{O}_E,\ v, \eta \in R^d)$$

   defines an $\mathcal{O}_E$-module structure on $R^d \times R^d$, and every formal biextension isomorphism $(\varphi, \psi, \lambda) : B \xrightarrow{\sim} B'$ induces an $\mathcal{O}_E$-linear isomorphism

$$\mathrm{transH}(\varphi, \psi, \lambda)(v, \eta) = \big(D\varphi(0)v,\ D\psi(0)^{\mathsf{t}}(\eta + \mathrm{bil}(\lambda)_{1,1}^{\mathsf{t}}v)\big)$$

from the module of B' to that of B. Gluing the stalk modules of S inside $V \times V^\vee$ defines a coherent $\mathcal{O}_E$-module $\mathcal{H}$ on X sitting in a locally split $\mathcal{O}_X$-exact sequence

$$0 \longrightarrow \mathcal{L}^\vee(\mathrm{conj}) \longrightarrow \mathcal{H} \longrightarrow \mathcal{L} \longrightarrow 0,$$

where $\mathcal{O}_E$ acts on $\mathcal{L}^\vee(\mathrm{conj})$ through conjugation, and at every point $q \in X$ the stalk $\mathcal{H}_q$ is free of rank $r = P + Q$ over $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathcal{O}_{X,q}$.

Outline. Denseness of U yields (1) and uniqueness of bi-rigidified maps. Local relative parameters build formal biextension germs whose $(1,1)$-jet modules glue to $\mathcal{H}$. At char-ℓ stalks, Chevalley descent, Dieudonné crystal freeness, strict henselisation, and Nakayama's lemma establish free rank r. The complete proof is on p. 1121. □

Lemma 35.9 (Formal-biextension conjugate sequence and degree zero on special-fibre curves)**.** *Keep the hypotheses and notation of Lemma 35.8. Let k' be any field extension of $k_0 = O/\varpi O$, and let $\Gamma \subset X_{k'}$ be a complete integral curve with normalisation $\nu : \widetilde{\Gamma} \to \Gamma$. Then every embedding $\sigma : E \hookrightarrow \Omega$ maps $\mathcal{O}_E$ into the image of O in Ω, and the invertible sheaf $\mathscr{D}_\sigma = \det(\mathcal{H} \otimes_{\mathcal{O}_E \otimes_{\mathbf{Z}} \mathcal{O}_X,\, \sigma\otimes 1} \mathcal{O}_X)$ satisfies $\deg_{\widetilde{\Gamma}}(\nu^* \mathscr{D}_\sigma) = 0$.*

Outline. Glues formal-biextension maps from $[\ell]_B$ via 35.8 into a conjugate exact sequence on $\widetilde{\Gamma}$. Comparing degrees with the Hodge sequence around each Frobenius cycle of idempotents forces $\deg_{\widetilde{\Gamma}}(\nu^* \mathscr{D}_\sigma) = 0$. The complete proof is on p. 1122. □

Lemma 35.10 (Finite-place order of the flat section)**.** *Assume resolution, semistable reduction, Lan extension, integral Siegel and good-prime ($\ell \nmid 2dD_E M'$) smooth integral PEL toroidal models (split fibres), and GAGA of 35.5. Let s be the horizontal generator (35.6) on Z/F' for host E (5.1, 8.3, prime $d \equiv 7 \pmod{16}$). For each finite place w of F' ($v_w(\varpi_w) = 1$), there is $\rho_w \in \mathbf{Q}$ such that every integral frame y on Z over a trait of ramification index e over w satisfies*

$$v_y(s(y)) = e\rho_w$$

($v_y(\varpi_y) = 1$). Moreover $\rho_w = 0$ for almost all w, and $\rho_w \in \mathbf{Z}$ whenever $w \nmid 2dD_E M'$. For the different-corrected line the finite coefficient at $w \mid \ell$ is

$$\rho_w - Q\, v_w(\ell) \left(\frac{v_\ell(|\operatorname{disc} E|)}{[E : \mathbf{Q}]} - \mathbf{1}_{\ell = d} \right).$$

Outline. Extends $\mathcal{L}_\tau$ of 35.6 to a normal model via 35.8, showing $\mathrm{div}_X(s)$ is vertical. Applies 35.9, Stein factorization, and 35.7 to obtain $v_y(s(y)) = e\rho_w$ and subtract the different defect. The complete proof is on p. 1123. □

Theorem 35.11 (Geometric inputs for the covolume estimate)**.** *In the notation of 35.19, assume resolution, principalisation, invariant Poincaré–Lelong, the joint arithmetic inequalities (A.422) and (A.423), 35.33, Fargues exactness, Zarhin's retraction, nef Riemann–Roch, both compactifications of 35.19 (with fan divisor and good-prime $\ell \nmid 2dD_E M'$ smooth integral PEL toroidal models with split fibres), Lie seminegativity, point separation, arithmetic degree, $\mathcal{H}_P$ (35.5), and Grothendieck deformation. Then 35.2 holds.*

Outline. Assembles $\mathcal{H}_X, \mathcal{H}_H$ and the section injection, proves deep goodness via 35.20, 35.21, and flat degree from 35.6, 35.10. Combines 35.35, 35.26, 35.94, 35.95, 35.22, 35.96, 35.97, 35.98, and 35.99. The complete proof is on p. 1127. □

Lemma 35.12 (Sections on the integral model and its normalisation)**.** *Let $y = (y_{ki})$ be homogeneous coordinates in a field Ω, in finitely many nonempty blocks indexed by k, with a nonzero coordinate in each block. Let P be the kernel of the homomorphism*

$$\mathbf{Z}[X_{ki}] \longrightarrow \Omega[T_k : k], \qquad X_{ki} \longmapsto y_{ki} T_k.$$

Suppose $P \cap \mathbf{Z} = 0$ and every element of Ω is a quotient of evaluations at y of integral multihomogeneous forms of the same multidegree. Let X be the closed subscheme defined by P in the product of projective spaces

over $\mathbf{Z}$*, let* $d = \dim X_{\mathbf{Q}}$*, and let* $\nu : X^{\nu} \to X$ *be its normalisation. For an integral multidegree* $a = (a_k)$*, write* $L = \mathcal{O}_X(a)$*.*

For a choice $c = (c_k)$ *with all* $y_{kc_k} \neq 0$*, put*

$$R_c = \mathbf{Z}[y_{ki}/y_{kc_k} : k, i], \qquad m_c = \prod_k y_{kc_k}^{a_k},$$

and let $\widetilde{R}_c$ *be the integral closure of* R_c *in* Ω*. For* $r \geq 0$ *put*

$$S_r = \{s \in \Omega : s/m_c^r \in R_c \text{ for every } c\}, \qquad \widetilde{S}_r = \{s \in \Omega : s/m_c^r \in \widetilde{R}_c \text{ for every } c\}.$$

These are the global sections of L^r *on* X *and of* $\nu^* L^r$ *on* X^{ν}*, respectively. Give* L *a smooth conjugation-invariant metric specified on the complex cone by a positive function* h *with* $h((t_k z_{ki})) = \prod_k |t_k|^{a_k} h(z)$*. The point norm of a form* g *of multidegree* ra *is* $|g(z)|/h(z)^r$*. For a multidegree with all* $a_k \geq 1$*, there is an integral multihomogeneous conductor form* F *of multidegree* qa*, for some* $q \in \mathbf{N}$*, with* $F(y) \neq 0$*, such that* $F(y)\widetilde{S}_r$ *consists of evaluations of integral forms of multidegree* $(r+q)a$ *for every* r*. For the following asymptotic comparison, also assume the arithmetic-volume limit theorem for the Hermitian lines on* X^{ν}*, including their conductor twists, and continuity and homogeneity of that volume. Write* $\widehat{h}^0$ *for the logarithm of the number of sections of sup norm at most one. Then*

$$0 \leq \widehat{h}^0(\widetilde{S}_r) - \widehat{h}^0(S_r) = o(r^{d+1}).$$

Both counts, divided by $r^{d+1}/(d+1)!$*, tend to the arithmetic volume of* $\overline{L} = (L, \|\cdot\|)$*.*

Suppose also that $P_0 \subseteq P$ *and that the Levi form of* $\log h$ *is nonnegative at points of the complex cone of* P *on all vectors tangent to the equations of* P_0*. There is a projective birational model with smooth generic fibre on which the pulled-back metric is smooth and semipositive.*

Outline. Glue the localised chart rings and use the conductor to compare section counts with a fixed twist via Moriwaki's convergence theorem. Stein factorisation and the projection formula preserve sections on the resolution, and cone lifts give semipositivity. The complete proof is on p. 1130. □

Lemma 35.13 (Detecting a divisor on a completed arithmetic stalk)**.** *Let* x *be a closed point of an integral normal scheme of finite type and flat over* $\mathbf{Z}$*. Put* $A = \mathcal{O}_{X,x}$*,* $R = \widehat{A}$*, and let* p *be the residue characteristic. Fix a dense open* U*. Test on local homomorphisms* $v : R \to \mathcal{O}_{\overline{\mathbf{Q}}_p}$ *whose generic point maps into* U*. Such tests exist. If* $a, s \in R$*,* $a \neq 0$*, and* $|v(s)| \leq |v(a)|$ *on all these tests, then* $a \mid s$ *in* R*. A rational function regular on* U *and bounded by one on all tests belongs to* A*.*

Outline. Prime avoidance and completion yield one-dimensional tests. If $s \notin aR$, localise $R[a/s]$ to get a test with generic point in the prescribed open and $|v(a)| < |v(s)|$, then use divisibility and faithful flatness to show that bounded rational functions are regular. The complete proof is on p. 1131. □

Lemma 35.14 (The Fourier coordinate at a boundary point)**.** *On a component of the unitary moduli space with its finite theta level, write* N *for the theta modulus and* rN *for the torsion level used to evaluate the sections. Here* $N = m^2$*, with* $m = \lambda^k$ *for an odd prime* λ *and* $k \geq 2$*,* $r = \lambda^j$ *for* $j \geq 1$*, and* rN *is invertible on the base. If* g *is the dimension of the principally polarised abelian variety, require*

$$1 + 2g\, 2^{g/2} < N, \qquad 2N^g g!\, g(3/2)^g 2^{g/2-1} N < rN, \qquad 6N^g g! < rN.$$

The theta data include a primitive $(rN)^2$*-th root of unity* ζ*. Take a normal projective model carrying the semiabelian extension and its rigidified theta line, on which the theta-null tuple and the nonzero finite Fourier tuples have a generating entry locally. Let* x *be a closed point,* $A = \mathcal{O}_{X,x}$*,* $R = \widehat{A}$*, and* X_* *the period lattice of the completed split degeneration. Choose a nonempty dense theta open* U_0 *on which every element of the theta coefficient ring is regular and every nonzero theta null and finite Fourier entry is a unit. Require a coherent lattice of theta-section coefficients which is the standard free lattice on* U_0 *and has the following compatibility at every local test into* $\mathcal{O}_{\overline{\mathbf{Q}}_p}$ *with generic point in* U_0*. Its evaluated span must equal the lattice generated by*

$$u\,\Pi_{\alpha,v} e_i, \qquad q_v(\alpha) \leq q_v(\alpha + N\beta) \ \ (\beta \in X_v), \qquad |u|_p \leq \exp(q_v(\alpha)/N).$$

Here $u \in \overline{\mathbf{Q}}_p$, X_v and q_v are the period lattice and quadratic period form of the test, e_i is a theta-basis coordinate vector, and $\Pi_{\alpha,v}$ is the character projector for the character determined by α through the level Weil pairing. Use a completed split uniformisation of the pulled-back semiabelian extension identifying its theta line with the uniformising theta line. For every $\alpha \in X_*$, require its cubical period section to be a unit on the pulled-back abelian locus, and let $s_\alpha \in R$ be its coefficient in an integral frame of its line. Two such frames change s_α by a unit.

For every $c \in X_*/NX_*$ there are a representative μ_c and a nonzero rational function e_c, the quotient of a finite Fourier sum by a theta null, such that

$$e_c^N \in A, \qquad e_c^N R = s_{\mu_c} R, \qquad s_\alpha \in s_{\mu_c} R \quad (\alpha \bmod NX_* = c).$$

Outline. The level pairing selects classes on tests from 35.13, and theta translates and torsion orthogonality give sharp Fourier bounds. The generating quotient defines e_c, whose N-th power is regular and divides the class coefficients, one up to a unit. The complete proof is on p. 1131. □

Lemma 35.15 (Order of the chosen quotient on a degeneration trait)**.** *Use the model, closed point x, and choices of Lemma 35.14. Let a degeneration trait mean the spectrum of a complete discrete valuation ring V of fraction characteristic zero, with a generic point in the moduli open, the given polarisation and level structure, and the semiabelian degeneration of that point. Suppose its centre on the model specialises to x and its generic point lifts to the chosen theta open. Write X_V for its period lattice, b_V for its monodromy pairing, and normalise the order by $\operatorname{ord}_V(\varpi) = 1$. There is a surjection $\pi_V : X_* \to X_V$. The choices in Lemma 35.14 satisfy*

$$\operatorname{ord}_V(e_c^{2N}) = b_V(\pi_V \mu_c, \pi_V \mu_c) \le b_V(\pi_V \alpha, \pi_V \alpha) \qquad (\alpha \bmod NX_* = c).$$

On a trait centred at x the period map is an isomorphism. In the unitary setting the resulting maps between the period lattices are $\mathcal{O}_E$-linear for the action on characters.

Outline. Restrict torus characters over an extension of ramification index one to get a period-lattice surjection. The biextension law and tame inertia compare orders, 35.14 gives the order formula and inequality, and positivity forces injectivity. The complete proof is on p. 1133. □

Lemma 35.16 (Finitely many cones for fixed minimum representatives)**.** *Let X be the lattice of a cusp, $N \ge 3$ the residue modulus, and $\mu_c \in c$ a representative for each $c \in X/NX$. Use the smooth polarised fan on the balanced forms on X. Suppose an integral positive definite balanced form b_0 takes its minimum on c at μ_c for every c. There is a finite list of cones in this fixed lattice whose union contains every integral positive semidefinite balanced form with the same minimum representatives.*

Outline. Minimal evaluation tuples bound the minimisers in each cone with positive definite generator sum. Fan finiteness and the spanning representatives give finitely many cones in the fixed lattice, and closure puts the limit of $(b_0 + jB)/j$ in one of them. The complete proof is on p. 1134. □

Lemma 35.17 (Generation of a boundary ideal from approximation on the base)**.** *Let $\pi : Y \to X$ be a proper birational morphism of integral normal schemes flat and of finite type over $\mathbf{Z}$. Let D be an effective Cartier divisor on Y, and let $U \subset Y \setminus |D|$ be a dense open. At a closed point $x \in Y$ of residue characteristic p, a test means a local map from $\widehat{\mathcal{O}}_{Y,x}$ to $\mathcal{O}_{\overline{\mathbf{Q}}_p}$ with generic point in U. Write $|a(y)|$ for the absolute value of a germ on a test y.*

For each closed x put $z = \pi(x)$, $A = \mathcal{O}_{X,z}$, and let $\mathfrak{m}$ be its maximal ideal. Suppose that there is a test y_0 at x with the following property. For each $n \ge 1$ there is $F_n \in A$ such that

$$|F_n(y_0)| = |s_x(y_0)|, \qquad |F_n(y)| \le \max\bigl(|s_{x'}(y)|, \rho_y^n\bigr)$$

for every test y at every closed x' over z. Here $s_{x'}$ is a local equation of D, and ρ_y is the maximum of the absolute values on y of a finite generating set of $\mathfrak{m}$. Then the ideal

$$J = \ker\bigl(\mathcal{O}_X \longrightarrow \pi_* \mathcal{O}_D\bigr)$$

satisfies $J\mathcal{O}_Y = \mathcal{O}_Y(-D)$.

Outline. Take a limit of the approximants in the completed base ring and use 35.13 to get divisibility along the fibre, with a unit quotient at the chosen point. Properness and flat base change put the limit in the completed ideal, and faithful flatness gives equality. The complete proof is on p. 1134. □

Lemma 35.18 (The ideal norm on a degeneration trait)**.** *Let $X_1 \to X$ be a finite dominant morphism of integral normal Noetherian schemes with function fields L/K, where K has characteristic zero and L/K is separable of degree h. For a coherent ideal I on X_1 there is an ideal J on X with the following property. For a trait $\operatorname{Spec} V \to X$ with V a complete discrete valuation ring whose fraction field has characteristic zero, there is an extension to a complete discrete valuation ring V' and h lifts $y_j : \operatorname{Spec} V' \to X_1$, counted with multiplicity, such that*

$$JV' = \prod_{j=1}^{h} y_j^{*} I\, \mathcal{O}_{\operatorname{Spec} V'}.$$

Here an ideal pulled back to a trait denotes its image in the valuation ring.

Outline. Generate the norm ideal by coefficients of products over the field embeddings, and extend the trait using the algebra generated by the conjugate rings. Over the extended valuation ring, coefficient ideals multiply, and a tuple of ideal generators gives the branch product. The complete proof is on p. 1135. □

Lemma 35.19 (The geometric and arithmetic premises)**.** *The fifteen premises of 35.11 hold for the application under the inputs of 39.1. The compactification and Hodge-bundle statements are quantified over their own moduli data before the choice of family and level.*

Outline. Use theta valuations and a normalised blow-up to identify the prescribed toroidal model. Combine resolution, nef Hilbert–Samuel on the model of 35.12, the volume bound 35.33, Hodge and deformation comparisons, and the inputs of 35.5. The complete proof is on p. 1135. □

At a nonterminal layer the product of full flags has dimension $d_{\mathrm{fl}} = f\binom{r}{2}$. Put $T_* = \lceil P_*^2 d_{\mathrm{fl}}^2 k/m \rceil_{\mathbf{N}}$ and $\mathcal{B} = \binom{T_*-1}{d_{\mathrm{fl}}}$. Keep the anchored weights u_W, u_M and write $\kappa = \|\lambda\|_1$. A quotient weight u_1 has $u_{1,\tau,k-2} = 0$ and, for $b' = a+1$,

$$u_{1,\tau,a} - u_{1,\tau,b'} = \left\lfloor \frac{u_{W,\tau,a} - u_{W,\tau,b'} + \begin{cases} -c_0 - (2r-2), & b' = P, \\ 2, & b' \neq P \end{cases}}{T_* + d_{\mathrm{fl}} + 1} \right\rfloor . \tag{35.6}$$

Write the remainders in these divisions as adjacent differences of the row anchored at zero in position $k-2$. Let u_2 be u_1 plus that row. For $u' = u_1 - 3r\eta_H$ and $V(u) = \prod_{\tau,a<b'}(u_{\tau,a} - u_{\tau,b'})$, set

$$\rho = 1 - V(u')/V(u_1), \qquad z = 7(d_{\mathrm{fl}}^2/T_* + d_{\mathrm{fl}}\rho). \tag{8.28a}$$

Lemma 35.20 (One size exponent for every layer)**.** *Fix eligible F_m, C_H. For the family size exponent C_t of Definition 33.3, retain the family's chosen precision constant $C_P \geq 1$, with $C_p = 1$ and $e_P = 12$, and put $A = 22 + |C_t|$ and $B = A + 2$. Beyond a threshold, at every admissible point and layer, with $\kappa = \sum_{\tau,a} |\lambda_{\tau,a}|$,*

$$1 \leq b \leq E_p \leq x^B b, \qquad |M_{p,i}| \leq x^B b, \qquad \max(b, |M_{p,i}|) \leq P_* \kappa. \tag{35.7}$$

At every nonterminal layer a quotient weight satisfying (35.6) exists. For every such weight,

$$T_* \geq 3d_{\mathrm{fl}} + 5, \qquad 0 \leq \rho < 1, \qquad 0 < z \leq \tfrac{1}{2}. \tag{8.28b}$$

Outline. Host bounds and row sums give a common size exponent, and summing floored adjacent differences from the zero anchor constructs the quotient weight. Gap and spacing bounds control the cross-wall Vandermonde loss and give $z \leq 1/2$. The complete proof is on p. 1146. □

Lemma 35.21 (The field and level of a deep cover)**.** *Assume the ray-class-field assertion of 35.19. For each eligible F_m, C_H there is a threshold beyond which every admissible point and every $i+1<m$ admit a good cover with $M'=g_0g_1 15^e$, $e=\lfloor 3L_x/\log 3\rfloor$, and coefficient field F' satisfying*

$$\log[F':\mathbf{Q}]+\log \mathrm{rd}(F')\le 391(frL_x)^2.$$

Outline. Splitting at 3 and 5 supplies complementary level ideals, and the ray class field supplies roots of unity and Weber values for compatible frames over the compositum. The ray class sequence and conductor-discriminant formula bound the field size. The complete proof is on p. 1147. □

Lemma 35.22 (The core and realised row metrics)**.** *Assume $\mathcal{H}_P$ of 35.11. For fixed eligible F_m, C_H, eventually at every admissible point and layer the core Petersson norm has a Gram frame. Its cusp lattice is finitely generated and spans the cusp space over the core coefficient field. The realised norm has a Gram frame, its ambient integral lattice Λ_{all} is finitely generated, and the good lattice Λ_{good} of (8.30a) is nonzero. At each embedding of the realisation field, the realised norm on the image of the cusp space is $c>0$ times the Petersson norm at the restricted embedding, with $|\log c|\le C_{\mathrm{meas}}E_p$ for an absolute C_{meas}. There is a family exponent $A_1\ge 0$ with $\mu_{\max}(\Lambda_{\mathrm{all}})\le x^{A_1}b$, where $\mu_{\max}$ is the maximum sublattice slope in the squared-norm convention.*

Outline. Uniformisation, lattice regularity and row realisation give the two frames. Tamagawa measures and Weyl dimensions bound their norm ratio using 35.20, and rescaling and Galois descent transfer the Petersson slope bound. The complete proof is on p. 1149. □

35.2 Arithmetic volume and component heights

For signature $(p_{\mathrm{sig}},q_{\mathrm{sig}})$ put $n=p_{\mathrm{sig}}+q_{\mathrm{sig}}$ and let Ψ be the CM type. Write $\|v\|_1$ and $\|v\|_\infty$ for the sum and maximum of the absolute weight entries, and $\mathrm{spr}(v)$ for the sum of the row spreads. A gap $g>0$ requires strict decrease in each row and $|v_{\tau,a}-v_{\tau,b}|\ge g$ for $a\ge p_{\mathrm{sig}}$ and $a\ne b$. Define

$$B(v,g)=4|\Psi|n^2(1+\|v\|_\infty/g),\quad \mathcal{E}_s(v,g)=(\log B(v,g)+s)\|v\|_1+q_{\mathrm{sig}}\,\mathrm{spr}(v),\qquad \ell(D)=\log \mathrm{rd}(E)+|L_E|. \tag{8.26a}$$

For a unitary moduli datum over a coefficient field F, a valuation family on its algebra of rules assigns to each height-one prime ideal λ of $\mathcal{O}_F$ a set of valuations over λ. The record valuation family assigns to λ the valuations $f\mapsto v(f(y))/e$ obtained by evaluating rules at algebraic integral frames y on complete discrete valuation traits over λ of ramification index e, in the sense of Definition 35.53. The height valuation family assigns to λ, above a rational prime ℓ, the valuations (normalised to extend v_λ) obtained from Hodge–Tate frames over traits above λ that trivialise the Lie eigenspaces through the integral Hodge–Tate map of the ℓ-divisible group over $\mathcal{O}_C$ for an algebraically closed ℓ-adically complete valued extension C of the trait fraction field. For $B=B(v,g)$ put

$$\begin{aligned} S_B&=\{\ell \text{ prime}:\ell\le B \text{ or } \ell\mid 2|\operatorname{disc}E|M'\},\quad h_E(\ell)=\frac{1}{\ell-1}+\frac{40v_\ell(|\operatorname{disc}E|)}{[E:\mathbf{Q}]},\\ c_2(B)&=\sum_{\ell\in S_B}(h_E(\ell)+\mathbf{1}_{\ell=2})\log\ell, \end{aligned} \tag{35.8}$$

and form the patched valuation family using the height valuation family at primes λ of $\mathcal{O}_F$ above S_B and the record valuation family elsewhere. On a component Z over F, let $n_j(t)$ count the degree-j cusp sections integral for this patched valuation family whose norm at every good embedding (one per conjugate pair) and frame on Z is at most e^{jt} in the host scalar scale $\xi_{\mathbf{R}}(D)=|\phi_0(\xi_0)|$, where ξ_0 is the host skew-Hermitian element and $\phi_0:E\hookrightarrow\mathbf{C}$ is the distinguished host embedding. Let $D_{\mathrm{fl}}=|\Psi|\binom{n}{2}$ be the dimension of the flag bundle Z_{fl} of Z and $\deg_Z(v)>0$ the top self-intersection on Z_{fl} of the line of weight v. Put

$$\mathrm{vol}_Z(t)=\limsup_{k\to\infty}\frac{D_{\mathrm{fl}}!\log n_{8k}(t)}{[F:\mathbf{Q}](8k)^{D_{\mathrm{fl}}+1}},\qquad \nu_Z(v;B)=\inf_{t\in\mathbf{R}}\big(\mathrm{vol}_Z(t)/\deg_Z(v)-t\big).$$

Lemma 35.23 (Horizontal curve heights and arithmetic self-intersection)**.** *Let X be the integral projective model cut out by birational multihomogeneous coordinates y of generic dimension $D_A \geq 0$ with horizontal prime ideal P as in Lemma 35.12, equipped with an arithmetic intersection system II satisfying item 4 of Definition 35.5. Let A be a smooth conjugation-invariant Hermitian line on X of generic degree d_A and top arithmetic self-intersection $I = II(P; A^{D_A+1})$. For a horizontal arithmetic curve W on X of complex generic degree $\deg_{\mathbf{C}} W > 0$, write $h_A(W) = II(W; A)/\deg_{\mathbf{C}} W$ for its normalised height. Assume characteristic-zero resolution, the arithmetic Kleiman inequality (A.422) and Theorem 35.33.*

1. *Suppose A is nef on geometric fibres and its cone metric has nonnegative Levi form at complex cone points of P along the cone-tangent directions of an ideal $P_0 \subseteq P$. If $h_A(W) \geq a$ for every horizontal arithmetic curve W on X, then*
$$I \geq (D_A + 1)ad_A. \tag{35.9}$$

2. *Suppose A is nef on geometric fibres and its cone metric has nonnegative Levi form at complex cone points of P along the cone-tangent directions of an ideal $P_0 \subseteq P$. If horizontal arithmetic curves W on X satisfying $h_A(W) \leq a$ form a Zariski-dense collection in $X_{\mathbf{Q}}$, then*
$$I \leq (D_A + 1)ad_A.$$

Outline. For (1), shifts the metric by $-a$, resolves via 35.12, and applies arithmetic Kleiman. For (2), assuming $I > (D_A + 1)ad_A$, shifts the metric and applies 35.33 for a small section whose restriction to a dense curve with $h_A(W) \leq a$ gives a contradiction. The complete proof is on p. 1153. □

Lemma 35.24 (Intersection system, semiabelian extension, Hodge–Tate annihilation, Hopf duality and polarised abelian quotients)**.** *Assume the first eight premises of Theorem 35.11 (resolution through nef Riemann–Roch).*

1. *Characteristic-zero resolution, principalisation and conjugation-invariant Poincaré–Lelong yield unrestricted Poincaré–Lelong on horizontal multihomogeneous arithmetic varieties together with an arithmetic intersection system II satisfying item 4 of Definition 35.5 (in particular, symmetric in smooth Hermitian lines, multilinear under tensor products, invariant under isometry, compatible with dominant generically finite pullback and smooth metric shifts, and satisfying the rational-section divisor recursion).*

2. *Let S be a locally Noetherian normal scheme, $U \subset S$ a dense open subscheme and G, G' semiabelian schemes over S. Every group-scheme homomorphism $f : G_U \to G'_U$ extends uniquely to a group-scheme homomorphism $F : G \to G'$.*

3. *Let p be prime and (C, v) an algebraically closed valued field of characteristic zero with multiplicative valuation $v : C \to \mathbf{R}_{\geq 0}$ satisfying $v(p) < 1$ and p-adically complete valuation ring $\mathcal{O}_C = \{x \in C : v(x) \leq 1\}$. For every p-divisible group G over $\mathcal{O}_C$, the linearised Hodge–Tate map $\alpha_G : T_pG \otimes_{\mathbf{Z}_p} \mathcal{O}_C \to \omega_{G^\vee}$ has kernel equal to the annihilator under the Tate pairing of $\ker(d\log : T_pG^\vee \otimes_{\mathbf{Z}_p} \mathcal{O}_C \to \omega_G)$, and every $\kappa \in \mathcal{O}_C$ satisfying $v(\kappa)^{p-1} \leq v(p)$ (in particular, for $p = 2$, every κ with $v(\kappa) \leq v(4)$) satisfies $\kappa\,\omega_{G^\vee} \subseteq \mathrm{im}(\alpha_G)$.*

4. *Every proper smooth $\mathbf{C}$-scheme decomposes into pairwise disjoint clopen subschemes of constant relative dimension on which compatible holomorphic local sections of a line bundle algebraize and glue uniquely. Over the valuation ring R of (C, v), for every finite abelian group N and character isomorphism $H \simeq \mathrm{Hom}(N, C^\times)$ with $\#H = \#N$, the Fourier transform maps the R-dual lattice of any R-Hopf order $\mathcal{A} \subset C^N$ onto a dual R-Hopf order $\mathcal{A}^\vee \subset C^H$ satisfying $(\mathcal{A}^\vee)^\vee = \mathcal{A}$ and $(cc')^2 = v(\#N)^{\#N}$ for determinant valuations c, c', while restriction to a subgroup $K \subseteq N$ and pullback along $N \to N/K$ yield subgroup and quotient R-Hopf orders.*

5. *Let R_0 be a commutative ring, A an abelian scheme over $S_0 = \operatorname{Spec} R_0$ equipped with a relatively ample line bundle L, and $G = \operatorname{Spec} B \hookrightarrow A$ a finite locally free subgroup scheme over S_0. There exist an abelian scheme A' over S_0, a relatively ample line bundle L' on A' and a finite, locally free, surjective group-scheme homomorphism $q : A \to A'$ such that for every R_0-algebra S' and every S'-point $z : \operatorname{Spec} S' \to A$, one has $q \circ z = 1_{A'}$ if and only if z factors through $G \hookrightarrow A$. When $R_0 = R$ is*

the valuation ring of (C, v) (over which every finite flat subgroup scheme is finite free), A' embeds as a direct-summand group-scheme retract of a principally polarised abelian scheme over S_0.

Outline. Proves (1) by a quadratic twist and divisor induction, (2) by DVR trait lifting and torsion density, and (3) from 35.11 and Fargues's theorem. Derives (4) from projective GAGA and Fourier duality, and (5) from finite flat quotients, norm bundles, and Zarhin's retraction. The complete proof is on p. 1154. □

Proposition 35.25 (Existence of good root towers)**.** *Assume the first eight premises of Theorem 35.11 (resolution through nef Riemann–Roch). Let E be an abelian CM host with CM type Ψ of size f. Take a core theta chain satisfying the chain laws and $m^2 \le k$, a layer $1 \le j < m$, and a deep cover Y good with parameters (e, o) and with at most fifteen prime divisors of its level, of split index $p_{\mathrm{sig}} = k + j - 1$ and rank $n = p_{\mathrm{sig}} + q_{\mathrm{sig}}$, where $p_{\mathrm{sig}}, q_{\mathrm{sig}} \ge 1$, $3q_{\mathrm{sig}} + 1 \le n$, and the bad-prime set has cardinality at most $4fn^2$. Then there is a good root tower $\mathscr{T}$ over E, with model parameter 4 (Definition 35.27), of split index p_{sig} and rank n, whose component set* Comp *is the set of algebraic components of Y, and whose root model $M(N, Z)$ at cutoff N on component Z carries a degree-d_0 extension $\Omega(Z_{\mathrm{fl}}) \hookrightarrow \Omega(M(N, Z))$ of the function field of the flag bundle of Z, with the universal and model flag root lines identified over valuation rings centred on the characteristic-zero flag charts, and with an effective boundary section ε of degree $q_0\nu$ and finite-fibre nef, cone-semipositive smoothed combinations converging to the top intersections, with an asymptotic denominator cost of at most $2c_2(B)\|v\|_1$ per unit degree in the transfer of model sections of integral weight v and gap $g > 0$, where $B = B(v, g)$, $N = \lfloor B \rfloor$, and $c_2(B)$ is given by (35.8).*

Outline. The assumed hypotheses with Lemmas 35.12, 35.23, 35.24, and 35.6 build flat generators and split cutoff root models. Degree comparison, flag heights, Hodge–Tate inclusions, and wall perturbation give clauses 1 to 3 of Definition 35.27. Frobenius relations, Definition 35.3, and regularised base metrics with Stokes comparison give clauses 4 to 6. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1155. □

Proposition 35.26 (The deep-cover mean estimate)**.** *Assume the first eight premises of Theorem 35.11 (resolution through nef Riemann–Roch). There is $C_{\mathrm{deep}} \ge 0$ such that for every datum D, every core theta chain T satisfying the chain laws and $m^2 \le k$, and every $1 \le j < m$, the following holds. Let Y be a deep cover good with parameters (e, o), for any $e \in \mathbf{N}$ and $o \in \mathbf{R}$. Suppose at most fifteen distinct primes divide its level M'. For each integral weight v with gap $g > 0$ there are real s_Z such that*

$$s_Z \le \nu_Z(v; B(v, g)), \qquad \frac{1}{\#\,\mathrm{Comp}(Y)} \sum_Z s_Z \ge -C_{\mathrm{deep}} \mathcal{E}_{\ell(D)}(v, g). \tag{8.26b}$$

Outline. Build a global PEL model from the first eight premises of Theorem 35.11 via Lemmas 35.23 and 35.24, attaching its good root tower by Proposition 35.25. Compare model slopes with $\nu_Z(v; B)$ via a trace basis and Lemma 35.24(3), bound $c_2(B)$, and average. The complete proof is on p. 1176. □

For the tower estimates write $r = p_{\mathrm{sig}}$, $s = q_{\mathrm{sig}}$, $n = r + s$, $f = |\Psi|$ and $\lambda_N = \log(N \operatorname{rd}(E)) + |L_E|$. Assume the first eight premises of Theorem 35.11 (resolution through nef Riemann–Roch), and assume $s \ge 1$ and $3s + 1 \le n$, so $n \ge 4$ and $s < r$. Relative to the arithmetic intersection system II of Lemma 35.24(1), let $\mathscr{T}$ be a good root tower of split index r over the abelian CM host in the sense of Definition 35.27 below, satisfying $\#\mathcal{B}_{\mathscr{T}} \le 4fn^2$, with $d = f\binom{n}{2}$, $e = frs$, L_v, D, h_{fl}, $P_{N,Z}$, P_N, V_Z and V as defined there.

Definition 35.27 (Root models, good root towers and flag functionals)**.** Fix a finite nonempty CM type Ψ of cardinality $f = |\Psi|$ and integers $1 \le r < n$ with $s = n - r$, $d = f\binom{n}{2}$ and $e = frs$. Let $\mathbf{R}[X_{\tau,i}]$ $(\tau \in \Psi,\ 0 \le i < n)$ be the real polynomial ring in the formal Chern roots. For $\sigma : \Psi \to S_n$ write $X^{\mathrm{perm}(\sigma)} = \prod_{\tau\in\Psi} \prod_{i=0}^{n-1} X_{\tau,i}^{\sigma_\tau(i)}$, and define the linear top-degree flag functional

$$D(F) = \sum_{\sigma:\Psi\to S_n} \Big(\prod_{\tau\in\Psi} \operatorname{sgn}(\sigma_\tau) \Big) [X^{\mathrm{perm}(\sigma)}] F,$$

which vanishes on homogeneous polynomials of degree different from d. For a real weight $v = (v_{\tau,i})$ put $L_v = -\sum_{\tau,i} v_{\tau,i} X_{\tau,i}$ and $\delta_{\mathrm{fl}}(v) = D(L_v^d)$. Call v dominant if $v_{\tau,a} \ge v_{\tau,b}$ whenever $0 \le a \le b < n$, and

regular dominant if $v_{\tau,a} > v_{\tau,b}$ whenever $0 \le a < b < n$. Set $p_{\tau,k} = \sum_{i=0}^{n-1} X_{\tau,i}^{k}$, $p_{\tau,k}^{<r} = \sum_{i<r} X_{\tau,i}^{k}$, and for $F \in \mathbf{R}[X_{\tau,i}]$ define $\mathrm{Sh}_r(\tau,F) = D(\sum_{i<r} \partial_{\tau,i} F)$ and $\mathrm{Tot}(\tau,F) = D(\sum_{i=0}^{n-1} \partial_{\tau,i} F)$.

Embed $\prod_{\tau\in\Psi} \mathrm{Fl}(\mathbf{Z}^n)$ as a horizontal multihomogeneous arithmetic variety of arithmetic dimension $d+1$ in the product of Plücker projective spaces $\prod_{\tau\in\Psi}\prod_{j=1}^{n-1} \mathbb{P}_{\mathbf{Z}}^{\binom{n}{n-j}-1}$. On the τ-factor let $\mathcal{O}^n = E_{\tau,0} \supset E_{\tau,1} \supset \cdots \supset E_{\tau,n} = 0$ ($\mathrm{rk}\, E_{\tau,j} = n-j$) be the universal decreasing flag equipped with the subquotient metrics induced by the standard Hermitian inner product on $\mathbf{C}^n$, and put $\overline{Q}_{\tau,i} = (\overline{E_{\tau,i}/E_{\tau,i+1}})^{\vee}$, so that $\bigotimes_{i=j}^{n-1} \overline{Q}_{\tau,i} \simeq (\overline{\det E_{\tau,j}})^{\vee}$ pulls back the Plücker Fubini–Study Hermitian line $\overline{\mathcal{O}_{\mathrm{Gr}(n-j,n)}(1)}$ on $\mathrm{Gr}(n-j,n)_{\mathbf{Z}}$. Relative to an arithmetic intersection system II satisfying item 4 of Definition 35.5, define the linear arithmetic flag height functional $h_{\mathrm{fl}} : \mathbf{R}[X_{\tau,i}] \to \mathbf{R}$ to vanish on homogeneous polynomials of degree different from $d+1$ and to map each monomial $\prod_{m=1}^{d+1} X_{\tau_m,i_m}$ of degree $d+1$ to $II\big(\prod_{\tau\in\Psi}\mathrm{Fl}(\mathbf{Z}^n); \overline{Q}_{\tau_1,i_1}, \dots, \overline{Q}_{\tau_{d+1},i_{d+1}}\big)$.

Relative to II, a root model M for (Ψ, n) consists of a finite block set K with finite coordinate sets $(\iota_k)_{k\in K}$, a function field Ω with birational multihomogeneous coordinates $y : \bigsqcup_{k\in K} \iota_k \to \Omega$ cutting out a horizontal prime ideal $\mathfrak{p}_M$ of arithmetic dimension $d+1$, multidegrees $x_{\deg}(\tau,i) \in \mathbf{Z}^K$, smooth conjugation-invariant Hermitian root lines $x(t,(\tau,i))$ ($t \in \mathbf{R}$) of multidegree $x_{\deg}(\tau,i)$, an integer $k_A > 0$, a smooth conjugation-invariant auxiliary Hermitian line A, and a positive integer degree factor δ. For the models with geometric origin in Proposition 35.25, δ is the coefficient-field degree times the generic flag-cover degree. When the right-hand limits exist, define $P_M : \mathbf{R}[X_{\tau,i}] \to \mathbf{R}$ by linearity, with value zero on monomials of degree different from $d+1$. For a monomial of degree $d+1$, substitute the root lines $x(t,(\tau,i))$, evaluate II on $\mathfrak{p}_M$, divide by δ, and take the limit as $t \to 0^+$; on a good root tower below, each model further exports an effective boundary section ε of degree $q_0\nu$, finite-fibre nefness and cone-semipositivity of the smoothed rational combination bundles for small $s > 0$ once $N = \lfloor B(v,g) \rfloor$, and convergence of their normalized top geometric and arithmetic intersections as $s \to 0$.

A root tower $\mathscr{T}$ of split index r consists of a finite nonempty component set Comp, a finite bad-prime set $\mathcal{B}_{\mathscr{T}} \subset \mathbf{N}$ with $2 \in \mathcal{B}_{\mathscr{T}}$, a rational weight ν, and root models $M(N,Z)$ for each cutoff $N \in \mathbf{N}$ and $Z \in \mathrm{Comp}$. Write $P_{N,Z} = P_{M(N,Z)}$, $P_N = \sum_{Z\in\mathrm{Comp}} P_{N,Z}$, V_Z for the geometric component volume on Z, $V = \sum_{Z\in\mathrm{Comp}} V_Z$, and when $\delta_{\mathrm{fl}}(v) \neq 0$ define the model slope

$$a_Z(v;N) = \frac{P_{N,Z}(L_v^{d+1})}{(d+1)V_Z\delta_{\mathrm{fl}}(v)}.$$

For an abelian CM host E with $\lambda_N = \log(N\,\mathrm{rd}(E)) + |L_E|$, call $\mathscr{T}$ a good root tower over E, with model parameter 4, if it is equipped with a Frobenius path map $(\ell,\tau,h) \mapsto \mathrm{Fr}_{\ell,h}(\tau) \in \Psi \sqcup \Psi$ ($\ell, h \in \mathbf{N}$, $\tau \in \Psi$) satisfying the following six laws.

1. Every $V_Z > 0$ takes a common value across $Z \in \mathrm{Comp}$, and the geometric top intersection functional on $M(N,Z)$ equals $V_Z D$.

2. For every dominant weight v one has $h_{\mathrm{fl}}(L_v^{d+1}) \ge 0$, with $h_{\mathrm{fl}}(L_v^{d+1}) = 0$ whenever $v_{\tau,a} = v_{\tau,a+1}$ for some $\tau \in \Psi$ and $0 \le a < n-1$; moreover $h_{\mathrm{fl}}(p_{\tau,1}Q) = 0$ for all $\tau \in \Psi$ and all Q homogeneous of degree d.

3. There exist tower-dependent constants $C \ge 0$ and $N_1 \in \mathbf{N}$ with $N_1 \ge 2$ and real sequences $(\alpha_M), (\beta_M)$ such that for every integer $M \ge N_1$ and every monomial m of degree $d+1$,

$$\Big| P_M(m) - V\Big(h_{\mathrm{fl}}(m) - (\alpha_M - \beta_M)\sum_{\tau\in\Psi}\mathrm{Sh}_r(\tau,m) - \beta_M \sum_{\tau\in\Psi}\mathrm{Tot}(\tau,m)\Big)\Big| \le \frac{C\log M}{M}.$$

4. Whether $\mathrm{Fr}_{\ell,h}(\tau)$ lies in the left or right copy of Ψ is independent of $\tau \in \Psi$. Write $P_{\ell,\tau,h+1,m} = \sum_{i<r} X_{\sigma,i}^{m}$ if $\mathrm{Fr}_{\ell,h}(\tau)$ is the left copy of $\sigma \in \Psi$, and $P_{\ell,\tau,h+1,m} = \sum_{i\ge r}(-X_{\sigma,i})^m$ if it is the right copy of σ. For all integers $2 \le N \le M$, $Z \in \mathrm{Comp}$, $\tau \in \Psi$, $1 \le k \le n$ and Q homogeneous of degree $d+1-k$,

$$P_{N,Z}(p_{\tau,k}Q) = P_{M,Z}(p_{\tau,k}Q) + V_Z \sum_{\substack{N<\ell\le M \text{ prime} \\ \ell\notin\mathcal{B}_{\mathscr{T}}}} \delta_\ell(\tau,k,Q),$$

where

$$\delta_\ell(\tau,k,Q)=k\log\ell\left(\sum_{h\ge0}\ell^{-k(h+1)}D\big(P_{\ell,\tau,h+1,k-1}Q\big)-\frac{D(p_{\tau,k-1}^{<r}Q)}{\ell-1}\right).$$

5. For $N\ge2$, $Z\in\mathrm{Comp}$, $\tau\in\Psi$ and Q homogeneous of degree d, $P_{N,Z}(p_{\tau,1}Q)=h_{N,Z,\tau}V_ZD(Q)$ with $|h_{N,Z,\tau}|\le4n\lambda_N$ whenever $N\ge\max(128,\#\mathcal{B}_{\mathscr{T}})$.

6. When $3s+1\le n$, putting $\mu=\sum_{\tau,i\ge r}X_{\tau,i}$, $J=\prod_\tau\big(\prod_{0\le a<b<r}(X_{\tau,b}-X_{\tau,a})\prod_{r\le a<b<n}(X_{\tau,b}-X_{\tau,a})\big)$ and $F_*=\mu^{e+1}J$, for each integer $N\ge\max(128,4fn^2,\#\mathcal{B}_{\mathscr{T}})$ and $Z\in\mathrm{Comp}$, $M(N,Z)$ admits a base Grassmannian PEL model of generic dimension e (as in Lemma 35.12) with an auxiliary ample Hermitian line A of normalised horizontal curve height at least $-K_A$ ($K_A\ge0$) such that: (i) its interior carries an abelian scheme of relative dimension fn with $\mathcal{O}_E$-action of signature (r,s) over Ψ whose adapted determinant metric on μ (defined by the local adapted frames of the component model, with Hodge–Tate discrepancy bounded by Lemma 35.24(3)) bounds the labelled fibre degree at closed interior horizontal points above by $fsX-ft\log\mathrm{rd}(E)+x+\frac12ft\log(4\pi)$ for some $0\le t\le s$, where $X=2\sum_{\ell\le N\text{ or }\ell\in\mathcal{B}_{\mathscr{T}}}(\log\ell)/(\ell-1)+2\log\mathrm{rd}(E)$ and the saturated Lie eigenlattice Arakelov degree sums x (of the rank-s eigenlattices over $\overline{\Psi}$) and y (of the rank-r eigenlattices over Ψ) satisfy $|x-y|\le c_{\mathrm{cm}}fn\lambda_N$ for an absolute constant $c_{\mathrm{cm}}\ge0$ independent of $E,\mathscr{T},N$ and Z; (ii) complex uniformisation presents each complex sheet over a fundamental set $\mathcal{F}$ of finite positive Haar measure for an arithmetic unitary group Γ acting on an $\mathcal{O}_E$-stable Hermitian lattice Λ of signature (r,s) (equipped with an integral trace pairing of modulus at most $\|v\|\|w\|$ in the positive majorant of first minimum λ_1) together with a nonnegative boundary cusp weight $U\le fs\big(2\log^+(1/\lambda_1)+\log(2fs\,\mathrm{rd}(E))\big)$ such that adding $\frac12U$ dominates the boundary logarithmic singularity of the adapted determinant metric; and (iii) for every pair of integers $1\le a\le b$, the base model carries smoothed cleared-denominator lines $\overline{M}_{N,a,b,t'}$ of generic class $b\mu+aA$, which for all sufficiently small $t'>0$ are smooth, conjugation-invariant, cone-semipositive along its defining ideal, nef on finite fibres, and ample on the generic fibre. Their denominator-normalised curve height is

$$h_{\mathrm{reg}}(W)=\frac{II(W;\overline{M}_{N,a,b,t'})}{b\deg_{\mathbf{C}}W}.$$

Thus $h_{\mathrm{reg}}(W)\ge e_0$ on all horizontal curves is equivalent to $II(W;\overline{M}_{N,a,b,t'})/\deg_{\mathbf{C}}W\ge be_0$. Write $I_{a,b,t'}$ for the top arithmetic intersection and

$$D_{a,b,t'}=b^e\deg_{\mathrm{gen}}(\mu+(a/b)A)>0.$$

For every $\eta\in\mathbf{R}$, if for each $1\le a\le b$ there is $t_0>0$ such that

$$(e+1)\left(\eta-\frac{a}{b}K_A\right)\frac{D_{a,b,t'}}{b^e}\le\frac{I_{a,b,t'}}{b^{e+1}}\qquad(0<t'\le t_0),$$

then

$$\eta\le\frac{P_{N,Z}(F_*)}{(e+1)V_ZD(\mu^eJ)}+\frac12\operatorname{mean}U.$$

Lemma 35.28 (Weyl degree formula, power-sum transition decomposition and flag height positivity)**.** *Adopt the notation of Definition 35.27 for a good root tower $\mathscr{T}$, and write $S(F)=\sum_{\tau\in\Psi}\mathrm{Sh}_r(\tau,F)$ and $T(F)=\sum_{\tau\in\Psi}\mathrm{Tot}(\tau,F)$.*

1. *For every real weight v,*

$$\delta_{\mathrm{fl}}(v)=d!\prod_{\tau\in\Psi}\prod_{0\le a<b<n}\frac{v_{\tau,a}-v_{\tau,b}}{b-a},$$

which is strictly positive if v is regular dominant and nonnegative if v is dominant.

2. *Every homogeneous $F\in\mathbf{R}[X_{\tau,i}]$ of degree $d+1$ decomposes as $F=\sum_{\tau\in\Psi}\sum_{k=1}^{n}p_{\tau,k}Q_{\tau,k}$ with $Q_{\tau,k}$ homogeneous of degree $d+1-k$. For $k\ge1$ and $\sigma,\tau\in\Psi$, $\mathrm{Sh}_r(\sigma,p_{\tau,k}Q)=kD(p_{\tau,k-1}^{<r}Q)$ if $\sigma=\tau$*

and 0 *otherwise, while* $\mathrm{Tot}(\sigma, p_{\tau,k}Q) = nD(Q)$ *if* $\sigma = \tau$ *and* $k = 1$*, and* 0 *otherwise. For each* $\tau_0 \in \Psi$ *there exist* Q_1 *homogeneous of degree* d *with* $D(Q_1) = 1$ *and* Q_2 *homogeneous of degree* $d-1$ *with* $D(p_{\tau_0,1}^{<r}Q_2) = 1$*.*

3. *Let* $\mathrm{rk}_{\mathrm{Lie}} : \Psi \sqcup \Psi \to \mathbf{R}$ *take the value* r *on the left copy and* s *on the right copy, fix* $\tau_0 \in \Psi$*, and put*

$$\kappa_\ell = \frac{\log \ell}{n} \sum_{h \geq 0} \ell^{-(h+1)} \, \mathrm{rk}_{\mathrm{Lie}}(\mathrm{Fr}_{\ell,h}(\tau_0)).$$

Set $\rho_\ell(\tau, 1, Q) = 0$ *and, for* $k \geq 2$*,*

$$\rho_\ell(\tau, k, Q) = k \log \ell \sum_{h \geq 1} \ell^{-hk} D(P_{\ell,\tau,h,k-1}Q).$$

Then for every prime $\ell \geq 2$*,* $\tau \in \Psi$*,* $k \geq 1$ *and* $Q \in \mathbf{R}[X_{\tau,i}]$*,*

$$\delta_\ell(\tau, k, Q) = -\frac{\log \ell}{\ell - 1} S(p_{\tau,k}Q) + \kappa_\ell T(p_{\tau,k}Q) + \rho_\ell(\tau, k, Q),$$

and $|\rho_\ell(\tau, k, Q)| \leq c(\log \ell)/\ell^2$ *for some* $c \geq 0$ *independent of* ℓ*, so the sum over primes* $\ell > N$*,* $\ell \notin \mathcal{B}_{\mathscr{T}}$ *converges absolutely.*

4. *Assuming the arithmetic Kleiman inequality* (A.422) *(and without assuming clause 2 of Definition 35.27), the arithmetic flag height functional* h_{fl} *defined on* $\prod_{\tau \in \Psi} \mathrm{Fl}(\mathbf{Z}^n)$ *in Definition 35.27 satisfies clause 2 of Definition 35.27.*

Outline. Derives (1) by S_n-alternation, (2) from flag-variety cohomology and symmetric invariance, and (3) from 35.27 and geometric tail bounds. Proves (4) by writing L_v in Plücker pullbacks, using arithmetic Kleiman for nonnegativity and partial-flag pullback on walls. The complete proof is on p. 1178. □

Lemma 35.29 (The Hodge test class)**.** *In the setting of Definition 35.27 with* $3s + 1 \leq n$*, put* $\mu = \sum_{\tau, i \geq r} X_{\tau i}$ *and let* J *be the product of* $X_{\tau b} - X_{\tau a}$ *over* $a < b$ *in each signature block. Set* $F_* = \mu^{e+1}J$ *and* $H_* = (e+1)D(\mu^e J) > 0$*. There is an absolute* $C_h \geq 0$ *such that for integers* $N \geq \max(128, 4fn^2, \#\mathcal{B}_{\mathscr{T}})$*,*

$$P_N(F_*) \geq -VH_*(C_h + 2)fs\lambda_N.$$

Outline. On each component of Definition 35.27(6), regularise μ by $\frac{1}{2}U$, perturb by an ample line, and apply Lemma 35.24(3). Maximal-slope and eigenlattice estimates bound horizontal heights below, while isotropic-cone unfolding gives $\mathrm{mean}\, U \leq 4fs\lambda_N$. Lemma 35.23(1) and Definition 35.27(6) then yield the bound upon summing over components. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1179. □

Lemma 35.30 (The two flag integrals for the Hodge test class)**.** *Use the test class* $F_* = \mu^{e+1}J$ *and the normalisation* $H_* = (e+1)D(\mu^e J)$ *of Lemma 35.29. Put* $a = rs$*, so that* $e = fa$*, where* f *is the number of embeddings. Write*

$$F_* = \sum_{\tau, j} p_{\tau j} Q_{\tau j}, \qquad p_{\tau j} = \sum_{i=0}^{n-1} X_{\tau i}^j, \qquad 1 \leq j \leq n,$$

with homogeneous summands. Fix $2 \leq k \leq n$*. At an output embedding* σ*, let* P *be the power sum of degree* $k-1$ *on the smaller signature block, with either overall sign. Then*

$$\frac{k|D(PQ_{\tau k})|}{H_*} \leq \begin{cases} \dfrac{2^{k-1}}{a+1}\dbinom{a+k}{k}, & \sigma = \tau, \\ \dfrac{2^{k-1}}{k!}\dfrac{a!}{(a-k+1)!}, & \sigma \neq \tau. \end{cases}$$

Outline. Use alternation and the Vandermonde to extract a single power-sum coefficient. The Frobenius formula restricts the Schur expansion to hooks at extraction, and tableau counts and multinomial ratios give the bounds for equal and distinct embeddings. The complete proof is on p. 1183. □

Lemma 35.31 (The two cutoff coefficients)**.** *For N in the range of 35.29, put $C_0 = C_h + 2$. Let F be homogeneous of degree $d+1$ and choose a decomposition $F = \sum_{\tau,k} p_{\tau k} Q_{\tau k}$, where $p_{\tau k} = \sum_i X_{\tau i}^k$, $1 \le k \le n$, and $Q_{\tau k}$ is homogeneous of degree $d+1-k$. Set*

$$S(F) = D\Big(\sum_{\tau, i<r} \partial_{\tau i} F\Big), \qquad T(F) = D\Big(\sum_{\tau, i} \partial_{\tau i} F\Big).$$

Write h_{fl} for the arithmetic flag height functional, and define the remainder functional $R_N(F) = \sum_{\tau,k} R_N(p_{\tau,k} Q_{\tau,k})$ by $R_N(p_{\tau,1} Q) = 0$ and

$$R_N(p_{\tau,k} Q) = \sum_{\substack{\ell > N \text{ prime} \\ \ell \notin \mathcal{B}_{\mathscr{T}}}} k \log \ell \sum_{h \ge 1} \ell^{-hk} D(P_{\ell,\tau,h,k-1} Q) \qquad (2 \le k \le n). \tag{35.10}$$

The cutoff decomposition

$$P_N(F)/V = h_{\mathrm{fl}}(F) - AS(F) - bT(F) + R_N(F) \tag{35.11}$$

has coefficients

$$A \ge -K\lambda_N, \qquad |rA + nb| \le 4n\lambda_N, \qquad K = 2(C_0 + 4) + 3.$$

Outline. Combines 35.27 and 35.28 as $M \to \infty$ on test classes to prove the decomposition and bound $|rA + nb|$. Evaluates it on F_*, using 35.23, 35.30, 35.29 to bound $h_{\mathrm{fl}}(F_*)$, $R_N(F_*)$, $P_N(F_*)$, and hence A below. The complete proof is on p. 1184. □

Proposition 35.32 (The mean of the model slopes)**.** *For a real weight v with gap $g > 0$ and an integer $N \ge \max(128, \#\mathcal{B}_{\mathscr{T}})$ with $B(v,g) < N+1$, the model slopes satisfy*

$$\frac{1}{\#\,\mathrm{Comp}} \sum_Z a_Z(v; N) \ge -C_T[\lambda_N \|v\|_1 + s\,\mathrm{spr}(v)], \qquad C_T = 2C_0 + 17. \tag{8.26c}$$

Outline. Applies 35.28 and 35.31 to L_v^{d+1}, bounding the main term via A and $rA + nb$. Bounds $R_N(L_v^{d+1})$ by Weyl alternation, perturbed Vandermonde reduction, and prime-tail summation, then averages over components using common volumes from 35.27. The complete proof is on p. 1186. □

Theorem 35.33 (Arithmetic volume in the semipositive case)**.** *Use the integral projective model X, homogeneous ideal P, and ordinary section lattice S_r of Lemma 35.12. Write $d = \dim X_{\mathbf{Q}}$. Let II satisfy the arithmetic intersection axioms and let A be the smooth conjugation-invariant Hermitian line with multidegree a and cone metric h. Assume that the line is nef on the geometric fibres, including the generic fibre, and that the Levi form of $\log h$ at the complex points of the cone of P is nonnegative on vectors tangent to P_0, where $P_0 \subseteq P$. Then*

$$II(P; A^{d+1}) \le \limsup_{r \to \infty} \frac{\log \#\{s \in S_r : \|s\|_{\sup} \le 1\}}{r^{d+1}/(d+1)!}. \tag{8.27a}$$

Outline. The modification of 35.12 gives a smooth semipositive metric and preserves fibrewise nefness. Moriwaki's volume inequality bounds the intersection, which descends by the projection formula, and 35.12 identifies the volume with the leading chart count. The complete proof is on p. 1188. □

Use the quotient weights and bounds of Lemma 35.20.

Lemma 35.34 (Height of a polynomial determinant)**.** *Let K be a number field. Let E be a Hermitian K-space of dimension $N \ge 1$, and let $w_1, \dots, w_n$ span E. Suppose that $\|w_i\|_\sigma \le \Lambda$ at every embedding $\sigma : K \hookrightarrow \mathbf{C}$, where $\Lambda \ge 1$. For a nonzero multihomogeneous polynomial G on a product of copies of E, define $h_0(G)$ as follows. At a finite place take the logarithm of the maximum absolute coefficient after substituting $\sum_i t_i w_i$ in each block, with separate variables for each block. At an embedding take the mean of $\log|G|$ on the product of the Hermitian unit spheres with their invariant probability measures. Sum with the usual product-formula weights, normalised by $[K : \mathbf{Q}]$. Write $e(G)$ for the sum of the block degrees. Then*

$$h_0(G) \ge -e(G)\left(\log(n\Lambda) + \frac{1 + \log n}{2}\right), \qquad h_0(G_1 G_2) = h_0(G_1) + h_0(G_2).$$

Let M be a full Hermitian lattice of rank v over $\mathcal{O}_K$, and fix a K-basis of its ambient space. Let $P_1,\dots,P_v$ be polynomial vectors on the same product of spaces. Suppose their coefficients after the finite-place substitutions lie in M, and that $\|P_j\|_\sigma \le L_j$ on every product of unit spheres, with $L_j > 0$. If $D = \det(P_1,\dots,P_v)$ in the chosen basis is nonzero and multihomogeneous, then the normalised Arakelov degree satisfies

$$\widehat{\deg} M \;\ge\; h_0(D) - \sum_{j=1}^{v} \log L_j.$$

Outline. Spherical submean, Cauchy's coefficient estimate and the product formula bound the height below, and finite coefficient norms and spherical means give additivity. Exterior products and Hadamard's inequality turn the determinant height into a lattice degree bound. The complete proof is on p. 1188. □

Proposition 35.35 (Component degrees, ranks and section degrees)**.** *Under the hypotheses of Theorem 35.11, beyond a family threshold take any admissible point, nonterminal layer, good cover Y and quotient u_1. Write Δ_Z for the degree and $h_Z = h_{\mathrm{FS}}(F_{\mathrm{Ch}})/((d_{\mathrm{fl}}+1)\Delta_Z)$ for the Petersson Chow height of the image of Z in weight u_1. Set $\Delta'_Z = \deg_Z(u') = v_{0,Z}V(u')$. Then*

$$0 < \Delta'_Z \le \Delta_Z \le \Delta'_Z/(1-\rho). \tag{35.12}$$

The Δ'_Z share a common value $\Delta_* > 0$. Let r_Z, d_Z be the rank and unsquared normalised degree of the Petersson cusp lattice on Z in weight u_M, and $\mu_Y(u_M)$ the cover slope. Then

$$(1-z)r_Z \le \mathcal{B}\Delta_Z \le r_Z. \tag{8.29a}$$

For each cover and quotient weight there exist $A > 0$ and $R \ge 0$ such that for every component Z,

$$\begin{aligned} d_Z &\ge \mathcal{B}\Delta_Z\big(T_* h_Z - 3(d_{\mathrm{fl}}+1)|h_Z| - A\log(2+|h_Z|) - R\big),\\ A\log\max(A,1) + R &\le C_{\mathrm{HS}} E_p, \end{aligned} \tag{8.29b}$$

where C_{HS} is uniform in the point, layer and cover. Moreover

$$2\frac{\sum_Z d_Z}{\sum_Z r_Z} \le \mu_Y(u_M). \tag{8.29c}$$

Outline. Top Chern proportionality and moving degrees compare component degrees, while nef section counts and jets bound ranks. Factor the cut determinant by the Chow form, use 35.34 and the size bound of 35.20, then compare the summed lattice degrees with the cover slope. The complete proof is on p. 1189. □

Lemma 35.36 (Short generators of the integral lattice)**.** *Let K be a number field of degree a, V a finite-dimensional K-space and $L \subset V$ a finitely generated $\mathcal{O}_K$-module. For each embedding σ let N_σ be a nonnegative seminorm with $N_\sigma(cx) = |\sigma c| N_\sigma(x)$. Suppose $S \subset L$ spans V over K and $N_\sigma(x) \le H$ for $x \in S$ and all σ. With $v = \dim_K V$, there are at most $2av$ generators of L over $\mathbf{Z}$, each of seminorm at most $\max(1, a^4 v\,\mathrm{rd}(K) H)$ at every embedding.*

Outline. Apply transference to the inverse different to find short independent integers, then multiply them by a short K-basis to form a rational basis. Reduce integral basis coordinates modulo integers and bound the resulting generators by the seminorm triangle inequality. The complete proof is on p. 1197. □

Lemma 35.37 (The short isotropic periods)**.** *Let E be a CM field of degree $2f$ and root discriminant R_E. Let V be a complex Hermitian space on which $\mathcal{O}_E$ acts, with the adjoint of a acting as $\bar a$. Write V_σ for its summand at the embedding σ. Let $L \subset V$ be a full $\mathcal{O}_E$-stable lattice, principally polarised by the imaginary part B of the Hermitian form. Let λ be the least length of a nonzero vector of L. Set $\Omega = 2f^{3/2}R_E$, and take $\theta > 0$ with $\Omega\theta^2 \le 1$.*

There is a saturated $\mathcal{O}_E$-stable isotropic lattice $I \subset L$. Put $W = I \otimes_{\mathbf{Z}} \mathbf{R}$ and define $u(x) \in W$ by $\langle u(x), w\rangle_{\mathbf{R}} = B(x,w)$ for $w \in W$. Then

$$\begin{aligned} x \in L \setminus I &\implies \operatorname{dist}(x, W) \ge \theta,\\ x \in L,\quad u(x) \ne 0 &\implies \|u(x)\| \ge (\Omega\theta)^{-1}. \end{aligned}$$

One may take $I = 0$ for $\lambda \geq \theta$. For $\lambda < \theta$, I contains every shortest vector. Its real rank is $2f\rho$, with $\rho \leq \dim_{\mathbf{C}} V_\sigma$ for every σ. If $x_1, \dots, x_t \in L$ and the $u(ax_j)$, $a \in \mathcal{O}_E$, span W over $\mathbf{R}$, then, in this latter case,

$$\frac{2f}{R_E^2 \lambda^2} \leq \frac{1}{2} \sum_{j=1}^{t} \|u(x_j)\|^2.$$

Outline. Use short integral scalars from 35.36 to adjoin nearby lattice vectors while preserving isotropy and the projection bound. Pair a shortest period with the spanning vectors and use the norm bound for the inverse different. The complete proof is on p. 1197. □

Lemma 35.38 (Curvature and interpolation on the flag chart)**.** *Let $0 < p < r$, and take the product, over f embeddings, of the unitary flag domains of signature $(p, r-p)$. At an orthogonal frame x use lower unitriangular coordinates $n = 1 + z$. Give the coordinates the Euclidean norm obtained by summing the squared moduli of all their entries. Let w be the metric potential of weight λ. Assume that the adjacent gaps $\lambda_i - \lambda_{i+1}$ within each signature block, and the gap $\lambda_{r-1} - \lambda_0$, are at least T at every embedding. Indices start at zero.*

Write $d(z)\,dz$ for invariant measure in this chart, normalised by $d(0) = 1$, and R_x for minus the sum of the logarithms of the absolute Gram determinants of the first p columns. Put $b_* = 4fr^2 + 1$ and $U = w - b_* R_x - \log d$. If $T \geq b_* + 2r$, U is plurisubharmonic and

$$\partial\bar\partial U(z)[v, \bar v] \geq \tfrac{1}{2}(T - b_* - 2r)\|v\|^2 \qquad \big(\|z\| \leq 1/(100r^2)\big).$$

Here and below the Levi form is written as a Hermitian form.

Put $\rho = 1/(200r^2)$ and $D = fr(r-1)/2$. Suppose the ball of radius $1/(100r^2)$ embeds in the arithmetic quotient by a discrete arithmetic group Γ, and that the Γ-translates of the chart ball $B(0,\rho)$ are pairwise disjoint modulo the Borel subgroup B_{Bor} (that is, $\gamma z b = y$ for $z, y \in B(0,\rho)$, $\gamma \in \Gamma$, $b \in B_{\mathrm{Bor}}$ forces $\gamma = 1$). Choose a fixed smooth $\chi : [0,\infty) \to [0,1]$ equal to one on $[0, 1/4]$ and zero on $[1,\infty)$. There is a constant $C \geq 0$, depending only on the fixed cutoff, such that the following holds. For $A \geq 1$ bounding $|\chi'|, |\chi''|$, suppose

$$T \geq \max\big\{b_* + 2r + 2\{D(13A + 20) + A^2\}/\rho^2,\ C(fr)^8\big\}.$$

There is a holomorphic seed F on the chart that extends holomorphically to a global Borel-equivariant section P_0 on the entire flag domain satisfying $P_0(x) = F(0) = 1$, $P_0(\gamma x) = 0$ for every $1 \neq \gamma \in \Gamma$, and the nonsingular L^2 bound

$$\int |F|^2 e^{-w + b_* R_x}\, d\mu \leq e^{C(\|\lambda\|_1 + (fr)^8)}\, \mu(B(0,\rho)).$$

The constant C depends only on the fixed cutoff.

Outline. Compute the Levi form from Gram minors and transport it by reorthogonalising the frame. A weighted $\bar\partial$ estimate on the pseudoconvex chart corrects a cutoff, with the singular limiting weight forcing the correction to vanish at orbit points. The complete proof is on p. 1198. □

Lemma 35.39 (The kernel sum for the peak section)**.** *In the flag domain of Lemma 35.38 let $K_x(y) = \exp(-b_* R_x(y))$, with R_x evaluated in a frame centred at x. Write B_x for the chart ball of radius $1/(100r^2)$. If its translates under the arithmetic group Γ are disjoint on the flag domain, there is an absolute C such that*

$$\sup_y \sum_{\gamma \in \Gamma} K_x(\gamma y) \leq e^{C(fr)^8}.$$

For a holomorphic seed F in the weight line with finite integral $\int \|F\|^2/K_x\, d\mu$, its periodisation with a unitary character has squared integral over a fundamental domain at most

$$e^{C(fr)^8} \int \|F\|^2/K_x\, d\mu.$$

Outline. Use the chart bounds of 35.38 to compare the kernel on disjoint orbit balls, then integrate using the matrix ball and compact flag measures. Weighted Cauchy–Schwarz and unfolding give the bound for the periodised seed. The complete proof is on p. 1199. □

Lemma 35.40 (The zero Fourier coefficient of the theta derivative)**.** *Consider a nonterminal layer of signature (P,Q). At the base level take an even number N of theta columns. At every CM embedding take a product of minors of the first j rows of the differential variables, with the minors of size j occurring to powers $a_{\tau,j}-a_{\tau,j+1}$. Here the row degrees are decreasing and the degree following the last row is zero. Suppose the differential is harmonic and that $a_{\tau,P}>0$ for some embedding in the CM type. Every theta derivative with this differential and finite level characteristics is cuspidal.*

Outline. In a symplectic basis adapted to a cusp, each zero-frequency term is constant in a common differential direction across all columns. The full-row minor annihilates these terms, and absolute convergence permits termwise averaging. The complete proof is on p. 1200. □

Definition 35.41 (Polarised CM period tori and admissible Gaussian parameters)**.** Let E be a CM field of degree $2f=[E:\mathbf{Q}]$ and root discriminant $R_E=\mathrm{rd}(E)$. A polarised CM period torus $\mathcal{T}=(V,L_0,H,\varepsilon)$ of integer rank $r\ge 1$ over E consists of the following data.

1. A multiplicity function $\tau\mapsto d_\tau\in\mathbf{N}$ on $\mathrm{Hom}(E,\mathbf{C})$ satisfying $d_\tau+d_{\bar\tau}=r$ for every τ, giving the complex space $V=\bigoplus_\tau \mathbf{C}^{d_\tau}$ of dimension $g_A=fr$ with its standard positive-definite Hermitian form H and coordinatewise $\mathcal{O}_E$-action $(\alpha\cdot v)_\tau=\tau(\alpha)v_\tau$.
2. A full $\mathcal{O}_E$-stable $\mathbf{Z}$-lattice $L_0\subset V$ on which $\operatorname{Im}H$ is integer-valued and unimodular, so that $x\mapsto \operatorname{Im}H(x,\cdot)$ maps L_0 isomorphically onto $\mathrm{Hom}_{\mathbf{Z}}(L_0,\mathbf{Z})$.
3. A quadratic refinement $\varepsilon:L_0\to\{\pm1\}$ satisfying $\varepsilon(x+y)=\varepsilon(x)\varepsilon(y)\cos(\pi\operatorname{Im}H(x,y))$ for all $x,y\in L_0$, and $\varepsilon(x)=1$ whenever $\operatorname{Im}H(\alpha x,x)=0$ for all $\alpha\in\mathcal{O}_E$.

Write $\lambda_1(\mathcal{T})=\inf\{\|x\|:0\ne x\in L_0\}$ and, for $\theta>0$, define the truncated phase $\phi_\theta(\mathcal{T})=2f/(R_E^2\lambda_1(\mathcal{T})^2)$ if $\lambda_1(\mathcal{T})<\theta$ and $\phi_\theta(\mathcal{T})=0$ otherwise.

For integers $r,q,m,R,N_c\ge1$ and real $R_0>0$, put $d_*=2fm$, $\Omega_*=2f^{3/2}R_E$, $M_1=90m^2d_*^2R_E$,

$$\theta(q,R_0)=\sqrt{\frac{fR_0^2R_E^{21}}{q}}>0,\qquad b_0(q,R_0)=\frac{q}{R_0^2R_E^{28}},\qquad t_0(q,R_0)=qb_0(q,R_0),$$

and define the archimedean theta-constant bound $\mathcal{P}(E,r,q,m,R,N_c)$ by setting $p_1=1+fr\log(1+6\sqrt{frq})$, $p_2=\log(2(3d_*)^{d_*})$, $p_3=r(p_1+p_2+d_*)$, $A=R^4rd_*/2$, $K_*=4R_E^7d_*^2$ and

$$\mathcal{P}(E,r,q,m,R,N_c)=2R^4\bigl(p_3+\log N_c+2fr^2\log R\bigr)+A\log(AK_*).$$

For integers $D,q,m\ge1$ and real $R_0>0$, call (q,R_0,m) an admissible Gaussian parameter tuple at differential degree D on $\mathcal{T}$ if $fr\ge2$, $\min_\tau d_\tau\le m$, $m^4\le f$, $f^2\le R_E$, $R_E\ge16$, $R_0^2\ge\max(270,90(fr+D))$ and

$$d_*^2\Omega_*\theta(q,R_0)^2<1,\qquad b_0(q,R_0)\ge1,\qquad M_1\le q/2,\qquad D\le q,$$
$$4D(7+\log(qD))+d_*+1\le t_0(q,R_0),$$
$$fr\log2+2d_*+3+(D+1)(r\log r+3r+3)\le t_0(q,R_0).$$

For integers $r,D,q,m\ge1$ and real $R_0>0$, call (E,r,D,q,R_0,m) uniformly admissible if (q,R_0,m) is an admissible Gaussian parameter tuple at differential degree D on every polarised CM period torus $\mathcal{T}$ of rank r over E satisfying $\min_\tau d_\tau\le m$.

Lemma 35.42 (Parameter admissibility and Appell–Humbert reduction of level-R theta rules)**.** *Keep the notation of Definition 35.41.*

1. *Suppose integers $D,q,m,r\ge1$ satisfy $fr\ge2$, $m^4\le f$, $f^2\le R_E$, $R_E\ge16$ and $90R_E^{28}(fr+D)Dr\le q$. Put $R_0=\sqrt{90(fr+D)}>0$. Then (E,r,D,q,R_0,m) is uniformly admissible. If in addition real $L\ge2$ satisfies $m\le r$, $\log R_E\le 2L$, $\log f\le 2L$, $\log r\le L$ and $\log q\le 70L$, then $P_0=\mathcal{P}(E,r,q,m,5,1)$ satisfies $0\le P_0\le 200000(frL)^2$.*
2. *Let (X_0,F_c) be a unitary moduli datum over E of signature $(p_{\mathrm{sig}},q_{\mathrm{sig}})$ with $r=p_{\mathrm{sig}}+q_{\mathrm{sig}}\ge1$, level modulus M, good complex embedding $\sigma:F_c\to\mathbf{C}$ and analytic uniformisation U. Let $R>1$ be an odd integer with $R^4\mid M$, let $(x_1,\dots,x_r)$ span the level-R^4 residue module over $\mathcal{O}_E$, and let f_0 be a*

single-term level-R theta rule for $(x_1,\dots,x_r)$. *Let* $D,q,m\ge 1$ *be integers and let* $R_0>0$. *At every component* k *of* U *and domain frame* y *on* k, *write* $\lambda_1(y)$ *for the length of the shortest nonzero lattice vector in the positive majorant at* y, *set* $\Phi=2f/(R_E^2\lambda_1(y)^2)$ *if* $\lambda_1(y)<\theta(q,R_0)$ *and* $\Phi=0$ *otherwise, and write* $\|f_0^2(y)\|$ *for the invariant point norm of* f_0^2 *in theta weight* $(0,2r)_\theta$ *(where* $(a,N)_\theta$ *counts theta columns* $N=2m$, *twice the central parameter* m *in a row weight* $(u,m)_{\mathrm{row}}$*). Right-multiplication by an upper-triangular Borel matrix* u_{B} *produces a domain frame* $x=yu_{\mathrm{B}}$ *on* k *whose distinguished Gram minors all have modulus* 1, *satisfying* $\lambda_1(x)=\lambda_1(y)$ *and* $\|f_0^2(y)\|=|f_0(x)|^2$. *At* x, *the complex abelian fibre of dimension* fr *carries a period map* $\mathrm{Pm}:\Lambda_k\to\mathbf{C}^{fr}$, *a positive-definite Appell–Humbert Hermitian form* H_m *on* $\mathbf{C}^{fr}$ *whose imaginary part on* $\mathrm{Pm}(\Lambda_k)$ *is the integer-valued Riemann form, a* $\mathbf{C}$*-linear isometry* $\psi:(\mathbf{C}^{fr},H_m)\xrightarrow{\sim}(V,H)$ *onto a polarised CM period torus* $\mathcal{T}=(V,L_0,H,\varepsilon)$ *of rank* r *over* E *with* $L_0=\psi(\mathrm{Pm}(\Lambda_k))$, $\min_\tau d_\tau=\min(p_{\mathrm{sig}},q_{\mathrm{sig}})$ *and* $\phi_{\theta(q,R_0)}(\mathcal{T})=\Phi$, *an entire principal theta function* $\theta_0:\mathbf{C}^{fr}\to\mathbf{C}$ *of unit* L^2 *norm against* $e^{-\pi\operatorname{Re}H_m(z,z)}(\operatorname{vol}\mathcal{F}_0)^{-1}\,dz$ *on a fundamental domain* $\mathcal{F}_0$ *for* $\mathrm{Pm}(\Lambda_k)$, *and vectors* $z_1,\dots,z_r\in\mathbf{C}^{fr}$ *such that* $R^2\psi(z_1),\dots,R^2\psi(z_r)$ *lie in* L_0 *and span* L_0/RL_0 *over* $\mathcal{O}_E$, *and*

$$|f_0(x)|=\prod_{j=1}^{r}|\theta_0(z_j)|\exp\bigl(-\tfrac{\pi}{2}\operatorname{Re}H_m(z_j,z_j)\bigr).$$

If (E,r,D,q,R_0,m) *is uniformly admissible and* $\min(p_{\mathrm{sig}},q_{\mathrm{sig}})\le m$, *then*

$$\|f_0^2(y)\|\le\exp\bigl((\mathcal{P}(E,r,q,m,R,1)-\Phi)/R^4\bigr).$$

Outline. Proves (1) by elementary parameter inequalities. For (2), Gram–Schmidt reduces to a unit-minor frame on a polarised CM period torus, bounding theta null-values directly for $\Phi=0$ or via Fourier expansion along the isotropic sublattice of 35.37 for $\Phi>0$. The complete proof is on p. 1201. □

Lemma 35.43 (Component Bergman lower bounds from supremum ratios and fundamental-set peaks)**.** *Let* (X_0,F_c) *be a unitary moduli datum of rank* r *at a good complex embedding* $\sigma:F_c\to\mathbf{C}$ *with analytic uniformisation* U, *adelic quotient* Q_W *with probability measure* $\mu_{Q,1}$ *and frame map* π_σ, *algebraic component* Z, *class set* $\mathcal{C}_Z=\{q\in Q_W:\pi_\sigma(q)\text{ lies over }Z\}$ *with conditioned measure* $\mu_Z=\mu_{Q,1}(\mathcal{C}_Z)^{-1}\mu_{Q,1}|_{\mathcal{C}_Z}$ *(zero if* $\mu_{Q,1}(\mathcal{C}_Z)=0$*), and domain frame* $y=(k_0,y_0)$ *over* Z. *For a section weight with associated model weight* $\lambda=(\lambda_k)_k$, *write* $\|P(z)\|_\lambda$ *for the invariant point norm and* $\|P\|_{2,Z}=\bigl(\int_{Q_W}\|P(\pi_\sigma(q))\|_\lambda^2\,d\mu_Z(q)\bigr)^{1/2}$, *and let* $G(y)\in[0,\infty]$ *be the supremum of* $\|P(y)\|_\lambda^2/\|P\|_{2,Z}^2$ *over holomorphic cuspidal families* $P=(P_k)_k$ *of weight* λ, *with the convention* $c/0=+\infty$ *for* $c>0$ *and* $0/0=0$.

1. *Suppose* $\pi_\sigma(q)$ *lies in the signature domain for every* $q\in Q_W$. *Let* P *be a holomorphic cuspidal family of weight* λ *with* $P_{k_0}(y_0)\neq 0$, *and let* $e>0$. *If* $e\,\|P(z)\|_\lambda\le\|P(y)\|_\lambda$ *for every domain frame* z *on* U, *then* $G(y)\ge e^2$.

2. *Write* $\mathrm{comp}(z)=k$ *for the analytic component of a frame* $z=(k,z_0)$. *Assume the uniformisation satisfies the level-geometry laws, meaning that* $\pi_\sigma(Q_W)$ *lies in the signature domains, all domain frames on a given analytic component* k *lie over the same algebraic component, extending a holomorphic cusp form of weight* λ_{k_0} *on the signature domain of* k_0 *by* 0 *outside that domain and on* $k\neq k_0$ *defines a holomorphic cuspidal family of weight* λ, *each analytic component* k *admits an orthogonal frame* $x_{0,k}$ *and measurable fundamental set* $\mathcal{F}_k$ *of finite invariant chart measure* $\mu_F(\mathcal{F}_k)<\infty$, *and there is a finite constant* $c_{\mathrm{vol}}>0$ *such that* $\mu_{Q,1}(\{q\in Q_W:\mathrm{comp}(\pi_\sigma(q))=k_0\})=c_{\mathrm{vol}}\mu_F(\mathcal{F}_{k_0})$ *and* $\int_{Q_W}\|P'(\pi_\sigma(q))\|_\lambda^2\,d\mu_{Q,1}(q)=c_{\mathrm{vol}}\int_{\mathcal{F}_{k_0}}\|P_0(x_{0,k_0}n(z))\|_{\lambda_{k_0}}^2\,d\mu_F(z)$ *for every such zero-extended family* P'. *Let* $\beta\ge 0$. *Suppose that for every orthogonal frame* x_0 *on* k_0 *and every measurable fundamental set* $\mathcal{F}$ *in the lower-unitriangular chart* $z\mapsto x_0n(z)$ *on* k_0 *with invariant chart measure* μ_F, *there exists a holomorphic cusp form* P_0 *of weight* λ_{k_0} *on* k_0 *such that* $P_0(y_0)\neq 0$ *and*

$$\beta\int_{\mathcal{F}}\|P_0(x_0n(z))\|_{\lambda_{k_0}}^2\,d\mu_F(z)\le\|P_0(y_0)\|_{\lambda_{k_0}}^2\,\mu_F(\mathcal{F}).$$

Then $G(y)\ge\beta$.

Outline. For (1), since μ_Z has mass at most 1, $\|P\|_{2,Z} \le \sup_z \|P(z)\|_\lambda \le e^{-1}\|P(y)\|_\lambda$. For (2), extends P_0 by zero off k_0 and unfolds $\mu_{Q,1}$ into $\mathcal{F}_k$ to bound $\|P'\|_{2,Z}^2$ by the $\mathcal{F}_{k_0}$-average. The complete proof is on p. 1203. □

Lemma 35.44 (Uniform bound for cuspidal theta derivatives)**.** *Fix a layer of the theta chain, with host field E, Hermitian space W of rank $r \ge 1$, level modulus M, coefficient field F_c, analytic uniformisations and adelic frame maps. Assume the level-geometry laws: the moduli datum represents this layer with its distinguished host root and CM type; the auxiliary level factor is one; at every good complex embedding the uniformisation and adelic frame map give the level-M arithmetic frame quotients of W; the distinguished Gram minors of adelic frames have absolute value one; the stabiliser characters of dominant theta weights with an even number of columns are trivial; and the Hermitian space on every analytic component is a model of W.*

Put

$$f = [E : \mathbf{Q}]/2, \qquad R_E = \mathrm{rd}(E), \qquad \xi_{\mathbf{R}} = |\varphi_0(\xi_0)| > 0,$$

where φ_0 and ξ_0 are the distinguished complex embedding and polarising scalar of the host. Fix a good embedding $\sigma : F_c \hookrightarrow \mathbf{C}$.

Let (a, N) be a theta weight: the row entries $a_{\tau,i}$ are nonnegative integers, nonincreasing in each row, and N is even. Write

$$D = \sum_{\tau,i} a_{\tau,i}.$$

Let $P \in \mathrm{harm}(a, N)$, with the integral coefficient condition of Definition 35.53(c) below. For a tuple $\mathbf{x}$ of N lattice characteristics, let $s = s_{\mathbf{x},P}$ be the theta rule formed from P using the canonical theta connection. Assume that s belongs to the cuspidal section space of weight (a, N) at every good complex embedding.

Let $L, \kappa, \rho \in \mathbf{R}$ satisfy

$$L \ge 1, \qquad r \le N, \qquad \log M \le 30L, \qquad \log R_E \le 3L, \qquad \log(fr) \le L,$$
$$D \le \kappa, \qquad frN \le 2\kappa, \qquad \log \kappa \le 20L, \qquad \sum_{\nu \in \mathrm{supp}\, P} |\sigma(P_\nu)| \le \rho.$$

Then every domain frame z at σ satisfies

$$(2\pi\xi_{\mathbf{R}})^{-D/2}\|s(z)\|_{(a,N)} \le \rho \exp\bigl(9000(frL)^2\kappa\bigr).$$

Outline. Put $S = \sup_z \|s(z)\|_{(a,N)} < \infty$ and assume $S > 0$ with $A = e^L$. Orthonormalising a frame z with $\|s(z)\|_{(a,N)} > S/2$ gives x with $S \le 2|s(x)|$, and a self-dual transvection model with weight spread at most 3κ yields $\lambda_1(x)^2 \ge A^{-80}$. The orthonormal pointwise theta bound at x with $\lambda = A^{-40}$ then gives the uniform estimate. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1203. □

Lemma 35.45 (Lower bound for a family of block theta derivatives)**.** *Retain the layer, full level-geometry laws, good embedding σ, and notation $f, r, R_E, \xi_{\mathbf{R}}$ of Lemma 35.44, writing q for the level modulus and $(p_{\mathrm{sig}}, q_{\mathrm{sig}})$ for the signature. In particular, every uniformised algebraic frame is obtained by base change from a frame over a test algebra, and the polarising scalar $\xi_{0,k}$ on each analytic component k satisfies $|\tau(\xi_{0,k})| = \xi_{\mathbf{R}}$ for every $\tau : E \hookrightarrow \mathbf{C}$. Let (a, N) be a dominant theta weight with N even. Let $P \in \mathrm{harm}(a, N)$, with the integral coefficient condition of Definition 35.53(c), be exactly a product of powers of the prescribed block determinants, with pairwise disjoint sets of columns for distinct blocks, and assume*

$$\deg P = D = \sum_{\tau,i} a_{\tau,i}.$$

Suppose $q \ge 3$ is odd, $\alpha \in \mathcal{O}_E$ satisfies $\alpha\bar{\alpha} = q$ and $(\alpha) + (\bar{\alpha}) = \mathcal{O}_E$, the half-level ideal is $\mathfrak{b} = (\bar{\alpha})$, and (E, r, D, q, R_0, m) is uniformly admissible in the sense of Definition 35.41, with $q_{\mathrm{sig}} \le m$. Write Λ for the $\mathcal{O}_E$-lattice of the level datum. For each of the q^{frN} tuples $u \in (\Lambda/\mathfrak{b}\Lambda)^N$, let s_u be the theta rule obtained from P and representatives of u using the canonical theta connection. Assume that every s_u belongs to the section space of weight (a, N).

Set

$$d_* = 2fm, \qquad K_6 = 6(2d_*)^{d_*/2}, \qquad C_\chi = 8100m^4 d_*^4 R_E^7,$$
$$A_0 = \frac{1}{2}\left\{\log\pi + N\left(\log 24 + \frac{d_*}{2}\log(2d_*)\right) + (r+2)\log 2\right\}, \qquad c_0 = \frac{\pi N C_\chi}{q}.$$

At a domain frame y, let $\lambda_1(y) > 0$ be the first minimum, put $\phi(y) = 2f/(R_E^2\lambda_1(y)^2)$, and let $\Phi_D(y)$ be the truncated phase with threshold $\theta(q, R_0)$ of Definition 35.41. Then, with

$$v_u(y) = (2\pi\xi_{\mathbf{R}})^{-D/2}\|s_u(y)\|_{(a,N)}, \qquad J_y = 4^{-N}K_6^{-N}2^{-(r+2)D}\exp\{-2\pi N C_\chi \Phi_D(y)/q\},$$

one has

$$J_y > 0, \qquad q^D \sum_u v_u(y)^2 \geq \pi^{-D} J_y.$$

For all real A, c with $A_0 \leq A$ and $c_0 \leq c$, there is a tuple u such that

$$v_u(y) \geq \exp\{-AD - c\phi(y) - (D + frN)\log q\} > 0.$$

Outline. Pass to an orthogonal frame and a polarised $\mathcal{O}_E$-torus with a unit-norm principal section and the tuple of Definition 35.41. Lifted translates give an orthogonal basis of level-q sections, while greedy isotropic splitting supplies a unit-norm test section with squared derivative at least J_y. Cauchy–Schwarz, characteristic matching, and maximising over the q^{frN} tuples give both bounds. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1205. □

Lemma 35.46 (The level-five numerator and the Bergman bounds)**.** *Fix eligible F_m, C_H and assume the packages $\mathcal{H}_P, \mathcal{H}_H$, deep goodness and flat-degree conditions of Theorem 35.11 and Definition 35.3, together with the size and good-cover conditions of Lemmas 35.20 and 35.21 and the stage-three family spacing exponent $C_{\mathrm{count}} \geq 4e_P{+}11$ (which gives, beyond the family threshold, $\min(4\Delta, H_0, h_0) \geq (T_*{+}d_{\mathrm{fl}}{+}2)(T_*{+}d_{\mathrm{fl}}{+}2{+}2r){+}1$ with $d_{\mathrm{fl}} = f\binom{r}{2}$). There are an absolute constant $C_{\mathrm{peak}} \geq 0$ and a family threshold with the following property. At any admissible point beyond the threshold take a nonterminal layer of signature $(p_{\mathrm{sig}}, q_{\mathrm{sig}})$, a good cover of level M and a quotient weight u_1 of (35.6). Write $r = p_{\mathrm{sig}} + q_{\mathrm{sig}}$, $f = [E : \mathbf{Q}]/2$, $R_E = \mathrm{rd}(E)$, $Q_0 = g_0 g_1$ and $T = T_*$. There are $P_0, B_0' \geq 0$ with*

$$TP_0 \leq 200000E_p, \qquad TB_0' \leq 835C_{\mathrm{peak}}E_p. \tag{35.13}$$

For every component Z of the cover, good embedding σ of the cover field and domain frame y over Z at σ, let λ_1 be the length of the shortest nonzero lattice vector in the positive majorant at y, put $\phi = 2f/(R_E^2\lambda_1^2)$, and let $G(y) \in [0, \infty]$ be the component Bergman supremum of Lemma 35.43 for the section weight $(u_1, 0)$. For every single-term level-five theta rule f_0 from an r-tuple spanning the level-5^4 residue module over $\mathcal{O}_E$ (as in Lemma 35.42(2) with $R = 5$), let $a = \|f_0^2(y)\|$ be the invariant point norm of f_0^2 at y. Everywhere $a \leq \exp(P_0/625)$. For $M\lambda_1^2 < b$ one has $a \leq \exp((P_0 - \phi)/625)$ and $G(y) \geq \exp(-\phi/625)$. For $M\lambda_1^2 \geq b$ one has $G(y) \geq \exp(-B_0')$.

Outline. Lemmas 35.20, 35.21, and 35.42 give the budget and level-five bounds. For $M\lambda_1^2 < b$, Lemmas 35.44, 35.45, and 35.6 feed Lemma 35.43; for $M\lambda_1^2 \geq b$, a seed Poincaré series does. The complete proof is on p. 1208. □

Lemma 35.47 (Order of a product of level-five theta constants)**.** *Let V be a complete discrete valuation ring with fraction field of characteristic zero. Write v for its additive valuation, with $v(\varpi) = 1$, and extend v to an algebraic closure. Take an integral frame of the polarised semiabelian degeneration over the host CM field E of 5.1 and 8.3 ($d \equiv 7 \pmod{16}$ prime), with full level 5^4 on its generic fibre. Let X be its period lattice and let $Q : X \to \mathbf{Z}_{\geq 0}$ be the order of the cubical period. Its polarisation*

$$B(x, y) = Q(x + y) - Q(x) - Q(y)$$

is symmetric and satisfies $B(ax, y) = B(x, \bar{a}y)$ for $a \in \mathcal{O}_E$. On the real space of symmetric forms with this property, let ϕ be the infimum of $\ell(B)$ over the real linear functionals ℓ which are integral on integral forms and strictly positive on every nonzero positive semidefinite form.

Let f_0 be the product of the level-five theta constants for a tuple spanning the full level module over $\mathcal{O}_E$. Then

$$v(f_0) \geq \phi/625,$$

with $v(0) = +\infty$.

Outline. Bound each theta valuation below by the minimum cubical period order on its label coset divided by 625. The minimizing periods span over E and define a positive integral functional whose value on the period form is at least ϕ. The complete proof is on p. 1212. □

Lemma 35.48 (The prime sum in the reference-weight budget)**.** *Use the nonterminal layer and quotient weight of 35.94. Put $L = L_x$, $T = T_*$, $r = k + m - 1$, and let S be the width in 35.20. Define*

$$B_0 = \lfloor 8b/T \rfloor, \quad J = 2B_0 + 1, \quad U = B_0 + 1, \quad V = f\{k(S + 2r)/T + 2q_{\mathrm{sig}}B_0\}, \quad A = fmU.$$

Let $D_E = |\operatorname{disc} E|$ and $\delta_\ell = v_\ell(D_E)/(2f)$. For a prime $\ell > J$ put $e_\ell = 0$. For primes $\ell \leq J$ use the first applicable line of the following formula, with $h_\ell = \log J/\log \ell$:

$$e_\ell = \begin{cases} A(1 + \delta_\ell/2) + V, & \ell \mid Q_S, \\ (frU + V)/(\ell - 1), & \ell \in \{q, d\}, \\ \{A(4 + h_\ell/2) + V\}/(\ell - 1) + f(m + 1)r^2(h_\ell + 1) + 1, & m\ell < k, \\ (6A + V)/(\ell - 1) + 1, & \text{otherwise.} \end{cases}$$

Let N be the even number of columns used for the theta weight in 35.46, M' the covering level, and $\mathcal{B}$ the primes dividing $2dD_EM'$. Set

$$a(\ell) = e_\ell + \mathbf{1}_{\ell \in \{g_0, g_1\}} + \mathbf{1}_{\ell \in \mathcal{B}}\big(1 + \tfrac{N}{2} fr\, \delta_\ell\big).$$

These nonnegative weights are supported on $\{\ell \text{ prime} : \ell \leq J\} \cup \{g_0, g_1\} \cup \mathcal{B}$. There is a family constant C_R such that, beyond one threshold,

$$T \sum_\ell a(\ell) \log \ell \leq C_R E_p.$$

Outline. Apply elementary prime sums with the width bound 35.20. Bound the extra prime contributions using the level bounds and 33.3, and the different contribution using the theta column count in 35.46. The complete proof is on p. 1213. □

Lemma 35.49 (A short period basis on a trait)**.** *On a split semiabelian uniformisation over a trait V (a complete discrete valuation ring with uniformiser ϖ_V), let X be the period lattice, $s = \operatorname{rank}_{\mathbf{Z}} X$, and Q_X the order of its cubical period, with valuation normalized by $v(\varpi_V) = 1$. Write B_X for the polarisation of Q_X and ϕ for the phase of Lemma 35.47. Put $n_E = [E : \mathbf{Q}]$ and $\kappa = \operatorname{rd}(E)^2/(2n_E)$. There is a basis $b_1, \ldots, b_s$ of X such that*

$$Q_X(b_i) \leq \Phi, \qquad |B_X(b_i, z)| \leq 2\sqrt{s}\, Q_X(z) \quad (z \in X), \qquad \Phi = \kappa s^3 \phi.$$

In the layer of signature (P, Q) with $P \geq Q$ one has $s \leq n_E Q$.

Outline. Bounds s from the toric Lie algebra, then uses AM–GM on balanced forms with 35.47 to lower-bound $\lambda_1(X^\vee)$. Banaszczyk's transference inequality bounds $\lambda_s(X)$, and lifting generators along a saturated flag gives the basis. The complete proof is on p. 1213. □

Lemma 35.50 (Integral jets in the theta expansion)**.** *In the notation of Lemma 35.49 over the complete discrete valuation ring V, let D be a nonnegative integer and let $M \geq 4D + 4$ satisfy $2D\sqrt{s} < M$. On any positive number of copies of the uniformisation, fix an integral frame of the theta line at the identity. After passing to a finite ramified discrete valuation ring extension V'/V (with v and Q_X kept normalized by $v(\varpi_V) = 1$ on a uniformiser of the base trait V), every integral jet of total order D, multiplied by a nonzero scalar in V' of valuation at most $D^2\Phi/M$, is the jet of a level-M theta section in the following lattice. A section with Fourier coefficient c_z at the period index z belongs to the lattice if*

$$v(c_z) \geq Q_X(z)/M$$

in integral frames on the abelian part. On a product, Q_X in this definition is the sum of the forms of the factors.

Outline. Uses 35.49 so degree-$\le D$ sums χ uniquely minimize Q_X mod MX. Prescribes abelian D-jets at χ and extends by period translation so germs span all D-jets by Nakayama, then scales by valuation $\lfloor D^2\Phi \rfloor / M$. The complete proof is on p. 1214. □

Lemma 35.51 (The reference depth outside a finite set for fixed data)**.** *Fix a nonterminal member, a good cover and u_1 as in Proposition 35.94, beyond the threshold of Lemma 35.46. There is a finite set Σ of primes, depending on these fixed data, such that for $\ell \notin \Sigma$, every algebraic integral frame y on a trait over a finite place dividing ℓ ($v(\ell) > 0$) of phase ϕ satisfies*

$$\inf_{f \text{ integral cusp section of weight } u_1} \frac{v(f(y))}{v(\ell)} \le \frac{\phi}{2500 v(\ell)}.$$

Outline. Builds cusp sections via 35.46 and 35.40, excluding level, $c_{\tau i}!$, and denominator primes via 35.10 and 35.7. For $\ell \notin \Sigma$, applies 35.50 to a monomial jet and decomposes into theta translates via 35.5. The complete proof is on p. 1215. □

Lemma 35.52 (The period estimate for either half-level translation)**.** *Use the base level $g_0 g_1$. Take $g = g_0,\ g' = g_1$, or $g = g_1,\ g' = g_0$. Let $\alpha \in \mathcal{O}_E$ be the half-level factor with $\alpha\bar{\alpha} = g$. With X, Q_X, s, Φ as in Lemma 35.49, every class $y + \alpha X$ contains z_0 such that*

$$\frac{Q_X(z_0)}{gg'} \le \frac{s\Phi}{4g'}, \qquad Q_X(z_0 + g'\alpha z) - Q_X(z_0) \ge g'(g'-1)Q_X(\alpha z) \quad (z \in X).$$

In a theta expansion supported on $z_0 + g'\alpha X$ whose orders are at least $Q_X(z)/(gg')$, a coefficient at z_0 attaining this bound at the point gives an evaluation of exactly that order.

Outline. Uses $Q_X(\alpha z) = g Q_X(z)$ and randomized rounding in the short basis b_i to bound $\min_{z_0 \in y + \alpha X} Q_X(z_0)$ by $gs\Phi/4$. Minimality yields $|B_X(z_0, \alpha z)| \le Q_X(\alpha z)$, giving the quadratic growth bound and ultrametric order. The complete proof is on p. 1216. □

Definition 35.53 (Block depths, local depths and reference depths)**.** Fix a nonterminal layer of signature (P, Q) over the host CM field E of 5.1 and 8.3 ($d \equiv 7 \pmod{16}$ prime) at base level $M = g_0 g_1$, with coefficient field F_c. Write $\bar{\tau}$ for the conjugate of an embedding τ of E, and put $\mu_\tau = P$ for τ in the CM type and $\mu_\tau = Q$ otherwise. A block is a pair (τ, i) with $1 \le i \le \mu_\tau$. A theta weight (a, N) consists of integers $a_{\tau,i} \ge 0$, nonincreasing in i, and an even number N of columns. Its gaps are $c_{\tau,i} = a_{\tau,i} - a_{\tau,i+1}$, with $a_{\tau,\mu_\tau+1} = 0$, and its degree is

$$D = \sum_{\tau,i} a_{\tau,i} = \sum_{\tau,i} i\, c_{\tau,i}.$$

Let λ be a finite place of F_c above the prime ℓ. A trait over λ is a complete discrete valuation ring V whose fraction field has characteristic zero, with an embedding $F_c \to \operatorname{Frac} V$ inducing λ. Its valuation v has value one on a uniformiser, so $v = e\, v_\lambda$ on F_c, where e is the ramification index. Let y be an integral frame on V and $\widehat{G}$ the formal group at the origin of its semiabelian scheme. For each τ the frame supplies integral Lie vectors $u_{\tau,1}, \dots, u_{\tau,\mu_\tau}$. They form a basis of a saturated summand of the integral Lie algebra, and $\mathcal{O}_E$ acts on that summand through τ. Let C be an algebraically closed complete valued field containing $\operatorname{Frac} V$, with valuation v_ℓ normalised by $v_\ell(\ell) = 1$ and with ring of integers $\mathcal{O}_C$. Choose integral formal coordinates of $\widehat{G}$, and let exp be its exponential over C. A power series in these coordinates is integral if its coefficients lie in $\mathcal{O}_C$. Such fields and coordinates exist. The infima below run over all of them.

(a) Let (τ, i) be a block and $k \ge 0$. The block depth $\nu_{\tau,i}(k; y)$ is the infimum of

$$v_\ell\Big(\det\big(\partial/\partial t_{ab}\big)_{a,b \le i}^{k}\, F\big(\exp(\textstyle\sum_a t_{a1} u_{\tau,a}), \dots, \exp(\sum_a t_{ai} u_{\tau,a})\big)\big|_{t=0}\Big),$$

taken over integral power series F on i copies of $\widehat{G}$. For exponents $c = (c_{\tau,i})$ put $\nu_c(y) = \sum_{\tau,i} \nu_{\tau,i}(c_{\tau,i}; y)$.

(b) Let $\varrho \ge 0$. The exponents c have oscillation at most ϱ at ℓ if every place $\lambda \mid \ell$ of F_c has an integer $A_\lambda \ge 0$ such that $A_\lambda \le \nu_c(y)$ for all integral frames over λ and $\nu_c(y) \le A_\lambda + \varrho$ for all algebraic integral frames over λ.

(c) The set $\operatorname{harm}(a, N)$ consists of the polynomials P with coefficients in $\mathcal{O}_{F_c}$, in variables $X_{\tau,i,b}$ with $1 \le b \le N$, that have the following three properties.

- Each monomial has degree $a_{\tau,i}$ in the variables of the row (τ,i).
- The polynomial lies in the span of products of leading minors $\det(X_{\tau,i',b_j})_{i',j\le i}$, as in Lemma 35.40.
- It is harmonic. This means that it is annihilated by $\sum_b \partial^2/\partial X_{\tau,i,b}\partial X_{\bar\tau,i',b}$ for every τ in the CM type and all i,i'.

Replacing $\mathcal{O}_{F_c}$ by a ring $R\subset \operatorname{Frac} V$ defines $\operatorname{harm}_R(a,N)$.

(d) Let P be a polynomial over $\operatorname{Frac} V$ in the variables $X_{\tau,i,b}$. Let $P(D)$ be the operator obtained by substituting for $X_{\tau,i,b}$ the invariant derivation of $\widehat{G}^N$ along $u_{\tau,i}$ in the b-th factor. Let h_L be the logarithm along $\widehat{G}$ of the canonical trivialisation of the theta line relative to its integral frame. It is the restriction of the canonical gauge of the formal Poincaré biextension along $x\mapsto(x,w_0x)$, where $w_0\in\mathcal{O}_E$ satisfies $w_0+\bar w_0=1$. For $k\ge 0$ the local depth $\delta_k(P;y)$ is the infimum of

$$v_\ell\Big(\big[P(D)\big(F\,e^{k\sum_b h_L(x_b)}\big)\big](0)\Big),$$

taken over integral power series F on $\widehat{G}^N$, where x_b is the point of the b-th factor. Thus $\delta_k(P;y)$ is the least normalised order at the origin of $P(\nabla)s$. Here s runs over the integral germs of the k-th power of the theta line on N copies, and ∇ is the canonical connection that defines theta derivatives.

(e) A cusp section f over F_c is integral if $v(f(y))\ge 0$ for every algebraic integral frame y on every trait over every finite place. Let w be a weight and $c\ge 0$. The reference depth $d^c_\ell(w)$ is the supremum of

$$\inf_f \frac{v(f(y))}{v(\ell)} - \frac{c\,\phi}{v(\ell)},$$

where f runs over the integral cusp sections of weight w. The supremum is taken over the places $\lambda\mid\ell$, the traits over λ, the algebraic integral frames y on them, and their phases ϕ (Lemma 35.47). The same definition applies on a cover with its own coefficient field. The number $d_q(u_1)$ in the proof of Proposition 35.94 is $d_q^{1/2500}(u_1)$ on the cover.

Lemma 35.54 (Bounds for one block)**.** *Let y be an integral frame over a place above ℓ, let (τ,i) be a block and let $k\ge 0$. Then*

$$0\le \nu_{\tau,i}(k;y)\le i\,v_\ell(k!)\le \frac{ik}{\ell-1}.$$

In particular $\nu_{\tau,i}(k;y)=0$ for $k<\ell$.

Outline. Rewrites $\partial/\partial t_{ab}$ on $\widetilde{F}$ as commuting invariant derivations preserving integral series, giving $\nu_{\tau,i}(k;y)\ge 0$. Evaluates on the diagonal monomial from a unit $i\times i$ frame minor and applies Legendre's formula. The complete proof is on p. 1216. □

Lemma 35.55 (Summation and rounding of block depths)**.** *Let $\lambda\mid\ell$ and let c be exponents. Suppose every block f has real numbers $A_f\le B_f$ such that*

- *$A_f\le\nu_f(c_f;y)$ for all integral frames over λ, and*
- *$\nu_f(c_f;y)\le B_f$ for all algebraic integral frames over λ.*

Put $A_\lambda=\lfloor\max(0,\sum_f A_f)\rfloor$. Then $A_\lambda\le\nu_c(y)$ for all integral frames over λ, and

$$\nu_c(y)\le A_\lambda+\sum_f(B_f-A_f)+1$$

for the algebraic ones. Moreover c has oscillation at most 0 at every prime $\ell>\max_f c_f$. At every prime it has oscillation at most $D/(\ell-1)$, where $D=\sum_{\tau,i} i\,c_{\tau,i}$.

Outline. Combining $\nu_c(y)\ge 0$ from 35.54 with $\sum_f A_f\le\nu_c(y)$ gives the lower bound; $\sum_f B_f$ and $\max(0,\sum_f A_f)<A_\lambda+1$ give the algebraic upper bound. Setting $A_\lambda=0$ and summing 35.54 over blocks yields both oscillation bounds. The complete proof is on p. 1217. □

Lemma 35.56 (Clearing denominators away from a place)**.** *Let f be a cusp section of weight w over F_c. Let m be a finitely supported real function on the finite places of F_c such that $v(f(y))/e \geq m(w')$ for every algebraic integral frame y on a trait over w' of ramification index e. Let λ be a finite place and k an integer such that $v(f(y))/e \geq k$ for the algebraic integral frames over λ. Then there is $\sigma \in F_c^{\times}$ with $v_\lambda(\sigma) = -k$ such that σf is integral.*

Outline. Sets $n_{w'} = -\lfloor m'(w') \rfloor$ on the finite support S of m' and applies the Chinese remainder theorem modulo primes dividing L to build $\sigma = \beta/L^j$ with $v_{w'}(\sigma) = n_{w'}$ on S and ≥ 0 outside, making σf integral. The complete proof is on p. 1217. □

Lemma 35.57 (Isotropic columns and a global differential)**.** *Let λ be a place of F_c above ℓ, and let V_λ be the completion of $\mathcal{O}_{F_c}$ at λ. An embedding τ also denotes the map $\mathcal{O}_E \to V_\lambda$ through the coefficient field. Let (a, N) be a theta weight with $N \geq \sum_{c_{\tau,i} \neq 0} 2i$.*

(a) For every embedding τ there is $\alpha_\tau \in V_\lambda$ with the following property. For every $n_0 \geq 0$ there are $u, t \in \mathcal{O}_E$ with

$$\tau(u) \equiv 1, \quad \tau(t) \equiv \alpha_\tau \pmod{\ell^{n_0} V_\lambda}, \qquad u\bar{u} + t\bar{t} \in \ell^{n_0}\mathcal{O}_E.$$

(b) Assign disjoint sets of $2i$ columns $b_1, \ldots, b_{2i}$ to the blocks (τ, i) with $c_{\tau,i} \neq 0$. Let $B_{\tau,i}$ be the $i \times N$ matrix whose j-th row has entry 1 in column b_j, entry α_τ in column b_{i+j}, and zeros elsewhere. The column differential

$$P_{\mathrm{col}} = \prod_{c_{\tau,i} \neq 0} \det\Big(\sum_b X_{\tau,i',b}(B_{\tau,i})_{j,b}\Big)_{i',j \leq i}^{c_{\tau,i}}$$

lies in $\mathrm{harm}_{V_\lambda}(a, N)$. For every n_0 there are $x_{j,b} \in \mathcal{O}_E$ with the following properties.

- *$x_{j,b} = 0$ outside the columns of the block.*
- *$\tau(x_{j,b}) \equiv (B_{\tau,i})_{j,b}$ modulo $\ell^{n_0} V_\lambda$.*
- *$\sum_b x_{j,b}\bar{x}_{j',b} \in \ell^{n_0}\mathcal{O}_E$ for all j, j'.*

(c) For every n_0 there is $P_1 \in \mathrm{harm}(a, N)$ with $P_1 - P_{\mathrm{col}} \in \ell^{n_0}\,\mathrm{harm}_{V_\lambda}(a, N)$. Through the canonical map $V_\lambda \to V$, the same identity holds on every trait V over λ.

Outline. Solves $u\bar{u}+t\bar{t} \equiv 0$ locally at $\mathfrak{q} \mid \ell$ and glues by Chinese remainder for α_τ. Checks $P_{\mathrm{col}} \in \mathrm{harm}_{V_\lambda}(a, N)$ via Cauchy–Binet, then approximates its coordinates in a $\mathbf{Z}$-basis of $W_{\mathbf{Z}}$. The complete proof is on p. 1217. □

Lemma 35.58 (Reference depths on a good cover)**.** *Let Y be a good cover of the base moduli at a nonterminal layer, as in Lemma 35.21, with coefficient field F'. For every prime ℓ, every theta weight $w = (a, N)$ and every $c \geq 0$, the reference depth $d_\ell^c(w)$ computed on Y is at most $d_\ell^c(w)$ computed on the base moduli.*

Outline. Forgets the cover level and restricts coefficients to map any algebraic integral frame y' on Y to a base frame y with the same valuation and phase. Pulling back base integral cusp sections preserves values and integrality, yielding the inequality on infima. The complete proof is on p. 1218. □

Lemma 35.59 (Theta derivatives at a place prime to the level)**.** *Let (a, N) be a theta weight of degree D, and let $P \in \mathrm{harm}(a, N)$. Let Λ be the marked lattice and $\mathfrak{b}$ its chosen half-level ideal, with $\mathfrak{b}\bar{\mathfrak{b}} = (M)$ and $\mathfrak{b} + \bar{\mathfrak{b}} = \mathcal{O}_E$. The characteristic module is $Q_M = \Lambda/\mathfrak{b}\Lambda$, with its level marking $\eta : Q_M \to A[\mathfrak{b}]$. For $x = (x_1, \ldots, x_N) \in Q_M^N$, let $\theta_{x,P}$ be the theta derivative with differential P. All subsequent quantifiers over characteristics, including those in Propositions 35.75 and 35.74, use this module. Over a field with rational M^2-torsion choose $t_b \in A[\mathfrak{b}^2]$ with $[M]t_b = \eta(x_b)$. Such lifts exist because $[M] : A[\mathfrak{b}^2] \to A[\mathfrak{b}]$ is surjective; the conjugate half-level acts invertibly on this primary part. Its value at an integral frame over any trait is*

$$\theta_{x,P}(y) = M^{-D}\big[P(\nabla)(s_{x_1} \boxtimes \cdots \boxtimes s_{x_N})\big](0),$$

where s_{x_b} is the section of the M^2-th power of the theta line attached to x_b. This section is the translate of the pullback under $[M]$ of the integral principal theta section, viewed in the M^2-th power, by a point of order dividing M^2 that lifts x_b, normalised at the frame. Now let $\lambda \mid \ell$ be a place of F_c with $\ell \nmid M$, and let y be an algebraic integral frame on a trait V over λ.

(1) Suppose $A \le \delta_k(P;y')$ for all k and all algebraic integral frames y' over λ. Then $v(\theta_{x,P}(y)) \ge A\,v(\ell)$ for every x.

(2) Suppose y has a split uniformisation, with s and Φ as in Lemma 35.49. If $4D+4 \le M$ and $2D\sqrt{s} < M$, then

$$\inf_x \frac{v(\theta_{x,P}(y))}{v(\ell)} \le \delta_M(P;y) + \frac{D^2\Phi}{M\,v(\ell)}.$$

Outline. For (1), passes to a split uniformization and checks integrality of s_{x_b} via 35.47 to bound by δ_{M^2}. For (2), lifts the D-jet of F via 35.50, expands in theta translates as in 35.51, and uses the ultrametric inequality. The complete proof is on p. 1218. □

Proposition 35.60 (Reference depth from a differential of controlled local depth)**.** *Let (a,N) be a theta weight of degree D with $N \ge 2r$ and $a_{\tau,P} > 0$ for every τ in the CM type. Let $c \ge 0$ satisfy*

$$4D+4 \le g_0 g_1, \qquad 2D\sqrt{2fm} < g_0 g_1, \qquad \kappa(2fm)^3 D^2 \le c\,g_0 g_1,$$

with κ as in Lemma 35.49. Let $\ell \notin \{g_0, g_1\}$, let $\lambda \mid \ell$ be a place of F_c, and let $\varrho \ge 0$. Assume the following.

(i) There are an integer $A \ge 0$ and a differential $P_1 \in \mathrm{harm}(a,N)$ with $A \le \delta_k(P_1;y) \le A+\varrho$ for all k and all algebraic integral frames y over λ.

(ii) For each characteristic tuple x there is a finitely supported function m_x on the finite places such that $v(\theta_{x,P_1}(y))/e \ge m_x(w)$ at every algebraic integral frame over each finite place w.

Then every algebraic integral frame y over λ with phase ϕ satisfies

$$\inf_f \frac{v(f(y))}{v(\ell)} - \frac{c\,\phi}{v(\ell)} \le \varrho,$$

where f runs over the integral cusp sections of weight (a,N). If (i) and (ii) hold at every place above ℓ, then $d_\ell^c(a,N) \le \varrho$.

Outline. Splits the uniformization, bounds s via 35.49, and applies 35.59(2) for an upper bound on θ_{x,P_1}. Checks cuspidality via 35.40, rescales θ_{x,P_1} integrally by 35.59(1) and 35.56, and applies 35.53(e). The complete proof is on p. 1219. □

Lemma 35.61 (Inversion in the ring of approximable coefficients)**.** *Let R be a Dedekind domain, let V be a discrete valuation ring, and let $\varphi : R \to V$ be a ring homomorphism. Let $q \in R$ have nonzero, nonunit image in V. Define*

$$S = \{t \in V : \text{ for every } n \ge 0 \text{ there exist } a \in R,\ z \in V \text{ with } \varphi(a) = t + \varphi(q)^n z\}.$$

Then S is a subring of V. If $u \in S \cap V^\times$, then $u^{-1} \in S$. Consequently, for $r \in \mathbf{N}$, if an $r \times r$ matrix M has entries in S and $\det M \in V^\times$, its inverse has entries in S.

Outline. The congruences make S a subring. For $u \in S \cap V^\times$, approximate u by $\varphi(a)$, use maximality of $\varphi^{-1}(\mathfrak{m}_V)$ for $ab + i = 1$, invert u mod $\varphi(q)^n V$ via a geometric sum, and use the adjugate for M^{-1}. The complete proof is on p. 1220. □

Lemma 35.62 (Cancellation of a frame with integral pullback)**.** *Let C be a valued field of characteristic zero, with valuation ring $\mathcal{O}_C$. For $h,r,s \in \mathbf{N}$, let $t = (t_1,\dots,t_h)$, $Y = (Y_1,\dots,Y_r)$ and $X = (X_1,\dots,X_s)$. Let*

$$\mathbf{e}(t) \in C[[t]]^r, \qquad \mathbf{c}(t) \in C[[t]]^s, \qquad \psi(Y) \in \mathcal{O}_C[[Y]]^s, \qquad \rho(X) \in \mathcal{O}_C[[X]]^r$$

have zero constant terms and satisfy

$$\psi(\mathbf{e}(t)) = \mathbf{c}(t), \qquad \rho(\mathbf{c}(t)) = \mathbf{e}(t).$$

Let $W \in C[[X]]$ have constant term 1, and suppose that $U = W(\psi(Y))$ has coefficients in $\mathcal{O}_C$. For every $F \in \mathcal{O}_C[[Y]]$ and integer $k \ge 0$, the series

$$G(X) = (FU^{-k})(\rho(X))$$

has coefficients in $\mathcal{O}_C$ and satisfies

$$G(\mathbf{c}(t))W(\mathbf{c}(t))^k = F(\mathbf{e}(t)).$$

The equality remains true after applying any polynomial in the partial derivatives with respect to t and evaluating at $t = 0$.

Outline. Recursive comparison in $UU^{-1} = 1$ makes U^{-1} integral, and substituting ρ keeps G integral. The curve identities cancel U^{-k} against W^k on $\mathbf{c}(t)$, and finite-jet linearity of polynomial derivatives at 0 finishes. The complete proof is on p. 1220. □

Lemma 35.63 (Prime idempotents and compatible approximants)**.** *Let R be a Dedekind domain, let V be a discrete valuation ring, and let $\varphi : R \to V$ be a ring homomorphism. Let $q \in R$ have nonzero, nonunit image. There are elements $\varepsilon_N \in R$, for $N \geq 0$, such that*

$$\varphi(\varepsilon_N) \in 1 + \varphi(q)^N V, \qquad \varphi(a) \in \varphi(q)^M V,\ M \leq N \implies \varepsilon_N a \in q^M R.$$

If $t \in V$ and $x_N \in R$ satisfy $\varphi(x_N) - t \in \varphi(q)^N V$, then

$$\varepsilon_n x_n - \varepsilon_m x_m \in q^m R \quad (m \leq n), \qquad \varphi(\varepsilon_N x_N) - t \in \varphi(q)^N V.$$

Outline. Factor $(q) = \mathfrak{p}^r J$ at $\mathfrak{p} = \varphi^{-1}(\mathfrak{m}_V)$ and lift V-divisibility to $\mathfrak{p}$-powers. Comaximality of $\mathfrak{p}^{k_N}$ and J^N gives $r_N + \varepsilon_N = 1$ with $\varepsilon_N \in J^N$, and telescoping finishes. The complete proof is on p. 1220. □

Lemma 35.64 (Limiting CM actions preserve integral formal points)**.** *Use the integral formal chart, C, $\mathcal{O}_C$, and v_ℓ of Proposition 35.66. Write $d[a]$ for the differential of $[a]$, and write $\exp_{\widehat{G}}$ and $\log_{\widehat{G}}$ for the mutually inverse formal exponential and logarithm. For each embedding τ, there is a sequence $\varepsilon_{\tau,N} \in \mathcal{O}_E$ with the following property. If $t \in V$ and $x_N \in \mathcal{O}_E$ satisfy $\tau(x_N) - t \in \ell^N V$, there is $A \in M_d(\mathcal{O}_C)$ such that*

$$v_\ell\big(A_{ij} - d[\varepsilon_{\tau,N} x_N]_{ij}\big) \geq N, \qquad Au_{\tau,a} = t\, u_{\tau,a}$$

for all N, i, j, a; here $d = \dim \widehat{G}$, and the coefficient maps to V and C are understood.

Let $\mathcal{B}$ be a finite set, and let $w_b \in (t)C[[t]]^d$, for $b \in \mathcal{B}$, where t now denotes a finite set of formal variables. Suppose that $\exp_{\widehat{G}}(w_b)$ has coefficients in $\mathcal{O}_C$. If matrices $A_b \in M_d(C)$ and elements $a_{N,b} \in \mathcal{O}_E$ satisfy

$$v_\ell\big((A_b - d[a_{N,b}])_{ij}\big) \geq N \quad \text{for all } N, b, i, j,$$

then $\exp_{\widehat{G}}(\sum_b A_b w_b)$ has coefficients in $\mathcal{O}_C$.

Outline. Lemma 35.63 at $q = \ell$ makes $d[\varepsilon_{\tau,N} x_N]$ entrywise Cauchy with integral limit A acting as t on τ-eigenvectors. Formal-logarithm additivity makes approximants integral, and polynomiality of each coefficient passes to the limit. The complete proof is on p. 1221. □

Lemma 35.65 (Integral frame along a limiting isotropic inclusion)**.** *In the chart of Proposition 35.66, let $\mathcal{B}$ be finite and let $w_b \in (t)C[[t]]^d$ satisfy $\exp_{\widehat{G}}(w_b) \in \mathcal{O}_C[[t]]^d$. Let $A_{b,j} \in M_d(C)$, for $b \in \mathcal{B}$ and $1 \leq j \leq n$. Suppose there are $a_{N,b,j} \in \mathcal{O}_E$ such that*

$$v_\ell\big((A_{b,j} - d[a_{N,b,j}])_{rs}\big) \geq N, \qquad \sum_{j=1}^{n} \overline{a_{N,b,j}}\, a_{N,b',j} \in \ell^N \mathcal{O}_E$$

for every N, b, b', r, s. Then

$$\exp\left(\sum_{j=1}^{n} h_L\left(\exp_{\widehat{G}}\Big(\sum_b A_{b,j} w_b\Big)\right)\right)$$

is an integral power series with constant term 1.

Outline. Biextension cocycles make $h(\exp_{\widehat{G}} w, \exp_{\widehat{G}} z)$ biadditive and adjoint modulo integral-unit exponentials, reducing each step to $U_N \exp(T_N)$. The finite-jet estimate of Lemma 35.64 on $\exp(T_N)$ and $A_{b,j}$ gives integrality. The complete proof is on p. 1221. □

Proposition 35.66 (Column-differential formal-group matching)**.** *Let $\lambda \mid \ell$ be a finite place of F_c, V a trait over λ, and y an integral frame on V with formal group $\widehat{G}$. There exist an algebraically closed complete valued field C containing $\operatorname{Frac} V$, with valuation v_ℓ normalised by $v_\ell(\ell) = 1$ and integer ring $\mathcal{O}_C$, and integral formal coordinates of $\widehat{G}$ over V from an integral formal biextension chart (meaning V-coordinates descending the formal group law, the $\mathcal{O}_E$-action with τ-eigenvectors $u_{\tau,i'}$, and the formal Poincaré biextension cocycle, symmetry, and adjointness data to V, in which the canonical biextension gauge h on $\widehat{G} \times \widehat{G}$ and $h_L(x) = h(x, [w_0]x)$ of Definition 35.53(d) have zero constant term and coefficients in $\operatorname{Frac} V$). Let (a, N) be a theta weight of degree D with gaps $c = (c_{\tau,i})$ and $N \geq \sum_{c_{\tau,i}\neq 0} 2i$, and let $P_{\mathrm{col}} \in \operatorname{harm}_V(a, N)$ be the image on V of the column differential of Lemma 35.57(b) formed from matrices $B_{\tau,i}$. Put $\mathcal{C} = \{(\tau, i, j) : c_{\tau,i} \neq 0,\ 1 \leq j \leq i\}$. In these V-coordinates over C, set*

$$\mathcal{D}_c(F) = \Big(\prod_{c_{\tau,i}\neq 0} \det\big(\partial/\partial t_{\tau,i;a,j}\big)_{a,j\leq i}^{c_{\tau,i}} \Big) F\Big(\Big(\exp\Big(\textstyle\sum_{a=1}^{i} t_{\tau,i;a,j} u_{\tau,a} \Big) \Big)_{(\tau,i,j)\in\mathcal{C}} \Big) \Big|_{t=0}$$

for an integral power series F over $\mathcal{O}_C$ on $\widehat{G}^{\mathcal{C}}$, and set

$$\mathcal{N}_k(G) = \big[P_{\mathrm{col}}(D)\big(G\, e^{k \sum_{b=1}^{N} h_L(x_b)} \big) \big](0)$$

for $k \geq 0$ and an integral power series G over $\mathcal{O}_C$ on $\widehat{G}^N$. Let $w_0 \in \mathcal{O}_E$ be as in Definition 35.53(d).

(i) For each $k \geq 0$ and each integral power series G over $\mathcal{O}_C$ on $\widehat{G}^N$, there exists $K \geq 0$ with the following property. Whenever $n' \geq 0$ and elements $x_{\alpha,b} \in \mathcal{O}_E$ ($\alpha = (\tau, i, j) \in \mathcal{C}$, $1 \leq b \leq N$) satisfy $v_\ell((B_{\tau,i})_{j,b} - \tau(x_{\alpha,b})) \geq n'$ and $v_\ell\big(\tau'\big(w_0 \sum_{b=1}^{N} \bar{x}_{\alpha,b} x_{\alpha',b}\big)\big) \geq n'$ for all $\alpha, \alpha' \in \mathcal{C}$ with $\alpha' = (\tau', i', j')$, there is an integral power series F over $\mathcal{O}_C$ on $\widehat{G}^{\mathcal{C}}$ such that $v_\ell(\mathcal{N}_k(G) - \mathcal{D}_c(F)) \geq n' - K$.

(ii) For each $k \geq 0$ and each integral power series F over $\mathcal{O}_C$ on $\widehat{G}^{\mathcal{C}}$, there is an integral power series G over $\mathcal{O}_C$ on $\widehat{G}^N$ such that $\mathcal{N}_k(G) = \mathcal{D}_c(F)$.

Outline. For (i), twist a pullback of G by a biextension unit and compare exponential jets via v_ℓ-Lipschitz continuity. For (ii), build ψ, ρ from Lemma 35.57(b) via Lemmas 35.61, 35.64, and 35.65, then apply Lemma 35.62. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1222. □

Lemma 35.67 (Local depth of a column differential)**.** *In the setting of Proposition 35.66, let $\lambda \mid \ell$ be a finite place of F_c, V a trait over λ, and y an integral frame on V. Let (a, N) be a theta weight of degree D with gaps $c = (c_{\tau,i})$ and $N \geq \sum_{c_{\tau,i}\neq 0} 2i$, and let $P_{\mathrm{col}} \in \operatorname{harm}_V(a, N)$ be the image on V of the column differential of Lemma 35.57(b). Then $\delta_k(P_{\mathrm{col}}; y) = \nu_c(y)$ for every $k \geq 0$.*

Outline. Reduces $\delta_k(P_{\mathrm{col}}; y)$ to integral series in the V-coordinates of 35.66, then matches $\inf_F v_\ell(\mathcal{D}_c(F))$ via 35.57(b) and 35.66. Product testing and monomial expansion decompose this across blocks via 35.54. The complete proof is on p. 1224. □

Lemma 35.68 (Conjugate contractions and denominator powers)**.** *Let T be a finite set with an involution $\tau \mapsto \bar{\tau}$. Let $N, r_\tau \in \mathbf{N}$, and put*

$$L = \{(\tau, i) : \tau \in T,\ 1 \leq i \leq r_\tau\}.$$

Let K be a field of characteristic zero, and let P be a polynomial over K in variables $X_{\ell,b}$, with $\ell \in L$ and $1 \leq b \leq N$. Suppose

$$\sum_{b=1}^{N} \frac{\partial^2 P}{\partial X_{(\tau,i),b}\, \partial X_{(\bar{\tau},j),b}} = 0 \qquad (\tau \in T,\ 1 \leq i \leq r_\tau,\ 1 \leq j \leq r_{\bar{\tau}}).$$

If $d_{\ell,\ell'} \in K$ vanishes unless $\ell = (\tau, i)$ and $\ell' = (\bar{\tau}, j)$, then

$$\sum_{\ell,\ell' \in L} d_{\ell,\ell'} \sum_{b=1}^{N} \frac{\partial^2 P}{\partial X_{\ell,b}\, \partial X_{\ell',b}} = 0.$$

Suppose every monomial of P has row degree $a_\ell \in \mathbf{N}$, and set $D = \sum_\ell a_\ell$. Let Π be a finite set, and let $c : T \to \Pi$ and $s : T \to \{+,-\}$ satisfy

$$c(\bar\tau) = c(\tau), \qquad s(\bar\tau) \neq s(\tau).$$

For a monomial X^μ in the support of P, define

$$n^{\pm}_{q,b}(\mu) = \sum_{\substack{\ell=(\tau,i)\in L\\ c(\tau)=q,\ s(\tau)=\pm}} \mu_{\ell,b}, \qquad e(\mu) = \sum_{b=1}^{N}\sum_{q\in\Pi} \min(n^{+}_{q,b}(\mu), n^{-}_{q,b}(\mu)).$$

Then $e(\mu) \le \lfloor D/2 \rfloor$.

Let V be a discrete valuation ring with fraction field K and valuation v, let e_0 be a positive integer, and let $\eta \ge 0$ be real. Put

$$J_\eta = \{z \in K : v(z)/e_0 \ge -\eta\}, \qquad v(0) = +\infty.$$

This is a V-submodule of K containing 1. For $m \in \mathbf{N}$, let J_η^m be the V-submodule generated by products of m elements of J_η, with $J_\eta^0 = V$. Then

$$z \in J_\eta^m \quad\Longrightarrow\quad v(z)/e_0 \ge -m\eta.$$

In particular,

$$z \in J_\eta^{\lfloor D/2\rfloor} \quad\Longrightarrow\quad v(z)/e_0 \ge -\eta D/2.$$

Outline. Each summand vanishes by nonconjugacy or harmonicity, and $2\min(A,B) \le A+B$ gives $2e(\mu) \le D$. The ultrametric inequality makes J_η a V-submodule containing 1, valuation additivity bounds J_η^m, and $\lfloor D/2\rfloor \le D/2$ finishes. The complete proof is on p. 1224. □

Lemma 35.69 (Ordered evaluation from the connection identities)**.** *Let V be a discrete valuation ring of characteristic zero with fraction field K, let $\mathcal{F}$ be a commutative d-dimensional formal group law over V, and let $m \in \mathbf{N}$. Put*

$$\mathscr{A} = K[[x_1,\dots,x_d,z_1,\dots,z_d]],$$

let $\iota f = f(x)$, and let ct *denote constant coefficient. For $u \in K^d$, define*

$$D_u f = \sum_{i,j} u_i \frac{\partial \mathcal{F}_j}{\partial y_i}(x,0)\frac{\partial f}{\partial x_j}, \qquad \beta((u,\phi),(u',\phi')) = \sum_i (\phi_i' u_i - \phi_i u_i').$$

Suppose a family of K-linear operators ∇_ξ on $\mathscr{A}$, linear in $\xi \in K^d \oplus K^d$, satisfies

$$[\nabla_\xi, \nabla_\eta] = m\beta(\xi,\eta)\,\mathrm{id}, \qquad \nabla_{(0,\phi)}\iota f = 0,$$

and, for every finite sequence $u_1,\dots,u_r$,

$$\mathrm{ct}\big(\nabla_{(u_1,0)}\cdots\nabla_{(u_r,0)}\iota f\big) = \mathrm{ct}(D_{u_1}\cdots D_{u_r} f).$$

Let I, Π be finite sets, fix an ordering of I, and let $c : I \to \Pi$ and $s : I \to \{+,-\}$. Put $I_\pm = s^{-1}(\pm)$. For a monomial Y^α, define the ordered operator

$$O_\xi(Y^\alpha) = \left(\prod_{\ell\in I_+} \nabla_{\xi_\ell}^{\alpha_\ell}\right)\left(\prod_{\ell\in I_-} \nabla_{\xi_\ell}^{\alpha_\ell}\right),$$

where each product follows the fixed ordering, composition acts from right to left, and the minus block acts first. Extend O_ξ linearly to polynomials. For $P = \sum_\mu P_\mu X^\mu \in K[X_{\ell,b} : \ell \in I, 1 \le b \le N]$, set $(\mu_b)_\ell = \mu_{\ell,b}$ and

$$\mathcal{E}_\xi(P; g_1,\dots,g_N) = \sum_\mu P_\mu \prod_{b=1}^{N} \mathrm{ct}\big(O_\xi(Y^{\mu_b})\iota g_b\big).$$

Say two vector families A_ℓ, B_ℓ satisfy opposite-pair vanishing if $\beta(A_\ell, B_{\ell'}) = 0$ whenever $c(\ell) \neq c(\ell')$ or $s(\ell) = s(\ell')$.

(a) Let $u_\ell, \phi_\ell \in K^d$. Suppose the families $((u_\ell, 0), (0, \phi_\ell))$ and $((0, \phi_\ell), (u_\ell, 0))$ satisfy opposite-pair vanishing. Set

$$\kappa_{\ell\ell'} = \begin{cases} m\beta((0,\phi_\ell),(u_{\ell'},0)), & c(\ell) = c(\ell'),\ s(\ell) = +,\ s(\ell') = -, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\Delta = \sum_{b,\ell,\ell'} \kappa_{\ell\ell'} \partial_{X_{\ell,b}} \partial_{X_{\ell',b}}.$$

If $\Delta P = 0$, then

$$\mathcal{E}_{(u,\phi)}(P; g_1, \dots, g_N) = \left[P(D) \prod_b g_b(x_b) \right]_{x_1 = \cdots = x_N = 0},$$

where $X_{\ell,b}$ acts as D_{u_ℓ} in the b-th copy.

(b) Let $J \subset K$ be a V-submodule containing 1. Choose $\xi^0_\ell = (u_\ell, \eta_\ell)$ with $u_\ell \in V^d$, and $\psi_\ell \in J^d$. Suppose opposite-pair vanishing holds for

$$(\xi^0, \xi^0), \qquad ((0,\psi), \xi^0), \qquad (\xi^0, (0,\psi)).$$

Let $E_0 \in K[[x]]$ have constant term 1, and suppose

$$(\iota E_0)^{-1} \nabla_{\xi^0_\ell}\big((\iota E_0) G\big) \in V[[x,z]] \qquad (G \in V[[x,z]],\ \ell \in I).$$

If P has coefficients in V and every μ in its support satisfies

$$\sum_{b,q} \min\left(\sum_{\substack{c(\ell)=q \\ s(\ell)=+}} \mu_{\ell,b}, \sum_{\substack{c(\ell)=q \\ s(\ell)=-}} \mu_{\ell,b} \right) \leq e,$$

then, for $f_b \in V[[x]]$,

$$\mathcal{E}_{\xi^0 + (0,\psi)}(P; E_0 f_1, \dots, E_0 f_N) \in J^e.$$

Here J^e is the V-submodule generated by products of e elements of J, and $J^0 = V$.

Outline. Commuting vertical operators past the base ones expands each column by $\Delta_0^r / r!$. For (a), horizontal evaluation and $\Delta P = 0$ leave $r = 0$; for (b), integral contraction bounds and conjugation invariance place the value in J^e. The complete proof is on p. 1224. □

Lemma 35.70 (Formal connection identities in an integral chart)**.** *Let V have positive residue characteristic, and use an integral formal biextension chart and canonical gauge h as in Proposition 35.66. Let $\mathcal{F}$ be its formal group law. Put $K = \operatorname{Frac} V$, $d = \dim \widehat{G}$, and*

$$\sigma_i(x) = -\frac{\partial h}{\partial y_i}(x, 0), \qquad S_{ij} = [x_j]\sigma_i(x), \qquad H_V = \{(u, \phi - Su) : u, \phi \in V^d\}.$$

For $m \in \mathbf{N}$, let $E_0(x) = \exp(m h_L(x))$. There is a family ∇_ξ on $K[[x,z]]$ satisfying the linearity, commutator, vertical annihilation, and horizontal evaluation hypotheses of Lemma 35.69. It also satisfies

$$(\iota E_0)^{-1} \nabla_\xi\big((\iota E_0) G\big) \in V[[x,z]] \qquad (\xi \in H_V,\ G \in V[[x,z]]).$$

Fix such a family and use the ordered products of Lemma 35.69.

Outline. Taking $\nabla_\xi = \mathcal{V}_\xi - m\lambda_u$ with commuting vector fields yields the commutator and vertical identities. Intertwining $\nabla_{(u,0)}$ with D_u gives horizontal evaluation, and differentiating the gauge laws gives integral conjugation. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1225. □

Proposition 35.71 (Harmonic valuation lower bound in an integral chart)**.** *For the host field E of conductor $Q_S qd$ of 5.1 and 8.3, put $D_E = |\operatorname{disc} E|$ and $\delta_p = v_p(D_E)/[E:\mathbf{Q}]$, and let (a, N) be a theta weight of degree D. Let V be a trait over a finite place of F_c, with uniformiser ϖ and valuation v normalised by $v(\varpi) = 1$, let y be an integral frame on V, and take integral formal coordinates of $\widehat{G}$ over V from an integral formal biextension chart as in Proposition 35.66. For every pair of integers $e > 0$ and $k \geq 0$, every polynomial $R \in \operatorname{harm}_V(a, N)$, and every power series F on $\widehat{G}^N$ with coefficients in V, the value $z = [R(D)(F\, e^{k\sum_{b=1}^{N} h_L(x_b)})](0)$ satisfies:*

(1) $v(z) \geq 0$ when $\operatorname{disc} E$ is a unit in V; and

(2) $v(z)/e \geq -(v(p)/e)\,\delta_p\, D/2$ when p is a prime dividing $Q_S qd$ that is not a unit in V.

Outline. Truncating F to its degree-D jet reduces both parts to products in separate variables. In (1), gauge independence covers residue characteristic zero, and trace-dual idempotents with Lemma 35.68, Lemma 35.69, and Lemma 35.70 cover positive residue characteristic. In (2), ramified eigenlifts and the same lemmas bound $v(z)/e$ below by $-\lfloor D/2 \rfloor (v(p)/e)\delta_p$. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1226. □

Lemma 35.72 (Chart lower bound and global differential of controlled local depth)**.** *Let (a, N) be a theta weight of degree D with gaps $c = (c_{\tau,i})$, and use $D_E, \delta_p, Q_S qd$ and the chart data $V, y, \widehat{G}, e, k, R, F, z$ of Proposition 35.71.*

(1) For every such V, y, e, k, R, F,

$$\frac{v(z)}{e} \geq -\sum_{p \mid Q_S qd} \frac{v(p)}{e}\,\delta_p\,\frac{D}{2}.$$

(2) Let $\lambda \mid \ell$ be a finite place of F_c, and suppose $N \geq \sum_\tau \sum_{i=1}^{\mu_\tau} 2i$. Let $A \leq B$ be real numbers such that $A \leq \nu_c(y)$ for all integral frames y over λ and $\nu_c(y) \leq B$ for all algebraic integral frames y over λ. Then there is $P_1 \in \operatorname{harm}(a, N)$ such that $A \leq \delta_k(P_1; y) \leq B$ for all $k \geq 0$ and all algebraic integral frames y over λ. In particular, if c has oscillation at most ϱ at ℓ, then hypothesis (i) of Proposition 35.60 holds at λ.

Outline. For (1), uses 8.3 and 35.71 to sum ramified bounds over $p \mid Q_S qd$. For (2), approximates P_{col} modulo ℓ^{n_0} via 35.57(c) and applies (1), 35.66, 35.67, and 35.53(b) to preserve local depth. The complete proof is on p. 1229. □

Lemma 35.73 (Mixed translates and the chart lower bound)**.** *Let V be a complete discrete valuation ring with fraction field K of characteristic zero and valuation v. Fix an integral frame y, an integral formal chart with frames compatible under truncation, and a split PEL uniformisation with its torsion identification. Let L be the rigidified theta line and s_0 its principal section in this frame.*

For a section s of L^k and $\gamma \in \mathcal{O}_E$ with $\gamma\bar{\gamma} = C \in \mathbf{N}$, write $\mathcal{P}_{\gamma,k} s$ for its pullback by $[\gamma]$, viewed as a section of L^{kC} through the isomorphism compatible with the rigidification. At odd level k, for a point z killed by k, write $\mathcal{T}_{z,k} s$ for the symmetric theta-group translate of s. Its lift has k-th power equal to the identity and is compatible with inversion. These normalisations fix both operations.

Let (a, n) be a theta weight of degree D, and put

$$\nu_a(\gamma) = \prod_\tau \tau(\gamma)^{\sum_i a_{\tau,i}},$$

using the coefficient-field maps followed by the map to K. For a germ h on n copies of the formal group, put

$$\mathcal{H}_k(P; h) = \left[P(D)\left(h \exp\left(k \sum_{b=1}^{n} h_L(X_b)\right)\right)\right](0).$$

The derivations, directions and h_L are those of Proposition 35.66. For sections $S_1, \ldots, S_n$, write $\mathcal{H}_k(P; S_1, \ldots, S_n)$ for this value on the product of their germs in the compatible frames.

Let $e \in \mathbf{N}$ with $e \geq 1$ and $\rho \in \mathbf{R}$, and suppose that

$$v(\mathcal{H}_k(R; h))/e \geq -\rho$$

for every $k \geq 0$, every $R \in \mathrm{harm}_V(a,n)$, and every integral formal series h on the product chart. Let $m, n', A, B \in \mathbf{N}$ and $\alpha, \beta \in \mathcal{O}_E$ satisfy

$$m \text{ odd}, \qquad m = \beta\alpha n', \qquad \alpha\bar{\alpha} = A, \qquad \beta\bar{\beta} = B, \qquad n' \in V^\times.$$

For $1 \leq b \leq n$, let $z_{1,b}, z_{2,b}$ be K-points satisfying

$$[m^2]z_{1,b} = [m^2]z_{2,b} = 0, \qquad [n'^2\alpha\beta]z_{1,b} = 0, \qquad [n'^2 A\beta]z_{2,b} = 0,$$

and set

$$F_b = \mathcal{T}_{z_{2,b},m^2}\mathcal{T}_{z_{1,b},m^2}\mathcal{P}_{m,1}s_0.$$

Then $m^2 = n'^2 AB$, and there are sections T_b of $L^{n'^2 A}$, regular on the integral semiabelian model, such that

$$F_b = \mathcal{P}_{\beta,n'^2 A}T_b.$$

Each T_b has a unique integral formal germ f_b. For every $P \in \mathrm{harm}(a,n)$ whose image in $K[X]$ lies in $\mathrm{harm}_V(a,n)$,

$$\nu_a(\alpha)m^{-D}\mathcal{H}_{m^2}(P;F_1,\dots,F_n) = n'^{-D}\mathcal{H}_{n'^2 A}(P;T_1,\dots,T_n).$$

Consequently the valuation of the left side, divided by e, is at least $-\rho$.

Outline. Build T_b from s_0 by four pullbacks and translates satisfying $\mathcal{P}_{\beta,n'^2 A}T_b = F_b$; these preserve the integral theta lattice, giving integral germs f_b. The pullback chain rule, $\nu_a(m) = m^D$, and the chart bound conclude. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1229. □

Proposition 35.74 (Split-uniformisation lower bound for theta derivatives)**.** *At the base level $M = g_0 g_1$, let $\pi_0, \pi_1 \in \mathcal{O}_E$ be the half-level generators satisfying $\pi_0\bar{\pi}_0 = g_0$ and $\pi_1\bar{\pi}_1 = g_1$, with half-level ideal $\mathfrak{b} = (\pi_0\pi_1)$. Let (a,N) be a theta weight of degree D, and for $\beta \in \mathcal{O}_E$ put*

$$\nu_a(\beta) = \prod_{\tau:E\to\mathbf{C}} \tau(\beta)^{\sum_{i=1}^{\mu_\tau} a_{\tau,i}} \in \mathcal{O}_{F_c},$$

where $\tau : \mathcal{O}_E \to \mathcal{O}_{F_c}$ is the coefficient-field map. Write $\theta_{x,P}$ for the theta derivative of Lemma 35.59, and use $Q_S qd$ and δ_p as in Proposition 35.71. Let V' be a trait of ramification index e' over a finite place w of F_c with residue characteristic ℓ_0, on which an algebraic integral frame y' has a split uniformisation and rational M^2-torsion. Put $\alpha = 1$ if $\ell_0 \nmid M$, $\alpha = \bar{\pi}_0$ if $\ell_0 = g_0$, and $\alpha = \bar{\pi}_1$ if $\ell_0 = g_1$. Then every $P \in \mathrm{harm}(a,N)$ and every characteristic tuple x satisfy

$$\frac{v'\big(\nu_a(\alpha)\,\theta_{x,P}(y')\big)}{e'} \geq -\sum_{p\mid Q_S qd} \frac{v'(p)}{e'}\,\delta_p\,\frac{D}{2}.$$

Outline. Factor $M = \beta\alpha n'$ by whether $\ell_0 \mid M$ and write s_{x_b} as a $\mathfrak{b}^2$-torsion translate of $\mathcal{P}_{M,1}s_0$. Lemma 35.73 with Lemma 35.72(1), Proposition 35.66, and Lemma 35.59 gives the bound. The complete proof is on p. 1231. □

Lemma 35.75 (A finitely supported lower bound for theta derivatives)**.** *At the base level $M = g_0 g_1$, let $\pi_0, \pi_1 \in \mathcal{O}_E$ be the half-level generators satisfying $\pi_0\bar{\pi}_0 = g_0$ and $\pi_1\bar{\pi}_1 = g_1$, with half-level ideal $\mathfrak{b} = (\pi_0\pi_1)$. For a theta weight (a,N) of degree D and $\beta \in \mathcal{O}_E$, set*

$$\nu_a(\beta) = \prod_{\tau:E\to\mathbf{C}} \tau(\beta)^{\sum_{i=1}^{\mu_\tau} a_{\tau,i}} \in \mathcal{O}_{F_c}, \qquad \nu = \nu_a(\bar{\pi}_0\bar{\pi}_1),$$

where $\tau : \mathcal{O}_E \to \mathcal{O}_{F_c}$ is the coefficient-field map, and write $\theta_{x,P}$ for the theta derivative as in Lemma 35.59. With $Q_S qd$ and δ_p as in Lemma 35.72, assume $\gcd(M, Q_S qd) = 1$. For each finite place w of F_c, put $\rho(w) = \sum_{p\mid Q_S qd} v_w(p)\,\delta_p\,D/2$ and $m(w) = -(\rho(w) + v_w(\nu))$.

(1) The function m has finite support on the finite places of F_c.

(2) For every $P \in \mathrm{harm}(a, N)$, every characteristic tuple x, every finite place w of F_c, and every algebraic integral frame y on a trait V of ramification index e over w, one has $v(\theta_{x,P}(y))/e \geq m(w)$.

(3) Whenever $w \mid g_0$ or $w \mid g_1$, one has $m(w) = -v_w(\nu)$.

Outline. Gets (1) and (3) from finite support of $v_w(\nu), v_w(p)$ and $\gcd(M, Q_S qd) = 1$. For (2), extends V to a trait with split uniformization and rational M^2-torsion, applies 35.74, and uses $v_w(\nu_a(\alpha)) \leq v_w(\nu)$. The complete proof is on p. 1231. □

In the host and nonterminal-layer setting of Definition 35.53 and Lemma 35.48 (with 8.3), write $\eta = \sqrt{-d} \in \mathcal{O}_E$, let $\eta_0 = \phi_0(\eta) \in \mathbf{C}$ be its image under the distinguished host embedding $\phi_0 : E \hookrightarrow \mathbf{C}$, and put

$$K = \mathbf{Q}(\eta), \qquad \Psi = \{\tau : E \hookrightarrow \mathbf{C} : \tau(\eta) = \eta_0\}, \qquad 2f = [E : \mathbf{Q}],$$

so that Ψ is the CM type of the host and layer. For a signature (P, Q), put $r = P + Q$. As in Definition 35.53, the Lie multiplicity is $\mu_\tau = P$ for $\tau \in \Psi$ and $\mu_\tau = Q$ for $\tau \notin \Psi$. A block (τ, i) has integer size $1 \leq i \leq \mu_\tau$, and its exponent is a nonnegative integer. All block depths below use the normalization $v_\ell(\ell) = 1$.

Lemma 35.76 (Oscillation at split places)**.** *Use the host and nonterminal-layer data of Lemma 35.48, with*

$$(P, Q) = (k + j - 1, m - j), \qquad 1 \leq j < m, \qquad k \geq 2.$$

Call $\beta = (\tau, i)$ big when $\tau \in \Psi$ and $i \geq k - 1$, and small otherwise. Let $c = (c_\beta)$ be a tuple of nonnegative integer exponents, and let $U, V \in \mathbf{R}$ satisfy

$$\sum_{i=k-1}^{P} c_{\tau,i} \leq U \quad (\tau \in \Psi), \qquad \sum_{\beta=(\tau,i)\ \text{small}} i c_\beta \leq V.$$

Let ℓ be a prime.

(a) Let $\delta \geq 0$, and assume the following split-place condition. For every algebraically closed field C of characteristic zero equipped with a real-valued nonarchimedean valuation, with valuation ring O containing $\mathbf{Z}_\ell$ and with ℓ a nonunit, and every injective map

$$\tau_O : \mathrm{Hom}_{\mathbf{Q}}(E, \mathbf{C}) \longrightarrow \mathrm{Hom}_{\mathbf{Z}}(\mathcal{O}_E, O)$$

satisfying

$$\tau(\eta) = \tau'(\eta) \quad \Longleftrightarrow \quad \tau_O(\tau)(\eta) = \tau_O(\tau')(\eta),$$

where $\eta \in \mathcal{O}_E$ is the host root, so that $\eta^2 = -d$, there is a decomposition

$$\mathbf{Z}_\ell \otimes_{\mathbf{Z}} \mathcal{O}_E \simeq \prod_{s=1}^{u} (R_s \times \bar{R}_s)$$

into primitive factors paired by conjugation, oriented so that the embeddings through the R_s are precisely $\tau_O(\Psi)$, and those through the $\bar{R}_s$ are precisely $\tau_O(\Psi^c)$. Here an embedding through R_s means a map $\sigma : \mathcal{O}_E \to O$ whose extension to $\mathbf{Z}_\ell \otimes \mathcal{O}_E$ factors through R_s; each embedding belongs to exactly one factor. Put

$$n_s = \mathrm{rank}_{\mathbf{Z}_\ell} R_s, \qquad \delta_s = \frac{v_\ell(\mathrm{disc}(R_s/\mathbf{Z}_\ell))}{n_s}.$$

The discriminant here is the determinant of the trace-pairing matrix $\left(\mathrm{Tr}_{R_s/\mathbf{Z}_\ell}(e_a e_b)\right)_{a,b}$ for a $\mathbf{Z}_\ell$-basis (e_a) of R_s, and require that this decomposition satisfy

$$\sum_{s=1}^{u} n_s = f, \qquad \sum_{s=1}^{u} n_s \delta_s = f\delta.$$

Then, for every place $\lambda \mid \ell$ of F_c, there is an integer $N_\lambda \geq 0$ such that

$$N_\lambda \leq \nu_c(y)$$

for every integral frame over λ, and

$$\nu_c(y) \le N_\lambda + fmU(1+\delta/2) + V$$

for every algebraic integral frame over λ. Thus c has oscillation at most $fmU(1+\delta/2)+V$ at ℓ.

(b) If $\ell \nmid 2\operatorname{disc} E$ and ℓ splits in K, then for every place $\lambda \mid \ell$ of F_c, one has $0 \le \nu_c(y)$ at every integral frame over λ and

$$\nu_c(y) \le \frac{fQU+V}{\ell-1} \le \frac{fmU+V}{\ell-1}$$

at every algebraic integral frame over λ; in particular, c has oscillation at most $(fQU+V)/(\ell-1) \le (fmU+V)/(\ell-1)$ at ℓ.

Outline. Lemma 35.54 gives $\nu_c(y) \ge 0$, so $N_\lambda = 0$. For (a), an integral-chart colength bound yields characters with $\nu_{\tau,i}(n;y) \le nt_\tau$ and $\sum_\tau t_\tau \le fQ(1+\delta/2)$, and Lemma 35.54 bounds the small blocks. For (b), split-place characters give $\nu_{\tau,i}(n;y) \le nQ/(\ell-1)$ on big blocks, and Lemma 35.54 again bounds the small blocks. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1232. □

Lemma 35.77 (Frobenius orbits and character minors)**.** *Use the setting of Definition 35.53. Let p be a prime unramified in E, let y be an integral frame over a trait above p, and choose an integral formal chart G and an algebraically closed valued extension C as in that definition. Write O for its valuation ring, $|\cdot|$ for its multiplicative valuation, and*

$$v(x) = \frac{\log|x|}{\log|p|} \quad (x \ne 0), \qquad v(0) = +\infty.$$

Thus $v(p) = 1$. Put $\mu_\sigma = P$ for $\sigma \in \Psi$ and $\mu_\sigma = Q$ otherwise. For each embedding τ there are an integer $h > 0$ and distinct embeddings τ_s, indexed by $s \in \mathbf{Z}/h\mathbf{Z}$, such that $\tau_0 = \tau$ and

$$\tau_{s+1}(a) \equiv \tau_s(a)^p \pmod{\mathfrak{m}_O} \qquad (a \in \mathcal{O}_E).$$

Here embeddings into C are obtained from the coefficient embedding of the trait. If p splits in K, membership of τ_s in Ψ is constant along this orbit. If p is inert in K, membership changes at every step.

Suppose also that $p \ne 2$ and y is algebraic. Let $u_{\tau,1}, \dots, u_{\tau,\mu_\tau}$ be its full integral τ-eigenframe and set

$$R_\tau = \sum_{m \ge 0} p^{-m-1} \mu_{\overline{\tau_{m+1}}}.$$

For every $1 \le i \le \mu_\tau$, there are integral formal characters $\chi_1, \dots, \chi_i$ of G, meaning

$$\chi_b(0) = 1, \qquad \chi_b(G(X,Y)) = \chi_b(X)\chi_b(Y),$$

for which

$$v\Big(\det\big(d\chi_b(u_{\tau,a})\big)_{a,b=1}^{i}\Big) \le R_\tau.$$

Outline. Bijective reduction to $O/\mathfrak{m}_O$ lets Frobenius define the orbit τ_s. For odd p and algebraic y, uniformization and a colength bound give s characters of minor valuation at most R_τ; Laplace expansion extracts i columns. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1233. □

Lemma 35.78 (Block depth at an odd unramified prime)**.** *In the setting of Definition 35.53 with signature (P,Q) and CM type Ψ, let $\ell \nmid 2\operatorname{disc} E$ be a prime. Let $\tau \in \Psi$, let $1 \le i \le P$ be an integer, and let $n \ge 0$ be an integer. For every algebraic integral frame y over a place above ℓ,*

$$0 \le \nu_{\tau,i}(n;y) \le \begin{cases} \dfrac{nQ}{\ell-1}, & \ell \text{ splits in } K, \\ n\left(\dfrac{Q}{\ell-1} + \dfrac{(P-Q)\ell}{\ell^2-1}\right), & \ell \text{ is inert in } K. \end{cases}$$

The lower bound holds for every integral frame. The expression $P-Q$ in the second upper bound is subtraction in $\mathbf{R}$.

Outline. Applying $\mathcal{D}^n$ to $\prod_{b=1}^{i}\chi_b(X^{(b)})$ from Lemma 35.77 gives $\nu_{\tau,i}(n;y)\le nR_\tau$. Summing R_τ in the split and inert cases yields the upper bounds, while Lemma 35.54 gives nonnegativity. The complete proof is on p. 1236. □

Lemma 35.79 (Typed formal factors)**.** *Let p be a prime unramified in E, and let y, G, C, O, v be as in the first paragraph of Lemma 35.77. For an i-tuple $u=(u_1,\dots,u_i)$, write $\operatorname{depth}_G(\det D_u^n)$ for the infimum of the valuations obtained by applying $\det(D_{u_a}^{(b)})^n$ to integral series on i copies of G and evaluating at the identity.*

Fix τ and its Frobenius orbit $(\tau_s)_{s\in\mathbf{Z}/h\mathbf{Z}}$. There are a commutative integral formal group G' of dimension d', with logarithm $\log_{G'}$ having identity linear term, a type function

$$t:\{1,\dots,d'\}\to\mathbf{Z}/h\mathbf{Z},$$

and a linear projection Π on Lie coordinates with the following properties. For every $\zeta\in C$ satisfying $\zeta^{p^h-1}=1$, there is an integral formal endomorphism $[\zeta]$ of G', with zero constant term, such that

$$(\log_{G'}[\zeta](X))_e=\zeta^{p^{\widetilde{t(e)}}}(\log_{G'}X)_e,$$

where $\widetilde{t(e)}\in\{0,\dots,h-1\}$ represents $t(e)$. Moreover,

$$\#\{e:t(e)=s\}=\mu_{\tau_s}.$$

For every flag step in the τ_s-eigenspace, its projected frame $u'_a=\Pi u_{\tau_s,a}$ is integral, has a unit maximal minor, is supported in coordinates of type s, and satisfies, for every integer $n\ge 0$,

$$\operatorname{depth}_{G'}(\det D_{u'}^n)=\operatorname{depth}_G(\det D_u^n).$$

These assertions hold for every integral frame.

Outline. Take the formal factor G' for the orbit component of $\mathcal{O}_E\otimes\mathbf{Z}_p$, with Teichmüller multiplicities μ_{τ_s}. Eigencharacter coordinates give the logarithm law, and integral projection gives Π preserving saturation and depths. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1236. □

Lemma 35.80 (Lower depth on an even Frobenius orbit)**.** *In the setting of Definition 35.53 with signature (P,Q) and CM type Ψ, put $r=P+Q$, let $\ell\nmid 2\operatorname{disc}E$ be an odd prime inert in K, fix a place $\lambda\mid\ell$ of F_c, and let $\tau\in\Psi$, $1\le i\le P$, $n\ge 1$, and $L=\log_\ell n$. For every integral frame y and every chart used to define its block depth,*

$$\operatorname{depth}_G(\det D^n_{u_{\tau,1},\dots,u_{\tau,i}})\ge\left(i-Q\left(\frac{3}{2}+\frac{L}{2}\right)\right)\frac{n\ell}{\ell^2-1}-i^2(L+1).$$

Outline. Apply Lemma 35.79 and Lemma 35.77 to pass to a typed factor G' of even orbit length with odd-type multiplicity Q. Inductively rebase integral ℓ-typical curves to depth K with $n\le\ell^{2K+1}$, eliminate low-depth columns, and solve the logarithm system with weighted order bounds. The divided-power content bound and Lemma 35.79 then give the estimate. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1237. □

Lemma 35.81 (Fine block oscillation at an odd inert prime)**.** *In the setting of Definition 35.53 with signature (P,Q) and CM type Ψ, put $r=P+Q$, let $\ell\nmid 2\operatorname{disc}E$ be a prime inert in K, and fix a place $\lambda\mid\ell$ of F_c. Let $\tau\in\Psi$, let $1\le i\le P$ be an integer, and let $n\ge 1$ be an integer. Put*

$$L=\log_\ell n=\frac{\log n}{\log\ell},\qquad W=\frac{n(r-2Q-i)\ell}{\ell^2-1}+\frac{nQ(3+L/2)}{\ell-1}+i^2(L+1).$$

There are real numbers $a\le b$, depending on λ,τ,i,n, such that

$$a\le\nu_{\tau,i}(n;y)$$

for every integral frame over λ, and

$$\nu_{\tau,i}(n;y)\le b$$

for every algebraic integral frame over λ, with $b-a\le W$. The subtraction in $r-2Q-i$ is taken in $\mathbf{R}$.

Outline. Set a and b to the frame-independent lower and upper bounds from Lemma 35.80 and the inert case of Lemma 35.78. Direct algebraic comparison of $s = \ell/(\ell^2-1) \le t = 1/(\ell-1)$ verifies $a \le b$ and $b - a \le W$. The complete proof is on p. 1241. □

Lemma 35.82 (Divided-power determinant content)**.** *For a prime p and integers $i, n \ge 0$, put*

$$H_{i,n}(Z) = \det(Z_{ab})_{a,b=1}^{i\,\,n} = \sum_{\alpha} c_\alpha Z^\alpha \in \mathbf{Z}[Z_{ab}], \qquad \alpha! = \prod_{a,b} \alpha_{ab}!.$$

For any characteristic-zero valued field (C, v) with $v(p) = 1$, define its divided-power content by

$$\operatorname{cont}_v(H_{i,n}) = \min_{c_\alpha \neq 0} v(c_\alpha \alpha!).$$

This is the same finite number

$$\kappa_{p,i,n} = \min_{c_\alpha \neq 0} v_p(c_\alpha \alpha!)$$

for every such field and valuation. If G is a commutative integral formal group and u is an integral i-frame with a unit maximal minor, then

$$\operatorname{depth}_G(\det D_u^n) \le \kappa_{p,i,n}.$$

Outline. Nonempty support and $v(m) = v_p(m)$ for $m \in \mathbf{Z} \setminus \{0\}$ give $\operatorname{cont}_v(H_{i,n}) = \kappa_{p,i,n}$. Dualizing a unit minor of u gives a test monomial whose derivative at 0 is $c_\alpha \alpha!$, bounding the depth. The complete proof is on p. 1241. □

Lemma 35.83 (Two radii for a typed determinant)**.** *Let p be prime, let C be algebraically closed of characteristic zero, and let v be an additive valuation with $v(p) = 1$, with valuation ring O. Let G be a commutative integral formal group of dimension d with a logarithm having identity linear term. Suppose $h > 0$, a function $t : \{1, \dots, d\} \to \mathbf{Z}/h\mathbf{Z}$, and integral formal endomorphisms $[\zeta]$ satisfy the root-of-unity formula of Lemma 35.79. Let $u_1, \dots, u_i$ be an integral frame with a unit maximal minor, supported in type τ. If at most j coordinates have type $\tau + 1$, then for every integer $n \ge 0$,*

$$\kappa_{p,i,n} \le \operatorname{depth}_G(\det D_u^n) + \frac{2in}{p(p-1)} + \frac{2jn}{p-1}.$$

Outline. The proof builds an adapted typical-curve basis with at most j curves having $L_{r(a),1} \neq 0$ and bounds their logarithm valuations inductively. Weighted-order induction on pullback coordinates z_b with two radii and Lemma 35.82 yield the depth inequality. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1241. □

Lemma 35.84 (Block oscillation at an inert prime)**.** *In the setting of Definition 35.53 with signature (P, Q) and CM type Ψ, let ℓ be a prime unramified in E and inert in K, and fix a place $\lambda \mid \ell$ of F_c. Let $\tau \in \Psi$, let $1 \le i \le P$ be an integer, and let $n \ge 0$ be an integer. There are real numbers $a \le b$, depending on λ, τ, i, n, with*

$$a \le \nu_{\tau,i}(n; y)$$

for every integral frame over λ, and

$$\nu_{\tau,i}(n; y) \le b$$

for every algebraic integral frame over λ, such that

$$b - a \le \frac{2in}{\ell(\ell-1)} + \frac{2Qn}{\ell-1}.$$

If k, m are nonnegative integers satisfying

$$i + 1 \le k + m, \qquad k \le m\ell, \qquad Q \le m,$$

the same endpoints satisfy

$$b - a \le \frac{5mn}{\ell - 1}.$$

For $n = 0$, one may take $a = b = 0$.

Outline. Lemma 35.54 handles $n = 0$; otherwise set $b = \kappa_{\ell,i,n}$ and $a = b - W_0$. Lemma 35.82 gives the upper bound, Lemmas 35.79 and 35.83 ($j = Q$) give the lower bound, and the size bounds yield $5mn/(\ell - 1)$. The complete proof is on p. 1243. □

Proposition 35.85 (The four local formal-group oscillation bounds)**.** *Use a nonterminal layer of signature $(P, Q) = (k + j - 1, m - j)$ with $1 \le j < m$ and $k \ge 2$, and the quantities $r, J, U, V, A, \delta_\ell, h_\ell$ of Lemma 35.48. Call a block (σ, i) big if σ lies in the CM type and $i \ge k - 1$, and small otherwise. Let $c = (c_{\sigma,i})$ be a gap tuple satisfying*

$$\sum_{i=k-1}^{P} c_{\tau,i} \le U \quad (\tau \text{ in the CM type}), \qquad \sum_{\text{small } (\sigma,i)} i\, c_{\sigma,i} \le V, \qquad \max_{\sigma,i} c_{\sigma,i} \le J.$$

Put $K = \mathbf{Q}(\sqrt{-d})$. Then at every prime $\ell \le J$:

(1) if $\ell \mid Q_S$ and $\mathbf{Z}_\ell \otimes_{\mathbf{Z}} \mathcal{O}_E$ decomposes into conjugate pairs $(R_s, \bar{R}_s)$ of primitive factors whose $\mathbf{Z}_\ell$-ranks n_s sum to f and whose local discriminant valuations $v_\ell(\mathrm{disc}(R_s/\mathbf{Z}_\ell))$ sum to $f\delta_\ell$, then c has oscillation at most $A(1 + \delta_\ell/2) + V$ at ℓ;

(2) if $\ell \nmid 2 \operatorname{disc} E$ splits in K, then $\nu_\beta(c_\beta; y) \le c_\beta Q/(\ell - 1)$ for every big block β and every algebraic integral frame y over a place above ℓ;

(3) if $\ell \nmid 2 \operatorname{disc} E$ is inert in K, then at each place $\lambda \mid \ell$ every big block β with $c_\beta \ge 1$ admits $A_\beta \le B_\beta$ as in Lemma 35.55 with

$$\sum_{\beta \text{ big},\, c_\beta \ge 1} (B_\beta - A_\beta) \le \frac{A(4 + h_\ell/2)}{\ell - 1} + f(m + 1)r^2(h_\ell + 1);$$

(4) if $\ell \nmid 2 \operatorname{disc} E$ is inert in K and $m\ell \ge k$, then at each place $\lambda \mid \ell$ every big block β admits $A_\beta \le B_\beta$ as in Lemma 35.55 with $B_\beta - A_\beta \le 5mc_\beta/(\ell - 1)$.

Outline. Part (1) orients conjugate pairs by an idempotent and Lemma 35.76. Parts (2)–(4) use Lemmas 35.78, 35.81, 35.84, and 35.54. The complete proof is on p. 1244. □

Lemma 35.86 (Exponent bounds and the four oscillation regimes)**.** *Use a nonterminal layer of signature $(P, Q) = (k + j - 1, m - j)$ with $1 \le j < m$ and $k \ge 2$, the quantities $T, r, S, B_0, J, U, V, A, \delta_\ell, h_\ell, e_\ell$ of Lemma 35.48, and the theta weight (a, N) constructed from u_1 in Lemma 35.46, with gaps $c = (c_{\tau,i})$ and degree D. Call a block (σ, i) big if σ lies in the CM type and $i \ge k - 1$, and small otherwise.*

(1) Suppose that, with zero-based column indices on u_1, each row $u_{1,\tau}$ is nonincreasing and satisfies $u_{1,\tau,k-2} = 0$, $|u_{1,\tau,s}| \le B_0$ for $0 \le s < r$, $u_{1,\tau,P-1} - u_{1,\tau,P} \ge N + 1$, and $\sum_{0 \le s < k-2} u_{1,\tau,s} \le k(S + 2r)/T$. Then

$$\sum_{i=k-1}^{P} c_{\tau,i} \le U \quad (\tau \text{ in the CM type}), \qquad \sum_{\text{small } (\sigma,i)} i\, c_{\sigma,i} \le V, \qquad \max_{\sigma,i} c_{\sigma,i} \le J,$$

and $D \le frU + V$.

(2) Assume the bounds of (1), and assume that $2 \mid Q_S$ and every prime $\ell \mid Q_S$ splits the quadratic subfield $K = \mathbf{Q}(\sqrt{-d})$ over $\mathbf{Z}_\ell$. Then every prime $\ell \mid Q_S$ decomposes $\mathbf{Z}_\ell \otimes_{\mathbf{Z}} \mathcal{O}_E$ into conjugate pairs $(R_s, \bar{R}_s)$ of primitive factors whose $\mathbf{Z}_\ell$-ranks $n_s = \mathrm{rk}_{\mathbf{Z}_\ell}(R_s)$ sum to f and whose rank-weighted local discriminant depths $n_s \cdot (v_\ell(\mathrm{disc}(R_s/\mathbf{Z}_\ell))/n_s) = v_\ell(\mathrm{disc}(R_s/\mathbf{Z}_\ell))$ sum to $f\delta_\ell$, and every prime $\ell \nmid Q_S$ outside $\{q, d\}$ is odd and does not divide $\operatorname{disc} E$. Moreover, c has oscillation at most e_ℓ at every prime ℓ.

Outline. For (1), telescopes 35.46 and sums by parts to bound big and small gaps and D. For (2), splits $\mathbf{Z}_\ell \otimes_{\mathbf{Z}} \mathcal{O}_E$ via 8.3 and K-idempotents, then combines 35.54, 35.55, and 35.85 across all prime regimes. The complete proof is on p. 1245. □

Lemma 35.87 (Harmonic minors on two column sets)**.** *Let $\mathcal{E} = \mathrm{Hom}_{\mathbf{Q}}(E, \mathbf{C})$, with conjugation $\tau \mapsto \bar{\tau}$, and use the multiplicities μ_τ of Definition 35.53. Let $a_{\tau,i}$, $1 \le i \le \mu_\tau$, be nonnegative integers that are nonincreasing in i. Put $a_{\tau,\mu_\tau+1} = 0$ and*

$$c_{\tau,i} = a_{\tau,i} - a_{\tau,i+1}, \qquad D = \sum_{\tau\in\mathcal{E}} \sum_{i=1}^{\mu_\tau} a_{\tau,i}.$$

Let $S \subset \mathcal{E}$ contain exactly one member of each conjugate pair. Choose $n_1, n_2 \in \mathbf{N}$ such that $\mu_\tau \le n_1, n_2$ for every τ, with $N = n_1 + n_2$ even. In the variables $X_{\tau,i,b}$, $1 \le b \le N$, define

$$\Delta^{(1)}_{\tau,j} = \det(X_{\tau,i,b})_{1\le i,b\le j}, \qquad \Delta^{(2)}_{\tau,j} = \det(X_{\tau,i,n_1+b})_{1\le i,b\le j},$$

and

$$P_S = \prod_{\tau\in S} \prod_{j=1}^{\mu_\tau} (\Delta^{(1)}_{\tau,j})^{c_{\tau,j}} \prod_{\tau\notin S} \prod_{j=1}^{\mu_\tau} (\Delta^{(2)}_{\tau,j})^{c_{\tau,j}}.$$

Then P_S has integer coefficients and belongs to the harmonic polynomial space $\mathrm{harm}(a, N)$ of Definition 35.53(c). It has degree D and row degree $a_{\tau,i}$. Its displayed factorisation is preserved by every change of coefficient ring.

Outline. Telescoping $c_{\tau,j}$ gives row degrees $a_{\tau,i}$, degree D, and row invariance of Definition 35.53(c). Disjoint columns for conjugate types kill the mixed second derivatives, and determinants commute with ring maps. The complete proof is on p. 1246. □

Lemma 35.88 (A theta family with a controlled harmonic value)**.** *Use the base level $M = gg'$, where g, g' are distinct odd primes, and half-level generators π, π' satisfying*

$$\pi\bar{\pi} = g, \qquad \pi'\bar{\pi}' = g', \qquad (\pi) + (\bar{\pi}) = (\pi') + (\bar{\pi}') = \mathcal{O}_E.$$

Assume that g splits completely in E. Let y be an algebraic integral frame over a trait V of residue characteristic g, with an integral chart compatible under truncation and a split PEL uniformisation with its torsion identification. Write X for the period lattice and Q_X for its cubical-period form. Let $b_1, \dots, b_s$ be a basis such that

$$Q_X(b_j) \le \Phi, \qquad |Q_X(b_j + z) - Q_X(b_j) - Q_X(z)| \le 2\sqrt{s}\, Q_X(z) \quad (z \in X), \qquad \Phi \ge 0.$$

Let F be an algebraically closed valued extension of $\mathrm{Frac}\, V$, of characteristic zero, whose additive valuation

$$v_F : F \to \mathbf{Q} \cup \{+\infty\}$$

extends v, whose valuation ring $\mathcal{O}_F$ is g-adically complete, and over which the PEL Tate module is identified with the fixed PEL datum, including the Weil pairings.

Let (a, N) be a theta weight of degree D, and write $N = n_1 + n_2 \ge 1$, with $\mu_\tau \le n_1, n_2$ for all τ. Put

$$S = \{\tau : v_F(\tau(\bar{\pi})) > 0\},$$

using the coefficient-field maps followed by $\mathrm{Frac}\, V \to F$. Let P_S be the polynomial of Lemma 35.87, with these two column sets. Assume

$$D < g - 1, \qquad 4D + 4 \le g', \qquad 2D\sqrt{s} < g'.$$

For a level-k theta family s on the product uniformisation, write $s \in \mathscr{T}_k^+$ when its coefficient at every $\chi \in X^N$, measured in integral frames on the abelian part, has valuation at least

$$\frac{Q_N(\chi)}{k}, \qquad Q_N(\chi) = \sum_{b=1}^{N} Q_X(\chi_b).$$

This is the theta lattice of Lemma 35.50.

There exist a family $s \in \mathscr{T}_{g'}^{+}$, its normalised pullback s_2 by $[\bar\pi]$ of level M, and a finite-order theta-group translate t of s_2 at a point of $\ker[\pi]^N$, whose germ h in the product integral frame satisfies

$$\rho := v_F(\mathcal{H}_M(P_S; h)) \in \mathbf{Q}, \qquad \rho \le \frac{v(g)D}{g-1} + \frac{n_1 s\Phi}{4g'} + \frac{2D^2\Phi}{g'}.$$

Here a finite-order theta-group translate means the action obtained from translation and a nonzero scalar for which some positive iterate acts as the identity on theta families. The notation $\mathcal{H}_M$ is that of Lemma 35.73.

Outline. Use complete splitting and multiplicative chart series u_j to build finite flat characters on $\ker[\bar\pi]$ with leading minor valuations bounded via $(g-1)v_F(c) = v_F(g)$. Construct eigen-column theta families for the first n_1 columns and jet families for the rest, then factor P_S to bound the column value. Taking external products and applying Lemma 35.87 transfers this bound to $\mathcal{H}_M(P_S; h)$. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1247. □

Lemma 35.89 (Integral expansion and the theta normaliser)**.** *Retain the field, frame, split uniformisation and half-level data of Lemma 35.88, with the base half-level ideal $\mathfrak{b} = (\pi\pi')$, and set*

$$\alpha = \bar\pi\bar\pi', \qquad \alpha\bar\alpha = M.$$

Let $s \in \mathscr{T}_{g'}^{+}$, let s_2 be its normalised $[\bar\pi]$-pullback, and let h be the compatible product germ of a finite-order translate of s_2 at a point killed by $[\pi]$ in every factor. Then there are a finite number q of scalars $a_i \in \mathcal{O}_F$, a principal theta family $\vartheta \in \mathscr{T}_1^{+}$ whose coefficient at period index 0 is the base change of an integral section not divisible by a uniformiser of V, and germs h_i such that

$$h = \sum_{i=1}^{q} a_i h_i.$$

Each h_i is the compatible germ of a finite-order translate of the normalised $[\alpha]$-pullback of ϑ, at a point killed by $[\bar\alpha] = [\pi\pi']$ in every factor. For each i there are a characteristic tuple x_i and a scalar $u_i \in \mathcal{O}_F^{\times}$ such that, for every $P \in \mathrm{harm}(a, N)$,

$$\mathcal{H}_M(P; h_i) = u_i\, \nu_a(\alpha)\, \theta_{x_i,P}(y).$$

In particular, if $\rho = v_F(\mathcal{H}_M(P; h)) < \infty$, some i satisfies

$$v(\theta_{x_i,P}(y)) + v(\nu_a(\alpha)) \le \rho.$$

Outline. Integral generation expands $h = \sum_i a_i h_i$ ($a_i \in \mathcal{O}_F$) via translates of the $[\alpha]$-pullback of ϑ. Frame splitting and the pullback chain rule give $u_i\nu_a(\alpha)\theta_{x_i,P}(y)$, and the ultrametric inequality concludes. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1250. □

Proposition 35.90 (Split-uniformisation upper bound at the level primes)**.** *Use the base level $M = g_0 g_1$, whose distinct odd primes g_0, g_1 split completely in E, and the half-level generators $\pi_0, \pi_1 \in \mathcal{O}_E$ satisfying $\mathfrak{b} + \bar{\mathfrak{b}} = \mathcal{O}_E$ as in Proposition 35.74. Let (a, N) be a theta weight of degree D with $N \ge 2r$, where $r = P + Q$, and put $\nu = \nu_a(\bar\pi_0\bar\pi_1)$. Take $(g, g') = (g_0, g_1)$ or (g_1, g_0), and let V' be a trait over a place $\lambda \mid g$ of F_c on which an algebraic integral frame y' has a split uniformisation with period lattice of rank s and short basis as in Lemma 35.49 of parameter Φ'. If $D < g-1$, $4D+4 \le g'$, and $2D\sqrt{s} < g'$, then there exist $P_1 \in \mathrm{harm}(a, N)$ and a characteristic tuple x satisfying*

$$v'\big(\theta_{x,P_1}(y')\big) + v'(\nu) \le \frac{v'(g)\,D}{g-1} + \frac{rs\Phi'}{4g'} + \frac{2D^2\Phi'}{g'}.$$

Outline. Over an algebraically closed g-adically complete extension F with a PEL Tate isometry, set $n_1 = r$ and form $P_1 = P_S$ by Lemma 35.87. Lemma 35.88 gives a germ h with bounded ρ, and Lemma 35.89 yields x. The complete proof is on p. 1251. □

Proposition 35.91 (Reference depth at the level primes). *Let (a,N) be a theta weight of degree D with $N \ge 2r$ and $a_{\tau,P} > 0$ for all $\tau \in \Psi$. Let $c \ge 0$ satisfy, for $g \in \{g_0, g_1\}$,*

$$4D+4 \le g, \qquad 2D\sqrt{2fm} < g, \qquad \kappa(2fm)^3\Big(\frac{r(2fm)}{4} + 2D^2\Big) \le c\,g,$$

with κ from Lemma 35.49, and g_0, g_1 distinct odd primes splitting completely in E and coprime to $Q_S qd$, with conjugate-coprime half-level generators $\pi_0, \pi_1 \in \mathcal{O}_E$ ($\mathfrak{b} + \bar{\mathfrak{b}} = \mathcal{O}_E$). Then $d_g^c(a,N) \le 1$ for $g = g_0, g_1$.

Outline. Passes to a split-uniformization extension V' and applies 35.49 and 35.90 to get a cusp section θ_{x,P_1} via 35.40. Clears denominators using 35.75 and 35.56, then takes the supremum in 35.53(e). The complete proof is on p. 1252. □

Proposition 35.92 (Weight-change inequality for reference depths). *Fix eligible F_m, C_H and assume the packages $\mathcal{H}_P, \mathcal{H}_H$, deep goodness, and flat-degree conditions of Theorem 35.11 and Definition 35.3, together with the size and good-cover conditions of Lemmas 35.20 and 35.21. At an admissible point beyond the common family threshold, at a nonterminal layer take a good cover Y of level M' as in Lemma 35.21, a quotient weight u_1, and the theta weight (a,N) of Lemma 35.46. Let $\mathcal{B}$ be the set of primes dividing $2dD_E M'$ and put $\delta_\ell = v_\ell(D_E)/(2f)$ as in Lemma 35.48. Then on Y, for every prime ℓ,*

$$d_\ell^{1/2500}(u_1) \le d_\ell^{1/2500}(a,N) + \mathbf{1}_{\ell\in\mathcal{B}}\big(1 + \tfrac{N}{2} f r\,\delta_\ell\big).$$

Outline. Set $m = N/2$ via Lemma 35.46 and multiply the flat section for $(-s_\tau)_\tau$ by the $(-m)$-th power of the determinant twisting section on each component. These reciprocal sections have integral lower bounds off $\mathcal{B}$ and spread at most $mfr\,\delta_\ell e_w$. Multiplying cusp sections by them and clearing denominators via Lemma 35.56 gives the bound. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1252. □

Proposition 35.93 (Combining the primes and changing to the quotient weight). *Fix eligible F_m, C_H and assume the packages $\mathcal{H}_P, \mathcal{H}_H$, deep goodness, and flat-degree conditions of Theorem 35.11 and Definition 35.3, together with the size and good-cover conditions of Lemmas 35.20 and 35.21. At an admissible point beyond the common threshold of Lemmas 35.20, 35.46, and 35.48 and Proposition 35.92, use a nonterminal layer, good cover Y of level M' as in Lemma 35.21, quotient weight u_1, theta weight (a,N) of Lemma 35.46 of degree D, and $L, T, J, e_\ell, a(\ell)$, bad-prime set $\mathcal{B}$, and constant C_R of Lemma 35.48.*

(1) Every prime ℓ satisfies

$$d_\ell^{1/2500}(a,N) \le e_\ell + \mathbf{1}_{\ell\in\{g_0,g_1\}}$$

both on the base moduli and on Y.

(2) Put $\mathcal{S}_{\mathrm{ref}} = \{\ell \text{ prime} : \ell \le J\} \cup \{g_0,g_1\} \cup \mathcal{B}$. Then $a(\ell) \ge 0$ for every prime ℓ, $a(\ell) = 0$ for $\ell \notin \mathcal{S}_{\mathrm{ref}}$, $d_\ell^{1/2500}(u_1) \le a(\ell)$ for every prime ℓ on Y, and $T\sum_{\ell\in\mathcal{S}_{\mathrm{ref}}} a(\ell)\log\ell \le C_R E_p$.

Outline. Bounds $d_\ell^{1/2500}(a,N)$ by 35.91 at g_0, g_1 and by 35.86, 35.72(2), 35.75, and 35.60 elsewhere using 35.46 and 8.3. Transfers to Y via 35.58, then combines 35.92 and 35.48 for (2). The complete proof is on p. 1254. □

Proposition 35.94 (Heights of the covering components). *Assume the self-intersection and section-injection comparisons and the packages $\mathcal{H}_X, \mathcal{H}_P, \mathcal{H}_H$ of 35.11, Grothendieck deformation, 35.26, and uniform deep goodness, size, cover degree and flat degree. Fix eligible F_m, C_H. There exist $C_{\mathrm{ht}} \ge 0$ and x_0 such that for every admissible point with $x \ge x_0$, every nonterminal layer, good cover Y and quotient weight u_1, there are $c_s, c_c, X \ge 0$ and real s_Z satisfying*

$$h_Z + c_s \ge 0, \qquad (1-\rho)\max(0, s_Z + c_s) \le h_Z + c_s + c_c, \qquad \frac{1}{\#\,\mathrm{Comp}(Y)}\sum_Z s_Z \ge -X \tag{8.28c}$$

and

$$T_*X,\ T_*c_c \le C_{\mathrm{ht}} E_p, \qquad c_s\big(6d_{\mathrm{fl}} + 10 + T_*(z+2\rho)\big) \le C_{\mathrm{ht}} E_p. \tag{8.28d}$$

Outline. Builds short generators via 35.36 for $h_Z + c_s \ge 0$ and a carrying section θ_Z via 35.46, 35.47, and 35.93(2). Maps small sections of u' from 35.26 into u_1 and compares volumes via the section-injection premise and Definition 35.5. The complete proof is on p. 1254. □

35.3 From component averages to the core lattice

Proposition 35.95 (Cover and core slopes)**.** *Under the uniform goodness, size and component estimates of 35.35 and 35.94,*

$$\mu_Y(u_M) \geq -C_U E_p, \qquad C_U = 2(3C_{\mathrm{ht}} + C_{\mathrm{HS}} + 4). \tag{8.29d}$$

Under the geometric hypotheses of 35.11 and the family conditions above, the core type lattice satisfies, eventually at every admissible point and layer,

$$\mu_{\mathrm{core}}(p,i) \geq -M_{p,i} - C_{\mathrm{core}} E_p, \tag{8.29e}$$

with

$$C_{\mathrm{core}} = \max\{C_{\mathrm{term}},\, 2(C_F + C_U) + 9\}, \tag{8.29f}$$

where $C_F = 1183$ and C_{term} is the terminal constant below.

Outline. Turn the mean height of 35.94 into a cover slope by degree and rank bounds. Descend via 35.4 and 35.20, using a translation determinant to inject into zero-shift lattices at nonterminal layers and the definite Gram comparison at the terminal layer. The complete proof is on p. 1258. □

35.4 Removing the bad part

In the rational realisation let $\Lambda_{\mathrm{type}} \subseteq \Lambda_{\mathrm{all}}$ be the type and ambient lattices. For the good projector e set

$$\begin{gathered} \Lambda_{\mathrm{good}} = \Lambda_{\mathrm{type}} \cap \operatorname{im} e, \qquad \Lambda_{\mathrm{bad}} = \Lambda_{\mathrm{type}} \cap \ker e, \\ \delta = \exp(-x^{e_z/7}), \quad e_z = \varepsilon/J, \qquad J \geq 1 \text{ the family exponent parameter.} \end{gathered} \tag{8.30a}$$

Proposition 35.96 (Rank of the bad lattice)**.** *Assume the geometric and section comparisons of 35.11, the cuspidal Green parametrisation and the uniform Gram, cover, size, flat degree and share conditions above. For each eligible F_m, C_H, eventually at every admissible point and layer,*

$$\operatorname{rk} \Lambda_{\mathrm{bad}} \leq 3\delta \operatorname{rk} \Lambda_{\mathrm{type}}. \tag{8.30b}$$

Outline. Use 32.53 and 32.60 to bound bad label shares. Orbit separation from 35.95 embeds nonterminal cover spaces in regular copies, and terminal summands already supply such copies, so projector traces bound the common good rank loss. The complete proof is on p. 1262. □

Lemma 35.97 (Slope loss under orthogonal truncation)**.** *Over a totally complex number field K, let V have Hermitian norms N_σ with a Gram frame. Take a finitely generated $\mathcal{O}_K$-lattice $M \subset V$, $0 \neq L \subseteq M$, and an idempotent e with*

$$N_\sigma(ex + y - ey)^2 = N_\sigma(ex)^2 + N_\sigma(y - ey)^2 \qquad (\sigma, x, y). \tag{8.31a}$$

Suppose $g \geq 1$ is integral, $ge(L) \subseteq L$, $0 \leq \delta \leq 1/12$, and

$$\operatorname{rk}(L \cap \ker e) \leq 3\delta \operatorname{rk} L, \qquad \mu_{\max}(M),\ -\mu(L),\ 2\log g \leq \mathcal{Q}. \tag{8.31b}$$

Then

$$\mu(L \cap \operatorname{im} e) \geq \mu(L) - 12\delta\mathcal{Q}. \tag{8.31c}$$

Outline. With $L_g = L \cap \operatorname{im} e$, $L_b = L \cap \ker e$, $gL \subseteq L_g \oplus L_b$ gives $\operatorname{rk} L = d_g + d_b$ and $1 - e$ embeds $L/(L_g \oplus L_b)$ in $g^{-1}L_b/L_b$. Orthogonality and $d_b \leq 4\delta d_g$ bound $\mu(L) - \mu(L_g)$ by $12\delta\mathcal{Q}$. The complete proof is on p. 1263. □

35.5 The layer inequality and the common constant

Give the lattices the rational-realisation norm. In the recursive comparison $d_W(i) = D_{\mathrm{norm}} > 0$ by the primitive row equality. In the selected basis the frame comparison carries the good norm to the recursive row norm with factor $1/\sqrt{n_P} \le 1$, where n_P is the positive row multiplicity. The primitive lattice inclusion gives the determinant comparison

$$\Lambda_{\mathrm{all}} \text{ finitely generated}, \qquad \Lambda_{\mathrm{good}} \neq 0, \qquad \frac{\mathrm{lcv}(X_{W,i}, \Lambda_i)}{D_{\mathrm{norm}}} \le -\mu(\Lambda_{\mathrm{good}}). \tag{8.32a}$$

Let G be the finite level group acting on the realised row lattice, and write its good projector as $e = \sum_{h \in G} a_h h$. The tied coefficients satisfy

$$\sum_{h \in G} a_h a_{h^{-1}k} = a_k, \qquad \sigma(a_{h^{-1}}) = \overline{\sigma(a_h)}, \qquad |G| a_h \in \mathcal{O}_K.$$

The action preserves the lattice and the inner product. The first two identities give idempotence and self-adjointness, hence (8.31a). Set $g = |G|$, which clears the denominators on Λ_{type}. The group-order bound gives $2 \log g \le b$.

By 35.20 and 35.22, eventually there are family exponents A, B with

$$\mu_{\max}(\Lambda_{\mathrm{all}}) \le x^A b, \qquad 1 \le b \le E_p \le x^B b, \qquad |M_{p,i}| \le x^B b. \tag{8.32b}$$

Let K be the realisation field. The type lattice Λ_{type} is the $\mathcal{O}_K$-span of the integral row forms lying in the type subspace. Under the semilinear identification these forms are exactly the images of the core type lattice. Field extension preserves the slope in the normalised core norm. The norm change in 35.22 costs at most $2C_{\mathrm{meas}}E_p$ in slope. With 35.95 this gives

$$\mu(\Lambda_{\mathrm{type}}) \ge \mu_{\mathrm{core}} - C_1 E_p, \qquad \mu_{\mathrm{core}} \ge -M_{p,i} - C_2 E_p, \tag{8.32c}$$

with $C_1 = 2C_{\mathrm{meas}}$ and $C_2 = C_{\mathrm{core}}$.

Lemma 35.98 (Covolume at one layer)**.** *At $x > 0$ assume the Gram, recursive, projector, bad-rank, maximum-slope, full size and two core bounds above, with $C_1, C_2 \ge 0$. If*

$$12 e^{-x^{e_z/7}} (x^A + x^B + C_1 + C_2 + 1) \le 1, \tag{8.32d}$$

then

$$\frac{\mathrm{lcv}(X_{W,i}, \Lambda_i)}{D_{\mathrm{norm}}} \le M_{p,i} + (C_1 + C_2 + 1) E_p. \tag{8.32e}$$

Proof. Put $H = x^A + x^B + C_1 + C_2 + 1$. By (8.32d) and $H \ge 1$, $\delta \le 1/12$. Equations (8.32b), (8.32c) and $2 \log g \le b$ bound the three quantities in (8.31b) by HE_p. Apply 35.97 to $\Lambda_{\mathrm{type}} \subseteq \Lambda_{\mathrm{all}}$ with $\mathcal{Q} = HE_p$. The cut costs at most $12\delta H E_p \le E_p$. Apply (8.32a) and (8.32c). □

Proposition 35.99 (One constant for the family)**.** *If the Gram, recursive, truncation, bad-rank, maximum-slope, size and two core bounds hold eventually on the eligible families, then 35.2 holds with*

$$C_0 = C_1 + C_2 + 1. \tag{8.33a}$$

Proof. Fix F_m, C_H and A, B, C_1, C_2. Since $e_z > 0$, $e^{-x^{e_z/7}} x^a \to 0$ for $a = A, B, 0$. Take the maximum of the input thresholds and a threshold for (8.32d). Apply 35.98 at all admissible points and layers beyond it. The same threshold works for $C \ge C_0$ by $E_p \ge 0$. □

35.6 Transfer to the selected indefinite realisation

Lemma 35.100 (Size of the field generated by the coefficient list)**.** *Let E be the abelian CM host, of degree $2f$, with distinguished quadratic subfield $K=\mathbf{Q}(\sqrt{-d})$ of discriminant $-d$. Let $F\subset\mathbf{C}$ be the field generated by the coefficient list of the constructed member in its specified embedding. Assume the ray class field theorem. Suppose $f\ge 8$, $r\ge 62$, $L\ge 4$, and*

$$\log f\le L,\quad \log \mathrm{rd}(E)\le 2L,\quad \log g_i\le 9L+1\ (i=0,1),\quad \log p_{\mathrm{de}},\ \log p'_{\mathrm{de}}\le L.$$

Then $\log[F:\mathbf{Q}]$ and $\log\mathrm{rd}(F)$ are each at most $19fr^2L+198f^2L$.

Outline. Place the coefficient field in a compositum of torsion, ray class, cyclotomic and Kummer fields. Use 26.29(2) for the CM class field, bound degrees and differents, and combine the bounds by the compositum inequalities of 35.21. The complete proof is on p. 1263. □

Theorem 35.101 (Covolume bounds on the linked bundles)**.** *Assume the discrete-series covering, Harish–Chandra module theory, cusp rapid decay, Casselman–Wallach and ray-class-field statements for the linked bundles, and 35.2. Let h_L give local Langlands parameters, N be a norm-choice extension, and S,Γ be as in 33.74. Assume a height-counting extension with its law for each datum admitted by N. Assume the twelve open conditions of 33.74 for the parameters determined by N,S,Γ. There is a condition R on members which eventually holds for every constructed member over the permitted base frames. There is also a condition $\mathcal{E}$ on bundles such that*

$$\textit{eventually every bundle linked to a good member satisfying } R \textit{ satisfies } \mathcal{E}. \tag{8.34a}$$

The same R and $\mathcal{E}$ work for every family class $\mathcal{A}$ and every compatible selection σ of such linked bundles satisfying $\mathcal{E}$. Compatibility here includes the seven family conditions of Definition 33.3 for every $F_m\in\mathcal{A}$, as in the linked row condition (33.8). For each such selection the following estimate is closed on $W_3(\mathcal{A})$ in the sense of 36.10. For every eligible F_m, C_H there is $C_0\ge 0$ such that for each $C\ge C_0$ there is x_0 for which, at every $x>x_1$ with $x\ge x_0$, every selected index $\iota\in I_x$ and every layer,

$$\frac{\mathrm{lcv}(X_{W,i},\Lambda_i)}{D_{\mathrm{norm}}}\le M_{p,i}+CE_p \tag{8.34b}$$

in the selected indefinite realisation.

Outline. Intersect the open rows of 33.74 to obtain a common witness and identify the selected coefficient field. Apply the degree and root discriminant bounds of 35.100, then use 35.2 on the selected indefinite realization. The complete proof is on p. 1265. □

The grid-strengthened height-counting extension of 33.1 gives the covolume row of 33.29 by 35.101. For the wordwise application, take a common linked selection satisfying the hypotheses of 36.8, 36.13 and 21.32, as well as the covolume condition just obtained. In particular, use the family local-error estimate and the eventual window bound at budget $\sqrt{b}$ required in 36.13, and its rank identities $d_{j,0}=D_0$, supplied by the surjective level projections and the compatible reference realization. Conjoin the closed conditions by 36.10 and take the maximum of their cutoffs. On this common selection, 36.13 bounds the local-error sum by (36.18) and bounds the weighted lengths of the lifts modulo their cuts. Proposition 36.8 bounds the sum of the indices at conjugate twist places by (36.12). Its error terms are nonnegative by (36.13), and their weighted sum together with the cut exponents satisfies (36.14) in every layer. Together with the Gram bound and the pairing-index and prime-square bounds these give the inputs of 21.28. Use $l_{3,i}=M_{p,i}+CE_p$ from (8.34b) in 21.29, with the passage budget of 21.30. Use 21.32 to identify the lattice and its metric and pairing. Its numerical bound and the identities $H^*-\bar\kappa=h_u$ and $2F_*=\mathrm{Fo}$ give the wordwise inequality

$$\mu(RR^\vee)+h_u-\mathrm{Fo}\le S_{\mathrm{div}}-S_a\log T_u+E_{\mathrm{div}}$$

of H2.

Chapter 36

The definite side and local errors

Use the data of 33.6. Write $\mathcal{L}$ for the size datum, K for the coefficient field of the member, $L = x^{\varepsilon}$, $b = e^{L}$, and $a = xb$. The definite calculation bounds point evaluation on the divided columns. The local calculation bounds the indices lost in projecting to the good Hecke eigenspaces.

36.1 The definite column condition

Definition 36.1 (Definite rows for the selected bundle)**.** Fix $D, O, \mathbf{b}, c_{\mathrm{sz}}, C, C_H$. For every choice of level-one primitive linkage, separation, covolume, surjectivity, isometry and passage data, form the reference and selected sides and the definite face D_c. Take $\lambda_0, c_0, c_e, S_w, C_1 \in \mathbf{R}$ satisfying the wordwise numerical conditions of Lemma 21.24 for every word, with the member's N_c, $C_0 = c_0$ and $C_h = c_{\mathrm{sz}}$. Let $d(z)$ be the dimension of column z of the definite face. Quantify over every family of $\mathcal{O}_K$-submodules $U_z \subset K^{d(z)}$ and every division datum I that agrees, after transport to these sides and face, with the division datum of the selected bundle.

The division ties identify the length and ranks m, r, t, finite group and its idempotent coefficients, tuple labels, infinite places and finite-place selectors. Write e_G for the group isomorphism and P_0, P_s for the reference and selected subgroups. Require $|P_0| = |P_s|(g_0 g_1)^{kf}$. The map e_G identifies P_0 with the level subgroup. For every $u \in P_0$, twist index i, and coefficient embedding $b_0 \in \{0, 1\}$, there is an element z of the common auxiliary coefficient field K_{aux} whose image in K is $\psi_0(u)^{-1}$ and whose image in the (i, b_0) local coefficient field is $\psi_{U,i,b_0}(e_G u)$. In the residue-field matrix models the reference subgroup and its character agree with their level models. On the selected side put $s = \max(\lfloor m/2 \rfloor - 1, 0)$. In zero-based indices, read entry (i, j) of the level model at $(\sigma(i), \sigma(j))$ of the selected matrix, where $\sigma(i) = k + i$ for $i < s$, $\sigma(i) = i - s$ for $s \le i < s + k$, and $\sigma(i) = i$ thereafter, with values taken modulo r. This identifies both the subgroup and its character.

The row maps identify the Hermitian lattices, forms, idempotents and integral tuple modules and commute with projection to the common row space. The reference layer maps identify the layer forms and lattices as well as the zero-summand spans. They commute with projected theta. The theta frames are bases of their complex spans and identify the integral layer lattices. The two sides have the same I_j and j_w. Their local errors η_J agree on the twist set and vanish off it. The assignment $u \mapsto w_U(u)$ from words to columns in the division datum is bijective. For each word u, the column isomorphism e_u from $K^{d(w_U(u))}$ to the ambient space of the divided lattice carries $U_{w_U(u)}$ onto the original integral column module and preserves the infinite-place forms. The pairings B agree under the row and column maps.

The definite column condition requires the following lattice, factorization and norm bounds for every such choice and every word u. Form the numerical table R_u from the rows of u, padded to length N_m, its level T_u, and the remainders of Lemma 21.25. Put $w = w_U(u)$, $R = R_u$, $T = T_u$, $N = N_m$, $L_a = \log(a \operatorname{rd} E)$, and $d_0 = d(w)$. Let N_u^0 denote the input submodule of word u before division, as in 19.1. Require $d_0 > 0$ and that $e_u^{-1}(N_u^0)$ be a full lattice in K^{d_0}.

Let J be the layers and Pl the types. The condition requires the existence of a height table (comprising a volume parameter V, size parameter g_*, count weights g_ℓ^0, log periods $\omega_{\tau,v}$, log theta norm factors ϑ_v, block bounds $M_{\ell,v}$, and corrections h_ℓ, h'_ℓ) and a terminal table (below) as follows. For the degrees $\nu_{\ell,\tau,i}$ put

$n_{\ell,\tau} = \sum_i \nu_{\ell,\tau,i}$ and $p_\ell = |\mathrm{Pl}| g_\ell^0$; bars denote means over infinite places.

$$\begin{aligned} \bar{\omega}_\tau &= \log \Omega, \qquad \bar{\vartheta} \le C_\theta L, \qquad g_* \le C_8 L, \\ M_{\ell,v} &\le C_2 f m^2 N^2 (g_* + \log T) + C_e f N^2 g_*, \\ h_\ell, h'_\ell &\le C_4 f N^2 (L + \log T + g_*), \qquad 0 \le g_\ell^0 \le N^2, \\ \log(8 + 32 p_\ell V) &\le c_5 L_a, \qquad \log(2 n_{\ell,\tau} + 2) \le c_6 L_a. \end{aligned} \tag{36.1}$$

Write $S, \eta, b', a_\tau, b_\tau, TT, \log\Omega$ for the size, loss, auxiliary weight, top weights, marked bottom weights, top set and mean log period of R. The terminal table, comprising an evaluation bound A, degree parameters n_τ, leading weights λ_τ, gaps $D_{0,\tau}, D_{1,\tau}$, branching bounds c_τ^*, ξ_τ^*, and evaluation allowance e_{ev}, satisfies

$$\begin{aligned} A &\le -\sum_\tau n_\tau \log T + \sum_\tau (2c_\tau^* + \xi_\tau^*) + e_{\mathrm{ev}}, \\ c_\tau^* &\le (N-1) D_{1,\tau} \log(2a), \qquad \xi_\tau^* \le \tfrac{1}{2} N(N-1)\log(2a), \\ n_\tau &= \lambda_\tau - D_{0,\tau}, \quad D_{1,\tau} \le 3 D_{0,\tau} + S + \eta, \quad D_{0,\tau} \le S + m\eta + \mathbf{1}_{TT}(\tau) b_\tau, \\ b_\tau &\le 2b' \quad (\tau \in TT), \qquad \lambda_\tau = a_\tau - \tfrac{N-k}{2} \quad (\tau \in TT), \qquad \lambda_\tau \ge S - \tfrac{N-1}{2} \quad (\tau \notin TT), \\ 0 &\le \log T \le C_T m L, \qquad e_{\mathrm{ev}} \le C_{\mathrm{ev}} f N^2 L_a. \end{aligned} \tag{36.2}$$

The bounding constants are nonnegative, and

$$C_h \ge \max\{2(16 + C_T + C_{\mathrm{ev}}),\ C_\theta + c_5 + 2c_6 + 1 + 2C_4 + 2C_e C_8,\ 2(C_2 + C_4)(C_T + C_8)\}. \tag{36.3}$$

Use $\log 0 = 0$ and $\log z = \log|z|$ for real $z \ne 0$. Put

$$\begin{aligned} s_{\ell,v} = \sum_\tau \Bigg[\sum_i (\nu_{\ell,\tau,i} \log \nu_{\ell,\tau,i} - \nu_{\ell,\tau,i}) - n_{\ell,\tau}\omega_{\tau,v} + 1 + 2 g_\ell^0 \log(2 n_{\ell,\tau} + 2) \Bigg] \\ + p_\ell \{\vartheta_v + \log(8 + 32 p_\ell V)\} + 2 M_{\ell,v} + h_\ell + h'_\ell. \end{aligned} \tag{36.4}$$

Choose a matrix chain $K^{d_0} \xrightarrow{K_0} \cdots \xrightarrow{K_{s_{\mathrm{ch}}-1}} K^{d_{s_{\mathrm{ch}}}}$ for the ordinary theta steps and the complementary step (absorbing the initial inclusion f_0 into K_0). At the infinite place v, use frames $X_{\alpha,v} \in \mathrm{GL}_{d_\alpha}(\mathbf{C})$, taking the given Hermitian frame of column w at the initial stage. Require K-linear coordinate maps to the actual form spaces at every stage, including the final complementary space. These maps must intertwine each matrix with the corresponding theta or complementary operator and identify $\|X_{\alpha,v}^{-1} z\|_2^2$ with the squared norm of the corresponding form at v. Assign each layer to a step by a map $\iota : J \to \{0, \dots, s_{\mathrm{ch}} - 1\}$ and a terminal map $P_{\mathrm{end}} : K^{d_{s_{\mathrm{ch}}}} \to (K^{n_0})^\vee$, where n_0 is the dimension of the reference base form. Write $P_{w,\mu_0}(\phi, F)$ for the point-evaluation pairing of column w with the reference vector F. Require

$$P_{w,\mu_0}(\phi, F) = P_{\mathrm{end}}(K_{s_{\mathrm{ch}}-1} \cdots K_0 \phi)(F), \tag{36.5}$$

$$(Y_v^{-1})^* Y_v^{-1} = v(H_{\mu_0}), \qquad Y_v \in \mathrm{GL}_{n_0}(\mathbf{C}). \tag{36.6}$$

The coefficient maps $\Phi_{\alpha p,v}$ to normed complex spaces and the continuous linear transitions $T_{\alpha pq,v}$ satisfy

$$\begin{aligned} 0 < \gamma_{\alpha p} \le 1, \quad \sum_p \gamma_{\alpha p} \|\Phi_{\alpha p,v}(z)\|^2 = \|X_{\alpha,v}^{-1} z\|_2^2, \qquad t_{\max}(\alpha, v) \ge 0, \\ \Phi_{\alpha+1,q,v}(v(K_\alpha) z) = \sum_p \gamma_{\alpha p} \epsilon_{\alpha pq,v} T_{\alpha pq,v} \Phi_{\alpha p,v}(z), \qquad |\epsilon_{\alpha pq,v}| = 1, \quad \|T_{\alpha pq,v}\|^2 \le t_{\max}(\alpha, v). \end{aligned} \tag{36.7}$$

If a_α counts coefficient groups and h_α^{cl} bounds their sizes, write $B_{\alpha,v} = a_\alpha a_{\alpha+1} h_\alpha^{\mathrm{cl}} h_{\alpha+1}^{\mathrm{cl}} t_{\max}(\alpha, v)$ for the squared transition bound. The required bounds are

$$\|P_{\mathrm{end}}\|^2_{X_{s_{\mathrm{ch}},v} \to Y_v^\vee} \le e^A, \tag{36.8}$$

$$\log B_{\alpha,v} \le \sum_{\iota(\ell) = \alpha} s_{\ell,v}. \tag{36.9}$$

Lemma 36.2 (Size conditions for the height table)**.** *For a member with its window size datum, the height size conditions at $Y_0 \in \mathbb{R}$ require an admissible choice of the window parameters on that same datum, as in 33.32, and the following inequalities. Write $N = N_m$, $t = t_{\mathrm{ch}}$, $r_d = \operatorname{rd} E$, and let D_1 be the quadratic logarithmic derivative in 21.12. Then*

$$\log r_d \le L, \qquad |D_1| \le L, \qquad \log(4x) \le L, \qquad \log g_i \le Y_0 L + 1 \quad (i = 0, 1),$$
$$4N^3 \le \eta, \qquad t \le \eta, \qquad m(t+1)\eta \le w'', \qquad kS_c \le w'', \qquad b_s \le 2b' \quad (s \in TT).$$

The spacing δ must be a natural number, and $w_j(\rho) \le b + \eta + 1$ is required at every thick ρ and $1 \le j \le m$. For the selected family these conditions hold at $Y_0 = C_{YL}$ for all sufficiently large indexed x, uniformly in its members.

Outline. Use the parameters of 33.32, host bounds from 8.17 and 8.16, and prime bounds from 8.12. Apply 38.3 to node sizes and 15.9 to w'', then use its ceiling, the box in 14.7, and the endpoint in 14.2. The complete proof is on p. 1266. □

Definition 36.3 (Column comparisons and kernel size bounds)**.** Use the coefficient embedding $\iota : K \hookrightarrow \mathbf{C}$. The real normalizations in Theorem 33.77(5) identify the scaled Haar measure of each signed unitary matrix group with the measure on $\mathrm{U}(P, Q)$ obtained by ordering its positive and negative coordinates. At each type τ, the specified discrete series exists with the prescribed formal degree for this measure, and every discrete series with these parameters has that formal degree. For every Hermitian space of the specified signatures the assigned adelic volume is the measure of a fundamental domain for the rational unitary group. The symmetric column forms have the following level-zero property. For each word u there is an integer $h_u > 0$ such that every value of its level-zero character belongs to $\iota(K)$ and has h_u-th power one.

For the point-evaluation comparison, identify the finite row group with the product of the residue-field matrix groups. Identify the selected subgroup and character with the selected level and its character, and the reference pair with the subgroup and character formed from the column additive characters. At a type prime these additive characters are nontrivial. At places over the same g_i their values on integers agree. The layer characters there are distinct and, in the residue-field coordinates, are the powers $2j$, $1 \le j \le m$, of one faithful character of $\mathbf{F}_{g_i}^{\times}$. Under the group identification the column character set consists of the absolutely irreducible K-valued characters χ for which

$$\sum_{g} e_0(g)\, \iota(\chi(g)) = \iota(\chi(1)),$$

where $e_0(g)$ is the coefficient of g in the finite-type idempotent.

The column Harish–Chandra parameters agree with those of the member at every word and type. Their row parameters and the two endpoints of the initial column are independent of the word. At each step $i < m$ there are integral determinant shifts and nonnegative shifts of the polynomial weights. Adding the latter to each coordinate of the natural polynomial weight gives the prescribed jet degree. The current determinant shift plus the character shift on the sharp side equals the nonnegative shift, and the next determinant shift is the negative of the divided-side character shift. These choices work for every tuple, and the adjacent-weight jet belongs to the specified jet space over K.

For every tuple, every intermediate index at $1 \le j \le m - 2$, and every selected auxiliary place v away from the type primes, the local transformation must give the same image of the layer lattice as a matrix of the following form. Write N_j for the old rank, c for the number of added coordinates, and q_v for the residue cardinality. The old diagonal block is the identity, the new diagonal entries are $q_v^{e_h}$ with $-j \le e_h \le 0$, and the only possible off-diagonal entries form the vector $q_v^{e_1} z$ above the first new diagonal entry, with $(z, 0)$ in the layer lattice. At each word there must also be a complex linear isomorphism from its row space to the complex row model. For a reference tuple with empty auxiliary set, every projected row-coordinate vector has a lift to a level-equivariant form with the prescribed restriction. The isomorphism takes its row image to the vector of values of that form. On the selected subspace the specified column evaluation itself is level-equivariant and has these same values.

Use the column weights for the kernel. Its ranks, host fields, CM type, conductor and primes must agree with those of the member, with the given unramified abelian extension, prime congruences and conjugation-compatible coefficient embedding. A single bijection of archimedean types identifies the marked positions

and, for every word, the row parameters and word coordinates. The global and local characters have the prescribed infinity types and Weil normalizations, and the selector chooses the specified place over each base place. The analytic data use the discriminant twelfth root of the ideal lattice, its ideal character and generators of the period ideals over a common number field. The reference tuple gives the trivial character and the sharp level. The kernel uses $m-2$ intermediate steps, with natural-number subtraction, and the jets for each word at adjacent ranks are defined over the prescribed ambient coefficient field at the selected places and level.

The *kernel size bounds at* (Y,U) require

$$2fk < g_i - 1, \qquad 2fN_m + 1 < 4a, \qquad 4a < T_u, \qquad \log T_u \le UmL.$$

At every word, layer $0 \le j \le m$ of rank N_j, and type τ, the dominant-weight spread $\lambda_{\tau,0} - \lambda_{\tau,N_j-1} + N_j$ is at most $2a$. For each word the numerical rows must also satisfy

$$\max\{\log \mathrm{rd}\, E, \log q, \log d, \log(4x), |D_1|\} \le L, \qquad (kb)^8 \le g_i, \quad \log g_i \le YL + 1, \quad fN_m \le b.$$

Every degree in a layer row, up to its unpadded length N_{j-1}, lies in $[0, 3a]$. At each type, every difference between a coordinate of the base tuple b and a coordinate of the layer tuple e has absolute value at most η; here b denotes the base auxiliary weight tuple (whose negatives are the negative-sheet Harish–Chandra coordinates). All reference v-coordinates and all interior word U-coordinates lie in $[0, S_c]$. At a marked position require $b' \le b_s \le 2b'$. At an unmarked position, for $k \ge 2$, the reference endpoints satisfy

$$U^0_{k-1} = v_{k-2} - \tfrac{1}{2}, \qquad S_c \le U^0_0 - v_0, \qquad U^0_0 \le 3S_c.$$

Theorem 36.4 (Columns from point evaluation)**.** *For $C_Y, C_T \in \mathbf{R}$ with $C_T \ge 0$, there is a threshold C_0 uniform in $D, O, c_{\mathrm{sz}}, \mathbf{b}, C, C_H$. Assume the real normalizations and symmetric column forms of Theorem 33.77(5), with the point-evaluation identifications and compatible kernel weights of Definition 36.3. Assume that for all C'_Y, C'_T the five conditions*

$$\begin{gathered} \text{the height size conditions of Lemma 36.2 at } C'_Y, \quad X' = \exp(x^{\varepsilon/3}) \ge 8, \quad x \ge 2, \\ \forall u\ (4a < T_u), \quad \forall u\ (\log T_u \le C'_T mL) \end{gathered} \tag{36.10}$$

imply the kernel size bounds of Definition 36.3 at (C'_Y, C'_T). Suppose they hold at C_Y, C_T, and

$$N'_u = I_u N^0_u \tag{36.11}$$

is a full lattice for every word, with $\operatorname{rank} N'_u > 0$*. If $c_{\mathrm{sz}} \ge C_0$, the selected bundle satisfies the definite column condition of Definition 36.1 at C, C_H.*

Outline. Factor column periods in Hermitian frames. Use 21.13 for evaluation, Cauchy–Schwarz for kernels, 21.22 for mean volume, and 36.3 for the threshold, and transport the lattice obtained from a multiple of N^0_u. The complete proof is on p. 1266. □

36.2 A common bound along the selected family

Theorem 36.5 (The definite condition along members)**.** *Fix any local Langlands witness h_L and any member parameters p for that witness. For every family class $\mathcal{A}$ and every selection linked to p, the definite column condition of Definition 36.1 is closed on $W_3(\mathcal{A})$ in the sense of Definition 36.10. Write x_1 for the family's selection cutoff and I_x for its indices at x, as in Definition 33.6. For every F_m, C_H meeting the thresholds in this closed assertion, and every $C \ge 0$, the condition holds eventually for all $x > x_1$ and $i \in I_x$; in particular the fixed family constant satisfies $c_{\mathrm{sz}} \ge C_0(C_{YL}, \max(C_{Tu}, 0))$ from 36.4. Separately, for any family satisfying the real second-layer identifications, the selected-family design of 33.1 with constants $3, C'_H$, and, beyond one cutoff, $b^{m/2} \le T_u$ and $\log T_u \le C_{Tu} mL$ for every member and word, the lattices N'_u are eventually full with* $\operatorname{rank} N'_u > 0$ *for every word.*

Outline. Linkage supplies evaluation and size, and 21.8 gives the conditions of 21.9. Clear denominators for fullness, use the lower rank bound for positivity, and apply 36.4 with size and level bounds from 36.2 and 37.10. The complete proof is on p. 1271. □

Definition 36.6 (Member points for the reference estimates)**.** Use the class $\mathcal{P}$ defined by the three member-point conditions in Definition 35.1, before its two additional arithmetic requirements. In the reference coordinates these conditions have the following description. The indefinite realization has the layer lattices, inclusions and theta maps of the primitive construction. At good coefficient embeddings the evaluation at the chosen vector is injective; at the other embeddings the conjugate evaluation identity holds. In every layer the action has a geometric realization, an extremal vector pairs to the prescribed delta vector, and the leaf map has the prescribed values and phase. The primitive row types are separated and satisfy the level-one conditions. The frames and the division maps identify the forms, lattices, projectors and pairings of the two realizations as in Definition 36.1. The group map respects the subgroup characters, and the dimension of the reference projected space is D_0. There are linear maps from the arithmetic layer spaces to the exported spaces which agree in layer coordinates and give the prescribed maps into the glued lattice. All finite types are good and every single-index place is twisted.

These identifications must come from the same recursion and spectral realization. The core and the realization use the same base prime and tame types. The exported series agree, the reference character and the local level character have the common coefficient representatives of Definition 36.1, and the exported sets contain every datum transforming by the prescribed level character. At a nonreference place the additive character takes its level-zero values in the embedded coefficient field. The auxiliary field contains the required ray values and the roots of unity of order dividing $g_i(g_i-1)$. Each tuple field contains the ray field of its level, and the twists agree with the chosen norm.

Let $\mathcal{R}$ consist of points satisfying the following extra requirements for the reference calculation. Every single-index place is twisted. Form the reference realization using the numerical data of the point for the two runs. Its core obeys the theta identities with the canonical levels, positive signature rank in every layer and the prescribed half-error estimates. The coefficient-field conditions hold at both recursion primes and the small-field bound holds for the member. The level ideal is the product of the designated prime ideals. For each residue characteristic of a twisted place there is a prime of the intermediate field below every selected prime for that place and tuple. Both determinant characters take values in the designated coefficient field. In this realization the series, inverse reference character and level eigendata agree with the exports. The additive characters agree with the norm choice and have coefficient-field values at the nonreference places. The small field has the prescribed valuations, ray values and roots of unity; the tame types and the ray fields of all tuple levels agree with the recursion.

Theorem 36.7 (One constant for both realizations)**.** *Assume the discrete-series, Harish–Chandra, rapid-decay and Casselman–Wallach inputs and the surjectivity and lower-weight rows. At the linked slot assume the primitive, divisor, goodness, geometric compatibility, third-layer covolume, functional, multiplicity, κ and defect rows. For a class $\mathcal{A}$ and raw selection satisfying the slot member condition, the following assertion is closed on $W_3(\mathcal{A})$. For every F_m, C_H in the closed assertion there is $C \geq 0$ such that the following bounds hold eventually at every selected member above x_1. On the indefinite side every single-index place is twisted. The reference covolume bound, the isometry, multiplicity and passage identities, and the κ-bounds hold. Under the division estimate* (36.18) *one also has the paired-index and functional bounds. If each layer has the prescribed single leaf map with its fixed values and phase, the selected-side lower weight bound holds. On the definite side the restricted bundle satisfies Definition 36.1 at the family's size constant and the same C, C_H.*

Outline. Use Miller–Schmid decay and 36.9 to extend the intertwiners. Use 33.6 to transport the data for 35.2, carry over 36.8 by linkage, and apply 36.10 on $W_3(\mathcal{A})$ to the eight rows, including definiteness from 36.5. The complete proof is on p. 1272. □

Proposition 36.8 (The paired indices in the reference row)**.** *In the reference realization let $D_R > 0$ be the common rank and* Tw *the twist set. Write T_{j,t_w} for the selected type lattice, θ_{j,t_w} for its layer map, and $A_{j,t_w} = \theta_{j,t_w}(T_{j,t_w})$ for its image in KM_j. Use the signed index of the two commensurable lattices as in 21.27. Write p_w for the residue characteristic, e_w for the ramification index, and $\ell_w = \log N\mathfrak{p}_w/[K:\mathbf{Q}]$. Put*

$$i_{jw} = [M_j : A_{j,t_w}]_w + [M_j : A_{j,t_w}]_{\bar{w}}, \qquad c_w = 2e_w/p_w^2, \qquad R_w = \left(D_R^{-1} \sum_j i_{jw} - 2e_w \sum_j F_{0,j,w} - c_w \right)_+ .$$

Here indices are positive for proper sublattices. In the transported reference coordinates of Definition 20.13, use $F_{0,j,w} = F_j^L(w) = F_j(w)$*, with the layer and place identifications and the base-term condition of 15.10. Set* $e'_{jw} = R_w/m$ *(stored error* $R_w/(e_w m)$*), let* $J_{0,j,w}$ *be the least normalized exponent carrying the lift into its cut,* $B_0 = fmbL$*, and* N_c *the size budget of 36.11. Assume at the family's member points: (1)* two-run, *two theta runs (at the distinguished characteristic and elsewhere) with subspace index remainders summing to at most* $n_j fmb\delta_2$ *(*$0 \le \delta_2 \le 1$*); (2)* rationality, *integral level-zero character and Hecke rationality over* K*; (3)* geometry, *compatibility of cusp-form embeddings, layer lattices and localized theta images; (4)* rank, $(1-y)n_j \le |U|d_{0j}$ *and* $(1-\delta_j)d_{0j} \le D_R$ *(*$y = 1/(4(fk)^3)$*,* $0 \le \delta_j \le 1/4$*); (5)* twist, $\xi_{vj} \ge 0$ *with* $\sum_j \xi_{vj} \le p_v^{-2}$*; (6)* share, $s = [P : U] \ge (1-y)|U|$*; (7)* bad-place, *local depth bounds at* g_0, g_1*, small primes,* q, d*, and ordinary* $p \le Y$*; (8)* exponent, $0 \le J_{0,j,w} \le J_j(p_w)$ *for admissible layer weights with thick set* $T_{2,j}$ *(*$|T_{2,j}| \le f$*, height cutoff* h*), non-thick top/thin width bounds, excluded primes, cut threshold* $l_J = mbL/[k(m+t)\log(3h)]$ *with* $h = b + S_c + \eta + t + 1$*, distinguished set* $S_j = \{q, d, p_{\mathrm{de}}, p'_{\mathrm{de}}\}$ *(*$|S_j| \le 5$*), ambient inclusion exponent* $J_A(p)$*, and coarse short-weight function* c_{sh} *bounding the local short-weight exponent at each prime* p*, setting* $J_j(p) = J_A(p)$ *if* $p \le l_J$*,* p *is excluded, or* $p \in S_j$ *and* $J_j(p) = \sum_{\tau \in T_{2,j}} \min(J_{\mathrm{ex}}(\tau,p), \lfloor c_{\mathrm{sh}}(\tau,p) \rfloor) + \sum_{\tau \notin T_{2,j}} J_{\mathrm{ex}}(\tau,p)$ *otherwise; and (9)* page, *eventual size bounds on* $W_j, S_e, T, k, b', Q_S, g_0, g_1, Y, X''$ *with error unit* $L(B_0 + N_c)$*. Then (36.18) gives eventually, uniformly for* C *above a threshold dominating* $|C_N| + 1$ *from (36.18) (independent of* (x,i) *and* D_R*):*

$$D_R^{-1} \sum_j i_{jw} \le \sum_j (2e_w F_{0,j,w} + e'_{jw}) + c_w \quad (w \in \mathrm{Tw}), \tag{36.12}$$

$$e'_{jw} \ge 0, \tag{36.13}$$

$$\sum_{w \in \mathrm{Tw}} \ell_w (e'_{jw} + e_w J_{0,j,w}) \le CL(B_0 + N_c) \quad (1 \le j \le m). \tag{36.14}$$

Outline. Apply the two-run remainder bound 20.7 and the projection estimate 20.12 to the lattice in 21.27. Pair conjugate indices, subtract the base terms of 15.10, and bound the averaged positive excess and cut exponents by weighted prime sums. The complete proof is on p. 1273. □

Proposition 36.9 (Sobolev bounds for unitary intertwiners)**.** *Let* r, s *be nonnegative integers,* $G = \mathrm{U}(r,s)$ *and* $K = \mathrm{U}(r) \times \mathrm{U}(s)$*. Let* (σ, H) *be irreducible strongly continuous unitary, with finite vectors* V *and smooth vectors* H^∞*. Let* (π, E) *be smooth complex Fréchet with continuous derived operators and defining seminorms* p_i *satisfying*

$$p_i(\pi(g)z) \le h(g)^{N_i} b_i(z), \qquad h(g) = 1 + \mathrm{tr}(g^* g) \tag{36.15}$$

for continuous seminorms b_i *and* $N_i \in \mathbf{N}$*. A linear map* $T_0 : H \to E$ *intertwining* $K, \mathfrak{g} = \mathrm{Lie}(G)$ *on* V *has a unique continuous equivariant extension* $T : H^\infty \to E$ *from* V*, with closed image. For* $T_0|_V \ne 0$ *and each finite list* S *of derived words, a continuous seminorm satisfies*

$$\sum_{D \in S} \|d\sigma(D)w\| \le q_S(Tw) \qquad (w \in H^\infty). \tag{36.16}$$

The empty word acts by the identity. The map on H^∞ *extends algebraically to* H*.*

Outline. Schwartz integration and Casselman–Wallach globalization extend the map. Hahn–Banach bounds coefficients on compact Casimir eigenspaces, and the quotient topology and Casimir Sobolev estimate give Hilbert bounds for derived vectors, with closed image by completeness. The complete proof is on p. 1276. □

36.3 The quantifiers and the local error

Definition 36.10 (Closed assertions on family classes)**.** A threshold datum $\mathcal{K}$ has a real pre-threshold $\mathcal{K}_{\mathrm{pre}}$ and two real list functions $\mathcal{K}_{\mathrm{mid}}, \mathcal{K}_{\mathrm{last}}$. In canonical coordinate order for a family F_m (with concatenation $\|$ and zero-based indexing), the 16-entry outer list is $\mathbf{o} = [C_t, C_c, C_{T_2}, C_{Lw}, C_{j_0}, c_w, C_{YL}, C_{p_{\mathrm{de}}}, C_{ccK}, C_P, C_p, e_P, c_{TO}, C_{Tu}, C_{\eta F}, \max(C_o, J)]$ ($\mathbf{o}[6] = C_{YL}$, $\mathbf{o}[13] = C_{Tu}$), the 10-entry middle list is $\mathbf{m}(C_H) = [C_H, C_\Gamma, C_{\mathrm{sz}}, \dots, C_{\mathrm{sz}}, c_{X,\mathrm{err}}, K_\Lambda, C_F]$ ($\mathbf{m}(C_H)[1] = C_\Gamma$, index 17 of $\mathbf{o} \,\|\, \mathbf{m}(C_H)$), and the last list $\mathbf{l}$ holds the stage-three constants controlling $C' = C_k$ and

$C_\ell = \min(C_f, C_w, C_\Delta, C_{\mathrm{count}}, C_r, C_{\mathrm{gap}})$. Say F_m meets $\mathcal{K}$ at C_H if every preliminary entry is $\geq \mathcal{K}_{\mathrm{pre}}$, every entry of $\mathbf{m}(C_H)$ is $\geq \mathcal{K}_{\mathrm{mid}}(\mathbf{o})$, and every entry of $\mathbf{l}$ is $\geq \mathcal{K}_{\mathrm{last}}(\mathbf{o} \,\|\, \mathbf{m}(C_H))$. Define P to be *closed on* $\mathcal{C}$ if there is $\varepsilon_1 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_1$ there is $\mathcal{K}$ such that $P(F_m, C_H)$ holds for every F_m of parameter ε and every $C_H \in \mathbf{R}$ with $F_m \in \mathcal{C}$ and F_m meeting $\mathcal{K}$ at C_H. For the closed local-error assertion, take $P(F_m, C_H)$ to be the family estimate in Definition 36.11. Conjunction uses the smaller cutoff and componentwise maximum thresholds.

Definition 36.11 (The local-error estimate)**.** At index i, let O_i be the local coefficient ring and ℓ_i its residue characteristic. Write $\lambda_i = \mathrm{length}_{O_i} / \mathrm{length}_{O_i}(O_i/\ell_i O_i)$, with infinite length and division by zero assigned value zero. For the lattice image M_i and the eigenspace sums V_Z, V_{Z^c} of $Z = Z_{\mathrm{good}}(i)$ and its complement, put

$$U_i = V_Z \cap (M_i + V_{Z^c}), \quad L_i = M_i \cap V_Z, \quad \eta_j(w) = \sum_{i \in I_{j,w}} \lambda_i(U_i/L_i), \quad \eta_j(\mathcal{L}) = \sum_{\substack{w \in S_{\mathcal{L}} \\ \mu(w) \neq \mu_0}} \ell_w e_w \eta_j(w).$$

Here $S_{\mathcal{L}}$ is the finite support of the size datum, $I_{j,w}$ is the index set of layer j at w, and $\ell_w = \log N\mathfrak{p}_w / [K : \mathbf{Q}]$. The finiteness hypothesis requires the lengths $s_{\ell,e}(E) = \mathrm{length}_{\mathbf{Z}_\ell} \mathrm{Sha}_{\ell,e}(E)$ of Definition 26.1, taken with the member's local data, to be finite for every prime ℓ, including 2, and every $1 \leq e < t$. Under this hypothesis put $s_\ell = \sum_{e=1}^{t-1} s_{\ell,e}(E)$ at each prime and $v_\ell(D) = (\log \ell) s_\ell$, with $v_\ell(D) = 0$ elsewhere. Write $d_{j,0} = \dim_{\mathbf{C}} Z_j(\mu_0)$. With $n = f/2$ and T_2 the host's thick set, the size budget is $N_c = |T_2| k b L^2 / n^2$. The point condition is

$$(v_\ell(D))_{\ell \text{ prime}} \text{ summable} \quad \Longrightarrow \quad \eta_j(\mathcal{L}) \leq d_{j,0} \left[C\{f m b L^2 + (\log x)^2 N_c\} + \sum_{\ell \text{ prime}} v_\ell(D) \right]. \tag{36.17}$$

The family local-error estimate requires constants $C, x_0 \in \mathbf{R}$ such that (36.17) holds for every $x \geq x_0$, every $i \in I_x$ whose member satisfies the finiteness hypothesis, and every $1 \leq j \leq m$.

Lemma 36.12 (The sum over twisted places)**.** *Use the place bijection ι from the arithmetic coordinates to the row coordinates and the type selector of Lemma 18.6. Number the arithmetic layers from* 1*, and write $a(j)$ for the arithmetic number of row layer j. Assume the division data identify the local quotients at each selected type in the two coordinates and that $\iota^{-1}(\mathrm{Tw})$ lies in the finite support $S_{\mathcal{L}}$. If a_{jv} is the sum of the ordinary lengths of the lift quotients at the selected type, computed over the base discrete valuation ring at $w = \iota^{-1} v$ before the splitting extension of Lemma 33.54, then*

$$\sum_{v \in \mathrm{Tw}} \ell_v a_{jv} \leq \eta_{a(j)}(\mathcal{L}).$$

Outline. For $w = \iota^{-1} v$, both coordinates share the prime, weight, and index e_w, so 36.11 gives $a_{jv} = e_w \eta_{a(j)}(w)$. Since 18.6 picks nonreference type on Tw, reindexing gives a subsum of $\eta_{a(j)}(\mathcal{L})$ with nonnegative terms. The complete proof is on p. 1277. □

Lemma 36.13 (From the window bound to the division input)**.** *For the host field E and the member's number t of scalar blocks, the window bound of Definition 26.1 at budget B gives the finiteness hypothesis of Definition 36.11, summability, and $\sum_\ell v_\ell(D) \leq B$. Suppose $\varepsilon > 0$ and the family local-error estimate holds with constant C_N. Suppose also that eventually every member satisfies the window bound at budget $\sqrt{b}$ and that each $d_{j,0}$ equals the dimension D_0 of the reference target. Then eventually*

$$\eta_j(\mathcal{L}) \leq D_0 b_N, \qquad b_N = (|C_N| + 1) L (B_0 + N_c), \qquad B_0 = f m b L. \tag{36.18}$$

Thus one constant and one cutoff give the division estimate in every layer of every member beyond that cutoff. In the tied reference coordinates, writing Tw *for the twisted support and a_{jv} for the cut lengths of lifts modulo their cuts at layer j and place v, one has $\sum_{v \in \mathrm{Tw}} \ell_v a_{jv} \leq D_0 b_N$.*

Outline. Apply 26.1 to finite sums and use the family local-error estimate. Transfer the bound to twisted lift lengths by 36.12 and the division tie, and use surjectivity in 33.6 to identify the layer image with the reference target. The complete proof is on p. 1277. □

36.4 The primes 2 and 3

Theorem 36.14 (Local cohomology)**.** *Let $F_v/\mathbf{Q}_p$ be finite and M a finite continuous G_{F_v}-module killed by ℓ^a, with ℓ prime and $a>0$. Cohomology is finite through degree 2 and zero above it. Cup product pairs $H^i(F_v,M)$ perfectly with $H^{2-i}(F_v,\mathrm{Hom}(M,\mu_{\ell^a}))$ into $H^2(F_v,\mu_{\ell^a})\simeq\mathbf{Z}/\ell^a$. Also*

$$|H^1(F_v,M)| = |H^0(F_v,M)||H^2(F_v,M)| \begin{cases} |M|^{[F_v:\mathbf{Q}_\ell]}, & p=\ell,\\ 1, & p\neq\ell. \end{cases} \tag{36.19}$$

The local finiteness theorem and Tate duality, together with the Euler characteristic formula, apply also to decomposition groups identified with local Galois groups [127, Chapter VII, §§2–3].

Definition 36.15 (Inputs for the small-prime calculation)**.** The inputs $\mathscr{H}_2$ include Fontaine's crystalline period comparison and fundamental sequence [191], global Artin reciprocity [127, Chapter VIII], and 36.14 at $p=\ell$. For $K/\mathbf{Q}_\ell$ finite let $H^1_f(K,V)$ be the kernel of the period map to $H^1(K,B_{\mathrm{cris}}\otimes V)$. Use its inverse image for a stable lattice $T\subset V$ and its image for V/T. A cocycle is finite when its class is finite. For V de Rham, the Bloch–Kato inputs [120, §3] give

$$\dim_{\mathbf{Q}_\ell} H^1_f(K,V) = \dim_{\mathbf{Q}_\ell} H^0(K,V) + \dim_{\mathbf{Q}_\ell} D_{\mathrm{dR}}(V)/\mathrm{Fil}^0, \tag{36.20}$$

and the finite conditions are exact annihilators for rational Tate duals and for an integral lattice and its divisible Tate dual. Over finite $K/\mathbf{Q}_\ell$, prismatic F-crystals realize crystalline lattices, and equivariant maps lift uniquely [192, Theorems 1.2 and 5.6]. In particular scalar multiplication lifts to the coefficient-ring action.

At each local index i (with local coefficient ring O_i and residue characteristic ℓ_i), let X_i be the finite generic Hecke spectrum carried by the local datum, $Z(i)\subseteq X_i$ the subset of spectral points realized by the layer's theta-lifted automorphic forms, and $Z_{\mathrm{good}}(i)\subseteq Z(i)$ the subset arising from good finite-type rows. Let $\chi_{i,1}:G_E\to O_i^\times$ be the base character and $\chi_{i,c}=\chi_{i,1}\varepsilon_{\ell_i}^{-(c-1)}$ $(1\le c\le t)$. Let S_i be the finite set of split places of E not dividing ℓ_i lying over g_0,g_1, or the auxiliary prime $p_*(i)$ (equal to p'_{de} if $\ell_i=p_{\mathrm{de}}$ and p_{de} otherwise), let Q_i be the finite set of split places of E disjoint from S_i and not dividing ℓ_i at which $\chi_{i,1}$ is ramified, fix a tame inertia generator $\gamma_{i,v}\in I_v$ at each $v\in S_i\cup Q_i$, and set $\zeta_{i,v}=\chi_{i,1}(\gamma_{i,v})\in O_i^\times$. Let $T_i\subseteq O_i^{X_i}$ be the Hecke O_i-algebra generated by the Hecke polynomial coefficients at split places outside $\{v\mid\ell_i\}\cup S_i\cup Q_i$. Let $\mathfrak{c}_{i,\pi}\subset O_i$ be the evaluation at π of the Hecke ideal of operators vanishing off $Z(i)$. Put $v_{cc}(i,\pi)=\lambda_i(O_i/\mathfrak{c}_{i,\pi})$. Let $s_\pi(\rho_i)$ be the normalized Selmer length of $\rho_i(\pi)\chi_{i,1}^{-1}$ with divisible coefficients, finite above ℓ_i, unrestricted at $S_i\cup Q_i$ away from ℓ_i, and unramified elsewhere. Write h_E for the class number and normalize $v_\ell(\ell)=1$. Set

$$\mathcal{B}_{i,\pi}(\rho;B) \iff v_{cc}(i,\pi)\le s_\pi(\rho_i)+B+\sum_{1\le e<t} s_{\ell_i,e}(E)+v_{\ell_i}(h_E)+\ell_i^{-2}. \tag{36.21}$$

Use r for the rank of ρ_i, t for the number of scalar blocks, $n_V=r+t$, $n=f/2$, $N=Q_S qd$ for the host conductor modulus, and $s_0(i)$ for the depth of the selected cyclotomic inertia element. Put $H_B=b+S_c+\eta_{\mathrm{sz}}+t+2$.

For the small-prime estimate, take data X giving both the full Galois representation $r_0(x)$ $(x\in X_i)$ and the rank-r representation $\rho_i(\pi)$ at each Hecke point π. Impose the following conditions. At every $x\in X_i$, $r_0(x)$ is semisimple over a finite extension of $\mathrm{Frac}(O_i)$ with pairwise non-isomorphic absolutely irreducible summands, all $r_0(x)$ $(x\in X_i)$ share a common residual characteristic polynomial, and the Hodge–Tate jump multiset of $r_0(x)$ at each embedding above ℓ_i is independent of $x\in X_i$. At an inert index i the column set is empty. For every place w, the local length allowance at each spectral point π equals the row allowance at (i,w). The row allowance is the long-orbit or the other inert allowance according to (A.528). At split indices it uses the canonical row and column declaration, chosen by the common embedding sign, with the canonical column length in (A.533). Its integral width is $\eta_X=\lfloor\eta_{\mathrm{sz}}\rfloor$. The canonical hub constants c_0,c_B (shared at unramified split $\ell\ge5$ in Lemma 36.92; distinct from $c_0^{\mathrm{cyc}},c_B^{\mathrm{cyc}}$ in 36.23) are given below, with $j(z)=\lfloor\log_{\ell_i}z\rfloor$ for $z>0$, $j(0)=0$, $h=\lfloor2b\rfloor$ and $a_0=4nn_V$,

$$c_0=5+j(2(h+1))+a_0[4+2j(2(h+1))],\qquad c_B=3+j(2h)+t[3+a_0(4+2j(2h))]+4t+3.$$

The small-prime setup bundles the memberwise norm-choice facts (the point, admissibility, Galois, compatibility, and localization-uniqueness conditions established on a good cut in Theorem 36.26(I) below), the canonical hub parameters above (c_0, c_B, η_X, and row/column allowances), the polynomial parameter bounds with $L \geq 8$ and host splitting cutoff at least 60, the Selmer and Q_i-killing bounds (36.26) for ρ, and odd-prime finiteness of the Tate–Shafarevich lengths $s_{\ell,e}(E)$ of Definition 36.11. The size bounds used below give $H_B \leq 2b$.

Theorem 36.16 (A family bound at 2 and 3)**.** *Assume $\mathscr{H}_2$. For every $\varepsilon > 0$ and every family F_m of parameter ε there are $C, x_0 \in \mathbf{R}$ with the following property. For each $x \geq x_0$ with $x > x_1$ and each $a \in I_x$, use the norm choice of the selected extension at (x, a). For every primewise Wach system, Fontaine datum and comparison, and every X, C_2 satisfying the preceding small-prime hypotheses (in particular, multiplicity-free semisimplicity with common residual characteristic polynomial and x-independent Hodge–Tate jump multisets across X_i), one has*

$$\mathcal{B}_{i,\pi}(\rho; Cfmb \log x + Cx^C L^2) \tag{36.22}$$

at every twisted i with $\ell_i = 2$ or with $\ell_i = 3$ and $3 \nmid Q_S$ for this member, for all $\pi \in Z_{\mathrm{good}}(i)$. This gives the linked closed bound on $W_3(\mathcal{A})$.

Outline. Insert the cyclotomic exponents of 36.22 into the split congruence bound of 36.23, using inertia valuations for the character and projector terms. Use degree sums, place counts, the tame bound, and parameter bounds to fit the family allowance. The complete proof is on p. 1278. □

Lemma 36.17 (The Wach lattice at the lower cyclotomic roots)**.** *Let ℓ be prime, let $K_0/\mathbf{Q}_\ell$ be finite unramified with residue field k', and put $K' = K_0(\mu_{\ell^s})$, where $s \geq 1$ and $s \geq 2$ if $\ell = 2$. Let L be a finite extension containing K', with integer ring O. Put*

$$A = O[[Z]], \qquad X = (1+Z)^{\ell^{s-1}} - 1, \qquad Q = \Phi_{\ell^s}(1+Z).$$

Let T be a finite free O-lattice with continuous $G_{K'}$-action such that $T[1/\ell]$ is crystalline. Its prismatic realization gives a free A-module $N_i(T)$ of rank $\operatorname{rank}_O T$ for each embedding $i : W(k') \hookrightarrow O$. These modules are functorial in equivariant lattice maps. Their completed étale base changes are the components of the cyclotomic (φ, Γ)-module of T.

Here $\Gamma = \mathrm{Gal}(K'(\mu_{\ell^\infty})/K')$. Let $\gamma \in \Gamma$ have cyclotomic value a with $v_\ell(a-1) = s$, where $v_\ell(\ell) = 1$. For each $\zeta^{\ell^{s-1}} = 1$, put $t_\zeta = \log((1+Z)/\zeta)$. Over $L[[t_\zeta]]$ the component $N_i(T)$ has a basis fixed by γ. In particular,

$$(\gamma - 1)N_i(T) \subset XN_i(T), \qquad (\gamma - a^j)X^j N_i(T) \subset X^{j+1}N_i(T) \quad (j \geq 0).$$

These inclusions also hold on the image of any equivariant lattice map.

Outline. Use 36.15 to evaluate the crystal on the cyclotomic prism and split it into free components. Construct a γ-fixed basis at 1, transport it by Frobenius to the lower roots, and use Weierstrass division to obtain the X-adic inclusions. The complete proof is on p. 1278. □

Lemma 36.18 (Integral comparison of two Frobenius lattices)**.** *Use the cyclotomic extension, coefficient ring O and components of Lemma 36.17. Set $A_i = O[[Z]]$, $A_* = \prod_i A_i$, and let B be the ℓ-adic completion of $A_*[1/Z]$. Put $X = (1+Z)^{\ell^{s-1}} - 1$ and $Q = \varphi(X)/X$. Let T be a crystalline O-lattice and $\mathcal{D}$ its étale module over B. Suppose $L, L' \subset \mathcal{D}$ are componentwise finite free, both base change to $\mathcal{D}$, and their linearized Frobenius maps are isomorphisms after inverting Q. Then there is $a \geq 0$ such that*

$$X^a L \subset L', \qquad X^a L' \subset L.$$

If both lattices carry the same ambient semilinear γ, acting as the identity modulo Z on each lattice, and $u = \chi_{\mathrm{cyc}}(\gamma)$ has depth s as in Lemma 36.17, then $L = L'$.

Outline. Use the étale comparison of 36.17 and Witt ring integrality to clear both transition matrices with a power of X. Cyclotomic equivariance excludes poles at zero, and Frobenius propagates regularity to every root of X. The complete proof is on p. 1279. □

Lemma 36.19 (Annihilators on a cyclotomic component)**.** *Let ℓ be prime, $N > 0$, and $4 \mid N$ at $\ell = 2$. Set $K' = \mathbf{Q}_\ell(\mu_{\mathrm{lcm}(\ell,N)})$, $s = \max(1, v_\ell(N))$, and let O be the integers of a finite coefficient field containing K'. Use the Wach components over K' constructed for the crystalline realization in Lemma 36.17, with K_0 the maximal unramified subfield of K'. The roots of unity of order prime to ℓ generate an unramified extension, so $K' = K_0(\mu_{\ell^s})$. Each unramified embedding has ring $A = O[[Z]]$, with components in Frobenius order. Set*

$$\varphi(Z) = (1+Z)^\ell - 1, \quad X = (1+Z)^{\ell^{s-1}} - 1, \quad Q = \sum_{j=0}^{\ell-1} (1+X)^j, \qquad \varphi(X) = XQ.$$

Let γ have cyclotomic value u of depth s. For a surjection $f : T_1 \to T_2$ of crystalline lattices with source jumps in $[-M_i, h_i]$ at component i, $M_i, h_i \in \mathbf{N}$, write N_{ji} for the Wach components and $P_i = fN_{1i}$. For $H = \max_i(M_i + h_i)$, $w = 2Hs + v_\ell((2H)!)$, one has

$$X^H N_{2i} \subset P_i, \qquad \ell^w N_{2i} \subset P_i. \tag{36.23}$$

Outline. Use the Wach comparisons of 36.17, and identify the saturated image with the target by 36.18. Frobenius descent gives the X^H bound, and interpolation in the cyclotomic action gives the ℓ^w bound. The complete proof is on p. 1280. □

Lemma 36.20 (Lifting a scalar cocycle at cyclotomic depth s)**.** *Let $K' = K_0(\mu_{\ell^s})$ as in Lemma 36.17. For $e \geq 0$, put $b = s + v_\ell(e+1)$. If $\bar{z}$ is a continuous cocycle of $G_{K'}$ with values in $(\mathbf{Q}_\ell/\mathbf{Z}_\ell)(-e)$, there is a continuous cocycle z with values in $\mathbf{Q}_\ell(-e)$ whose reduction is $\ell^b \bar{z}$ at every group element.*

Outline. The cyclotomic valuation shows that ℓ^b kills the dual invariants, so local duality in 36.14 gives the same bound for the lifting obstruction. Adjust the rational lift by a coboundary to obtain pointwise agreement. The complete proof is on p. 1282. □

Lemma 36.21 (Convergence of the cyclotomic primitive)**.** *Let O be the integers of a finite extension $L/\mathbf{Q}_\ell$, with $|\ell| = \ell^{-1}$. Take $s \geq 1$, with $s \geq 2$ for $\ell = 2$, and $u \in \mathbf{Z}_\ell^\times$ with $v_\ell(u-1) = s$. On $L[[Z]]$ let γ and φ act by the substitutions $(1+Z)^u - 1$ and $(1+Z)^\ell - 1$, respectively. Put*

$$X = (1+Z)^{\ell^{s-1}} - 1, \qquad Q = \varphi(X)/X, \qquad t_0 = \log(1+Z).$$

On a finite cyclic set of components indexed by i, suppose the matrices of two commuting semilinear operators are

$$A_i = \begin{pmatrix} G_i & z_i \\ 0 & 1 \end{pmatrix}, \qquad B_i = \begin{pmatrix} P_i/Q^m & y_i/\Lambda \\ 0 & 1 \end{pmatrix}, \qquad \Lambda = \prod_{j \in J} \varphi^j(Q),$$

where J is a finite multiset of nonnegative integers, $m \geq 0$, and G_i, P_i, z_i, y_i have coefficients in $O[[Z]]$. Assume $A_i(0) = 1$, $B_i(0)$ is invertible over L, and there exist an integer $k \geq 0$ and matrices P_i' over $O[[Z]]$ such that $P_i P_i' = Q^k I$ for every i. Assume also $A_i \gamma(B_i) = B_i \varphi(A_{i-1})$. The solution with $w_i(0) = 0$ of

$$G_i \gamma(w_i) - w_i = z_i$$

is unique, and $t_0^{m+|J|} w_i$ converges on $|Z| < 1$.

Outline. Solve for the invariant matrix coefficient by coefficient, bounding the divisors $1 - u^j$ to get convergence on a small disc. Clear the Frobenius poles with a power of $\log(1+Z)/X$ and iterate the contracting Frobenius substitution. The complete proof is on p. 1282. □

Lemma 36.22 (Three cyclotomic denominators)**.** *Use $\mathscr{H}_2$ and the comparison data assumed in Theorem 36.16. Let $K \subset K' \subset K_{\mathrm{coeff}}$ be local fields with K', s, N as in 36.19, and $O = \mathcal{O}_{K_{\mathrm{coeff}}}$. Put $j(z) = \lfloor \log_\ell z \rfloor$ for $z > 0$, $j(0) = 0$. For $h, a, t \in \mathbf{N}$, set*

$$D_h = s + j(2h) + j(N) + 1, \quad E_h = 32(2hs+1) + D_h, \quad I_{h,a} = 20(a+1)D_h, \quad T_{h,a,t} = 40(t+1)(a+1)D_h.$$

For a surjection of crystalline lattices with source jumps in $[-h, h]$, ℓ^{E_h} takes finite target classes into the image of finite source classes. For a crystalline lattice T with jumps in a set of at most a integers in $[-h, h]$ and zero absent, $\ell^{I_{h,a}}$ kills $H^0(K, T[1/\ell]/T)$.

Suppose $t \le h$, Y is crystalline with stable lattice Y_0 and jumps in such a set allowing zero, and $p : Y \to K_{\mathrm{coeff}}$ is invariant linear with $p(Y_0) \subset O$ and $p(y_0) = 1$ for $y_0 \in Y_0$. For $0 \le e < t$ let $\phi : Y \to (K_{\mathrm{coeff}}/O)(-e)$ be O-linear and equivariant on $Y_0 \cap \ker p$. Then the cocycle $\ell^{c_B}\phi(gy_0 - y_0)$ is finite for $c_B = T_{h,a,t}$. On the divisible character write $Z_D, F_D, B_D, Z_{\mathrm{ur}}$ for cocycles, the finite condition, coboundaries and cocycles zero on inertia. Then $Z_D \cap F_D = B_D$ for $e > 0$, and $Z_D \cap F_D = Z_D \cap (Z_{\mathrm{ur}} + B_D)$ for $e = 0$. With λ_O the length normalized by the length of $O/\ell O$,

$$\lambda_O(\{z \in Z_D : \ell^{c_B} z \in F_D\}/(Z_D \cap F_D)) \le ([K : \mathbf{Q}_\ell] + 1)c_B. \tag{36.24}$$

Outline. Use 36.21 and 36.18 to compare closed Wach pairs with finite classes, then correct the lifts from 36.19 by a polynomial in γ. Separate the weights to kill invariants, and use 36.20, corestriction and duality for the character bound. The complete proof is on p. 1283. □

Lemma 36.23 (The split congruence quotient)**.** *In the setup of 36.16 with $H_B \le 2b$ (in particular, across $x \in X_i$, $r_0(x)$ is semisimple over a finite splitting extension with pairwise non-isomorphic absolutely irreducible summands, shares a common residual characteristic polynomial, and has x-independent Hodge–Tate jump multisets above $\ell = \ell_i$), let i be twisted at 2, at 3 outside the small set, or at an odd small prime with split embedding types. There is a choice of column places A and conjugate row places $A' = \{\bar{u} : u \in A\}$ such that the following holds for every $\pi \in Z_{\mathrm{good}}(i)$. Put $\ell = \ell_i$, $K_i = \mathrm{Frac}(O_i)$, E_u for the completion of E at u, $d_u = [E_u : \mathbf{Q}_\ell]$, and W for the normalized rank-r lattice of $\rho_i(\pi)\chi_{i,1}^{-1}$. Let*

$$h_W(u) = \lambda_i H^0(E_u, W \otimes_{O_i} (K_i/O_i)), \quad \delta = \sum_{1 \le e < t} (s_0(i) + v_\ell(e)), \quad h_0 = \max_{u \in A,\, 1 \le e < t} \lambda_i H^0(E_u, (K_i/O_i)(-e)),$$

with empty maximum zero. Use the bounding cyclotomic exponents $e' = E_{\lfloor H_B \rfloor}$, $c_0^{\mathrm{cyc}} = I_{\lfloor 2b \rfloor + 1, 4nn_V}$, $c_B^{\mathrm{cyc}} = T_{\lfloor 2b \rfloor, 4nn_V, t}$ of 36.22. Then $h_W(u) \le r c_0^{\mathrm{cyc}}$ and

$$\begin{aligned} v_{cc}(i,\pi) \le s_\pi &+ \sum_{1 \le e < t} s_{\ell,e}(E) + v_\ell(h_E) + t \sum_{v \in Q_i} \lambda_i(O_i/(\zeta_{i,v} - 1)) \\ &+ \sum_{u \in A} \left(e' + d_u m(2e' + \delta) + 2\delta + h_W(u) + r c_0^{\mathrm{cyc}}\right) \\ &+ t c_B^{\mathrm{cyc}} \sum_{w \in A'} (d_w + 2) + 4n_V(|A| + 1)\delta + t|A|h_0. \end{aligned} \tag{36.25}$$

Outline. Use 36.22 for cyclotomic bounds on column invariants, finite lifts, and character blocks. Bound the relation determinant by elementary divisors and sum the lengths from the local images of Galois difference cocycles and their common global kernel. The complete proof is on p. 1290. □

36.5 Cutting to the good part

Lemma 36.24 (Degree-one Chebotarev)**.** *Let F be a number field with algebraic closure Ω, $U \triangleleft G_F = \mathrm{Gal}(\Omega/F)$ open, $g \in G_F$, and A a finite prime set of F. There is $v \notin A$ of residue degree and ramification index one over $\mathbf{Q}$ with a prolongation to Ω whose inertia lies in U and a Frobenius representative σ with $\sigma U = gU$. Existence follows from Chebotarev [38, §1A]. Primes of residue degree at least two have Dirichlet density zero and finitely many primes ramify over $\mathbf{Q}$; conjugating the prolongation specifies the representative in G_F/U.*

Lemma 36.25 (The two type primes and a residual characteristic)**.** *Use g_0, g_1 and their disjoint sets $B(g_k, r)$ of 32.58. Fix an index i of nonempty good spectrum and write $\ell = \ell_i$. A residual realization is a rank-r representation over the residue field of O_i with open kernel. For every spectral point x, every split v outside $\{v \mid \ell\} \cup S_i \cup Q_i$, every prolongation of v and every Frobenius representative there, its Frobenius polynomial times $\prod_{c=1}^{t}(U - \bar{\chi}_{i,c}(\mathrm{Frob}_v))$ equals the reduced full Hecke polynomial at x, v.*

Assume the label and residual tame eigenvalue identifications, split Frobenius density in finite quotients, and residual Brauer–Nesbitt. The depth-zero labels of realized parameter lists have degrees equal to their local

dimensions. For every $x \in Z_{\mathrm{good}}(i)$ and every parameter list Σ for x of good signature, assume there is an integral ρ realizing Σ and a character ξ such that $\rho \otimes \xi$ realizes x in rank r. Require $\xi = 1$ on inertia above each $g_k \neq \ell$. Then each residual realization is absolutely irreducible. At every $v \mid g_k \neq \ell$ its polynomial on the prescribed tame generator is nonzero at the value of $\bar{\chi}_{i,1}$ there.

Outline. Use squarefreeness for multiplicity one and the good-label conditions of Definition 33.5 to exclude multiple parts and proper stable subspaces by divisor degrees and tame Frobenius. Test inertia by roots, then cancel common character factors and transfer by split Frobenius density and Brauer–Nesbitt. The complete proof is on p. 1293. □

A good cut keeps the indices with nonempty good spectrum and spectral sets X_i with $Z_{\mathrm{good}}(i) \subset X_i$ and $Z(i) \cap X_i \subset Z_{\mathrm{good}}(i)$, where the inclusions allow equality. An integral Hecke-commuting α_i acts by M_0, the order of the finite row group of Definition 36.1, on good eigenspaces and by zero outside X_i. For every lift into X_i and every $w \mid g_k$, let a_w be the dimension of its local parameter. Require zero monodromy and characteristic polynomial $(U-1)^{a_w}$ on every wild inertia element. For every tame generator, each root z of its inertia polynomial of multiplicity at least the threshold must satisfy $z^{Nw-1} = 1$. The threshold is 2 if p_w splits completely in E and either $a_w = 1$ or $16a_w^2 < p_w$. It is 1 otherwise.

The *four local compatibility conditions* require at each index i: (1) *paired characters*, matching the local character pair to the type-prime labels; (2) *depth-zero type and eigenspan*, zero monodromy, label factorization $f(X)(X-u)^t$ at type primes, and eigenspan containment under the good paired-character idempotent e_0; (3) *inertia polynomial*, matching the tame-generator inertia polynomial to the character pin; and (4) *local series*, identifying the label polynomial from the nonzero action of its central operator. The *first Galois provision* requires, at each i: integral full realizations $r_0(x)$ ($x \in X_i$) with Hecke–Galois compatibility and permutation-unique base-change parameter lists at equal spectral points; crystallinity off $\{g_0, g_1\}$ with x-independent level-zero Hodge–Tate jump multisets across X_i (and potential crystallinity at $\{g_0, g_1\}$ under zero monodromy); multiplicity-free semisimplicity of $r_0(x)$ over a finite splitting extension with common residual characteristic polynomial across X_i; unramifiedness outside $\{v \mid \ell_i\} \cup S_i \cup Q_i$; vanishing local invariants for normalized cuspidal rank-r blocks and their duals; and positive theta scalars with cuspidal parameter lists on good rows. The *second Galois provision* consists of three parts: the *fifth provision* (for every good full realization r_0 at i: at each $w \in Q_i$ and each auxiliary-prime place $w \mid p_*(i)$ off ℓ_i, $(r_0(x)(\gamma) - 1)(r_0(x)(\gamma) - \chi_{i,1}(\gamma)) = 0$ for all $x \in X_i$, $\gamma \in I_w$ and $r_0(x)|_{I_w}$ is conjugate over K_i to $\mathrm{diag}(1^r, \chi_{i,1}^t)$; at each type-prime place $w \in S_i$, some $P_p \in T_i[U]$ with $P_p(\chi_{i,1}(\gamma_{i,w})) \in T_i^\times$ satisfies $P_p(r_0(x)(\gamma_{i,w}))(r_0(x)(\gamma) - \chi_{i,1}(\gamma)) = 0$ for all $x \in X_i$, $\gamma \in I_w$; and at each $w \in S_i$, character-block cocycle functionals vanish on I_w and local $H^0 = H^1 = H^2 = 0$ for every residual cyclotomic twist of $\bar{\rho} \otimes \bar{\chi}_{i,1}^{-1}$ and its Tate dual), the *sixth provision* (whenever $x \in X_i$ admits a nonzero G_E-invariant vector in $r_0(x) \otimes \chi_{i,1}^{-1}$ over K_i, one has $x \in Z(i)$ and every parameter list at x is cuspidal), and the *Z-multiplicity identity* (injective theta transfer of parameter blocks onto good eigenspaces with $\sum_{x \in Z_{\mathrm{good}}(i)} \mathrm{mult}(i,x) = d_M(i)$), accompanied by admissible jumps and residual existence. A family F_m is *twist-good* if eventually (for all $x \geq x_0$ with $x > x_1$ and $i \in I_x$) its norm-choice twist packages are conjugation-symmetric with opposite signs at distinct conjugate places and satisfy the integral Hecke-form goodness bounds at every nonreference place. The Galois hypotheses for the chosen norm consist of the point and admissibility conditions, the first Galois provision, Z-multiplicity, residual irreducibility, tame eigenvalues, the weight and block dictionaries, unramifiedness, depth zero, base change, weight use, localization uniqueness, coefficient largeness, and the norm and same-twist ties. For every parameter list on a good row the triple of good signature supplies a twist place w' of nonbase type and a coefficient embedding ι_a such that $\iota_a(z)$ lies in the maximal ideal $\mathfrak{m}_{O_0(a)}$ of the local coefficient ring $O_0(a)$ at index a exactly when $v_{w'}(z) > 0$, for all $z \in \mathcal{O}_{K_{\mathrm{aux}}}$. It supplies a nonzero type-idempotent vector with the prescribed Hecke eigenvalues for $1 \leq k \leq r$ at split places outside its excluded set. Its finite-type labels are good for the characters and their inverses.

Theorem 36.26 (Good cuts and restoration of the family bound)**.** *Assume 36.24, the decomposition-group inputs of 36.14, and the preceding Galois hypotheses and four local compatibility conditions.*

(I) Good cuts and type constants. *There is a good cut c on which the four local compatibilities persist. At its induced norm choice the cut satisfies the point and admissibility conditions, the first and second Galois provisions, residual irreducibility and tame eigenvalue conditions, the weight, block and de Rham*

jump conditions, and the unramifiedness, depth-zero, base-change and weight identities; localization is unique, the coefficients satisfy the required largeness and realization conditions, and the norm, good triples, finite types and common twist agree with the cut data. Moreover, under $\mathscr{H}_2$, there exist a type threshold $C_T \geq 40$ and a paired-character constant $C' \geq 8C_T$ depending only on $\mathscr{H}_2$ (fixed before selecting the cuts, with any integer $C_T \geq 40$ and real $C' \geq 8C_T$ admissible) for the type-prime paired-character estimates on the cut.

(II) Restoration of the family bound. *If also $\mathscr{H}_2$ holds, F_m is twist-good, and the Galois hypotheses and four local compatibilities hold at the chosen norm for every $x > x_1$ and $i \in I_x$, then F_m satisfies the family local-error estimate of Definition 36.11.*

Outline. Cut by the good character idempotent, using 40.11 for depth zero and 36.25 with local duality for residual cohomology vanishing. Sum local costs on the cut and use the integral projector and 20.7 to absorb the transfer error into the family constant. The complete proof is on p. 1294. □

Lemma 36.27 (The selected twist and the multiplicity sum)**.** *On a twist-good member with norm-choice facts, take the tied representations and let $\nu_\ell = \min\{v_\ell(\varepsilon_\ell(g)^{tr}-1) : g \in G_E,\ \varepsilon_\ell(g)^{tr} \neq 1\}$, with ε_ℓ cyclotomic. For each twisted i and good π the Selmer length is finite and*

$$s_\pi \leq \nu_\ell + \ell^{-2}, \qquad t \sum_{v \in Q_i} \lambda_i(O_i/(\zeta_{i,v} - 1)) \leq \ell^{-2}. \tag{36.26}$$

For every finite prime set S,

$$\sum_{\ell \in S} \nu_\ell \log \ell \leq \log Q_S + \log(tr). \tag{36.27}$$

Under theta scalars, the Z-multiplicity identity, internal isotypic decomposition and localization uniqueness, every layer j and twisted place w satisfy

$$\sum_{i \in I_{j,w}} \sum_{\pi \in Z_{\mathrm{good}}(i)} \mathrm{mult}(i, \pi) \leq \dim Z_j(\mu_w). \tag{36.28}$$

Outline. Cancel common characters and use 36.24 for trace equality and 36.25, matrix density, and trace pairing for integral conjugacy and the twist. Bound Selmer and tame costs by signature and conductor roots, valuations by cyclotomic image, and multiplicity by theta. The complete proof is on p. 1296. □

Lemma 36.28 (The Frobenius remainder at a cut)**.** *Let ℓ be odd, $q_0 = \ell - 1$, $A = O[[Z]]$ for the integers O of a finite extension of $\mathbf{Q}_\ell$, and*

$$\varphi(Z) = (1+Z)^\ell - 1 = ZQ.$$

Index the components by $i \in \mathbf{Z}/d\mathbf{Z}$, with $d > 0$. Let $P_i \subset N_i$, where N_i is a free A-lattice, and suppose $Z^{H_*} N_i \subset P_i$ for an integer $H_* \geq 0$. Suppose a cyclotomic generator acts semilinearly by $\gamma(Z) = (1+Z)^u - 1$, trivially modulo Z on both N_i and P_i. Here $u \bmod \ell$ has order q_0 and $v_\ell(u^{q_0} - 1) = 1$.

Suppose linearized Frobenius is injective on $\varphi^* N_{i-1}$ in the ambient rational module. For nonnegative integers M_i, h_i, write $H_i = M_i + h_i$, and assume

$$Q^{M_i} \varphi N_{i-1} \subset N_i, \qquad Q^{h_i} P_i \subset A\varphi P_{i-1}.$$

Let c be a component, $J \geq 0$, and $g, h \geq 0$ with $H_* \leq 2h$ and $H_{c+k} \leq 2g$ for $1 \leq k \leq J$. Put $C_h = \lceil 1 + \log_\ell(2h+1) \rceil$. Then $\ell^{\rho_c} N_c \subset P_c$ for an integer $\rho_c \geq 0$ satisfying

$$\rho_c \leq C_h + \frac{\ell}{q_0^2} \left((2g + C_h + 3) \frac{q_0}{q_0 - 1} + \frac{2h}{q_0^J} \right).$$

Outline. Use the cyclotomic action to trade powers of Z for powers of ℓ, counting the valuations of $u^k - 1$. Expand powers of Q and use Frobenius to bound each least Z-exponent in terms of its successor, then iterate the bound over J steps. The complete proof is on p. 1296. □

Lemma 36.29 (Simultaneous lifting at two downward positions)**.** *Let O be a complete discrete valuation ring of residue characteristic ℓ, with uniformizer ϖ. Put $A = O[[z]]$. On an interval of length v, let p_j, $0 \le j < v$, be monic distinguished polynomials of positive degree, pairwise relatively prime over $\mathrm{Frac}(O)$. Multiplication of a p_j by a unit of A is allowed. For integers $H_j \ge 0$, put $\Lambda = \prod_j p_j^{H_j}$. Mark two distinct positions $j_0 \ne j_1$ and put $D_i = p_{j_i}^{b_i}$, $b_i \ge 0$, for $i = 0, 1$.*

For a finite free O-module V, choose saturated submodules V_0, V_1, and write $V_A = A \otimes_O V$. Define

$$\mathcal{N}(V) = \{\Lambda x : x \in V_A,\ x \in (V_i)_A + D_i V_A\ (i = 0, 1)\}.$$

Define $\mathcal{I}(V)$ by allowing the residue of x at each marked position, modulo $(V_i)_A + D_i V_A$, to be the image of a constant vector after inverting ϖ. Thus in this definition $x \in V_A$, and the two rational constant vectors may differ.

Give a second finite free module W the same data, with its own saturated submodules. Suppose $F : W \to V$ is O-linear and integers $r_0, a_0 \ge 0$ with $r_0 \le 2a_0$ satisfy $\ell^{r_0} V \subset F(W)$. Then for every $y \in \mathcal{I}(V)$ there are $y_1 \in \mathcal{I}(W)$ and $\nu \in \mathcal{N}(V)$ with

$$\ell^{2a_0} y = F_A(y_1) + \ell^{2a_0} \nu.$$

Outline. Distinguished division gives integral constant residues, and Smith coordinates express their difference as a sum of filtration vectors and a vector divisible by ϖ^α. Interpolate the residues and lift both constant vectors after scaling by ℓ^{2a_0}. The complete proof is on p. 1297. □

Lemma 36.30 (The numerical factor at a ramified prime)**.** *Let ℓ be odd and let e, s, n be positive integers with $es = \ell - 1$. Let h, S, η, r, m be nonnegative integers and $1 \le h' \le h + S + 1$. Set $\log_\ell 0 = 0$ in the following formulas. Write $L(k) = k + v_\ell(k!)$, and put*

$$\begin{aligned} x &= h'/s, \quad R = L(2\lceil 2x \rceil - 1), \quad D = L(\lceil 4x \rceil - 1), \\ Y &= 2R + 4D + e\big(L(S) + \lfloor \log_\ell h' \rfloor\big) + \max\{2D, \lceil \eta + 1/2 + \log_\ell h' \rceil\}, \\ B &= eh/\ell + eS + e \log_\ell h + 1 + \eta. \end{aligned}$$

Let $J \subset [-h, 2h] \cap \mathbf{Z}$ exclude zero, and set

$$E' = 2L(4h) + \sum_{j \in J} (1 + v_\ell(j)) + L(h).$$

Let C be a quotient of finite lattice cohomology over a coefficient discrete valuation ring O above ℓ. Suppose C is generated by at most $2(ne + 1)r$ elements and has a submodule C_0 generated by at most $ne(e + 1)m$ elements, with $\ell^Y C \subset C_0$ and $\ell^{E'} C_0 = 0$. Then its length, divided by $\mathrm{length}_O(O/\ell O)$, is at most

$$160\big((ne + 1)rB + ne(e + 1)m(h + 1)\big).$$

Outline. Count factorial valuations and bound the logarithmic terms to obtain $Y \le 80B$ and $E' \le 18h$. Apply the generator bounds to C/C_0 and C_0 and add their normalized lengths. The complete proof is on p. 1298. □

Lemma 36.31 (The coefficient field and the crystalline representations at q, d)**.** *Let $\ell \in \{q, d\}$, let $w \mid \ell$ in the host E, and put $K = E_w$. If K_0 is the maximal unramified subfield of $K/\mathbf{Q}_\ell$, then*

$$K = K_0(\sqrt{-\ell}), \qquad K \subset K_0(\mu_\ell), \qquad [K_0(\mu_\ell) : K] = (\ell - 1)/2.$$

For the representation assertions, assume Fontaine's crystalline period comparison from Definition 36.15. At a twisted index i of residue characteristic ℓ with nonempty good spectrum on the good cut of Theorem 36.26(I), write R_x for the full representation at $x \in X_i$, and put $V_x = R_x \otimes \chi_{i,1}^{-1}$. At a good point π, put $W_\pi = \rho_i(\pi) \otimes \chi_{i,1}^{-1}$. Over K, every V_x and both W_π and $W_\pi^\vee(1)$ are crystalline. Every subrepresentation of a finite direct sum of the $V_x^\vee(1)$ is crystalline. Its jumps at an embedding τ belong to the union of the sets

$$\{-1 - a : a \text{ is a jump of } V_x \text{ at } \tau\}$$

over the summands in that sum.

Outline. Use the quadratic Gauss sum of 8.3 and the cyclotomic Eisenstein polynomial to compute the local field and index. Apply 36.26 and dimension additivity for crystalline constituents, then use the filtration and Tate twist convention of 39.4 for the jumps. The complete proof is on p. 1298. □

Definition 36.32 (Cyclic carries and position data)**.** Let ℓ be prime and let $d', t \geq 1$ be integers. For $h : \mathbf{Z}/d'\mathbf{Z} \to \mathbf{N}$ and $i \in \mathbf{Z}/d'\mathbf{Z}$, the cyclic carry of h at i is

$$\sigma_h(i) = \Big\lfloor \sum_{j \geq 1} h(i+j)\, \ell^{-j} \Big\rfloor.$$

Position data at ℓ of width t consist of maps $h, w, \eta' : \mathbf{Z}/d'\mathbf{Z} \to \mathbf{N}$, signs $\varsigma_i \in \{1, -1\}$ and finite multisets U_i of integers, $i \in \mathbf{Z}/d'\mathbf{Z}$, with the following six properties. Put $\sigma_i = \sigma_h(i)$. Let the low window W_i be $[1, \eta'_i]$ if $\varsigma_i = 1$ and $[h_i - \eta'_i, h_i - 1]$ if $\varsigma_i = -1$. Put $\bar{x} = x$ if $\varsigma_i = 1$ and $\bar{x} = h_i - x$ if $\varsigma_i = -1$. (P1) Every $u \in U_i$ satisfies $1 \leq u \leq h_i - 1$. (P2) $1 \leq w_i$ and $w_i + t \leq h_i$. (P3) $w_i + t - 1 \leq \eta'_i$ if $\varsigma_i = 1$, and $h_i - \eta'_i \leq w_i$ if $\varsigma_i = -1$. (P4) $\eta'_i + 4\sigma_i + 6 < \ell$. (P5) $|x - y| > \sigma_i + 2$ for every $x \in U_i \cap W_i$ and every integer $y \in [w_i, w_i + t - 1]$. (P6) $\bar{x} \not\equiv m \pmod{\ell}$ for every $x \in U_i$ outside W_i and every integer $m \in [-2\sigma_i - 2,\ \eta'_i + 2\sigma_i + 3]$.

Lemma 36.33 (Position data at one embedding)**.** *Let ℓ be prime, $t \geq 1$ an integer, $\varsigma \in \{1, -1\}$, and let F_l, F_c be finite sets of integers. Put $c_t = (t-1)/2$ and $\delta(z) = \varsigma(z - c_t)$. Let η, S, B, W_0 be real numbers and σ, σ_P integers such that*

$$t \leq \eta, \quad B \geq 0, \quad S \geq 0, \quad 2B + \tfrac{t}{2} + 3 \leq W_0, \quad \sigma \leq \min(2B, \sigma_P), \quad 2\eta + 8B + 7 < \ell.$$

Suppose every $z \in F_l$ satisfies $2B + t/2 + 3 \leq |z - c_t| \leq \eta$, every $z \in F_c$ satisfies $W_0 \leq \delta(z) \leq W_0 + S + \eta + 1$, and ℓ divides no integer in

$$[W_0 - \eta - 2\sigma_P - 3,\ W_0 + S + 2\eta + 2\sigma_P + 4]. \tag{36.29}$$

Let m_l be the least element of $F_l \cup \{0\}$ and M_l the largest element of $F_l \cup \{t-1\}$. Define m_a, M_a in the same way with $F_l \cup F_c$ in place of F_l. Put $\eta' = M_l - m_l + 1$, $U = \{z + w : z \in F_l \cup F_c\}$ and

$$(w, h) = (1 - m_l,\ M_a + 2 - m_l) \ \text{ if } \varsigma = 1, \qquad (w, h) = (1 - m_a,\ M_l + 2 - m_a) \ \text{ if } \varsigma = -1.$$

Then $h \leq W_0 + S + 2\eta + 3$, the multiplicity of $b + w$ in U is 1 for $b \in F_l \cup F_c$ and 0 for other integers b, and $(h, w, \eta', \varsigma, U)$ satisfies (P1)–(P6) of Definition 36.32 at one index, with σ in place of σ_i.

Outline. Bounds m_l, M_l, η' from the F_l inequalities to verify (P4), (P5), and the multiplicities. Treating $\varsigma = \pm 1$ separately checks (P1)–(P3) and the bound on h, while any congruence violating (P6) places an ℓ-multiple in the excluded interval. The complete proof is on p. 1298. □

Definition 36.34 (Admissible cocycles and the large-prime statement)**.** Let $\ell \geq 5$ be prime and $K/\mathbf{Q}_\ell$ finite unramified of degree d'. Let L' be a finite extension of $\mathbf{Q}_\ell$ containing the images of all embeddings of $K(\mu_\ell)$, with integers O and uniformizer ϖ. Number the embeddings $\tau_i : K \to L'$, $i \in \mathbf{Z}/d'\mathbf{Z}$, in Frobenius order as in Lemma 36.19. For a continuous representation V of G_K over L', write $m_V(\tau, a)$ for the multiplicity of the jump a at τ, with $\mathbf{Q}_\ell(1)$ of jump -1. Write χ_ℓ for the cyclotomic character.

Fix $t \geq 1$ and position data $(h, w, \eta', \varsigma, U)$ at ℓ of width t. Let V be a continuous crystalline representation with $m_V(\tau_i, a) = \mathrm{mult}_{U_i}(a)$ for all i, a, let $T_0 \subset V$ be a stable O-lattice, and let $\chi_1, \dots, \chi_t : G_K \to O^\times$ be continuous crystalline characters with $\chi_c = \chi_1 \chi_\ell^{1-c}$ and with the single jump $w_i + c - 1$ at τ_i. For $g \in G_K$ put

$$P_g(U) = \det(U - g \mid V) \prod_{c=1}^{t} (U - \chi_c(g)), \qquad \tilde{m}_i(a) = \mathrm{mult}_{U_i}(a) + \mathbf{1}_{[w_i, w_i + t - 1]}(a).$$

A full representation for these data is a continuous crystalline representation V' of G_K over L' with $m_{V'}(\tau_i, a) = \tilde{m}_i(a)$ for all i, a, such that $\det(U - g \mid V') - P_g(U)$ has coefficients in ϖO for every g.

Let L be a stable O-lattice in V or in one of the characters χ_c, let $\mathcal{M}$ be a finite O-module, and let $z : G_K \to L \otimes_O \mathcal{M}$ be a continuous cocycle for the action of g by $\chi_1(g)^{-1} g$. For $a \geq 1$ with $\varpi^a \mathcal{M} = 0$, let $\mathcal{E}_{z,a} = (L \otimes \mathcal{M}) \oplus O/\varpi^a$ with

$$g(v, s) = (gv + s\chi_1(g) z(g),\ \chi_1(g) s).$$

The cocycle z is admissible if there is an integer $a_0 \geq 1$ such that for every integer $a \geq a_0$ with $\varpi^a \mathcal{M} = 0$ the $O[G_K]$-module $\mathcal{E}_{z,a}$ is isomorphic to L_1/L_2 for stable O-lattices $L_2 \subset L_1$ in a finite direct sum of full representations.

Statement (T) asserts, for all such data and every finite O-module $\mathcal{M}$, the following. (T1) A cocycle $z : G_K \to T_0 \otimes \mathcal{M}$ is admissible exactly when $z = \sum_j \xi_j \otimes m_j + z_0$, where $m_j \in \mathcal{M}$, z_0 is a coboundary, and the ξ_j are cocycles of $\chi_1^{-1} V$ with values in T_0 whose classes are finite in the sense of Definition 36.15. (T2) For $2 \leq c \leq t$ and a stable lattice L of χ_c, a cocycle with values in $L \otimes \mathcal{M}$ is admissible exactly when it is a coboundary. (T3) For a stable lattice L of χ_1, a cocycle with values in $L \otimes \mathcal{M}$ is admissible exactly when its restriction to the inertia group of K is a coboundary there.

Definition 36.35 (Admissible finite modules and cohomology classes)**.** Fix data $(K, L', O, \varpi, \tau_i, t, h, w, \eta', \varsigma, U, V, T_0, \chi_1, \dots, \chi_t)$ at a prime $\ell \geq 5$ as in Definition 36.34. A finite $O[G_K]$-module N is *admissible* for these data if there exist a finite direct sum $\mathcal{V}$ of full representations, stable O-lattices $L_2 \subset L_1 \subset \mathcal{V}$, and an $O[G_K]$-equivariant surjection $p : L_1 \twoheadrightarrow N$ with $\ker p = L_2$. Let W be T_0 or $O(\chi_c)$ for $1 \leq c \leq t$, write $W' = \chi_1^{-1} W$, and let $\mathcal{M}$ be a finite O-module. Since O is a complete discrete valuation ring and ϖ is a uniformizer, there exists an integer $a \geq 1$ with $\varpi^a \mathcal{M} = 0$. Define $\mathrm{Adm}(W, \mathcal{M}) \subset H^1(K, W' \otimes_O \mathcal{M})$ to be the subset of cohomology classes represented by continuous cocycles $z : G_K \to W \otimes_O \mathcal{M}$ (for the action $g \mapsto \chi_1(g)^{-1} g$) for which there exists $a_0 \geq 1$ such that $\mathcal{E}_{z,a}$ is an admissible finite $O[G_K]$-module for every integer $a \geq a_0$ with $\varpi^a \mathcal{M} = 0$.

Definition 36.36 (Cyclotomic scalar data and shifted kernels)**.** Let $\ell \geq 5$ be prime and let k be a field of characteristic ℓ. Put $A = k[[X]]$ and $B = k((X))$. Scalar Frobenius and cyclotomic operators are commuting k-algebra endomorphisms $\varphi, \gamma : B \to B$ preserving A, with

$$\varphi(X) = u X^{\ell}, \qquad \gamma(X) = X(1 - \kappa X), \qquad u \in A^{\times}, \quad \kappa \in A, \quad \kappa(0) = 1.$$

To specify the cyclotomic condition, introduce a second variable π, put $A_\pi = k[[\pi]]$, $B_\pi = k((\pi))$, and set

$$\varphi_\pi(f)(\pi) = f(\pi^{\ell}), \qquad \gamma_\pi(f)(\pi) = f((1+\pi)^{1+\ell} - 1).$$

For $z \in \mathbf{Z}_\ell$, the series $(1+\pi)^z$ means $\sum_{j \geq 0} \overline{\binom{z}{j}} \pi^j$, with coefficients reduced from $\mathbf{Z}_\ell$ to k. Let $\Delta = \{ z \in \mathbf{Z}_\ell^{\times} : z^{\ell - 1} = 1 \}$; it acts on B_π by the substitutions $\delta_z(\pi) = (1+\pi)^z - 1$. The scalar data are cyclotomic if there is a k-algebra embedding $\iota : B \hookrightarrow B_\pi$ such that

$$\iota(A) = A_\pi^{\Delta}, \quad \iota(X) = v \pi^{\ell - 1} \ (v \in A_\pi^{\times}), \quad \iota\varphi = \varphi_\pi \iota, \quad \iota\gamma(a) = \gamma_\pi \iota(a) \quad (a \in A)$$

(hence $\iota(B) = B_\pi^{\Delta}$ and $\iota\gamma = \gamma_\pi \iota$ on B). For a finite-dimensional B-space E, an A-lattice is a finitely generated A-submodule $M \subset E$ with $BM = E$. For $a \in \mathbf{Z}$, write $X^a M = \{ X^a m : m \in M \}$. For a φ-semilinear map $f : E \to E$, put $f^* M = A f(M)$, the A-span of its image. For a k-linear map $\Psi : E \to E$, put

$$K_a(M, \Psi) = X^a M \cap \ker \Psi.$$

For $b \in B \setminus \{0\}$ and $y \in B_\pi \setminus \{0\}$, let $v_X(b)$ and $v_\pi(y)$ be their least Laurent exponents (with $v_X(0) = v_\pi(0) = +\infty$), and write $M_n(R)$ for the ring of $n \times n$ matrices over a ring R. Frobenius contraction refers to the equality $v_X(\varphi(b)) = \ell v_X(b)$, or equivalently to multiplication of the X-adic order by ℓ.

Lemma 36.37 (Scalar identities)**.** *Under the hypotheses of Definition 36.36, the substitutions δ_z define a Δ-action on A_π and B_π, and $v_X(\varphi(b)) = \ell v_X(b)$ for every $b \in B \setminus \{0\}$. When the scalar data are cyclotomic with embedding $\iota : B \hookrightarrow B_\pi$, one has $\iota(B) = B_\pi^{\Delta}$ and $\iota\gamma = \gamma_\pi \iota$ on B. For any A-lattice M in a finite-dimensional B-space E, any $a \in \mathbf{Z}$, and any k-linear map $\Psi : E \to E$, the shift $X^a M$ is an A-lattice and $K_a(M, \Psi)$ is a k-subspace.*

Outline. The binomial identities $\delta_z \delta_w = \delta_{zw}$ give the action. An invariant of B_π times a power of $\iota(X)$ lies in $\iota(A)$, and writing $b = X^m a$ with $a \in A^{\times}$ gives $v_X(\varphi(b)) = \ell v_X(b)$. The complete proof is on p. 1299. $\square$

Lemma 36.38 (The exact ψ-shift)**.** *For cyclotomic scalar data as in Definition 36.36, every $y \in B_\pi$ has a unique expansion*

$$y = \sum_{j=0}^{\ell-1} (1+\pi)^j \varphi_\pi(y_j), \qquad y_j \in B_\pi.$$

Let $\psi_\pi(y) = y_0$. There is a k-linear map $\psi : B \to B$, determined by $\iota\psi = \psi_\pi \iota$, with

$$\psi(\varphi(b_1)b_2) = b_1\psi(b_2) \quad (b_1, b_2 \in B), \qquad \psi\varphi = 1, \qquad \psi\gamma = \gamma\psi, \qquad \psi(X^a A) = A \quad (2-\ell \le a \le 1).$$

Let E be a finite-dimensional B-space, let $f : E \to E$ be φ-semilinear with $Bf(E) = E$, and let $g_0, g : E \to E$ be γ-semilinear maps satisfying $fg_0 = gf$. There is a k-linear map $\Psi : E \to E$ such that

$$\Psi(bf(x)) = \psi(b)x \quad (b \in B,\ x \in E), \qquad \Psi g = g_0 \Psi.$$

For every A-lattice $M_0 \subset E$, put $P = f^* M_0$. Then

$$\Psi(X^a P) = M_0 \qquad (2-\ell \le a \le 1).$$

If $(g_0 - 1)M_0 \subset XM_0$, then

$$(g-1)K_a(P,\Psi) = K_{a+1}(P,\Psi) \qquad (2-\ell \le a \le 0). \tag{36.30}$$

Outline. Group Laurent coefficients modulo ℓ in the $(1+\pi)^j$-basis via Definition 36.36 to construct ψ_π, descend it to ψ, and verify $\psi(X^a A) = A$. Applying ψ in a Frobenius-image basis of f^*M_0 defines Ψ. Lifting $g-1$ to B_π splits it into bijections on the $j \ge 1$ components of $\ker \psi_\pi$, and Δ-invariants give the exact shift. The complete proof is on p. 1299. □

Definition 36.39 (Residue models and their extension classes)**.** Retain the scalar data of Definition 36.36, let $d' \ge 1$ be an integer, and put $I = \mathbf{Z}/d'\mathbf{Z}$, $A_I = \prod_{i\in I} k[[X]]$, and $B_I = \prod_{i\in I} k((X))$. A cyclotomic module over B_I is a family of finite-dimensional B-spaces D_i, with φ-semilinear maps $\varphi_i : D_{i-1} \to D_i$ whose images span D_i, and bijective γ-semilinear maps $\gamma_i : D_i \to D_i$, satisfying

$$\varphi_i \gamma_{i-1} = \gamma_i \varphi_i;$$

a morphism $(D_i) \to (D_i')$ is a family of B-linear maps $f_i : D_i \to D_i'$ intertwining φ_i and γ_i. A lattice in this module is a family $N_i \subset D_i$ of A-lattices, regarded as the A_I-module $\prod_i N_i$. Trivial cyclotomic action modulo X means $(\gamma_i - 1)N_i \subset XN_i$ for every i.

Let $K/\mathbf{Q}_\ell$ be a finite extension and let $\mathcal{C}$ be a class of finite-dimensional continuous $k[G_K]$-representations closed under subquotients and finite direct sums and containing k. A residue realization D is a k-linear exact functor from $\mathcal{C}$ to these cyclotomic modules, preserving dimension at each component, together with elements $\mathbf{1}_i \in D(k)_i \setminus \{0\}$ satisfying

$$\varphi_i(\mathbf{1}_{i-1}) = \mathbf{1}_i, \qquad \gamma_i(\mathbf{1}_i) = \mathbf{1}_i.$$

It has the following fixed-vector property: if $x_i \in D(U)_i$ satisfy $\varphi_i(x_{i-1}) = x_i$ and $\gamma_i(x_i) = x_i$, then there is a G_K-equivariant map $v : k \to U$ with $D(v)_i(\mathbf{1}_i) = x_i$.

Fix a collection $\mathscr{B}$ of pairs (U_b, N^b), where $U_b \in \mathcal{C}$ and N^b is a lattice in $D(U_b)$. A model lattice of $U \in \mathcal{C}$ is a lattice obtained as follows. Choose $(U_b, N^b) \in \mathscr{B}$, an injective equivariant map $j : U_1 \hookrightarrow U_b$, and a surjective equivariant map $p : U_1 \twoheadrightarrow U$. Set

$$N_i = D(p)_i\big(D(j)_i^{-1}(N_i^b)\big). \tag{36.31}$$

In the setting of Definition 36.34 with $k = O/\varpi O$, choose $\gamma \in \mathrm{Gal}(K(\mu_{\ell^\infty})/K(\mu_\ell))$ with $\chi_\ell(\gamma) = 1+\ell$, identify $A_\pi = k[[\pi]]$ with $O[[Z]]/\varpi O[[Z]]$ via $\pi = \bar{Z}$, and embed $A = k[[X]]$ into A_π by $\iota(X) = -\prod_{z\in\Delta}((1+\pi)^z - 1)$, which gives cyclotomic scalar data with $u = 1$. Let T be a stable O-lattice in a crystalline L'-representation with $\overline{T} = T/\varpi T \in \mathcal{C}$. Let $D_{\mathrm{et}}(T)$ be the étale cyclotomic module of T over $K(\mu_\ell)$ with Wach lattice $N(T)$ as in Lemmas 36.17–36.18 (at $s = 1$), on which Δ acts via the Teichmüller lift $\omega \circ \bar{\chi}_\ell : \mathrm{Gal}(K(\mu_\ell)/K) \xrightarrow{\sim} \Delta$ of the modulo-ℓ cyclotomic character. A residue presentation of D on T is a family of surjective additive maps

$\jmath_i : D_{\mathrm{et}}(T)^{\Delta} \to D(\overline{T})_i$ compatible with Frobenius, γ, and scalar multiplication through $O[[Z]]^{\Delta} \to A_{\pi}^{\Delta} = \iota(A)$ (at the i-th component) such that its reduced base lattice

$$\overline{N}_i(T) = \{\jmath_i(x) : x \in N(T),\ \delta x = x \text{ for every } \delta \in \Delta\}$$

spans $D(\overline{T})_i$ over B. Require that these maps belong to a single system $\jmath_{T,U,q,i} : D_{\mathrm{et}}(T)^{\Delta} \to D(U)_i$, defined for every stable O-lattice T and every equivariant O-linear surjection $q : T \twoheadrightarrow U$ onto a finite-dimensional continuous $k[G_K]$-representation, with every such U belonging to $\mathcal{C}$. Each map is surjective and satisfies the preceding scalar and operator compatibilities. The displayed $\jmath_i$ is the map for $T \twoheadrightarrow \overline{T}$. Write e_i for the source component idempotent used by $\jmath_{T,U,q,i}$. If $f : T' \hookrightarrow T$ is an equivariant lattice map with image $\ker q$, require

$$\jmath_{T,U,q,i}(x) = 0 \quad \Longleftrightarrow \quad e_i x \in \operatorname{im} D_{\mathrm{et}}(f).$$

For equivariant maps $f : T \to T'$ and $g : U \to U'$ with $q'f = gq$, require

$$\jmath_{T',U',q',i} \circ D_{\mathrm{et}}(f) = D(g)_i \circ \jmath_{T,U,q,i}.$$

For the trivial lattice and every equivariant surjection onto k, the image of every Δ-invariant generator of its Wach lattice must generate $A\mathbf{1}_i$ at component i. For the canonical modulo-ϖ realization $D(U)_i$ obtained by taking Δ-invariants of the étale (φ, Γ)-module of U over $\prod_{i \in I} k((\pi))$, reduction modulo ϖ supplies $\jmath_i$.

Let $U \in \mathcal{C}$. A cocycle is a continuous function $z : G_K \to U$ with $z(gh) = z(g) + gz(h)$. It defines the extension $E_z = U \oplus k$ with action

$$g(u, a) = (gu + az(g), a).$$

Such a cocycle has a model if $E_z \in \mathcal{C}$ and E_z has a model lattice of the form (36.31). Coboundaries are the cocycles $z(g) = gu - u$, with $u \in U$.

When $E_z \in \mathcal{C}$, to define the Herr extension class of z, choose $e_i \in D(E_z)_i$ mapping to $\mathbf{1}_i$. Via the injection $D(U)_i \hookrightarrow D(E_z)_i$, put

$$y_i = \varphi_i(e_{i-1}) - e_i, \qquad t_i = \gamma_i(e_i) - e_i.$$

They satisfy $(\gamma_i - 1)y_i = \varphi_i(t_{i-1}) - t_i$. The Herr extension class is the class of $(y_i, t_i)_i$ in

$$\frac{\{(y_i, t_i)_i : (\gamma_i - 1)y_i = \varphi_i(t_{i-1}) - t_i \text{ for all } i\}}{\{(\varphi_i(x_{i-1}) - x_i,\ \gamma_i(x_i) - x_i)_i : x_i \in D(U)_i\}}.$$

For an integer multiset J_i and an integer shift w_i, define its reduced jump multiplicity at $\alpha \in k$ by

$$r_i(J_i, w_i; \alpha) = \#\{u \in J_i : \overline{u - w_i} = \alpha\},$$

with multiplicities counted in J_i. The reduced jump multiset is $\{\overline{u - w_i} : u \in J_i\}$.

Lemma 36.40 (Residue lattices and the Herr class)**.** *Under the hypotheses of Definition 36.39, the modules in* (36.31) *are lattices. In the unramified level-one crystalline setting, canonical reduction gives the presentation maps and each $\overline{N}_i(T)$ is a lattice. The Herr class of a cocycle with $E_z \in \mathcal{C}$ is independent of the lifts e_i.*

Outline. Exactness and finite generation over A make the modules of (36.31) lattices. In the unramified level-one setting, a Δ-invariant Wach basis and averaging over Δ give the reduction maps and the lattice $\overline{N}_i(T)$; the cocycle law and the commutation of φ_i and γ_i give the extension and the independence of the Herr class from the lifts. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1301. □

Lemma 36.41 (The geometric carry bound)**.** *Let $d' \geq 1$, $\ell \geq 2$ and $H \geq 0$ be integers. Let $h : \mathbf{Z}/d'\mathbf{Z} \to \mathbf{N}$ satisfy $h_i \leq H$ for every i, and put*

$$\sigma_i = \left\lfloor \sum_{j \geq 1} h_{i+j} \ell^{-j} \right\rfloor.$$

Then $\sigma_i \leq H/(\ell - 1)$ for every i.

Proof. For every $j \geq 1$, $0 \leq h_{i+j}\ell^{-j} \leq H\ell^{-j}$. The geometric series on the right converges and has sum $H/(\ell-1)$. Comparison gives convergence of the series defining σ_i and

$$\sigma_i \leq \sum_{j\geq 1} h_{i+j}\ell^{-j} \leq \sum_{j\geq 1} H\ell^{-j} = \frac{H}{\ell-1}.$$

□

Lemma 36.42 (Excluding false roots in a downward window)**.** *Let $\ell, d', t, h, w, \eta', \varsigma, U$ be as in Definition 36.32, and assume (P1)–(P4) and (P6). Put $J_i = U_i + \{w_i, w_i+1, \ldots, w_i+t-1\}$, with multiset addition. If $\varsigma_i = -1$, $u \in J_i$, and a is an integer with $0 \leq a \leq h_i - w_i$, then*

$$\ell \mid a - (u - w_i) \quad \Longrightarrow \quad a = u - w_i.$$

Outline. Set $m = h_i - w_i - a \in [0, \eta_i']$ by (P3), so $h_i - u \equiv m \pmod{\ell}$. Condition (P6) rules out $u \in U_i$ outside the low window; otherwise (P1)–(P4) give $|(h_i - u) - m| \leq \eta_i' < \ell$, forcing $h_i - u = m$. The complete proof is on p. 1302. □

Lemma 36.43 (Transporting independent admissible residue classes)**.** *Retain the data of Definition 36.34, let $k = O/\varpi O$, and put $\overline{W} = (\chi_1^{-1}T_0)/\varpi(\chi_1^{-1}T_0)$. Suppose the residue realization of Definition 36.39 carries every χ_1^{-1}-twisted admissible finite module to a module with a model lattice. Assume also that admissibility of a residue class is equivalent to admissibility of its extension at exponent 1. Given s linearly independent classes in $\mathrm{Adm}(T_0, k)$, one can choose continuous cocycles $z_1, \ldots, z_s : G_K \to \overline{W}$ whose extensions have model lattices and whose classes modulo coboundaries are linearly independent. Their Herr extension classes are linearly independent.*

Outline. Quotienting and twisting identifies coefficients with $\overline{W}$, giving modelled cocycles and preserving independence. A zero Herr class yields φ_i- and γ_i-fixed lifts splitting the extension, so its coefficients vanish. The complete proof is on p. 1302. □

Lemma 36.44 (Character quotients and the two residual vanishings)**.** *Let $K/\mathbf{Q}_\ell$ be a finite extension. Let k be a finite field of characteristic ℓ, let T be a finite-dimensional continuous $k[G_K]$-representation, and let $\omega = \bar{\chi}_\ell$. Suppose that, for each $e \in \{-1, 0\}$, every finite-dimensional representation A, every injective equivariant map $j : A \hookrightarrow T$, and every equivariant map $p : A \to k(\omega^{-e})$, one has $p = 0$. Then*

$$\mathrm{Hom}_{G_K}(k, T) = 0, \qquad \mathrm{Hom}_{G_K}(T, k(\omega)) = 0.$$

Under local Tate duality these imply $H^0(K, T) = H^2(K, T) = 0$.

Proof. A nonzero equivariant map $k \to T$ is injective. Taking $A = k$, $e = 0$, and $p = \mathrm{id}_k$ contradicts the hypothesis. For the second assertion take $A = T$, $j = \mathrm{id}_T$, and $e = -1$. The first vanishing says $T^{G_K} = 0$. Local Tate duality identifies the dual of $H^2(K, T)$ with $H^0(K, T^\vee(1))$, and the latter is the second displayed Hom space. This proves both cohomological vanishings. □

Lemma 36.45 (Unramified extensions and the residue dimension one test)**.** *Retain the setting of Definition 36.34. Assume the crystalline sum and unramified twist rules, closure of admissible modules under subquotients and finite direct sums, and admissibility of $O(\chi_1) \otimes_O X$ for every finite O-module X. Assume also that a class is admissible whenever its extension is admissible at one exponent annihilating its coefficient module. Then*

$$H^1_{\mathrm{ur}}(K, X) \subset \mathrm{Adm}(O(\chi_1), X) \qquad \text{for every finite } X.$$

Suppose, in addition, that these admissible classes form submodules, are functorial in X, satisfy $i_*^{-1}(\mathrm{Adm}(O(\chi_1), X')) \subset \mathrm{Adm}(O(\chi_1), X)$ for every injective O-linear map $i : X \to X'$, and have residue dimension at most one. Then

$$\mathrm{Adm}(O(\chi_1), X) = H^1_{\mathrm{ur}}(K, X), \qquad \dim_k \mathrm{Adm}(O(\chi_1), k) = 1.$$

Outline. A lattice in $\chi_1\mu \oplus \chi_1$ for $\mu = (1+q)^\nu$ pushes out to show every unramified class is admissible. Subtracting the unramified part and applying Nakayama's lemma to any nonzero remainder contradicts the residue dimension bound. The complete proof is on p. 1303. □

Lemma 36.46 (The carry recurrence and the standard residue counts)**.** *Let the position data satisfy (P1)–(P6) of Definition 36.32. Put*

$$d_i = \sum_{u<w_i} \mathrm{mult}_{U_i}(u), \qquad \rho_i(\alpha) = r_i(U_i, w_i; \alpha) \quad (\alpha \in k),$$

where k has characteristic ℓ and r_i is defined in Definition 36.39. Then

$$\sigma_i + 1 \le \ell(\sigma_{i-1} + 1) - h_i.$$

No $u \in U_i$ is congruent modulo ℓ to an integer in

$$[w_i - \sigma_i - 2,\ w_i + t + 1 + \sigma_i].$$

In particular, $\rho_i(\bar a) = 0$ for $0 \le a \le \sigma_i + 1$. The two block counts are

$$\sum_{j=0}^{w_i-1} \rho_i(\overline{-w_i + j}) = d_i \quad (\varsigma_i = 1), \qquad d_i + \sum_{a=0}^{h_i-w_i} \rho_i(\bar a) = |U_i| \quad (\varsigma_i = -1).$$

Outline. The identity $\ell S_{i-1} = h_i + S_i$ for $S_i = \sum_{j\ge1} h_{i+j}\ell^{-j}$ gives the carry bound. Conditions (P4)–(P6) exclude cloud jumps and turn low-window congruences modulo ℓ into equalities, proving the excluded band and block counts. The complete proof is on p. 1303. □

Lemma 36.47 (A dimension bound for bounded Herr pairs)**.** *Let ℓ be a prime and let $(h, w, \eta', \varsigma, U)$ be position data at ℓ of width t on $\mathbf{Z}/d'\mathbf{Z}$, as in Definition 36.32, with cyclic carries σ_i and $|U_i| = r$ for every i. Put $d_i = \sum_{u<w_i} \mathrm{mult}_{U_i}(u)$. Let k be a field of characteristic ℓ with cyclotomic scalar data as in Definition 36.36, and let $(D_i, \varphi_i, \gamma_i)_{i\in\mathbf{Z}/d'\mathbf{Z}}$ be a cyclotomic module over B_I as in Definition 36.39 with $\dim_B D_i = r$ for every i.*

For an A-lattice $N_i \subset D_i$ with $(\gamma_i - 1)N_i \subset XN_i$, the operator $\nabla_i = X^{-1}(\gamma_i - 1)$ maps N_i and XN_i into themselves. For $\alpha \in k$, the Sen multiplicity $\mathrm{sen}_i(N;\alpha)$ is the k-dimension of

$$\{x \in N_i : (\nabla_i - \alpha)^j x \in XN_i \text{ for some } j \ge 0\}/XN_i,$$

that is, of the generalised α-eigenspace of the endomorphism of N_i/XN_i induced by ∇_i.

Let $N = (N_i)_i$ be a lattice in D such that, for every i,

$$X^{h_i-w_i}N_i \subset A\varphi_i(N_{i-1}) \subset X^{-w_i}N_i, \qquad (\gamma_i - 1)N_i \subset XN_i,$$

and such that the following three conditions hold.

(S1) $\mathrm{sen}_i(N;\bar a) = 0$ *for every integer* $0 \le a \le \sigma_i + 1$.

(S2) If $\varsigma_i = 1$, *then* $\sum_{j=0}^{w_i-1} \mathrm{sen}_i(N; \overline{j - w_i}) = d_i$.

(S3) If $\varsigma_i = -1$, *then* $d_i + \sum_{a=0}^{h_i-w_i} \mathrm{sen}_i(N;\bar a) = r$, *and there is* $f_i \in k[Y]$ *with* $f_i(0) \ne 0$ *and* $\nabla_i f_i(\nabla_i)N_i \subset X^{h_i-w_i+1}N_i$.

An N-bounded Herr pair is a pair $(b, g) \in \prod_i D_i \times \prod_i D_i$ such that, for every i,

$$(\gamma_i - 1)b_i = \varphi_i(g_{i-1}) - g_i, \qquad g_i \in XN_i,$$

with $b_i \in X^{-w_i}N_i$ if $\varsigma_i = 1$, and, if $\varsigma_i = -1$, with $b_i \in X^{-(h_i-w_i)}A\varphi_i(N_{i-1})$ and some $x_i \in N_i$ satisfying $g_i - (\gamma_i - 1)x_i \in X^{h_i-w_i+2}N_i$. The Herr boundaries are the pairs $((\varphi_i(m_{i-1}) - m_i)_i, ((\gamma_i - 1)m_i)_i)$ with $m \in \prod_i D_i$. If s N-bounded Herr pairs have k-linearly independent classes modulo the Herr boundaries, then

$$s \le \sum_{i\in\mathbf{Z}/d'\mathbf{Z}} d_i.$$

Outline. The proof gauge-shifts N-bounded Herr pairs using Lemma 36.46 to obtain $g_i' \in X^{\sigma_i+3}N_i$ and polar elements c_i. It bounds $\dim_k Z_i/(Z_i \cap N_i) \le d_i$ via Sen weights at upward indices and Lemma 36.38 at downward indices, then sums. The complete proof is on p. 1303. □

Lemma 36.48 (Model lattices of the canonical residue realization)**.** *Assume $\mathscr{H}_2$ and the data of Definition 36.34. Put $k = O/\varpi O$, $\omega = \bar{\chi}_\ell$ and $\overline{W} = (\chi_1^{-1}T_0)/\varpi(\chi_1^{-1}T_0)$, and use the canonical residue realization D of Definition 36.39 on the class $\mathcal{C}$ of all finite-dimensional continuous $k[G_K]$-representations, with the cyclotomic scalar data of that definition. Take the base pairs to be $(\overline{T}, \overline{N}(T))$ for stable O-lattices T in finite sums of χ_1^{-1}-twisted full representations, and use Sen multiplicities and bounded Herr pairs as in Lemma 36.47. Write $J_i = U_i + \{w_i, \dots, w_i + t - 1\}$, with multiset addition. A Herr representative of a cocycle z is a pair $(y_i, t_i)_i$ obtained from lifts e_i as in Definition 36.39.*

(a) Every model lattice N of a representation in $\mathcal{C}$ satisfies $X^{h_i-w_i}N_i \subset A\varphi_i(N_{i-1}) \subset X^{-w_i}N_i$ and $(\gamma_i - 1)N_i \subset XN_i$, and at a downward index there is $f_i \in k[Y]$ with $f_i(0) \neq 0$ and $\nabla_i f_i(\nabla_i)N_i \subset X^{h_i-w_i+1}N_i$. For every base pair $(\overline{T}, \overline{N}(T))$ there is an integer $\mu \ge 0$ such that $\overline{T}$ has the composition factors of μ copies of $\overline{W} \oplus \bigoplus_{0 \le e < t} k(\omega^{-e})$ and $\mathrm{sen}_i(\overline{N}(T); \alpha) = \mu\, r_i(J_i, w_i; \alpha)$ for all i and $\alpha \in k$.

(b) The reduced Wach lattice N^{std} of $\overline{W}$ is a model lattice, and $\mathrm{sen}_i(N^{\mathrm{std}}; \alpha) = r_i(U_i, w_i; \alpha)$.

(c) For $-1 \le e < t$ there is a lattice $C^{(e)}$ of $D(k(\omega^{-e}))$ with $A\varphi_i(C_{i-1}^{(e)}) = X^e C_i^{(e)}$ and $(\gamma_i - 1)C_i^{(e)} \subset XC_i^{(e)}$, whose Sen multiplicity is 1 at $\bar{e}$ and 0 at every other $\alpha \in k$. For $0 \le e < t$ it is a model lattice, every model lattice of $k(\omega^{-e})$ equals $C^{(e)}$, and $C_i^{(0)} = A\mathbf{1}_i$.

(d) Every representation $\chi_1^{-1}M$, with M an admissible finite $O[G_K]$-module killed by ϖ, has a model lattice. If $j : A_1 \hookrightarrow A$ is injective and equivariant and N is a model lattice of A, then $D(j)^{-1}(N)$ is a model lattice of A_1; if $p : A_1 \twoheadrightarrow A_2$ is surjective and equivariant and N is a model lattice of A_1, then $D(p)(N)$ is a model lattice of A_2.

(e) Every representation with a model lattice has a model lattice $N^{\max}$ with $N_i \subset N_i^{\max}$ for every model lattice N and every i.

(f) For every $A \in \mathcal{C}$ and every cocycle $z : G_K \to A$ whose extension E_z has a model lattice, there are a model lattice N of A and a Herr representative of z which is an N-bounded Herr pair.

(g) If N is a model lattice of $\overline{W}$ with $N_i^{\mathrm{std}} \subset N_i$ for every i, then N satisfies (S1), (S2) and the equality in (S3) of Lemma 36.47.

Outline. Wach properties from Lemma 36.17, Lemma 36.18, and Lemma 36.42 verify the base-pair height, Sen, and polynomial properties, giving (a), (b), and (d). Height comparisons and Lemma 36.46 exclude maps from subrepresentations of $\overline{W}$ to $k(\omega^{-e})$, proving (c). Noetherianity gives (e), lifting $\mathbf{1}_i$ gives (f), and filtering N/N^{std} proves (g). A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1305. □

Lemma 36.49 (A common model for the residual Herr representatives)**.** *Assume $\mathscr{H}_2$ and the data of Definition 36.34. Put $k = O/\varpi O$ and $\overline{W} = (\chi_1^{-1}T_0)/\varpi(\chi_1^{-1}T_0)$, and use the canonical residue realization of Definition 36.39. Take the base pairs to be the reductions of stable lattices in finite sums of χ_1^{-1}-twisted full representations. There is a model lattice $N^{\max}$ of $D(\overline{W})$ containing every model lattice of $\overline{W}$, including its reduced Wach lattice N^{std}.*

Put $P_i^{\max} = A\varphi_i(N_{i-1}^{\max})$. Every admissible residue class has a Herr representative $(b_i, g_i)_i$ satisfying

$$(\gamma_i - 1)b_i = \varphi_i(g_{i-1}) - g_i, \qquad g_i \in XN_i^{\max},$$

and

$$b_i \in X^{-w_i}N_i^{\max} \quad (\varsigma_i = 1), \qquad b_i \in X^{-(h_i-w_i)}P_i^{\max} \quad (\varsigma_i = -1).$$

At a downward index there is moreover $x_i \in N_i^{\max}$ with $g_i - (\gamma_i - 1)x_i \in X^{h_i-w_i+2}N_i^{\max}$. These conditions say that (b, g) is an $N^{\max}$-bounded Herr pair in the sense of Lemma 36.47. The common lattice satisfies

$$X^{h_i-w_i}N_i^{\max} \subset P_i^{\max} \subset X^{-w_i}N_i^{\max}, \qquad (\gamma_i - 1)N_i^{\max} \subset XN_i^{\max}.$$

At a downward index, with $\nabla_i = X^{-1}(\gamma_i - 1)$, there is $f_i \in k[Y]$ such that

$$f_i(0) \neq 0, \qquad \nabla_i f_i(\nabla_i) N_i^{\max} \subset X^{h_i - w_i + 1} N_i^{\max}.$$

An independent family of admissible residue classes gives an independent family of these Herr classes.

Outline. Reducing an extension from Definition 36.34 modulo ϖ gives an admissible exponent-one extension. Lemma 36.48 then supplies a bounded Herr representative for a model lattice N, the largest model lattice $N^{\max} \supset N$ preserving the bounds, and the lattice conditions. Linearity, Definition 36.39, and Lemma 36.43 preserve independence. The complete proof is on p. 1309. □

Lemma 36.50 (The character obstruction)**.** *Assume $\mathscr{H}_2$ and the data of Definition 36.34. Put $k = O/\varpi O$ and $\overline{W} = (\chi_1^{-1} T_0)/\varpi(\chi_1^{-1} T_0)$ and $\omega = \bar{\chi}_\ell$. For every integer $-1 \leq e < t$, every finite-dimensional continuous $k[G_K]$-representation E, every injective equivariant map $j : E \hookrightarrow \overline{W}$, and every equivariant map $p : E \to k(\omega^{-e})$, one has $p = 0$.*

Outline. If $p \neq 0$, Lemma 36.48(b), (d) builds a model lattice N^q of $k(\omega^{-e})$. Comparing Sen weights of N^q, $C^{(e)}$, and N^{std} via Lemma 36.48(a), (c) forces a jump into the excluded band of Lemma 36.46. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1310. □

Lemma 36.51 (The residue character bounds)**.** *Assume $\mathscr{H}_2$ and the data of Definition 36.34, and put $k = O/\varpi O$. Then*

$$\mathrm{Adm}(O(\chi_c), k) = 0 \quad (2 \leq c \leq t), \qquad \dim_k \mathrm{Adm}(O(\chi_1), k) = 1.$$

Outline. Reducing extensions from Definition 36.34 modulo ϖ gives admissible exponent-one extensions. For $1 \leq e < t$, Lemma 36.48 gives a $C^{(e)}$-bounded Herr pair, Definition 36.36 removes poles, Frobenius contraction yields a boundary, and Lemma 36.43 splits it. For $e = 0$, pairs reduce to constant tuples of dimension at most one, and an unramified twist gives a nonzero class. The complete proof is on p. 1310. □

Lemma 36.52 (The ψ-shift dimension estimate and admissible cohomology classes)**.** *Assume $\mathscr{H}_2$. Fix data $(K, L', O, \varpi, \tau_i, t, h, w, \eta', \varsigma, U, V, T_0, \chi_1, \dots, \chi_t)$ at a prime $\ell \geq 5$ as in Definition 36.34. At each embedding $\tau_i : K \to L'$ ($i \in \mathbf{Z}/d'\mathbf{Z}$), the Hodge–Tate jumps $u \in U_i$ of V and $w_i + c - 1$ of χ_c ($1 \leq c \leq t$) lie in $[1, h_i - 1]$, with no upper bound of the form $h_i < \ell$ on the spread h_i; relative to ℓ, the position data $(h, w, \eta', \varsigma, U)$ satisfy conditions (P1)–(P6) of Definition 36.32 in terms of the cyclic carries $\sigma_i = \sigma_h(i) = \lfloor \sum_{j \geq 1} h_{i+j} \ell^{-j} \rfloor$, the low windows W_i ($[1, \eta_i']$ if $\varsigma_i = 1$ and $[h_i - \eta_i', h_i - 1]$ if $\varsigma_i = -1$), and the reflected jumps $\bar{x}$ (x if $\varsigma_i = 1$ and $h_i - x$ if $\varsigma_i = -1$):*

$$\eta_i' + 4\sigma_i + 6 < \ell, \qquad |x - y| > \sigma_i + 2 \quad (x \in U_i \cap W_i,\ y \in [w_i, w_i + t - 1]),$$

and $\bar{x} \not\equiv m \pmod{\ell}$ for every $x \in U_i \setminus W_i$ and every integer $m \in [-2\sigma_i - 2,\ \eta_i' + 2\sigma_i + 3]$. Let $k = O/\varpi O$ be the residue field, put $r = \dim_{L'} V$, $W_0 = \chi_1^{-1} T_0$, and

$$d_- = \sum_{i \in \mathbf{Z}/d'\mathbf{Z}} \sum_{u < w_i} \mathrm{mult}_{U_i}(u),$$

the total multiplicity of negative Hodge–Tate jumps of $\chi_1^{-1} V$ across all embeddings τ_i. For a finite O-module $\mathcal{M}$ with trivial G_K-action, write $H^1_{\mathrm{ur}}(K, \mathcal{M}) = \ker\big(H^1(K, \mathcal{M}) \to H^1(I_K, \mathcal{M})\big)$, where $I_K \subset G_K$ is the inertia subgroup. Then:

(a) Admissible finite $O[G_K]$-modules are closed under submodules, quotients, and finite direct sums, and $T_0 \otimes_O \mathcal{M}$ and $O(\chi_c) \otimes_O \mathcal{M}$ ($1 \leq c \leq t$) are admissible for every finite O-module $\mathcal{M}$. For $W = T_0$ or $W = O(\chi_c)$ ($1 \leq c \leq t$) and $W' = \chi_1^{-1} W$, the subset $\mathrm{Adm}(W, \mathcal{M}) \subset H^1(K, W' \otimes_O \mathcal{M})$ coincides with the set of classes represented by cocycles z such that $\mathcal{E}_{z,a}$ is admissible for some (equivalently, every) integer $a \geq 1$ with $\varpi^a \mathcal{M} = 0$; it is an O-submodule, functorial in $\mathcal{M}$, and satisfies $i_*^{-1}(\mathrm{Adm}(W, \mathcal{M}')) = \mathrm{Adm}(W, \mathcal{M})$ for every injective O-linear map $i : \mathcal{M} \hookrightarrow \mathcal{M}'$. A cocycle $z : G_K \to W \otimes_O \mathcal{M}$ is admissible in the sense of Definition 36.34 if and only if its cohomology class belongs to $\mathrm{Adm}(W, \mathcal{M})$.

(b) Over the residue field k, one has

$$\dim_k \mathrm{Adm}(T_0,k) \le d_-, \qquad \dim_k \mathrm{Adm}(O(\chi_c),k) = 0 \quad (2 \le c \le t), \qquad \dim_k \mathrm{Adm}(O(\chi_1),k) = 1.$$

(c) Furthermore, $H^0(K, W_0/\varpi W_0) = H^2(K, W_0/\varpi W_0) = 0$, the O-module $H^1(K, W_0)$ is finite free of rank $d'r$ with $H^1_f(K, W_0)$ an O-direct summand of rank d_-, the natural map $H^1(K, W_0) \otimes_O \mathcal{M} \to H^1(K, W_0 \otimes_O \mathcal{M})$ is an isomorphism for every finite O-module $\mathcal{M}$, and

$$\mathrm{Adm}(T_0,\mathcal{M}) = \mathrm{im}\big(H^1_f(K,W_0)\otimes_O \mathcal{M} \to H^1(K, W_0\otimes_O \mathcal{M})\big),$$
$$\mathrm{Adm}(O(\chi_c),\mathcal{M}) = \{0\} \quad (2 \le c \le t), \qquad \mathrm{Adm}(O(\chi_1),\mathcal{M}) = H^1_{\mathrm{ur}}(K,\mathcal{M}).$$

Outline. Lattice presentations, fiber products, and pushouts give (a). For (b), bound independent Herr pairs via Lemma 36.47 and Lemma 36.38, apply this to the common model of Lemma 36.49 and Lemma 36.48, and use Lemma 36.51. For (c), Lemma 36.50 and Lemma 36.44 give $H^0 = H^2 = 0$, and length induction with Lemma 36.45 identifies all admissible submodules. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1312. □

Proposition 36.53 (Admissible cocycles at large primes)**.** *Assume $\mathscr{H}_2$. Then statement (T) of Definition 36.34 holds for all data of that definition.*

Outline. Identify twisted-action and twisted-coefficient cocycles, applying Lemma 36.52(a),(c) to $W_0 = \chi_1^{-1}T_0$ for (T1) via Definition 36.15. Transport along $O(\chi_c) \simeq L_c$ and apply Lemma 36.52(a),(c) to obtain (T2) and (T3). The complete proof is on p. 1316. □

Lemma 36.54 (Zero local loss off the exceptional primes)**.** *Work on the twist-good cut family of Theorem 36.26(I). At a member put $n = f/2$ and $\eta = \eta_{\mathrm{sz}}$, let $N(m)$ be the terminal rank, and set*

$$Z = \frac{mb}{nt^2(N(m)+t)L^2 + kn^2(S_c + t + L)}, \qquad a_* = \lfloor 2b/Z \rfloor.$$

The base sizes are b, $w''(n)$ and $b_T + \lfloor b'/x \rfloor$ at top positions T, where $w''(n), b_T, b'$ are the ordinary lower endpoint, top-position size and auxiliary scale. A prime $\ell > Z$ is exceptional if it divides an integer in one of the intervals

$$[W - \eta - 2a_* - 3,\ W + S_c + 2\eta + 2a_* + 4] \tag{36.32}$$

with W a base size.

(a) There is a family cutoff beyond which every prime $\ell > Z$ satisfies $\ell \ge 11$, $t+1 < \ell$ and $\ell \notin \{q, d\}$, lies outside the small set, and is unramified in E. If moreover ℓ is not exceptional, then for every layer $1 \le j \le m$, every $d' \ge 1$, and every choice of signs ς_i and weight types at $i \in \mathbf{Z}/d'\mathbf{Z}$, there are position data $(h, w, \eta', \varsigma, U)$ at ℓ of width t with $\mathrm{mult}_{U_i}(b + w_i) = \mathbf{1}_{J_i}(b)$ for all i and b. Here, for layer j and the weight type τ at i, the jump set $J_i = F_l(i) \cup F_c(i)$ consists of the low jumps $F_l(i)$ (indexed by the m row parameters of τ) and the cloud jumps $F_c(i)$ (indexed by its $k-1$ column parameters) at sign $\varsigma_i = 1$, transported by $u \mapsto t - 1 - u$ when $\varsigma_i = -1$.

(b) Assume $\mathscr{H}_2$ and statement (T) of Definition 36.34. Beyond the cutoff of (a), let i be a twisted index whose residue characteristic $\ell = \ell_i > Z$ is not exceptional and lies outside $\{g_0, g_1\}$. Let $\pi \in Z_{\mathrm{good}}(i)$ and $w \mid \ell$. Let L' be a finite extension of the fraction field K_i of O_i containing the images of all embeddings of $E_w(\mu_\ell)$, with integers O and uniformizer ϖ. For $x \in X_i$ write R_x for the full representation at x and $V_x = R_x \otimes \chi_{i,1}^{-1}$, as in Lemma 36.31. For an $O_i[G_E]$-lattice M of rank k_M with action μ, a cut cocycle of M at w is a map $G_{E_w} \to M \otimes K_i/O_i$ of the form $g \mapsto \phi(gv_0 - v_0)$. Here $x_1, \dots, x_{a'} \in X_i$, $v_0 \in \bigoplus_j V_{x_j}$, $\mathcal{Y} \subset \bigoplus_j V_{x_j}$ is a finitely generated O_i-submodule stable under G_E and containing $gv_0 - v_0$ for all $g \in G_E$, the map $\phi : \mathcal{Y} \to M \otimes K_i/O_i$ is O_i-linear with $\phi(gy) = \mu(g)\phi(y)$, and G_{E_w} acts through the decomposition embedding. Let M be the normalized rank-r lattice W_π of $\rho_i(\pi)\chi_{i,1}^{-1}$, or the character $\chi_{i,c}\chi_{i,1}^{-1}$ with $1 \le c \le t$. Let κ be a cut cocycle of M at w and $\bar\kappa$ its image in $M \otimes L'/O$. Then $\bar\kappa$ is a finite cocycle in the sense of Definition 36.15 if $M = W_\pi$, a coboundary on G_{E_w} if $c \ge 2$, and a coboundary on the inertia group of E_w if $c = 1$.

Outline. For (a), combines family size bounds with 8.3 and 36.33 to build position data. For (b), twists by a Lubin–Tate character (36.31), extends congruences via 36.25 and 36.24, proves admissibility by Artin–Rees, and applies statement (T). The complete proof is on p. 1317. □

Remark 36.55 (Range of Hodge–Tate weights relative to ℓ in the global application). In Lemma 36.54, Lemma 36.52 and Proposition 36.53 are applied at primes $\ell > Z$ (with $\ell \geq 11$) outside the exceptional prime divisors of (36.32). There the shifted jump spread satisfies $h_i \leq b+S_c+2\eta_{\mathrm{sz}}+3$, so for $Z < \ell \leq b+S_c+2\eta_{\mathrm{sz}}+3$ the Hodge–Tate weight spread can exceed ℓ, i.e. lie outside the Fontaine–Laffaille range (while (P4) at $i-1$ gives $\lfloor h_i/\ell \rfloor \leq \sigma_{i-1} \leq (\ell-7)/4$, hence $h_i < \ell(\ell-3)/4$); nonetheless $\sigma_i \leq a_* = \lfloor 2b/Z \rfloor$ and $\eta_i' \leq 2\eta_{\mathrm{sz}}+1$ satisfy $\eta_i' + 4\sigma_i + 6 \leq 2\eta_{\mathrm{sz}} + 8b/Z + 7 < Z < \ell$, and excluding prime divisors of (36.32) enforces (P6).

Definition 36.56 (The annihilator statement at a ramified prime)**.** Let ℓ be odd, $K_0/\mathbf{Q}_\ell$ finite unramified of degree n_K, and $K \subset K_0(\mu_\ell)$ an extension of K_0 of degree e. Put $s = (\ell-1)/e$. Let L' be a finite extension of $\mathbf{Q}_\ell$ containing the images of all embeddings of $K_0(\mu_\ell)$, with integers O. For a stable O-lattice T in a continuous representation of G_K over L', let $Z_f(T)$ be the O-module of continuous cocycles $G_K \to T$ whose classes are finite in the sense of Definition 36.15, and $B(T)$ the module of coboundaries.

Admissible data at K consist of the following. Crystalline lattices T_1, T_2 and an equivariant surjection $F : T_1 \to T_2$, with $r = \operatorname{rank} T_2$ and $I_F = F(Z_f(T_1)) + B(T_2)$. Integers $S, \eta, m, h \geq 0$. For each embedding $\tau : K \to L'$ a sign ς_τ and an integer $\beta_\tau > \eta$. They are required to satisfy three conditions. The jumps of T_1 at τ lie in $[-h, h]$, and in $[\beta_\tau, \beta_\tau + S] \cup [-\eta, \eta]$ if $\varsigma_\tau = 1$, in $[-\beta_\tau - S, -\beta_\tau] \cup [-\eta, \eta]$ if $\varsigma_\tau = -1$. At each τ, the lattice T_2 has at most m jumps in $[-\eta, \eta]$, counted with multiplicity. If $e \neq 2$, the sign ς_τ depends only on the restriction of τ to K_0.

Statement (A) at K asserts the following for all admissible data at K. There are an integer h' with $1 \leq h' \leq h + S + 1$ and a set Σ_0 of at most $n_K e(e+1)m$ elements of $Z_f(T_2)$ with

$$\ell^Y Z_f(T_2) \subset O\Sigma_0 + I_F,$$

where Y is the integer of Lemma 36.30 for $(\ell, e, s, h', S, \eta)$.

Lemma 36.57 (Normalization, Δ-invariant ring relations, and component splitting)**.** *Assume $\mathscr{H}_2$. Let $\ell, K_0, K, n_K, e, s, L', O$ be as in Definition 36.56, put $K' = K_0(\mu_\ell)$, $\Delta = \mathrm{Gal}(K'/K)$, and $\Gamma = \mathrm{Gal}(K'(\mu_{\ell^\infty})/K')$, and fix a topological generator $\gamma \in \Gamma$ with $v_\ell(\chi_\ell(\gamma) - 1) = 1$.*

(a) The degree $e = [K : K_0]$ divides $\ell - 1$, so $es = \ell - 1$ and $s \geq 1$. For any admissible data $(T_1, T_2, F, r, S, \eta, m, h, \varsigma_\tau, \beta_\tau)$ at K, every jump of T_2 at an embedding $\tau : K \to L'$ is a jump of T_1 at τ, and there exist integers $0 \leq \eta_1 \leq \eta$, $1 \leq h' \leq h + S + 1$, and β_τ' ($\tau : K \to L'$) with $\eta_1 < \beta_\tau'$ and $\beta_\tau' + S \leq h'$ such that at every τ, the jumps of both T_1 and T_2 lie in $[\beta_\tau', \beta_\tau' + S] \cup [-\eta_1, \eta_1]$ when $\varsigma_\tau = 1$ and in $[-\beta_\tau' - S, -\beta_\tau'] \cup [-\eta_1, \eta_1]$ when $\varsigma_\tau = -1$ (hence in $[-h', h']$), and T_2 has at most m jumps in $[-\eta_1, \eta_1]$, counted with multiplicity.

(b) Let $A_* = \prod_{c \in \mathbf{Z}/n_K\mathbf{Z}} O[[Z]]$ be the level-one Wach ring over K'. There exist a Δ-invariant parameter $\pi' = (\pi'_c)_{c \in \mathbf{Z}/n_K\mathbf{Z}} \in A_*^\Delta$ and degree-e polynomials $E_c \in O[U]$ ($c \in \mathbf{Z}/n_K\mathbf{Z}$) such that $A_*^\Delta = \prod_{c \in \mathbf{Z}/n_K\mathbf{Z}} O[[\pi'_c]]$, π' is a unit of A_* times Z^s, the cyclotomic polynomial $Q = \Phi_\ell(1+Z)$ is a unit of A_* times $E(\pi') = (E_c(\pi'_c))_c$, $\varphi(\pi')$ is a unit times $\pi' E(\pi')^s$, and $(\gamma - 1)\pi'$ is a unit times $\pi' E(\pi')$.

(c) For $h' \geq 1$, put $M = \lceil 4h'/s \rceil$, $L(k) = k + v_\ell(k!)$, $\rho_A = L(2\lceil 2eh'/(\ell-1) \rceil - 1)$, and $\sigma_A = L(M-1)$. Let $\mathscr{E}$ be the ℓ-adic completion of $A_*[Z^{-1}]$. For a crystalline O-lattice T with jumps in $[-h', h']$, let $D(T)$ be its étale (φ, Γ)-module over $\mathscr{E}$, with Wach lattice $N(T)$. Let ϵ_c be the idempotent of the c-th factor of A_*, and put $E_\pi = E(\pi')$ and $N_c(T)^\Delta = \epsilon_c N(T)^\Delta$. Define

$$\begin{aligned}
\Phi_c(T) &= A_*\text{-}span\{\varphi z : z \in \epsilon_{c-1} N(T)\},\\
\Lambda_c^P(T) &= \{w \in \epsilon_c D(T)^\Delta : E_\pi^{h'} w \in \Phi_c(T)\},\\
U_k(T) &= \pi'^k N(T)^\Delta,\\
\mathcal{P}_c^{(k)}(T) &= \frac{\{w \in \Lambda_c^P(T) : (\gamma-1)w \in U_k(T)\}}{\pi'^k N_c(T)^\Delta} \qquad (k \geq 1).
\end{aligned}$$

Here $N_c(T)^\Delta \subset \Lambda_c^P(T)$, and the denominator is contained in the numerator. For every equivariant map $f : T \to T'$ of crystalline O-lattices with jumps in $[-h', h']$, write $f_* : \mathcal{P}_c^{(k)}(T) \to \mathcal{P}_c^{(k)}(T')$ for the map induced by $D(f)$. A Herr pair *is a pair $(w, z) \in D(T)^\Delta \times D(T)^\Delta$ with $(\gamma-1)w = (\varphi-1)z$. Under the étale comparison $D(T) \subset (W(C^\flat) \otimes_{\mathbf{Z}_\ell} T)^{G_{K'(\mu_{\ell^\infty})}}$ of Lemma 36.18, choosing $\tilde{b} \in W(C^\flat) \otimes_{\mathbf{Z}_\ell} T$ with $(\varphi-1)\tilde{b} = w$ and letting $g \mapsto z_g \in D(T)^\Delta$ be the continuous 1-cocycle on G_K factoring through $\Delta \times \Gamma$ with $z_g = 0$ for $g \in \Delta$ and $z_\gamma = z$ defines a cocycle $g \mapsto z_g - (g-1)\tilde{b}$ in T; its cohomology class $\mathrm{cl}_{K,T}(w, z) \in H^1(K, T)$ is O-linear, natural in T, and vanishes on $((\varphi-1)v, (\gamma-1)v)$ $(v \in D(T)^\Delta)$. The map $\varphi - 1 : U_M(T) \to U_M(T)$ is bijective. Every numerator vector w for $\mathcal{P}_c^{(M)}(T)$ therefore has a unique $z \in U_M(T)$ completing it to a Herr pair, and that pair has a class in $H^1_f(K, T)$, the image of $Z_f(T)$ in $H^1(K, T)$. This defines an O-linear map*

$$\mathrm{cl}_c^H : \mathcal{P}_c^{(M)}(T) \longrightarrow H^1_f(K, T), \qquad [w] \longmapsto \mathrm{cl}_{K,T}(w, z),$$

which is natural under equivariant maps $f : T \to T'$ of such crystalline lattices ($f_* \circ \mathrm{cl}_c^H = \mathrm{cl}_c^H \circ f_*$). Let $\mathcal{J}_c(T)$ be the O-span of all $\kappa \in Z_f(T)$ for which there are such w, z with $[\kappa] = \mathrm{cl}_{K,T}(w, z)$. Then

$$\ell^{\rho_A + \sigma_A} Z_f(T) \subset \sum_{c \in \mathbf{Z}/n_K\mathbf{Z}} \mathcal{J}_c(T) + B(T).$$

Outline. Tower multiplicativity and setting $\eta_1 = \min(\eta, h)$, $\beta'_\tau = \min(\beta_\tau, h+1)$ prove (a), and tame descent gives (b). Height bounds on E_π make $\varphi - 1$ bijective on $U_M(T)$, while the period-primitive criterion and Δ-averaging define cl_c^H. Lifting unit vectors by ℓ^{ρ_A}, clearing γ-terms by ℓ^{σ_A}, and summing pairs proves the splitting. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1319. □

Definition 36.58 (Coordinate lattices and evaluation maps)**.** Use the local data of Lemma 36.57, fix an integer $h' \geq 1$ with $M = \lceil 4h'/s \rceil$, fix $c \in \mathbf{Z}/n_K\mathbf{Z}$, and let T be a crystalline O-lattice over K with jumps in $[-h', h']$. Put $A^\Delta = A_*^\Delta$, $E_\pi = E(\pi')$, and $V = T[1/\ell]$. Fix compatible primitive roots $(\zeta_{\ell^n})_{n \geq 1}$ as in the proof of Lemma 36.17 (at depth 1), so $\zeta_\ell \in K'$ identifies $W(k')[[Z]]/(Q)$ with $\mathcal{O}_{K'}$ via $1 + Z \mapsto \zeta_\ell$, and let $B_{\mathrm{dR}} = \mathrm{Frac}(B_{\mathrm{dR}}^+)$ be Fontaine's de Rham period field, where B_{dR}^+ is the $\ker\theta$-adic completion of $W(\mathcal{O}^\flat)[1/\ell]$ with uniformizer $t_{\mathrm{cris}} = \log[\epsilon]$. Put $\mathscr{V}_T = V \otimes_{\mathbf{Q}_\ell} B_{\mathrm{dR}}$; then $D_{\mathrm{dR}}(V) = \mathscr{V}_T^{G_K} = \bigoplus_{\tau : K \to L'} D_{\mathrm{dR}}(V)_\tau \subset \mathscr{V}_T$, with τ-component projections $\mathrm{pr}_\tau : D_{\mathrm{dR}}(V) \to D_{\mathrm{dR}}(V)_\tau$. Let $\iota_c : K_0 \to L'$ be the embedding indexing the component c, and let ϵ_c be its idempotent in A_*. An embedding $\tau : K \to L'$ is *above* c if $\tau|_{K_0} = \iota_c$. An *evaluation pair above* c is a pair (ζ, τ) for which an embedding $\widetilde{\tau} : K' \to L'$ satisfies

$$\widetilde{\tau}|_{K_0} = \iota_c, \qquad \widetilde{\tau}|_K = \tau, \qquad \widetilde{\tau}(\zeta_\ell) = \zeta.$$

For $a = (a_i)_i \in A_*$, write

$$\mathrm{ev}_{c,\zeta}(a)(t) = a_c(\zeta \exp(t) - 1) \in L'[[t]],$$

with coefficientwise ℓ-adic evaluation. Under the level-one prismatic comparison of Lemmas 36.17–36.18 ($1 + Z \mapsto [\epsilon^{1/\ell}] = \zeta_\ell \exp(t_{\mathrm{cris}}/\ell)$ in B_{dR}^+), completed along $Z = \zeta\exp(t) - 1$ via $\widetilde{\tau}$ with t tracking t_{cris}/ℓ, each $z \in N(T)$ has a Laurent expansion $\mathrm{ev}^N_{c,\zeta,T}(z) \in \mathscr{V}_T((t))$ (lying in $D_{\mathrm{dR}}(V)((t))$ when $z \in N(T)^\Delta$). Write $[t^k]F$ for the coefficient of t^k in a Laurent series F (with $[t] = [t^1]$). For $z \in D(T)$, if there is a nonnegative integer k with $Q^k\varphi z \in N(T)$, take the least such k and set

$$v_{c,\zeta,T}(z) = \big([t]\,\mathrm{ev}_{c,\zeta}(Q)\big)^{-k}[t^k]\,\mathrm{ev}^N_{c,\zeta,T}(Q^k\varphi z).$$

Set this value equal to zero when there is no such k. Define the decreasing filtration

$$\mathscr{F}^a_\tau(T) = \mathrm{Fil}^a D_{\mathrm{dR}}(V)_\tau + \sum_{\tau' \neq \tau} D_{\mathrm{dR}}(V)_{\tau'} \subset \mathscr{V}_T \qquad (a \in \mathbf{Z}).$$

The residue module and its scalar extension are

$$L_c(T) = \frac{N_{c-1}(T)^\Delta}{N_{c-1}(T)^\Delta \cap \pi' N(T)}, \qquad \mathscr{B}_c(T) = \mathscr{E} \otimes_O L_c(T).$$

For $z \in N_{c-1}(T)^{\Delta}$, write $\bar{z}$ for its image in $L_c(T)$. For an O-submodule $S_0 \subset L_c(T)$, set

$$R_c(S_0) = \operatorname{im}\bigl(A^{\Delta} \otimes_O S_0 \longrightarrow \mathscr{B}_c(T),\ a \otimes s \longmapsto \epsilon_c a \otimes s\bigr),$$
$$\Lambda_c^L(T; S_0) = \{x \in \mathscr{B}_c(T) : E_\pi^{h'} x \in R_c(S_0)\}, \qquad \Lambda_c^L(T) = \Lambda_c^L(T; L_c(T)).$$

Multiplication by A^{Δ} acts on the first tensor factor, as does $\gamma_L = \gamma_{\mathscr{E}} \otimes 1$, where $\gamma_{\mathscr{E}}$ is the γ-action on $\mathscr{E}$. Let $\Phi_c(T)$ and $\Lambda_c^P(T)$ be the modules in Lemma 36.57(c), and put

$$P_c^{\varphi}(T) = \Phi_c(T) \cap \epsilon_c D(T)^{\Delta},$$
$$J_c^L(T) = \pi'^M E_\pi^{3h'} R_c(L_c(T)), \qquad J_c^P(T) = \pi'^M E_\pi^{3h'} P_c^{\varphi}(T),$$
$$\overline{\Lambda}_c^L(T) = \Lambda_c^L(T)/(\Lambda_c^L(T) \cap J_c^L(T)),$$
$$\overline{\Lambda}_c^P(T) = \Lambda_c^P(T)/(\Lambda_c^P(T) \cap J_c^P(T)).$$

Write q_L and q_P for these quotient maps. An equivariant map $f : T \to T'$ of crystalline O-lattices induces $f_P : D(T) \to D(T')$, its reduction $f_0 : L_c(T) \to L_c(T')$, and $f_L = 1 \otimes f_0 : \mathscr{B}_c(T) \to \mathscr{B}_c(T')$.

Lemma 36.59 (Evaluation and induced maps)**.** *Under the hypotheses of Definition 36.58, for every evaluation pair (ζ, τ) above c, the evaluation map $\mathrm{ev}_{c,\zeta}$ converges coefficientwise, $[t]\,\mathrm{ev}_{c,\zeta}(Q) = \ell/(\zeta - 1) \neq 0$, and $v_{c,\zeta,T}(z)$ is well defined for every $z \in D(T)$. The coordinate modules and quotients $L_c(T)$, $\overline{\Lambda}_c^L(T)$, and $\overline{\Lambda}_c^P(T)$ are well defined. Every equivariant map $f : T \to T'$ of crystalline O-lattices induces f_0 and f_L from f_P.*

Outline. The substitution converges since $v_\ell(\zeta - 1) > 0$, and $Q(Z) = ((1+Z)^\ell - 1)/Z$ gives the linear coefficient $\ell/(\zeta - 1)$. The coordinate modules are images and inverse images under O-linear maps, and f_P descends to f_0 and f_L. The complete proof is on p. 1321. □

Definition 36.60 (Model lattices, Frobenius-horizontal frames, and slot spaces)**.** Use Definition 36.58, with $h' \geq 1$, $M = \lceil 4h'/s \rceil$, and $\sigma_A = L(M-1)$.

(a) For a crystalline O-lattice T over K with jumps in $[-h', h']$, a *representation* of $x \in \mathscr{B}_c(T)$ consists of finitely many $a_j \in A^{\Delta}$ and $z_j \in N_{c-1}(T)^{\Delta}$ such that

$$E_\pi^{h'} x = \sum_j \epsilon_c a_j \otimes \bar{z}_j.$$

Its numerator expansion at (ζ, τ) is

$$G_{c,\zeta}(a, z; t) = \sum_j \mathrm{ev}_{c,\zeta}(a_j)(t) v_{c,\zeta,T}(z_j).$$

The *model lattice* $\mathcal{M}_c(T)$ is the O-span of the elements admitting such a representation for which

$$[t^i] G_{c,\zeta}(a, z; t) \in \mathscr{F}_\tau^{h'-i}(T) \quad \text{for every evaluation pair above } c \text{ and every } i \in \mathbf{Z}.$$

Coefficients with negative index in a power series are zero.

(b) A *Frobenius-horizontal frame* is a family of O-linear maps

$$\alpha_T : \Lambda_c^L(T) \longrightarrow \overline{\Lambda}_c^P(T), \qquad \beta_T : \Lambda_c^P(T) \longrightarrow \overline{\Lambda}_c^L(T),$$

for crystalline lattices T with jumps in $[-h', h']$, subject to the following conditions. They kill $\Lambda_c^L(T) \cap J_c^L(T)$ and $\Lambda_c^P(T) \cap J_c^P(T)$, respectively. Their induced quotient maps satisfy

$$\bar{\beta}_T \bar{\alpha}_T = \ell^{2\sigma_A}\,\mathrm{id}, \qquad \bar{\alpha}_T \bar{\beta}_T = \ell^{2\sigma_A}\,\mathrm{id}\,.$$

They satisfy the following representative compatibilities whenever the arguments lie in the indicated domains. If $\alpha_T(x) = q_P(y)$, then

$$\alpha_T(ax) = q_P(ay) \quad (a \in A^\Delta), \qquad \alpha_T(\gamma_L x) = q_P(\gamma y).$$

If $\beta_T(y) = q_L(x)$, then

$$\beta_T(ay) = q_L(ax) \quad (a \in A^\Delta), \qquad \beta_T(\gamma y) = q_L(\gamma_L x).$$

For every equivariant map $f : T \to T'$ of crystalline O-lattices with jumps in $[-h', h']$, these identities are natural under f_L, f_P: the first equality $\alpha_T(x) = q_P(y)$ implies $\alpha_{T'}(f_L x) = q_P(f_P y)$, and the second equality $\beta_T(y) = q_L(x)$ implies $\beta_{T'}(f_P y) = q_L(f_L x)$. Finally, whenever $\alpha_T(x) = q_P(y)$, $(a_j, z_j)_j$ is a representation of x, and $k \geq 0$ satisfies $Q^k E_\pi^{h'} y \in N(T)$, they satisfy

$$[t^i]\,\mathrm{ev}^N_{c,\zeta,T}(Q^k E_\pi^{h'} y) = \ell^{\sigma_A}[t^i]\big(\mathrm{ev}_{c,\zeta}(Q^k) G_{c,\zeta}(a, z; t)\big) \qquad (i < k + 4h')$$

for every evaluation pair above c.

(c) For an O-submodule $S_0 \subset L_c(T)$, define

$$\mathcal{I}^L_c(T; S_0) = \{x \in \Lambda^L_c(T; S_0) : (\gamma_L - 1)x \in \pi' \mathcal{M}_c(T)\},$$
$$\mathcal{R}_c(T; S_0) = \frac{\mathcal{I}^L_c(T; S_0)}{\mathcal{I}^L_c(T; S_0) \cap \pi' \mathcal{M}_c(T)}, \qquad \mathcal{H}_c(T) = \mathcal{R}_c(T; L_c(T)).$$

Given a crystalline O-lattice T_2 with jumps in $[-h', h']$ and a chosen O-submodule $L_r \subset L_c(T_2)$, the *residual module* is $\mathcal{I}_{\mathrm{rest}} = \mathcal{R}_c(T_2; L_r)$.

(d) Fix the normalized windows $(S, \eta_1, \varsigma_\tau, \beta'_\tau)$ of Lemma 36.57(a); call τ (and any evaluation pair (ζ, τ)) *upward* if $\varsigma_\tau = 1$ and *downward* if $\varsigma_\tau = -1$. A scalar $f \in \mathscr{E}$ is an *admissible slot coefficient* if there exist $a, g \in A^\Delta$ with

$$\epsilon_c a = a, \qquad E_\pi^{h'} f = a, \qquad E_\pi^{h'}(\gamma f - f) = \pi' g,$$

and, at every evaluation pair above c, the following coefficient conditions hold:

$$\begin{aligned} &\varsigma_\tau = 1: && [t^i]\,\mathrm{ev}_{c,\zeta}(a) = 0 \quad (0 \leq i < h' - \beta'_\tau), \\ &\varsigma_\tau = -1: && \text{there is } b_{\zeta,\tau} \in L' \text{ such that} \\ & && [t^i]\,\mathrm{ev}_{c,\zeta}(a) = b_{\zeta,\tau}[t^i]\,\mathrm{ev}_{c,\zeta}(E_\pi^{h'}) \quad (0 \leq i < h' + \beta'_\tau + S). \end{aligned}$$

An *elementary slot tensor* for T is $f \otimes \bar{z}$, where f is such a coefficient, $z \in N_{c-1}(T)^\Delta$, and

$$v_{c,\zeta,T}(z) \in \mathscr{F}_\tau^{\beta'_\tau}(T) \quad \text{whenever } (\zeta, \tau) \text{ is above } c \text{ and } \varsigma_\tau = 1.$$

Let $\mathcal{T}_c(T) \subset \Lambda^L_c(T)$ be their O-span. When $\mathcal{T}_c(T) \subset \mathcal{I}^L_c(T; L_c(T))$, write $\mathcal{C}_{\mathrm{lift}}(T, c)$ for its image in $\mathcal{H}_c(T)$.

Lemma 36.61 (Model comparison and invariant coordinates)**.** *Assume $\mathscr{H}_2$, and let T be crystalline with jumps in $[-h', h']$. Use Definitions 36.58 and 36.60.*

(a) The module $\mathcal{M}_c(T)$ is stable under A^Δ and γ_L, and

$$E_\pi^{h'} R_c(L_c(T)) \subset \mathcal{M}_c(T) \subset \Lambda^L_c(T), \qquad J^L_c(T) \subset \pi'^M \mathcal{M}_c(T).$$

It is ℓ-saturated in $\Lambda^L_c(T)$: if $x \in \Lambda^L_c(T)$, $a \geq 0$, and $\ell^a x \in \mathcal{M}_c(T)$, then $x \in \mathcal{M}_c(T)$. The quotient

$$\mathcal{Q}_c(T) = \Lambda^L_c(T)/(\Lambda^L_c(T) \cap \pi' \mathcal{M}_c(T))$$

is finite free over O.

(b) Suppose $x \in \Lambda_c^L(T)$ *and* $(\gamma_L - 1)x \in \mathcal{M}_c(T)$. *For a representation* $(a_j, z_j)_j$ *of* x, *let* $f(t) \in \mathscr{V}_T((t))$ *satisfy*

$$\mathrm{ev}_{c,\zeta}(E_\pi^{h'})f(t) = G_{c,\zeta}(a, z; t).$$

Then $[t^i]f(t) \in \mathscr{F}_\tau^{-i}(T)$ *for every evaluation pair* (ζ, τ) *above* c *and every* $i \neq 0$.

(c) For any Frobenius-horizontal frame, $\alpha_T(\mathcal{M}_c(T))$ *has representatives in* $N_c(T)^\Delta$, *and* $\beta_T(N_c(T)^\Delta)$ *has representatives in* $\mathcal{M}_c(T)$. *For* $1 \le j \le M$, $x \in \Lambda_c^L(T)$, *and* $y \in \Lambda_c^P(T)$,

$$\alpha_T(x) = q_P(y),\ x \in \pi'^j\mathcal{M}_c(T) \Longrightarrow y \in \pi'^j N_c(T)^\Delta,$$
$$\beta_T(y) = q_L(x),\ y \in \pi'^j N_c(T)^\Delta \Longrightarrow x \in \pi'^j\mathcal{M}_c(T).$$

Moreover, for $x \in \Lambda_c^L(T)$ *and* $y \in \Lambda_c^P(T)$, $\alpha_T(x) = q_P(y)$ *with* $x \in \mathcal{I}_c^L(T; L_c(T))$ *implies* $(\gamma - 1)y \in \pi' N_c(T)^\Delta$, *and* $\beta_T(y) = q_L(x)$ *with* $(\gamma - 1)y \in \pi' N_c(T)^\Delta$ *implies* $x \in \mathcal{I}_c^L(T; L_c(T))$. *Consequently* $\alpha_{1,T}([x]) = [y]$ *(when* $\alpha_T(x) = q_P(y)$*) and* $\beta_{1,T}([y]) = [x]$ *(when* $\beta_T(y) = q_L(x)$*) define natural* O*-linear maps*

$$\alpha_{1,T} : \mathcal{H}_c(T) \longrightarrow \mathcal{P}_c^{(1)}(T), \qquad \beta_{1,T} : \mathcal{P}_c^{(1)}(T) \longrightarrow \mathcal{H}_c(T),$$

with $\alpha_{1,T}\beta_{1,T} = \ell^{2\sigma_A}\,\mathrm{id}$ *and* $\beta_{1,T}\alpha_{1,T} = \ell^{2\sigma_A}\,\mathrm{id}$.

Outline. Filtration stability, $\gamma(E_\pi) = u_\gamma E_\pi$, division by $\pi_c' E_c(\pi_c')^{2h'}$, and ℓ-saturation prove (a). Inducting on Laurent coefficients of $f(\chi_\ell(\gamma)t) - f(t)$ gives (b). Definition 36.60(b), scalar compatibility, and γ-commutation then match π'^j-powers and descend the frame to $\alpha_{1,T}$ and $\beta_{1,T}$ in (c). A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1321. □

Lemma 36.62 (Division at an integral evaluation root)**.** *Use the local ring data of Lemma 36.57(b) and the evaluation maps of Definition 36.58. Thus* O *is the ring of integers of the finite extension* $L'/\mathbf{Q}_\ell$, E_c *is the polynomial defining* E_π *on component* c, *and each evaluation uses a primitive* ℓ*-th root of unity. Assume* $e = 2$, *put* $s = (\ell - 1)/2$, *and let* $r \in O$ *satisfy* $\ell \mid r^2$. *For every* $a \in A^\Delta$, *every component* c, *and every integer* $d \ge 0$, *there are* $p_0, \dots, p_{d-1} \in O$ *and* $a_d \in A^\Delta$ *such that, at every evaluation pair above* c,

$$\mathrm{ev}_{c,\zeta}(a) = \sum_{j<d} p_j\big(\mathrm{ev}_{c,\zeta}(\pi') - r\big)^j + \big(\mathrm{ev}_{c,\zeta}(\pi') - r\big)^d\,\mathrm{ev}_{c,\zeta}(a_d).$$

If $r = [t^0]\,\mathrm{ev}_{c,\zeta}(\pi')$, *then* $\mathrm{ev}_{c,\zeta}(\pi') - r = tU(t)$ *with* $U(0) \neq 0$. *Consequently, vanishing through degree* $d-1$ *at one such evaluation is equivalent to* $p_0 = \cdots = p_{d-1} = 0$, *and then holds at every evaluation with the same value of* r.

Outline. Repeated division of $F(\pi')$ by $\pi' - r$, convergent in O since $\ell \mid r^2$, gives the expansion. The simple zero of $\Phi_\ell(\zeta\exp(t))$ at $t = 0$ yields $\mathrm{ev}_{c,\zeta}(\pi') - r = tU(t)$ ($U(0) \neq 0$), proving the last claim. The complete proof is on p. 1322. □

Lemma 36.63 (The quadratic root separation and its denominator cost)**.** *Continue with* $e = 2$. *Let* $r_U, r_D \in O$ *be distinct evaluation values of* π' *above a component* c, *with* $\ell \mid r_U^2$ *and* $\ell \mid r_D^2$, *and put* $\delta_0 = r_D - r_U$. *Then*

$$E_c(X) = a_2(X - r_U)(X - r_D), \qquad a_2 \in O^\times, \qquad \delta_0^2 \mid \ell.$$

In particular $v_\ell(\delta_0) \le 1/2$. *Let* b, B *be integers with* $1 \le B \le b$, *put* $d = b - B + 1$, *and let* $x \in L'$ *satisfy* $o_0 = x\binom{b}{B-1}\delta_0^d \in O$. *If an integer* $y \ge 0$ *satisfies*

$$d + 2v_\ell\binom{b}{B-1} \le 2y,$$

then $\ell^y x \in O$. *More precisely, write* $v = v_\ell\binom{b}{B-1}$ *and* $\binom{b}{B-1} = \ell^v c'$, *where* $c' \in O^\times$, *and choose* $r_0 \in O$ *with* $\ell = \delta_0^2 r_0$. *Then*

$$\ell^y x = o_0 (c')^{-1}\delta_0^{2(y-v)-d} r_0^{y-v}.$$

Outline. Dividing the quadratic polynomial E_c by $X-r_U$ and $X-r_D$ and comparing constant coefficients in $Q=w'E_\pi$ shows that $a_2\in O^\times$, while comparing linear coefficients at the root of unity for r_U and squaring yields $\delta_0^2\mid\ell$. Under $d+2v\le 2y$, substituting $o_0=x\ell^v c'\delta_0^d$ and $\ell=\delta_0^2 r_0$ gives the integral expression for $\ell^y x$. The complete proof is on p. 1322. □

Proposition 36.64 (Frames, slot lifting, and the component bound)**.** *Assume $\mathscr{H}_2$. In the setting of Definitions 36.58 and 36.60 and Lemma 36.61, fix normalized data $(T_1,T_2,F,S,\eta_1,m,h',\varsigma_\tau,\beta'_\tau)$ as in Lemma 36.57(a), with the sign condition of Definition 36.56, and fix $c\in\mathbf{Z}/n_K\mathbf{Z}$. Put*

$$\kappa_A=e\big(L(S)+\lfloor\log_\ell h'\rfloor\big),\qquad y_c(\eta_1)=\big\lceil\eta_1+\tfrac12+\log_\ell h'\big\rceil,\qquad y_1=\rho_A+\max\big(2\sigma_A,y_c(\eta_1)\big).$$

Define the finite set of integers

$$\Omega_c=\bigcup_{\substack{\tau\ above\ c\\ \varsigma_\tau=1}}\{i\in\mathbf{Z}:-\beta'_\tau-S\le i<-\beta'_\tau\}\cup\bigcup_{\substack{\tau\ above\ c\\ \varsigma_\tau=-1}}\{i\in\mathbf{Z}:\beta'_\tau\le i<\beta'_\tau+S\},\qquad d_c=\prod_{i\in\Omega_c}(1-\chi_\ell(\gamma)^i)\in O.$$

Then the following statements hold.

(a) A Frobenius-horizontal frame exists. For $j=1,2$, the reduction $r_c:\mathcal{P}_c^{(M)}(T_j)\to\mathcal{P}_c^{(1)}(T_j)$ induced by numerator inclusion $U_M(T_j)\subset U_1(T_j)$ is injective and its image contains $\ell^{\sigma_A}\mathcal{P}_c^{(1)}(T_j)$. There is a unique O-linear map lift_c with $r_c\,\mathrm{lift}_c=\ell^{\sigma_A}\mathrm{id}$. It satisfies $\mathrm{lift}_c\,r_c=\ell^{\sigma_A}\mathrm{id}$ and commutes with F_*.

(b) For any such frame there are an O-submodule $L_r\subset L_c(T_2)$, an integer $k\ge0$, and vectors $z_1,\dots,z_k\in N_{c-1}(T_2)^\Delta$ such that

$$\operatorname{rank}_O L_r\le em,\qquad O\bar z_1+\cdots+O\bar z_k+L_r=L_c(T_2),$$

and $v_{c,\zeta,T_2}(z_j)\in\mathscr{F}_\tau^{\beta'_\tau}(T_2)$ at every upward evaluation pair above c. The module L_r is finite free. For every $w\in\mathcal{P}_c^{(1)}(T_2)$, there are admissible slot coefficients $f_1,\dots,f_k$ and $x_r\in\mathcal{I}_c^L(T_2;L_r)$ such that each $f_j\otimes\bar z_j$ belongs to $\mathcal{I}_c^L(T_2;L_c(T_2))$ and

$$\beta_{1,T_2}(d_c w)=\sum_{j=1}^{k}[f_j\otimes\bar z_j]+[x_r]\quad\text{in }\mathcal{H}_c(T_2).$$

Here square brackets denote the quotient by $\pi'\mathcal{M}_c(T_2)$.

(c) For every O-submodule $S_0\subset L_c(T_2)$, $\mathcal{R}_c(T_2;S_0)$ is finite free and

$$\operatorname{rank}_O\mathcal{R}_c(T_2;S_0)\le(e+1)\operatorname{rank}_O S_0.$$

In particular, $\operatorname{rank}_O\mathcal{I}_{\mathrm{rest}}\le e(e+1)m$ for the L_r of (b). Inclusion of the numerators induces an injection $\mathcal{R}_c(T_2;S_0)\hookrightarrow\mathcal{H}_c(T_2)$.

(d) There is $a_c\in O$ such that $a_c d_c=\ell^{\kappa_A}$. Every elementary slot tensor for T_2 belongs to $\mathcal{I}_c^L(T_2;L_c(T_2))$. For any Frobenius-horizontal frame, if $f\otimes\bar z$ is such a tensor and $y\in\Lambda_c^P(T_2)$ satisfies $\alpha_{T_2}(f\otimes\bar z)=q_P(y)$, then

$$(\gamma-1)y\in\pi' N_c(T_2)^\Delta,\qquad \ell^{y_1}y-F_P(y'_1)\in\pi' N_c(T_2)^\Delta$$

for some $y'_1\in\Lambda_c^P(T_1)$ with $(\gamma-1)y'_1\in\pi' N_c(T_1)^\Delta$. This holds when all signs above c are downward, when they are all upward, and when $e=2$. Equivalently,

$$\ell^{y_1}\alpha_{1,T_2}\big(\mathcal{C}_{\mathrm{lift}}(T_2,c)\big)\subset F_*\big(\mathcal{P}_c^{(1)}(T_1)\big).$$

(e) There is a finite subset $\Sigma_c\subset Z_f(T_2)$, of cardinality at most $e(e+1)m$, such that

$$\ell^{\sigma_A+2\sigma_A+\kappa_A+y_1}\mathcal{J}_c(T_2)\subset I_F+O\Sigma_c.$$

Outline. Backward Frobenius and graded $\gamma - 1$ lifting give (a), filtration subspaces give (b) via Definition 36.60, and Lemma 36.61(a) with a γ_L-stable filtration gives (c). Wach lifting via Lemma 36.57, Lemma 36.61(c), Lemma 36.62, and Lemma 36.63 proves (d). Completing lifts to Herr pairs yields (e). A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1323. □

Proposition 36.65 (The annihilator statement at q and d)**.** *Assume $\mathscr{H}_2$. Then statement (A) of Definition 36.56 holds for every local field K of that definition, and in particular at every completion $K = E_w$ of the host E above q or d.*

Outline. Normalize data and split $Z_f(T_2)$ into $\mathcal{J}_c(T_2)$ via Lemma 36.57 (with Lemma 36.31 at q, d), applying Proposition 36.64(e) at each c for Σ_0. Comparing exponents with Lemma 36.30 via $\eta_1 \le \eta$ gives (A). The complete proof is on p. 1328. □

Lemma 36.66 (The local lifting bound at a ramified prime)**.** *Assume $\mathscr{H}_2$ and statement (A) of Definition 36.56 at K. For admissible data at K, put $B = eh/\ell + eS + e\log_\ell h + 1 + \eta$, with $\log_\ell 0 = 0$. Then*

$$\lambda_O\big(Z_f(T_2)/I_F\big) \le 160\big((n_K e + 1)rB + n_K e(e+1)m(h+1)\big),$$

where λ_O is the length normalized by the length of $O/\ell O$.

Outline. Bounds generators of $Z_f(T_2)/B(T_2)$ by $2(n_K e+1)r$ via 36.14, Nakayama's lemma, and Bloch–Kato dimension. Lifts finite classes at exponent E' over $K_0(\mu_\ell)$ via 36.19, 36.22, and corestriction, then applies (A) and 36.30. The complete proof is on p. 1329. □

Definition 36.67 (The paired-character statement at the type primes)**.** Let $C_T \ge 40$ be the type threshold and $C' \ge 8C_T$ the paired-character constant of Theorem 36.26(I). At a member of the family, let i be an index with $\ell = \ell_i = g_k$, $k \in \{0,1\}$. Statement (P) at the member asserts the following for every such i. Let $(R_x)_{x\in X_i}$ be full representations realizing the points of X_i, and $\rho = (\rho_x)$ rank-r representations with $R_\pi \simeq \rho_\pi \oplus \bigoplus_c \chi_{i,c}$ for $\pi \in Z(i)$. For an integer $\eta_0 \ge 0$ and $\pi \in Z_{\mathrm{good}}(i)$ put $s_\pi = s_\pi(\rho)$, $\delta_X = \sum_{0<e<t}(s_0(i) + v_\ell(e))$ and

$$R_i(\pi) = 4fn_V C' n_V^2(n_V + \eta_0 + t) + 2fr^2 + s_\pi + \sum_{0<e<t} s_{\ell,e}(E) + v_\ell(h_E) + 4n_V\delta_X + t\sum_{v\in Q_i}\lambda_i(O_i/(\zeta_{i,v}-1)).$$

Let $\eta_0, s, S \in \mathbf{N}$. Let sg assign a sign to each place $w \mid \ell$ of E, with opposite signs at distinct conjugate places, and let $U_w \subset \mathbf{Z}$ be finite. Suppose the following three conditions hold. (i) $\eta_0 + t + 1 < s$, $\log g_{1-k} \le 2\log g_k$, and $C_T((s+S)(n_V+1)^2 + (n_V+1)^3) < g_k$. (ii) For every $w \mid \ell$, $|U_w| \le n_V$, every jump at w of every $R_x\chi_{i,1}^{-1}$ lies in U_w, and every $a \in U_w$ satisfies $a \notin \{-1, t\}$ together with one of $0 \le a < t$, $|a - (t-1)/2| \le \eta_0 + 1$, or $0 < \mathrm{sg}(w)(a - (t-1)/2) \le s + S - (t-1)/2$. (iii) The lengths s_π, $\pi \in Z_{\mathrm{good}}(i)$, and $s_{\ell,e}(E)$, $0 < e < t$, are finite. Then

$$\lambda_i(U_i/L_i) \le \sum_{\pi\in Z_{\mathrm{good}}(i)} \mathrm{mult}(i,\pi)R_i(\pi),$$

with U_i, L_i as in Definition 36.11.

Lemma 36.68 (Congruence length from a family of endomorphisms)**.** *Fix an index i, and let M_i be the integral Hecke lattice of Definition 36.11. Put $K_i = \mathrm{Frac}(O_i)$, and write $V_x \subset M_i \otimes_{O_i} K_i$ for the eigenspace on which every $f \in T_i \subset O_i^{X_i}$ acts by $f(x)$. For a subset $\mathcal{S} \subset X_i$, put*

$$V_{\mathcal{S}} = \bigoplus_{x\in\mathcal{S}} V_x, \quad U_{\mathcal{S}} = (M_i + V_{\mathcal{S}^c}) \cap V_{\mathcal{S}}, \quad L_{\mathcal{S}} = M_i \cap V_{\mathcal{S}}.$$

Extend endomorphisms of M_i to its scalar extension when applying them to V_x. Let $\mathcal{A}$ be a family of O_i-linear endomorphisms of M_i commuting with T_i, and suppose every member of $\mathcal{A}$ vanishes on V_x for $x \notin \mathcal{S}$. For every $\pi \in \mathcal{S}$, suppose that $a_\pi \in O_i \setminus \{0\}$ and that some $\alpha_\pi \in \mathcal{A}$ acts by a_π on V_π. Then

$$\lambda_i(U_{\mathcal{S}}/L_{\mathcal{S}}) \le \sum_{\pi\in\mathcal{S}} \dim_{K_i} V_\pi\, \lambda_i(O_i/a_\pi O_i).$$

Outline. Each $\alpha \in \mathcal{A}$ preserves V_x and maps $U_{\mathcal{S}}$ into $L_{\mathcal{S}}$. Filtering by $F_j = \bigoplus_{a \le j} V_{\pi_a}$ gives $a_{\pi_j} p_j(U_j) \subset p_j(L_j)$, and summing exact-sequence lengths proves the bound. The complete proof is on p. 1330. □

Lemma 36.69 (An endomorphism supported on the good spectrum)**.** *Use the Hecke algebra and eigenspaces of Lemma 36.68. Suppose $\ell_i = g_k$, and put*

$$M_0 = |\operatorname{GL}_r(\mathbf{F}_{g_0})|^f |\operatorname{GL}_r(\mathbf{F}_{g_1})|^f.$$

Let $\pi \in Z_{\mathrm{good}}(i)$, and let $Y \subset X_i$ contain π, with $Y \cap Z(i) \subset Z_{\mathrm{good}}(i)$. Let $\beta \in \operatorname{End}_{O_i}(M_i)$ commute with the Hecke algebra, vanish on V_x for $x \notin Y$, and act by a nonzero $b \in O_i$ on V_π. Suppose

$$\lambda_i(O_i/bO_i) \le v_{g_k}(M_0).$$

Define

$$\mathfrak{c}_{Y,\pi} = \{f(\pi) : f \in T_i,\ f(x) = 0 \text{ for all } x \in Y \setminus Z(i)\}, \qquad v_{Y,\pi} = \lambda_i(O_i/\mathfrak{c}_{Y,\pi}).$$

There are $g \in T_i$ and $c \in O_i \setminus \{0\}$ such that $\mathfrak{c}_{Y,\pi} = cO_i$, $g(\pi) = c$, and g vanishes on $Y \setminus Z(i)$. The endomorphism $\alpha = g\beta$ commutes with the Hecke algebra, vanishes on every V_x for $x \notin Z_{\mathrm{good}}(i)$, and acts by $a = cb \neq 0$ on V_π. Moreover,

$$\lambda_i(O_i/aO_i) \le v_{Y,\pi} + v_{g_k}(M_0).$$

Outline. A product of separating elements in T_i shows $\mathfrak{c}_{Y,\pi} = cO_i \neq 0$, supplying $g \in T_i$. Then $\alpha = g\beta$ vanishes outside $Z_{\mathrm{good}}(i)$, acts by cb on V_π, and length additivity bounds $\lambda_i(O_i/cbO_i)$. The complete proof is on p. 1330. □

Lemma 36.70 (Length after absorption of finite pushouts)**.** *Assume the local cohomological inputs of Definition 36.15. Let $K/\mathbf{Q}_\ell$ and $L/\mathbf{Q}_\ell$ be finite, let $O = \mathcal{O}_L$, and let X, U be continuous finite-dimensional de Rham L-representations of G_K. Write $U' = U^\vee(1)$. Fix dual stable lattices $L_U \subset U$, $L_{U'} \subset U'$, put $r_U = \operatorname{rank}_O L_U$, and normalize length by $\lambda_O(O/\ell O) = 1$.*

For each stable lattice $L_X \subset X$ and equivariant O-linear map $\phi : L_X \to U/L_U$, push forward the finite integral cocycles of L_X. Let $S \subset Z^1(K, U/L_U)$ be the O-span of all these pushouts, and let $Z^1_f(K, U/L_U)$ denote the cocycles whose classes are finite. Let $e \ge 0$ be an integer and assume the absorption condition $\ell^e S \subset Z^1_f(K, U/L_U)$. Then

$$\lambda_O\big((S + Z^1_f)/Z^1_f\big) \le e\big(\operatorname{rank}_O H^1_f(K, L_{U'}) + r_U\big).$$

Outline. Local Tate duality and Definition 36.15 inject $(S + Z^1_f)/Z^1_f$ into $\operatorname{Hom}_O(A/\ell^e A, L/O)$ for $A = H^1_f(K, L_{U'})$. Bounding $A[\varpi]$ via $H^0(K, L_{U'}/\varpi L_{U'})$ and the structure theorem over O give the length bound. The complete proof is on p. 1330. □

Lemma 36.71 (The local lifting input at a type prime)**.** *Assume $\mathscr{H}_2$, let ℓ be prime, and let $C_T \ge 40$ be an integer. Fix an algebraic closure $\Omega/\mathbf{Q}_\ell$, a finite Galois intermediate extension $K'/\mathbf{Q}_\ell$, and a finite extension $L/\mathbf{Q}_\ell$ contained in Ω, with ring of integers O. Let K'_0 be the maximal unramified subfield of K', and let $\iota : K'_0 \to B_{\mathrm{cris}}$ be its embedding in the chosen period datum. Assume that its image is stable under $G = G_{\mathbf{Q}_\ell}$. Put*

$$H' = G_{K'}, \qquad I = \{g \in G : g\iota(c) = \iota(c) \text{ for every } c \in K'_0\}.$$

Choose a Frobenius element F for this period datum, and a finite set Σ of characters $I \to L^\times$ stable under conjugation by F. Let T_1, T_2 be stable O-lattices, write $V_j = T_j \otimes_O L$, and let $f : T_1 \twoheadrightarrow T_2$ be equivariant. Suppose V_1, V_2 are crystalline over K', and that

$$\chi|_{I \cap H'} = 1, \quad \gcd(\operatorname{ord}\chi, \ell) = 1 \quad (\chi \in \Sigma), \qquad D_{\mathrm{cris},K'}(V_j) = \bigoplus_{\chi \in \Sigma} D_{\mathrm{cris},K'}(V_j)[\chi] \quad (j = 1, 2),$$

where the bracket denotes the I-eigenspace. Let $N, \eta', H \in \mathbf{N}$ and $J \subset \mathbf{Z}$ satisfy

$$1 \in \Sigma, \quad |\Sigma| \le N, \quad N \ge 2, \quad 0 \in J \subset [-\eta', H], \quad |J| \le N + 1, \quad 1 \le \eta' < H, \quad C_T(HN^2 + N^3) < \ell.$$

Suppose all Hodge–Tate jumps of both representations, at every embedding, belong to J. Let r_{orb} be the maximum length of a conjugation orbit of a ratio $\chi\psi^{-1}$, for $\chi, \psi \in \Sigma$. Then

$$4r_{\mathrm{orb}} \le N^2, \qquad \ell^{C_T(Nr_{\mathrm{orb}}+\eta'(N+r_{\mathrm{orb}}))} Z^1_f(T_2) \subset f_* Z^1_f(T_1) + B^1(T_2).$$

Here Z^1_f consists of integral cocycles whose rational class is finite, and B^1 consists of integral coboundaries.

Outline. The proof bounds Frobenius orbit lengths of ratios $\chi\psi^{-1}$ by $N^2/4$ using $\mathrm{lcm}(a,b) \le (a+b)^2/4$. It extends coefficients for tame descent, applies the good-lattice lifting bound via Definition 36.15 and Theorem 36.14, and descends. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1331. □

Lemma 36.72 (Passing the lifting bound to a family)**.** *Assume $\mathscr{H}_2$. Let $(V_x)_{x\in A}$ be a family of continuous L-representations of $G_{\mathbf{Q}_\ell}$ of a common finite dimension, crystalline over one extension K', with the extension and period stability of Lemma 36.71. Suppose the family has a common Frobenius-stable character set Σ and jump set J, with the inertia and jump conditions of that lemma for every member. Retain its integers C_T, N, η', H. Put*

$$r' = \lfloor N^2/4 \rfloor, \qquad e_{\mathrm{fam}} = C_T\big(Nr' + \eta'(N + r')\big).$$

Let W_1, W_2 be subrepresentations of finite direct sums of members of the family. For every pair of stable lattices $L_j \subset W_j$ and equivariant surjection $f : L_1 \twoheadrightarrow L_2$,

$$\ell^{e_{\mathrm{fam}}} Z^1_f(L_2) \subset f_* Z^1_f(L_1) + B^1(L_2).$$

Outline. Exactness and strictness of Fontaine comparison transfer crystallinity over K' and the jump set J to W_1 and W_2, while character averaging idempotents preserve the Σ-eigenspace decomposition. Lemma 36.71 then applies to $L_1 \twoheadrightarrow L_2$, and $r_{\mathrm{orb}} \le r'$ bounds its lifting exponent by e_{fam}, yielding the inclusion. The complete proof is on p. 1334. □

Lemma 36.73 (The local cut-cocycle bound from either family)**.** *Assume $\mathscr{H}_2$. Fix a type-locus spectral set $X_s \subseteq X_i$ with a good Galois family $(R_x)_{x\in X_s}$, characters $\chi_{i,1}, \dots, \chi_{i,t}$, and invariant-complement semisimplicity (rank-r part ρ_π at $\pi \in X_s \cap Z(i)$). Fix $w \mid \ell$ with $E_w = \mathbf{Q}_\ell$, put $V_x = R_x\chi_{i,1}^{-1}$, and choose $\pi \in X_s \cap Z(i)$. Let U be either the normalized rank-r constituent $\rho_\pi\chi_{i,1}^{-1}$, with $k = r$, or the character constituent $\chi_{i,c}\chi_{i,1}^{-1}$, with $k = 1$ and $1 \le c \le t$. The semisimple decomposition of V_π embeds U into it. Fix a stable lattice $L_U \subset U$. Suppose the family $(V_x)_{x\in X_s}$ is crystalline over a common extension and has a common finite character set satisfying the extension, period-stability and inertia conditions of Lemma 36.71. Suppose that every lattice surjection in subrepresentations of finite sums of the family, or every such surjection for its Tate-dual family, lifts finite cohomology after multiplication by ℓ^e. Let $S_{w,U}$ be the span of the cut cocycles in U/L_U, with all spectral points in X_s. Then*

$$\lambda_i\big((S_{w,U} + Z^1_f)/Z^1_f\big) \le 2ke.$$

Outline. Fontaine comparison puts cut cocycles into finite pushouts from a local span X. The direct or dual lifting hypothesis and the Bloch–Kato dimension formula give the bound $2ke$ on finite collections, extending to the full span via direct sums. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1334. □

Proposition 36.74 (The paired-character estimate at the type primes)**.** *Assume $\mathscr{H}_2$. For every integer $C_T \ge 40$ and every real $C' \ge 8C_T$, statement (P) of Definition 36.67 with type threshold C_T and paired-character constant C' holds at every member satisfying the norm-choice conditions and four local compatibility conditions of Theorem 36.26(I). Because any fixed integer $C_T \ge 40$ (such as $C_T = 40$) and $C' \ge 8C_T$ depend on $\mathscr{H}_2$ alone, they may be chosen before the cuts in Theorem 36.26(I), and statement (P) then holds at every member above x_1 of the twist-good cut family of Theorem 36.26(I).*

Outline. Lemma 36.71, Lemma 36.70, Lemma 36.22, and Theorem 36.14 supply the local inputs, while Lemma 36.24 transports block projectors and central type idempotents build type-locus cuts. On each locus, Lemma 36.72, Lemma 36.73, and global Artin reciprocity bound $v_{cc}(i, X_s, \pi)$. Lemma 36.69 and Lemma 36.68 then bound $\lambda_i(U_i/L_i)$. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1336. □

Lemma 36.75 (The local error at the type primes)**.** *Consider a member in the small-prime setup preceding Theorem 36.16, with hub width η_X and type threshold $40 \le C_T \le b$, at which statement (P) of Definition 36.67 holds. Let i be a twisted index with $\ell_i = g_k$. Then $\lambda_i(U_i/L_i) \le \sum_{\pi \in Z_{\mathrm{good}}(i)} \mathrm{mult}(i,\pi) R_i(\pi)$ with $\eta_0 = \eta_X$.*

Outline. If $Z_{\mathrm{good}}(i)$ is empty, $U_i = L_i = 0$ by 36.11. Otherwise the parameter split bound, the type-prime inequality $g_{1-k} \le 2g_k$, and the finiteness hypothesis of 36.11 verify conditions (i)–(iii) of 36.67, and statement (P) gives the bound. The complete proof is on p. 1341. □

Definition 36.76 (The coefficient-chain statement for words with every letter thick)**.** Let $\ell \ge 11$ be prime and $q_\ell = \ell - 1$. Let $K/\mathbf{Q}_\ell$ be unramified of even degree d', and let L' be a finite extension of $\mathbf{Q}_\ell$ containing a primitive ℓ-th root of unity and the images of all embeddings of $K(\mu_\ell)$, with integers O. Number the embeddings τ_i, $i \in \mathbf{Z}/d'\mathbf{Z}$, in Frobenius order. Let $F : T_1 \to T_2$ be an equivariant surjection of crystalline O-lattices, $r = \operatorname{rank} T_2$, and use Z_f, B and I_F as in Definition 36.56. Let $p_0 \in \{0,1\}$ and $S, \delta, m, \beta_i \ge 0$ be integers with the following properties. The jumps of T_1 at τ_i lie in $[-\beta_i - S, -\beta_i] \cup [-\delta, \delta]$ when $i + p_0$ is even, for the representative of i in $[0, d')$, and in $[\beta_i, \beta_i + S] \cup [-\delta, \delta]$ otherwise. At each τ_i, the lattice T_2 has at most m jumps in $[-\delta, \delta]$. With

$$h = \max_i \beta_i + S, \quad H = h + \delta, \quad \lambda = 1 + \log_\ell(2H), \quad A = \frac{H(2\ell - 1)}{q_\ell^2} + 3 + \lambda,$$

with $\log_\ell 0 = 0$, one has $h/2 \le \beta_i$ and $2A + \delta + 1 \le \beta_i$ for every i. Put

$$\sigma = \frac{\lfloor h/q_\ell \rfloor + 1 + \delta}{q_\ell} + 1, \qquad \sigma_2 = \frac{2H}{q_\ell^2} + 1.$$

Statement (R) asserts the following for all such data. There are an integer $a_1 \le A+1$, real numbers s_1, E_1, e_2 and an integer $e \ge 0$ with

$$s_1 \le \frac{2a_1 + 2\delta + S}{q_\ell} + 2\lambda, \quad E_1 \le \frac{3a_1 + S + 2\delta}{q_\ell} + 3\lambda, \quad e_2 \le \frac{h}{\ell q_\ell} + E_1 + 2\sigma_2 + 2S + 5 + \lambda,$$

and $e \le s_1 + 2\sigma + e_2 + S + E_1 + \sigma_2 + 1$. There are also a finite free O-module F_0 of rank at most $(d'+1)r$, an O-linear map $\jmath : F_0 \to Z_f(T_2)$ surjective modulo $B(T_2)$, a chain $F_0 \supset F_1 \supset \cdots \supset F_k$, and a set $\mathcal{G}$ of at most $4d'$ indices below k with $k - |\mathcal{G}| \le 3d'/2$. For $i \in \mathcal{G}$, F_i lies in the sum of F_{i+1} and the span of at most m of its elements. For $i < k$ outside $\mathcal{G}$, the length of $F_i/(F_i \cap (F_{i+1} + \jmath^{-1}(I_F)))$, normalized by that of $O/\ell O$, is at most $rS + 2m((H+1)/q_\ell + 1)$. Finally $\ell^e \jmath(F_k) \subset I_F$.

Lemma 36.77 (Coordinate-subset basis refinement for a saturated submodule)**.** *Let O be a discrete valuation ring, M a finite free O-module with finite basis $(b_i)_{i \in I}$, and $S \subset I$ a subset. Let $P \subset \sum_{i \in S} Ob_i$ be an O-saturated submodule of M, meaning that whenever $a \in O \setminus \{0\}$, $x \in M$, and $ax \in P$, one has $x \in P$. Then there exists an O-basis $(b'_i)_{i \in I}$ of M such that $b'_i = b_i$ for every $i \notin S$, $\sum_{i \in S} Ob'_i = \sum_{i \in S} Ob_i$, and $P = \sum_{i \in I,\, b'_i \in P} Ob'_i$.*

Outline. Restrict to $Q = \sum_{i \in S} Ob_i$, where P is O-saturated; over the DVR O, P is a direct summand of Q with adapted basis $(c_j)_{j \in S}$. Replacing $(b_i)_{i \in S}$ by $(c_i)_{i \in S}$ while fixing b_i for $i \notin S$ gives the basis of M. The complete proof is on p. 1342. □

Definition 36.78 (Components and coefficient chains for thick words)**.** Use the data of Definition 36.76, and put $n = d'$, $q = \ell - 1$, and

$$J_- = \{c \in \mathbf{Z}/n\mathbf{Z} : c + p_0 \text{ is even for the representative of } c \text{ in } [0, n)\}.$$

Thus $|J_-| = n/2$. Put $\Delta = \mathrm{Gal}(K(\mu_\ell)/K)$, and choose a topological generator γ of $\mathrm{Gal}(K(\mu_{\ell^\infty})/K(\mu_\ell))$. Write $\mathcal{A} = \prod_{i \in \mathbf{Z}/n\mathbf{Z}} O[[Z]]$, and let e_i be its i-th idempotent. For a crystalline O-lattice T, use the Wach lattice $N(T) \subset \mathcal{D}(T)$ and its associated étale module $\mathcal{D}(T)$ as in Lemmas 36.17–36.19 (at $s = 1$), and let $\mathrm{cmp}_T : N(T) \to D_{\mathrm{cris}}(T[1/\ell]) = (B_{\mathrm{cris}} \otimes_{\mathbf{Q}_\ell} T[1/\ell])^{G_K}$ be the specialization map modulo Z in the proof of Lemma 36.17 composed with scalar extension to L'. Write φ for Frobenius on these modules, and F_D for

the map induced by F. Choose the invariant parameter π_0 at unramified level one, and put $Q_0 = \pi_0 + \ell$. The parameter is normalized by $\mathcal{A}^\Delta = \prod_i O[[\pi_0]]$; the cyclotomic polynomial $\Phi_\ell(1+Z)$ is a unit times Q_0.

For $k \geq 0$, put $P_k = \prod_{j=0}^{k-1} \varphi^j(Q_0)$, with $P_0 = 1$. Thus $P_k^{a_0} x \in N(T)$ for some $k, a_0 \geq 0$ means that x has finite poles along these Frobenius translates. Define

$$\begin{aligned}
\mathcal{D}_i^\Delta(T) &= \{x \in \mathcal{D}(T) : e_i x = x,\ \varepsilon x = x \text{ for every } \varepsilon \in \Delta\},\\
\mathcal{N}_i(T) &= N(T) \cap \mathcal{D}_i^\Delta(T), \qquad \mathcal{U}^j(T) = \pi_0^j N(T)^\Delta \quad (j \geq 0),\\
\mathcal{R}_i^{(j)}(T) &= \mathcal{A}\text{-span}\{\varphi^j x : x \in \mathcal{N}_i(T)\},\\
\mathcal{V}_i(T) &= \{x \in \mathcal{D}_i^\Delta(T) : (\gamma-1)x \in \mathcal{U}^1(T),\ \text{and } P_k^{a_0} x \in N(T) \text{ for some } k, a_0 \geq 0\},\\
\mathcal{W}_i(T) &= \{x \in \mathcal{V}_i(T) : Q_0^\delta x \in \mathcal{R}_{i-1}^{(1)}(T)\},\\
\mathcal{C}_i(T) &= \mathrm{cmp}_T(\mathcal{N}_i(T)),\\
\mathcal{C}_i(F) &= \{\mathrm{cmp}_{T_2}(F_D x) : x \in \mathcal{N}_i(T_1),\ F_D x \in N(T_2)\}.
\end{aligned}$$

A Herr cocycle of T is a pair $(z, v) \in \mathcal{D}(T)^\Delta \times \mathcal{D}(T)^\Delta$ satisfying $(\gamma-1)z = (\varphi-1)v$. Write $\mathcal{H}(T)$ for their O-module and put $\mathrm{cl}_T = \mathrm{cl}_{K,T} : \mathcal{H}(T) \to H^1(K, T)$ for the natural integral Herr class map of Lemma 36.57(c) (with $e = 1$). Its kernel consists of the pairs $((\varphi-1)t, (\gamma-1)t)$ with $t \in \mathcal{D}(T)^\Delta$. The map $F_{\mathcal{H}} : \mathcal{H}(T_1) \to \mathcal{H}(T_2)$ applies F_D to both coordinates.

Set

$$a = \lceil A \rceil, \qquad M = \lfloor h/q \rfloor + 1, \qquad \widehat{\sigma} = \lceil \sigma \rceil, \qquad C_{\mathrm{len}} = rS + 2m\left(\frac{H+1}{q} + 1\right),$$

and define the real numbers

$$\begin{aligned}
B_1 &= \frac{2a + 2\delta + S}{q} + \lambda, \qquad E_G = \frac{3a + S + 2\delta}{q} + \lambda,\\
B_{\mathrm{red}} &= B_1 + 2\sigma + 2,\\
B_{\mathrm{vec}} &= \frac{h}{\ell q} + \frac{S+1}{q} + S + E_G + 2\sigma_2 + 6 + \lambda,\\
B_{\mathrm{int}} &= S + E_G + \sigma_2 + \lambda.
\end{aligned}$$

Here $A, h, H, \lambda, \sigma, \sigma_2$ have the meanings fixed in Definition 36.76.

For submodules $U' \subset U$ of a finite free O-module, a *generator step* means that U/U' is generated by at most m elements. A *length step* means that $\lambda_O(U/U') \leq C_{\mathrm{len}}$, where $\lambda_O = \mathrm{length}_O / \mathrm{length}_O(O/\ell O)$. Equivalently, in the *obstruction-map formulation*, a step $U' \subset U$ is a generator step (respectively, a length step) if there is an O-linear map $U \to Q$ with kernel U' such that Q is generated over O by at most m elements (respectively, $\lambda_O(Q) \leq C_{\mathrm{len}}$). A chain with at most a' generator steps and b' length steps is a finite descending chain whose steps can be assigned these two types with the indicated counts.

Lemma 36.79 (Obstruction maps for a step)**.** *Under the hypotheses of Definition 36.78, the quotient and obstruction-map conditions are equivalent for each generator step and each length step.*

Proof. The obstruction-map formulation of a step gives an injection $U/U' \hookrightarrow Q$. If Q is generated by at most m elements, choose a surjection $O^m \to Q$. The inverse image of U/U' is a submodule of O^m, hence is free of rank at most m, and maps onto U/U'. If $\lambda_O(Q) \leq C_{\mathrm{len}}$, the injection gives the same bound for U/U'. Conversely, take $Q = U/U'$ and the quotient map. Thus the two formulations of coefficient chains agree. □

Lemma 36.80 (Cocycle reduction on the negative components)**.** *Assume $\mathscr{H}_2$ and the data of Definitions 36.76 and 36.78. There is an integer $u_1 \geq 0$ with $u_1 \leq B_1$ such that*

$$s_{\mathrm{red}} = u_1 + 2\widehat{\sigma} \leq B_{\mathrm{red}}$$

has the following property. For every finite free O-module P and O-linear map $g : P \to Z_f(T_2)$, there are a submodule $P_2 \subset P$, a chain from P to P_2 with at most n generator steps and no length steps, and O-linear maps

$$w_c : P_2 \longrightarrow \mathcal{D}(T_2) \qquad (c \in J_-)$$

with $w_c(P_2) \subset \mathcal{W}_c(T_2)$. Write $Z^1(K,T_2)$ for the O-module of continuous cocycles $G_K \to T_2$. For each $f \in P_2$ and $c \in J_-$, one can choose an integral cocycle $\kappa_c(f) \in Z^1(K,T_2)$ and a Herr cocycle $X_c(f) = (z_c(f), v_c(f))$ such that

$$v_c(f) \in \mathcal{U}^M(T_2), \qquad \mathrm{cl}_{T_2}(X_c(f)) = [\kappa_c(f)],$$

for a sign $\epsilon(f) \in \{1,-1\}$,

$$\ell^{s_{\mathrm{red}}} g(f) - \epsilon(f) \sum_{c \in J_-} \kappa_c(f) \in B(T_2),$$

and whenever $e_* \geq 0$ is an integer and $u \in \mathcal{V}_c(T_1)$ satisfies $\ell^{e_*} w_c(f) - F_D u \in \mathcal{U}^1(T_2)$, there exists $Y \in \mathcal{H}(T_1)$ with $\mathrm{cl}_{T_1}(Y) \in H^1_f(K,T_1)$ and $\ell^{e_*} X_c(f) - F_{\mathcal{H}} Y \in \mathcal{U}^M(T_2) \times \mathcal{U}^M(T_2)$.

Outline. Crystalline extensions and Herr comparison on a basis of P give linear extension coordinates, which polynomial operators in $D = \pi_0^{-1}(\gamma - 1)$ and uniformizer truncation correct to w_c. Two rounds of constant-coefficient kernels over J_- produce P_2 in n generator steps. Ordinary truncation and bijectivity of $\varphi - 1$ on $\mathcal{U}^M(T_2)$ then construct X_c, κ_c, and the lift Y. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1342. □

Lemma 36.81 (A recursive letter and its interpolation cost)**.** *Assume $\mathscr{H}_2$ and the data of Definitions 36.76 and 36.78. For $i \notin J_-$, put*

$$E_i = \left\lfloor \frac{3a + S + 2\delta - 1}{q} + \log_\ell H \right\rfloor, \qquad s_2 = \lceil \sigma_2 \rceil.$$

Then $E_i \geq 0$ and

$$E_i \leq \frac{3A + S + 2\delta}{q} + \lambda \leq E_G.$$

For every $y \in \mathcal{N}_{i-2}(T_2)$ such that $d = \varphi^2 \,\mathrm{cmp}_{T_2}(y) \in \mathrm{Fil}^{\beta_i}_{\mathrm{cris}}$, there is $z \in \mathcal{N}_i(T_1)$ with

$$F_D z \in N(T_2), \qquad \mathrm{cmp}_{T_2}(F_D z) = \ell^{E_i + s_2 + \beta_{i-1} + S - \beta_i} d.$$

The exponent on the right is an integer and may be negative.

Outline. Two-Frobenius truncation of y, Laurent coefficient bounds, and the filtered Wach criterion produce $x \in \mathcal{N}_i(T_2)$ with comparison value $\ell^{s_2 + \beta_{i-1} + S - \beta_i} d$. The gamma-derivation identity and Δ-averaged Wach lifting construct source lifts of $P_I(D_2)x$ and $P_J(D_2)x$, and polynomial interpolation at exponent E_i combines them into z. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1346. □

Lemma 36.82 (A Wach basis adapted to the central jumps)**.** *Assume $\mathscr{H}_2$ and the data of Definitions 36.76 and 36.78. Let $i \in J_-$, put $h_i = \beta_i + S$, and let $e_1, \dots, e_r$ be an $O[[\pi_0]]$-basis of $\mathcal{N}_{i-1}(T_2)$. There are a matrix $B \in \mathrm{GL}_r(O)$, an integer $m' \leq m$, and integers $-\delta \leq u_j \leq \delta$, $1 \leq j \leq m'$, such that*

$$e'_j = \sum_{k=1}^{r} B_{jk} e_k$$

is an $O[[\pi_0]]$-basis and

$$\iota\varphi \,\mathrm{cmp}_{T_2}(e'_j) \in \mathrm{Fil}^{u_j} D_{\mathrm{dR}} \quad (1 \leq j \leq m'), \qquad \sum_{u=-\delta}^{\delta} \mathrm{mult}_{\tau_i}(u)(h_i + u) \leq \sum_{j=1}^{m'} (h_i + u_j).$$

The basis change and its inverse have integral entries, so their determinant valuations are zero.

Outline. Pulling back $\mathrm{Fil}^u D_{\mathrm{dR}}$ via the basis $w_k = \iota\varphi \,\mathrm{cmp}_{T_2}(e_k)$ gives saturated submodules $F^u \subset O^r$. Iterating Lemma 36.77 produces $B \in \mathrm{GL}_r(O)$ and u_j matching the jump multiplicities. The complete proof is on p. 1347. □

Lemma 36.83 (First Frobenius correction of the chosen coordinates)**.** *Assume $\mathscr{H}_2$ and the data of Definitions 36.76 and 36.78. There is an integer $e_{\mathrm{vec}} \ge 0$ with $e_{\mathrm{vec}} \le B_{\mathrm{vec}}$ such that, for every $c \in J_-$, every finite free O-module P, every submodule $U \subset P$, and every O-linear map $w : U \to \mathcal{D}(T_2)$ with $w(U) \subset \mathcal{W}_c(T_2)$, there are a submodule $U' \subset U$, an O-linear map $y_c : U' \to \mathcal{D}(T_1)$, and a chain from U to U' with at most three generator steps and one length step, such that*

$$y_c(f) \in \mathcal{V}_c(T_1), \qquad r_c(f) := \ell^{e_{\mathrm{vec}}} w(f) - F_D y_c(f) \in \mathcal{N}_c(T_2) \quad (f \in U').$$

Outline. Constant-coefficient reduction links $D_f(0)$ to $x_f \in \mathcal{N}_{c-2}(T_2)$, while Lemma 36.81 and level-one truncation build source lifts $v_d \in \mathcal{V}_c(T_1)$. An adapted Wach basis from Lemma 36.82 and a determinant length bound give one length step and one generator step, and two filtration kernels yield U' and y_c. The filtered Wach criterion then places $\ell^{e_{\mathrm{vec}}} w - F_D y_c$ in $\mathcal{N}_c(T_2)$. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1348. □

Lemma 36.84 (Restricting a coefficient module through a quotient)**.** *Let ℓ be a prime, let O be the ring of integers of a finite extension of $\mathbf{Q}_\ell$, and normalize length by $\lambda_O(Q) = \mathrm{length}_O(Q)/\mathrm{length}_O(O/\ell O)$. Let $a, m \ge 0$ be integers with $a \le m$, let $C \ge 0$ be real, let $K \subset X \subset N$ be O-modules, and let $s_1, \dots, s_a \in N$. Suppose*

$$\lambda_O\big(N/(X + \textstyle\sum_{j=1}^{a} O s_j)\big) \le C.$$

Let U be an O-module. For an O-linear map $\psi : U \to N/K$, put

$$U_1 = \psi^{-1}\big((X + \textstyle\sum_j O s_j)/K\big), \qquad U_2 = \psi^{-1}(X/K).$$

Then $U \supset U_1 \supset U_2$ has one length step of normalized length at most C and one generator step with at most m generators.

Proof. The map induced by ψ embeds U/U_1 in $N/(X + \sum_j O s_j)$, proving the length estimate. It embeds U_1/U_2 in $(X + \sum_j O s_j)/X$, which is generated by the images of the s_j. A submodule of a module generated by a elements over a discrete valuation ring is generated by at most a elements: lift it to a submodule of O^a, take a basis of that submodule, and project. This proves the generator estimate. □

Lemma 36.85 (Explicit reconstruction after three crystalline comparisons)**.** *Use the modules of Definition 36.78 and their level-one Wach properties. Explicitly, specialization kills precisely $\pi_0\mathcal{N}_i(T_j)$ in each $\mathcal{N}_i(T_j)$, commutes with maps of lattices, and satisfies*

$$\mathrm{cmp}_{T_j}(Q_0^k \varphi z) = \ell^k \varphi\, \mathrm{cmp}_{T_j}(z)$$

whenever $z \in \mathcal{N}_{i-1}(T_j)$, $k \ge 0$, and $Q_0^k \varphi z \in \mathcal{N}_i(T_j)$. Also $Q_0(0) = \ell$ and $(\gamma - 1)\mathcal{N}_i(T_j) \subset \pi_0\mathcal{N}_i(T_j)$.

Let c be a component and let $B \ge 0$ be an integer. Suppose $Q_0^B \varphi \mathcal{N}_{c-1}(T_j) \subset \mathcal{N}_c(T_j)$ for $j = 1, 2$. Let $r \in \mathcal{N}_c(T_2)$, let $d_1, d_2, d_3 \in D_{\mathrm{cris}}(T_2[1/\ell])$, and let s_1, s_2, b_0, h_0, e, t be nonnegative integers and $k \in \mathbf{Z}$ satisfying

$$\begin{aligned} \ell^{s_1}\, \mathrm{cmp}_{T_2}(r) &= \ell^B \varphi d_1, \\ \ell^{b_0} d_1 &= \varphi d_2, \\ \ell^{s_2} d_2 &= \ell^{h_0} \varphi d_3, \\ e + h_0 &= t + k + b_0. \end{aligned}$$

Suppose $z \in \mathcal{N}_{c-1}(T_1)$ satisfies

$$F_D z \in N(T_2), \qquad \mathrm{cmp}_{T_2}(F_D z) = \ell^k \varphi^2 d_3.$$

Then

$$x = \ell^t Q_0^B \varphi z \in \mathcal{V}_c(T_1), \qquad \ell^{e+s_1+s_2} r - F_D x \in \mathcal{U}^1(T_2).$$

Outline. Height bounds give $x \in \mathcal{V}_c(T_1)$ and $F_D x \in \mathcal{N}_c(T_2)$. Weighted specialization and the identities equate specializations of $F_D x$ and $\ell^{e+s_1+s_2} r$, putting the difference in $\pi_0\mathcal{N}_c(T_2) \subset \mathcal{U}^1(T_2)$. The complete proof is on p. 1352. □

Lemma 36.86 (Integral residual correction and compatible Herr classes)**.** *Assume $\mathscr{H}_2$ and the data of Definitions 36.76 and 36.78. Put*

$$e_{\mathrm{int}} = S + \lceil E_G + \sigma_2 \rceil.$$

Then $e_{\mathrm{int}} \le B_{\mathrm{int}}$. For every $c \in J_-$, every finite free O-module P, every submodule $U \subset P$, and every O-linear map $r : U \to \mathcal{D}(T_2)$ with $r(U) \subset \mathcal{N}_c(T_2)$, there are a submodule $U'' \subset U$ and a chain from U to U'' with at most three generator steps and two length steps such that, for each $f \in U''$, there is an element $x_f \in \mathcal{V}_c(T_1)$ satisfying

$$\ell^{e_{\mathrm{int}}} r(f) - F_D x_f \in \mathcal{U}^1(T_2).$$

When $U = U'$ and $r = r_c$ are obtained in Lemma 36.83 from $w = w_c|_V$ on any submodule $V \subset P_2$ of Lemma 36.80, set $u_f = \ell^{e_{\mathrm{int}}} y_c(f) + x_f$ for $f \in U''$. Then

$$\ell^{e_{\mathrm{vec}}+e_{\mathrm{int}}} w_c(f) - F_D u_f \in \mathcal{U}^1(T_2),$$

and there is a Herr cocycle $Y_c(f) \in \mathcal{H}(T_1)$ such that

$$\mathrm{cl}_{T_1}(Y_c(f)) \in H^1_f(K, T_1), \qquad \ell^{e_{\mathrm{vec}}+e_{\mathrm{int}}} X_c(f) - F_{\mathcal{H}} Y_c(f) \in \mathcal{U}^M(T_2) \times \mathcal{U}^M(T_2).$$

Consequently,

$$\ell^{e_{\mathrm{vec}}+e_{\mathrm{int}}} \kappa_c(f) \in I_F.$$

Outline. An adapted basis via Lemma 36.82 and determinant bound, two applications of Lemma 36.84, and a filtration kernel produce U'' in three generator steps and two length steps. On U'', Lemma 36.81 and Lemma 36.85 construct $x_f \in \mathcal{V}_c(T_1)$. Lemma 36.80 then lifts u_f to $Y_c(f)$, and bijectivity of $\varphi - 1$ on $\mathcal{U}^M(T_2)$ makes the residual a boundary. The complete proof is on p. 1352. □

Lemma 36.87 (A basis adapted to a saturated submodule)**.** *Let ℓ be prime, let $L'/\mathbf{Q}_\ell$ be finite, let O be its ring of integers, let M be a finite free O-module, and let $(b_i)_{i \in I}$ be an O-basis with I finite. A submodule $P \subset M$ is called saturated if*

$$a \in O \setminus \{0\}, \quad x \in M, \quad ax \in P \quad \Longrightarrow \quad x \in P.$$

If P is saturated, there are an O-basis $(c_i)_{i \in I}$ of M and a subset $J \subset I$ such that

$$P = \bigoplus_{i \in J} O c_i.$$

The change of basis and its inverse have entries in O. Thus both matrices have entries of nonnegative valuation, and their determinants have valuation zero, for the valuation normalized by $v_\ell(\ell) = 1$.

Outline. Smith normal form gives a basis (c_i) of M and $p_j = a_j c_{\alpha(j)}$ spanning P with $a_j \neq 0$. Saturation gives $c_{\alpha(j)} \in P$, so $J = \alpha(J_0)$ spans P, and the O-transition matrices are inverses with unit determinants. The complete proof is on p. 1355. □

Proposition 36.88 (The coefficient-chain statement for thick words)**.** *Assume $\mathscr{H}_2$. Then statement (R) of Definition 36.76 holds for all data of that definition.*

Outline. Theorem 36.14, the rank formula for $H^1_f(K, T_2)$, and multiplication by ϖ yield F_0 of rank at most $(n+1)r$ and $\jmath : F_0 \to Z_f(T_2)$ surjective modulo $B(T_2)$. Lemma 36.80 reduces $\jmath$ to J_-, and iterating Lemma 36.83 and Lemma 36.86 over J_- gives a chain with $\ell^e \jmath(F_k) \subset I_F$. Elementary inequalities then verify Definition 36.76. The complete proof is on p. 1355. □

Lemma 36.89 (Separated-cluster lifting and colength bound at unramified primes)**.** *Assume $\mathscr{H}_2$. Let $\ell \ge 11$ be prime and $q_\ell = \ell - 1$. Let $K/\mathbf{Q}_\ell$ be finite unramified of degree $d_u \ge 1$, and let L' be a finite extension of $\mathbf{Q}_\ell$ containing the images of all embeddings of $K(\mu_\ell)$, with integers O. Number the embeddings $\tau_a : K \to L'$, $a \in \mathbf{Z}/d_u\mathbf{Z}$, in Frobenius order. Let $F : T_1 \to T_2$ be an equivariant surjection of crystalline O-lattices over K with $r = \mathrm{rank}_O T_2$, use Z_f, B, and I_F as in Definition 36.56, and write λ_O for length normalized by $\mathrm{length}_O(O/\ell O)$. Let $b_*, w_*, S, \delta, J \ge 0$ be integers with $1 \le J < d_u$, and for each $a \in \mathbf{Z}/d_u\mathbf{Z}$*

let $\varsigma_a \in \{1,-1\}$ *and* $s_a \in \mathbf{N}$ *be such that the jumps of* T_1 *at* τ_a *lie in* $[s_a, s_a+S]\cup[-\delta,\delta]$ *when* $\varsigma_a = 1$ *and in* $[-s_a-S,-s_a]\cup[-\delta,\delta]$ *when* $\varsigma_a=-1$. *Call* $a \in \mathbf{Z}/d_u\mathbf{Z}$ thick *if* $b_* \le s_a \le b_*+\delta$ *and* nonthick *if* $s_a \le w_*$. *Assume every* $a \in \mathbf{Z}/d_u\mathbf{Z}$ *is thick or nonthick. Assume also this two-case separation condition: for every thick* a, *either no* $a-k$ *with* $1\le k\le J$ *is thick, or there is an integer* d *with* $1\le d\le J$ *such that* $a-d$ *is thick and no* $a-k$ *with* $1\le k\le d+J$ *and* $k\ne d$ *is thick. All positions here are read in* $\mathbf{Z}/d_u\mathbf{Z}$. *Put* $g=w_*+S+\delta$, $g_*=\max(1,g)$, *and* $h=b_*+\delta+S$ *if a thick position exists,* $h=g$ *otherwise, with* $\log_\ell 0=0$.

There is a threshold $C_0\ge 1$, *depending on* $\mathscr{H}_2$ *alone, such that whenever* $C_2\ge C_0$ *and* $C_2(J+1)(w_*+S+\delta+1)\le b_*$, *there exists an integer* $e_{\mathrm{cl}}\ge 0$ *with* $e_{\mathrm{cl}}\le 800\{h_1/q_\ell^{J+1}+g_*(3J+1)+\log_\ell h_1\}$, *where* $h_1=\max(h,1)$, *satisfying* $\ell^{e_{\mathrm{cl}}}Z_f(T_2)\subset I_F$ *and*

$$\lambda_O\big(Z_f(T_2)/I_F\big)\le C_2(d_u+1)r\big\{h/q_\ell^{J+1}+(g+1)(J+1)+\log_\ell h\big\}. \tag{36.33}$$

Outline. The two-case separation condition yields cuts with backward lengths at most $3J+1$ and at most two thick positions per interval, and Lemma 36.28 bounds the cut remainders. Propagating frames across intervals and lifting downward positions by Lemma 36.29 gives the exponent e_{cl}. The generator bound from Proposition 36.88 then yields the colength estimate. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1356. □

Lemma 36.90 (Unrestricted and short-interval coordinate lifting at unramified primes)**.** *Assume* $\mathscr{H}_2$. *Let* $\ell\ge 5$ *be prime,* $q_\ell=\ell-1$, $K/\mathbf{Q}_\ell$ *unramified of degree* $d_u\ge 1$, L', O, τ_a *(*$a\in\mathbf{Z}/d_u\mathbf{Z}$*),* $F: T_1\to T_2$, $r=\operatorname{rank}_O T_2$, Z_f, B, I_F, *and* λ_O *be as in Lemma 36.89.*

(a) Suppose the jumps of T_1 *at* τ_a *lie in* $[-M_a,h_a]$ *with* $M_a,h_a\in\mathbf{N}$, *and set* $H'=\max_{a\in\mathbf{Z}/d_u\mathbf{Z}}(M_a+h_a)$. *Then there is an integer* $e'\ge 0$ *satisfying* $\ell^{e'}Z_f(T_2)\subset I_F$ *and*

$$e'\le 23\{2+H'/\ell+\log_\ell(H'+1)\}. \tag{36.34}$$

(b) Assume $\ell\ge 11$. *Let* $b_*,w_*\ge 1$ *and* $S,\delta,m\ge 0$ *be integers with* $20w_*,20\delta,20S\le b_*$. *For each* $a\in\mathbf{Z}/d_u\mathbf{Z}$, *let* $\varsigma_a\in\{1,-1\}$ *and* $s_a>\delta$ *be integers with* $\varsigma_{a+1}=-\varsigma_a$ *such that the jumps of* T_1 *at* τ_a *lie in* $[s_a,s_a+S]\cup[-\delta,\delta]$ *when* $\varsigma_a=1$ *and in* $[-s_a-S,-s_a]\cup[-\delta,\delta]$ *when* $\varsigma_a=-1$, *and* T_2 *has at most* m *jumps in* $[-\delta,\delta]$ *counted with multiplicity. Assume every* $a\in\mathbf{Z}/d_u\mathbf{Z}$ *is either thick (*$b_*\le s_a\le b_*+\delta$*) or nonthick (*$s_a\le w_*$*), at least one* $a\in\mathbf{Z}/d_u\mathbf{Z}$ *is nonthick, and no three consecutive positions in* $\mathbf{Z}/d_u\mathbf{Z}$ *are thick. Put* $h=b_*+\delta+S$ *if a thick position exists and* $h=w_*+S$ *otherwise, set* $U_0=h/\ell^2+(w_*+S+\delta)/\ell+\log_\ell(2h)+1$, *and let* $I=1$ *when some nonthick position* a *adjacent to a thick position satisfies*

$$|\ell s_a-b_*|\le 320\{\ell\delta+\ell S+b_*/\ell+s_a+\ell\log_\ell(2h)\}, \tag{36.35}$$

and $I=0$ *otherwise. Then, for every integer* $e'\ge 0$ *with* $\ell^{e'}Z_f(T_2)\subset I_F$ *(by (a), applied with* $(M_a,h_a)=(\delta,s_a+S)$ *when* $\varsigma_a=1$ *and* (s_a+S,δ) *when* $\varsigma_a=-1$, *such an* e' *exists and satisfies (36.34)),*

$$\lambda_O\big(Z_f(T_2)/I_F\big)\le 320(d_u+1)r\big\{h/\ell^2+(w_*+\delta)/\ell+S+1+\log_\ell(2h)+\delta I\big\}+320d_u m e'. \tag{36.36}$$

Outline. For (a), two level-one Wach lifts and a shifted-weight correction give $\ell^{2e_W+e_U}Z_f(T_2)\subset I_F$, and valuation estimates bound e'. For (b), the case $b_*/4<40U_0$ follows from this box count and the generator bound of Proposition 36.88. Otherwise, cuts before nonthick positions separate at most $4d_um$ classes killed by $\ell^{e'}$ from a quotient killed by ℓ^E. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1359. □

Lemma 36.91 (The non-place part of the row bound)**.** *Fix a cut datum at a prime* ℓ *outside the small set (so* $0\le\delta_{\mathrm{pr}}\le 2t$*), with* $f,n,t,n_V,L\ge 1$ *and* $m,r\le t$. *Let* $\mathcal{R}$ *and* $\mathcal{C}$ *be the row and column sets, put* $d_u=[E_u:\mathbf{Q}_\ell]$, *and assume* $\sum_{w\in\mathcal{R}}d_w\le 2f$ *and* $\sum_{u\in\mathcal{C}}d_u\le 2f$. *Write* $h_W(u)$ *and* h_0 *for the normalized invariant lengths of Lemma 36.23, and rename its projector valuation* δ_{pr}. *Let* $c\ge 1$, *and suppose*

$$c_B\le ctnn_VL,\qquad h_0\le ct,\qquad c_0(u)\le cnn_VL,\qquad h_W(u)\le crnn_VL\quad(u\in\mathcal{C}).$$

Put

$$D_0 = tc_B \sum_{w\in\mathcal{R}} (d_w + 2) + 4n_V(|\mathcal{C}| + 1)\delta_{\mathrm{pr}} + t|\mathcal{C}|h_0,$$
$$D_{\mathrm{sp}} = D_0 + \sum_{u\in\mathcal{C}} \big(d_u m\delta_{\mathrm{pr}} + 2\delta_{\mathrm{pr}} + h_W(u) + rc_0(u)\big),$$
$$B_c = cfnt^2 n_V L.$$

Then $D_0 \le 32B_c$ and $D_{\mathrm{sp}} \le 48B_c$. In particular, for $c = 120$, both are at most $7200fnt^2n_VL^2$.

Outline. The degree sum $2f$ bounds $|\mathcal{C}| \le 2f$ and $\sum_{w\in\mathcal{R}}(d_w+2) \le 6f$. With $m, r \le t$ and $\delta_{\mathrm{pr}} \le 2t$, termwise bounds in B_c give $D_0 \le 32B_c$, $D_{\mathrm{sp}} \le 48B_c$, and $c = 120$. The complete proof is on p. 1362. □

Lemma 36.92 (Inert word dichotomy and final-row congruence passage)**.** *Assume $\mathscr{H}_2$, and work on the twist-good cut family of Theorem 36.26(I) with the norm-choice facts and four ties. Fix $C_2 \ge \max(1, C_0)$, where C_0 is the threshold of Lemma 36.89. At a member satisfying the finiteness hypothesis of Definition 36.11, use the size width S_c, error width $\eta = \eta_{\mathrm{sz}}$, $n = f/2$, $n_V = r + t$, $H_B = b + S_c + \eta + t + 2$, thick density $\delta_2 = |T_2|/n > 0$, progression-window length L_w, order o_ℓ of ℓ modulo 2^{e+2} up to sign for the host tower exponent e, and base sizes b, $w''(n)$, $b_T + \lfloor b'/x\rfloor$ $(T \in TT)$. Put*

$$Y = \frac{kL}{m\delta_2}, \quad \eta_3 = \eta + t + 2, \quad w_B = \max\big\{w''(n) + \eta,\ \max_{T\in TT}(b_T + b'/x + \eta)\big\}, \quad w_P = w_B + \tfrac{t+1}{2}.$$

Eventually along the family, for every member and every twisted index i whose residue characteristic $\ell = \ell_i \ge 5$ does not divide $Q_S q d g_0 g_1$, the following hold:

(a) Suppose ℓ is inert in $\mathbf{Q}(\sqrt{-d})$. Then $\ell \ge 11$, and for every place $u \mid \ell$ of E the completion $K = E_u$ is unramified of degree d_u over $\mathbf{Q}_\ell$. Set $T_0 = (t+1)/2$, $b_* = \lceil b\rceil - T_0$, $S = \lfloor S_c\rfloor$, $\delta = \lfloor\eta\rfloor + t + 2$, and $J = \max(1, \lceil \log_{\ell-1} Y\rceil)$. At every $\pi \in Z_{\mathrm{good}}(i)$, over a finite splitting extension L' of $\mathbf{Q}_\ell$ containing the images of all embeddings of $E_u(\mu_\ell)$, with integers O, there exist an integer $w_* \in \mathbf{N}$ and, for each $a \in \mathbf{Z}/d_u\mathbf{Z}$, a sign $\varsigma_a \in \{1, -1\}$ and an integer $s_a \in \mathbf{N}$ such that for every equivariant surjection $F : T_1 \to T_2$ from a stable crystalline O-lattice T_1 in a subrepresentation of a finite sum of twisted dual full representations onto a stable crystalline O-lattice T_2 in the rank-r Tate-dual target representation over E_u, the jump hypotheses of Lemma 36.89 hold, T_2 has at most m jumps in $[-\delta, \delta]$ whenever $s_a > \delta$, every $a \in \mathbf{Z}/d_u\mathbf{Z}$ is either thick ($b_* \le s_a \le b_* + \delta$) or nonthick ($s_a \le w_*$, with $|s_a - y| \le T_0$ for some $y \in [w''(n), w_B]$), and with $(M_a, h_a) = (\delta, \max(\delta, s_a + S))$ for $\varsigma_a = 1$ and $(\max(\delta, s_a + S), \delta)$ for $\varsigma_a = -1$,

$$M_a + h_a \le 2H_B, \quad b_* + \delta + S \le H_B, \quad w_* \le w_P, \quad S \le S_c, \quad \delta \le \eta_3. \tag{36.37}$$

If $\ell < Y$ and $o_\ell > 4L_w$, then $2J + 2 \le L_w$, $1 \le J < d_u$,

$$C_2(J+1)(w_* + S + \delta + 1) \le b_*, \tag{36.38}$$

and the thick positions satisfy the two-case separation condition of Lemma 36.89. Otherwise $\varsigma_{a+1} = -\varsigma_a$ for all $a \in \mathbf{Z}/d_u\mathbf{Z}$, and either d_u is even and every $a \in \mathbf{Z}/d_u\mathbf{Z}$ is thick with $\frac{1}{2}(\max_{a'} s_{a'} + S) \le s_a$ and $2A + \delta + 1 \le s_a$ (where A is given by Definition 36.76 for $\beta_a = s_a$ and $d' = d_u$), or

$$1 \le b_*, w_*, \quad 20w_*, 20\delta, 20S \le b_*, \qquad s_a > \delta \quad (\text{all } a), \tag{36.39}$$

at least one position in $\mathbf{Z}/d_u\mathbf{Z}$ is nonthick, and no three consecutive positions in $\mathbf{Z}/d_u\mathbf{Z}$ are thick.

(b) Let $L_{i,u} \in \mathbf{Q}_{\ge 0}$ $(u \mid \ell)$ be place allowances such that when ℓ splits in $\mathbf{Q}(\sqrt{-d})$, $L_{i,u} = (2d_u m + 1)e_A$ on the canonical column set of i and 0 outside it, where $e_A = \lceil 23\{2 + 2\lfloor H_B\rfloor/\ell + \log_\ell(2\lfloor H_B\rfloor + 1)\}\rceil$, and when ℓ is inert in $\mathbf{Q}(\sqrt{-d})$, $L_{i,u}$ bounds $\lambda_O(Z_f(T_2)/I_F)$ at u uniformly across all $\pi \in Z_{\mathrm{good}}(i)$ and all equivariant surjections $F : T_1 \to T_2$ as in (a) (with $L_{i,u} = 0$ if $Z_{\mathrm{good}}(i) = \varnothing$). Then for every $\pi \in Z_{\mathrm{good}}(i)$,

$$\mathcal{B}_{i,\pi}\Big(\rho;\ 7200fnt^2n_VL^2 + \sum_{u\mid\ell} L_{i,u}\Big).$$

Outline. For (a), large-scale bounds under the hypotheses of Theorem 36.26 yield a common cyclic word giving the separation of Lemma 36.89 on long orbits and otherwise Definition 36.76 or Lemma 36.90. For (b), bounding the invariants of Definition 36.15 and Lemma 36.23 feeds into the split and inert lattice-index bounds, and Lemma 36.91 gives the $\mathcal{B}_{i,\pi}$ inequality. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1362. □

Lemma 36.93 (Local costs on the cut)**.** *Assume $\mathscr{H}_2$. On the twist-good cut family of 36.26(I) with the norm-choice facts and four ties, there is one family constant $C > 0$ such that eventually, at every member satisfying the finiteness hypothesis of Definition 36.11, there exist nonnegative costs A_ℓ such that*

$$\eta_{j,\ell} \le d_{j,0} \log \ell \big(A_\ell + \nu_\ell + 2\ell^{-2} + \sum_{1 \le e < t} s_{\ell,e}(E) + v_\ell(h_E) + v_\ell(M_0) \big), \tag{36.40}$$

$$6 + \log(Q_S t r) + \log h_E + \log M_0 + \sum_{\ell \in S} A_\ell \log \ell \le 2000 (C+1)^3 f m b L^2 \qquad (S \text{ any finite set of primes}). \tag{36.41}$$

Here $\eta_{j,\ell}$ is the part of $\eta_j(\mathcal{L})$ above ℓ. In applying (36.41) to the family estimate of Definition 36.11, take S to be the characteristics of the twisted support.

Outline. Bound $\eta_{j,\ell}$ by cases via Theorem 36.16, Lemmas 36.54, 36.66, 36.75, 36.89, 36.90, 36.92, and Propositions 36.65, 36.88. Summing $A_\ell \log \ell$ via integral comparisons gives the global bound. The complete proof is on p. 1366. □

36.6 The closed local condition in the witness

Lemma 36.94 (Artin action on the ratio of toric sums)**.** *Use the host $E \supset K = \mathbf{Q}(\sqrt{-d})$ and the toric data of 1.15(3). Thus E/E^+ is unramified at finite places, the CM embeddings restrict to one embedding of K, $s, \kappa_i \ge 1$, and χ has the prescribed base conductor $\mathfrak{f}$. Let Q be a finite set of split primes prime to $\mathfrak{f}$. Write $T(q)$ for the class term at $q \in \mathrm{Pic}(O_Q)$, and $T_0 \neq 0$ for the sum at the base level with trivial Picard twist. Fix an algebraic closure $\overline{K}$ of K in $\mathbf{C}$. For every character ψ of the finite group $\mathrm{Pic}(O_Q)$, the number*

$$R_\psi = \frac{\sum_q \psi(q)^{-1} T(q)}{T_0}$$

belongs to $\overline{K}$. Suppose $g \in \mathrm{Aut}(\overline{K})$ fixes the embedded CM-type values and the finite character values $\chi_f(y)|y|_{E,f}^{s}$. Choose $t \in \mathbb{A}_{K,f}^{\times}$ with $\mathrm{Art}_K(t)^{-1} = g|_{K^{\mathrm{ab}}}$, and write $[t]$ for its image in $\mathrm{Pic}(O_Q)$. Then

$$g R_\psi = (g\psi)([t])^{-1} R_{g\psi}.$$

Outline. Apply 1.15(3), proved in 1.19, to the twisted and base sums with a common period. Translate the Picard classes by $[t]^{-1}$ in the transport formula, then divide to cancel the common period and transport factors. The complete proof is on p. 1371. □

Lemma 36.95 (The denominator of the finite Fourier expansion)**.** *Let L be the completed finite tensor lattice of a self-dual source and target pair in the Weil representation. Write $\langle\,,\,\rangle$ for its alternating pairing and ψ for the additive character, normalized so that L is its own annihilator. At a collection of split places let e select one factor and $\bar{e}$ the conjugate factor. Put $L_+ = eL$, $L_- = \bar{e}L$. Let $M \ge 1$ be prime to $Q_S q\, \mathfrak{N}\mathfrak{c}_K$ for the host datum, and suppose that every selected split place divides M. Let R be a subring of $\mathbf{C}$ containing the M-th roots of unity. Let Φ be an R-valued function supported on L_- and constant modulo ML_-. Write f_Φ for its vector in the lattice model and f_0 for the vacuum. There is an integer K, fixed by the original tensor lattice, such that $M^K f_\Phi$ belongs to the R-span of the Heisenberg translates of f_0 by $M^{-1}L_+$. In particular, at a place over ℓ, for $M = M_0 d'$ with $\ell \nmid d'$, the power $\ell^{K\,\mathrm{ord}_\ell(M_0)}$ suffices over the local ring containing the values of Φ and the roots of unity.*

Outline. Project integral lattice generators to obtain the Fourier characters. Self-duality and character orthogonality express $M^K f_\Phi$ as a sum of vacuum translates, and the prime-to-ℓ part of M^K is a local unit. The complete proof is on p. 1372. □

Lemma 36.96 (Two denominators independent of the auxiliary level)**.** *Fix a prime ℓ. Let J be a complex linear map into a polynomial space, and let L_0 be a finitely generated additive lattice in its domain. There is $c_1 \geq 0$ such that, in every embedded number field containing the coefficients of $J(L_0)$, all coefficients of $\ell^{c_1} J(L_0)$ are integral at every place over ℓ.*

For the level factor, let E be the CM field, $\mathcal{L}$ the integral marking lattice of $\mathbf{Z}$-rank s, r the target rank, n the source rank and D the degree of the harmonic polynomial. Put $M = M_0 d'$ and $N = Mu$, where $M_0, d', u \geq 1$ and $\ell \nmid d'u$. Let $\mathfrak{b}$ be the marking ideal with $M\mathcal{O}_E \subseteq \mathfrak{b}$. For a lattice $\Lambda \subset E^n$ satisfying $N\mathcal{O}_E^n \subseteq \Lambda \cap \mathcal{O}_E^n$, put

$$I = [\Lambda \cap \mathcal{O}_E^n : N\mathcal{O}_E^n], \qquad A = \mathcal{L}/\mathfrak{b}\mathcal{L}, \qquad H = I^r |A|^n (2|\operatorname{disc} E|N)^D.$$

Then $\operatorname{ord}_\ell H \leq c_3$, where

$$c_3 = ([E : \mathbf{Q}]nr + sn) \operatorname{ord}_\ell M_0 + D \operatorname{ord}_\ell(2|\operatorname{disc} E|M_0).$$

Outline. Clear the ℓ-denominators of the polynomial coefficients on additive generators of L_0. Bound the level denominator using $I \mid N^{[E:\mathbb{Q}]n}$, $|A| \mid M^s$, and the equal ℓ-valuations of M, N, M_0. The complete proof is on p. 1373. □

Lemma 36.97 (The different in descent to the coefficient lattice)**.** *Let $O \subset \mathbf{C}$ be a discrete valuation ring, let ℓ be a rational prime in its maximal ideal, and put $e = \operatorname{ord}_O(\ell)$. Let $K \subset L \subset \operatorname{Frac}(O)$ be number fields. Denote the places selected by O on K, L by v, w. Use integer-valued valuations on both fields. For $y \in L$, suppose there is $b \geq 0$ such that*

$$\operatorname{Tr}_{L/K}(ay) \in \mathcal{O}_{K,v} \quad \text{for every } a \in \mathcal{O}_L \text{ with } a \in (w')^b \text{ at every } w' \mid v,\ w' \neq w.$$

Here w' in an ideal power denotes the prime ideal. Then $\ell^{e^2} y \in \mathcal{O}_{L,w}$. If L_w/K_v is unramified, $y \in \mathcal{O}_{L,w}$.

Outline. Use the Chinese remainder theorem and the completed trace pairing to place y in the local inverse different. Bound its exponent using the trace from the maximal unramified subfield, and use the perfect residue trace pairing in the unramified case. The complete proof is on p. 1373. □

Lemma 36.98 (The short theta value on an integral frame)**.** *Use the lattice and marking of Lemma 36.96 over the host CM field E containing $w_0 = (1+\sqrt{-d})/2 \in \mathcal{O}_E$ ($w_0 + \bar{w}_0 = 1$) with principal line $L \simeq (1, w_0)^* \mathcal{P}_L$ (Lemma 35.8(b)). Write n for the number of theta columns, r for the target rank, $A = \mathcal{L}/\mathfrak{b}\mathcal{L}$, and $\Delta = \operatorname{disc} E$. Suppose that Λ is self-dual, agrees with the standard lattice at the primes dividing M, and that N is an odd multiple of M with $N\mathcal{O}_E^n \subseteq \Lambda \cap \mathcal{O}_E^n$. Fix a place of the coefficient field K. Let $w : A^n \to K$ and the coefficients of a homogeneous harmonic polynomial P of degree D be integral there. On every integral frame of good reduction over that place the short theta value v, with class coefficient one, satisfies*

$$I^r |A|^n (2\Delta N)^D v \ \in V, \qquad I = [\Lambda \cap \mathcal{O}_E^n : N\mathcal{O}_E^n].$$

Here V is the complete discrete valuation ring of the test frame. Good reduction means that its generic abelian variety with its endomorphisms and polarization extends to an abelian scheme over V.

Outline. Rationalizes N^2-torsion and checks integrality of the germs F_{qzb}. Adjusts the logarithmic gauge by a balanced symmetric form (killed by harmonic P) via a trace-dual basis, then proves by induction that 2Δ clears each covariant derivative. The complete proof is on p. 1373. □

Lemma 36.99 (Testing a cusp section at a degenerating frame)**.** *Let M be the level of a uniformized unitary moduli space over the host CM field E of Lemma 36.98 (containing $w_0 = (1+\sqrt{-d})/2 \in \mathcal{O}_E$), let $\lambda \mid \ell$ be a coefficient place, and suppose that $M/\ell^{\operatorname{ord}_\ell M} \geq 3$. Let f be a cusp section of nonnegative dominant weight and nonnegative theta degree, cuspidal at every designated complex embedding, and let a be in the coefficient field. If af is integral at every good-reduction integral frame over λ on complete discrete valuation rings with finite residue field, it is integral at every integral frame on those rings over λ.*

Outline. Excludes definite degeneration and lifts a split degeneration to a toroidal arc giving $h(t)/(\pi t)^d$. Kills negative t-powers via Cauchy's formula on the complex arc, and proves $\pi^d \mid h/t^d$ by specializing over $V[[t]]_{(\pi)}$ to finite-residue traits. The complete proof is on p. 1375. □

Proposition 36.100 (One clearing power on the ray covers). *Fix a standard layer, its pinned base frame, its lowest-weight tensor and its coefficient map. There is a phase u, independent of the cover, with the following property. Fix a rational prime ℓ, a finite source translate s, and a finitely generated additive lattice L_0 of coefficient vectors. On all the uniformized tower covers (Section 1 of Chapter 35, Definition 35.2 and Lemma 35.4) with squarefree split auxiliary level, twists, places, and integral tests specified in Theorem 36.106, the reading multiplied by $\ell^c u$ belongs to the indicated integral cusp-section lattice, for a single integer $c \geq 0$. On a good rational ray cover the phase-normalized reading is rational for every rational test and rational coefficient vector, also after the allowed changes of splitting.*

Outline. Fixes u from the base pin, expands tests via 36.96 and 36.95, flips half-lattices, and moves the point to split primes by unitary reflections. Forms integral cusp sections via 36.98, 35.40, and 36.99, then deduces rationality. The complete proof is on p. 1377. □

Lemma 36.101 (Integral descent of an invariant reading). *Let $K \subset L$ be the base and cover coefficient fields of a uniformized tower cover of base level M_0, pinned at the nonnegative dominant weight and nonnegative theta degree in question. Let F be a rational cover cusp reading invariant under the base compact subgroup. Fix a rational prime ℓ with $M_0/\ell^{\mathrm{ord}_\ell M_0} \geq 3$, and let S be a set of places of L over ℓ where F is integral. If every $w \in S$ is unramified over K, then F is a finite sum of base cusp readings integral at the places below S, with coefficients in L integral at S. For the assertion at a possibly ramified place, assume also that the last weight in the positive block is positive at every CM-type embedding where that block is nonempty, that the holomorphic section space at the designated base embedding is spanned over $\mathbf{C}$ by readings of base algebraic sections lying in that space, and that the base adelic multiplier is right invariant under the base compact subgroup. At a single such place $w \mid \ell$ where F is integral, if K, L are in the fraction field of the fixed valuation ring O of Lemma 36.97 and $\mathrm{ord}_O(\ell) = e$, the same conclusion holds at $S = \{w\}$ for $\ell^{e^2} F$.*

Outline. Étale descent and flatness write F in a local integral base cusp basis (36.99). Lifting integral frames gives integral traces so 36.97 bounds the coefficients at each $w \in S$, and the Chinese remainder theorem glues the decompositions. The complete proof is on p. 1380. □

Lemma 36.102 (Transport of the theta generator by a Hecke isogeny). *Let B be a marked geometric fibre of the unitary moduli space in characteristic zero over the host CM field E of Lemma 36.98 (with $w_0 = (1+\sqrt{-d})/2 \in \mathcal{O}_E$ and principal rigidified theta line $L \simeq (1, w_0)^* \mathcal{P}_L$). Let P be a maximal ideal of $\mathcal{O}_E$ above the rational prime p, with $P + \bar{P} = \mathcal{O}_E$ and p prime to the full level. Write $g = \dim B$ and $q_P = \mathrm{N}P$. For an integer order $i \geq 0$, the kernels are the $\mathcal{O}_E$-stable subgroups C satisfying*

$$B[(p)P^{-1}] \subseteq C \subseteq B[(p)\bar{P}], \qquad e_{p^2}|_{C \times C} = 1, \qquad |C| = p^{2g}, \qquad [B[p] : C \cap B[p]] = q_P^i,$$

where $B[J]$ is annihilated by J and e_{p^2} is the Weil pairing of the principal polarization. There are finitely many such kernels. For each of them there is a unique quotient framed point with the prescribed transported marking. If $\Psi : B \to B' = B/C$ is the quotient, its tangent basis is $p^{-1} d\Psi$ of the source basis. For a generator s of $H^0(B, L)$, its theta generator s' is uniquely determined by

$$\Psi^* s' = \sum_{c \in C/(C \cap B[p])} t_c^* \big(\beta^{-1} [p]^* s\big).$$

In this formula $[p]^* L$ is identified with L^{p^2} by rigidification, $\beta : \Psi^* L' \simeq L^{p^2}$ is rigidified, L' is the marked principal line on B', and t_c denotes translation by any representative. The summands on the right are sections of $\Psi^* L'$, with its natural kernel linearization.

Outline. Descends the polarisation and marking to B/C from isotropy of $C \subset B[p^2]$ and identifies $\Psi^* L' \simeq L^{p^2}$ via biextensions. Descends the C-invariant trace sum by faithful flatness, and proves nonvanishing by $B[p]$-character orthogonality. The complete proof is on p. 1381. □

Lemma 36.103 (Hecke frames and integral values away from p). *Let $\lambda \nmid p$ be a place of the coefficient field. Start with an algebraic integral test frame over a complete discrete valuation ring V above λ with finite residue field. In an algebraic closure of its fraction field take any Hecke target of Lemma 36.102. After finite valued extension it is obtained from an integral target frame by a triangular frame change with unit diagonal,*

including a unit on the theta line. If the source has good reduction the target frame may be taken of good reduction. A section integral on the algebraic test frames at λ has value integral over V at this target.

Outline. Extends Ψ, Ψ' to semiabelian models so $p^{-1}d\Psi$ identifies integral Lie lattices. Proves regularity of the trace in 36.102 via Fourier bounds ($v(a) \geq 0$), then uses a reverse trace along $[p]\Psi'$ and orthogonality to get $v(a) = 0$. The complete proof is on p. 1382. □

Proposition 36.104 (The Hecke transform on sections and cusps)**.** *Use the kernels of Lemma 36.102. For a marked unitary space of positive rank r, $0 \leq i \leq r$, nonnegative even theta degree n, scalar b in the coefficient field and section f of weight a and theta degree n, there is a unique section of the same weight and degree whose value at a geometric frame y is*

$$(T_{P,i,b}f)(y) = b\, q_P^{-in/2} \sum_C f(y_C).$$

Here y_C is the transported frame of Lemma 36.102. In a uniformized layer satisfying the moduli and section laws of the tower, at a split prime outside the full level and for $0 \leq i \leq r$, any section transform of dominant weight given by this kernel formula preserves the cusp subspace. At $\lambda \nmid p$ the displayed operator preserves integral sections if b is integral at λ.

Outline. Realizes the kernels of 36.102 as a finite étale correspondence, builds the section by trace, and checks integrality via 36.103. Bounds $|t_C| \leq q_P^i$ by Cauchy–Schwarz and dominates the transform by finitely many rational isometries for cuspidality. The complete proof is on p. 1383. □

Proposition 36.105 (The Hecke reading in the pinned frame)**.** *Fix the designated embedding ι of a pinned layer j and put $n = n_V$, its even source rank. At an admissible split Hecke prime $P \mid p$, take the element α of order i in the split matrix model and a set $\mathcal{R}$ of representatives of $K\alpha K/K$, where K is the member's compact level. Write $\mu_j(P)$ for the reference unramified character at a uniformizer of P. For a section f of the layer weight let $R_f(h)$ be its evaluation in the frame at h, multiplied by the layer's Lie normalization, its pinned component constant and its prescribed adelic multiplier. If f' is the transform of Proposition 36.104, then*

$$R_{f'}(h) = \iota(b)\mu_j(P)^{-i} \sum_{a \in \mathcal{R}} R_f(ha).$$

Outline. Bijects the kernels of 36.102 with $K\alpha K/K$, matching frames up to unipotent Lie changes. Compares the Weil action $\omega(1,a)f_0$ with the trace in 36.102 at a rational period vector as in 36.100 to find t^n, then sums over 36.104. The complete proof is on p. 1385. □

Theorem 36.106 (Series laws for the third-layer construction)**.** *Use the third-layer nonendoscopic inputs and the geometric fields of P in 33.1. Over a nonarchimedean local field, Bernstein's unitary irreducibility theorem is used for the normalized parabolic induction of two smooth irreducible unitarizable blocks [15, §8.2]. For two smooth irreducibles with disjoint unramified twist classes of cuspidal support, the normalized product is irreducible by [16, Proposition 8.5]. In this case any two segments from opposite multisegments have disjoint cuspidal lines, so they are unlinked. The local and Galois compatibilities include 36.24 and the residual Brauer–Nesbitt input of 36.25. At each embedding τ over a de Rham place w of a rank-a representation V, the geometric jump clause is $\sum_i \dim \mathrm{gr}^i D_{\mathrm{dR}}(V)_\tau = a$.*

For each pinned standard layer the phase-normalized series is rational on every good rational ray cover, including for the permitted determinant twist and change of splitting in Proposition 36.100. For a fixed standard layer with its pinned frame, extremal tensor and coefficient map, the integral assertion uses a phase $u \in \mathbf{C}$, $|u| = 1$. For each prime ℓ, finite source translate s and finitely generated additive lattice L_0 of coefficient vectors, there is an integer $c \geq 0$. It works on every uniformized tower cover (Section 1 of Chapter 35, Definition 35.2 and Lemma 35.4) whose squarefree auxiliary level is a product of primes splitting completely in E and prime to $\ell Q_S q d g_0 g_1 \operatorname{cond}(\chi) \mathrm{N}\mathfrak{c}_K$, pinned at every dominant weight of the prescribed degree. The cover field must contain the ray field of the full level in the designated embedding, the coefficients of L_0, and the normalized finite values of the reference splitting character of weight $-1/2$. Let S be any set of its places over ℓ. For every level test with coefficient function integral at S, every $v \in L_0$, and every determinant twist ξ defined over the cover field on norm-one idèles and trivial there on rational

and auxiliary-congruence elements, the function $g \mapsto \ell^{c} u\, \xi(\det g_f)\Theta(s, g; v)$ is the reading of a cusp section integral at S. Here Θ uses the indicated level test and coefficient map. The descent assertion concerns a cover integral cusp form invariant under the base compact level. If the cover coefficient field is unramified at every place of S over the base coefficient field, the form is a finite linear combination of base cusp forms integral at the places below S, with coefficients in the cover field integral at S. At each split Hecke prime outside the level, the transported frames and their values satisfy Lemmas 36.102 and 36.103, the latter at coefficient places away from that prime. Proposition 36.104 gives the section transform and preserves cuspidality. At the designated member embedding its reading is the normalized double-coset sum of Proposition 36.105. For a nonzero base toric sum T_0, the split ray-level character sums divided by T_0 are algebraic, with the action of Lemma 36.94. In particular, for a ray character pulled back to a Picard character ψ and an integer h, the action is $gR_{\psi^h} = (g\psi)([t])^{-h} R_{(g\psi)^h}$ whenever g, t satisfy the hypotheses of that lemma. This follows by applying its formula to ψ^h, since g commutes with integer powers.

Outline. Verifies the hypotheses of 36.100 for rational and integral series readings and uses 36.101 for unramified descent. Obtains the Hecke laws from 36.102, 36.103, 36.104, and 36.105, and the toric action from 36.94. The complete proof is on p. 1386. □

Theorem 36.107 (The closed family local-error estimate)**.** *Assume $\mathscr{H}_2$, the third-layer and geometric hypotheses of Theorem 36.106, depth-zero support, and the decomposition-group inputs of 36.14. For a local Langlands witness h_L and $c_{\mathrm{Dep}} \in \mathbf{R}$, take the list-extension parameters $p = p_0(h_L, c_{\mathrm{Dep}})$ of 33.1, with the default geometric closure. For these parameters, the linked closed estimate at $2, 3$ implies that the family local-error estimate of Definition 36.11 is closed on $W_3(\mathcal{A})$ for every linked selection.*

Outline. Verifies the hypotheses of 36.26 from the third-layer, series, and depth-zero data via 36.25, then applies 36.26 and 36.16 on each cut family. Strengthens and projects to the selection of 33.30 using 36.106. The complete proof is on p. 1386. □

Chapter 37

Primes in short intervals, masses and counts

Let E be the CM field of the arithmetic datum D, with the Artin map of 5.1. Put $F = E^{+}$ and $f = [F : \mathbf{Q}]$. Write m, k, r for the chain length and the starting and spectral ranks, respectively. Let T_u be the level prime at word u. Put $L = x^{\varepsilon}$ and $b = e^{L}$, with $0 < \varepsilon < 1/52$.

37.1 Uniform estimates for Dirichlet L-functions

For primitive $\chi \bmod N$, put $\psi_0(\chi, u) = \sum_{n \le u} \Lambda(n)\chi(n) - \mathbf{1}_{\chi=1} u$. Write $m_\chi(\rho)$ for zero multiplicity. The sets $Z_\chi(T)$ and $Z_\chi[t]$ consist of distinct zeros in the open critical strip with $|\operatorname{Im}\rho| < T$ and $|\operatorname{Im}\rho - t| \le 1$, respectively.

Definition 37.1 (The uniform analytic input)**.** The uniform analytic input consists of Theorems 37.2 and 37.3 with positive constants C_1, c, C_7, C_{11} independent of the arithmetic datum D.

Theorem 37.2 (Density near 1 and logarithmic zero regions)**.** *There are $C_1, c > 0$ such that, for $Q \ge 3$, every finite set of triples (N, χ, ρ) with primitive $\chi \bmod N$, $1 \le N \le Q$, and*

$$L(\rho, \chi) = 0, \qquad \operatorname{Re}\rho \ge 399/400, \qquad |\operatorname{Im}\rho| \le Q$$

has total multiplicity at most $C_1 Q^{1/4}$. For primitive $\chi \bmod N$, $N \ge 3$, the region $\operatorname{Re} s \ge 1 - c/\log(\max(N, N|\operatorname{Im} s|))$ contains at most one zero. Such a zero is real and simple and χ is real. Every open-strip zero of ζ has $\operatorname{Re}\rho \le 1 - c/\log(|\operatorname{Im}\rho| + 3)$.

Proof. Montgomery's density estimate gives $O(Q^{1/8})$ occupied cells $(N, \chi, \lfloor \operatorname{Im}\rho \rfloor)$ at this threshold [40, Ch. 12]. By 37.3 their total multiplicity is $O(Q^{1/8}\log Q)$, hence $O(Q^{1/4})$. The two regions use a common constant in Davenport's zero-free regions [89, §§13–14]. □

Theorem 37.3 (Explicit formula and unit-window zero count)**.** *There are $C_7, C_{11} > 0$ such that for every primitive $\chi \bmod N$, $N \ge 1$, and $2 \le T \le u$, the set $Z_\chi(T)$ is finite and*

$$\left| \psi_0(\chi, u) + \sum_{\rho \in Z_\chi(T)} m_\chi(\rho) \frac{u^\rho}{\rho} \right| \le C_7 \left(\frac{u \log^2(NuT)}{T} + \log(Nu) + \sum_{\rho \in Z_\chi(1)} \frac{m_\chi(\rho)}{|\rho|} \right). \tag{37.1}$$

For each $t \in \mathbf{R}$, $Z_\chi[t]$ is finite and

$$\sum_{\rho \in Z_\chi[t]} m_\chi(\rho) \le C_{11} \log\bigl(N(|t| + 2)\bigr). \tag{37.2}$$

Proof. The first bound replaces Davenport's signed low-zero correction by absolute values; for $u \geq 2$ the jump $|\psi_0 - \psi_0^*| \leq \log u$ and the $s = 0$ trivial zero for even primitive characters are absorbed into $C_7 \log(Nu)$. The conductor-one formula and the unit-window count use the corresponding estimates for zeta and primitive characters [89, §§15–17 and §19, (9)]. □

Let q, d and Q_S be the two primes and the prime-to-qd factor of the Artin modulus in 5.1. Write $N_E = Q_S qd$ for that modulus and put $\lambda_E = \max(1, \log N_E)$. Let s_D be the odd part of Q_S. A primitive Dirichlet character χ belongs to D if there is a character $\omega : \mathrm{Gal}(E/\mathbf{Q}) \to \mathbf{C}^\times$ such that $\chi(a) = \omega(\mathrm{Art}_D(a))$ for every positive integer a coprime to N_E. For $y \geq 1$ define

$$g_\chi(y) = \sum_{n \leq y} \Lambda(n)\chi(n)(y/n - 1) + \begin{cases} yL'(1,\chi)/L(1,\chi), & \chi \neq 1, \\ -y(\log y - 1 - \gamma), & \chi = 1. \end{cases} \tag{37.3}$$

Put $X_1(Y) = e^{Y^{\varepsilon/5}}$, $X_2(Y) = e^{Y^{\varepsilon/5}(\log Y)^2}$. Let $\mathcal{Z}(H)$ be the set of real primitive characters of conductor at most H for which $L(s,\chi) \neq 0$ throughout $\mathrm{Re}\, s > 399/400$, $|\mathrm{Im}\, s| \leq H$. The zero-free box hypothesis for a family of exponent J requires that for all sufficiently large x and every member $i \in I_x$ there be a $Y \in [x^{1/J}, x]$ with the following properties. For every real character χ of conductor N belonging to the datum at (x, i),

$$\begin{aligned} d \mid N &\implies \chi \in \mathcal{Z}(8 s_D X_1(Y) X_2(Y)), \\ q \mid N,\ d \nmid N &\implies \chi \in \mathcal{Z}(8 s_D X_1(Y)). \end{aligned} \tag{37.4}$$

At this same Y, put $\chi_q = (-q/\cdot)$, $\chi_d = (-d/\cdot)$, and $\chi_8 = (2/\cdot)$. For $\eta = 1, \chi_8$, every primitive character agreeing with $\eta\chi_q$ on integers coprime to $8qd$ must belong to $\mathcal{Z}(8X_1(Y))$. Every primitive character agreeing there with $\eta\chi_d$ or $\eta\chi_d\chi_q$ must belong to $\mathcal{Z}(8X_1(Y)X_2(Y))$. The hypothesis on the auxiliary primes requires that for all sufficiently large x and every $i \in I_x$ there be a $Y \in [x^{1/J}, x]$, possibly different from the one used for the boxes, such that

$$\begin{gathered} q \equiv 3 \bmod 8, \quad X_1(Y)^{1/2} < q \leq X_1(Y), \qquad d \equiv 7 \bmod 16, \quad X_2(Y)^{1/2} < d \leq X_2(Y), \\ (-d/q) = 1, \qquad (-d/p) = 1 \quad (p \leq Y^{\varepsilon/5},\ p \text{ prime}), \qquad \log N_E \leq 2Y^{\varepsilon/5}(\log Y)^2, \end{gathered} \tag{37.5}$$

where the symbols are Kronecker symbols.

Proposition 37.4 (The character-sum estimate on a family)**.** *Assume the uniform analytic input of Definition 37.1. Let F_m be a family with starting scale $x_1 > 1$ and fixed family constant C_o satisfying $o \leq C_o$ and $Q_S \leq C_o|A|$ for all $x > x_1$ and $i \in I_x$ (as built into the family structure for the host group A, character order $o \geq 1$, and 2-exponent $e \in \mathbf{N}$ of 5.1, where $o = [A : A_0]$ and $2^{e+2} \mid Q_S$). Since h is injective on A_0 and its target has order 2^e, one has $|A| = |A_0|\, o \leq 2^e o$. Suppose F_m satisfies the zero-free box hypothesis, including the six bounds following (37.4), and the auxiliary-prime hypothesis (37.5). There are $c_6 > 0$ and $x_0 > x_1$ such that for every $x \geq x_0$, every member $i \in I_x$, and every character χ belonging to the datum at (x, i),*

$$|g_\chi(y)| \leq y e^{-c_6 x^{\varepsilon/8}} \qquad (y \geq e^{x^{\varepsilon/3}}/3). \tag{37.6}$$

In particular, for every family class $\mathcal{C}$ satisfying 37.1 and the bounds above, the implication from the zero-free box hypothesis, (37.5), and some $C_5 > 0$ with $|g_\chi(x)| \leq x^{1/2+\varepsilon}$ and $|L'(1,\chi)/L(1,\chi)| \leq C_5 \log x$ (for primitive χ agreeing with $\chi_q, \chi_d, \chi_q\chi_d$ coprime to $8qd$) to $c_6 > 0, x_0 > x_1$ giving (37.6) is closed on $\mathcal{C}$ (Definition 36.10).

Outline. Combine the zero-free boxes and 37.2 with reflection and unit-window counts to bound the reciprocal zero weights. Apply the explicit formula at $T = e^{\sqrt{\log u}}$, then use partial summation and the conductor bound to estimate the tail integral. The complete proof is on p. 1388. □

37.2 Standard masses and hyperplane volumes

For a CM field K put $h_M(z,w) = \sum_a \overline{z_a} w_a$, $\Lambda_M = \mathcal{O}_K^M$, $f_K = [K^+ : \mathbf{Q}]$, and $v_K = |\mathrm{disc}\, K|^{1/(2f_K)}/2$. Use the involution induced by complex conjugation in every embedding $K \hookrightarrow \mathbf{C}$. The finite unramifiedness

hypothesis says that for every maximal ideal $\mathfrak{P} \subset \mathcal{O}_K$ some $a \in \mathcal{O}_K$ satisfies $a - \bar{a} \notin \mathfrak{P}$. Equivalently, K/K^+ is unramified at every finite place. The mass and Euler factor are

$$\operatorname{mass}(h, \Lambda) = \sum_{[\Lambda'] \in \operatorname{gen}(\Lambda)} |\operatorname{Aut}(h, \Lambda')|^{-1}, \tag{37.7}$$

$$L_\eta(K, M) = \prod_{v < \infty} (1 - \eta_v^M q_v^{-M})^{-1}, \tag{37.8}$$

where v runs over the finite places of K^+, q_v is its residue cardinality, and $\eta_v = 1$ at split places and -1 at inert places. Set $M_j = \operatorname{mass}(h_j, \mathcal{O}_E^j)$.

Theorem 37.5 (The adjacent-rank mass formula)**.** *For a CM field K unramified over K^+ at every finite place and every $M \geq 3$,*

$$\frac{\operatorname{mass}(h_M, \Lambda_M)}{\operatorname{mass}(h_{M-1}, \Lambda_{M-1})} = \frac{\big((M-1)!(v_K/\pi)^M\big)^{f_K}}{(2v_K)^{f_K/2}} L_\eta(K, M). \tag{37.9}$$

Outline. Apply the Gan–Hanke–Yu unramified standard-lattice mass formula [200] and rewrite the adjacent-rank quotient $2^{-f_K}|L(1-M, \eta^M)|$ via the functional equation for η^M at even and odd M. The complete proof is on p. 1389. □

Call a Hermitian space W over E/F *admissible of constant signature* (P, Q) if E/F is unramified at all finite places (Definition 34.46(1)), W carries a self-dual $\mathcal{O}_E$-lattice, and W has signature (P, Q) at every real place of F. The quotient-volume hypothesis on D asks that each such W admit Haar measure on $\mathrm{U}(W)(\mathbb{A}_F)$ (compact-dual normalization at infinity, unit lattice-stabilizer measure) with a $\mathrm{U}(W)(F)$-fundamental domain of volume $\operatorname{vol}_D(W)$.

Theorem 37.6 (The volume ratio for a unit hyperplane)**.** *Assume 34.17 and the quotient-volume hypothesis on D. Let W be an admissible Hermitian space of rank $M \geq 3$ and constant signature (P, Q). For $s = 1$ if $P \geq 1$, or $s = -1$ if $Q \geq 1$, there is a vector of norm s with complement W' of signature $(P-1, Q)$ or $(P, Q-1)$, respectively. With $v_E = |\operatorname{disc} E|^{1/(2f)}/2$,*

$$\frac{\operatorname{vol}_D(W')}{\operatorname{vol}_D(W)} = \left(\frac{(2\pi)^M}{(M-1)!}\right)^f \frac{(2v_E)^{f(1/2-M)}}{L_\eta(E, M)}. \tag{37.10}$$

Outline. Choose a unit coordinate vector in the standard signature form. Compare the one-vector Gram mean of 34.17 with its unfolding along the vector stabilizer, computing the real fibre by differentiation and the finite densities by residue norm counts and lifting. The complete proof is on p. 1389. □

Proposition 37.7 (The definite genus mass)**.** *For every datum D, let W be an admissible Hermitian space of constant signature $(M, 0)$, with $M \geq 1$. If h is its Hermitian form and Λ_0 its self-dual base lattice, then* $\operatorname{mass}(h, \Lambda_0) = M_M$.

Outline. Checking local equivalence at inert, split, and real places lets Landherr's theorem give a rational isometry with h_M. Jacobowitz's local classification puts the transported self-dual lattice in the standard genus, and the isometry preserves automorphism orders. The complete proof is on p. 1390. □

Lemma 37.8 (Degree of a separated row)**.** *Use the family's compact-dual measure and signature $(p, q) = (p_Y, q_Y)$, $p, q \geq 1$, $r = p + q$. Put $F_s = \prod_{a=0}^{s-1} a!$. For the decreasing Harish–Chandra parameter λ at column ρ, put $\Delta(\lambda) = \prod_{a<b}(\lambda_a - \lambda_b)$, $V(\lambda) = \Delta(\lambda)/F_r$, and $H = \max_{a,b} |\lambda_a - \lambda_b|$. For $e^{-r} \leq t < 1/2$, assume that opposite-sign differences have absolute value at least $8192 \min(p, q)(pq)^3 \log r / t^2$. Then the formal degree and lowest-type dimension satisfy*

$$(1 - t)V(\lambda) \leq d_{\Sigma,\rho} \leq 4 d_\tau (3pqH)^{pq}. \tag{37.11}$$

Outline. The coordinate increment formula keeps the degree proportional to the Vandermonde along a ray of increasing gaps. Compare the Gaussian limit from Schur orthogonality and the minor bounds with the compact Weyl dimension formula to fix the constant and bound the degree. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1390. □

Lemma 37.9 (Volume at the starting signature). *Assume the adjacent-rank formulas of 37.5 and 37.6, and the definite rank-two volume formula of 37.7. Let W be an admissible Hermitian space of constant signature (P,Q), with $P \geq 2$. Then $\mathrm{vol}_D(W) = M_{P+Q}$. In particular $\mathrm{vol}_D(Y_0) = M_r$ for the starting row space.*

Outline. Take successive orthogonal complements of unit vectors using 37.6, removing negative directions first. By 37.5 each step preserves the volume-to-mass ratio, which is one by the definite rank-two formula of 37.7. The complete proof is on p. 1394. □

Proposition 37.10 (The scale of T_u). *Assume 37.6 and the rank-two case of 37.7. For every member condition and bundle association, with type-prime exponent C_{YL} and*

$$c_{T,\min}(y) = \max(11, 16 + 3\max(5, y+1)), \tag{37.12}$$

let a selected family in $W_1(\mathcal{A})$ (§33, quantifiers of Definition 36.10) carry the built point sizes, volume ties, and prime selection $T_u^\star \leq T_u \leq (1+(fk)^{-1})T_u^\star$. If $C_{Tu} \geq c_{T,\min}(C_{YL})$, then for all sufficiently large x, every $i \in I_x$ and every word u satisfy

$$b^{m/2} \leq T_u, \qquad \log T_u \leq C_{Tu} m L. \tag{37.13}$$

Outline. Insert the starting volume from 37.9 into the product formula. Above the cutoff of 21.34, use the product and denominator bounds of 21.35 to bound $\log T_u$ on both sides. The complete proof is on p. 1394. □

Lemma 37.11 (Selecting a prime at the word scale). *Fix $0 < \varepsilon < 1/52$ and $c_s, C_Y > 0$. For sufficiently large x, let $d \geq 100$ be prime with $d \equiv 3 \bmod 4$, and take a positive multiple N of d with $\log N \leq 3x^{\varepsilon/5}(\log x)^2$. Let g_0, g_1, m be positive integers with $\log g_i \leq C_Y x^\varepsilon$ and $m \geq x^{1/3}$. For $1 \leq V \leq x^{12}$ and $\log S \geq c_s m x^\varepsilon$, there is a prime $S < T \leq (1+1/V)S$ with*

$$T = \pi_T \bar{\pi}_T, \qquad \pi_T \equiv 1 \bmod M\mathcal{O}_K, \qquad M = N(g_0 g_1)^2, \quad K = \mathbf{Q}(\sqrt{-d}).$$

Outline. Bound $\log(4dM^2)$ and $\log V$ by $c\sqrt{\log S}$ and apply 8.11 with width $1/V$. The resulting quadratic representation and congruences give an integral π_T of norm T congruent to 1 modulo $M\mathcal{O}_K$. The complete proof is on p. 1395. □

37.3 Two counting statements on the same selected data

Write $\mathcal{D}_3 = W_3(\mathcal{A})$ for the selected family class of §33, and ℓ_{out} for the family's list of outer constants in Definition 36.10.

Lemma 37.12 (The labelled space before counting). *Fix a point with the label realization of 33.74. Let X denote that realization, including its unit lowest-type vectors, level subgroup, Haar measures and finite level projector e. Write σ for the product representation at infinity, w for the product unit vector, and T_ϕ for the real convolution of 33.28. Assume that the rank is at least 1801 and the adjacent auxiliary weight gaps are at least $16r^{19}$, where r is the rank.*

A label t specifies the finite type at every place over both distinguished primes. For a set S of such labels, let e_S be e multiplied, in the finite convolution algebra, by the sum of the type projectors for $t \in S$. Put

$$V_S = \mathrm{im}(R_{e_S} T_\phi), \qquad d_t = \dim_{\mathbf{C}} V_{\{t\}}.$$

Then V_S is finite dimensional and is identified by evaluation at w with the image of R_{e_S} on the continuous G_∞-intertwiners from σ into $L^2(X)$. There is a full index set with d_t indices labelled t. Its good indices can be identified, preserving the label, with the selected indices of the point. For every S,

$$\dim_{\mathbf{C}} V_S = \sum_{t \in S} d_t.$$

Outline. Coefficient integrability from 33.25 permits 33.28, and Schur orthogonality identifies the image by evaluation at the cyclic vector. The orthogonal projectors of 32.44 split it by label, and 33.74 identifies the good basis. The complete proof is on p. 1395. □

Theorem 37.13 (Counting constants from the label condition)**.** *Assume 34.17 and the label row of 33.74 for K_P, Q, Q', L_k, enlargement R, and condition $Q' \wedge R$. There are a constant $c_{\mathrm{Ic}} > 0$ and, for each finite real list ℓ, a tuple $c_{30}(\ell) = (C_{\mathrm{res}}, n_0, A, C_b, C, c, c_b)$. At $\ell = \ell_{\mathrm{out}}$ its entries satisfy*

$$0 \le C, \qquad 2(C_{YL} + 1) \le A. \tag{37.14}$$

On $\mathcal{D}_3$, with the quantifiers of Definition 36.10, these constants give the estimate of Lemma 37.14 on the specified framed Hermitian realization whenever $\max_v \min(p_v, q_v) \ge 1$. Here the frames identify the base lattice and its constant signature. The real measures have the compact-dual normalization, quotient volumes agree in equal rank, and in definite signature the finite class decomposition has sum of reciprocal stabilizer orders equal to the standard mass. The level-character idempotents in this assertion have the form $e = \mathbf{1}_H \chi^{-1}/\mu_f(H)$, where H is a compact open subgroup of the lattice stabilizer with unipotent reduction at $g_\circ$ and χ is a unitary character of H.

At each sufficiently large point there is an arithmetic row system A on the selected base data with a common nontrivial additive character at each distinguished prime. It satisfies the join conditions for the selected real parameters and for the split level prime and finite projectors. The selected indices inject into the full label index set of 37.12, with image exactly the good indices. For any subset S' of label pairs, $A(S')$ is the projected intertwiner space of that realization. Its dimension is the number of full indices with label in S'. The total finite idempotent e' has $D^\star_A(e') > 0$. For $k_0 = 0, 1$ and a subset S of types at k_0, put $\mathrm{cyl}_{k_0} S = \{t : t_{k_0} \in S\}$, with idempotent $e_{\mathrm{cyl}_{k_0} S}$. Its main term is nonnegative and

$$\left|\dim A(\mathrm{cyl}_{k_0} S) - D^\star_A(e_{\mathrm{cyl}_{k_0} S})\right| \le e^{-c_{\mathrm{Ic}} r f \log r} D^\star_A(e_{\mathrm{cyl}_{k_0} S}). \tag{37.15}$$

Outline. Use 33.6 and 33.78 for the gap test of 33.25 and the size bounds, combining thresholds by 36.10. Form cylinder projectors using 32.44 on the labelled space of 37.12, lift them, and use 37.14 for the constants and counts. The complete proof is on p. 1396. □

Let I be a rank-n Hermitian datum with self-dual lattice over E/F unramified at finite places. Put $\mathrm{rd} = |\operatorname{disc} E|^{1/(2f)}$. Write U for its unitary group, K_f for the finite lattice stabilizer and $K(g)$ for its congruence kernel modulo g. Use product Haar measure on $U(\mathbb{A}_F)$ with finite factor μ_f, and let $\mathscr{F}$ be a fundamental domain for $U(F)$, of volume V. Write $L^2 = L^2(U(F)\backslash U(\mathbb{A}_F))$. At each real place take a discrete series σ_v with parameter λ_v and unit lowest-type vector w_v. Let $\sigma = \widehat{\bigotimes}_v \sigma_v$ on H_σ, $w = \bigotimes_v w_v$, $d_\infty = \prod_v d(\sigma_v)$ in the prescribed measures, and $G_\infty = \prod_{v|\infty} U(F_v)$. For right convolution R_e by e, put

$$D_{\mathrm{count}} = \dim_{\mathbf{C}}\big(R_e \circ \mathrm{Hom}^{\mathrm{cont}}_{G_\infty}(H_\sigma, L^2)\big), \qquad D^\star = d_\infty V \operatorname{Re} e(1). \tag{37.16}$$

For $x \in U(\mathbb{A}_F)$, Λ_x is the lattice cut out by the finite translate x_f, with Euclidean norms in the real signature frames at x_∞. The size of a nonzero rank-a lattice B, with induced or quotient metrics, is

$$\mathrm{size}(B) = \left(\frac{2^{fa}\,\mathrm{covol}(B)}{|\operatorname{disc} E|^{a/2}}\right)^{1/(2fa)}. \tag{37.17}$$

Write $\mathrm{least}(x)$ for the infimum of nonzero sublattice sizes and $\tau_1(x) = \min(1/2, \mathrm{least}(x))$. The largest size of a lattice is the supremum of its nonzero saturated quotient sizes. In the Harder–Narasimhan filtration $G_{\le t}$ is the sum of sublattices of largest size at most t. For $y \in U(F_v)$ let $d_v(y) = (\sum_a (\log s_a(y))^2)^{1/2}$ in terms of singular values $s_a(y)$. For common signature (p, q), size parameter b and level g, set

$$H = nf \log n, \quad N = n^2 f, \quad X = nf\,\mathrm{rd}\, gb, \quad \eta = (10^6 n^8 f^2)^{-1}, \quad \tau_* = X^{-11q}.$$

For $q \ge 1$, call $\mathrm{least}(x) \ge \tau_*$ the moderate region and its complement the deep region.

Lemma 37.14 (Counting idempotents)**.** *Assume the finite-volume and nonsingular Gram-density assertions of 34.17. For each $A \in \mathbf{R}$ there are $n_0 \in \mathbf{N}$, $C_0 \ge 0$ such that for any additional level predicate P, $C \ge C_0$, and $c = c_b = 1/16$, the following holds. Take*

$$\begin{gathered} n \ge n_0, \quad b \ge 2, \quad \log\log \mathrm{rd} \le n, \quad |\lambda_{v,a}| \le b^2, \quad (p_v, q_v) = (p, q), \quad q^2 \le n, \\ g \in \mathbf{N}, \quad 1 \le g \le b^A, \qquad 4n < g_\circ, \quad g_\circ \text{ prime}. \end{gathered} \tag{37.18}$$

Require $g_\circ$ to split completely and unramified in E. The locally constant, self-adjoint convolution idempotent e is of positive type, supported in K_f, bi-invariant under $K(g)$, and has unipotent reduction on its support modulo $g_\circ$. Assume $P(I, \mu_f, g_\circ, e)$. Each λ_v decreases strictly in $(n-1)/2+\mathbf{Z}$, with absolute opposite-sign differences at least h, where

$$Cn^{19} f^5 \log(b\,\mathrm{rd}) \le h, \qquad \log f \le \sqrt{n}, \qquad \log\log b \le c_b n f \log n. \tag{37.19}$$

Then

$$|D_{\mathrm{count}} - D^\star| \le e^{-cnf\log n} D^\star. \tag{37.20}$$

Outline. Unfold the projector trace, with only the identity surviving in definite signature by the congruence. In indefinite signature use 37.16 for decay and packing, 37.17 for flag densities and deep volume, and 37.20 for the deep trace. The complete proof is on p. 1397. □

Lemma 37.15 (Uniqueness of the lowest-type germ)**.** *Let S be the lowest type of a unitary discrete series of $U(p,q)$, $n=p+q$, with compact group $K=U(p)\times U(q)$. Let the decreasing parameter λ have opposite-sign gap at least $g>2n$. Let $\mathfrak{p}_{\mathbf{C}}$ be the span of opposite-sign matrix units and Λ_S the highest weight of S. Let $V_-\subset \mathfrak{p}_{\mathbf{C}}\otimes S$ be the sum of isotypic components of highest weight $\Lambda_S+e_a-e_b$ for $b<a$ of opposite signs. The homogeneous Schmid equation kills the first derivative contracted against V_-. Two analytic $\mathrm{End}(S)$-valued germs at 1, with lowest-type left and right covariance, agree if they have the same value and both satisfy this equation.*

Outline. Induct on noncompact derivatives, using covariance and commutators to make the first possible nonzero derivative symmetric. The differentiated Schmid equation removes the lowering components, and the compact Casimir gap forces that derivative to vanish. The complete proof is on p. 1399. □

Lemma 37.16 (Local decay and arithmetic balls)**.** *At a real place of 37.14 with $q=\min(p_v,q_v)\ge 1$ and $h\ge 120nq^{3/2}$, the lowest-type compression of $\sigma_v(y)$ has norm at most $e^{-(h/8)\min(d^2,d/\sqrt{2q})}$, $d=d_v(y)$. For $n\ge 1$, differences of distinct rational operators preserving Λ_x have geometric mean of real operator norms at least $\tau_1(x)^2$. The arithmetic ball in $x^{-1}U(F)x$ with finite component in K_f and radii $d_v\le R_v$ ($R_v\ge 0$) has cardinality at most*

$$A_x e^{2n^2\sum_v R_v}, \qquad A_x=(3\sqrt{n}/\tau_1(x)^2)^{2N}. \tag{37.21}$$

Outline. Use 37.15 to identify the compression with a compact average, then bound its minor weights by Cauchy–Binet. For the arithmetic ball, determinants give separation and rescaling the real radii gives the packing bound. The complete proof is on p. 1399. □

Lemma 37.17 (Two volume estimates)**.** *Use the quotient of 37.14, put $r=\mathrm{rd}$, and let $c_v\mu_v$ be Haar measure with Hilbert–Schmidt density one at the identity. Write $P_0=\prod_v c_v$, $k_R=\mu_f(K(g))$. Assume 34.17, $n=p+q\ge 8$, $q^2\le n$, $Z\ge 2$, $r\le Z$, $2fr\le Z^2$, $g\ge 1$. Then*

$$P_0 V \le Z^N \mu_f(K_f) \le Z^N g^{2N} k_R. \tag{37.22}$$

If also $\log n\ge 15$, $\log r\le e^n$, $\log f\le\sqrt{n}$, the probability measure of filtration flags of type $(j_1,\dots,j_a)$, $j_i\ge 1$, $1\le J=\sum j_i\le q$, is bounded by

$$\kappa\prod_\lambda \lambda^{m_\lambda}\prod_i 2fj_i\,\frac{dt_i}{t_i} \qquad (0<t_1<\cdots<t_a<1/2), \tag{37.23}$$

with $\kappa\le e^{-JH/2}$ and scales (λ,m_λ) given by

$$\lambda\in\{t_i/t_k,\ t_it_k\ (i<k);\ t_i;\ t_i^2\},\quad m_\lambda\in\{2fj_ij_k\ \text{each};\ 2fj_i(n-2J);\ fj_i^2\}. \tag{37.24}$$

In particular $\Pr(\text{least}<1/2)\le e^{-H/3}$.

Outline. Iterate the unit-hyperplane calculation of 37.6 and bound the rank-two class count by Minkowski's theorem. Unfold isotropic flags and bound the resulting Levi volumes, zeta factors and compact volume ratios to estimate the flag density. The complete proof is on p. 1400. □

Lemma 37.18 (The deep-region exponent)**.** *Let $n \ge 8,\ f, q \ge 1,\ q^2 \le n$ be integers and $r, g \ge 1,\ b \ge 2$. Put $X = nfrgb,\ N = n^2 f,\ k = 8qN$. For $0 < \tau \le X^{-11q}$, define*

$$\rho = \frac{n^2\sqrt{f}\,gr}{4}(2\tau)^{1/q}, \quad B_j = \sqrt{n}\,b^2 + \tfrac{3}{2}n^{3/2} + 3j, \quad B = B_{k+N}, \qquad U = (2\rho\sqrt{f}B)^{2k}(6B)^N e^f (17\sqrt{n}/\tau^2)^{2N}.$$

Then $\rho \le \log 2$ and $Ug^{2N}X^N \le e^{-N}$.

Proof. Put $Y = X^q$. Since $X \ge 16$, $(2\tau)^{1/q}X^{11} \le 2$ and $n^2\sqrt{f}\,gr \le X^2$, one has $\rho \le 1/2$. The rank bounds give $B \le 30X^3$ and $n^2 fgrB/2 \le 15X^5 \le X^6$. Thus $(2\rho\sqrt{f}B)^{16q} \le Y^{96}(2\tau)^{16}$. With $Bng^2X \le Y^{10}$ and $\tau Y^{11} \le 1$,

$$(Ug^{2N}X^N)^{1/N} \le 2^{16}\cdot 6\cdot 289\,e\,Y^{96}\tau^{12}Bng^2X \le 2^{16}\cdot 6\cdot 289\,e\,Y^{-26} \le e^{-1}.$$

□

Lemma 37.19 (A finite family of cuspidal isometries)**.** *Use the data of 37.14 with $n_0 \ge 8,\ C \ge \max(0, 8\cdot 10^{14}(A+4))$ and $q \ge 1$. Let $T_i : H_\sigma \to L^2$, $1 \le i \le D$, be isometric intertwiners with cuspidal images. Let V_j be the span of derivatives through order j of lowest-type tensors, taken one place at a time. Suppose $T_i u$ has continuous right $K(g)$-invariant representative $F_i(u)$ on these spaces, with right derivative $F_i(d\sigma(Z)u)$ in direction Z. Put $F(u,y) = (F_i(u)(y))_i$. At $\tau = \mathrm{least}(x) < X^{-11q}$, use 37.18 with $r = \mathrm{rd}$. Then*

$$\|F(w,x)\|_2^2 \le UDP_0/k_R. \tag{37.25}$$

Outline. Casimir derivative bounds and local cube integration with the packing estimate of 37.16 bound evaluation. A gap in the small flag gives a short cover of a unipotent orbit, where vanishing constant terms allow repeated integration of derivatives. The complete proof is on p. 1402. □

Corollary 37.20 (Mass of the counting projector)**.** *Under the hypotheses of Lemma 37.19, the counted space is finite dimensional. Its projector has continuous kernel and diagonal (for an orthonormal basis (F_ν) of continuous representatives)*

$$K(x,y) = \sum_{\gamma \in U(F)} \phi(x^{-1}\gamma y), \tag{37.26}$$

$$K(x,x) = \sum_\nu |F_\nu(x)|^2, \qquad D_{\mathrm{count}} = \int_{\mathscr{F}} K(x,x)\,dx, \tag{37.27}$$

and satisfies

$$\int_{\mathrm{deep}} K(x,x)\,dx \le e^{-N} D_{\mathrm{count}}. \tag{37.28}$$

Outline. Smooth averaging and coefficient decay give continuous cuspidal vectors. Use 37.19, 37.18 and the volume bound for the deep estimate, then use Bessel's inequality on the moderate region for finiteness and unfold to identify the kernel. The complete proof is on p. 1403. □

37.4 Exact class counts

For the upper unitriangular subgroup $U_k(\mathbf{F}_g)$ put

$$I_k(g) = [\mathrm{GL}_k(\mathbf{F}_g) : U_k(\mathbf{F}_g)], \qquad I_{01} = \prod_{\nu=0}^{1} I_k(g_\nu)^f. \tag{37.29}$$

A rational prime p splits completely in F precisely when F has $f = [F : \mathbf{Q}]$ primes above p, each with residue field $\mathbf{F}_p$.

Theorem 37.21 (The class number at constructed points)**.** *Use the quantifiers of 36.10 on $W_1(\mathcal{A})$ for every member condition and bundle association. At each constructed point $x > x_1$, $i \in I_x$, suppose both g_0 and g_1 split completely in E (and hence in F) and*

$$\exists \nu \in \{0,1\}, \qquad 2fk < g_\nu - 1. \tag{37.30}$$

For every word u, $T_u \equiv 1 \bmod Q_S qd$ and T_u splits completely in E and F. Write $\mathcal{C}_{u,0}$ for the initial column class set of the word u. Every class in $\mathcal{C}_{u,0}$ has trivial rational stabilizer, and its cardinality is

$$\#\mathcal{C}_{u,0} = M_k I_{01}(T_u^k - 1)^f. \tag{37.31}$$

Outline. Use Definitions 34.46(4)(b) and 5.1 for splitting and compute the level index from the unitriangular and vector stabilizers. The cyclotomic degree bound forces the genus automorphism actions on level cosets to be free, so their orbit count is the index times the mass of 37.7. The complete proof is on p. 1404. □

37.5 Bounded real rows and reciprocity data

Lemma 37.22 (Descent of a complex spanning relation)**.** *Let K be a field, $\sigma : K \hookrightarrow \mathbf{C}$ an embedding, M a K-vector space, X a complex vector space, $C \subseteq M$ a K-subspace, and $L : M \to X$ a σ-semilinear map. Suppose that for every finite K-linearly independent family $b_1, \dots, b_m$ in C, the family $L(b_1), \dots, L(b_m)$ is linearly independent over $\mathbf{C}$. For $S \subseteq C$ and $f \in C$,*

$$L(f) \in \operatorname{span}_{\mathbf{C}} L(S) \quad \Longrightarrow \quad f \in \operatorname{span}_K S.$$

Proof. Choose a finite subset $S_0 \subseteq S$ whose images span $L(f)$, and choose a K-basis $b_1, \dots, b_m$ of $\operatorname{span}_K S_0$ from S_0. Semilinearity gives $L(f) \in \operatorname{span}_{\mathbf{C}}\{L(b_1), \dots, L(b_m)\}$. If $f \notin \operatorname{span}_K S$, then $f, b_1, \dots, b_m$ is K-linearly independent. All its members lie in C, so the hypothesis makes $L(f), L(b_1), \dots, L(b_m)$ linearly independent over $\mathbf{C}$, a contradiction. □

Lemma 37.23 (Independence of comparison values)**.** *Fix the moduli, uniformization and comparison data at a selected layer of Proposition 37.24. Let $K \subseteq \mathbf{C}$ be its second comparison field and C_K its space of algebraic cusp sections of the specified weight and degree. For an algebraic section s, let $R(s)(k, x)$ be its value in the complex frame x of component k, extended by zero outside the frame domain. If the comparison assigns the component and frame $(\alpha(g), x_g)$ to an adelic point g, put*

$$E(s)(g) = R(s)(\alpha(g), x_g).$$

For every finite K-linearly independent family $s_1, \dots, s_m$ in C_K, the functions $E(s_1), \dots, E(s_m)$ are linearly independent over $\mathbf{C}$.

Outline. Base change and uniformization identify complex test evaluations with scalar multiples of frame readouts $R(s)(k, x)$. In characteristic zero, the Nullstellensatz and K-linear functionals descend complex relations among the $R(s_i)$ to K-relations among the s_i, giving independence of $R(s_i)$. Arithmetic-group and Borel equivariance then pass this independence to $E(s_i)$. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1404. □

Proposition 37.24 (Bounded rows of the indefinite realization)**.** *Assume the following four inputs of the real theta construction.*

1. *For $p, q \geq 0$ with $p + q \geq 1$, use the determinant double cover*

$$\widetilde{U}(p,q) = \{(g, z) : g \in U(p,q),\ z \in \mathbf{C}^\times,\ z^2 = \det g\},$$

realized as a topological group with a chosen Haar measure and compatible exponential map. Put $I = \{1, \dots, p\} \sqcup \{1, \dots, q\}$, and write $a \prec_c b$ when a, b lie in the same block of I with $a < b$. An allowed parameter is an injective map $\lambda : I \to \mathbf{Q}$ such that $a \prec_c b$ implies $\lambda_b < \lambda_a$, with $2\lambda_a \in \mathbf{Z}$ and $\lambda_a - \lambda_b \in \mathbf{Z}$.

There is a family Π_λ of nonzero irreducible unitary discrete-series representations indexed by all allowed parameters, representing every such unitary equivalence class. Each includes an injective $\mathbf{C}$-linear Lie-algebra module realization onto the compact-finite vectors intertwining derivatives along the exponential map, and a formal degree $d_\Pi > 0$ satisfying

$$\int_{\widetilde{U}(p,q)} |\langle y, \Pi(g)x\rangle|^2 \, dg = \frac{\|x\|^2\|y\|^2}{d_\Pi} \qquad (x, y \in H_\Pi).$$

Define the Blattner weight $b(\lambda)_a = \lambda_a + \rho_{n,a}(\lambda) - \rho_{c,a}(\lambda)$, where $2\rho_{n,a}(\lambda) = \#\{b \text{ opposite block} : \lambda_b < \lambda_a\} - \#\{b \text{ opposite block} : \lambda_a < \lambda_b\}$ and $2\rho_{c,a}(\lambda)$ is the same-block count; the cone $\mathcal{C}_\lambda = \{b(\lambda) + \sum_{\lambda_b<\lambda_a} m_{ab}(e_a - e_b) : m_{ab} \in \mathbf{N}\}$; the Casimir $C = \sum_{a,b\in I} E_{ab}E_{ba}$; and $q_{p,q}(\lambda) = \sum_{a\in I}\lambda_a^2$, $r_{p+q} = \sum_{i=0}^{p+q-1}((p+q-1)/2 - i)^2$. The $b(\lambda)$-highest-weight space of Π_λ's compact-finite module has dimension one, C acts by $q_{p,q}(\lambda) - r_{p+q}$, and every highest weight μ of that module lies in $\mathcal{C}_\lambda$.

2. *Use the Harish–Chandra module realization, admissibility and irreducibility conventions of the real theta construction. For every such cover and irreducible unitary representation, all compact-finite vectors are smooth, there is a Harish–Chandra module realization, and every such realization is admissible and irreducible. Two irreducible unitary representations with isomorphic realized complex Lie-algebra modules are unitarily equivalent. Conversely, a unitary representation with an admissible irreducible Harish–Chandra module realization is irreducible.*

   There is also the following analytic assertion for the matrix unitary groups. For every $n \geq 0$, sign vector $\epsilon \in \{\pm 1\}^n$, and nonzero complex Hilbert space carrying a continuous irreducible unitary representation σ of

$$G_\epsilon = \{g \in \mathrm{GL}_n(\mathbf{C}) : g^* J_\epsilon g = J_\epsilon\}, \qquad J_\epsilon = \mathrm{diag}(\epsilon),$$

   the orbit map $g \mapsto \sigma(g)v$ is real analytic for every vector finite under the standard maximal compact subgroup.

3. *For every arithmetic datum and Hermitian space V, choose compatible infinite embeddings and an exponential family agreeing with matrix exponentiation at each infinite place, and let $H_V(g)$ (product of local operator-norm heights) and $\overline{H}_V(g) = \inf_{\gamma\in U(V)(F)} H_V(\gamma g)$ denote the adelic and quotient heights. Suppose φ is a continuous function on $U(V)(\mathbb{A}_F)$, left-$U(V)(F)$-invariant, smooth at infinity, and right-invariant under the finite part of an open adelic subgroup U, whose integral along every rational unipotent radical of an isotropic frame vanishes (for all translates, compatible Haar measures, and fundamental domains). For a word $X = (X_1, \ldots, X_s)$ in the infinite Lie algebra, let D_X be its word derivative. Assume uniform moderate growth:*

$$\exists N \in \mathbf{N} \quad \forall X \quad \exists C_X \in \mathbf{R} \quad \forall g, \qquad |D_X\varphi(g)| \leq C_X H_V(g)^N.$$

   Then

$$\forall A \in \mathbf{N} \quad \exists C_A \in \mathbf{R} \quad \forall g, \qquad |\varphi(g)|\,\overline{H}_V(g)^A \leq C_A.$$

   The rapid-decay reference used here is Miller and Schmid, On the rapid decay of cuspidal automorphic forms *[27].*

4. *Let G_ϵ be as above, with Lie algebra $\mathfrak{g}_\epsilon$ and standard maximal compact subgroup K_ϵ. Let σ be a continuous unitary representation on a complex Hilbert space H whose closed invariant subspaces are 0 and H. Write H_{K_ϵ} and H^∞ for its compact-finite and smooth vectors. Let E be a complete Hausdorff complex vector space whose topology is generated by a countable family of seminorms (p_i). Suppose $\pi(g)$ and $d\pi(X)$ are continuous complex-linear operators on E, with $\pi(1) = 1$, $\pi(gh) = \pi(g)\pi(h)$, and continuous orbit maps. Require $d\pi(X)v$ to be the differentiated action along the prescribed exponential in direction X, for every $X \in \mathfrak{g}_\epsilon$ and $v \in E$. Require the growth condition*

$$\forall i \quad \exists q_i, N_i \quad \forall g \in G_\epsilon,\ v \in E, \qquad p_i(\pi(g)v) \leq \left(1 + \sum_{a,b} |g_{ab}|^2\right)^{N_i} q_i(v),$$

where q_i is a continuous seminorm and $N_i \in \mathbf{N}$. For a finite list $\mathcal{S}$ of Lie-algebra words, put

$$Q_{\mathcal{S}}(w) = \sum_{(X_1,\dots,X_s)\in\mathcal{S}} \|d\sigma(X_1)\cdots d\sigma(X_s)w\|,$$

with the empty word acting as the identity.

Let $T_0 : H \to E$ be complex-linear and intertwine the K_ϵ-action and the derived $\mathfrak{g}_\epsilon$-action on H_{K_ϵ}. There is a complex-linear map $T : H \to E$ agreeing with T_0 on H_{K_ϵ} and intertwining G_ϵ on H^∞, such that

$$\forall i \quad \exists \mathcal{S}, C \quad \forall w \in H^\infty, \qquad p_i(Tw) \le C Q_{\mathcal{S}}(w).$$

Every other map with these properties agrees with T on H^∞, and $T(H^\infty)$ is closed in E. If $T_0 v \neq 0$ for some $v \in H_{K_\epsilon}$, then for every finite list $\mathcal{S}$ there is a continuous seminorm $q_{\mathcal{S}}$ on E such that

$$Q_{\mathcal{S}}(w) \le q_{\mathcal{S}}(Tw) \qquad (w \in H^\infty).$$

The globalization reference used here is Wallach, Real Reductive Groups II, *Chapter 11 [25, Ch. 11].*

For every member condition and bundle association the following hold at each selected point.

For every $j < m$ there are smooth unit extremal vectors u_ρ of the specified weights in the lowest types and a unit vector w in the paired space such that the Fock-model pairing of $\bigotimes_\rho u_\rho$ with w is the prescribed product of minors Δ_j. Let K_{c} be the coefficient field of the selected point, with its fixed embedding $\iota : K_{\mathrm{c}} \hookrightarrow \mathbf{C}$. At the reference tuple $t = \mu_0$, the comparison uses the field $\iota(K_{\mathrm{c}})$. The component coefficients c_j and phase ϕ_j have modulus one. There are a central level element z and a root of unity $\zeta_t \in \iota(K_{\mathrm{c}})$. For each allowed input x, its cuspidal theta class has, for every admissible representative f and every $h \in G_{\mathbf{Q}}(W)\backslash G_{\mathbb{A}}(W)$, where W is the layer-j target Hermitian space of the moduli datum on which the theta function is evaluated, the identity

$$\phi_j \Theta_{j,t}(x)(h) = \zeta_t c_j(\alpha(h))\, a_j\, J_j(z\cdot f)(h).$$

Here $\Theta_{j,t}(x)$ is the adelic theta function of the input, $\alpha(h)$ is the component of its comparison frame at h, and $J_j(f)(h)$ is the adelic evaluation of the section of f in that frame before multiplication by c_j. If A_j is the sum of the section's weight exponents over all embeddings and frame coordinates, then $a_j = (2\pi\xi)^{-A_j/2}$, where $\xi = (\sqrt{d}\,\mathrm{N}\mathfrak{c}_K)^{-1}$ and $\mathfrak{c}_K$ is the host curve ideal. Thus the component-normalized section is $\varphi_j(c_j, f)(h) = c_j(\alpha(h))J_j(f)(h)$. Every layer lift (g, k_ℓ) preserves the cusp space and satisfies $\varphi_j(c_j, gf)(h) = \chi_{\det}(g)\varphi_j(c_j, f)(hk_\ell)$, where $\chi_{\det}$ is the product of the designated determinant characters. For each reference input there are finitely many $a_\nu \in K_{\mathrm{c}}$ and algebraic cusp forms f_ν of the specified weight such that

$$\phi_j \mathcal{T}_j(x)(h) = a_j \sum_\nu \iota(a_\nu)\, \varphi_j(c_j, f_\nu)(h) \tag{37.32}$$

for every h, where $\mathcal{T}_j$ uses the reference vector. At every weight and degree the base forms span over the comparison field. Scalar extension holds over each common coefficient algebra and the level action preserves coefficients.

For a tuple t, write Q_t for its prescribed set of primes of E. The prime compatibility condition requires that any two primes in the same Q_t over one rational prime have the same contraction to $\mathcal{O}_{\mathbf{Q}(\sqrt{-d})}$, and that no prime in any Q_t divide $Q_S q\,\mathrm{N}(\mathfrak{c}_K)$. Under this condition one comparison at layer j, over a field containing $\iota(K_{\mathrm{c}})$, realizes the prescribed lattices, level actions and normalized theta class sums for all tuples. The theta identity holds for every tuple with the same central element z and phase ϕ_j, and with a root of unity $\zeta_t \in \iota(K_{\mathrm{c}})$ depending on the tuple. Its component factor is the prescribed unit multiplier for sections at layer $j+1$, taken at the embedding of the base field induced by the enlarged comparison field. The scalar-extension and level identities hold over the new field.

Outline. The four real inputs and Fock-model injectivity produce the extremal vectors u_ρ. At the reference tuple, Hermitian Gaussian-theta reproducing-kernel unfolding and a rational unitary move to split-supported lattices express complex readouts of C_K via theta kernels of rational lifts. Lemma 37.23 and Lemma 37.22 descend spanning to $K = \iota(K_{\mathrm{c}})$, and prime compatibility extends the comparison to all tuples. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1405. □

Lemma 37.25 (The unramified CM model and its inert parameter)**.** *Let $d > 0$ be squarefree, $d \equiv 3 \bmod 4$, and let a complex period lattice be stable under $\omega = (1+\sqrt{-d})/2$. For $p \geq 5$, $p \nmid d$, fix a field isomorphism $\mathbf{C} \simeq \mathbf{C}_p$. Put $W = W(\overline{\mathbf{F}}_p)$, $K^{\mathrm{ur}} = \operatorname{Frac} W$. In the homothety class of the given lattice there are short Weierstrass invariants in W of discriminant one. If p is inert in $\mathbf{Q}(\sqrt{-d})$, and f is the logarithm of this good model with tangent one, there are $\epsilon \in W^{\times}$ and mutually inverse integral parameters v, h with*

$$f(v(T)) = \epsilon \ell(T), \qquad \ell(h(T)) = \epsilon^{-1} f(T), \qquad \ell(T) = \sum_{k \geq 0} p^{-k} T^{p^{2k}}.$$

Outline. Use CM reciprocity and 36.24 for unramifiedness and the auxiliary kernel choice, then use abscissa products for the valuation divisibilities. At an inert prime reduce the logarithm to type $p - a\tau$ and apply Hazewinkel's lemma. The complete proof is on p. 1414. □

Proposition 37.26 (Reciprocity supplies the twisting and kernel conditions)**.** *Use the weak local correspondence of Theorem 40.1 in the extension parameters of (33.1). Take a bundle association for which each associated reciprocity datum is identified with the compatible datum of a constructed point. The following assertions hold for the good members of that construction.*

Use the family quantifiers of Definition 36.10 on $\mathcal{D}_3$. Beyond one family threshold, at every $x > x_1$ and every member $i \in I_x$, choose levels and local lifts on the prescribed fillings for all words and twists. At the used twists and permitted places outside the twist primes these choices have the comparisons of Definition 14.15 and the following property:

$$\begin{gathered}\text{the local systems satisfy the stabilizer and complementary-orbit bounds} \\ \text{of Theorem 16.21 with initial saving one.}\end{gathered} \tag{37.33}$$

At every member beyond the selection threshold with $x > x_1$, adjacent class scalars have norm one at every level-free w, for every word and tuple. For each word u and used twist μ, the adjacent pair maps preserve rational row spaces at such places. On the reference chain and on each used twisted chain, the lift sends the complex span of the embedded rational class functions at layer j into the corresponding span at layer $j+1$. Each of these chains has trivial initial stabilizers.

For the selected entries at this member there are coordinate scalars, orbit phases and signs $(e_w, \beta_w, \sigma_w)_w$. Write p for the residue characteristic of w. A place is level-free for (u, μ) when $p \neq T_u$, $p \notin \{g_0, g_1\}$ and $p \notin Q'(\mu)$. At each such place the following assertions hold.

- *If p is in the inert class with h_P of even order, or is in the signal class, every set of orbit representatives satisfies the letter conditions of Proposition 15.5 for these data. For every $j < m$ with $N_j N_{j+1}$ even, all class-pair entries have the all-place expansion of Proposition 16.12.*
- *At an inert-class prime and every layer of even rank product, the entries have the one-position expansion when $h_P = 0$, and the all-real orbit expansion when $h_P \neq 0$ has odd order.*
- *If $p \neq d$, the minor evaluations of Definition 10.18 have integral coefficients at every class pair and every layer of even rank product.*

At every level-free place in the discriminant class, each entry at $j < m$ has the form

$$\Theta = \exp\left(-S \sum_t u_t \sum_{a,b} X_{(0,t),a,b} X_{(1,t),a,b}\right) \Theta_0, \quad |S|_p \leq p^{1/2}, \quad |u_t|_p \leq 1, \quad \left| \left(\prod_{\alpha} \nu_\alpha! \right) [X^\nu] \Theta_0 \right|_p \leq 1 \tag{37.34}$$

for every multi-index ν.

If $5 \leq p \leq X''$ is in the inert class, $h_P \neq 0$, and the orbit length is even, then for every word u, every $j < m$, every selected first-sheet orbit representative $(0,t)$, and every series with the prescribed expansion on that orbit, all coefficients of its minor evaluation at zero have norm at most $p^{-\max(g_0(t,j+1),0)}$ for every column choice. The gain g_0 and the minor evaluation are those of 14.3 and 16.10.

Outline. Use reciprocity 1.22, the norm choices of 40.1 and the model of 37.25. Combine the chain of 37.31 with the cyclic bound of 15.5, then use 37.29, 37.30, 37.28, four-copy local normalization, and the phase defects for expansions and jet bounds across all primes. The complete proof is on p. 1415. □

Lemma 37.27 (The pairing of the CM sigma series)**.** *Let $K = \mathbf{Q}(\sqrt{-d})$ be the host quadratic field, $\mathfrak{c}$ a nonzero curve ideal of norm prime to $2d$, and $\Lambda = \delta\mathfrak{c}$ of discriminant one. Let $p \geq 5$ be inert in K with $p \nmid \mathrm{N}\mathfrak{c}$, and fix $\lambda : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$. Put $W = W(\overline{\mathbf{F}}_p)$ and $K_0 = \operatorname{Frac} W$. Let σ be Witt Frobenius. Use a good Weierstrass model over W in the prescribed homothety class, as supplied by Lemma 37.25, with its parameter at the origin defined over W and logarithm $l'(0) = 1$, and write $m_a = l^{-1}(al)$, $t \oplus u = l^{-1}(l(t) + l(u))$.*

For a period ω, let $\eta_\Lambda(\omega)$ be the increment of Weierstrass zeta. For a positively oriented basis ω_1, ω_2, put

$$s_2 = \frac{\eta_\Lambda(\omega_1)\bar{\omega}_2 - \eta_\Lambda(\omega_2)\bar{\omega}_1}{\omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1}, \qquad \Sigma_\Lambda(z) = \sigma_\Lambda(z)\exp(-s_2 z^2/2).$$

For model invariants $e^4\lambda(g_2), e^6\lambda(g_3)$, with g_2, g_3 those of Λ, let $\Sigma(z) = e^{-1}(\lambda_*\Sigma_\Lambda)(ez)$. Its coefficients are independent of the extension of λ used in the transfer. Write

$$P_E(t) = -\log(\Sigma(l(t))/t), \qquad B_E(t,u) = P_E(t \oplus u) - P_E(t) - P_E(u).$$

Here $A \sim B$ means $A - B$ has zero constant term and $\exp(A - B)$ is integral. Then $P_E \in K_0[[t]]$, B_E is biadditive modulo $\sim$, and m_a is integral for $a \in W^{\sigma^2}$, with

$$P_E(m_a(t)) \sim a\sigma(a)P_E(t), \qquad B_E(m_a(t), u) \sim B_E(t, m_{\sigma(a)}(u)).$$

Let h be an integral parameter isomorphism with $\ell(h(t)) = e_0 l(t)$, where ℓ is as in 10.5 and $e_0 \in W^\times$. Put $v = h^{-1}$, $\epsilon = e_0^{-1}$, $F(X,Y) = \ell^{-1}(\ell(X) + \ell(Y))$, and

$$P(T) = -\log\frac{\Sigma(\epsilon\ell(T))}{\epsilon T}, \qquad b(X,Y) = B_E(l^{-1}(\epsilon X), l^{-1}(\epsilon Y)).$$

Then $b \in XYK_0[[X,Y]]$ satisfies 10.6, and $b^\ell = B_E(v(X), v(Y)) \sim B_P$, where $B_P(X,Y) = P(F(X,Y)) - P(X) - P(Y)$. The norm and adjoint laws hold for P, B_P with $[a] = \ell^{-1}(a\ell)$.

Outline. The sigma addition and isogeny formulas give biadditivity and the CM norm law modulo integral exponentials. Continuity and polarization give the scalar and adjoint laws, which transfer through the height-two parameter to the pairing of 10.6. The complete proof is on p. 1420. □

Lemma 37.28 (The ordinary gain on a standard orbit series)**.** *Let w have $5 \leq p \leq X''$, inert class (14.1), $h_P \neq 0$, and even orbit length $d_0 = n_P = 2h \geq 2$, with $K_{\delta,1} \geq 5$, $K_{\delta,2} \geq 4$ (Definition 14.2). At stage $j+1$ $(j < m)$ with word-coordinate spacing $\mathcal{D}$ (33.7), scale $a_* = xb$ (14.2), and filling bound $B_j(z)$ ((16.17)), let w_z, s_z be the weight and diagram size, $r_{z,0}, r_{z,1}$ its first two rows, and $c_z = r_{z,0}$ off the top or $r_{z,1}$ on the top; assume*

$$\begin{gathered} w_z > 0, \quad 0 \leq s_z - (k-1)w_z \leq 4w_z \ \text{(off top)}, \\ r_{z,0} \geq 2w_z, \quad 0 \leq s_z - r_{z,0} - (k-3)w_z \leq 4w_z \ \text{(top)}, \\ s_z \leq a_*^2, \ 2\log_5(a_*^2) + 2 \leq k-1, \ 1 < a_*, \ X'' \leq a_*, \\ c_z \leq \min(b + \eta + 1 + 12k\mathcal{D}, B_j(z) + c_G), \ \bar{\alpha}_t \leq r_{(0,t),0} - r_{(0,t),1} \leq 2a_*, \end{gathered} \tag{37.35}$$

Here $c_G = c = 4S$ is the global row correction of 14.2, and the last inequality is at marked first-sheet positions. For a word u and selected first-sheet orbit representative $(0,t)$ at p, let b_{pair} satisfy 10.6, let L_i^X, L_i^Y be the σ^{d_0}-fixed trace-dual forms of 10.13, let β be the average of $\sum_i b_{\mathrm{pair}}(L_i^X, L_i^Y)$ under $Z_s \mapsto \zeta^{(-1)^s p^s} Z_s$ $(\zeta^{p^{d_0}-1} = 1)$, and put $\mathcal{L}_s(Z) = \sum_{k\geq 0} Z_{s-2k}^{p^{2k}}/p^k$ (cyclic indices). For every $|\xi|_p = 1$ and every series Θ with matrix-entry expansion

$$\Theta(\xi\mathcal{L}(Z)) = U(Z)\exp\left(-\sum_{a',b'}\beta(\mathcal{L}(Z_{\bullet,a',b'}))\right), \qquad U \in \mathcal{O}_{\mathbf{C}_p}[[Z]], \tag{37.36}$$

each coefficient of the column-minor evaluation of 16.10 has norm at most $p^{-\max(0, g_0(t,j+1))}$.

Outline. Undo the trace-dual change by integral substitution and remove the torus average modulo integral exponentials via 10.6. Use 10.7 for exponent zero and 11.14 for g_A, then compare the caps and digit errors in 11.1 and 14.2. The complete proof is on p. 1421. □

Lemma 37.29 (The corrected Taylor series in an orthonormal basis)**.** *A complex period lattice stable under* $(1+\sqrt{-d})/2$*, d squarefree and $d \equiv 3 \bmod 4$, has for every $p \geq 5$ prime to d and isomorphism $\mathbf{C} \simeq \mathbf{C}_p$ a homothetic short Weierstrass model over the completed unramified extension with integral coefficients and unit discriminant.*

Take inert $p \geq 5$ in $K = \mathbf{Q}(\sqrt{-d})$, a theta datum for a nonzero ideal $\mathfrak{c}$ of norm prime to $2d$, and $\Lambda = \delta\mathfrak{c}$ of discriminant one. Its character χ is multiplicative on norm-one ideals with $\chi((z)) = \prod_\tau \tau(z)^{-1}$ on unitary z, for the prescribed CM type of E. Fix $\lambda : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$. Let L, L' be self-dual of even ranks. Let F be a finite linear combination with coefficients in $\mathcal{O}_{\mathbf{C}_p}$ of the transferred Taylor series of translates of the sections in Lemma 12.11. For each summand the two lattices agree with L, L' after localization at p, and some positive integer prime to p carries its shift into $P(L, L')$.

Use any integral Weierstrass model W of discriminant one obtained from the transferred period equation by a Weierstrass change of variables. Write l_W for its logarithm of tangent one and e for the inverse of the leading scale of the change, so that $e\, l_W$ is the period differential coordinate. Put $A(T) = \sigma_\Lambda^*(e\, l_W(T))/(eT)$, with $\sigma_\Lambda^* = \Sigma_\Lambda$ transferred by λ. Let $B_1, \dots, B_r$ be a K-linear basis localized at p of $\mathfrak{a}_0 \operatorname{Hom}(L, L')$, where $\mathfrak{a}_0 = \mathfrak{b}^{-1}$ with $\mathfrak{b}$ from Lemma 12.11. So the B_i form an $\mathcal{O}_{K_p}$-basis of the completed module. Take it orthonormal for $\operatorname{Tr}_{E/K} \sum_{a,b} x_{ab}\overline{y_{ab}}$. Then

$$F\left(\left(e\sum_i \lambda(\tau(B_{i,ba}))l_W(T_i)\right)_{\tau,b,a}\right)\prod_i A(T_i)^{-1} \in \mathcal{O}_{\mathbf{C}_p}[[T_1, \dots, T_r]]. \tag{37.37}$$

Outline. Use 37.25 and the sigma product for prime-to-p translates, with 37.27 for unitary basis changes. The four-copy identity of 17.14 gives integrality coefficientwise, and 12.11 and 12.15 handle the endpoint moves. The complete proof is on p. 1422. □

Lemma 37.30 (The change to paired variables)**.** *Use the height-two phase P and pairing b of 37.27, or any phase with the same biadditivity, norm and adjoint laws and a pairing satisfying 10.6 with $b^\ell \sim B_P$. Let $f_e, f'_e \in (W^{\sigma^2})^r$ for a finite index set e. For $\langle u, v\rangle = \sum_i \sigma(u_i)v_i$, assume $\langle f_e, f_h\rangle = \langle f'_e, f'_h\rangle = 0$, $\langle f_e, f'_h\rangle = \delta_{eh}$. Put $L_i = \sum_e (f_{ei}X_e + f'_{ei}Y_e)$, $S = \sum_e b(X_e, Y_e)$. If $H(T) = \Theta((\ell(T_i))_i)\exp(\sum_i P(T_i))$ is integral, then $\Theta(L)\exp S$ is integral after logarithmic substitution. The phase S is even modulo the equivalence of 10.5.*

Outline. Expand the phase of the formal sum using the norm, adjoint and biadditivity laws. The Gram relations leave only the matched cross-pairings, giving integrality after substitution, and adjointness for -1 gives evenness. The complete proof is on p. 1423. □

Proposition 37.31 (The kernel on the selected chain)**.** *Fix a constructed arithmetic column, a word u, a used twist μ, and a coefficient place w of residue characteristic p. Assume the permission condition of Theorem 13.8, $p \notin Q'(\mu)$, and $2r_G \leq k$. Use the prescribed filling and the initial stabilizer condition of Theorem 16.21. The curve with its prescribed period lattice and embedding has the good model and parameter of Lemma 37.25.*

There are levels and a chain on that filling for which the arithmetic kernel satisfies Definition 10.10. One embedding extending the coefficient place, one family of target readings and one family of normalized entry series give the class expansions at every layer.

On these same levels and chain there are semilinear comparisons on the scalar extensions of the rational lift spaces. They agree with the initial comparison, intertwine adjacent lifts and transfer terminal local valuation bounds to the terminal rational lattice with the same exponent. On the selected levels and chain the system has $d_G(0) = 0$ and $d_G(j) = v_p(D_{\Gamma,u})$ for $j > 0$, with $D_{\Gamma,u}$ the common stabilizer divisor.

Outline. Use the good model 37.25, finite tests 12.13 and paired variables 37.30 to build the chart series. Transport contractions by 10.17 and sum with stabilizer weights to intertwine lifts, then use 16.18 to transfer the terminal bound. The complete proof is on p. 1424. □

Chapter 38

The chain length

Fix a scale $s = (x, \varepsilon, J) \in \mathbf{R}^3$ with $x > 1$, $0 < \varepsilon < 1/52$ and $J \geq 1$. Put

$$L = x^{\varepsilon}, \qquad \delta_1 = x^{-\varepsilon/8}, \qquad \alpha = \sqrt{x\delta_1} = x^{1/2-\varepsilon/16}, \qquad \beta = x^{9/2+10\varepsilon}.$$

Definition 38.1 (The prescribed chain length)**.** For $y \in \mathbf{R}$ let $\lceil y \rceil_+ = \max(0, \lceil y \rceil)$ and set $e(y) = 2\lceil y/2 \rceil_+$. The prescribed chain length (the number of lifts) is $m_*(s) = e(\alpha)$. Theorem 38.3 uses the same rounding formula to choose m and the starting rank k on a prescribed frame.

For $y > 0$ we have $y/2 \leq \lceil y/2 \rceil_+ < y/2 + 1$, so $y \leq e(y) < y + 2$. Every even natural number at least y is $2a$ with $a \geq \lceil y/2 \rceil_+$. Thus $m_*(s)$ is the least even natural number at least α, with $\alpha \leq m_*(s) < \alpha + 2$.

Definition 38.2 (Rank data and rank selection)**.** For a frame φ as in 5.1, let A be its finite abelian group, $a_0 \in A$ its distinguished element, $n = |A|$, $f = 2n$, and $o_\varphi = [A : \langle a_0 \rangle]$. *Rank data* on φ consist of a triple $(k, m, t) \in \mathbf{N}^3$ with k, m even, t odd, and $r = k + m - 1$, satisfying

$$2 \leq m, \quad m + 3 \leq k, \quad r + 5 \leq t, \quad 3m < t,$$
$$m \leq f \leq k, \qquad m^2 \leq k + m(t + 1).$$

Fix nonnegative *family constants* $C = (c_W, c_T, c_o)$ (Theorem 38.3 specializes to $c_W = 10$ and $c_T = 377$). For $n \in \mathbf{N}$ and $f = 2n$, define

$$k_* = e(\beta), \qquad r_* = k_* + m_* - 1,$$
$$w_C = 8\left\lceil \frac{c_W r_*}{8}\left(f + \frac{f^2(k_* - 1)^3 L^2}{m_*}\right)\right\rceil_+, \quad t_C = \lceil x^{c_T} \rceil_+ w_C + 8fr_* + 4 - r_*,$$

with subtraction in $\mathbf{N}$ truncated at zero. Rank data (k, m, t) on φ are defined to satisfy the family's *rank selection condition* at scale s for C when

$$x^{2+8\varepsilon} \leq n \leq 2x^{2+8\varepsilon}, \qquad k = k_*, \quad m = m_*, \quad t = t_C, \quad o_\varphi \leq c_o.$$

The triple (k_*, m_*, t_C) is defined to be *admissible* for order n when t_C is odd and the six inequalities above hold for (k_*, m_*, t_C). Because k_* and m_* are even by definition of e, adjoining an admissible triple to any frame of order n gives rank data on that frame.

Theorem 38.3 (Ranks at a window point)**.** *For each $0 < \varepsilon < 1/52$ one may take $x_0 = 16$. For every scale s with this ε and $x \geq x_0$, and every arithmetic frame φ of order n satisfying $x^{2+8\varepsilon} \leq n \leq 2x^{2+8\varepsilon}$, there are rank data (k, m, t) on φ with k, m even and t odd such that*

$$\alpha \leq m \leq 2\alpha, \qquad \beta \leq k \leq 2\beta, \qquad r + t = 8fr + 4 + TW,$$

where $f = 2n$, $r = k + m - 1$, and

$$T_O = T = \lceil x^{377} \rceil_+, \qquad n_W = W = 8\left\lceil \frac{10r}{8}\left(f + \frac{f^2(k-1)^3L^2}{m}\right)\right\rceil_+.$$

Outline. Bound m, k by 38.1 and the order window, then set $t = 8fr + 4 + TW - r$ and check the rank inequalities. The complete proof is on p. 1427. □

Part IX

Galois representations and local Langlands in the archive

Chapter 39

Galois representations attached to polarized cuspidal representations

39.1 The global theorem

Let E/F be a CM extension with conjugation c. Fix an algebraic closure Ω of E, a prime ℓ, and an isomorphism $\iota : \bar{\mathbf{Q}}_\ell \simeq \mathbf{C}$. Put $G_E = \mathrm{Gal}(\Omega/E)$. A polarized input consists of a cuspidal π on $\mathrm{GL}_a(\mathbf{A}_E)$, $a \geq 1$, with distinct infinitesimal weights in $(a-1)/2+\mathbf{Z}$, and a Hecke character χ of F with $\chi_v(-1) = (-1)^a$ at each real place and

$$\pi^c \simeq (\pi \otimes (\chi \circ N_{E/F})^{-1})^\vee.$$

For a finite place w write E_w for its completion, $k(w)$ for its residue field, and $q_w = \#k(w)$. For a Satake multiset B at w put $A_a(w,B) = q_w^{(a-1)/2} B$ and $P_{w,B}(X) = \prod_{\alpha \in A_a(w,B)} (X - \alpha)$. We use inverse arithmetic Frobenius in the Satake formulas and transport coefficients by ι. A prolongation of w to Ω specifies the restriction r_w of a representation r to the local Galois group. Write W_w, I_w and P_w for the Weil group, inertia and wild inertia. The degree on W_w sends arithmetic Frobenius to 1 and has kernel I_w. Galois representations are continuous and semisimple unless specified otherwise.

Definition 39.1 (Global and local inputs)**.** Let h denote Theorem 39.2 and L the local existence theorem 40.1. The three inputs, in order, are (h, L, h). Theorem 26.4 uses the global theorem to prove the window bound of Definition 26.3 with $C_t = 401$ for every primitive Dirichlet character.

Theorem 39.2 (Global theorem)**.** *For each polarized input there exist $r : G_E \to \mathrm{GL}_a(\bar{\mathbf{Q}}_\ell)$, an integer w_{gt} and integer multisets H_τ as in 39.4. These data satisfy 39.5 at every $w \nmid \ell$ and 39.6 at every $w \mid \ell$, for every prolongation and every map of 40.2.*

Proof. Apply 39.3. □

Proposition 39.3 (Assembly)**.** *Theorem 39.9 and Proposition 39.60 imply 39.2.*

Proof. The geometric seed gives 39.36, and the approximations of 39.40 give the odd-rank limit of 39.44. The stable-range theorem gives the two lifts of 39.63. Lemmas 39.64–39.65 and Propositions 39.66–39.67 recover the even-rank data by subtracting the added lines. The spherical theorem 39.54 gives weak attachment over CM fields. Apply 39.8, with the tame places restricted as in 39.5. □

Definition 39.4 (Weights)**.** For an embedding $\tau : E \hookrightarrow \bar{\mathbf{Q}}_\ell$, form H_τ by replacing each infinitesimal weight z of π at $\iota \circ \tau$ by $(a-1)/2 - z$, with its multiplicity. The weight w_{gt} satisfies $H_{\tau c} = w_{\mathrm{gt}} - H_\tau$.

For a cuspidal Π of rank $a \geq 2$ over a CM field E, cohomological for an irreducible algebraic coefficient representation and satisfying $\Pi^c \simeq \Pi^\vee$, let $R_\ell(\Pi)$ be the representation of Shin and Chenevier–Harris [210, 116]. Caraiani proves that at each finite place w,

$$\iota\,\mathrm{WD}(R_\ell(\Pi)|_{G_{E_w}})^{\mathrm{F\text{-}ss}} \simeq \mathcal{L}_{a,E_w}(\Pi_w),$$

where the geometric normalization is

$$\mathcal{L}_{a,E_w}(\Pi_w) = \mathrm{rec}_{\mathrm{cl}}\Big(\Pi_w^\vee \otimes |\det|_w^{(1-a)/2}\Big),$$

with $\mathrm{rec}_{\mathrm{cl}}$ the correspondence compatible with local L- and ϵ-factors. The isomorphism includes monodromy at every finite place [212, 211, Thm. 1.1]. Theorem 39.2 allows general polarizations and uses the following local tests.

Definition 39.5 (Away from ℓ)**.** Write a Weil–Deligne parameter z as (ρ_z, N_z), with Weil action ρ_z and monodromy N_z. For $w \nmid \ell$, a prolongation and a map R of 40.2, put $x = \mathrm{WD}_\iota(r_w)$ and $y = R(\pi_w \otimes |\det|_w^{(1-a)/2})$. Here WD_ι denotes the Frobenius-semisimplified Weil–Deligne class with its monodromy, transported by ι.

The toral test requires characteristic polynomial $(X-1)^a$ on every Weil commutator acting on y. The tame test requires this polynomial on P_w. For each lift $\gamma \in I_w$ of a topological generator of I_w/P_w, it also requires $z^{q_w-1} = 1$ for every repeated root z of $\det(X - \rho_y(\gamma))$. The full tame test adds a continuous isomorphism $E_w \simeq \mathbf{Q}_p$, where p is the residue characteristic, and the condition $a = 1$ or $p > 16a^2$. The local comparison of x with y requires that they have the same rank. If $N_y = 0$, it requires equality of classes under the toral test, and $N_x = 0$ with equality of all inertia characteristic polynomials under the full tame test.

Definition 39.6 (Above ℓ)**.** At $w \mid \ell$, take any prolongation. Use any period-ring data P, A_{inf} admitting the following identifications. The crystalline and de Rham rings of P admit $G_{\mathbf{Q}_\ell}$-equivariant ring isomorphisms with B_{cris} and B_{dR} that preserve the images of $\mathbf{Z}_\ell$. The crystalline isomorphism respects Frobenius and the map from A_{inf}. The de Rham isomorphism carries each filtration step for every finite extension of $\mathbf{Q}_\ell$ to the corresponding step of B_{dR}. For $z \in \bar{\mathbf{Q}}_\ell$, the image of the coefficient z lies in B_{dR}^+ and reduces to z under θ. The distinguished crystalline period maps in B_{dR} to $\log([\varepsilon])$ for the chosen system ε of ℓ-power roots of unity. Under these identifications the map $B_{\mathrm{cris}} \to B_{\mathrm{dR}}$ agrees with the natural map on the uncompleted divided-power envelope and sends A_{cris} into B_{dR}^+. In each finite filtration quotient of B_{dR}^+, a single power of ℓ carries the image of A_{cris} into the image of A_{inf}. Take any period bridge, that is, a comparison map inducing the specified local embedding. For a map R of 40.2 put $y = R(\pi_w \otimes |\det|_w^{(1-a)/2})$. If π_w is spherical, r_w is crystalline and de Rham with jumps H_τ at every induced embedding τ. If $N_y = 0$ and either the toral or the full tame test of 39.5 holds, r_w is potentially semistable and de Rham with those jumps. Its Frobenius-semisimplified period parameter x satisfies the local comparison with y in Definition 39.5.

Lemma 39.7 (Parallel infinity exponent and polarized weights)**.** *For every polarized input (π, χ) of rank $a \geq 1$, there is an integer b such that $\chi_v(x) = x^b$ for all $x \in \mathbf{R}_{>0}$ at every real place v of F. With $w_{\mathrm{gt}} = a - 1 - b$, the integer multisets H_τ of Definition 39.4 satisfy $H_{\tau c} = w_{\mathrm{gt}} - H_\tau$ for every $\tau : E \hookrightarrow \bar{\mathbf{Q}}_\ell$.*

Outline. Polarization pairs infinitesimal weights above v by $z \mapsto b_v - z$, making $b_v \in \mathbf{Z}$. Because χ kills a finite-index subgroup of totally positive units, Dirichlet's unit theorem forces all $b_v = b$, giving $H_{\tau c} = a - 1 - b - H_\tau$ in 39.4. The complete proof is on p. 1427. □

Proposition 39.8 (Local clauses)**.** *Assume 39.54 over CM fields. Every polarized input has common witnesses to 39.4–39.6. In both local clauses the wild-inertia and repeated-root tests give the tame conclusion for every rank $a \geq 1$ when $E_w \simeq \mathbf{Q}_p$ continuously and $N_y = 0$. The toral clause applies over every E_w.*

Outline. Takes w_{gt}, H_τ of 39.4 from 39.7 and r from the spherical theorem. Uses genericity for the toral clause; for the tame clause over $E_w \simeq \mathbf{Q}_p$, kills wild inertia and compares inertia polynomials and jumps after an unramified lift. The complete proof is on p. 1427. □

39.2 Geometric seeds in odd rank

Let a be odd, $x^2 = -d$ in E with $d \in \mathbf{Z}_{>0}$, $-d \in (\mathbf{Q}_\ell^\times)^2$, and $[F : \mathbf{Q}]$ even. Assume every nonsplit finite place is unramified over F and lies away from ℓ. Places are separated if their restrictions to F are distinct. A split place over p is usable if $-d \in (\mathbf{Q}_p^\times)^2$.

Take separated $w_0, v' \mid \ell$ inducing the same place on $\mathbf{Q}(x)$. Let U be a definite-unitary automorphic representation of rank a, unramified at nonsplit places off a finite nonsplit set T, and spherical at v' and at split places separated from w_0 and every member of a finite set S. At w_0 assume U contains an elliptic type given by a nontrivial character of a torsion-free compact open J through a quotient $J \twoheadrightarrow \mathbf{Z}/\ell^m$, with $m \geq 1$, that forces cuspidality. For a dominant weight $\lambda = (\lambda_i)_{0 \leq i < a}$ put $J(\lambda) = \{i - \lambda_i : 0 \leq i < a\}$. A finite set is lacunary if disjoint nonempty subsets have different means.

Theorem 39.9 (Geometric seed)**.** *Suppose $J(\lambda)$ is lacunary for a weight of U at an embedding over v'. There are a dimension d_ρ and a representation $\rho : G_E \to \mathrm{GL}_{d_\rho}(\bar{\mathbf{Q}}_\ell)$ with a lacunary subset of size a in an actual jump multiset at v'. At usable $w \nmid \ell$ off a finite set, inertia is trivial and $P_{w,B}$ annihilates inverse Frobenius for every Satake multiset B of U there and every prolongation.*

For a specified usable spherical $w \nmid \ell$ one can require the same local conclusion at w. At a spherical $w \mid \ell$ separated from w_0, v' and agreeing with v' on $\mathbf{Q}(x)$, one can require crystallinity with every prescribed jump at every induced embedding. Alternatively, one can require the crystalline divisibility of 39.27 for all $R = A_a(w, B)$ and all compatible period data. Each choice gives its own witness to the first paragraph.

Outline. Twist the semisimplified eigenspace of 39.21 by the Hecke character in 39.13. Use 39.23 for annihilators and 39.26 and 39.27 for Hodge degrees, then shift the jumps and scale the roots by the twist. The complete proof is on p. 1428. □

Lemma 39.10 (Extension of the unitary eigensystem to similitudes)**.** *Use $E/F, x, a, U, w_0$ of 39.9, and put $I = \mathbf{Q}(x)$. On E^a take the standard definite Hermitian form. Its group of finite adelic similitudes with rational adelic multiplier is*

$$H = \{(g, s) \in \mathrm{GL}_a(\mathbf{A}_{E,f}) \times \mathbf{A}_{\mathbf{Q},f}^\times : {}^t\bar{g} g = s\, 1_a\}.$$

Write $\nu(g, s) = s$, $H^1 = \ker \nu$, and let Γ be the subgroup of rational points with positive rational multiplier. For a finite idèle y of E, put

$$\zeta(y) = \big(\overline{N_{E/I} y}\, 1_a, N_{E/\mathbf{Q}} y\big).$$

This lies in the centre of H, since the product of $N_{E/I} y$ and its conjugate is $N_{E/\mathbf{Q}} y$.

For each infinite place v of E, choose the embedding τ_v used to describe the weight of U. Write $\lambda_v = (\lambda_{v,i})_{0 \leq i < a}$ for its highest weight, W_v for the corresponding algebraic representation of $\mathrm{GL}_a(\mathbf{C})$, and $W = \bigotimes_v W_v$. Define

$$b_v = \sum_i |\lambda_{v,i}|, \qquad b = \sum_v \sum_i (\lambda_{v,i} + b_v), \qquad \xi(g, s) = s^b \bigotimes_v W_v^\vee(\tau_v g) \quad ((g, s) \in \Gamma).$$

Choose a nonzero algebraic form in the weight space of U with the prescribed type at w_0 and the prescribed spherical eigenvalues. There is a nonzero form $\Phi : H \to W^\vee$ with left transformation law ξ and the same type and spherical eigenvalues for the multiplier-one operators. There is an algebraic Hecke character κ of E such that

$$\Phi(h\zeta(y)) = \kappa_f(y)^{-1} \Phi(h).$$

Its exponent at an embedding $\tau : E \hookrightarrow \mathbf{C}$ is

$$m_\tau = b - \sum_{\{v :\, \tau_v|_I = \bar{\tau}|_I\}} \sum_i \lambda_{v,i}.$$

Outline. Extend the unitary form by zero outside its rational and level translates, preserving the multiplier-one eigenvalues. Take a common eigenvector in the finite span of central translates and read the character's infinity exponents from the rational central action. The complete proof is on p. 1428. □

Lemma 39.11 (The ambient PEL datum and the standard definite summand)**.** *Under the field hypotheses of 39.9, let X_0 be a finite set of rational primes with $-d \in (\mathbf{Q}_q^\times)^2$ for every $q \in X_0$, let v_0 be a distinguished real place of F, and let $\omega \in \mathcal{O}_I \subseteq \mathcal{O}_E$ generate the ring of integers of $I = \mathbf{Q}(x)$ over $\mathbf{Z}$.*

1. *There exist an E/F-Hermitian form h_{PEL} on E^a of signature $(1,a-1)$ at v_0 and $(0,a)$ at every real place $v\neq v_0$ of F, an $\mathcal{O}_E$-lattice $\Lambda\subset E^a$, and a positive integer e prime to $\prod_{q\in X_0} q$ such that the alternating form $e_\Lambda(u,v)=\mathrm{Tr}_{E/\mathbf{Q}}(x\,h_{\mathrm{PEL}}(u,v))$ takes values in $\mathbf{Z}$, satisfies $e\,\Lambda^\vee\subseteq\Lambda$, and at every rational prime $q\nmid e$ with $-d\in(\mathbf{Q}_q^\times)^2$ (including every $q\in X_0$) the form e_Λ is perfect over $\mathbf{Z}_q$ and the minimal polynomial of ω has distinct roots modulo q.*

2. *For every finite set X_1 of rational primes, there is a rational prime $p_0\notin X_1$ with $p_0\nmid e$ and $-d\in(\mathbf{Q}_{p_0}^\times)^2$. Specializing $(E^a,\mathcal{O}_E,\Lambda,e_\Lambda,\omega)$ at p_0 gives a local PEL datum satisfying the hypotheses of Proposition 39.15 with $m=1$ and $n=a$, local similitude group G_{p_0}, compact open stabilizer $K_{p_0}\subset G_{p_0}$ of $\Lambda\otimes\mathbf{Z}_{p_0}$, and an away-from-$p_0$ trivialization identifying the away-from-p_0 PEL similitude group with the away-from-p_0 factor H^{p_0} of H in Lemma 39.10.*

3. *There is $\mu\in F^\times$ with $\mu_{v_0}>0$ and $\mu_v<0$ for every real $v\neq v_0$ such that $h_{\mathrm{std}}=\mu\,1_a$ on E^a has signature $(a,0)$ at v_0, signature $(0,a)$ at every real $v\neq v_0$, the same finite local invariants as h_{PEL} at every finite place of F, and rational positive-multiplier similitude group $\Gamma=\{(g,s)\in\mathrm{GL}_a(E)\times\mathbf{Q}_{>0}:{}^t\bar g g=s\,1_a\}$.*

Outline. Builds $h_{\mathrm{std}}=\mu 1_a$ by Chebotarev's theorem and global norm reciprocity, and h_{PEL} of matching finite invariants by weak approximation and Definition 1.20. Forms unimodular lattices at X_0 and picks p_0 split in I via Chebotarev. The complete proof is on p. 1429. □

Lemma 39.12 (The auxiliary prime and regular Iwahori operator)**.** *Keep the notation of Lemma 39.11, let Φ be the similitude form of Lemma 39.10, let $\mathcal{H},\mathcal{C}$ be disjoint finite sets of usable split places, and fix the model prime p_0 away from ℓ, the residue characteristics of $\mathcal{H}\cup\mathcal{C}$, the bad level places of Φ, and a finite set outside which a chosen argument g_0 with $\Phi(g_0)\neq 0$ is integral and unimodular. There exist a rational prime $r_0\notin\{p_0,\ell\}$ outside the residue characteristics of $\mathcal{H}\cup\mathcal{C}$, the bad level places of Φ, and the prime factors of e, together with a nonempty finite set $\mathcal{W}$ containing one split place from each conjugate pair above r_0 on the selected side, such that for the geometric-coordinate transported form $\Phi^\sharp(g,s)=\Phi({}^tg^{-1},s^{-1})$ and every principal level $N\geq 3$ prime to p_0 with $r_0\mid N$ whose congruence subgroup $K(N)\subset H^{p_0}$ is contained in the prescribed transported compact open level K_Φ defined below (with w_0-component inside $\ker\psi_{\mathrm{typ}}\subseteq J_{\mathrm{typ}}$, the transported elliptic type), the away-from-p_0 Hecke algebra at level $K(N)$ contains an idempotent e_{typ} and centrally translated Iwahori tests u,u' with the following properties. Products below mean convolution, and T_g ranges over the selected deck operators and spherical Hecke operators away from $\mathcal{H},r_0,p_0$.*

1. *e_{typ} fixes the transported form $\Phi^\sharp$, and*

$$e_{\mathrm{typ}}^2=e_{\mathrm{typ}},\qquad e_{\mathrm{typ}}T_g e_{\mathrm{typ}}=T_g e_{\mathrm{typ}},\qquad T_gT_{g'}e_{\mathrm{typ}}=T_{g'}T_g e_{\mathrm{typ}},$$

$$e_{\mathrm{typ}}u=ue_{\mathrm{typ}}=u,\qquad uT_g e_{\mathrm{typ}}=T_g ue_{\mathrm{typ}},\qquad u'u=e_{\mathrm{typ}}.$$

2. *The inverse of every element in the support of $u^n e_{\mathrm{typ}}$ preserves the fixed away-from-p_0 lattice, for every $n\geq 1$. For every word polynomial b in the selected operators, every element in the support of $b(T)u^n e_{\mathrm{typ}}$ has distinct eigenvalue valuations at each $W\in\mathcal{W}$, and hence has separable characteristic polynomial of degree a.*

Outline. Chebotarev's theorem chooses r_0 and $W\mid r_0$ to define e_{typ} and the Iwahori operator u. Bernstein's presentation gives u' and support in $I_W t_W^n I_W$, where unit minors give a distinct Newton-polygon slopes. The complete proof is on p. 1429. □

For an ambient PEL datum as in Lemma 39.11 (prime p_0, principal level $N\geq 3$ prime to p_0, antitone weight λ of rank n), a *plan* (or *dominant-weight plan*) for (U,ι) consists of deck and spherical Hecke operators T_g at level $K(N)$, an eigensystem $(\theta_g)_g$ over $\bar{\mathbf{Q}}_\ell$, an algebraic Hecke character κ of E with infinity exponents $(m_\tau)_\tau$, and an integral away-from-p_0 test f satisfying Proposition 39.13. At a model embedding t (with conjugate $\bar t$ and auxiliary weight sum b_0), the two *Hodge exponent vectors* ($0\leq i<n$) of a weight λ with constant entry m on the complementary side are

$$h_i^{(1)}=(n-1)m+b_0+\lambda_{t,i}+n-1-i,\qquad h_i^{(2)}=m+b_0+\sum_{j\neq i}\lambda_{\bar t,j}+i. \tag{39.1}$$

A plan satisfies the *Hodge condition* at v if κ is unramified at v, the operators descend to a common smooth projective lower model over $\mathcal{O}_{E,(v)}$ with finite étale first projections, and for each $\tau \mid v$ some $\sigma \in \mathrm{Aut}(\mathbf{C})$ sends $\iota \circ \tau$ to t so every prescribed jump j of U at τ equals $h_i - m_{\iota\circ\tau}$ for an entry h_i of (39.1) for $\sigma(\lambda)$.

Proposition 39.13 (Geometric plans)**.** *Under the hypotheses of 39.9, there is a plan with a fibre-power tower and an algebraic character κ, whose Hecke operators T_g have prescribed eigensystem $(\theta_g)_g$ over $\bar{\mathbf{Q}}_\ell$ and commute on words applied to f. A word polynomial is a finite linear combination of operator products evaluated at θ; for every s with $s(\theta) \neq 0$, some word polynomial a_0 gives a regular test $F = a_0 s f$ with nonzero trace $D(F)$ on the definite similitude form spaces at the transported weight. Marked-root reducible geometric orbital terms and non-regular-irreducible definite orbital terms vanish.*

The plan satisfies the Hodge condition at v'; at every Hodge place the correspondences descend to a smooth proper model, κ is unramified, and at each τ above it the transported weights give the degrees $h_i = j + m_{\iota\circ\tau}$ of (39.1) and 39.26. Off finitely many usable $w \nmid \ell$, κ is unramified with correspondence roots $\kappa(w)^{-1} A_a(w, B)$ for all B; one may prescribe this root identity at a usable spherical $w \nmid \ell$, and either the Hodge condition or the root identity at $w \mid \ell$ as in 39.9.

Outline. Apply Lemma 39.11, Lemma 39.10, and Lemma 39.12 to construct the PEL datum, form Φ with character κ, and operators e, u, u'. The primary projector for $T_g e$ and a power u^n give the definite trace test, while cuspidality at w_0 kills reducible orbital terms. Integral models, weight transport, and spherical coset calculations yield the Hodge and root properties. The complete proof is on p. 1431. □

Definition 39.14 (PEL objects)**.** Fix the base datum over R with CM field F_{PEL}, order $\mathcal{O}$, and maps $t_i : \mathcal{O}_{F_{\mathrm{PEL}}} \to R$ taking the distinguished element ω to a common scalar ρ. A PEL object is an abelian scheme A/S with an $\mathcal{O}$-action. For every R-algebra R' and map $x : \operatorname{Spec} R' \to S$ over R, each tangent vector at the identity of x^*A on which ω acts by the image of ρ must be a sum of vectors on which $\mathcal{O}_{F_{\mathrm{PEL}}}$ acts through the respective maps t_i followed by $R \to R'$. At each geometric point the t_i-parts have the prescribed Lie multiplicities.

The polarization is given by a symmetric line bundle P on $A \times_S A$, trivializable on the axes and relatively ample on the diagonal. For every order element b it satisfies $(b \times 1)^* P \simeq (1 \times b^*)^* P$, where b^* is the involution of the order. The level is an $\mathcal{O}$-linear map $\eta : \Lambda/N\Lambda \to A[N](S)$ from the datum lattice modulo N. At every geometric point it must be the level-N evaluation of a Tate module isomorphism compatible with the specified Weil pairings for some compatible system of primitive roots of unity in the geometric residue field. The universal object classifies these objects by pullback up to the prescribed isomorphism.

Proposition 39.15 (The integral PEL moduli problem)**.** *Let p be a prime. Write K_0 for the CM field of the PEL datum, $\mathcal{O}$ for the order with involution $*$, an $\mathcal{O}_{K_0}$-algebra whose involution restricts to complex conjugation on $\mathcal{O}_{K_0}$, Λ for its lattice, and e_Λ for the alternating form on Λ, satisfying $e_\Lambda(bx, y) = e_\Lambda(x, b^* y)$ for all $b \in \mathcal{O}$ and $x, y \in \Lambda$. Suppose e_Λ is perfect over $\mathbf{Z}_p$ and*

$$\mathcal{O} \otimes \mathbf{Z}_p \simeq M_m(\mathcal{O}_{K_0} \otimes \mathbf{Z}_p).$$

Write $\operatorname{rank}_{\mathbf{Z}} \Lambda = mn[K_0 : \mathbf{Q}]$ with integers $m, n \geq 1$. Let $\omega \in \mathcal{O}_{K_0}$ generate an imaginary quadratic order whose defining polynomial splits into two distinct linear factors in $\mathbf{F}_p[X]$. Assume $\mathbf{R} \otimes_{\mathbf{Z}} \Lambda$ admits an $\mathcal{O}$-linear complex structure J ($J^2 = -1$) with $e_\Lambda(Jx, Jy) = e_\Lambda(x, y)$ and $e_\Lambda(Jx, x) > 0$ for $x \neq 0$, whose Lie-trace signature on the selected ω-side is $(1, n-1)$ at the b chosen archimedean places and $(0, n)$ at the remaining places (each with Morita multiplicity m). Use the polarization, Lie and level conditions of 39.14.

Let R be a Noetherian $\mathbf{Z}_{(p)}$-algebra. On the chosen side of ω, let $t_1, \ldots, t_b : \mathcal{O}_{K_0} \to R$ be the structural maps. They take ω to the same root. For $i \neq j$, the differences $t_i(c) - t_j(c)$, as c varies, generate R, and the reductions of t_i, t_j at each geometric point of characteristic p have distinct kernels in $\mathcal{O}_{K_0}$. On this side the prescribed Lie multiplicity at each t_i is m.

For $N \geq 3$ prime to p, use the objects of 39.14 with the following integral conditions. The kernel of the polarization homomorphism defined by the bundle on $A \times A$ is killed by an integer prime to p. At a geometric point the level is induced by an isomorphism of the prime-to-p Tate modules compatible with the Weil pairings for some compatible system of primitive roots of unity in the geometric residue field. Impose the Lie condition of 39.14 after every scalar extension, including extensions to rings with nilpotents. This functor is represented by a smooth quasi-projective R-scheme X_N of relative dimension $b(n-1)$.

If R is a domain of characteristic zero, X_N is projective under either of the following conditions:

$$2b < [K_0 : \mathbf{Q}], \qquad \text{or} \qquad \mathcal{O} \otimes \mathbf{Q} \text{ is a division algebra and } n < 2m.$$

Outline. Rigidifies over the Siegel scheme with a level-M frame, represents PEL data via bounded homomorphism graphs, and descends for $N \geq 3$. Proves smoothness by Dieudonné freeness and Grothendieck–Messing, and projectivity via the valuative criterion ruling out a torus. The complete proof is on p. 1434. □

Theorem 39.16 (Representability)**.** *Over a Noetherian $\mathbf{Q}$-algebra, the PEL datum of 39.15 with its structural maps and level $N \geq 3$ prime to p represents the moduli problem of 39.14 by a scheme.*

Outline. Apply 39.15 in characteristic zero. Restrict the full Tate frame away from p, and reconstruct it by adjoining dual bases at p while preserving the level sections, identifying the two moduli functors. The complete proof is on p. 1436. □

Proposition 39.17 (Fibre powers)**.** *Over a characteristic-zero field, compatible compact PEL data give smooth projective fibre powers of every nonnegative exponent with finite étale first Hecke projections of prescribed constant degree.*

Outline. Use 39.15 and 39.16 for the base and universal abelian scheme, and use exterior products of relatively ample bundles on its fibre powers. The Hecke projections come from finite étale level forgetting and the factorwise isogeny. The complete proof is on p. 1437. □

Proposition 39.18 (Integral models and the two Hecke projections at a selected place)**.** *Let w be a place selected in 39.13, p its residue characteristic, $R = \mathcal{O}_{E,(w)}$, and $q = |k(w)|$. Identify K_0 with E by the embedding in the plan and write $\mathfrak{w}$ for the inverse image of w. The lower model at w and its abelian fibre powers are smooth projective over R.*

At a root place and for $0 \leq j \leq a$, the lower Hecke correspondence on the moduli scheme has an integral model

$$X \xleftarrow{s_j} Y_j \xrightarrow{t_j} X.$$

Both arrows are finite, Y_j is flat over R, and both arrows are étale on the generic fibre. The universal isogeny has polarization multiplier q^j. On the chosen side its generic kernel is a j-dimensional $k(w)$-subspace of $A[\mathfrak{w}]$. On a fibre power the first projection is finite, is étale after inverting p, and then has degree $\#\operatorname{Gr}_j(k(w)^a)$. At a Hodge place the selected operators admit correspondences on one smooth projective lower model with finite étale first projections over R.

Outline. Use the chart of 39.13, the model of 39.15, and fibre powers as in 39.17. Parametrize root-place kernels by Grassmannians and use the tangent bound to obtain finite projections on normalization, with étale subgroup indices at Hodge places for 39.27. The complete proof is on p. 1437. □

Definition 39.19 (Total cohomology)**.** For a fibre power Z/E, use geometric pullback on the total rational cohomology $\bigoplus_j \mathbf{Q}_\ell \otimes_{\mathbf{Z}_\ell} \varprojlim_m H^j_{\text{ét}}(Z_\Omega, \mathbf{Z}/\ell^m)$ [214].

Lemma 39.20 (Correspondences)**.** *For a correspondence $Z \xleftarrow{s} Y \xrightarrow{t} Z$ with s finite étale of constant degree, set $T = \operatorname{Tr}_s t^*$, where Tr_s is the finite étale trace. If a Galois cover b of Z splits s into branches σ_j viewed as maps to Y, then $b^*Tu = \sum_j (t \circ \sigma_j)^* u$ on cohomology.*

Proof. Apply finite étale trace base change with rational ℓ-adic coefficients [214]. □

Proposition 39.21 (Eigenspaces)**.** *In 39.19 tensored with $\bar{\mathbf{Q}}_\ell$, the common eigenspace for the correspondences and eigenvalues of 39.13 is Galois-stable with finite-dimensional continuous semisimplification.*

Outline. Decompose the smooth projective geometric fibre into finitely many integral components and apply 39.34 for finite dimension, with continuity from inverse-limit cohomology. Correspondences over E commute with Galois, and a composition flag gives the semisimplification. The complete proof is on p. 1438. □

Lemma 39.22 (Ordinary points). *At an ordinary point of the integral PEL model at w, let T_j be the branch multiset indexed by j-planes of $H = k(w)^a$. For residue Frobenius F_* there are multisets A_j with $A_0 = T_0$, $A_a = 0$ and $T_{j+1} = F_* A_j + q_w^{j+1} A_{j+1}$ for $0 \le j < a$. Consequently*

$$\sum_{j=0}^{a} (-1)^j q_w^{\binom{j}{2}} F_*^{a-j} T_j = 0.$$

Let Y be the plan's integral PEL model, and let V be a valuation ring over its base with algebraically closed fraction field and residue characteristic p nonzero in V. If $y : \operatorname{Spec} V \to Y$ sends the closed point to a generic point of the special fibre, the reduced branches satisfy the same identity in the free abelian group on residue-field points.

Outline. Use deformations on the lower model of 39.18 to get generic connected height one. Planes containing the reduction kernel line yield Frobenius, and maps into the line count transverse lifts, giving a telescoping recurrence. The complete proof is on p. 1438. □

Proposition 39.23 (Annihilator). *At $w \nmid \ell$ the root identity of 39.13 gives trivial inertia and annihilator $P_{w,B}$ at inverse Frobenius on the piece of 39.21 twisted by κ_ℓ.*

Outline. On the lower model of Proposition 39.18, canonical trace weights and Lemma 39.22 give a weighted special-fibre endpoint identity. Dense vanishing yields the Frobenius relation on special-fibre cohomology, which smooth proper comparison transfers to inverse Frobenius with trivial inertia. Pulling back to the upper eigenspace and twisting by κ_ℓ via Proposition 39.13 gives $P_{w,B}$. The complete proof is on p. 1439. □

Theorem 39.24 (Holomorphic Lefschetz). *Let $Z/\mathbf{C}$ be smooth projective of dimension D and $Z \xleftarrow{c_1} Y \xrightarrow{c_2} Z$ a finite étale correspondence with finitely many transverse fixed points. At each fixed point y, put $A_y = dc_2 (dc_1)^{-1}$ on the tangent space of Z at $c_1(y) = c_2(y)$. For $j \ge 0$,*

$$\sum_{b=0}^{D} (-1)^b \operatorname{Tr}(\operatorname{Tr}_{c_1} c_2^* \mid H^b(Z, \Omega_Z^j)) = \sum_{y} \frac{[T^j] \det(1 + T A_y)}{\det(1 - A_y)}.$$

Proof. The trace along c_1 sums the pullbacks by the local branches of $c_2 \circ c_1^{-1}$. The Dolbeault trace kernel localizes where a branch meets the diagonal. At a transverse fixed point y, the local Lefschetz calculation [213] gives denominator $\det(1 - A_y)$ and fibre trace $\operatorname{tr}(\bigwedge^j A_y^*) = [T^j] \det(1 + T A_y)$. Summing over the fixed branches gives the formula. □

Lemma 39.25 (The orbital weight of a marked eigenline). *Use the odd rank a, the single indefinite place, and the regular test F of 39.13. Give a complex base fixed point of characteristic polynomial P weight $c(P, \mu)$, where μ is the eigenvalue on its positive line at the indefinite place. Use the same finite-level and stabilizer weights as for the Hecke trace. The sum of these weights is the sum over the regular irreducible definite classes with polynomial P, with their definite orbital weights multiplied by*

$$\sum_{P(\mu) = 0} c(P, \mu).$$

The roots are taken at the chosen complex embedding.

Outline. Uniformization identifies fixed points with similitudes preserving a positive line; reducible terms vanish by 39.13. For irreducible P, global reciprocity in $E[\gamma]/B^+$ twists the form to match definite orbital weights of 39.11 for each root μ. The complete proof is on p. 1441. □

Lemma 39.26 (Hodge degrees of a spectral test). *For a plan of 39.13, the fibre exponent k is uniform across conjugates of the weight. For each τ over a Hodge place, choose $\sigma \in \operatorname{Aut}(\mathbf{C})$ from the plan's weight transport; for each prescribed jump j, some datum embedding i_0 and index i in its Hodge exponent vector h of (39.1) give $h_i = j + m_{\iota\circ\tau}$, and on the fibre power Z over $\sigma \circ \iota \circ \tau$, some $H^b(Z, \Omega_Z^{h_i})$ has a nonzero common eigenvector with eigenvalues $\sigma(\iota(\theta_g))$.*

Outline. The proof interpolates fibre multiplications in 39.24, evaluates marked sums via Vandermonde quotients, and applies 39.25 to get trace $D(F)\sum_i y^{h_i}$. It uses 39.13 to isolate a nonzero ϑ-primary space and extract a common eigenvector. The complete proof is on p. 1443. □

Proposition 39.27 (Crystalline data)**.** *At $w \mid \ell$, a surjective level cover from the upper generic fibre to the smooth projective lower model of 39.18 with deck eigenvalues one gives crystallinity of the semisimplified piece ρ of 39.21. If the correspondences descend to one model with the prescribed multiplicities, a common Hodge eigenclass of degree h gives jump h at the induced embedding. Under the root identity of 39.13, write R for the correspondence root multiset before twisting. The piece is crystalline, and for each compatible period datum, prolongation and bridge there are a Weil element g of degree -1 and an integer $m \geq 0$ such that the Weil polynomial of ρ at g divides $(\prod_{\alpha\in R}(X-\alpha))^m$.*

Outline. Descending to the lower model of Proposition 39.18 embeds the piece into crystalline cohomology, and Hodge–Tate comparison with Lemma 39.26 gives jump h. Splitting the branches of Lemma 39.22 and comparing crystalline and residual Frobenius yields a shifted period relation whose shift cancels. Transporting this nilpotent Hecke polynomial to semistable periods gives the Weil polynomial divisibility. A longer outline, ending with the Lean declaration that contains the complete proof, is on p. 1445. □

39.3 Cohomological inputs

Write $H^j(Z,i) = H^j_{\text{ét}}(Z, \mathbf{Z}/b(i))$ for the Tate twist i with $b \geq 1$ invertible on the base. The notation includes simplicial presheaves.

Definition 39.28 (Cohomological core)**.** The cohomological core comprises Theorems 39.29, 39.31, and 39.32 together with Propositions 39.30 and 39.33. These results enter the proofs of generic-fibre acyclicity in Proposition 39.30, smooth projective duality in Theorem 39.34, and crystalline comparison in Proposition 39.27.

Theorem 39.29 (Affine curves)**.** *For a finite-type Dedekind k-algebra A over an algebraically closed field k, $H^j(\operatorname{Spec} A, i) = 0$ for $j \geq 2$.*

Proof. Apply the affine-curve cohomological dimension bound for Tate coefficients [214]. □

Proposition 39.30 (Generic restriction)**.** *For an absolutely integrally closed local domain T with fraction field K and an étale $T[t]$-algebra B, restriction from $\operatorname{Spec} B$ to $\operatorname{Spec}(K \otimes_T B)$ is an isomorphism in every degree and twist.*

Outline. The local-to-global criterion for $\mathbf{Z}/b(i)$ reduces the claim to a strictly henselian ind-étale $T[t]$-algebra A. Negative degrees vanish, and in degree zero connectedness together with a primitive b-th root of unity in T gives an isomorphism. In positive degrees the source is acyclic, while the generic fibre vanishes in degree one via the Kummer sequence and in higher degrees by generic-fibre vanishing. The complete proof is on p. 1452. □

Theorem 39.31 (Affine lines)**.** *Projection from the affine line over any ring induces isomorphisms in every degree and twist.*

Proof. Apply étale homotopy invariance for invertible Tate coefficients [215]. □

Theorem 39.32 (Laurent lines)**.** *For strictly henselian local A, the Laurent line $\operatorname{Spec} A[t,t^{-1}]$ has degree-one group $\mathbf{Z}/b$ additively and vanishing in degrees at least two, in every twist.*

Proof. Apply the Laurent-line formula over the given strictly henselian base [214]. □

Proposition 39.33 (Projective bundles)**.** *For a rank-N bundle V on a simplicial presheaf $\mathcal{E}$, let $\pi : \mathbf{P}(V) \to \mathcal{E}$ be its bundle of lines and L its tautological line. If b is invertible on schemes with zero-simplices, the map $(u_j)_j \mapsto \sum_j \pi^* u_j c_1(L)^j$ identifies $\bigoplus_{0\leq j<N} H^{n-2j}(\mathcal{E}, i-j)$ with $H^n(\mathbf{P}(V), i)$.*

Outline. For the trivial bundle, apply the scheme projective bundle formula to the map of complexes given by Chern-power cup products. Apply this in each simplicial degree with compatible Chern cocycles, then compare the bounded-below total complexes. The complete proof is on p. 1454. □

Theorem 39.34 (Duality). *For smooth projective integral Z over an algebraically closed field k with ℓ invertible in k, rational ℓ-adic cohomology is finite-dimensional in each degree and vanishes for $j > 2\dim Z$. The cup product followed by trace in degree $2\dim Z$ pairs complementary degrees perfectly after Tate trivialization.*

Proof. Apply smooth projective finiteness and Poincaré duality over k with rational ℓ-adic coefficients [214]. The pairing is cup product followed by the top-degree trace under the chosen Tate trivialization. □

Theorem 39.35 (Specialization). *Let Y be smooth proper over a ring O, and let V be a valuation ring over O with algebraically closed fraction and residue fields. Restriction from $Y_V = Y \times_O \operatorname{Spec} V$ to either fibre is an isomorphism for moduli invertible in the residue field. Rational specialization at such a prime ℓ respects pullbacks and compatible automorphisms.*

Proof. Apply smooth proper specialization over V [215, Exp. XVI, §2]. □

39.4 Rank extraction and congruence limits

For a dominant weight $\lambda = (\lambda_0, \dots, \lambda_{a-1})$, write $j_i = i - \lambda_i$. The ordered jumps have the stronger gaps if $j_{i+1} - j_0 > a(j_i - j_0)$ for $0 \le i < a-1$. A multiset of a nonzero roots is in general position if the following polynomial is nonzero in every enumeration.

$$\prod_{i\neq j}\prod_{h=1}^{a}(\alpha_i^h - \alpha_j^h)\prod_{j\neq i,\ j\neq k}(\alpha_i\alpha_k - \alpha_j^2)\prod_{\substack{\sum_i m_i = a,\ m_i\in\mathbf{Z}_{\ge 0}\\ (m_i)\neq(1,\dots,1)}}\left(\prod_i \alpha_i^{m_i} - \prod_i \alpha_i\right). \tag{39.2}$$

For a spherical Satake multiset B the test may be applied to either B or $A_a(w, B)$, since each factor is homogeneous.

Proposition 39.36 (Seed tests). *In the odd-rank field setting of 39.9, fix an isomorphism $\iota : \bar{\mathbf{Q}}_\ell \simeq \mathbf{C}$, separated split $w_0, v' \mid \ell$, a reference representation U, a finite set T of nonsplit places, and finite sets S, Y of places. There is a finite set Z_0 of usable places away from ℓ separated from w_0, S with the following property. Suppose U' is unramified at nonsplit places off T, has elliptic type at w_0, agrees with the weights of U at embeddings inducing places above ℓ separated from $\{w_0, v'\}$, and is spherical at v' and at split places separated from w_0, S. Suppose it has stronger gaps at an embedding inducing v' and is in general position on Z_0.*

There is a continuous semisimple $r : G_E \to \mathrm{GL}_a(\bar{\mathbf{Q}}_\ell)$ and there are a dense open set $V \subset G_E$ and a finite set X of places with the following properties. At each usable $w \nmid \ell$ separated from S and outside X, for each Satake multiset B there, r is unramified and the transported polynomial at s^{-1} is $P_{w,B}$ for every arithmetic Frobenius lift $s \in V$. At every usable $w \nmid \ell$ in Y separated from S, a general-position B gives unramifiedness and this polynomial at every inverse arithmetic Frobenius lift. At spherical $w \mid \ell$ in Y separated from w_0, v', S, r is crystalline and de Rham with the prescribed jumps. For each general-position B there, its unramified Weil polynomial at $\sigma \in W_{E_w}$ has roots $\alpha^{-\deg\sigma}$ for $\alpha \in A_a(w, B)$. The place statements hold for every prolongation and the period statements for every compatible datum and bridge.

Outline. Use the gaps and split coordinates to apply 39.9, normalize the determinant by 40.5, and extract rank by 39.37. Use degree-one Chebotarev in 36.24 to compare the witnesses and 39.38 to recover multiplicities and local data. The complete proof is on p. 1454. □

Proposition 39.37 (Rank extraction). *Let ρ be a continuous semisimple $\bar{\mathbf{Q}}_\ell$-representation of G_E which is de Rham at a prolongation of a place above ℓ. Suppose its jump multiset at one induced embedding for that prolongation contains a lacunary subset of size $a \ge 1$. Suppose that on a dense set $\rho(g)$ has a monic degree-a annihilator Q_g with $(-1)^a Q_g(0) = \eta(g)$ for a continuous character η. There is a continuous semisimple rank-a representation r of determinant η, with dense open separable locus and $r \hookrightarrow \rho \hookrightarrow r^{\oplus m}$ for some m.*

Outline. Pass the minimal-polynomial bound to the Lie algebra and use Sen's theorem and lacunarity to obtain scalar corners. Take one standard module per matrix factor and use the dense separable locus to recover characteristic polynomials and the determinant. The complete proof is on p. 1455. □

Lemma 39.38 (Multiplicities)**.** *Let S be a size-a multiset, and let P be a monic polynomial of degree a that divides a power of the root polynomial of S, with the product of its roots equal to $\prod_{\alpha\in S}\alpha$. If S is the unique size-a multiset supported in* $\operatorname{supp} S$ *with this product, then P is the root polynomial of S.*

Outline. Divisibility places the roots of P, with multiplicity, in $\operatorname{supp} S$. Their number and product are those of S, so the assumed uniqueness identifies the two multisets. The complete proof is on p. 1456. □

Lemma 39.39 (Approximation with one varying level)**.** *Let E/F be CM, G the definite unitary group of rank $a\ge 1$, and G_∞ its archimedean factor. Fix $\iota:\bar{\mathbf{Q}}_\ell\simeq\mathbf{C}$, a split $w_0\mid\ell$, an algebraic weight V, and a compact open level K away from the place below w_0, embedded in $G(\mathbf{A}_{F,f})$ with component 1 there. Let D be a finite set of split places separated from w_0 where K has hyperspecial factors. For each $w\in D$ take a nonzero polynomial $\Phi_w\in\mathbf{C}[X_1,\dots,X_a]$. A spherical system is good at w if Φ_w is nonzero on every ordering of its Satake roots.*

Let $f(v,g)$ be an algebraic weight form fixed by K, linear for $v\in V$, left invariant under $G(F)$ and smooth at finite places, with $f(v,gx)=f(xv,g)$ for $x\in G_\infty$. For every $\epsilon>0$ there is a finite sum F of K-fixed forms of weight V from representations good throughout D and a scalar function e such that

$$f(v,g)-F(v,g)=e(g)f(v,g),\qquad |\iota^{-1}e(g)|_\ell\le\epsilon\quad(v\in V,\ g\in G(\mathbf{A}_F)).\tag{39.3}$$

Outline. Use spherical Hecke polynomials of small Plancherel norm and deep-level separation to bound the exceptional span. On the finite ℓ-group, use the determinant estimate on translates of an annihilating row and iterate on the pointwise error. The complete proof is on p. 1456. □

Proposition 39.40 (Hecke approximation)**.** *Let E/F be CM, $a\ge 1$ odd, ℓ prime, and $\iota:\bar{\mathbf{Q}}_\ell\simeq\mathbf{C}$. Let T be a finite nonsplit set and U an automorphic representation of the definite unitary group of rank a, unramified at nonsplit places off T. Fix split $w_0,v'\mid\ell$ above distinct places of F. There are compact open subgroups K_w at split w separated from w_0. At each split w separated from both places, $U_w^{K_w}\neq 0$, and $K_w=\mathrm{GL}_a(\mathcal{O}_{E_w})$ if U_w is spherical.*

For finite sets Z,W of split spherical places of U separated from both places, with W above ℓ, there exist nonzero complex $c_{w,n}$ for $(w,n)\in W\times\mathbf{Z}$. For any index set J, let f_j be finitely supported local operators at split $w_j\nmid\ell$, with $|\iota^{-1}f_j(g)|_\ell\le 1$ on their coefficients, each acting by a scalar on the K_{w_j}-invariants of each smooth irreducible representation. Write θ_j for its scalar on U, transported by ι^{-1}. Augment the tuple by

$$\theta_{w,n}=\iota^{-1}\left(c_{w,n}\sum_{\alpha\in A_a(w,B_w)}\alpha^n\right),\qquad (w,n)\in W\times\mathbf{Z},$$

where B_w is the Satake multiset of U. The augmented tuple satisfies 39.41 with the augmented tuples of representations U' for the same operators and constants. Each U' is unramified at nonsplit places off T, has nonzero K_w-invariants at split places separated from the pair, and has the weights of U at embeddings inducing places separated from the pair. It has elliptic type at w_0, is spherical at v' with the stronger gaps at some embedding over v', and is in general position at every place of $Z\cup W$. If every f_j is rational-valued, a single finite extension $L/\mathbf{Q}_\ell$ contains all θ_j for $j\in J$.

Outline. Scale general-position and Laurent operators (39.39) to contract the lattice norm, and apply 39.39. Composing polynomial weight-shift and congruence elliptic-type steps gives the target tuples; finite Galois orbits bound the rational eigenvalue field. The complete proof is on p. 1458. □

Definition 39.41 (Polynomial approximation)**.** For $\vartheta\in k^J$ and $\mathcal{S}\subset k^J$ over an ultrametric field k, we say that ϑ admits polynomial approximation by $\mathcal{S}$ if for each $\delta>0$ there are a finite set $s\subset\mathcal{S}$ and weights $c_\eta\in k$ such that

$$\left|P(\vartheta)-\sum_{\eta\in s}c_\eta P(\eta)\right|\le\delta\qquad(P\in\mathbf{Z}[X_j:j\in J]).$$

Proposition 39.42 (Trace limit)**.** *Let $x : J \to \Gamma$ have dense range in a compact group and $\vartheta : J \to L$ with $[L : \mathbf{Q}_\ell] < \infty$. If the samples of a family of continuous rank-a traces over $\bar{\mathbf{Q}}_\ell$ give 39.41, then ϑ extends to the trace of a continuous semisimple rank-a representation, with 39.41 on Γ for the same family.*

Outline. Applying 39.41 to $X_i - X_j$ gives Cauchy convergence in L, extending ϑ continuously to Γ with 39.41. Testing $X_1 - a$ at the identity and taking limits of cyclic and degree-$(a+1)$ alternating trace identities lets 39.43 apply. The complete proof is on p. 1461. □

Theorem 39.43 (Pseudocharacters)**.** *Over an algebraically closed field of characteristic zero, every dimension-a group pseudocharacter is the trace of a semisimple rank-a representation, unique up to isomorphism. Over a topological coefficient field this representation is continuous if its trace is continuous.*

Proof. The characteristic-zero trace realization is the group case of Taylor [218, Thm. 1]; see also Rouquier [217, §4, especially Lemmas 4.1 and 4.3 and the discussion after Corollary 4.4]. For continuity, choose a basis of the semisimple image algebra from group elements. The trace pairing is nondegenerate, since on each matrix factor it is the ordinary trace pairing times the positive multiplicity of its standard module. Its dual basis expresses the matrix coordinates as linear combinations of translates of the trace. □

Joint approximation. Let $r : G_E \to \mathrm{GL}_a(\bar{\mathbf{Q}}_\ell)$ be continuous and semisimple, with $a \geq 1$, and let C be a set of continuous representations of the same group and rank. Fix a prolongation at $w \mid \ell$, compatible period data and a bridge for which every $\rho \in C$ is crystalline with the same jumps J_τ at every induced embedding τ. For $\sigma \in W_{E_w}$ put $s_j(\rho) = \iota^{-1}\operatorname{tr}(\mathrm{WD}_\iota(\rho_w)(\sigma)^j)$ for $j \geq 0$. Given $S_j \in \bar{\mathbf{Q}}_\ell$, joint approximation means that for every $N \geq 0$ and $\epsilon > 0$ there are a finite $H \subset C$ and weights $b_\rho \in \bar{\mathbf{Q}}_\ell$ such that

$$\begin{aligned} \left| \operatorname{tr} r(g) - \sum_{\rho \in H} b_\rho \operatorname{tr} \rho(g) \right|_\ell &\leq \epsilon \quad (g \in G_E), \\ \left| S_j - \sum_{\rho \in H} b_\rho s_j(\rho) \right|_\ell &\leq \epsilon \quad (0 \leq j \leq N). \end{aligned} \tag{39.4}$$

Proposition 39.44 (Removing gaps)**.** *Under the odd-rank field hypotheses of 39.9, take split $w_0, v' \mid \ell$ separated over F, and automorphic U unramified at nonsplit places off the finite nonsplit set T. There is a continuous semisimple $r : G_E \to \mathrm{GL}_a(\bar{\mathbf{Q}}_\ell)$. At every usable spherical $w \nmid \ell$ it is unramified and has transported polynomial $P_{w,B}$ at every inverse arithmetic Frobenius lift. At every spherical $w \mid \ell$ separated from the pair it is crystalline and de Rham with the jumps prescribed by the weight of U. For each such $w \mid \ell$, each Satake multiset B, compatible period datum, prolongation, bridge and Weil element σ, there are seeds with the common jumps and the joint approximation (39.4), with*

$$S_n = \iota^{-1}\left(\sum_{\alpha \in A_a(w,B)} \alpha^{-n \deg \sigma} \right).$$

Outline. Approximate by 39.40 and 39.36 on dense samples from 36.24, then pass 39.41 to the limit and use 39.42 to realize the trace. Use 39.43 for inertia, Newton's identities for Frobenius, and 39.48 above ℓ. The complete proof is on p. 1461. □

Lemma 39.45 (The integral action and finite de Rham congruences)**.** *Let $K/\mathbf{Q}_\ell$ be finite, with residue field k and uniformizer π. Write C for its completed algebraic closure and $G_K = \mathrm{Gal}(\bar{K}/K)$. Choose compatible ℓ-power roots $\pi^\flat$ of π. Put*

$$\mathfrak{S} = W(k)[[u]], \qquad A_{\mathrm{inf}} = W(\mathcal{O}_{C^\flat}), \qquad \mathcal{W} = W(C^\flat).$$

Embed $\mathfrak{S}$ into A_{inf} by $u \mapsto [\pi^\flat]$. Let $E \in W(k)[u]$ be the monic Eisenstein polynomial of π, and use E also for its image, which generates the kernel of $\theta : A_{\mathrm{inf}} \to \mathcal{O}_C$. Write ϕ for Witt Frobenius, with $\phi(u) = u^\ell$.

Let Λ be a finite free $\mathbf{Z}_\ell$-lattice of rank d with continuous action ρ of G_K, and suppose $\Lambda[1/\ell]$ is crystalline. Choose a basis of its Breuil–Kisin module and a basis of Λ. Let $T \in \mathrm{GL}_d(\mathcal{W})$ have as columns the images of the first basis in the second under étale comparison. Put

$$\varepsilon_g = T^{-1}\rho(g)g(T).$$

Then $\varepsilon_g - I_d \in uM_d(A_{\mathrm{inf}})$ for every $g \in G_K$. For each $j \geq 1$ there are $k_j \geq 0$ and $B \in M_d(\mathfrak{S})$ such that

$$B \equiv \ell^{k_j} I_d \pmod{E\mathfrak{S}}, \qquad \phi(\varepsilon_g)g(B) \equiv B \pmod{E^j A_{\mathrm{inf}}} \quad (g \in G_K).$$

Outline. Use the crystalline lattice theorem and Tate twists to get the identity action matrix modulo u. On the Frobenius pullback, normalize the constant isocrystal comparison to the identity modulo E and clear denominators at each finite order. The complete proof is on p. 1462. □

Lemma 39.46 (A common denominator at fixed order)**.** *Keep $K, \pi^\flat, \mathfrak{S}, A_{\mathrm{inf}}, E, \theta$ from Lemma 39.45. For each $J \geq 1$ there is an integer $c_J \geq 0$ with the following property. Let d be any rank. Take a crystalline lattice of rank d with a Breuil–Kisin comparison frame as in Lemma 39.45, and put $P_h = \phi(\varepsilon_h)$ for $h \in G_K$. Suppose that for some $k \geq 0$ a matrix $B \in M_d(\mathfrak{S})$ satisfies*

$$B \equiv \ell^k I_d \pmod{E}, \qquad P_h h(B) \equiv B \pmod{E^J A_{\mathrm{inf}}} \quad (h \in G_K).$$

Then a matrix over $\mathfrak{S}$ satisfies both congruences with c_J in place of k. The exponent is chosen before d, the matrices P_h, and k.

Outline. Use the cyclotomic periods of 39.45 to choose g with $\lambda = \theta(g(E)/E)$ of infinite order. Induct on the powers of E, bounding the denominator at step i by the valuation of $\lambda^i - 1$. The complete proof is on p. 1463. □

Lemma 39.47 (Uniform denominator for a Weil element)**.** *Use the representations, period data, bridge and Weil element of the joint approximation setup, and assume (39.4). The restriction r_w is crystalline. There are integral matrix lattices for the limit and the finite layers of seeds, with projection matrices Π_m to the limit and identity columns e, e_m. Their Weil operators have integral numerators V, V_m with a common denominator $s \in A_{\mathrm{cris}}$ invertible in B_{cris}. For each precision L some $m \geq L$ satisfies*

$$V\Pi_m \equiv \Pi_m V_m, \qquad \Pi_m e_m \equiv e \pmod{\ell^L}. \tag{39.5}$$

Here the operator on invariant columns represents the action at the fixed Weil element after extension to the crystalline period ring.

Outline. Use 39.43 and 39.48 to project seed lattices to the limit and 39.45 and 39.46 to pass de Rham congruences to it. Iterate Frobenius to construct crystalline bases and compare Weil operators with a common denominator using 39.6. The complete proof is on p. 1464. □

Proposition 39.48 (Crystalline limits)**.** *Assume the joint approximation conditions of (39.4) for r, C, w, σ and the targets S_j, with the period data, bridge and common jumps specified there. Then r_w is crystalline and de Rham with jumps J_τ at each induced embedding, and its Weil trace at σ^j is $\iota(S_j)$ for every $j \geq 0$.*

Outline. Use 39.47 for crystallinity and the element of 39.46 to encode jumps as powers of an infinite-order scalar. Recover the jumps from limiting power sums by Newton's identities and pass the weighted Weil traces to the limit using 39.47. The complete proof is on p. 1469. □

39.5 Descent to general linear groups

Lemma 39.49 (Unitary character with prescribed infinity type)**.** *Let E/F be CM and unramified at finite places, with quadratic idèle class character $\eta_{E/F}$. Given a finite set S of finite places of E and odd integers e_v at its complex places, there is a unitary Hecke character μ of E restricting to $\eta_{E/F}$ on the idèles of F, unramified on S, with*

$$\mu_w = (-1)^{\mathrm{ord}_w} \quad (w \text{ nonsplit finite}), \qquad \mu_v(z) = (z/|z|)^{e_v}.$$

The exponents are relative to the chosen embeddings at the complex places.

Outline. Prescribe the infinity and quadratic characters on the two idèle subgroups, using the auxiliary congruence and odd parity for agreement. Extend by divisibility of the unit circle and read the nonsplit values from units and a uniformizer. The complete proof is on p. 1471. □

Lemma 39.50 (The terms of the split Hecke test)**.** *Let E/F and n be as in 26.5. Let π be a regular algebraic conjugate self-dual cusp form on $\mathrm{GL}_n(\mathbf{A}_E)$, spherical at every nonsplit finite place. Choose μ as in 26.5. There exist a weight λ and a section Σ of finite places over F, with a finite set $Z \subset \Sigma$ of split places containing representatives of all pairs where π or μ ramifies. For $w \notin Z \cup \bar{Z}$, let $B(w)$ be the Satake multiset of π_w. At $w \in Z$ there are tests f_w with $\operatorname{tr} \pi_w(f_w) \neq 0$, one with regular support.*

On a finite set $T \subset \Sigma$ of split places disjoint from Z take spherical Hecke polynomials $e_{1,w}, e_{2,w}$. For any family of multisets X put

$$v(X) = \prod_{w \in T} e_{1,w}(X(w)) e_{2,w}(X(\bar{w})^{-1}),$$

where inversion is entrywise. Let f_k use the fixed tests at Z and the k-th power of the product of the two Hecke polynomials at each place in T. Use the unit elsewhere. Denote by $S_i(k)$ the stable terms of 26.5 before their signs. There are finite lists, chosen before T and the polynomials, with the following properties.

The normalized unitary trace is $\sum_h A_h v(B_h)^k$, where h ranges over automorphic summands unramified at every nonsplit place and B_h is their split Satake family. On a proper block with $a_i b_i > 0$ the list consists of pairs of cuspidal lists of ranks a_i, b_i, with Satake unions X_1, X_2. At a good place w with uniformizer ϖ_w put $X(w) = \mu_w(\varpi_w)^{b_i} X_1(w) \sqcup \mu_w(\varpi_w)^{a_i} X_2(w)$. If $|v(X)| < \rho$ for every entry, with $\rho > 0$, then

$$|S_i(k)| \leq C(1+k)^N \rho^k \qquad (k \geq 0). \tag{39.6}$$

For $(n, 0)$ the list consists of cuspidal lists of total rank n omitting π. Under the same strict inequality on that list,

$$|S_{n,0}(k) - P v(B)^k| \leq C(1+k)^N \rho^k, \qquad P = c \prod_{w \in Z} \operatorname{tr} \pi_w(f_w), \qquad c \neq 0. \tag{39.7}$$

Here c is fixed before the varying tests, whereas C, N may depend on the tests and ρ. The $(0, n)$ term equals $S_{n,0}(k)$. The competing rank-n lists, including the twisted unions from proper blocks, have regular infinitesimal characters and inversion-invariant Satake unions at all but finitely many nonsplit places.

Outline. Use the local matching in 26.5 and finiteness in 26.16 to form the Hecke lists. Spectral separation and the trace estimates of 26.11 bound competitors, leaving the same cuspidal contribution at both ends. Nonsplit tests enforce inversion invariance. The complete proof is on p. 1472. □

Lemma 39.51 (Weak descent)**.** *Suppose E/F is CM and unramified at finite places. A regular algebraic conjugate self-dual cuspidal π of positive rank, spherical at every nonsplit finite place, has a definite-unitary automorphic match at almost every split place, unramified at every nonsplit finite place.*

Outline. Choose μ by 39.49 and use split Hecke tests on the lists of 39.50 to isolate the nonzero end contributions in 26.5. The Jacquet–Shalika bound makes the factors of a matching list unitary, and Rankin–Selberg orthogonality forces one to be π. The complete proof is on p. 1474. □

Proposition 39.52 (Unitary descent)**.** *For $\chi = \eta_{E/F}^a$, where $\eta_{E/F}$ is the quadratic idèle class character of E/F, assume E contains an imaginary quadratic subfield and $[F : \mathbf{Q}]$ is even. Suppose every nonsplit finite place lies away from ℓ and is unramified over F. Assume π is spherical at these places. Given a finite set Z, there is a definite-unitary automorphic U matching π at split places, with weights giving its infinitesimal characters. At nonsplit places U is unramified off a finite nonsplit set T disjoint from Z.*

Outline. Take the weak match by 39.51 and apply Labesse's base change. The Jacquet–Shalika bound forces every Speh length to be one, and the split Rankin–Selberg argument of 39.51 identifies the transfer with π, giving full split and infinitesimal matching. The complete proof is on p. 1475. □

Lemma 39.53 (A ramified place for quadratic base change)**.** *Let M/K be quadratic with K totally complex and let π be cuspidal of rank $a \geq 1$ over K. If M/K ramifies over a spherical place of π, there is a cuspidal lift. At $w \mid v$ it has multiset $\{\alpha^{f(w/v)} : \alpha \in B_v\}$ whenever B_v exists, with residue degree $f(w/v)$. Infinitesimal characters pull back by restriction of embeddings. Local components agree under continuous completion isomorphisms extending $K \to M$.*

Proof. Twisting by the quadratic character changes the unramified cuspidal support at the designated place, so π has no quadratic self-twist. Prime-degree cyclic base change gives the cuspidal lift, and the local norm raises unramified parameters to the residue degree [93, Ch. III, Thms. 4.2 and 5.1]. □

Proposition 39.54 (Spherical theorem)**.** *Call a continuous semisimple $r : G_E \to \mathrm{GL}_a(\bar{\mathbf{Q}}_\ell)$ weakly attached to π if outside a finite set of w there is a Satake multiset B_w such that inertia acts trivially and every inverse arithmetic Frobenius has polynomial P_{w,B_w} after applying ι. Every polarized input has a weakly attached r. For every such r and every Satake multiset B at a spherical $w \nmid \ell$, inertia acts trivially and the inverse-Frobenius polynomial is $P_{w,B}$. At a spherical $w \mid \ell$, r_w is crystalline and de Rham with the prescribed jumps. For every compatible period datum and bridge its Weil polynomial at $h \in W_{E_w}$ is $\prod_{\alpha \in A_a(w,B)}(X - \alpha^{-\deg h})$ for every Satake multiset B there.*

Outline. For quadratic polarization over split fields, applies 39.52, 39.44, and 39.48 in odd rank and the even-rank case, descending via 39.53 and index-two patching. Reduces general polarization by an algebraic Hecke twist from 39.7. The complete proof is on p. 1476. □

Lemma 39.55 (Exponents of unlinked characters)**.** *Let (π, χ) be a polarized input of rank a, $K = E_w$, $\nu_K = |\cdot|_K$, and $s = (1-a)/2$. Suppose the cuspidal support of $\sigma^* = \pi_w \otimes |\det|_K^s$ consists of unlinked smooth characters $\xi_1, \dots, \xi_a$. For characters, unlinking means $\xi_i \neq \xi_j \nu_K$ for every i, j. With the weight of 39.4 there are real numbers e_j such that*

$$|e_j| < \tfrac{1}{2}, \qquad |\xi_j(z)| = \nu_K(z)^{e_j - w_{\mathrm{gt}}/2} \quad (z \in K^\times). \tag{39.8}$$

Outline. The central-character exponent $t = (a - 1 - w_{\mathrm{gt}})/2$ makes $\sigma^* |\det|^{w_{\mathrm{gt}}/2}$ unitarizable and generic. In the generic unitary dual, blocks have twists $|e| < 1/2$; unlinking forces length one, characters force rank one, and removing the twist gives the bound. The complete proof is on p. 1477. □

Lemma 39.56 (The support polynomial in the local functional equation)**.** *Let k be a nonarchimedean local field of characteristic zero, q its residue cardinality, and ϖ a uniformizer. Let τ be an irreducible generic representation of $\mathrm{GL}_n(k)$, S its multiset of supercuspidal support for normalized induction, and λ a smooth character of $k^\times$. Define*

$$D_k(S, \lambda; T) = \prod_{\substack{\mu \in S,\ \mathrm{rank}\,\mu = 1 \\ (\mu\lambda)|_{\mathcal{O}_k^\times} = 1}} \big(1 - (\mu\lambda)(\varpi)T\big).$$

For any nontrivial additive character of k, the standard local γ-factor of $\tau \otimes (\lambda \circ \det)$, in the variable $T = q^{-s}$, has the form

$$cT^d \, \frac{D_k(S, \lambda; T)}{D_k(S, \lambda; qT)} \qquad (c \in \mathbf{C}^\times,\ d \in \mathbf{Z}).$$

Outline. Multiply the local gamma factors over the twisted support. Higher-rank supercuspidals and ramified characters give epsilon monomials, and Tate's functional equation gives the quotient factors for unramified characters. The complete proof is on p. 1477. □

Lemma 39.57 (A spherical lift fixing a prescribed Weil element)**.** *Assume the hypotheses of 39.55 and let r be weakly attached to π. Fix a prolongation of w to Ω and the reciprocity map $b_K : W_K \to K^\times$ of 40.5. For each $g \in W_K$ of nonzero degree there exist a finite CM extension M/E in Ω, fixed pointwise by g, and a polarized cuspidal lift Π of π. Then $r|_{G_M}$ is weakly attached to Π. At the place u determined by the prolongation put $L = M_u$. The component Π_u has a Satake multiset B such that for every $h \in W_L \subset W_K$,*

$$\prod_{\alpha \in A_a(u,B)} (X - \alpha^{-\deg_L h}) = \prod_{j=1}^{a} (X - \xi_j(b_K(h))).$$

In particular the formula applies to g.

Outline. Use reciprocity and Grunwald–Wang for a CM tower that unramifies the characters and excludes self-twists by trace nonvanishing. Compare functional equations by 39.56 for support multiplicities and use the window of 39.55 for spherical induction. The complete proof is on p. 1478. □

Proposition 39.58 (Comparison on unlinked character support). *With r, σ^*, ξ_j as in 39.57, put $Y = \bigoplus_{j=1}^{a}(\xi_j \circ b_K, 0)$. If $w \nmid \ell$, then $\mathrm{WD}_\iota(r_w) = Y$. If $w \mid \ell$, then r_w is potentially semistable and de Rham with the jumps of 39.4 at every induced embedding, and its Frobenius-semisimplified period parameter for every compatible datum and bridge equals Y. In both cases $R(\sigma^*) = Y$ for every map R of 40.2.*

Outline. Use 39.57 and 39.54 to compare Weil polynomials and descend zero monodromy and the jumps. Recover inertia traces by Cayley–Hamilton and identify the Weil action by Brauer–Nesbitt and the parameter by 40.5 and 40.2. The complete proof is on p. 1480. □

Lemma 39.59 (An unramified extension for depth-zero support). *Let $(\pi, \chi), r, w$ be as in 39.54 and fix a prolongation. Suppose $K = E_w \simeq \mathbf{Q}_p$ continuously. Suppose π_w has first principal congruence invariants and unlinked cuspidal support, with its terms of rank at least two pairwise inequivalent up to unramified twist. Given any R of 40.2, there is a CM tower M/E with a polarized cuspidal lift Π weakly attached to $r|_{G_M}$. Its completion $L = M_u$ at the prolongation is unramified of degree $f = a!$ over K. The normalized component $\Pi_u^* = \Pi_u |\det|_L^{(1-a)/2}$ has unlinked character support. There is an upper map R' of 40.2 such that for every $t \in I_K = I_L$,*

$$\det(X - \rho_{R'(\Pi_u^*)}(t)) = \det(X - \rho_{R(\pi_w |\det|_K^{(1-a)/2})}(t)).$$

The infinitesimal character of Π at an embedding of M is that of π at its restriction to E.

Outline. Use a cyclic tower as in 39.57 and split Green orbits by Shintani matching with the labels of 32.31 and 40.6. Compute inertia by 40.4 and 40.2 and compare with the lifted characters by 40.1. The complete proof is on p. 1480. □

39.6 Two theta lifts and even rank

Proposition 39.60 (Stable range). *Let E/F be CM unramified at finite places with an imaginary quadratic subfield, and let its inert places have odd residue characteristic and norm greater than three. Choose a complex embedding of E above each real place of F and an F-embedding $E \hookrightarrow F_v$ for each v in a designated finite set P of split finite places. Use an additive character trivial on F and equal to $\exp(-2\pi i \operatorname{Tr}(x))$ at infinity. Require it to be unramified at P and the inert places. Equip the definite Hermitian source X with matrix 1_n, $n \geq 1$, and a self-dual lattice. Choose compatible archimedean Cartan data and split bases at P. There is a bound d_0 for definite skew-Hermitian targets W with matrix $\delta 1_d$, where $\bar{\delta} = -\delta$ and $\delta \neq 0$. Equip the target with a lattice self-dual for the trace pairing and choose compatible archimedean Cartan data and split bases at P.*

For $d \geq d_0$ use splitting characters χ_X, χ_W restricting to the quadratic powers n, d with infinity exponents m_X, m_W. A definite-unitary automorphic U with $2\lambda_{v,k} \geq d + m_W(v)$ for $0 \leq k < n$ has a lift U' of weights

$$2\lambda'_{v,k} = n + m_X(v) + \begin{cases} 2\lambda_{v,k} - d - m_W(v) & k < n, \\ 0 & n \leq k < d. \end{cases}$$

For every finite place v and every F-embedding $t : E \hookrightarrow F_v$, the components τ, τ' of U, U' at the place determined by t admit a nonzero pairing $\beta : \mathcal{S}(M_{n,d}(F_v)) \times \tau \to \tau'$. It is linear in the first slot and conjugate-linear in the second, with $\beta(\omega(g,h)\Phi, \tau(g)u) = \tau'(h)\beta(\Phi, u)$ for the split Weil action ω.

Outline. Use the doubling embedding and determinant bounds to control height-layer volumes and get integrability at sufficiently high powers. Pad the partition and restore the determinant shift, apply 39.61 for nonvanishing, and project to obtain the split local pairings. The complete proof is on p. 1482. □

Lemma 39.61 (Integral criterion). *For the unitary pair and splitting data of Proposition 39.60, write $[G] = \mathrm{U}(X)(F)\backslash \mathrm{U}(X)(\mathbf{A}_F)$. Let C_Φ be the Weil matrix coefficient of a polynomial times finite Schwartz vector Φ, and M_φ the right-translation matrix coefficient of a continuous form φ on $[G]$. Suppose the doubling height satisfies $\int_{\mathrm{U}(X)(\mathbf{A}_F)} h^t < \infty$ for $t \geq t_0$, with $t_0 < d/2$. Then $I = \int C_\Phi \overline{M_\varphi}$ converges absolutely and vanishes for a zero lift. If $d \geq 2n$ and intertwiners from one finite-dimensional irreducible archimedean module take a common vector to nonzero polynomial f_0 and nonzero φ in irreducible automorphic U, some finite Schwartz data give a nonzero lift.*

Outline. Apply 34.16 and unfold to a matrix-coefficient integral, using the height bound to reach the theta point. Use translated finite vacua and Schur orthogonality so the positive integral dominates the height tail. The complete proof is on p. 1483. □

Lemma 39.62 (Split spherical parameters of a lift)**.** *At a split place w, suppose U, U' of ranks $n, n+t$ have the local pairing of 39.60, with $t > n$ and unramified unitary splitting characters. Write $x = \chi_X(w)$, $y = \chi_W(w)$, $q = q_w$. A Satake multiset B of U gives the Satake multiset*

$$(xy^{-1})B \sqcup \{xq^{(t-1)/2-j} : 0 \le j < t\}$$

of U' at the same place.

Outline. Filter the split Schwartz model by rank and use strict contraction to force the pairing onto the full-rank orbit. Apply Frobenius reciprocity at its stabilizer and take the contragredient to obtain the twisted Satake multiset and the determinant-character string. The complete proof is on p. 1484. □

Proposition 39.63 (Two lifts)**.** *Under the field conditions of 39.60, with $[F : \mathbf{Q}]$ even and inert $v \nmid \mathfrak{D}_{F/\mathbf{Q}}$, take definite-unitary automorphic U of positive even rank n with weights λ_τ at every embedding and a split place w_0. For some odd t there are two lifts U_i of rank $n+t$ and algebraic characters κ, ψ_{ij} with disjoint ψ-families and κ, ψ_{1j} unramified at w_0. At split w where κ and the i-th family are unramified, each Satake multiset B of U admits B_i with*

$$A_{n+t}(w, B_i) = \kappa(w)^{-1} A_n(w, B) \sqcup \{\psi_{ij}(w)\}_{0 \le j < t}.$$

For infinity exponents $m_\tau, p_{ij,\tau}$, the jumps of U_i are $(J(\lambda_\tau) + m_\tau) \sqcup \{-p_{ij,\tau}\}_j$.

Outline. Choose odd excess rank by 39.60 and an unramified character by 39.49. Use two pairs of powers with the same quotient in 39.60, rescale 39.62 to get the two added families, and use infinity exponents to separate them and compute jumps. The complete proof is on p. 1485. □

Lemma 39.64 (Frobenius identity)**.** *Let Q be split places with dense inverse Frobenius away from ℓ after every finite removal. Suppose continuous equal-rank r_i and continuous characters χ_{ij} are cofinitely unramified there, with inverse-Frobenius roots of r_i a common multiset plus its own family's values. Then for all $g \in G_E$,*

$$\det(X - r_1(g)) \prod_j (X - \chi_{2j}(g)) = \det(X - r_2(g)) \prod_j (X - \chi_{1j}(g)).$$

Proof. At each inverse Frobenius in the dense set both sides multiply the common polynomial by both line products. Apply coefficientwise continuity. □

Lemma 39.65 (Line subtraction)**.** *Under 39.64 with semisimple r_i and disjoint families, $r_1 \simeq s \oplus \bigoplus_j \chi_{1j}$. For line roots c_j and jumps $m_{j,\tau}$, multisets $S \sqcup \{c_j\}_j$ and $J_\tau \sqcup \{m_{j,\tau}\}_j$ for r_1 become S, J_τ for s. At $w \mid \ell$ suppose r_1 and the lines are crystalline and their Weil polynomials at each g use their respective roots raised to $-\deg g$. Then s is crystalline with polynomial $\prod_{\alpha \in S}(X - \alpha^{-\deg g})$.*

Proof. Brauer–Nesbitt identifies the enlarged sums in 39.64. On the second side the first-family lines project trivially to the second-family lines, so on the first side they occur in r_1 with at least their listed multiplicities. Take their semisimple complement s. A direct summand of a crystalline representation is crystalline. Cancel the line factors in the Frobenius and Weil polynomials and subtract their jumps by additivity. □

Proposition 39.66 (One place)**.** *Assume the density of 39.64 and odd-rank spherical compatibility on Q. Suppose there are two lifts of rank-n U as in 39.63 at $w_0 \in Q$. There is a rank-n representation r weakly compatible on Q. At all but finitely many $w \in Q$ away from ℓ, it is unramified and has polynomial $P_{w,B}$ for every Satake multiset B there. For every Satake multiset B of U at w_0 the following local assertions hold. If $w_0 \nmid \ell$, r is unramified there with polynomial $P_{w_0,B}$. If $w_0 \mid \ell$, it is crystalline de Rham with jumps $J(\lambda_{\iota\circ\tau})$ at induced τ and the Weil polynomials of 39.65 for $S = A_n(w_0, B)$ and every compatible period datum.*

Outline. Apply odd-rank compatibility to both lifts, using injectivity of character realization to keep line families disjoint and cofinite Satake unions for 39.64. Combining 39.63 at w_0, 39.65 subtraction, and κ_ℓ-twisting gives r and compatibility. The complete proof is on p. 1485. □

Proposition 39.67 (All places)**.** *With 39.66 available at every place of Q, weakly compatible representations exist and all have its local data at every spherical place in Q.*

Proof. Take the witness of 39.66 at the target. Remove the exceptional sets of the witness and the given weakly compatible representation. Also remove the finitely many split places where U has no Satake multiset. Use the same multiset for both representations at every remaining place. Density and Brauer–Nesbitt identify them. □

Chapter 40

A weak local Langlands correspondence

Let $K = E_w$ be a finite completion of a number field E, with residue field $\mathbf{F}_q$ of characteristic p. Fix a valuation ring A over w in an algebraic closure Ω of E. The Weil group W_K has inertia I_K and wild inertia P_K. The degree map $\deg : W_K \to \mathbf{Z}$ sends arithmetic Frobenius to 1. Write $\mathcal{I}(K)$ for the smooth irreducible complex classes x of $\mathrm{GL}_{d(x)}(K)$, $d(x) \geq 0$. Cuspidal means supercuspidal of positive rank. In rank n, put $K_0 = \mathrm{GL}_n(\mathcal{O}_K)$ and $K_1 = \ker(K_0 \to \mathrm{GL}_n(\mathbf{F}_q))$.

The complex Weil–Deligne classes $V = (\rho_V, N_V)$ in $\mathcal{W}(K)$ have elementwise semisimple Weil action and open inertia kernel. The nilpotent N_V satisfies $\rho_V(\sigma) N_V \rho_V(\sigma)^{-1} = q^{\deg \sigma} N_V$. Irreducibility refers to ρ_V, and tameness to trivial P_K-action.

Put $\nu_K = |\cdot|_K$ and $\nu_W = q^{\deg}$. Twists of x use the determinant. Write $V(s) = V \otimes \nu_W^s$ for $s \in \mathbf{R}$. The self-dual Sp_m has one monodromy block and Weil characters $\nu_W^{j-(m-1)/2}$, $0 \leq j < m$.

40.1 The existence theorem and its laws

Theorem 40.1 (Existence)**.** *There is a map* $\mathrm{rec}_T : \mathcal{I}(K) \to \mathcal{W}(K)$ *satisfying Definition 40.2.*

Outline. Extend 40.8 by 40.9. The cuspidal product gives the degree and determinant laws of 40.2, block operations give twist and duality on good classes (40.3), indecomposable blocks recover good segments, and compact induction with exactness gives tameness and the type law. The complete proof is on p. 1486. □

Definition 40.2 (The weak correspondence)**.** A map $R : \mathcal{I}(K) \to \mathcal{W}(K)$ is a *weak local Langlands correspondence* if there exists a local reciprocity map $a : W_K \to K^\times$ as in Theorem 40.5 for which the following laws hold. For every x, we have $\dim R(x) = d(x)$ and $R(x\nu_K^s) = R(x)(s)$ for all $s \in \mathbf{R}$. In rank one, $R(\chi) = (\chi \circ a, 0)$ for every smooth character χ of $K^\times$. For the Langlands multisegment $\mathfrak{m}$ of x,

$$R(x) = \bigoplus_{[u,m] \in \mathfrak{m}} R(u)((m-1)/2) \otimes \mathrm{Sp}_m \,. \tag{40.1}$$

Cuspidal parameters are irreducible. For every x, $|\det \rho_{R(x)}| = |\omega_x \circ a|$. For good x and smooth χ,

$$R(x \otimes \chi) = R(x) \otimes (\chi \circ a), \qquad \det \rho_{R(x)} = \omega_x \circ a, \qquad R(x^\vee) = R(x)^\vee.$$

If $R(x) = R(y)$ and y is good, then $x = y$. Good classes with nonzero K_1-invariants have tame parameters. The map satisfies the type law of 40.4.

The full correspondence combines Scholze's cuspidal bijection [229, Thm. 1.2(c–d)] with Rodier's multisegment extension [230, §4.4].

Definition 40.3 (Good cuspidal classes and good support)**.** Depth zero means nonzero K_1-invariants. Let $\mathcal{C}_K \subseteq \mathcal{I}(K)$ consist of the rank-one classes and, when $K \simeq \mathbf{Q}_p$ topologically, all classes $u_0 \otimes (\chi \circ \det)$ with $u_0 \in \mathcal{I}(K)$ of depth zero and χ a smooth character of $K^\times$. A class is *good*, or has *good cuspidal support*, if every term of its cuspidal support belongs to $\mathcal{C}_K$. For a cuspidal class u, this means $u \in \mathcal{C}_K$, since its support is $\{u\}$.

Definition 40.4 (The depth-zero type law)**.** Fix a continuous isomorphism $i : \mathbf{Q}_p \simeq K$ and $n \geq 1$. Take a finite extension E'/E in Ω with place w' induced by A, $p \notin (w')^2$ and $Nw' = p^n$. Let a' be reciprocity at w' and $\mathrm{red} : \mathcal{O}_{E'} \twoheadrightarrow k_n$ reduction with kernel w', where k_n is the residue field of order p^n. Take a character $\theta : k_n^\times \to \mathbf{C}^\times$ of Frobenius orbit length n and a cuspidal irreducible τ of $\mathrm{GL}_n(\mathbf{F}_p)$ with

$$\operatorname{tr} \tau(g) = (-1)^{n-1} \sum_{j=0}^{n-1} \theta(z^{p^j}) \tag{40.2}$$

when g has irreducible characteristic polynomial with root $z \in k_n^\times$. If $\pi \in \mathcal{I}(K)$ of rank n contains the inflation of τ via i on $\mathrm{GL}_n(\mathbf{Z}_p)$, then $N_{R(\pi)} = 0$. For $t \in I_{E'_{w'}}$, let t_E be its image in I_K. For $b \in \mathcal{O}_{E'}$ with $|a'(t) - b|_{w'} < 1$ and $z = \mathrm{red}(b) \in k_n^\times$,

$$\det(X - \rho_{R(\pi)}(t_E)) = \prod_{j=0}^{n-1} (X - \theta(z^{p^j})). \tag{40.3}$$

Formula (40.2) is Green's elliptic character formula [20]. The law uses the cuspidal case over $\mathbf{Q}_p$ of the depth-zero inertia formula [220, Addendum (A.1)].

40.2 Cuspidal parameters

Theorem 40.5 (Local reciprocity)**.** *There is a unique surjective homomorphism $a : W_K \to K^\times$ with the following properties. Its restriction to I_K is continuous, and $\mathrm{ord}_K(a(\sigma)) = -\deg \sigma$ for all σ. Every smooth Weil character factors as $\chi \circ a$ with χ smooth. For every geometrically normalized global Artin map $\mathrm{Art}_E : \mathbf{A}_E^\times \to \mathrm{Gal}(\Omega/E)^{\mathrm{ab}}$ (sending a uniformizer at each finite place to the inverse arithmetic Frobenius coset, as in Definition 1.15) and $\sigma \in W_K$,*

$$\mathrm{Art}_E(\mathrm{loc}_w(a(\sigma))) = \bar{\sigma},$$

where loc_w is the local idèle inclusion and $\bar{\sigma}$ is the abelianized image.

Outline. Existence is local reciprocity in this Frobenius convention. For injectivity, separate u by a character of a finite quotient and extend it to an idèle class character by Grunwald–Wang. The complete proof is on p. 1486. □

Theorem 40.6 (Parameters on the good cuspidal classes)**.** *Fix the local reciprocity map $a : W_K \to K^\times$ of Theorem 40.5. Over $K \simeq \mathbf{Q}_p$ continuously, each depth-zero cuspidal u has a unique irreducible tame parameter V_u of dimension $d(u)$ satisfying the type law for u and $\det \rho_{V_u} = \omega_u \circ a$. The map $u \mapsto V_u$ is injective. The formulas*

$$r_0(\chi) = (\chi \circ a, 0), \qquad r_0(u_0 \otimes \chi) = V_{u_0} \otimes (\chi \circ a)$$

define an assignment on good cuspidals that commutes with smooth twists. Here χ is smooth. In the second formula $K \simeq \mathbf{Q}_p$ continuously and u_0 is depth-zero cuspidal.

Outline. Compact induction and Green's formula 32.31 give the type and central character. In 40.4, reciprocity gives the inertia orbit, its Frobenius determinant fixes the cycle, Newton's identities give the type law, and the label and central character track twists. The complete proof is on p. 1486. □

Lemma 40.7 (Auxiliary irreducible parameters)**.** *For $n \geq 2$ there is an irreducible J_n of dimension n with unitary determinant. If $K \simeq \mathbf{Q}_p$, one can require $J_n \neq V \otimes \xi$ for every tame parameter V and smooth Weil character ξ.*

Outline. Reciprocity on principal units and the trace pairing give a full Frobenius orbit of inertia characters. A determinant-one Frobenius cycle gives a finite-image irreducible parameter, and its nonscalar wild inertia action rules out all character twists of tame parameters. The complete proof is on p. 1487. □

Proposition 40.8 (The assignment on all cuspidal classes)**.** *Fix J_n as in 40.7. For cuspidal u of rank n, define $e(u) \in \mathbf{R}$ by $|\omega_u| = \nu_K^{ne(u)}$. The assignment*

$$r(u) = \begin{cases} r_0(u), & u \in \mathcal{C}_K, \\ J_{d(u)}(e(u)), & \text{otherwise} \end{cases} \tag{40.4}$$

satisfies the cuspidal laws of Definition 40.2. In particular, $r(u) = r(v)$ with v good forces $u = v$.

Outline. Real twisting and 40.7 give the second branch its laws, and wild inertia separates the branches. On good classes, 40.6 gives the higher-rank laws and injectivity, inversion gives duality, and reciprocity gives rank-one tameness and injectivity. The complete proof is on p. 1487. □

40.3 Multisegments and uniqueness

Theorem 40.9 (Langlands extension)**.** *Each x has a unique Langlands multisegment. A segment $[u, m]$ consists of $u\nu_K^j$, $0 \le j < m$, with u cuspidal and $m \ge 1$. Formula (40.1), using $r(u)$ for each first term, extends any cuspidal assignment r. The empty multisegment gives zero.*

Outline. Order the segments of Rodier's parametrization so that none precedes a later one. The normalized induction of the segment quotients then has x as its unique irreducible quotient, and Zelevinsky gives the submodule form. The complete proof is on p. 1488. □

Theorem 40.10 (Uniqueness on good representations)**.** *Let R, R' satisfy 40.2. Then $R(x) = R'(x)$ for every good x.*

Proof. By 40.5 the maps agree on characters. Over $K \simeq \mathbf{Q}_p$ continuously they agree on depth-zero cuspidals by 40.6 and on their twists by 40.2. Apply (40.1) to the first terms of good x. □

A cuspidal support s is unlinked if $u \ne v\nu_K$ for every $u, v \in s$. Two terms are inertially equivalent if one is an unramified character twist of the other.

Lemma 40.11 (Support under the local tests)**.** *Let x be generic with $N_{R(x)} = 0$ for a map R as in 40.2. If every Weil commutator has polynomial $(X-1)^{d(x)}$ on $R(x)$, then x has unlinked rank-one support. If $K \simeq \mathbf{Q}_p$ continuously and P_K acts trivially on $R(x)$, then x has unlinked depth-zero support and $x^{K_1} \ne 0$. In this second case, assume also that for each lift $\gamma \in I_K$ of a topological generator of I_K/P_K, every repeated root z of $\det(X - \rho_{R(x)}(\gamma))$ satisfies $z^{q-1} = 1$. Then the support terms of rank at least two are pairwise inertially inequivalent. Both support conclusions give goodness by 40.3.*

Outline. Genericity and $N_{R(x)} = 0$ give an unlinked length-one multisegment, where the commutator test forces rank one. In the tame case, counting orbits against cuspidals from Theorem 40.6 gives depth-zero support, and Proposition 26.25 lifts Jacquet invariants to $x^{K_1} \ne 0$. Under the repeated-root test, inertially equivalent terms have abelian Weil image and rank one, and Definition 40.3 gives goodness. The complete proof is on p. 1488. □

In 39.8 put $x = \pi_w \nu_K^{(1-d(\pi_w))/2}$ with $N_{R(x)} = 0$. Whenever 40.11 gives good support, 40.10 gives $R'(x) = R(x)$ for any weak correspondence R'. For unlinked character support, 39.58 gives equality with the Galois parameter. Over $K \simeq \mathbf{Q}_p$ continuously, the wild-inertia and repeated-root tests give K_1-invariants and unlinked support with pairwise inertially inequivalent terms of rank at least two. The normalizing twist preserves these properties. Apply 39.58 on the unramified lift of 39.59. Restriction keeps the monodromy operator and inertia group, so the Galois parameter downstairs has zero monodromy and the inertia polynomials of $R(x)$.

For any $x \in \mathcal{I}(K)$, suppose the inertia polynomials of $R(x)$ agree with those of a sum of $d(x)$ Weil characters. For $s, t \in W_K$, the commutator $[s, t]$ belongs to I_K, and every one-dimensional character takes value 1 at $[s, t]$. By (A.618) (in the proof of Theorem 40.1) and semisimplicity, each commutator acts trivially on each cuspidal parameter. Irreducibility gives rank-one support, so 40.10 applies.

Part X

Appendix: proofs

This appendix gives the proofs that the main text summarises. Each is headed by the statement it proves.

Proofs for Chapter 1: Five statements and the critical line

Proof of 1.16

Proof. For rational $t > r$ choose c with $v(c) = t$ by taking a root of a power of ℓ. The hth elementary symmetric function S_h of $(cH_i)^{12}$ has weight $12hw$ for the full modular group. Translation by one eliminates fractional powers of q. The modular function S_h/Δ^{hw} is holomorphic on the upper half-plane and has pole order at most hw at the cusp. Its Laurent coefficients lie in R, since Δ/q has integral coefficients and constant term one. Cancel the negative powers in succession by subtracting multiples of powers of j. The multipliers lie in R since j has leading coefficient one and integral coefficients. The remaining holomorphic function on $X(1)$ is its constant coefficient. Thus $S_h/\Delta^{hw} \in R[j]$.

At τ, the numbers $(cH_i(\tau)/\omega^w)^{12}$ are roots of a monic polynomial over R. A root of negative valuation would make the leading term the unique term of least valuation. Thus $v(H_i(\tau)/\omega^w) \geq -t$. Let t decrease to r. □

Proof of 1.17

Proof. Put $X(P) = \wp_\Lambda(P)/\Omega^2$. By homothety take $\Lambda = \Lambda_\tau$ and $\omega = \Omega/p$. Since the curve has CM, $j(\tau)$ is an algebraic integer [7]. Write $P = (a\tau + b)/N$ with $0 \leq a, b < N$ and $\zeta = \exp(2\pi i_{\mathbf{C}} b/N)$. In the partial fractions for $\wp$ expand $z/(1-z)^2 = \sum_{h\geq 1} hz^h$. Then $(G_2 + \wp_{\Lambda_\tau}(P))/p^2$ has constant term zero for $a \neq 0$ and $\zeta/(1-\zeta)^2$ for $a = 0$. Its coefficient at $q^{u/N}$, $u > 0$, is

$$\sum_{h|u} h\left(\mathbf{1}_{u/h\equiv a\ (N)}\zeta^h + \mathbf{1}_{u/h\equiv -a\ (N)}\zeta^{-h}\right). \tag{A.1}$$

When $a = 0$ one has $\zeta \neq 1$ and $N/(1-\zeta) = -\sum_{h=0}^{N-1} h\zeta^h$. Thus all coefficients are integral at ℓ if $\ell \nmid N$. The series G_2/p^2 has constant term $-1/12$ and coefficient $2\sum_{h|j} h$ at q^j, $j > 0$. The families $12\wp(P)/p^2$ and $(\wp(P) - \wp(P'))/p^2$ are permuted in weight two by changes of period basis at a common level prime to ℓ. Lemma 1.16 therefore gives

$$12X(P) \in R, \qquad X(P) - X(P') \in R \tag{A.2}$$

for all such nonzero points P, P'.

For a nonzero multiplier α of Λ put $h = \alpha\bar{\alpha}$. Cancel the poles in $\alpha^2\wp_\Lambda(\alpha z) - \sum_{T\in\ker\alpha}\wp_\Lambda(z+T)$. The resulting constant is $-\sum_{T\neq 0}\wp_\Lambda(T)$ by comparison at $z = 0$. Integrate using $\zeta'_\Lambda = -\wp_\Lambda$ and take the increment by a period u. The coefficients of $\bar{u}$ cancel and the coefficients of u give

$$(\alpha^2 - h)s_2(\Lambda) = \sum_{0\neq T\in\ker\alpha} \wp_\Lambda(T).$$

When $\ell \nmid h$ the kernel points are killed by h, so

$$(\alpha^2 - h)U - (\alpha^2 - 1)X(P) = \sum_{T\neq 0}\big(X(T) - X(P)\big) \in R. \tag{A.3}$$

In each case $\alpha^2 - h \neq 0$, which gives algebraicity of U.

For $\ell \neq d$ take $\alpha = w_0$ to get $-2dU + (d+1)X(P) \in R$. This suffices outside $6d$. At 2, $v(X(P)) \geq -2$ and $v(d+1) \geq 2$ for $d \equiv 3 \pmod 4$, while $v(d+1) = 1$ for $d \equiv 1 \pmod 4$. At 3 the same equation suffices if $3 \mid d+1$. Otherwise $d \equiv 1 \pmod 3$. Use $\alpha_\pm = 1 \pm w_0$ of norm $1+d$. Multiply each relation by $\alpha_\mp^2 - 1$, subtract and multiply by w_0 to get $-4d^2U \in R$. At $\ell = d$ use $\alpha = 1 + w_0$. Now 12 and $1+d$ are units and $\alpha^2 - (1+d) = 2w_0(1+w_0)$ has valuation $1/2$. □

Proof of 1.18

Proof. Put $\theta = (-1+w)/2$ if $d \equiv 3 \pmod 4$ and $\theta = w$ otherwise. Then $\tau = \phi_i(\theta)$ is independent of i and $\Lambda = \mathbf{Z}\tau + \mathbf{Z}$. Set $D = -(\theta - \bar\theta)^2$, equal to d or $4d$, and $h = v_\ell(D)$. Write $\sigma_i = \phi_i|_F$.

Choose positive integers A, B prime to ℓ with $AY \subset \mathcal{O}_E$ and $B\ell^a\mathcal{O}_E \subset Y$. The inclusions hold locally above ℓ by the conductor exponents, and away from ℓ by clearing lattice denominators. Put

$$L = \{(u,v') \in F^2 : u\theta + v' \in ADY\}, \qquad m_0 = \ell + 1, \qquad N = m_0 ADB\ell^a.$$

The coordinates of $z = (u\theta + v')/D \in AY$ are

$$u = D\frac{z - \bar z}{\theta - \bar\theta}, \qquad v' = D\frac{\theta\bar z - \bar\theta z}{\theta - \bar\theta}.$$

Both coordinates and their squares divided by D are integral. Together with $B\ell^a\mathcal{O}_E \subset Y$ this gives

$$N\mathcal{O}_F^2 \subseteq L \subseteq \mathcal{O}_F^2, \qquad v(\sigma_i(u)), v(\sigma_i(v')) \geq h/2 \quad ((u,v') \in L). \tag{A.4}$$

Choose the nonzero vector $y_0 = (N/m_0, N/m_0) \bmod N$, killed by m_0.

Lean: `GlobalTheory.RoadE.passl` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.MemPassScaled`, lines 42–116)

For a lattice L' between $N\mathcal{O}_F^2$ and $\mathcal{O}_F^2$, let $V_{L'}((b_i'),(a_i'))$ be c_k times the sum of $\prod_i(\sigma_i(u)b_i' + \sigma_i(v')a_i')^{-k}$ over $(L' \setminus 0)/\mathcal{O}_F^\times$, with the same continuation at $k = 2$. For a multi-index ν put $|\nu| = \sum_i \nu_i$, $W_\nu = n + 2|\nu|$, and

$$D_\nu(L'; t, C, T) = \partial_z^\nu V_{L'}\big((t + z_i(Ct+T))_i, (1 + Cz_i)_i\big)\big|_{z=0}.$$

If $ADt' = u\theta + v'$, the deformed ith coordinate of t', multiplied by AD, is $\sigma_i(u)(\tau + z_i\bar\tau) + \sigma_i(v')(1 + z_i)$. The substitution respects the real units, so

$$\partial^\kappa g(0) = (AD)^n D_\kappa(L; \tau, 1, \bar\tau - \tau). \tag{A.5}$$

For $k = 2$ the regularized sums give the additional factor $(AD)^{2f\sigma}$, equal to one at zero. The Fourier expansion justifies continuation and differentiation at zero.

Lean: `GlobalTheory.HilbFam.iterD_defSum_eq_jet` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.DeformedLatticeSumDerivativesEqEisensteinJets`, lines 21–53)

For $\alpha \in \mathrm{GL}_2(\mathbf{Z}/N)$ let L_α be the inverse image of $(L \bmod N)\alpha$ in $\mathcal{O}_F^2$, and set $y_\alpha = y_0\alpha$. Since $L_\alpha = L\widetilde\alpha + N\mathcal{O}_F^2$ for an integral lift $\widetilde\alpha$, it satisfies (A.4). Write $y_\alpha = (a', b')$ and $P_\alpha(t) = (a't + b')/N \bmod \Lambda_t$. This point is nonzero and killed by m_0. Put

$$C_\alpha(t) = \frac{G_2(t) + \wp_{\Lambda_t}(P_\alpha(t))}{p^2}, \qquad H_{\alpha,\nu}(t) = p^{-W_\nu} D_\nu(L_\alpha; t, p^2 C_\alpha(t), -p).$$

For $\gamma = \left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ put $j_\gamma = C't + D'$. The transformation of G_2 and homogeneity of $\wp$ give $C_\alpha(\gamma t) = j_\gamma^2 C_{\alpha\bar\gamma}(t) - C'j_\gamma/p$ [7, Ch. 2]. Multiplying the coordinates at γt by j_γ multiplies the lattice row by γ and replaces (C,T) by $(C - TC'j_\gamma, Tj_\gamma^2)$. For $(C,T) = (p^2C_\alpha(\gamma t), -p)$ this is $j_\gamma^2(p^2C_{\alpha\bar\gamma}(t), -p)$. The coordinates contribute j_γ^n and differentiation contributes $j_\gamma^{2|\nu|}$. Thus the family is permuted in weight W_ν.

Fix $L' = L_\alpha$. Let $\mathfrak{b}$ be its first-coordinate image and $0 \times \mathfrak{a}^\vee$ its vertical kernel, where $\mathfrak{a}^\vee = \mathfrak{D}_F^{-1}\mathfrak{a}^{-1}$ and $\mathfrak{D}_F$ is the different. Choose $b_j \in \mathfrak{b}$, $c_j \in \mathfrak{b}^{-1}$ with $\sum_j b_jc_j = 1$ and lifts $(b_j, t_j) \in L'$. Then $x = \sum_j c_jt_j$ satisfies $(u, ux) \in L'$ for all $u \in \mathfrak{b}$, and $L' = \{(u, ux + v') : u \in \mathfrak{b},\ v' \in \mathfrak{a}^\vee\}$. Put $W_\mathfrak{a} = |D_F|^{1/2}\,\mathrm{Nm}\,\mathfrak{a}$. For $x = 0$ the expansion of $V_{L'}((t_i),1)/(p^nW_\mathfrak{a}^k)$ is $\sum_\xi A_\xi \exp(p\sum_i \sigma_i(\xi)t_i)$. General x shifts each t_i by $\sigma_i(x)$. The coefficients are

$$A_0 = 2^{-f} z([\mathfrak{a}]^{-1}), \qquad A_\xi = \sum_{u \in \mathfrak{a},\ b \in \mathfrak{b},\ ub = \xi} \left(\frac{|N_{F/\mathbf{Q}}u|}{\mathrm{Nm}\,\mathfrak{a}}\right)^{k-1} \quad (\xi \gg 0), \tag{A.6}$$

with zero coefficients elsewhere. Here $\xi \gg 0$ means totally positive, and the pairs are taken modulo $(u,b) \mapsto (\epsilon^{-1}u, \epsilon b)$ for $\epsilon \in \mathcal{O}_F^\times$. The integral ideal $(u)\mathfrak{a}^{-1}$ runs through divisors of $(\xi)(\mathfrak{a}\mathfrak{b})^{-1}$, so this is a finite sum of powers of ideal norms.

For $b \neq 0$, apply Poisson summation on $\mathfrak{a}^\vee$, with trace dual $\mathfrak{a}$. For even k, the real Fourier transform of $(t+z)^{-k}$ at u vanishes for $u \operatorname{Im} z < 0$ and for $u \operatorname{Im} z > 0$ equals

$$\operatorname{sgn}(\operatorname{Im} z) \frac{(-p)^k u^{k-1}}{(k-1)!} \exp(puz).$$

Closing the contour in the half-plane containing $-z$ in the second case gives the residue formula. Use $z = \sigma_i(b) t_i$ at each embedding. The resulting support is $\xi = ub \gg 0$. The inverse covolume is $W_{\mathfrak{a}}$, and multiplication of the transforms with c_k gives the nonconstant terms. The row $b = 0$ sums over nonzero elements of $\mathfrak{a}^\vee$ modulo units and contributes the partial zeta function of $C = [\mathfrak{D}_F \mathfrak{a}]$ at k. The partial zeta functional equation with trivial sign and zero additive shift identifies $|D_F|^{z/2} \Gamma_{\mathbf{R}}(z)^f \zeta_F(z, C)$ at $z = k$ with the corresponding value at $1-k$ for $C' = [\mathfrak{a}]^{-1}$, since $CC' = [\mathfrak{D}_F]$ [8]. Substituting $\Gamma(s) = (s-1)!$ and $\Gamma(1/2 - s) = (-4)^s s! \sqrt{\pi}/(2s)!$ gives A_0.

For $k = 2$ first apply Poisson summation with the regularizing factor for $\operatorname{Re} \sigma > 0$. The zero frequency from the rows $b \neq 0$ is a regular factor times

$$\zeta_F(1 + 2\sigma, [\mathfrak{b}]^{-1}) \, c(\sigma)^f, \qquad c(\sigma) = \int_{\mathbf{R}} (t + i_{\mathbf{C}})^{-2} |t + i_{\mathbf{C}}|^{-2\sigma} \, dt.$$

The integral is holomorphic near zero and $c(0) = 0$ by the antiderivative $-(t + i_{\mathbf{C}})^{-1}$. The partial zeta function has at most a simple pole, so $f \geq 2$ makes this contribution vanish at zero.

For the nonzero frequencies, fix compact sets of σ in a neighborhood of zero and of (t_i) in the product of upper half-planes. Fix a derivative order in the real coordinates of (t_i). In the Fourier integrals rescale each real integration variable by $|\sigma_i(b)|$. The imaginary parts of the rescaled parameters stay away from zero. Repeated integration by parts in each real variable gives, for sufficiently large B, a bound

$$C_B \prod_i \frac{(|\sigma_i(b)| + |\sigma_i(b)|^{-1})^m}{(1 + |\sigma_i(ub)|)^B}$$

for every derivative of the fixed order, with m independent of B. After rescaling, differentiation of the integrand in the parameters gives integrable functions with any fixed number of integrable real-variable derivatives, uniformly on the compact sets. The scale factors in b are bounded by the displayed numerator. Integration by parts gives inverse powers for $|\sigma_i(ub)| \geq 1$, and the absolute integral bounds the remaining frequencies.

Choose one b per unit class with $|\sigma_i(b)| \leq C_1 |\sigma_j(b)|$ for all i, j, by moving its logarithms into a bounded fundamental domain for the unit lattice after subtracting their mean. Replace u by the inverse unit multiple when changing b. This preserves ub and permutes the nonzero points of $\mathfrak{a}$. Write $S(b) = \sum_i |\sigma_i(b)|$ and $S(u) = \sum_i |\sigma_i(u)|$. Lattice discreteness bounds $S(b)$ below by a positive constant, so $|\sigma_i(b)| + |\sigma_i(b)|^{-1} \ll S(b)$. Also $\prod_i (1 + |\sigma_i(ub)|) \geq \sum_i |\sigma_i(ub)| \geq S(b) S(u)/(f C_1)$. With $B \geq mf + f + 1$ the bound is at most a constant times $S(b)^{mf - B} S(u)^{-B}$. Each factor is summable over its lattice by the $O(R^f)$ count of points with $S \leq R$. This gives normal convergence for the nonzero frequencies and their derivatives. Evaluating at $\sigma = 0$ gives (A.6) and its holomorphic derivatives.

Lean: `GlobalTheory.HeckeTwo.heckeWeightTwo_of_fe` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.HeckeWeightTwoContinuationOfFunctionalEquation`, lines 22–63)

Lean: `GlobalTheory.HilbertEis.eisExp` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.HilbertEisensteinExpansionOfLipschitzHeckeZeta`, lines 61–99)

Put $b_0 = h + a$, $d'_F = v_\ell(|D_F|)$ and $\rho = 2c + 2a + h/2$. One has $d'_F/f \leq c$ since conductor primes are unramified in E and $|D_F|^2 \mid |D_E|$. Since $N \in \mathfrak{a}^\vee$, $Nu \in \mathfrak{D}_F^{-1}$ for $u \in \mathfrak{a}$. As $F/\mathbf{Q}$ is Galois, the exponents of the different are constant above each rational prime. Taking norms gives $\mathfrak{D}_F^f = (|D_F|)$. Using (A.4) for $b \in \mathfrak{b}$ gives

$$v(\sigma_i(\xi)) \geq -b_0 - d'_F/f + h/2 \geq -\rho \quad (\xi \neq 0, \ A_\xi \neq 0). \tag{A.7}$$

Also $N \mathfrak{D}_F \mathfrak{a}$ is integral, so $v_\ell(\operatorname{Nm} \mathfrak{a}) \geq -f b_0 - d'_F$ and $v(W_{\mathfrak{a}}^k) \geq -s d'_F - n b_0$.

Write $r(k, j, t) = (-1)^{j-t} \binom{j}{t} (k + t)_{j-t}$ for $0 \leq t \leq j$, with $(x)_u = x(x+1)\cdots(x+u-1)$ and $(x)_0 = 1$. For a scalar v', replacing C by $C + v'$ in D_ν gives a sum over $\mu \leq \nu$ with coefficients $\prod_i r(k, \nu_i, \mu_i)(v')^{|\nu| - |\mu|}$. In one coordinate use $z \mapsto z/(1 + v'z)$ and the factor $(1 + v'z)^{-k}$. The order t term contributes $j!/t!$ times the coefficient of z^{j-t} in $(1 + v'z)^{-k-t}$, which equals $r(k, j, t)(v')^{j-t}$.

Starting with direction $(0,-p)$, define

$$B_{\mu,m}=\sum_{N\operatorname{Tr}_{F/\mathbb{Q}}\xi=m}A_\xi\exp(p\operatorname{Tr}_{F/\mathbb{Q}}(\xi x))\prod_i\sigma_i(\xi)^{\mu_i}.$$

The sum is over the support of (A.6), with exponent zero giving the factor one. It is finite since a fixed trace bounds all totally positive coordinates. The indices are integers by $N\mathfrak{a}\mathfrak{b}\subseteq\mathfrak{D}_F^{-1}$, and the phase is a root of unity. Differentiating the exponential in direction $-p$ gives $-p^2\sigma_i(\xi)$ at each differentiation. The change of direction with $v'=p^2C_\alpha$ gives

$$H_{\alpha,\nu}=W_{\mathfrak{a}}^k\sum_{\mu\le\nu}\left(\prod_i r(k,\nu_i,\mu_i)(-1)^{\mu_i}\right)C_\alpha^{|\nu|-|\mu|}\sum_{m\ge0}B_{\mu,m}q^{m/N}.$$

Equations (A.6) and (A.7) give $v(B_{\mu,m})\ge -e-|\mu|\rho$. The coefficients of C_α are integral by (A.1) since m_0 is prime to ℓ, even when $\ell\mid N$. Each coefficient of $H_{\alpha,\nu}$ therefore has valuation at least $-R_\nu$, where

$$R_\nu=e+sd'_F+nb_0+|\nu|\rho. \tag{A.8}$$

Lean: `GlobalTheory.HilbFam.sz_hfun_coef` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.HilbertEisensteinCoefficientValuationBound`, lines 226–361)
Lean: `GlobalTheory.HilbFam.isFam_hfun_GL` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.HilbertFamilyWeierstrassTwist`, lines 27–35)

At the CM point $\omega=\Omega/p$ satisfies $\omega^{12}=\Delta(\tau)$ and $j(\tau)$ is integral. Lemma 1.16 gives $v(H_{1,\nu}(\tau)/\omega^{W_\nu})\ge -R_\nu$. Put $V=\pi/\operatorname{Im}\tau$, so $V(\bar\tau-\tau)=-p$. The change of direction now gives

$$\frac{D_\kappa(L;\tau,1,\bar\tau-\tau)V^{K_\kappa}}{p^{W_\kappa}\omega^{W_\kappa}}=\sum_{\nu\le\kappa}\left(\prod_i r(k,\kappa_i,\nu_i)\right)\left(\frac{V/p^2-C_1(\tau)}{\omega^2}\right)^{K_\kappa-|\nu|}\frac{H_{1,\nu}(\tau)}{\omega^{W_\nu}}. \tag{A.9}$$

The parenthesis is $-(s_2(\Lambda)+\wp_\Lambda(P_1(\tau)))/\Omega^2$. Lemma 1.17 bounds its valuation below by $-\rho$. At $\ell=d$ one has $h=1$. At $\ell=2$ one has $c\ge1/2$, since 2 ramifies in the Galois field E and ramification index $e_0\ge2$ gives different exponent at least e_0-1 [9, Ch. III, § 6]. Thus $\rho\ge1$ for $d\equiv3\pmod4$ and $\rho\ge2$ for $d\equiv1\pmod4$, where $h=2$. At the other primes $\rho\ge0$ suffices. Hence the left side has valuation at least $-R_\kappa$.
Lean: `GlobalTheory.HilbFam.sz_jet_conj` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.SzJetConj`, lines 24–57)

Let $\delta_\ell=\frac12\mathbf{1}_{\ell=d}$ and choose algebraic γ_ℓ with valuation $c+\delta_\ell$. Take $a_i=\Omega/\ell^a$ and $b_i=2\pi\gamma_\ell/(\sqrt{d}\,a_i)$. These give the first two bounds of (1.7). By (A.5), its normalized derivative is the left side of (A.9) multiplied by

$$(AD\ell^a)^n\left(\frac{2\operatorname{Im}\tau}{\sqrt{d}}\gamma_\ell\ell^{2a}\right)^{K_\kappa}.$$

The ratio $2\operatorname{Im}\tau/\sqrt{d}$ is 1 or 2 according as d is 3 or 1 mod 4, and its valuation is $h/2-\delta_\ell$. The denominator exponent is at most

$$R_\kappa-n(h+a)-K_\kappa(h/2+c+2a)=e+sd'_F+K_\kappa c\le e+(n+K_\kappa)c.$$

The coordinates depend on the lattice and period root and work for every κ. □

Proof of 1.19

Proof. Siegel–Klingen gives rational $z(C)$ before the conductors, weights or local embeddings are chosen [10, 11]. For $A\ge2$ prime to ℓ, apply Deligne–Ribet at weight k to the ordinary class indicator on ideals prime to ℓ and the ideal (A) [11]. The indicator takes values in $\mathbf{Z}_\ell$. It is constant on ray classes modulo (ℓ) and invariant under translation by (A). Thus A^n-1 times the partial zeta value with the primes above ℓ removed is integral at ℓ, since $\operatorname{Nm}(A)=A^f$. For each omitted $\mathfrak{p}$, the removed Euler factor is $1-(\operatorname{Nm}\mathfrak{p})^{k-1}S$, with S translation by $[\mathfrak{p}]^{-1}$ on classes. If $S^j=1$, its inverse is the finite geometric sum divided by $1-(\operatorname{Nm}\mathfrak{p})^{(k-1)j}$, an ℓ-adic unit. Restore the factors and divide by 2^f to obtain the first inequality of (1.5).

For odd ℓ, a primitive root modulo ℓ^2 attains the minimum. If $\ell-1 \mid n$, the valuation is $1+v_\ell(n)$ by binomial expansion along powers of ℓ. Otherwise it is zero already modulo ℓ. For odd A and even n,

$$v_2(A^n-1) = v_2(A-1)+v_2(A+1)+v_2(n)-1.$$

The minimum is attained at $A=3$. In particular $e_\ell = 0$ if $\ell > n+1$, and if $\ell-1$ is a proper divisor of n then $\ell \le n/2+1$. The product of primes up to u is at most 4^u. For $u=2m$ the primes above m divide $\binom{2m}{m}$, and for $u=2m+1$ those above $m+1$ divide $\binom{2m+1}{m}$. Both binomial coefficients are at most 4^m, so induction gives the bound. Separating a possible prime $n+1$ now gives

$$\sum_\ell e_\ell \log \ell \le \log n + (f+2)\log 2 + \log(n+1) + (n/2+1)\log 4 \le n\log(n+1) \qquad (n \ge 16).$$

The last step uses $2f \le n$ and $\log(n+1) \ge 4\log 2$.

Lean: `GlobalTheory.ZetaDenom.zetaDenominators` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.PartialZetaDenominatorsCMFieldOfSiegelKlingenDel`, lines 146–164)

For the toric bound choose y equal to one above ℓ and use Lemma 1.18. Put

$$J_\kappa = \partial^\kappa g(0) \prod_i \frac{(b_i/a_i)^{\kappa_i}}{a_i^k}, \qquad A_i = a_i/\Omega, \qquad B_i = \frac{2\pi}{a_i b_i \sqrt{d}}.$$

Differentiation gives $\partial^\kappa g(0) = c_k(-1)^{K_\kappa}\prod_i (k)_{\kappa_i} S_y$. At $k=2$ the regularized holomorphic derivatives have factors $(k+\sigma)_{\kappa_i}$. The continued Fourier expansion gives their evaluation at zero. Since $(k)_j = \Gamma(k+j)/(k-1)!$,

$$x_\Omega(S_y) = (-1)^{K_\kappa} J_\kappa \prod_i B_i^{\kappa_i} A_i^{k+2\kappa_i}. \tag{A.10}$$

If $c_\ell = 0$, then $\ell \ne d$, since d ramifies in the quadratic subfield and the conductor primes are unramified in E. The coordinate estimates give denominator exponent $e_\ell + Ma_\ell$. For $c_\ell > 0$, dividing J_κ by $\prod_i \kappa_i!$ gives denominator exponent at most $e_\ell + nc_\ell + K_\kappa(c_\ell+1)$, since $\sum_i v_\ell(\kappa_i!) = \sum_{i,j\ge 1}\lfloor \kappa_i/\ell^j \rfloor \le K_\kappa$, where i ranges over the CM type. Restore the integral factorials; this factorial normalization matches the divided-power intermediate bound while leaving the direct bound $e_\ell + (n+K_\kappa)c_\ell$ with $1+\mathbf{1}_{\ell=2}$ slack against the $K_\kappa(c_\ell+1)$ term. The B_i contribute at most $K_\kappa(c_\ell + \frac{1}{2}\mathbf{1}_{\ell=d})$ and the A_i at most Ma_ℓ. This proves $v(x_\Omega(S_y)) \ge -b_\ell$.

Every order class has a representative equal to one above ℓ. Weak approximation first moves these components into the open local order units, which can then cancel them. At a split conductor place an order unit is a diagonal real-field unit times a pair congruent to one modulo the conductor. The character is trivial on both factors by its real-field restriction and conductor. Away from the conductor it is unramified. Thus χ_f and its Picard twists are trivial on $\widehat{O}_Q^\times$. A principal change of representative acts on Y_y by multiplication. The infinity character and the product formula give the inverse of the resulting scaling of S_y in the factor $(\chi_f(y)|y|_{E,f}^s)^{-1}$.

Let H_y be the class order and write $y^{H_y} = tv'$ with $t \in E^\times$ and $v' \in \widehat{O}_Q^\times$. For $u = (\chi_f(y)|y|_{E,f}^s)^{-1}$ one has $u^{H_y} = \chi_\infty(t)|N_{E/\mathbb{Q}}t|^s$. Since t is a unit above ℓ, each embedded conjugate of t is an ℓ-unit. Thus u is an algebraic ℓ-unit, and the normalized class term $ux_\Omega(S_y)$ has the same bound. Picard character values are roots of unity. The ultrametric inequality for the finite twisted sum and the twelfth power give (1.2). The argument applies at every embedding above ℓ.

Lean: `GlobalTheory.ToricDenomRow.toricDenominator` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ToricDenominatorOfZetaDenominators`, lines 39–95)

For transport, choose a basis of the integer lattice of K and a root $\Omega_0^{12} = D(\Lambda_K)$. The elliptic main theorem of complex multiplication gives the conjugate algebraic elliptic curve and a differential that determines the period scale Ω_1. In Proposition 26.29(1) the arithmetic Artin symbol of t is $\mathrm{Art}_K(t)^{-1}$. This is $g|_{K^{\mathrm{ab}}}$ by the choice of t. The theorem of complex multiplication therefore sends the lattice $\mathfrak{a}$ to $t^{-1}\mathfrak{a}$ and the torsion marking z to $t^{-1}z$ on that lattice [7, Ch. 6].

Lean: `GlobalTheory.LatVal.exists_globalArtinMap` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.PeriodExistenceCm`, lines 32–79)

Lean: `GlobalTheory.LatVal.periodEx_of_mainTheorem` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.PeriodExistenceCm`, lines 81–102)

The scales work at every weight. The Fourier calculation uses $[F : \mathbf{Q}] \geq 2$. In the twisted assertion, if $[F : \mathbf{Q}] = 1$, then $F = \mathbf{Q}$ and finite unramifiedness gives $|D_E| = |D_F|^2 = 1$, which is impossible for an imaginary quadratic field.

Choose a common finite level N divisible by 2 (so that $N \geq 2$ and a nonzero torsion marking exists) after clearing the denominators of the two coordinates of Y_y. The construction of $H_{\alpha,\nu}$ uses the resulting full lattice and a nonzero torsion marking; the jet identity (A.9) requires only $[F : \mathbf{Q}] \geq 2$, without assuming $E/\mathbf{Q}$ abelian. Put $g\zeta_N = \zeta_N^a$ and $d_a = \mathrm{diag}(1, a)$ modulo N. In the coefficient (A.1), g preserves the row's first coordinate and multiplies the second by a. In the Hilbert expansion the constant term and ideal norms are rational. The phase $\exp(p \operatorname{Tr}_{F/\mathbf{Q}}(\xi x))$ goes to the phase with x replaced by ax. The lattice substitution $(b, bx + v') \mapsto (b, abx + av')$ modulo level leaves the first-coordinate ideal and vertical kernel unchanged. The equality $g\phi_i = \phi_{\pi(i)}$ permutes the factors $\sigma_i(\xi)^{\mu_i}$, with new order $\mu'_{\pi(i)} = \mu_i$. Thus g sends the coefficients of the family at α to those of the conjugate family at αd_a.

Let M_{CM} be the torsion marking matrix from CM reciprocity, and write $\Lambda_j = c_j(\mathbf{Z}\tau_j + \mathbf{Z})$ for the two period lattices. Put $p = 2\pi i$ and $\omega_j = c_j\Omega_j/p$. Here the transported invariant differential from the Elliptic CM Main Theorem fixes the period pair (Ω_0, Ω_1) directly for all $d > 0$, including the extra-automorphism cases $d = 1, 3$. Indeed, lattice homothety gives $g_2(c\Lambda) = c^{-4}g_2(\Lambda)$ and $g_3(c\Lambda) = c^{-6}g_3(\Lambda)$. Hence $D(\Lambda_j) = c_j^{-12}p^{12}\Delta(\tau_j)$. CM reciprocity carries $g_2(\Lambda_0)/\Omega_0^4$ and $g_3(\Lambda_0)/\Omega_0^6$ to the corresponding invariants at Λ_1. Their discriminant at Λ_0 is $D(\Lambda_0)/\Omega_0^{12} = 1$, so $D(\Lambda_1)/\Omega_1^{12} = g(1) = 1$. Hence $\Omega_j^{12} = c_j^{-12}p^{12}\Delta(\tau_j)$. Thus $\omega_j^{12} = \Delta(\tau_j)$ at both lattices. For a nonzero torsion point represented by z on $\mathbf{Z}\tau_j + \mathbf{Z}$, homothety gives

$$\frac{\wp_{\Lambda_j}(c_j z)}{\Omega_j^2} = \frac{\wp_{\mathbf{Z}\tau_j+\mathbf{Z}}(z)/p^2}{\omega_j^2}, \qquad \frac{\wp'_{\Lambda_j}(c_j z)}{\Omega_j^3} = \frac{\wp'_{\mathbf{Z}\tau_j+\mathbf{Z}}(z)/p^3}{\omega_j^3}.$$

Lean: `GlobalTheory.TransportProof.disc_j_of_torsTransport` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.HilbertClassFieldLatticeTransport`, lines 54–72)
Lean: `GlobalTheory.WpFam.Transp.main` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.EisensteinWeierstrassTorsionConjTransport`, lines 20–49)

CM reciprocity transports $E_4(\tau_j)/\omega_j^4$ and $E_6(\tau_j)/\omega_j^6$, with their usual weight normalizations, and transports the normalized $\wp, \wp'$ values at a torsion row x to those at xM_{CM}. To pass from the action on coefficients to the action on values, use the two basis torsion rows. Their pairs of $\wp, \wp'$ values, normalized in weights two and three, determine the ordered basis on the elliptic curve. Thus they separate the labels α. In the family with conjugate coefficients the label giving the transported values of these four functions is $\beta = \alpha M_{\mathrm{CM}} d_a^{-1}$, since $\beta d_a = \alpha M_{\mathrm{CM}}$. The same label works for every normalized form. Adjoin the four separating functions to the family. Use the elementary symmetric functions of linear combinations with independent indeterminates to form the polynomial of its values over the labels. The homogeneous coefficients are full-level modular forms. A coefficient of odd weight is zero, since the action of $-I$ changes its sign. Every coefficient of even weight is a polynomial in E_4, E_6, so its normalized value is a polynomial in the normalized E_4, E_6. Conjugating the coefficients and specializing the separating functions identifies the factor at α with that at β in the conjugate family. Consequently the CM value at α transports to the conjugate value at αM_{CM}. For the correction in (A.9), $\wp$ transports by the same marking. The isogeny identity of Lemma 1.17 for a nonreal multiplier in K holds for all $d > 0$ (without requiring $d > 4$ prime) and expresses s_2 as a sum of $\wp$ values on its kernel divided by a nonzero coefficient in K. Thus the normalized correction transports as well.

At the initial lattice use $c_0 = 1$, and at the inverse ideal choose the basis so that c_1 is a positive rational. Put t_E for the image of t in the finite idèles of E. On the finite completions the marking multiplies $y\widehat{O}_Q$ by t_E^{-1}. In the basis $(\tau_1, 1)$ its coordinates therefore describe $c_1^{-1}Y_{yt_E^{-1}}$. Multiplying an element by c_1^{-1} multiplies the undeformed summand of weight (k, κ') by c_1^{kf}, since at each embedding the two exponents of this real scalar sum to $-k$. Write $n_0 > 0$ for the common integer clearing the two lattice coordinates. Put $h = K_\kappa$, $M = kf + 2h$, and $q_j = 2\operatorname{Im}\tau_j/\sqrt{d}$. Let X_j be the right side of (A.9) at the marking τ_j, evaluated with its coordinate lattice and torsion row. Its total period denominator is ω_j^M. Each term of derivative order ν has total weight $2(h - |\nu|) + (kf + 2|\nu|) = M$. Its integer coefficient is preserved by permutation of the embeddings. The transport of the correction of weight two and of the modular family of weight $kf + 2|\nu|$ therefore gives $gX_0 = X_1$. No action on p is needed in this equality.

For the coordinate lattice at j, denote its sum before clearing denominators by S_j^{coord}. Thus $S_0^{\mathrm{coord}} = S_y$, and S_1^{coord} is the sum for $c_1^{-1}Y_{yt_E^{-1}}$ with the permuted derivative orders. Substitution into the left side of

(A.9) gives

$$x_{\Omega_j}(S_j^{\mathrm{coord}}) = (-1)^h n_0^{kf} q_j^h c_j^M X_j.$$

Clearing the coordinates contributes n_0^{kf} and converting the derivative contributes $(-1)^h q_j^h$. Since $p\omega_j = c_j\Omega_j$, replacing the denominator $(p\omega_j)^M$ by Ω_j^M multiplies the value by c_j^M. On the continued sum the change $t \mapsto c_1^{-1}t$ contributes $c_1^{kf+2f\sigma}$ before evaluation at $\sigma = 0$. Thus $S_1^{\mathrm{coord}} = c_1^{kf} S'_{yt_E^{-1}}$. Dividing by c_1^{kf} leaves $c_1^{M-kf} = c_1^{2h}$. Since $c_0 = 1$ and q_0 is rational, the two identities are

Lean: `GlobalTheory.LatConjEll.sum_transport` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LatticeValueConjugationOfEllipticCMTheorems`, lines 140–164)

$$gx_{\Omega_0}(S_y) = (-1)^{K_\kappa} n_0^{kf} (2\operatorname{Im}\tau_0/\sqrt{d})^{K_\kappa} X_1,$$
$$x_{\Omega_1}(S'_{yt_E^{-1}}) = (-1)^{K_\kappa} n_0^{kf} (2c_1^2 \operatorname{Im}\tau_1/\sqrt{d})^{K_\kappa} X_1.$$

To see the rationality used here, write the integral element δ with $\delta^2 = -d$ as $a\tau_0 + b$ in the embedded integer lattice. Taking imaginary parts gives $q_0 = 2/|a|$. The positive determinant taking a basis of $\mathcal{O}_K$ to a basis of the inverse ideal is its fractional ideal norm. Computing the same determinant by areas gives

$$A = |t|_{K,f} = \frac{c_1^2 \operatorname{Im}\tau_1}{\operatorname{Im}\tau_0}.$$

Multiplication of the first displayed value by A^{K_κ} gives the second, also when $X_1 = 0$. Hence $gx_{\Omega_0}(S_y) = A^{-K_\kappa} x_{\Omega_1}(S'_{yt_E^{-1}})$. Permuting the derivative orders preserves their sum K_κ and the product $\prod_i \Gamma(k+\kappa_i)$, hence also C_0.

Lean: `GlobalTheory.LatConjEll.value_eq` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LatticeValueConjugationOfEllipticCMTheorems`, lines 166–202)
Lean: `GlobalTheory.LatConjEll.latConj` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LatticeValueConjugationOfEllipticCMTheorems`, lines 204–395)
Lean: `GlobalTheory.LatConjEll.covol` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LatticeValueConjugationOfEllipticCMTheorems`, lines 55–113)
Lean: `GlobalTheory.LatConjEll.ratio_rat` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LatticeValueConjugationOfEllipticCMTheorems`, lines 115–138)
Lean: `GlobalTheory.T.latticeValueIsAlgebraic_of_ellipticCM_theorems` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LatticeValueIsAlgebraicOfEllipticCMTheoremsStatement`, lines 9–10)
Lean: `GlobalTheory.T.latticeValueConjugation_of_ellipticCM_theorems` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LatticeValueConjugationOfEllipticCMTheoremsStatement`, lines 9–10)

The normalized modular form values and correction are algebraic. The class order calculation shows that the finite class factors are algebraic as well. In the twisted sum, after transport the term from y carries $[\chi'_f(y)|y|^s_{E,f}]^{-1}\psi'([y])^{-1}$. With $y' = yt_E^{-1}$ this is the factor at y' times $[l_\chi\psi'([t])]^{-1}$. Multiplication by t_E^{-1} permutes the order classes, so summing gives (1.4). Both periods come from the elliptic lattices and are chosen before the extension, type, conductor or twist.

Lean: `GlobalTheory.TwistedTransportProof.twistedTransport_of_latClaims` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.TwistedTransportOfLatClaims`, lines 189–202)

For ordinary transport the Hilbert class field theorem gives $K(j(\Lambda_K))$ with one conjugate j for each ideal class [7]. Choose t_g by CM reciprocity and use its inverse ideal for Λ_g. Restriction of embeddings of L over K to the Hilbert class field has fibres of cardinality $[L : K(j(\Lambda_K))]$. Set the Picard character to one and take twelfth powers in the transport identity to obtain the first line of (1.3). Since $|t_g|_{E,f} = A^{[E:K]} = A^f$ and χ'_f is unitary, one has $|l_\chi| = A^{sf}$. The absolute value therefore gives the factor $A^{-12(K_\kappa+sf)} = A^{-6M}$. For equal weights, the periods and A^{-K_κ} cancel when dividing the two normalized values. A field automorphism carries a nonzero denominator to a nonzero value, which proves the ratio assertion.

Lean: `GlobalTheory.TransportProof.transport_of` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.HilbertClassFieldLatticeTransport`, lines 160–193)

For reflection take $C_{F,0} = C'_0 = x_0 = 0$. Fix a square-recursive size datum of 34.49, host, CM-type and arithmetic data of 34.46, and a level-character datum and grid of 34.47 as in Definition 1.15(5). The host conditions of 34.46 supply the CM type, and the splitting conditions and conductor exponents in 34.46–34.49 give the base conductor conditions on $\chi_0 = \chi_{0,i,j}$. For each t, write $\xi = \xi_t$. By Definition 1.15(5)(a), ξ is trivial on $\mathbb{A}_F^\times$, has finite order, and is tame, and the coefficient-field condition supplies all its required local

values. By Definition 1.15(5)(b), at each place v where ξ ramifies and χ_0 is unramified, the residue field has prime order $q_v \mid Q_n(t)$ with $v \in W_t$ and $d_v \geq 1$, so the root condition supplies all q_vth roots of unity in K, and these places occur in conjugate pairs under E/F (since E/F is unramified at finite places, no such prime-order place can be self-conjugate). By Definition 1.15(5)(c), at each place where both ξ and χ_0 ramify, the residue field has prime order, the place is unramified over its rational prime, and χ_0 has conductor exponent two.

By Definition 1.15(5)(a) and (d), χ_0 and $\chi_0\xi$ share the $(\bar{\tau}/\tau)$-infinity exponents $m_\tau = \kappa_\tau(h)+h > h \geq e_0$, where h is half the sum of the grid ranks at (i,j) and $(i+1,j+1)$. Thus $|m_\tau| + s \geq 1 + m_\tau - e_0 > 0$ and $|m_\tau| + 1 - s > 0$, so the archimedean gamma factors at s and $1-s$ are finite and nonzero throughout $1 - e_0 \leq s \leq 0$.

Apply the local epsilon-factor formulas [12]. At an unramified twist place, with reference conductor exponent a_v, local different exponent δ_v and uniformizer ϖ_v, the multiplier is $\xi_v(\varpi_v^{a_v+\delta_v})$. At a place where the twist has conductor one and χ_0 is unramified, the conductor-one sum gives $\chi_0(\varpi_v)$ times a finite-order character value, a power of q_v and a nontrivial Gauss sum. Its additive character takes values in the stipulated roots of unity. For the residue character, still denoted ξ, additive orthogonality gives $G(\xi)G(\xi^{-1}) = \xi(-1)q_v$. Hence $G(\xi)$ is a unit away from q_v. At a place where both ramify, write units modulo ϖ_v^2 as residue lifts times $1+\varpi_v x$. The sum over x selects a single residue unit, since χ_0 has conductor exponent two. The tame twist is constant on the principal units, so the quotient is its value at the selected unit times the uniformizer factor.

The places where the twist ramifies and the reference character is unramified occur in conjugate pairs. The two reference factors $\chi_0(\varpi_v)$ in each pair are inverse by triviality on the real-field idèles. The remaining product is $\iota_D(u)$ for a unit away from $Q_n(t)$. In the two global Hecke–Tate equations the infinite factors agree. Triviality on real-field idèles identifies each inverse character with its conjugate under E/F, and conjugation permutes the Euler factors of its Hecke function. Cross-multiply the two functional equations and cancel the common nonzero gamma factors to obtain (1.6).

Lean: `GlobalTheory.SqTop.refl_square` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ReflectionOfFunctionalEquation`, lines 156–250)
Lean: `GlobalTheory.SqTop.refl_of_functionalEquation` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ReflectionOfFunctionalEquation`, lines 252–261)

□

Proof of 1.21

Proof. A simple module M of at most countable complex dimension over a $\mathbf{C}$-algebra has scalar commutant. Indeed a nonscalar endomorphism A has $A - z$ invertible for all $z \in \mathbf{C}$ by Schur's lemma. For $m \neq 0$, the vectors $(A-z)^{-1}m$ are $\mathbf{C}$-linearly independent, since clearing denominators in a finite relation gives a nonzero polynomial in A killing m, and its linear factors are invertible; this contradicts countable dimension.

For a linearly independent tuple in such a simple module M, its orbit spans M^b, where b is the tuple length. A proper span would admit a nonzero module map $M^b \to M$ killing the tuple, by semisimplicity, and the scalar commutant would give a linear dependence. Thus the acting algebra prescribes the images of any independent finite family. In $\bigotimes_{i=1}^{r} M_i$, apply this in each slot to isolate a nonzero pure tensor from a finite sum and send it to arbitrary pure tensors; for $r = 0$ the empty product $\mathbf{C}$ is simple. In the restricted tensor product, any nonzero vector and any target vector lie in $\bigotimes_{v\in S} M_v$ for a common finite set $S \supseteq S_0$ by adjoining the nonzero vectors s_v, so simplicity follows from the finite case on S. □

Proof of 1.22

Proof. For the polarized pair take r, the weight datum w_{gt} and jump multisets H_τ from the field of O in 39.2. On a cuspidal representation of GL_n over the totally complex field, inverse transpose inverts the unramified Satake parameters. Cuspidal rigidity identifies this action with the contragredient [13].

Lean: `PositivityTheory.JMult1.gelfandKazhdanDual_of_rigidity` (`Development.WeilBound.CoreStatements.TemperedRepresentationTheory.MultiplicityOneRigidity.GelfandKazhdanDualOfRigidity`, lines 72–78)

For the second representation field use the dual local–global field of O on conjugate self-dual regular algebraic cuspidal representations of rank $a \geq 2$, with polarization $\chi_{E/F}^{a}$. Fix $r, w_{\mathrm{gt}}, H_\tau$ before the place $w \mid \ell$, prolongation, period datum and bridge. For each correspondence of Definition 40.2, suppose its

normalized local parameter passes the inertia test of Definition 1.20. Wild inertia acts trivially since its image is finite and all its elements are unipotent. When $d_w = 2$, complete splitting gives $E_w \simeq \mathbf{Q}_\ell$, and the repeated-root condition gives the full tame test of Definition 39.5. When $d_w = 1$, every eigenvalue of a tame generator has order dividing $q_w - 1$. Its image is semisimple of finite order, so its q_wth power equals itself. Frobenius therefore centralizes the inertia image, which gives the toral test. In both cases Definition 39.6 gives the stated comparison.

The shared local fields use Bernstein–Zelevinsky finiteness, Bernstein's unitary product theorem and Zelevinsky's polynomial ring on segments [14, 15, 16]. Use 34.16–34.17 for the global Weil and doubling formulas. For the unramified local integral, the spherical section restricted to the doubling embedding is constant on each K double coset. The zonal coefficient has the same sums over its right K cosets as the Borel spherical function with parameter α. Thus with $\operatorname{vol} K = 1$ the integral has the same double-coset sums as that spherical calculation. In the length generating series the variable is $X = q^{-s-a/2}$. By the unramified local doubling calculation of Piatetski-Shapiro and Rallis [30, Part A, §3, Theorem 3.1] and Lapid and Rallis [83, Proposition 3] for the split unitary group GL_a, in terms of X the resulting rational function is the quotient of the standard local L-factor $L(\alpha, s + 1/2)L(\alpha^{-1}, s + 1/2)$ by the unramified degenerate principal-series normalizing factor $\prod_{j=1}^{a}(1 - q^{-2s-j})^{-1}$, namely

$$\frac{\prod_{j=1}^{a}(1 - q^{a-j}X^2)}{\prod_{i=1}^{a}(1 - \alpha_i q^{(a-1)/2}X)(1 - \alpha_i^{-1} q^{(a-1)/2}X)}.$$

The double-coset series is absolutely summable for $|X| < \delta$ for some $\delta > 0$ depending on the local parameters. The zonal coefficient is constant on each double coset, so its absolute contribution is the absolute value of its coset sum times the absolute section value. Hence the same summability bound applies to the integral. Substitute $X = q^{-s-a/2}$ and take $s_0 = -\log_q \delta - a/2$ to obtain the asserted two Satake factors and finite product.

The Harish–Chandra results give the fixed-level finiteness, discrete series and formal degree, and square-integrable Schur orthogonality in the forms stated above. Use 36.9 for the smooth globalization and the seminorm bound for the intertwiner. For cuspidal decay at fixed open level, we use the following quantitative form of the reduction argument. Here h is the height used for the quantitative estimate and $q(g) = \inf_{\gamma \in U(V)(F)} h(\gamma g)$, with $h, q \geq 1$. A letter $i = (\tau, X)$ is a real-place Lie direction, and D_i is differentiation along right translation by the corresponding one-place exponential. For a word $l = (i_1, \dots, i_s)$ put $D_l = D_{i_1} \cdots D_{i_s}$, with the empty word acting as the identity.

Claim 1. Fix the arithmetic datum, the compatible infinite embeddings, V, and an open subgroup U of the full adelic group. For every pair of nonnegative integers A, N, a finite set L of words and a constant C can be chosen independently of the function. Let φ be a continuous, left rational-invariant cuspidal function that is right invariant under U intersected with the finite adelic subgroup. Assume smoothness in the finite products of one-place exponentials, and an estimate $|D_l\varphi(g)| \leq c_l h(g)^A$ for every word with some constant c_l independent of g. For any choice of the bounds c_l on the selected words $l \in L$, one has

$$|\varphi(g)| q(g)^N \leq C \sum_{l \in L} c_l \qquad (g \in U(V)(\mathbb{A}_F)).$$

Write $\mathcal{C}$ for the class with the stated smoothness, invariances, cuspidality and exponent A. Right Lie differentiation preserves this class. It commutes with left translation and the finite subgroup, and appending one derivative preserves the common exponent for all words. Differentiating the vanishing constant term gives the vanishing constant term of the derivative.

Lean: `GlobalTheory.T.rightDerivative_smoothCuspidal_same_level` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.RightDerivativeSmoothCuspidalSameLevelStatement`, lines 10–11)

We now give the uniform geometric input. At the fixed finite level there are finitely many finite adelic class representatives. Intersect their conjugates of the level subgroup, so one finite neighbourhood works for every representative. If $\dim_E V = 0$, the group is trivial and the empty derivative word gives Claim 1. In positive rank let $\lambda_1(g) > 0$ be the length of a shortest nonzero integral vector in the resulting real lattice. Choose fixed small $s_1, \eta_1 > 0$, with $\eta_1 \leq 1$, and put $\delta = \min\{\eta_1, (2\lambda_1(g)/s_1)^{1/(n+1)}\}$, where $n = \dim_E V$. Then every nonzero integral vector has length at least $s_1\delta^{n+1}/2$. Quantitative reduction gives a rational

translate with finite component in one fixed compact set and real matrix entries bounded by $C\delta^{-(n+1)b_0}$, for fixed C, b_0. The matrix height comparison therefore gives $q(g)\delta^b \le K$ for a fixed integer b. If $\delta < \eta_1$, a shortest vector has length at most $s_1\delta^{n+1}$. In a reduced basis, compare the successive minima with the $n+1$ scales $s_1\delta^k$. For some scale $t \le s_1$, the vectors of length at most δt span a nonzero E-space W containing every integral vector of length less than t. Indeed the spans at successive scales are nested, and each strict inclusion increases their dimension. Taking s_1 below the nonzero integral pairing threshold makes W isotropic. The root coordinates of its unipotent radical are ratios across this gap, or products of such ratios for the long roots, and are bounded by $C\delta$ after conjugation. The bounded fundamental set in these coordinates therefore gives real displacements of size at most $C\delta$. Strong approximation in that radical puts the remaining finite displacement in the chosen level subgroup. These choices depend on the lattice data and the level, and are independent of the function.

Fix a real basis of the infinite Lie algebra. On a sufficiently small identity neighbourhood, exponential coordinates have bounded derivatives and inverse derivatives. The mean-value formula therefore uses the same finite set B of basis directions for every such displacement. Taking maxima over the finitely many finite adelic representatives gives constants $K \ge 1, \delta_0 > 0$ and the same integer $b \ge 0$, independent of the function. For each g they give $0 < \delta \le 1$ with $q(g)\delta^b \le K$. When $\delta < \delta_0$, they also give a set Ω_g on which $K^{-1}q(g) \le q(w) \le Kq(g)$ and

$$|\psi(g)| \le K\delta \sum_{i\in B} u_i \qquad \text{if } \psi \in \mathcal{C} \text{ and } |D_i\psi(w)| \le u_i \text{ on } \Omega_g \text{ for nonnegative } u_i.$$

Indeed the cusp scale writes each translate ng from the adelic unipotent radical of the chosen isotropic frame as γguk, with γ rational, $k \in U$ finite, and the real displacement u of size bounded by a constant times δ. Both invariances remove γ, k. The mean-value estimate compares $\psi(gu)$ and $\psi(g)$ on a fixed compact right thickening of g, whose quotient heights are comparable by the height inequalities for right multiplication and its inverse. Averaging the difference over the corresponding rational quotient with its probability measure and using the zero constant term gives the displayed inequality.

Lean: `GlobalTheory.T.adelicUnitary_every_point_has_cusp_scale` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.AdelicUnitaryEveryPointHasCuspScaleStatement`, lines 10–11)
Lean: `GlobalTheory.T.meanValue_ineq_rightTranslate_realPoint_near_one` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.MeanValueIneqRightTranslateRealPointNearOneStatement`, lines 10–11)
Lean: `GlobalTheory.CuspGeo.norm_le_of_unipRad_variation` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.NormLeOfCuspidalVariationUnipotentRadical`, lines 15–36)
Lean: `GlobalTheory.CuspGeo.cuspDecayBound` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.CuspidalModerateGrowthRapidDecay`, lines 20–105)

By left invariance the derivative word estimates descend to quotient height, with constants at most $2^A c_l$, by using a translate of height at most $2q(g)$. Put $m = b+1$, $Q = q^{1/m}$, and take L_k to be the words in B of length at most k, including the empty word. With bounds $d_l q^A$ on these derivatives, each d_l is nonnegative since $q \ge 1$, and induction gives

$$|\psi(g)|Q(g)^k \le C_k q(g)^A \sum_{l\in L_k} d_l.$$

At $k = 0$ this is the empty-word bound. If $\delta \ge \delta_0$, then $Q(g)\delta \le K^{1/m}$, so $Q(g)$ is bounded and the empty-word bound suffices at the next step. Otherwise apply the induction hypothesis to $D_i\psi$ on Ω_g. Its constants for words $l \in L_k$ are d_{li}, where li appends i on the right. Height comparability bounds $D_i\psi$ there by a constant times $q(g)^A Q(g)^{-k} \sum_{l\in L_k} d_{li}$. Inserting these bounds into the one-step inequality and using $Q(g)\delta \le K^{1/m}$ proves the induction. In the double sum each term belongs to L_{k+1}; bounding the multiplicity by $|B||L_k|$ suffices. Taking $k = m(A+N)$ and absorbing 2^A proves the claim.

Lean: `GlobalTheory.iterated_gain_rapid_decay` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.IteratedGainRapidDecay`, lines 153–211)
Lean: `GlobalTheory.Published.CuspDecayBound` (`Development.AbelianVarietyTheory.PublishedTheorems.TwistedTraceFormula.UnifModGrowthOfIsUnifModerate`, lines 17–31)

To obtain the rapid-decay field, take any compatible exponential family and any smooth uniformly moderate cuspidal automorphic function for that family. The real-points description identifies the family with the componentwise exponential, and smoothness in those exponential coordinates gives smoothness in products of one-place exponentials. The comparison of heights changes the word constants while retaining one exponent A for every word; a one-place word is a word in the full infinite Lie algebra by inserting zeros

in the other components. Take U from the automorphic-function level condition. For each N, Claim 1 therefore gives a bound in q^N. The quotient height defined using the compatible embeddings is bounded by a constant times q, so one obtains the rapid-decay inequality at that height for each N. This is the adelic quantitative form of the decay argument; for the classical Siegel-set statement see Miller and Schmid [27, Theorem A].

Lean: `GlobalTheory.Grow.cuspRapidDecay_of_bound` (`Development.AbelianVarietyTheory.PublishedTheorems.TwistedTraceFormula.UnifModGrowthOfIsUnifModerate`, lines 88–97)
Lean: `GlobalTheory.Published.CuspRapidDecay` (`Development.AbelianVarietyTheory.PublishedTheorems.AutomorphicForms₂.ExistsNormLeHeightPow`, lines 28–32)

For the Sobolev form of the estimate, expand the derivative in a real-place Lie direction X as the sum of the complex-linear right derivatives in the matrix-unit directions at the same place, with coefficients X_{ij}. A word of length at most d thus has growth seminorm bounded by a constant times the sum of growth seminorms over the matrix-unit words of length at most d. This follows by expanding one letter at a time and applying the triangle inequality in the growth seminorm. Take d to be the maximum length in L, with $d = 0$ if L is empty, and apply Claim 1. The constants in these expansions depend on the selected words; their sum is absorbed in the uniform constant. The same argument applies to each derivative of a smooth uniformly moderate cusp form, since differentiation preserves its level, cuspidality and common growth exponent.

Lean: `GlobalTheory.Size.exists_growthSize_rWord_le` (`Development.AbelianVarietyTheory.PublishedTheorems.TwistedTraceFormula.ExistsGrowthSobNeTopOfIsAutFunSm`, lines 85–106)
Lean: `GlobalTheory.Size.cuspDecaySobolev_of_bound` (`Development.AbelianVarietyTheory.PublishedTheorems.TwistedTraceFormula.ExistsGrowthSobNeTopOfIsAutFunSm`, lines 108–121)

For a smooth compactly supported test f, the derivatives of $R(f)u$ are convolutions of u against derivatives of f, with support contained in $\operatorname{supp} f$. The quotient kernel for such a test is the sum of its translates by rational points. The number of contributing translates on the support is bounded by a power of the group height, uniformly in the evaluation point. By Cauchy–Schwarz on the quotient, these convolutions have moderate-growth bounds with constants $C_l\|u\|_2$ and a common exponent determined by the support. The convolution has a fixed open level and is cuspidal. Apply the finite-word decay estimate with decay order zero to obtain $|R(f)u(g)| \le C_f\|u\|_2$ at every g. For any finite orthonormal system (u_i), Bessel's inequality for evaluation of the convolution gives $\sum_i |R(f)u_i(g)|^2 \le C_f^2$. Integration over the finite-volume quotient gives a bound independent of the orthonormal system, hence $R(f)$ is Hilbert–Schmidt. Dixmier–Malliavin factorization, with the first support in the prescribed identity neighborhood, writes f as a finite sum of convolutions [17]. At the finite places use normalized averaging at a common open level of the test. Thus $R(f)$ is a finite sum of products of two Hilbert–Schmidt operators and is trace class.

Flath's theorem gives restricted tensor factorization of the admissible irreducible adelic representation, with spherical factors almost everywhere [18]. Strong approximation applies to the special unitary group in the stated noncompact range [19]. Theorem 32.31 gives the regular Green field of P from 32.5, Theorem 32.37 supplies the polynomial-classification entry 32.6, and Green's classification supplies the cuspidal and series entries in F [20]. Use Clozel for cohomologicality and purity in the regular algebraic range [21]. Apply 26.22 with finite unramifiedness and rank $n \ge 1$ to obtain the single definite base change at all places. The geometric-lemma filtration and the segment classification give the unlinked products and standard cosocle. Use depth-zero support and Jacquet injectivity for the first-congruence fields. The generic unitary dual gives the complementary products, and the strict Satake bounds and cuspidal rigidity are the Jacquet–Shalika results in the unitary cuspidal case [13].

For the finite Galois prime-existence field, apply 36.24 to the kernel of restriction to the extension and a lift of the specified automorphism. The resulting prime has residue degree and ramification index one over $\mathbf{Q}$. Its Frobenius restricts to the automorphism and its inertia acts trivially on the extension. For the second field use the prescribed open normal subgroup and finite avoidance set in 36.24.

Artin reciprocity gives the ℓ-adic and ray-character fields. Above ℓ, de Rham comparison for algebraic characters and potential semistability give the Weil–Deligne comparison. For every compatible local component ξ and every admissible correspondence R, local reciprocity gives the character parameter of ξ with zero monodromy. The rank-one law of 40.2 identifies it with $R(\xi)$. For the integral crystalline lattices, use the prismatic comparison of 36.15 and its functoriality on lattice maps. The CM curve clause uses Gross's $A(d)$ over $\mathbf{Q}(j(\mathcal{O}_K))$ [26, Ch. 5]. Its minimal discriminant ideal gives the stated local equations and good reduction. For the Weber field use the elliptic main theorem of complex multiplication at modulus (n) [7, Ch. 6]. This is the principal-modulus specialization of the ray class field theorem. The two formulas in

34.17 give the convergent Siegel–Weil fields, the second with its volume and Gram factors. The Schrödinger formula on the Fock vector gives the indicated special value and 34.16 gives continuation.

For the nonendoscopic hypotheses, combine the local and global results above with the segment classification, the depth-zero results, local converse and the existence of the local correspondence in Definition 39.1. The cohomology factor uses the local and global finiteness and Euler characteristics, Poitou–Tate, local duality and unramified orthogonality, Greenberg–Wiles and local Bloch–Kato. Attach Lemma 1.19 and the Hecke–Tate functional equation [12]. The Weil action and the discrete-series results, together with trace class, the two factorizations, the integral formulas and partial Hecke continuation, give the analytic hypotheses in Definition 1.20. For its de Rham count, fix a compatible embedding inducing w and take its de Rham component over a local coefficient field containing the image of the completion at w. De Rham comparison gives dimension a for that component. Its filtration is exhaustive and separated, so the dimensions of its graded pieces sum to a. These dimensions are the jump multiplicities. For the remaining local cohomology fields use 36.14. Fontaine's B_{cris} theorem, global reciprocity, local Bloch–Kato, Tate duality, local Euler characteristic and finiteness, and Bhatt–Scholze integral comparison give $\mathscr{H}_2$ as in Definition 36.15.
Lean: `UpperTheory.LiteratureTheorems.literature_nonexistence` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin`, lines 639–644)

For the prime bound in F, apply the general progression estimate of Thorner–Zaman for $q > 2$ and $\gcd(a, q) = 1$. Their relative error is less than one at a sufficiently large fixed power of q, and the main term is positive [28, §1.2]. For $q = 1, 2$, take $p = 3$ and enlarge the absolute constant. Use the CM theorem for j [7], finite-field cuspidal mirabolic restriction in the stated Kirillov model and Weyl branching. The convergent Rallis formula of 34.17 uses theta convergence, doubling and the first convergent Siegel–Weil formula. Take the Hermitian classification and Hasse principle, and quadratic class field theory. The Borel–Casselman isomorphism gives the Iwahori projection for admissible GL_n over the nonarchimedean local field [22, 23]. Radford's theorem gives the polynomial shuffle ring on the Lyndon words in characteristic zero [24]. Hecke continuation and nonvanishing at one give the analytic Hecke fields [12].

For (1.8), a primitive χ of conductor $q > 1$ is nonprincipal. Its L-function is entire and nonzero on $\operatorname{Re} z \geq 1$. Use the concentric disks about 2 of radii 7/2 and 15/4. On the larger disk the bound for $\operatorname{Re} z \geq -7/4$ gives $|L(z, \chi)| \ll q^4$ uniformly, while the Euler product gives $|L(2, \chi)| \geq 1/2$. Write D for the smaller closed disk and N_D for the zero count there with multiplicities. Jensen's formula gives $N_D \leq C_1 \log(2q)$. The logarithmic derivative formula on nested disks gives

$$\left| \frac{L'}{L}(1, \chi) - \sum_{\rho \in D} \frac{\operatorname{ord}_\rho L}{1 - \rho} \right| \leq C_1 \log(2q),$$

with C_1 enlarged if necessary. Divide by the Blaschke factors for the zeros in D, with respect to the larger disk. The quotient is zero-free in the smaller disk. Its logarithmic modulus has the same upper bound on the larger boundary, and its modulus at the centre is at least $|L(2, \chi)|$. The interior derivative bound for a holomorphic logarithm gives the estimate at 1. The derivatives of the denominators of the Blaschke factors contribute $O(N_D)$ since 1 is in a fixed interior disk. Every zero in the sum of (1.8) belongs to D. Each other zero in D has $\operatorname{Re} \rho \leq 0$ or $|\operatorname{Im} \rho| > 1$, so $|1 - \rho| \geq 1$. Removing its contribution costs at most N_D in total. This proves the bound with $C = 2C_1$.

For the real unitary fields use the discrete-series lowest types and the holomorphic chamber. The closure of an admissible $(\mathfrak{g}, K)$-submodule of a unitary $\mathrm{U}(p, q)$-representation is invariant and has that submodule as its compact-finite space. An isomorphism of the compact-finite modules of two irreducible unitary representations extends, after positive scalar normalization, to a unitary intertwiner [25]. These supply the admissible closure and infinitesimal equivalence fields. Cuspidal boundedness for the form and its invariant derivatives follows by the decay estimate above.
Lean: `GlobalTheory.gkGlobalisation_holds` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.InfinitesimalEquivalenceUnitaryGroupPQ`, lines 737–738)

Use the two real metaplectic realizations and the adelic Weil representation for the CM unitary pair, with Stone–von Neumann and general linear Howe duality. For the split local member the quadratic extension, nontrivial unitary additive character and the restriction of the splitting character give Kudla's action with the Levi multiplier in Definition 1.20 [29]. Conjugating translation by this action gives $H(w)\omega(g) = \omega(g)H(wg)$. In the Levi formula the square of the norm multiplier is the Jacobian of $x \mapsto xa$, which cancels in the integral. The Fourier operators are isometric with the self-dual measure

in their construction. Every additive Haar measure is a scalar multiple of that measure, so the isometry holds for each of them. For rational theta invariance on $\mathrm{Hom}_E(V,V') = V^* \otimes_E V'$ (where $V = E^N$ and $V' = E^{N'}$ carry the standard forms with identity Gram matrix and their standard lattices $\mathcal{O}_E^N$, $\mathcal{O}_E^{N'}$, NN' is even, and $(\gamma,\gamma') \in \mathrm{U}(V) \times \mathrm{U}(V')$ acts by $x \mapsto \gamma' x \gamma^{-1}$), global Poisson summation gives $\Theta(\omega(\gamma,\gamma')\varphi) = \Theta(\varphi)$. Since E is a CM field, these standard forms are definite at every archimedean place, so the archimedean components of rational γ,γ' lie in compact unitary groups, which preserve the Gaussian line. When $\varphi = \varphi_{\infty,0} \otimes \varphi_f$ uses the archimedean Gaussian $\varphi_{\infty,0}$, Kudla's splitting [29, Theorem 3.1 and Proposition 4.3] acts on $\varphi_{\infty,0}$ at each CM-type embedding τ by the partner splitting-character exponents $\epsilon' n'_\tau/2$ on $\det\gamma_\tau$ and $\epsilon n_\tau/2$ on $\det\gamma'_\tau$, multiplied by the metaplectic Gaussian half-determinant character of $x \mapsto \gamma'_\tau x \gamma_\tau^{-1}$, which contributes $-N'/2$ on $\det\gamma_\tau$ (right multiplication by γ_τ^{-1} across N' rows) and $+N/2$ on $\det\gamma'_\tau$ (left multiplication by γ'_τ across N columns), matching Lemma 17.14 and Definition 34.48(2)(d) (differing from the covariant model $X \otimes_E W$ of Proposition 39.60 by the contragredient sign on V^*). Thus $\omega_\infty(\gamma,\gamma')\varphi_{\infty,0} = \big(\prod_\tau \tau(\det\gamma)^{a_\tau}\tau(\det\gamma')^{a'_\tau}\big)\varphi_{\infty,0}$ with $2a_\tau = \epsilon' n'_\tau - N'$ and $2a'_\tau = \epsilon n_\tau + N$. Writing $F(z) = \Theta(W_z\varphi_{\infty,0}\otimes\varphi_f)$ for the archimedean Heisenberg translation operator W_z on $\prod_\tau M_{N'\times N}(\mathbf{C})$, the intertwining law $\omega_\infty(\gamma^{-1},\gamma'^{-1})W_z = W_{\gamma'^* z\gamma}\omega_\infty(\gamma^{-1},\gamma'^{-1})$ together with global theta invariance yields $\Theta(W_z\varphi_{\infty,0}\otimes\omega_f(\gamma,\gamma')\varphi_f) = \Theta(\omega_\infty(\gamma^{-1},\gamma'^{-1})W_z\varphi_{\infty,0}\otimes\varphi_f) = \big(\prod_\tau \tau(\det\gamma)^{-a_\tau}\tau(\det\gamma')^{-a'_\tau}\big)F(\gamma'^* z\gamma)$.
Lean: `GlobalTheory.Published.ThetaRationalInvariance` (`Development.AbelianVarietyTheory.PublishedTheorems.GlobalTheta.ThetaRationalInvarianceHolds`, lines 10–20)
Lean: `Formalism.MainNotions.StandardLatticesGL3.GA` (`Formalism.SharpRestriction.GlobalSchurBounds.StandardLatticesGL3`, lines 374–376)
Lean: `Formalism.MainNotions.StandardLatticesGL3.stdL` (`Formalism.SharpRestriction.GlobalSchurBounds.StandardLatticesGL3`, lines 395–396)

Use the uniqueness of the Hermitian discrete-series realization, the one-orbit ideal invariant and the Weil representation for the good models.

For the Siegel-set estimate, integrate in Iwasawa coordinates $g = pak$ with p in the modulus-one parabolic. In these coordinates Haar measure has the factor $\delta_P(t)^{-1}$ and positive torus measure $\prod_i dt_i/t_i$. Bound the test by $h(t)$ and integrate over the compact parabolic part and K. Their measures and the Haar normalizations give one constant C for all f,h. The reduction-theory height comparison on this set gives C', A uniformly for $g \in \mathfrak{S}$ and rational γ. Applying cuspidal decay in the quotient height and using this comparison gives the asserted decay on the set.

At an infinite place w let $\mathcal{P}_w$ be the polynomial oscillator module for the real unitary pair with the specified Weil splitting, with action ω_w. Divide by the span of $d\omega_w(X)f$ for source Lie algebra elements X and $\omega_w(k)f - f$ for elements of the source maximal compact. Call the quotient target module D_w. On the oscillator double cover this is the theta module for the genuine one-dimensional character corresponding under the splitting to the trivial source character. The pair is reductive with compatible Cartan involutions. If $D_w \neq 0$, the character occurs as a source quotient, and Howe gives an admissible finite-length target module with irreducible head [6, Theorems 1A and 2.1].

In polynomial coordinates the harmonics for a compact group are the simultaneous kernels of the constant-coefficient quadratic differential operators in its symplectic centralizer. Take joint harmonics for the two compact groups in the prescribed source compact character. This character has a single compact type. Howe's Lemmas 3.3 and 4.1(c) give the corresponding target type τ_w [6]. Pass the two generation identities of equation (4.5) to the source-character quotient defining D_w. Taking compact isotypic components is exact, since compact averaging lifts an isotypic vector through a quotient. The identities therefore give generation of D_w by the image of the joint type under the target enveloping algebra, and generation of $D_w[\tau_w]$ by that image under the source enveloping algebra. On the source-character quotient the latter algebra acts by scalars. Thus $D_w[\tau_w]$ is one copy of τ_w and generates D_w.

For the product of real groups the polynomial space is the algebraic tensor product of the $\mathcal{P}_w$. Quotienting by the source relations in all slots gives $D = \bigotimes_w D_w$. If some $D_w = 0$, the assertions hold for $D = 0$. Otherwise the tensor of the generating compact types occurs once and generates D. Projection to this isotypic space preserves compact-stable subspaces. Each proper target submodule has zero component of this type, as does the sum of two proper submodules. Each local composition factor is a subquotient of the countable-dimensional polynomial space $\mathcal{P}_w$, so their external tensor products are simple by Lemma 1.21. Tensoring the finite composition filtrations therefore bounds the length. The Fock map onto the polynomial test space identifies stable subspaces containing the source relations with submodules of D. Pullback preserves strict inclusions and sums. This gives the chain and proper-sum assertions.

Lean: `GlobalTheory.howeDualityReal_holds` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.HoweDualityRealHolds`, lines 18–47)

□

Proof of 1.23

Proof. Fix $h \in \mathcal{R}$. Theorem 26.4 gives $\mathcal{V}$. Write h_L for the conclusion of Theorem 40.1. For every number field, every finite place and every prolongation to an algebraic closure, there exists a local correspondence satisfying the laws of Definition 40.2. Put $O = (h, h_L, h)$ in 39.1. Proposition 1.22 gives P, F. Theorem 32.37 applied to the regular-character field gives $\mathcal{G}$.

Apply Theorem 34.7 with the doubling continuation and definite Siegel–Weil value of 34.16. Its local quotient maps supply the matched tests. For these maps, Theorem 34.7 directly supplies the full-rank decay (34.3), the proportional theta-integral identity (34.4) with a unique $c \in \mathbf{C}^{\times}$, and the step formulas (34.6)–(34.7).

Apply 34.25 with F, the vacuum generation of 34.18 and the first formula of 34.17, in the range $a < b \le 2a$ and $a + \mathrm{Witt}(B) < b$. Lemmas 34.30 and 34.33 give a column chain with nonzero bottom period. By the fixed-depth test calculation of Lemma 34.22 and the two exchanges of Lemma 34.23, every nonzero row t and prescribed nonzero bottom vector w have a nonzero period on an allowed column chain. Local Bessel functional dimensions are at most one, so the period factors as $A_w(\rho, \xi) c_{t,\Psi,\Phi}$ with real test pair (ρ, ξ) and column tests (Ψ, Φ). In a nonzero row space choose the real pair using one nonzero row, and for each nonzero row choose a column with $c_{t,\Psi,\Phi} \neq 0$. In a zero row space any real pair works. Theorem 34.41 uses this separation with the integral identity and local matching to supply the tests in 33.1.

Apply Theorem 35.11 with the fifteen proofs of Lemma 35.19, available before choosing the Galois and local correspondence data in O. The Siegel model uses one polarized fan for every odd level $N \ge 3$ of a fixed genus $g \ge 1$. For the unitary model use a polarized fan without self-intersection. Its function ϕ gives the weighted boundary BD, with Cartier ideal $(\prod_i t_i^{d\phi(\gamma_i)})$ in the boundary coordinates t_i, where $\gamma_i = M'\delta_i$ for the primitive generators δ_i of the chart cone and the level parameter M'. Here $d, k \ge 1$ are chosen with d clearing the denominators of ϕ. If $\det \omega$ is a line bundle, then $(\det \omega)^k \otimes \mathcal{O}(-BD)$ is ample. Lemmas 35.20 and 35.21 give the common size exponent and the nonterminal covers. Proposition 35.32 supplies C_T uniform in the field and tower. Take $C_{\mathrm{deep}} = C_T + 120$ in 35.26. The resulting bound h_X of 35.2 supplies $C_0 \ge 0, x_0$ for each eligible F_m, C_H, with the same x_0 valid for every $C \ge C_0$ at every eligible point and layer. The transfer of 35.101 uses the functional row of 34.69 on the same extension.

Lean: `UpperTheory.LiteratureTheorems.published_results_arithmetic` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin`, lines 781–809)
Lean: `SelfIntersectionVolumeEstimates.topCovolSlim2_use_alt_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.TopCovolSlim2Use`, lines 16–22)

For each $c \in \mathbf{R}$, use the parameters $p_0(h_L, c)$ of Chapter 33 (and Theorem 33.74), as entered in Theorem 33.1. At each point the norm choices carry the rationality law for the period maps and a grid of parameters $(1, c, b)$. The coefficient field contains the grid's prescribed layer-character values and roots of unity, and its abelian character list agrees with the two sides of the twisting data. Use the crystalline lattice realization in $\mathscr{H}_2$ supplied by P. On the cyclotomic extensions in the small-prime calculation, Lemma 36.17 gives the functorial Wach lattices and the integral action at the lower roots used in Lemma 36.19. Apply Theorem 36.16 with these components. This gives the linked bound at twisted indices above 2 and also above 3 when 3 is outside the small-prime set of the point's host. Theorem 33.1 gives an element of $\mathcal{H}$ from this bound together with P, F, Theorem 34.7 (with the paired theta-transfer conclusions), h_X, and h_L. Apply Theorem 1.24. □

Proof of 1.25

Proof. Take $h \in \mathcal{H}$. Apply Theorem 32.64 with Green's assertion $\mathcal{G}$ and the family class, selection and estimates supplied by h. This gives the connected-family assertion used in Theorem 2.2, with chain exponents $(e_D, e_L) = (2, 0)$. Indeed, fix $\lambda > 1$ and a request A for which hosts exist. Consider the families F_m with λ-windows that satisfy A at every point indexed by $i \in I_{F_m}(x)$ for $x > x_1(F_m)$. Suppose each such family has a threshold x_0 such that, for all $x \ge x_0$ and all $i \in I_{F_m}(x)$, the host field E, truncation index t and weight b at that point satisfy Definition 26.1 with bound $\sqrt{b}$. Thus for every choice of local data the lengths $s_{\ell,e}(E)$

are finite for $1 \le e < t$, and every finite prime sum in (26.1) is at most $\sqrt{b}$. Then for every sufficiently small $\varepsilon > 0$ there is a family of parameter ε in this class and a choice of chain bounds valid at every sufficiently large window point, as in Theorem 24.1.

Lean: `GlobalTheory.Fc.forall_goal_of_shape_of_hyps` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ShapeOf`, lines 31–32)

For every primitive ψ, Theorem 8.21 uses $\mathcal{V}$ to supply the required hosts with $\lambda = 3/2$.

Lean: `UpperTheory.Top.supplyChi` (`Development.AutomorphicArithmetic.CharacterBoxes.SupplyChi`, lines 8–16)

Apply Theorem 2.2. □

Proofs for Chapter 2: How arithmetic windows enter the zero-free criterion

Proof of 2.2

Proof. Fix a primitive ψ and take $A = A_\psi$ in the selection principle. At an indexed scale (x, ε, J) write t for its auxiliary rank and TT for its set of top positions. The arithmetic clause of A_ψ gives the arithmetic requirement of Definition 8.19 for the host and TT at the cyclic level $\mathcal{L}_\psi$, with $C = 4$. Let E be the number field of this host and put $b = e^{x^\varepsilon}$. For every algebraic closure of E and every compatible family of local data from 26.1, the request also requires finite $\mathbf{Z}_\ell$-lengths of $\mathrm{Sha}_{\ell,e}(E)$ for every prime ℓ and $1 \le e < t$, with $\sum_{\ell \in S} \log \ell \sum_{1 \le e < t} \mathrm{length}_{\mathbf{Z}_\ell} \mathrm{Sha}_{\ell,e}(E) \le \sqrt{b}$ for every finite set S of rational primes.

For a family in $\mathcal{C}_{\lambda,A}$, each indexed scale has coordinates x, ε when $x \ge x_1(\mathcal{F}) + 1$. Thus $x_1(\mathcal{F}) + 1$ serves both as the threshold in $E_{\mathcal{L}_\psi}(\mathcal{F})$, with $C = 4$, and as the threshold x_V for the length bound. Family selection now gives (2), since each selected family belongs to $\mathcal{C}_{\lambda,A_\psi}$ and therefore has λ-windows.

To prove (1), fix $e_D, e_L \in \mathbf{R}$, $\varepsilon > 0$, and an arithmetic parameter family $\mathcal{F}$ at ε satisfying the two hypotheses of (1). For all sufficiently large $x \in W(\mathcal{F})$, the tied chain gives an index i with the chain hypothesis of 6.12 at $D_{x,i}$ (with the rank and chain-length window $m \ge x^{1/2-\varepsilon}$ from Definition 33.3(2) and the completed-host datum of 5.1 supplied via Theorem 8.17). The condition $E_{\mathcal{L}_\psi}(\mathcal{F})$ gives the arithmetic hypothesis at the same index. Thus 6.12 gives (1). Apply 2.3 and 2.6. □

Proof of 2.4

Proof. Choose e_D, e_L, ε_1 from (2). Fix a kept character and s with $\sigma = \mathrm{Re}\, s > 1/2$. Set

$$b_0 = \tfrac{1}{2} \min(\sigma - \tfrac{1}{2}, \tfrac{1}{2}), \qquad \theta = \tfrac{1}{2} + b_0, \qquad D = e_D + e_L + 7,$$
$$\varepsilon = \min\left(\tfrac{\varepsilon_1}{2}, \tfrac{b_0}{|D|+1}\right).$$

Then $0 < b_0 \le 1/4$, $1/2 < \theta < \min(\sigma, 1)$, $0 < \varepsilon < \varepsilon_1$, and $D\varepsilon \le b_0$. Take the family from (2) and the threshold x_1 in (1). On its allowed scales above $\max(x_1, 1)$,

$$G_m^{\mathcal{L}}(y) \le y^{1/2 + D\varepsilon} \le y^\theta.$$

Choosing each family window beyond $\max(Y_0, x_1, 1)$ gives $\mathcal{H}_\lambda(\{y \in [1, \infty) : G_m^{\mathcal{L}}(y) \le y^\theta\})$. The assumed window implication gives nonvanishing throughout $\mathrm{Re}\, s > \theta$, which contains the chosen point since $\theta < \sigma$. □

Proof of 2.7

Proof. Index F by $\mathcal{X}_{\mathcal{L}}$, with its characters and moduli, weight 1 at modulus one and $w_{\mathcal{L}}(\chi) > 0$ elsewhere. A primitive principal character has modulus 1, so admissibility holds.

On the support, $1 \le n \le \lfloor y \rfloor_+$ gives $n \le y$, hence $\max(y/n - 1, 0) = y/n - 1$. For $y < 1$ the support is empty. At modulus one, $\chi(n) = 1$ for all n, and the centred term is $G_z(y)$. For each nonprincipal χ, both definitions give $S_\chi(y) + yL'(1, \chi)/L(1, \chi)$, with S_χ as in 2.10. The nonvanishing assertions have identical indices. □

Proof of 2.13

Proof. Given $\eta > 0$ and $U_0 \in \mathbf{R}$, use $\mathrm{WH}(F;\theta,\delta)$ to choose $Y \geq \max(e^{U_0},1)$ and its envelope R. Put $U = \log Y$ and $R_U(u) = R(e^u)$. Then $U \geq U_0$, and exp maps I_U bijectively onto J_Y, since $e^{(1+\delta)U} = Y^{1+\delta}$. The envelope inequalities follow. Apply Folland's change-of-variables theorem [31, Theorem 2.47(a)] to $u \mapsto e^u$ and $y \mapsto R(y)y^{-\theta-1}$ on J_Y, extended by zero outside J_Y. This specialization gives integrability and

$$\begin{aligned}\int_{I_U} R(e^u)e^{-\theta u}\,du &= \int_{I_U} R(e^u)(e^u)^{-\theta-1}e^u\,du \\ &= \int_{J_Y} R(y)y^{-\theta-1}\,dy \leq Y^\eta = e^{\eta U}.\end{aligned}$$

□

Proofs for Chapter 3: Zeros become poles

Proof of 3.1

Proof. By 3.9, g is Borel measurable and $|g(u)| \leq (\sum_i a_i C_{\chi_i})e^{2u}$ for $u \geq 0$. By 3.16, $2\operatorname{Re} G_{\chi_i} = G_{\chi_i} + G_{\chi_i^{-1}}$. Multiply by e^{-su} and integrate termwise for $\operatorname{Re} s > 2$ by 3.8. Both characters satisfy 3.5, since $\chi_i^{-1} = 1 \iff \chi_i = 1$. Fix z with $\operatorname{Re} z > 0$. By 3.5 and finiteness of I, there is a punctured neighbourhood of z on which $L(w,\chi_i)L(w,\chi_i^{-1}) \neq 0$ for every $i \in I$. By 3.16, this neighbourhood misses Z_F. If $\operatorname{Re} s > 0$ and $s \notin Z_F$, each summand is analytic at s by 3.5 and 3.16. By definition Z_F lies in $\operatorname{Re} s > 0$, and 3.17 gives the residues. □

Proof of 3.5

Proof. For $\chi \neq 1$, apply 3.7 with $b = 0$ to the entire function $f = L(\,\cdot\,,\chi)$, which has $f(1) \neq 0$. Use 3.12 and 3.8 for the integral.

If $\chi = 1$, then $N = 1$. The entire function $f(s) = (s-1)\zeta(s)$ for $s \neq 1$ has $f(1) = 1$ and $f'(1) = \gamma$ by the Laurent expansion [5, Section 25.2]. On $\operatorname{Re} s > 2$, the Euler product gives $\zeta(s) \neq 0$, so $f'/f = \zeta'/\zeta + 1/(s-1)$. With $b = 1$, 3.7 gives the numerator $\zeta'/\zeta + 1/(s-1) - s\gamma + 1 - s$ of 3.14. Apply 3.8 for the integral. The factor $s-1$ preserves zero orders away from 1. At 1, $f(1)$ and the assigned $\zeta(1)$ are both nonzero. □

Proof of 3.7

Proof. Since $N(1) = 0$, $T(s) = N(s)/(s-1)$ extends analytically with $T(1) = N'(1)$. Near 1 use $\Phi_f(s) = -T(s)/s$. Away from $0,1$ use the stated quotient. The identity theorem and $f(1) \neq 0$ give isolated zeros. At a zero $\rho \neq 0,1$, 3.6 gives $D(s) = m/(s-\rho) + h(s)$, with $m = \operatorname{ord}_\rho f$. For $Q(s) = -1/[s(s-1)]$,

$$\begin{aligned}\Phi_f(s) = \frac{mQ(\rho)}{s-\rho} + m\frac{Q(s)-Q(\rho)}{s-\rho} \\ + Q(s)\big(h(s) - sD(1) + b(1-s)\big).\end{aligned}$$

The second term extends with value $mQ'(\rho)$, and the third is analytic. The residue is $mQ(\rho) = m/[\rho(1-\rho)]$. If $\rho \neq 0$ and $f(\rho) \neq 0$, take $m = 0$ and $h_\rho = \Phi_f$. □

Proof of 3.15

Proof. The meromorphic Hurwitz function has a unique pole, at 1, with residue 1 [5, Section 25.11]. Since J is entire and $J(1) = 1$, subtracting $J(s)/(s-1)$ removes this pole. This proves the existence of the defining limit for $H_a(1)$ and the holomorphy of R_a at 1. Away from 1, both terms defining R_a are holomorphic. Thus R_a is entire.

Conjugation for the gamma function and the positivity of π give

$$J(\bar{s}) = \overline{J(s)}, \qquad \Gamma_{\mathbf{R}}(\bar{s}) = \overline{\Gamma_{\mathbf{R}}(s)}$$

away from the gamma poles; the rule $\Gamma(\bar{s}) = \overline{\Gamma(s)}$ follows from Euler's integral [5, (5.2.1)] on $\operatorname{Re} s > 0$ by analytic continuation. At $s = -2n$, the latter identity holds by the assigned value zero, while the former holds by continuity of the entire reciprocal.

For $\operatorname{Re} s > 1$, the absolutely convergent series

$$H_a(s) = \sum_{n=0}^{\infty} (n + a_*)^{-s}$$

has positive real bases, so $H_a(\bar{s}) = \overline{H_a(s)}$. It follows that the entire functions $R_a(s)$ and $\overline{R_a(\bar{s})}$ agree on that half-plane. The identity theorem gives equality on $\mathbf{C}$. Adding back $J(s)/(s-1)$ proves conjugation for H_a when $s \neq 1$; at 1, use $H_a(1) = R_a(1)$. □

Proof of 3.16

Proof. For $\operatorname{Re} s > 1$, absolute convergence permits grouping the Dirichlet series by residue classes modulo N, giving the displayed finite Hurwitz expansion [4, Chapter 12, (8), p. 255]. Meromorphic continuation gives the identity for $s \neq 1$. If χ is nonprincipal, then $\sum_j \chi(j) = 0$; consequently the polar terms $J(s)/(s-1)$ in the finite sum cancel, and its value at 1 is $N^{-1} \sum_j \chi(j) H_{j/N}(1)$. If χ is principal, the same formula at 1 is its assigned value. Thus the expansion holds at every $s \in \mathbf{C}$.

Since $N > 0$, conjugation preserves $N^{-s} = \exp(-s \log N)$. Conjugating the finite expansion and using Lemma 3.15 and $\overline{\chi(j)} = \chi^{-1}(j)$ proves

$$L(\bar{s}, \chi^{-1}) = \overline{L(s, \chi)}.$$

Conjugation preserves zero, which gives the equivalence for ρ.

For nonprincipal χ, the functions $L(s, \chi)$ and $L(s, \chi^{-1})$ are holomorphic and nonzero at 1 [33, Sections 4, 13–14]. Differentiating

$$L(s, \chi^{-1}) = \overline{L(\bar{s}, \chi)}$$

at the real point 1, and then dividing by the nonzero values, gives

$$D_{\chi^{-1}}(1) = \overline{D_\chi(1)}.$$

Each ramp in Definition 2.10 is a finite sum with real weights $\Lambda(n) \max(y/n - 1, 0)$. The character identity therefore gives $S_{\chi^{-1}}(y) = \overline{S_\chi(y)}$. For a principal character, the centring term is real. For a nonprincipal character, use the identity for $D_\chi(1)$. These two cases give the asserted conjugation identity for $G_\chi(y)$. □

Proofs for Chapter 5: Counting primes class by class

Proof of 5.2

Proof. The map $\{\pm 1\} \to \mathbf{Z}/2\mathbf{Z}$ displayed above is an isomorphism. Read the two signs in the order d, q and write the last coordinate additively. The inverse reads the signs in the order q, d. The formulas for the distinguished elements follow by substitution. Under T, the condition $\sigma_d = 1$ defining Ψ_d becomes $v_d = 0$. Composition with an isomorphism is a bijection of character groups, with inverse composition with T^{-1}. Evaluating $\theta_\alpha = \alpha \circ T$ at $\tau_1(s)$ for $s \in A_0$ gives the assertion about trivial restrictions. Evaluating it at ι_d gives the correspondence for X_+. □

Proof of 5.3

Proof. The divisibility $2^{e+2} \mid Q_S$ gives $2 \in S$. Also $q \in S$ and $d \nmid Q_S$, while $q \neq d$ because their residues modulo 8 differ. Thus $d \notin S$. If p is prime and $p \notin S \cup \{d\}$, then $p \nmid \mathfrak{m}$. The local unit subgroup defining inertia is trivial, so $I_p = \{0\}$.

Suppose $p \in S$. Since $d \mid \mathfrak{m}^{(p)}$, a unit defining inertia is 1 modulo d. Its d-sign is therefore 1. A unit defining Frobenius is p modulo d, and its d-sign is (p/d). For odd p, quadratic reciprocity and $d \equiv 3 \pmod 4$ give

$$\left(\frac{p}{d}\right) = \left(\frac{-d}{p}\right) = 1.$$

This uses the splitting condition on Q_S when $p \mid Q_S$, and the host condition when $p = q$. For $p = 2$ the same conclusion follows from $d \equiv 7 \pmod 8$. Thus $F_p \in \Phi$ and $I_p \subseteq \Phi$.

At d, the local units are those congruent to 1 modulo $Q_S q$, so their Artin images lie in $\{1, \iota_d\}$. The Chinese remainder theorem supplies such a unit which is -1 modulo d. Its d-sign is $(-1/d) = -1$, so its image is ι_d. Lemma 5.2 now gives $I_d = \{0, \epsilon_d\}$. Addition by ϵ_d exchanges Φ and its complement. □

Proof of 5.6

Proof. For $n > N$ and $\rho \in \Phi$ the positive part $[a/n - w(\rho)]_+$ is zero. Expand the von Mangoldt sum into prime powers, using $\Lambda(p^i) = \log p$, and interchange the finite sums. Lemma 5.5 makes the contribution of a prime power p^i equal to

$$\frac{2\log p}{|I_p|} \sum_{\substack{g \in iF_p + I_p \\ g \notin \Phi}} [a/p^i - w(r_t(g))]_+.$$

For $p \in S$, the coset $iF_p + I_p$ is contained in Φ, so the inner sum is empty. For $p \notin S \cup \{d\}$, inertia is trivial and the contribution is

$$2\log p\, \mathbf{1}_{iF_p \notin \Phi} [a/p^i - w(r_t(iF_p))]_+.$$

If $F_p \in \Phi$, all its multiples lie in Φ. If $F_p \notin \Phi$, its first binary coordinate is 1, so $iF_p \notin \Phi$ precisely when i is odd. At d, inertia has order 2, which gives the second line of the definition of ASK_N.

Since $p^i \le N$ implies $p, i \le N$, the rectangular cutoff retains every nonzero term. When $d > N$, every d-term vanishes. □

Proof of 5.7

Proof. For the trivial character every inertia average equals 1, so $v_{\mathbf{1}}(n) = 1$. Fix $g \notin \Phi$ and put $W = w(r_t(g))$. Since $r_t(g) \in \Phi$, we have $W > 0$ and $a/W \le N$. Then

$$\sum_{n=1}^{N} \Lambda(n) v_\alpha(n) [a/n - W]_+ = W G_\alpha(a/W) + R_\alpha(W),$$

where

$$R_\alpha(W) = \begin{cases} a\log a - a\log W - (1+\gamma)a, & \alpha = \mathbf{1}, \\ -a\ell_\alpha, & \alpha \neq \mathbf{1}. \end{cases}$$

Indeed the terms with $n > a/W$ vanish, and multiplication of the defining sum for $G_\alpha(a/W)$ by W gives the displayed formula.

The half-group character sum is

$$\sum_{g \notin \Phi} \overline{\alpha(g)} = \begin{cases} f, & \alpha = \mathbf{1}, \\ -f, & \alpha = \chi_d, \\ 0, & \alpha \notin \{\mathbf{1}, \chi_d\}. \end{cases}$$

Since $G \setminus \Phi = \epsilon_d + \Phi$, the character sum is $\overline{\alpha(\epsilon_d)} \sum_{h \in \Phi} \overline{\alpha(h)}$, which vanishes when $\alpha|_\Phi$ is nontrivial. The two characters trivial on Φ are $\mathbf{1}$ and χ_d, whose values on the other coset are 1 and -1.

Substitution in Lemma 5.6 gives $\Delta(a, t, w)$ and the remainders. The coset sum leaves $a\ell_{\chi_d}$ from the nontrivial characters. In the trivial remainder, change variables by $\rho = r_t(g)$ to obtain $a\log a - (1+\gamma)a - (a/f)\sum_{\rho \in \Phi} \log w(\rho)$. Together these give $\mathrm{Main}(a, w)$. □

Proof of 5.8

Proof. The identity $-e_1(s + u - t) = e_1(t) + \tau(-s) + \tau(-u)$ gives

$$\sum_{u \in A_0} \alpha(-e_1(s+u-t)) = \alpha(e_1(t))\alpha(\tau(-s)) \sum_{u \in A_0} \alpha(\tau(-u)).$$

For $\alpha \in P$ the last two factors are 1 and $|A_0|$. Otherwise the restriction to A_0 is a nontrivial character. Translation by an element on which it differs from 1 shows that its sum is zero. Multiply the identity for each α by $bG_\alpha(y)/f$, then sum over $s \in T_2$ and $\alpha \in \widehat{G}$. The surviving coefficient is $|A_0|b|T_2|/f = |A_0|b\delta/2$. □

Proof of 5.9

Proof. The condition on a/b gives $0 < (x-1)b \le a \le (x+1)b$. The inverse image of the thick position $((0,1), s+u)$ under r_t is $e_1(s+u-t)$. Since $\overline{\alpha(g)} = \alpha(-g)$, replacing $w(r_t(g))G_\alpha(a/w(r_t(g)))$ by $bG_\alpha(a/b)$ at those positions, and by zero at the other positions, gives $\Delta_{\mathrm{thick}}(u,t;a/b)$.

At a thick position, (5.2) bounds the change by

$$\frac{3a}{b}|w(\rho) - b| \le 3(x+1)(\eta+1).$$

At a position $\rho \in \Phi \setminus \mathcal{T}_u$, we have

$$\frac{X'}{3}w(\rho) \le \frac{2}{3}b'X' \le \frac{2}{3}xb \le (x-1)b \le a.$$

Thus $a/w(\rho) \ge X'/3$, and (5.1) gives

$$|w(\rho)G_\alpha(a/w(\rho))| \le a\theta \le (x+1)b\theta.$$

There are f positions and $|\widehat{G}| = |G| = 2f$ characters. Divide the sum of these bounds by f and use $|\alpha(g)| = 1$. The resulting bound on $|\Delta(a,t,w) - \Delta_{\mathrm{thick}}(u,t;a/b)|$ gives the inequality of real parts. □

Proof of 5.10

Proof. Write $\Psi(z) = \sum_{1 \le n \le z} \Lambda(n)$. Rosser and Schoenfeld prove $\Psi(z) < 1.03883z$ for $z > 0$ [34, Theorem 12]. For $z > 0$ this implies $\Psi(z) \le 3z$, which also holds at $z = 0$ since $\Psi(0) = 0$. This bound gives (5.2) for every D and ℓ.

Fix $a > 0$. For $\alpha \ne \mathbf{1}$ and $v > 0$, multiplication of the definition of G_α by v gives

$$vG_\alpha(a/v) = \sum_{n \ge 1} \Lambda(n)v_\alpha(n)[a/n - v]_+ + a\ell_\alpha.$$

The constant $a\ell_\alpha$ cancels in the difference for v, v'. For $n \le a/\min(v,v')$, each positive part changes by at most $|v - v'|$. All other summands vanish. Summing the changes with $|v_\alpha(n)| \le 1$ gives at most $\Psi(a/\min(v,v'))|v - v'|$. The Chebyshev bound gives (5.2).

For the trivial character, take $0 < v \le v'$ and let B be the decrease of $\sum_n \Lambda(n)[a/n - v]_+$ when v is replaced by v'. Then

$$0 \le B \le 3(a/v)(v' - v), \qquad 0 \le a\log(v'/v) \le (a/v)(v' - v).$$

The difference $vG_z(a/v) - v'G_z(a/v')$ equals $B - a\log(v'/v)$. Both nonnegative quantities are bounded by $3(a/v)(v'-v)$, so their difference has absolute value at most that bound. Interchanging v, v' covers the other order.

Lean: `Formalism.LFunctionChainParameters.ThickSetMeanBounds.AN.psi_le_three` (`Formalism.SizeWindowBrackets.ChainSlopeWindows.LipschitzWeightAndMinimum`, lines 668–690)
Lean: `Formalism.LFunctionChainParameters.ThickSetMeanBounds.lipschitz_weight_all` (`Formalism.SizeWindowBrackets.ChainSlopeWindows.LipschitzWeightAndMinimum`, lines 957–964)

For each $t \in TT$ the hypotheses imply $a_t > 0$. Theorem 5.7 and Lemma 5.9 therefore give, for every $u \in A_0$ and $0 \le j < m$,

$$\operatorname{Re}\Delta_{\mathrm{thick}}(u,t;a_t/b) \le \mathrm{ASK}_N(a_t,t,w_{u,j}) - \operatorname{Re}\mathrm{Main}(a_t,w_{u,j}) + \frac{2b}{x}.$$

The two cost inequalities bound each error contribution by b/x. After summation over t, j, the assumed bound on the gap yields

$$\sum_{t\in TT}\sum_{j=0}^{m-1}\operatorname{Re}\Delta_{\mathrm{thick}}(u,t;a_t/b)\le\mathscr{E}+\frac{2Q}{x}.$$

Sum over $u\in A_0$. By Lemma 5.8, the left side becomes

$$|A_0|\,mb\,\frac{\delta}{2}\sum_{t\in TT}\operatorname{Re}\sum_{\alpha\in P}\alpha(e_1(t))G_\alpha(a_t/b).$$

Divide by $|A_0|$. Multiplying the assumed lower bound for the sum over $t\in TT$ by $mb\delta/2\ge 0$ gives $Q(\delta/2)Y_*\le\mathscr{E}+2Q/x$. □

Proofs for Chapter 6: Reading off the target sum

Proof of 6.2

Proof. For $\alpha=1$, sum $G_z(y_t)\ge G_z(x)-C_d$ from 6.1 and use $C_d\le R$. For $\alpha\ne 1$ in K_p, $\alpha(e_1(t))=\alpha(0,t)$. Put $w=w_c(\alpha)$ and $S_\alpha=\sum_{t\in TT}\alpha(0,t)$. Then $|S_\alpha-wn_T|\le\rho n_T$ by hypothesis, with equality to zero on the left when $\alpha|_{\{0\}\times A}=1$. We have

$$\begin{aligned}\sum_{t\in TT}\alpha(0,t)G_\alpha(y_t)-wn_TG_\alpha(x)=&\sum_{t\in TT}\alpha(0,t)(G_\alpha(y_t)-G_\alpha(x))\\&+(S_\alpha-wn_T)G_\alpha(x).\end{aligned}$$

By 6.1, its absolute value is at most $n_T\{\log(x+1)+C_d+K_o+\rho x(\log x+C_d+K_o)\}\le n_TR$. Taking real parts gives the bound.

For $\alpha\in P\setminus K_p$, 6.1 gives $|G_\alpha(y_t)|\le|G_\alpha(x)|+\log(x+1)+C_d+L\le|G_\alpha(x)|+R$. Since $|\alpha(e_1(t))|=1$, each real part is at least $v_\alpha-R$. For $K_p\subseteq P$, sum v_α separately over K_p and $P\setminus K_p$ to obtain the identity. □

Proof of 6.3

Proof. Fix $u\in U$. By 24.12, $\mathrm{ASK}_{\mathscr{D}}(u)-\mathrm{Main}_{\mathscr{D}}(u)\le E_8(u)$, with E_8 defined in 24.11. The bijection e_T gives $|\mathcal{T}|=n_T$. Apply 25.2 with the assumed bounds on B_0,N_c to get

$$E_8(u)\le x^{C_{\mathrm{err}}\varepsilon}\bigl[n_Ta_E(1+m^2/k)+m(B_0+N_c)\bigr]\le x^{C_{\mathrm{err}}\varepsilon}\mathcal{B}.$$

By 25.3, the weight and constant identifications and $f=|\Phi|$ give the arithmetic main sum after reindexing by e_T,e_P. Subtract it from the assumed expression for $\mathrm{ASK}_{\mathscr{D}}(u)$. The result is $\mathrm{gap}(\mathrm{tr}(u))$ by 5.10. □

Proof of 6.5

Proof. Each local factor in 5.4 is an average of complex numbers of modulus one. Hence their product satisfies $|V_\alpha(n)|\le 1$ for $n\ge 1$. The positive part $[y/n-1]_+$ changes by at most $|y-z|/n$ between y and z, and both positive parts vanish for $n>\max(y,z)$. Thus for $\alpha\ne 1$ and $y,z\ge 1$,

$$|G_\alpha(y)-G_\alpha(z)|\le|y-z|\left(\sum_{n\le\max(y,z)}\frac{\Lambda(n)}{n}+|\ell_\alpha|\right).$$

Apply 6.4 with $r=\lfloor\max(y,z)\rfloor$ to obtain (6.1) with $C_d\ge\log 4+4$. On $(r,r+1)$ for an integer $r\ge 1$, differentiation of 2.10 gives

$$G_z'(y)=\sum_{n\le r}\frac{\Lambda(n)}{n}-\log y+\gamma.$$

By 6.4 and $1 \le y/r \le 2$, its absolute value is at most $C_d = \log 4 + 4 + \log 2 + |\gamma|$. Each new summand vanishes at the integer where it enters, so G_z is continuous there. Partition $[\min(y,z), \max(y,z)]$ at its interior integers. On each piece the mean value theorem bounds the absolute increment by C_d times the length. Summing gives (6.2).

Lean: `Formalism.LFunctionChainParameters.ThickSetMeanBounds.derivative_integral_all` (`Formalism.SizeWindowBrackets.ChainSlopeWindows.LipschitzWeightAndMinimum`, lines 966–981)

Put $A_1 = 5 + C_d + |K_o| + |M'|(4 + C_d + |K_o|) + 4|C_E|$ and $R = A_1 x^{1/4+\varepsilon}$. Choose $x_1 \ge 3$ so that for $x \ge x_1$,

$$26 \le x^{3\varepsilon/8}, \qquad |M'|A_1 \le x^{1/4}. \tag{A.11}$$

Use $y_t = a_t/b$, $\rho = |M'|x^{-1+\varepsilon}$, and the bounds $|K_o|$ and $L = |C_E| \log x$ in 6.2. They satisfy the character hypotheses by (6.4). The inequality $\log u \le u - 1$ for $u > 0$ gives $\log x \le 4x^{1/4}$ at $u = x^{1/4}$ and $\log 2 \le 1$. Hence $\log(x+1) \le 5x^{1/4}$, and

$$\rho x(\log x + C_d + |K_o|) \le |M'|(4 + C_d + |K_o|)x^{1/4+\varepsilon}.$$

Since $x^\varepsilon \ge 1$, these bounds give the three requirements on R in 6.2.

Put $G_* = G_m^A(x) - \Sigma_A(x)$. Sum 6.2 over P and use its identity for $\sum_{\alpha\in P} v_\alpha$. By (6.4) and (A.11), $|P|R \le x^{1/2+\varepsilon}$. In particular,

$$n_T(G_* - 4x^{1/2+\varepsilon}) \le \sum_{t\in TT} \operatorname{Re} \sum_{\alpha\in P} \alpha(e_1(t)) G_\alpha(a_t/b). \tag{A.12}$$

Apply 5.10 with (A.12) and the assumed error $x^{C\varepsilon}\mathcal{B}$. This gives

$$Q\frac{\delta_2}{2}(G_* - 4x^{1/2+\varepsilon}) \le x^{C\varepsilon}\mathcal{B} + \frac{2Q}{x}. \tag{A.13}$$

In 25.12 take $G_x = G_*$ and $E = x^{C\varepsilon}\mathcal{B}$. The size conditions follow from (6.5) and $T_2 \subseteq A$, and the threshold condition is (A.11). Thus $G_* \le x^{1/2+(C+3)\varepsilon}$. □

Proof of 6.6

Proof. Put $M_H = Q_S qd$ and $r_H = T \circ \mathrm{Art}$, with T as in 5.2. On units modulo M_H, $\theta_\alpha \circ \mathrm{Art} = \alpha \circ r_H$. The Dirichlet character given by this pullback has primitive representative χ_α.

Let $f_\alpha \mid M_H$ be its conductor. For a prime p, inertia I_p is the image under r_H of the kernel of reduction of units modulo the prime-to-p part of M_H. Thus $\alpha|_{I_p} = 1$ precisely when the pullback factors through this part, or equivalently when $p \nmid f_\alpha$. In that case the Frobenius unit is congruent to p modulo f_α, so for $i \ge 1$,

$$\frac{1}{|I_p|} \sum_{h\in I_p} \alpha(iF_p + h) = \chi_\alpha(p)^i.$$

When $p \mid f_\alpha$, translation by an $h_0 \in I_p$ with $\alpha(h_0) \ne 1$ multiplies the sum by $\alpha(h_0)$, so the sum is zero, as is $\chi_\alpha(p)$. Multiplication over prime powers gives $V_\alpha(n) = \chi_\alpha(n)$ for $n \ge 1$. Compatibility supplies the logarithmic derivative for $\alpha \ne 1$.

Equal primitive representatives induce equal characters on units modulo M_H, so Artin surjectivity gives equal host characters. The trivial character reduces to the primitive trivial character. By injectivity and the character isomorphism of 5.2, $\chi_\alpha = 1$ exactly when $\alpha = 1$. In that case both sums equal G_z. Otherwise, substitute the local values and the logarithmic derivative into 5.4. In 2.10, $[y/n - 1]_+$ equals $y/n - 1$ for $1 \le n \le y$ and vanishes for $n > y$, so the sums agree. □

Proof of 6.8

Proof. Each $\chi \in \mathcal{K}$ has a representative in X_+ by 8.19, unique by 6.6. Thus $K_H \subseteq P$ and each member takes value 1 on ϵ_d. The trivial character belongs to K_H. By 6.7 and 8.19, $[A : A_0] = o$ and $|P| = 4o$.

Put $N_\beta = |TT \cap \beta|$ for $\beta \in A/A_0$, and $h = x^{-1+\varepsilon}$. The shares of 8.19 give

$$|N_\beta - \lambda_\beta n_T| \le h n_T, \qquad \lambda_0 = \frac{4}{3o+1}, \quad \lambda_\beta = \frac{3}{3o+1} \quad (\beta \ne 0).$$

For $\alpha \in P$ nontrivial on A, $\bar\alpha(t + A_0) = \alpha(0, t)$ defines a nontrivial character of A/A_0. Translation by a coset on which $\bar\alpha$ differs from 1 shows that $\sum_\beta \bar\alpha(\beta) = 0$. Hence

$$\sum_\beta \bar\alpha(\beta)\lambda_\beta = \frac{3\sum_\beta \bar\alpha(\beta) + 1}{3o+1} = c,$$

and

$$\left|\sum_{t \in TT} \alpha(0,t) - c n_T\right| = \left|\sum_\beta \bar\alpha(\beta)(N_\beta - \lambda_\beta n_T)\right| \le o h n_T \le 4 o h n_T.$$

For retained $\alpha \ne 1$, 6.6 identifies $|\ell_\alpha|$ with a summand of K_o. For $\alpha \in P \setminus K_H$, apply the logarithmic derivative bound of 8.19 using (6.6) and 6.6 to get $|\ell_\alpha| \le C \log x$. □

Proof of 6.9

Proof. By 5.2, 6.6, and 8.19, the map $\alpha \mapsto \chi_\alpha$ gives a bijection $K_H \simeq \mathcal{K}$. A retained α has $\alpha(\epsilon_d) = 1$ by 8.19. If it is trivial on A, it factors through E_0, where this condition leaves 1 and χ_q. By 5.2, the host character θ_{χ_q} is the projection $((\sigma_q, \sigma_d), a) \mapsto \sigma_q$.

The trivial character contributes $G_z(y)$ with weight 1. When χ_{sp} is designated, 8.19 gives $\pi_H(\theta_{\chi_q}) = \chi_{\mathrm{sp}}$, so its contribution also has weight 1. In the absence of a designated character the same condition puts χ_q outside K_H. Every other retained α is nontrivial on A, so $w_c(\alpha) = c$. By injectivity in 6.6, its primitive representative is distinct from 1 and from the designated character when one is present. Hence $w_\Lambda(\chi_\alpha) = c$. Apply 6.6 and reindex by the bijection. The cyclic level has $\mathcal{K} = \mathcal{X}$ and no designated character. □

Proof of 6.12

Proof. Put $C' = \max(e_D + e_L, 0) + 3$. Take the threshold of 6.5 for C', $c = (3o+1)^{-1}$, $M' = 4o$, $C_E = C$, and $K_o = \sum_{\chi \in \mathcal{K}} |\kappa_\chi|$. Here $C' \ge 0$ and $|c| \le 1$. Since $\varepsilon > 0$ and $\log x / x^\varepsilon \to 0$, choose $x_1 \ge 3$ above this threshold so that for all $x \ge x_1$,

$$\log x \le x^\varepsilon, \qquad 200(1+K)^2(1+c_{HH})(1+e_K) \le x^\varepsilon, \qquad 1 + 4o \le x^\varepsilon. \tag{A.14}$$

Take a datum and compatible ℓ from 6.11. The window conditions of 25.2 give $e_{\mathrm{Div}}, e_{\mathrm{Loss}}, c_{HH}(\mathcal{C}), e_K \ge 0$. Thus $e_D, e_L \ge 0$ and $C' = e_D + e_L + 3$. Since $c_{HH}(\mathcal{C}) \le c_{HH}$, (A.14) gives the large-x condition of 6.3 for $\mathcal{C}$. For each $s \in A_0$, choose $u \in U$ with $\mathrm{tr}(u) = s$. Apply 6.3 and use $x \ge 1$ and $\mathcal{B} \ge 0$ to get

$$\mathrm{gap}(s) \le x^{(e_{\mathrm{Div}} + e_{\mathrm{Loss}} + 3)\varepsilon}\mathcal{B} \le x^{C'\varepsilon}\mathcal{B}.$$

The same function $a : A \to \mathbf{R}$ serves every translation, as required in 5.10.

Use the local conditions of 5.3 and the size and weight conditions of 6.11. Apply 6.5 with $K_p = K_H$, using 6.8 for the character conditions (6.4). Rewrite its left side by 6.9 to get

$$G_{m,\Lambda}(x) \le x^{1/2 + (e_D + e_L + 6)\varepsilon} + \Sigma_A(x).$$

By 6.10, $\Sigma_A(x) \le 4o\, x^{1/2+\varepsilon}$. Since $e_D + e_L \ge 0$, the right side is at most $(1 + 4o) x^{1/2 + (e_D + e_L + 6)\varepsilon}$. The last bound in (A.14) gives $G_{m,\Lambda}(x) \le x^{1/2 + (e_D + e_L + 7)\varepsilon}$. □

Proofs for Chapter 7: Choosing the auxiliary primes q and d

Proof of 7.2

Proof. Corollary 1.1 of the revised paper of Benli, Goel, Twiss, and Zaman [37] gives explicit constants for $M_0 > 400{,}000$, $T \geq 4$, and $\operatorname{Re}\rho > 1/2$. Take $T' = \max\{T, 4\}$ and $c_0 < 1/2$. Choose $c_1 \leq 1/10$. Then $1 - \beta_1 < c_1/\mathcal{L} \leq 1/(10 \log M_0)$, as required by the corollary. In this interval β_1 is a simple zero of $\prod_{\chi \bmod M_0} L(s, \chi)$ by the zero-free region in [37, (1.2)]. All factors are holomorphic at β_1, so χ_1 is the unique character whose L-function vanishes there. Nonvanishing at 1 for nonprincipal characters and the exclusion of the pole pair give $\rho \neq 1$. Thus every pair (χ, ρ) in the statement gives a zero $\rho \neq \beta_1$ of the product. Its denominator $10 \log M_0 + \log T' + 107$ is comparable to $\mathcal{L}$, so the corollary gives the bound with $B = 1$ after decreasing c_1 and c_2. For the finitely many smaller moduli, nonvanishing at 1 for nonprincipal real characters lets us decrease c_1. Taking $B = 1$ therefore establishes the stated bound. □

Proof of 7.3

Proof. Use McCurley's zero-free region for the product of all Dirichlet L-functions at one modulus, in the form of [37, (1.2)]. Take $k = 3m$, so that the result applies also when $m = 1, 2$. For $t \in \mathbf{R}$,

$$\max\{k, k|t|, 10\} \leq \big(m(|t| + 2)\big)^4.$$

Hence $c = 1/50$ puts the stated closed region inside McCurley's open region at modulus k. There the product has at most one zero, a simple real zero of a real nonprincipal character. Induction from modulus m to modulus k is injective on characters. It multiplies each L-function by Euler factors that are nonzero for $\operatorname{Re} s > 0$, so it preserves the zeros in the stated region and their multiplicities. The conclusion follows on restricting the induced characters to modulus m. □

Proof of 7.4

Proof. The first assertion is Page's exceptional-character theorem [39], expressed as equality of induced character values. Equality on integers coprime to $m_1 m_2$ gives equality at that common level, hence equality of the primitive representatives. Conversely, a common primitive representative gives the displayed equality. To prove the first assertion of Theorem 7.4, take $c_{\mathrm{P}} = c/3$ in terms of the constant of Lemma 7.3. Induce both characters to modulus $m = m_1 m_2$. Induction multiplies each L-function by finitely many Euler factors, so each induced character retains its zero β_i. Since $2m \leq D^3$, both zeros satisfy $\beta_i > 1 - c/\log(2m)$. Uniqueness in Lemma 7.3 identifies the induced characters, so $\chi_1(n) = \chi_2(n)$ whenever $(n, m) = 1$.
Lean: `Formalism.ZeroFreeDirichlet.zFRDirichlet` (`Formalism.LandauZeroFreeBounds.NoZeroComplex`, lines 32–33)

Davenport [33, Chapter 21] gives Siegel's theorem. Thorner–Zaman [28, Remark 1.2, (1.2)] state the bound for the exceptional zero with an effectively computable c_ν for $\nu \geq 1/2$. Zeros outside the exceptional interval satisfy the same bound after decreasing c_ν, since $1/\log(3m_0) \gg_\nu m_0^{-\nu}$. □

Proof of 7.5

Proof. Because $8 \mid m_0$, we have $m_0 \geq 8$ and $\log m_0 \geq 2$. Let $c_0^{\mathrm{ZF}} > 0$ be the uniform strip constant of Lemma 7.3 and let $c_Z = \min\{c_0^{\mathrm{ZF}}, 1/20\}$. Its closed region at modulus m_0 is

$$\mathcal{D}_Z = \{s : \operatorname{Re} s \geq 1 - c_Z/\log(m_0(|\operatorname{Im} s| + 2))\}.$$

Call (χ_1, β_1) an exceptional pair modulo m_0 if $\chi_1 \bmod m_0$ satisfies $\chi_1^2 = 1$, $\chi_1 \neq 1$, $L(\beta_1, \chi_1) = 0$, and $1 - 1/(8 \log m_0) < \beta_1 < 1$. Select the optional exceptional pair modulo m_0 by the two-case rule:

1. if there exists an exceptional pair (χ_1, β_1) with $\beta_1 \in \mathcal{D}_Z$, choose that pair;
2. otherwise, if at least one exceptional pair modulo m_0 exists (so every such pair satisfies $1 - 1/(8 \log m_0) < \beta_1 < 1 - c_Z/\log(2m_0)$), choose one such pair (χ_1, β_1); if none exists, choose no exceptional pair.

By construction, choosing no exceptional pair occurs precisely when no exceptional pair modulo m_0 exists, and whenever an exceptional zero lies in $\mathcal{D}_Z$ it is the chosen pair. Set $\eta(m_0) = \min\{1, (1-\beta_1)\log m_0\}$ when a pair (χ_1, β_1) is chosen and $\eta(m_0) = 1$ otherwise; Siegel's theorem gives an absolute constant $c_{\mathrm{S}} > 0$ with $\eta(m_0) \ge c_{\mathrm{S}} m_0^{-1}$ for all $m_0 \ge 8$.

Put $\psi(y,\chi) = \sum_{n\le y} \Lambda(n)\chi(n)$ and $\mathrm{main}(y,\chi) = \mathbf{1}_{\chi=\chi_0} y - \mathbf{1}_{\chi=\chi_1} y^{\beta_1}/\beta_1$, dropping the second term when no pair is selected. The near-range input is the following character estimate. For the fixed c_Z, there are $c_{\mathrm{n}} > 0$ and $C_{\mathrm{n}} \ge 0$ such that, for $y \ge 2$ and $q \le e^{c_{\mathrm{n}}\sqrt{\log y}}$, one has

$$\left|\psi(y,\chi) - \mathbf{1}_{\chi=\chi_0} y + \sum_{\beta\in B_\chi} y^\beta/\beta\right| \le C_{\mathrm{n}} y\, e^{-c_{\mathrm{n}}\sqrt{\log y}}.$$

Here q is any positive modulus and B_χ is a set of at most one real zero of a nonprincipal χ in the closed strip with modulus q and constant c_Z. Its elements must be simple, and the set must account for every regular zero of χ in this strip. The character estimate uses the smoothed contour formula and the separate principal-character estimate [33, Chapters 17, 19–20].

Lean: `Formalism.ZeroFreeDirichlet.PsiChiBound` (`Formalism.LandauZeroFreeBounds.SiegelWalfiszAP`, lines 22–27)
Lean: `Formalism.ZeroFreeDirichlet.psiChiBound` (`Formalism.LandauZeroFreeBounds.SiegelWalfiszAPOfSiegelLower`, lines 23–24)

Take $s = \sqrt{\log y}$ and $c_A = \min(c_{\mathrm{n}}/4, c_Z) > 0$. In the near range $\log m_0 \le c_A s$, the modulus condition holds. By the regular-zero guard in Lemma 7.3, the sets B_χ are all empty except for the selected character when its selected zero lies in $\mathcal{D}_Z$; for that character the set is the singleton of that zero. If the selected zero lies outside $\mathcal{D}_Z$, the desired subtraction differs from the character estimate by at most $2ye^{-s/2}$, since $\beta_1 \ge 1/2$ and $(1-\beta_1)\log y > c_Z \log y/\log(2m_0) \ge s/2$. The sum of the errors is thus at most $m_0 y(C_{\mathrm{n}} e^{-c_{\mathrm{n}} s} + 2e^{-s/2})$. The lower bound $\eta(m_0) \ge c_{\mathrm{S}}/m_0$ permits replacing m_0 by $m_0^2\eta(m_0)/c_{\mathrm{S}}$. Now $m_0^2 \le e^{2c_A s}$, $2c_A \le c_{\mathrm{n}}/2$, and $2c_A \le 1/10$. The near-range error is therefore at most $(C_{\mathrm{n}}+2)y\eta(m_0)e^{-c'_{\mathrm{n}} s}/c_{\mathrm{S}}$, where $c'_{\mathrm{n}} = \min(c_{\mathrm{n}}/2, 2/5)$.

In the far range $\log y \ge C_0 \log m_0 \log(\log m_0 + 2)$ and $\log m_0 > c_A\sqrt{\log y}$, let $c_0 > 0$ be the strip-width constant in the lossy Deuring–Heilbronn input and set

$$\kappa = \min\{c_0, 1/20\}, \qquad T = m_0^{12}, \qquad \delta = m_0^{-5}.$$

Claim 1. For this κ there is a constant $C_1 > 0$ such that for every integer $M \ge 3$, every $T_0 \ge 2$ and every character $\chi \bmod M$, there exist a finite set $Z(\chi)$ of regular zeros of $L(s,\chi)$, positive integer multiplicities $m_\chi(\rho)$ and a constant $K_\chi \in \mathbf{C}$ with the following properties. Every $\rho \in Z(\chi)$ satisfies $1-\kappa < \mathrm{Re}\,\rho < 1$ and $|\mathrm{Im}\,\rho| \le T_0 + 1$, and $Z(\chi)$ contains every regular zero of $L(s,\chi)$ with $\mathrm{Re}\,\rho \ge 1-\kappa/2$ and $|\mathrm{Im}\,\rho| \le T_0$. Every $\rho \in Z(\chi)$ with $L'(\rho,\chi) \ne 0$ has $m_\chi(\rho) = 1$, and for every $t \in \mathbf{R}$ the multiplicities in $Z(\chi)$ with $|\mathrm{Im}\,\rho - t| \le 1$ sum to at most $C_1 \log(M(|t|+2))$. For every $u \ge 2$,

$$\Big|\frac{1}{u}\sum_{1\le n\le u}\Lambda(n)\chi(n)(u/n-1) - K_\chi - \mathbf{1}_{\chi=\chi_0}(\log u - 1) + \sum_{\rho\in Z(\chi)} m_\chi(\rho)\frac{u^{\rho-1}}{\rho(\rho-1)}\Big|$$
$$\le C_1\Big((MT_0)^{C_1}u^{-\kappa/2} + \frac{\log u\,\log^2(MT_0)}{T_0}\Big).$$

Indeed, let $m_\chi(\rho)$ be the analytic zero order of $L(s,\chi)$ at ρ, which is a positive integer and equals 1 when $L'(\rho,\chi) \ne 0$. Standard local zero counting in unit discs gives the stated multiplicity bound. Let $\mathcal{Z}$ contain the zeros with $\mathrm{Re}\,\rho \ge 9/10$ and $|\mathrm{Im}\,\rho| \le T_0 + 3$. With $L = \log(M(T_0+2))$, its cardinality is at most $C(T_0+1)L$. For a finite set of N real numbers, an interval of length h contains a point at distance at least $h/(2(N+1))$ from that set, by considering $N+1$ equally spaced interval centres. Apply this to the real parts in a closed subinterval of $(1-\kappa, 1-3\kappa/4)$ of length $\kappa/8$, and to $|\mathrm{Im}\,\rho|$ in $[T_0+1/4, T_0+3/4]$. This gives σ_1, T' and a common separation $\delta \gg_\kappa ((T_0+1)L)^{-1}$ from all relevant zeros. Zeros outside $\mathcal{Z}$ are a fixed distance from the corresponding edges. The local logarithmic-derivative expansion, after removing the principal pole at 1, is consequently $O_\kappa(L/\delta) = O_\kappa((T_0+1)L^2)$ on these edges. After division by u, the left edge contributes at most $C_\kappa(T_0+1)L^2u^{-3\kappa/4}$, which is bounded by $(MT_0)^{C_\kappa}u^{-\kappa/2}$. On each horizontal edge, $|s(s-1)| \gg T_0^2$ and $u^{\mathrm{Re}\,s-1} \le e$ up to the original line $1+1/\log u$; hence its contribution is $O_\kappa(L^2/T_0)$. The truncated tails are $O(\log u\, L^2/T_0)$. All constants are independent of M, χ, T_0, u. Shifting the contour for the

kernel $u^s/(s(s-1))$ to the rectangle bounded by $\operatorname{Re} s = \sigma_1$ and $|\operatorname{Im} s| = T'$ yields the residue sum over the enclosed zeros $Z(\chi)$ weighted by their orders $m_\chi(\rho)$, while the residue at $s = 1$ gives $K_\chi + \mathbf{1}_{\chi=\chi_0}(\log u - 1)$ after division by u, and the left edge, horizontal edges and truncated tails give the displayed error.
Lean: `Formalism.HermitianLattices.Ext.PrimesInProgressions.smoothed_explicit_formula` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTExplMain`, lines 15–97)

Apply Claim 1 with $M = m_0$ and $T_0 = T$ to obtain, for each character χ, the finite zero set $Z(\chi)$, the positive integer weights $m_\chi(\rho)$ and the constant K_χ. The zeros used below are regular zeros with $1 - \kappa < \operatorname{Re}\rho < 1$ and $|\operatorname{Im}\rho| \le T + 1$, and $Z(\chi)$ contains every zero at a regular point of $L(s,\chi)$ with $\operatorname{Re}\rho \ge 1 - \kappa/2$ and $|\operatorname{Im}\rho| \le T$. Define $\epsilon_{\chi,\rho} = 1$ when the chosen exceptional pair is (χ,ρ), and $\epsilon_{\chi,\rho} = 0$ otherwise; here the real exceptional zero is regarded as a complex number. Also put $\epsilon_\chi = 1$ when χ is the chosen exceptional character, and $\epsilon_\chi = 0$ otherwise. Set

$$W(\chi) = \sum_{\rho \in Z(\chi)} \big(m_\chi(\rho) - \epsilon_{\chi,\rho}\big) y^{\operatorname{Re}\rho - 1}.$$

These are finite sums with no additional primed-sum convention. The subtraction removes one copy of the chosen exceptional zero when it belongs to $Z(\chi)$.

With $C_1 > 0$ from the smoothed explicit formula of Claim 1, put

$$E = C_1 \left((m_0 T)^{C_1} y^{-\kappa/2} + \frac{\log(2y)\log^2(m_0 T)}{T} \right).$$

Differencing that formula between y and $(1+\delta)y$, and separating the chosen exceptional term, gives

$$\begin{aligned} |\psi(y,\chi) - \operatorname{main}(y,\chi)| \le{}& \frac{4yE}{\delta} + 4(\delta y + 1)\log(2y) + 22 y W(\chi) \\ &+ \epsilon_\chi\big(20\delta y + 2y^{1-\kappa/2}\big). \end{aligned}$$

After increasing C_0, summing the terms other than $22yW(\chi)$ over the at most m_0 characters gives $O(y m_0^{-2})$. The lower bound $\eta(m_0) \gg m_0^{-1}$ and the far-range inequality give

$$y m_0^{-2} \ll y\eta(m_0) \exp\left(-c_A^2 \frac{\log y}{\log m_0} \right).$$

Lean: `Formalism.HermitianLattices.Ext.PrimesInProgressions.psi_relative_far_range` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTBAsm`, lines 15–166)
Lean: `Formalism.HermitianLattices.Ext.PrimesInProgressions.one_char` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTBAsmB`, lines 110–153)

Claim 2. For the finite zero sets and multiplicities above, there are absolute $A', c' > 0$ such that

$$\sum_{\chi \bmod m_0} W(\chi) \le A'\eta(m_0)\exp\big(-c'\log y/\log m_0\big)$$

when $\log y \ge C_0 \log m_0 \log(\log m_0 + 2)$, after increasing C_0.

We use two bounds for different counting measures. For every $t \in \mathbf{R}$, the sum of the multiplicities in $Z(\chi)$ with $|\operatorname{Im}\rho - t| \le 1$ is at most $C_1 \log(m_0(|t|+2))$. A zero in $Z(\chi)$ with nonzero $L'(\rho,\chi)$ has multiplicity one. Both multiplicity properties are part of Claim 1. The density bound counts unweighted pairs. There are absolute $a \ge 1$, $B_0 \ge 0$, and $C_{\mathrm{dens}} \ge 1$ such that, for $T_* \ge 2$, $\ell_* = \log(m_0 T_*)$ and $v \ge 0$, a set of regular zero pairs with

$$\tfrac{1}{2} \le \operatorname{Re}\rho \le 1, \qquad |\operatorname{Im}\rho| \le T_*, \qquad \ell_*(1 - \operatorname{Re}\rho) \le v$$

has cardinality at most $C_{\mathrm{dens}} e^{av} \ell_*^{B_0}$, provided distinct pairs with the same character have imaginary parts separated by at least one. Indeed the separated zero-density estimate [40, Theorem 12.1] uses the abscissa $\max\{1/2, 1 - v/\ell_*\}$; normalizing the constants gives this form.
Lean: `Formalism.HermitianLattices.Ext.PrimesInProgressions.ExplicitS` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.DeuringHeilbronnM`, lines 53–66)
Lean: `Formalism.HermitianLattices.Ext.PrimesInProgressions.Dens.log_density_bound` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTDens`, lines 19–109)

Lean: Formalism.HermitianLattices.Ext.PrimesInProgressions.density_norm (Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTZSumB, lines 11–71)

Put $A_1 = 12$, $\ell = \log m_0$, $T_* = m_0^{A_1} + 3$, $\ell_* = \log(m_0 T_*)$, $Y_* = \log y$, and $c_* = c_Z/(4(A_1+2))$. Here $2 \le \ell \le \ell_* \le (A_1+2)\ell$. Let $Z_0(\chi)$ consist of the pairs in $Z(\chi)$ outside the region $\mathcal{D}_Z$ defined above. Inside $\mathcal{D}_Z$, Lemma 7.3 gives at most one pair, real, nonprincipal and simple. Its real part exceeds $1 - 1/(8\ell)$ since $c_Z \le 1/20$. Thus if that pair occurs, it is the selected pair and its summand in $W(\chi)$ is zero. For the pairs remaining in $Z_0(\chi)$, discard the exceptional subtraction for an upper bound, keeping the full multiplicity. Each such pair satisfies

$$w = \ell_*(1 - \operatorname{Re}\rho) \ge c_Z, \qquad 0 < w \le \ell_*.$$

Write $c_1^{\mathrm{DH}}, c_2^{\mathrm{DH}} > 0$ and $N_{\mathrm{DH}} \in \mathbf{N}$ for the two constants and the logarithmic exponent in Theorem 7.2. There is $0 < \vartheta \le 1$ such that

$$c_2^{\mathrm{DH}} \log(1/\vartheta) \le w \quad (\rho \in Z_0(\chi)), \qquad \eta_0 \vartheta \le \eta(m_0)\ell_*^{N_{\mathrm{DH}}}, \qquad \eta_0 = \min\{c_Z/2, c_1^{\mathrm{DH}}/(A_1+2)\}.$$

If no pair was selected, take $\vartheta = 1$. If the selected pair was outside $\mathcal{D}_Z$, then $1 - \beta_1 > c_Z/\log(2m_0)$, so $\eta(m_0) \ge c_Z/2$ and again take $\vartheta = 1$. For a selected pair inside $\mathcal{D}_Z$, also take $\vartheta = 1$ if either $(1-\beta_1)\ell_* \ge c_1^{\mathrm{DH}}$ or $(1-\beta_1)\ell_*^{N_{\mathrm{DH}}} \ge c_1^{\mathrm{DH}}$. In the first case use $\ell_* \le (A_1+2)\ell$; in the second use $\ell \ge 1$ and $\ell_*^{N_{\mathrm{DH}}} \ge 1$. In the remaining case put $\vartheta = (1-\beta_1)\ell_*^{N_{\mathrm{DH}}}/c_1^{\mathrm{DH}}$. The threshold $(1-\beta_1)\ell_* < c_1^{\mathrm{DH}}$ permits Theorem 7.2 at height T_*: each remaining zero is outside $\mathcal{D}_Z$, so is distinct from the selected pair, lies in the required strip since $\kappa \le c_0$, and has $|\operatorname{Im}\rho| \le T_*$. This proves the lower bound on w. The bound on $\eta_0\vartheta$ follows from $\eta_0 \le c_1^{\mathrm{DH}}$, $\ell \ge 1$, and $\eta_0\vartheta \le 1 \le \ell_*^{N_{\mathrm{DH}}}$.

For each integer $j \ge 0$, group pairs with $j < w \le j+1$. In each group choose a maximal set with imaginary parts separated by at least one within each character. Each pair of the group lies within distance one in imaginary part of a chosen pair for its character, by maximality. The density bound with $v = j+1$ bounds the number of chosen pairs by $C_{\mathrm{dens}} e^{a(j+1)}\ell_*^{B_0}$. Summing the unit-height multiplicity bounds centered at these pairs gives total multiplicity at most $2|C_1|C_{\mathrm{dens}}e^{a(j+1)}\ell_*^{B_0+1}$. The factor two is harmless, even if some neighborhoods overlap.

To bound the exponential weight in the group, set $g = \log(\ell+2)$, $\mu = Y_*/((A_1+2)\ell)$. Enlarge C_0 to ensure

$$\mu \ge g\left(a + 1 + 2/c_2^{\mathrm{DH}} + 4(B_0+1)/c_Z\right).$$

Since $g \ge \log 4$, $j < w$, $w \ge c_Z$ and $w \ge c_2^{\mathrm{DH}}\log(1/\vartheta)$, the three terms $\frac{3}{4}wg(a+1)$, $\frac{3}{4}wg(2/c_2^{\mathrm{DH}})$, and $\frac{3}{4}wg\,4(B_0+1)/c_Z$ dominate respectively $(a+\log 2)j$, $\log(1/\vartheta)$, and $(B_0+1)g$. The remaining $w\mu/4$ is at least c_*Y_*/ℓ. Using $Y_*/\ell_* \ge \mu$ and $\log\ell_* \le \log(A_1+2) + g$ gives

$$\ell_*^{B_0+1}e^{-wY_*/\ell_*} \le (A_1+2)^{B_0+1}e^{-(a+\log 2)j}\vartheta\, e^{-c_*Y_*/\ell}.$$

Multiplying by the multiplicity bound and summing the convergent geometric majorant $\sum_{j\ge0}2^{-j} = 2$ gives $\sum_\chi W(\chi) \ll \vartheta e^{-c_*Y_*/\ell}$.

Lean: Formalism.HermitianLattices.Ext.PrimesInProgressions.zeroSum_core (Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTZSum, lines 240–572)

Finally use $\vartheta \le \eta(m_0)\ell_*^{N_{\mathrm{DH}}}/\eta_0$. Since $\ell_* \le (A_1+2)\ell$, once $Y_* \ge (2N_{\mathrm{DH}}/c_*)\ell\log(\ell+2)$ one has $\ell^{N_{\mathrm{DH}}}e^{-c_*Y_*/(2\ell)} \le 1$. This gives the claim with $c' = c_*/2$.

Lean: Formalism.HermitianLattices.Ext.PrimesInProgressions.pow_absorb (Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTZSumL, lines 12–28)

Lean: Formalism.HermitianLattices.Ext.PrimesInProgressions.zero_sum_bound_lossy (Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTZSumL, lines 30–193)

The two range estimates give common constants $c > 0$ and $A \ge 0$ such that their combined error is at most

$$Ay\,\eta(m_0)\left(e^{-c\log y/\log m_0} + e^{-c\sqrt{\log y}}\right)$$

for $y \ge 2$ above the far-range logarithmic threshold. Take the smaller of the two exponential constants and the sum of the nonnegative error constants; one exponential suffices in either branch. For the final counting estimate take t with $\log t \ge C_0 \log m_0 \log(\log m_0 + 2)$, where C_0 is at least twice the common threshold coefficient and at least 16. Then the common estimate applies on all of $[\sqrt{t}, t]$, and $L = \log t \ge 16\ell$, where $\ell = \log m_0$.

Lean: `Formalism.HermitianLattices.Ext.PrimesInProgressions.glue_ranges` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTGlue`, lines 22–56)

Write $\phi = \varphi(m_0)$, $\eta = \eta(m_0)$, and put $a = \chi_1(a_0)$, $\beta = \beta_1$ if a pair was selected; otherwise put $a = 0, \beta = 1$. Thus $|a| \le 1$ and $1/2 \le \beta \le 1$. Define

$$F(u) = \frac{u - au^{\beta}/\beta}{\phi}, \qquad c_3 = \min(c/2, c^2/2), \quad E = e^{-c_3 L/\ell} + e^{-\sqrt{c_3 L}}, \quad D = A\eta E/\phi.$$

Let $\Psi(u) = \sum_{n \le u,\ n \equiv a_0 \bmod m_0} \Lambda(n)$. Character orthogonality transfers the error bound to $|\Psi(u) - F(u)| \le Du$ on $[\sqrt{t}, t]$. In the exceptional term $a = \chi_1(a_0)^{-1}$ as well, since the value is 1 or -1. The exponent comparison follows from $\log u \ge L/2$ on this interval.

Lean: `Formalism.HermitianLattices.Ext.PrimesInProgressions.exp_sum_le_min_exponent` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTPiC`, lines 53–72)

Abel summation for $\sum_{2 \le n \le t,\ n \equiv a_0 \bmod m_0} \Lambda(n)/\log n$ gives $\Psi(t)/L + \int_2^t \Psi(u)/(u \log^2 u)\, du$. Subtract the expression for F, integrating $F'(u)/\log u$ and using $v = u^{\beta}$ for the second term. The difference from $H_{a_0}(t)$ is

$$\frac{\Psi(t) - F(t)}{L} + \int_2^t \frac{\Psi(u) - F(u)}{u \log^2 u}\, du + \frac{F(2)}{\log 2} + \frac{a}{\phi} \operatorname{li}_2(2^{\beta}).$$

On $[2, \sqrt{t}]$, Chebyshev's bound $\Psi(u) \le 6u$ and $|F(u)| \le 3u$ bound the absolute integral by $36(\sqrt{t} - 2)$, since $\log 2 > 1/2$. The two endpoint terms at 2 have absolute values at most 12 and 4, respectively; for the second, $2^{\beta} \ge \sqrt{2}$ and $\log v \ge (\log 2)/2$ between 2^{β} and 2. On $[\sqrt{t}, t]$ the absolute integral is at most $4Dt/L^2$, and the term at t is at most Dt/L. These bounds give an error at most $5Dt/L + 40\sqrt{t}$. Replacing the sum by the prime count costs at most $4\sqrt{t}\, L$. Indeed the other terms come from $p^k \le t$ with $k \ge 2$ and contribute at most $\sum_{2 \le k \le L/\log 2} t^{1/k} \le 2\sqrt{t}\, L$, even without their weights $1/k$.

Lean: `Formalism.HermitianLattices.Ext.PrimesInProgressions.abel_core` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTPiB`, lines 212–328)

Lean: `Formalism.HermitianLattices.Ext.PrimesInProgressions.abs_piAP_sub_le` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTPiA`, lines 44–100)

Claim 3. Put $P = t\eta/(2\phi L)$. Then $H_{a_0}(t) \ge P$, and for the constant c_{S} in the lower bound $\eta \ge c_{\mathrm{S}}/m_0$ one has

$$44\sqrt{t}\, L \le \frac{5632 e^{2c_3}}{c_{\mathrm{S}}} EP.$$

For $a \le 0$, use $\operatorname{li}_2(t) \ge (t-2)/L \ge t/(2L)$, $t^{\beta} \ge 2$, and $\eta \le 1$. For $a > 0$, one has $a = 1$ and $(1-\beta)L \ge \eta$. Then $t - t^{\beta} \ge t(1 - e^{-\eta}) \ge t\eta/2$; integrating from t^{β} to t gives the lower bound. For the second assertion, $\phi \le m_0$ and $2\ell \le L/8$ give $\eta/\phi \ge c_{\mathrm{S}} e^{-L/8}$. Also $\sqrt{c_3 L} \le L/8 + 2c_3$ and $L^2 e^{-L/4} \le 64$. Hence $EP \ge c_{\mathrm{S}} e^{-2c_3} e^{3L/4}/(2L)$, and the stated factor follows from $88 \cdot 64 = 5632$.

Since $5Dt/L = 10AEP$, the accumulated error is bounded by $(10A + 5632 e^{2c_3}/c_{\mathrm{S}}) E H_{a_0}(t)$, giving the final relative estimate. For the related log-weighted prime estimate, see Thorner and Zaman [28, Theorem 1.1 and Corollary 1.4].

Lean: `Formalism.HermitianLattices.Ext.PrimesInProgressions.main_lower` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTPiC`, lines 11–51)

Lean: `Formalism.HermitianLattices.Ext.PrimesInProgressions.small_terms` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTPiC`, lines 74–112)

Lean: `Formalism.HermitianLattices.Ext.PrimesInProgressions.pi_core` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTPiC`, lines 114–212)

Lean: `Formalism.HermitianLattices.Ext.PrimesInProgressions.psi_relative_near_range` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTRangeA`, lines 161–239)

Lean: `Formalism.HermitianLattices.Ext.PrimesInProgressions.psi_relative_far_range` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTBAsm`, lines 15–166)

Lean: `Formalism.HermitianLattices.Ext.PrimesInProgressions.glue_ranges` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTGlue`, lines 22–56)

Lean: `Formalism.HermitianLattices.Ext.PrimesInProgressions.prime_count_of_psi` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PrimeCountOfPsi`, lines 12–91)

□

Proof of 7.6

Proof. Apply the second assertion of Theorem 7.4 with $\nu = 1$. Decrease its constant to obtain $0 < c_* \le 1$ such that $1 - \beta_1 \ge c_*/m_0$ for every exceptional pair modulo m_0.

For $\log U \ge 4$, the corrected main term from Theorem 7.5 satisfies

$$H_{a_0}(U) \ge \frac{c_* U}{2m_0^2 \log U}. \tag{1}$$

Integration over $[U/2, U]$ gives $\mathrm{li}_2(U) \ge U/(2\log U)$. For a selected exceptional zero, $\beta_1 > 1/2$ gives $U^{\beta_1} \ge \sqrt{U} > 2$. This proves (1) when the pair is absent or $\chi_1(a_0) = -1$. If $\chi_1(a_0) = 1$, the gap $\mathrm{li}_2(U) - \mathrm{li}_2(U^{\beta_1})$ is bounded below by $(U - U^{\beta_1})/\log U$. The inequality $1 - e^{-v} \ge \frac{1}{2}\min\{v, 1\}$ for $v \ge 0$, together with $1 - \beta_1 \ge c_*/m_0$, gives $U - U^{\beta_1} \ge c_* U/(2m_0)$. Finally $\varphi(m_0) \le m_0$, which proves (1).

Write $L = \log Y$ and $u = \log U$. Admissibility gives

$$\log m_0 \le \frac{4u}{L^2}, \qquad Y^{\alpha/6} \le u \le Y^{\alpha/5} L^2. \tag{2}$$

Thus $\log(\log m_0 + 2) = O_\alpha(L)$ and $u/\log m_0 \ge L^2/4$. For sufficiently large Y, these inequalities place U in the range of Theorem 7.5 and make its relative error at most $1/2$. Consequently $\pi(U; m_0, a_0) \ge \frac{1}{2} H_{a_0}(U)$.

Uniformly in the allowed u,

$$\frac{1}{u}\log\left(\frac{c_* U}{2m_0^2 \log U}\right) \ge 1 - \frac{8}{L^2} - \frac{\log u + O(1)}{u} \longrightarrow 1.$$

Increase Y_0 so that $H_{a_0}(U) \ge 4U^{9/10}$. Removing primes at most $\sqrt{U}$ removes at most $\sqrt{U}$ integers. Hence

$$|P(m_0, a_0, U)| \ge 2U^{9/10} - U^{1/2} \ge U^{9/10}.$$

□

Proof of 7.7

Proof. Put $L = \log Y$, $u = \log U$, $D = m_0 Y^A$, and $\ell = \log D$. Choose the optional exceptional pair of Theorem 7.5 for every modulus $m_0 s$. For an allowed s, denote by $\lambda_s(v) = (v/s)$ the quadratic character on $(\mathbf{Z}/s\mathbf{Z})^\times$.

Call s exceptional if the selected pair (χ_s, β_s) modulo $m_0 s$ satisfies

$$1 - \beta_s < \frac{c_{\mathrm{P}}}{\ell}, \qquad \chi_s(v) = \lambda_s(v) \quad \big(v \equiv 1 \pmod{m_0},\ (v, s) = 1\big). \tag{4}$$

There is at most one such s. Indeed, Theorem 7.4 makes two corresponding exceptional characters agree on integers coprime to the product of their moduli. Their restrictions in (4) would therefore make λ_{s_1} and λ_{s_2} agree on integers congruent to 1 modulo m_0 and coprime to $s_1 s_2$. If $s_1 \ne s_2$, choose a prime dividing exactly one of them. At that prime choose a quadratic nonresidue, and at every other prime dividing $s_1 s_2$ choose 1. The Chinese remainder theorem, also imposing $v \equiv 1 \pmod{m_0}$, gives an integer on which the two Jacobi symbols differ. Take n_* to be that s, or 0 if none exists.

Fix $s \ne n_*$, and apply Theorem 7.5 modulo $m_0 s$. Let

$$\mathcal{B}_s = \{b \in (\mathbf{Z}/m_0 s\mathbf{Z})^\times : b \equiv a_0 \pmod{m_0}\}.$$

The Chinese remainder theorem identifies $\mathcal{B}_s$ with $(\mathbf{Z}/s\mathbf{Z})^\times$. Therefore

$$\sum_{b \in \mathcal{B}_s} \lambda_s(b) = 0. \tag{5}$$

For the exceptional contribution, character orthogonality on the kernel of reduction modulo m_0 gives

$$\sum_{b \in \mathcal{B}_s} \lambda_s(b)\chi_s(b) = 0$$

unless the restriction of χ_s to that kernel is λ_s. In this aligned case, $s \neq n_*$ gives $\beta_s \leq 1 - c_{\mathrm{P}}/\ell$. Since $2 \leq U^{\beta_s} \leq U$, the bounds $\mathrm{li}_2(U^{\beta_s}) \leq 2U^{\beta_s}$ and $\mathrm{li}_2(U) \geq U/(2u)$ give

$$\frac{|\mathrm{li}_2(U^{\beta_s})|}{\mathrm{li}_2(U)} \leq r, \qquad r = 4u \exp(-c_{\mathrm{P}} u/\ell). \tag{6}$$

Put $k = A_0(e^{-cu/\log(m_0 s)} + e^{-\sqrt{cu}})$. Write H_b for the corrected main term in class b, and put $A_U = \mathrm{li}_2(U)/\varphi(m_0 s)$. These main terms are nonnegative. In the aligned case, $r \leq 1/2$ gives $H_b \geq (1-r)A_U$, while the exceptional term in each class has magnitude at most rA_U. Its signed sum is therefore bounded by $r(1-r)^{-1}\sum_b H_b \leq 2r\sum_b H_b$. The signed sum vanishes when there is no exceptional pair and, by orthogonality, in the unaligned case. For $|k| \leq 1/2$, the relative estimate gives

$$\sum_{b \in \mathcal{B}_s} H_b \leq 2 \sum_{b \in \mathcal{B}_s} \pi(U; m_0 s, b) \leq 2(|P| + \sqrt{U}),$$

where $P = P(m_0, a_0, U)$. The summed errors contribute at most $|k|\sum_b H_b$, and the primary terms cancel by (5). Primes dividing s, and the passage from primes at most U to P, are covered by the primes at most $\sqrt{U}$. Here $s \leq Y^A < \sqrt{U}$ for large Y. It follows that

$$\left| \sum_{q \in P} \lambda_s(q) \right| \leq (4r + 2|k|)(|P| + \sqrt{U}) + \sqrt{U}. \tag{7}$$

Since $s \leq Y^A$, we have $\log(m_0 s) \leq \ell = \log m_0 + AL$. For large Y, admissibility gives $\log m_0 \leq 4u/L^2$ and $AL \leq 4u/L^2$, so $u/\ell \geq L^2/8$. Also $\log u \leq (\alpha/5 + 2)L$ and $u \geq Y^{\alpha/6}$. These bounds place every $m_0 s$ in the range of the progression theorem and yield, after increasing Y_0,

$$4r + 2|k| \leq \tfrac{1}{4} Y^{-B}, \qquad U^{-2/5} \leq \tfrac{1}{4} Y^{-B}.$$

Lemma 7.6 gives $\sqrt{U} \leq U^{-2/5}|P|$. Substitution in (7) proves (3). □

Proof of 7.8

Proof. For a positive integer v, let

$$r(v) = \prod_{\substack{p \mid v \\ v_p(v) \text{ odd} \\ p \nmid m_0}} p.$$

The product contains each prime at most once and excludes every divisor of m_0, including 2. Quadratic reciprocity gives, for $1 \leq n, n' \leq N$,

$$\left| \sum_{q \in P} \left(\frac{n}{q}\right)\left(\frac{n'}{q}\right) \right| = \left| \sum_{q \in P} \left(\frac{q}{r(nn')}\right) \right|. \tag{8}$$

The reciprocity sign and the factors contributed by primes dividing m_0 are constant on the progression. The factor at 2 is constant because $8 \mid m_0$. The condition $q > N^2$ prevents a candidate prime from dividing nn'.

Expand the square and use $|\kappa(n)| \leq 1$. The resulting upper bound is

$$\sum_{n,n' \leq N} c_y(n) c_y(n') \left| \sum_{q \in P} \left(\frac{n}{q}\right)\left(\frac{n'}{q}\right) \right|. \tag{9}$$

The coefficients vanish outside prime powers. If $n = p^a$, its kernel after removing primes dividing m_0 is p when a is odd and $p \nmid m_0$, and is 1 otherwise.

Pairs with $r(nn') = 1$ either have the same prime base or have both entries in the set counted by R_0. Their total coefficient is at most

$$\sum_p s_p(y,N)^2 + R_0(y,m_0,N)^2.$$

For a fixed kernel $r > 1$, two distinct nontrivial kernels p, p' can produce r only when $r = pp'$. These pairs have total weight at most $2B^2$. For prime r, the two kernel orders are $(r,1)$ and $(1,r)$. Together they contribute at most $2s_r(y,N)R_0(y,m_0,N) \le 2BR_0(y,m_0,N)$. Thus the exceptional nontrivial kernel has total coefficient at most $2B^2 + 2BR_0$. If $n_* \le 1$, there is no exceptional nontrivial kernel.

The total coefficient of all remaining pairs is at most $(\sum_{n\le N} c_y(n))^2$. Their character sums have magnitude at most $\eta|P|$, so their contribution to (9) is at most $\eta|P|(\sum_{n\le N} c_y(n))^2$. Combining these bounds with (9) proves the assertion. □

Proof of 7.9

Proof. For a prime p,

$$s_p(y,N) \le \log p \sum_{a\ge 1} \min\{1, y/p^a\}.$$

If $p \le y$, split the sum at $p^a \le y$. The initial part contributes at most $\log y$, and the geometric tail contributes at most $2\log p$. If $p > y$, the whole sum is at most $2y\log p/p \le 2\log y$, since $(\log t)/t$ decreases for $t \ge 3$. This proves the first bound.

To sum the squares, first consider $p \le \sqrt{y}$. Their contribution is $O(\sqrt{y}(\log y)^2) = O(y\log y)$. For $\sqrt{y} < p \le y$, the bound $s_p \le 3\log y$ and $\log y \le 2\log p$ give $s_p^2 \ll (\log y)\log p$. The elementary Chebyshev bound $\sum_{p\le t}\log p \le t\log 4$ handles this range. For $p > y$, we have $s_p^2 \le 4y^2(\log p)^2/p^2$. Partial summation of the same Chebyshev bound gives

$$\sum_{p>y} \frac{(\log p)^2}{p^2} \ll \frac{\log y}{y}.$$

This proves the second bound in (10).

For the even powers, write $p^{2a} = (p^a)^2$. Their contribution is bounded by

$$\sum_{v\ge 1} \Lambda(v)\min\{1, y/v^2\} \ll \sqrt{y}.$$

The last estimate follows by splitting at $v = \sqrt{y}$ and applying $\sum_{v\le t}\Lambda(v) \ll t$. There are at most w prime bases $p \mid m_0$ with $p \le N$. Each contributes at most $3\log y$. This proves (11).

Finally, $c_y(n) \le y\Lambda(n)/n$, and the elementary Mertens upper bound $\sum_{n\le Z}\Lambda(n)/n \ll \log Z$ proves (12). □

Proof of 7.10

Proof. Apply Lemma 7.7 with its parameters $A = 800K$ and $B = K+1$. It supplies the cancellation hypothesis of Lemma 7.8 with $\eta = Y^{-(K+1)}$, since $\lfloor Z\rfloor_+^2 \le Y^{800K}$. For large Y, admissibility gives $\log U \ge Y^{\alpha/6} \ge 1600K\log Y$. Therefore $\sqrt{U} \ge Z^2$, and every candidate q exceeds $\lfloor Z\rfloor_+^2$.

Put $w = Y^{\alpha/5}$. By (13), every prime divisor $p \mid m_0$ with $p \le \lfloor Z\rfloor_+$ is at most w. Since $\alpha \le 1$ and $y \ge Y$, we have $w \le y^{1/5}$. For large Y, the estimates of Lemma 7.9 consequently give $R_0 \le 5C_1\sqrt{y}$ and $B = C_2\log y$. Thus $R_0^2 + 2B^2 + 2BR_0 \ll y\log y$ with an absolute implied constant.

For the remaining term, use $\log Z = 400K\log Y$ and $(400K\log Y)^2 \le Y$. Then

$$\eta\left(\sum_{n\le Z} c_y(n)\right)^2 \le C_4^2 Y^{-(K+1)} y^2 (400K\log Y)^2$$
$$\le C_4^2 y^2 Y^{-K} \le C_4^2 y \le C_4^2 y\log y.$$

Insert these estimates and $\sum_p s_p^2 \le C_3 y\log y$ into Lemma 7.8. Take C_{m} to be the sum of the absolute constants in the resulting bounds. □

Proof of 7.11

Proof. For fixed Z, q, κ, the function $S_q(y) = S(Z, y, q, \kappa)$ is continuous and piecewise linear. Its right derivative is

$$D_q(v) = \sum_{\substack{1\le n\le Z\\ n>v}} \frac{\Lambda(n)\kappa(n)}{n}\left(\frac{n}{q}\right).$$

For $v > 0$,

$$vD_q(v) = S\big(Z, v, q, \kappa\, 1_{\{n>v\}}\big). \tag{14}$$

The coefficient function on the right has modulus at most 1. The full quantifier over κ in the mean-square hypothesis therefore gives

$$\sum_{q\in P} |D_q(v)|^2 \le C\frac{\log v}{v}|P| \qquad (Y \le v \le X). \tag{15}$$

Consider $T \le W \le 2T$ with $[T, W] \subseteq [Y, X]$. Let B_T consist of candidates with a witness in $[T, W]$. For every $q \in P$, choose $y_q \in [T, W]$ and take it to be a witness when $q \in B_T$. The fundamental theorem of calculus and Cauchy–Schwarz give

$$|S_q(y_q)|^2 \le 2|S_q(T)|^2 + 2T\int_T^W |D_q(v)|^2\, dv.$$

After summing and using (15),

$$\sum_{q\in P} |S_q(y_q)|^2 \le 4CT\log(2T)|P|. \tag{16}$$

For $q \in B_T$, the chosen witness satisfies $|S_q(y_q)|^2 > T^{1+2\tau}/4$. Hence

$$|B_T| \le 16C\log(2T)T^{-2\tau}|P| \le \frac{32C}{\tau}T^{-\tau}|P|, \tag{17}$$

where $T \ge 2$ and $\log(2T) \le 2T^\tau/\tau$.

If $X < Y$, the witness hypothesis gives $B = \varnothing$. Otherwise, for each integer $i \ge 0$ with $T_i = 2^i Y \le X$, put $W_i = \min\{2T_i, X\}$. Let B_i consist of the $q \in B$ with a witness in $[T_i, W_i]$. These intervals cover $[Y, X]$, so $|B| \le \sum_i |B_i|$. Summing (17) gives

$$|B| \le \frac{32C}{\tau}\frac{Y^{-\tau}}{1-2^{-\tau}}|P|.$$

Choose Y so that the coefficient is at most η. □

Proof of 7.12

Proof. Set $s_0 = 399/400$. Let $Q_0 \ge 20$ be an absolute threshold chosen below; for $3 \le Q < Q_0$, there are finitely many characters of conductor at most Q_0, which are absorbed into C_{z}. Assume $Q \ge Q_0$. Put $X = \lceil Q\rceil$, $Y = X^4$, $J = 8(\lfloor \log_2 X\rfloor + 1)$, $N = 2^J \ge X^8$, and $K = YN = Y2^J$. Remove the at most one character of conductor 1 from $\mathcal{E}$, leaving a subset $\mathcal{E}'$ of nonprincipal primitive characters. For each $\chi \in \mathcal{E}'$ of conductor $q_\chi \le Q$, fix a zero ρ_χ of $L(s, \chi)$ with $s_0 < \operatorname{Re}\rho_\chi < 1$ and $|\operatorname{Im}\rho_\chi| \le Q$.

For each $\chi \in \mathcal{E}'$, define the mollified Dirichlet polynomial coefficients $c_n = \sum_{d|n,\, d\le Y,\, n/d\le N} \mu(d)$. Expanding the product of the partial sum of $L(\rho_\chi, \chi)$ up to N and the Möbius mollifier up to Y gives the mollifier identity

$$\left(\sum_{n=1}^{N}\chi(n)n^{-\rho_\chi}\right)\left(\sum_{d=1}^{Y}\mu(d)\chi(d)d^{-\rho_\chi}\right) = 1 + \sum_{Y<n\le YN} c_n\chi(n)n^{-\rho_\chi}.$$

Because $L(\rho_\chi, \chi) = 0$ and partial sums of χ are bounded by q_χ, partial summation gives

$$\left|\sum_{n=1}^{N}\chi(n)n^{-\rho_\chi}\right| \le 2(q_\chi+1)\left(1+\frac{|\rho_\chi|}{\operatorname{Re}\rho_\chi}\right)N^{-\operatorname{Re}\rho_\chi} \le 2(2X)(3X)(X^8)^{-15/16} \le \frac{3}{XY} \le \frac{1}{2Y},$$

using $X \geq 20$ and $\operatorname{Re}\rho_\chi > 15/16$. Since the mollifier sum has modulus at most Y, their product has modulus at most $1/2$, and therefore $\left|\sum_{Y<n\leq YN} c_n\chi(n)n^{-\rho_\chi}\right| \geq 1/2$. Splitting $(Y, YN]$ into the J dyadic blocks $(U_j, 2U_j]$ with $U_j = Y2^j$ $(0 \leq j < J)$, each $\chi \in \mathcal{E}'$ selects an index j with

$$\left|\sum_{U_j<n\leq 2U_j} c_n\chi(n)n^{-\rho_\chi}\right| \geq \frac{1}{2J}.$$

Fix $0 \leq j < J$, put $U = U_j$, and let $T \subseteq \mathcal{E}'$ be the set of characters assigned to block j. Write $b_n = c_n n^{-s_0}$ and $v_\chi(n) = \chi(n)n^{-(\rho_\chi - s_0)}$. The divisor bound $|c_n| \leq d(n) \leq C_1 n^{1/400}$ gives

$$\sum_{U<n\leq 2U} |b_n|^2 \leq C_1^2 U^{-99/100} \leq C_1^2 U^{-197/200}.$$

For distinct $\chi, \chi' \in T$, the product $\chi\overline{\chi'}$ is a nonprincipal character modulo $q_\chi q_{\chi'} \leq Q^2$, so its partial sums are bounded by $q_\chi q_{\chi'} \leq Q^2$. Putting $w = (\rho_\chi - s_0) + \overline{\rho_{\chi'} - s_0}$ (which has $\operatorname{Re} w > 0$ and $|w| \leq 1 + 2Q$), summation by parts yields

$$\left|\sum_{U<n\leq 2U} v_\chi(n)\overline{v_{\chi'}(n)}\right| \leq 2q_\chi q_{\chi'}(1 + |w|) \leq 8Q^3,$$

while the diagonal sum $\chi = \chi'$ is at most U. By the Halász–Montgomery duality inequality,

$$\frac{|T|}{4J^2} \leq \sum_{\chi\in T}\left|\sum_{U<n\leq 2U} b_n v_\chi(n)\right|^2 \leq \left(\sum_{U<n\leq 2U}|b_n|^2\right)\big(U + 8Q^3|T|\big) \leq C_1^2 U^{3/200} + 8Q^3 C_1^2 U^{-197/200}|T|.$$

Since $U \geq Y \geq Q^4$, we have $U^{-197/200} \leq Q^{-197/50}$, so $8Q^3C_1^2U^{-197/200} \leq 8C_1^2Q^{-47/50} \leq 1/(8J^2)$ once Q_0 is chosen sufficiently large (as $J \leq 32\log Q$). Subtracting $|T|/(8J^2)$ gives $|T| \leq 8J^2C_1^2U^{3/200} \leq 8J^2C_1^2K^{3/200}$. Summing over $0 \leq j < J$ and restoring the conductor-1 character gives

$$|\mathcal{E}| \leq 1 + 8J^3C_1^2K^{3/200}.$$

Because $J \leq 32\log Q$ and $K = Y2^J \ll Q^{12}$, we have $J^3K^{3/200} \ll (\log Q)^3Q^{9/50} \ll Q^{1/4}$, yielding $|\mathcal{E}| \leq C_{\mathrm{z}}Q^{1/4}$.

Lean: `Formalism.HermitianLattices.Ext.PrimitiveCharacterCasts.ZD.mollifier_identity` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.CountZMoll`, lines 79–131)
Lean: `Formalism.HermitianLattices.Ext.PrimitiveCharacterCasts.ZD.zero_detection` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.CountZDetect`, lines 43–162)
Lean: `Formalism.HermitianLattices.Ext.PrimitiveCharacterCasts.ZD.coefficient_square_bound` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.CountZMoll`, lines 133–191)
Lean: `Formalism.HermitianLattices.Ext.PrimitiveCharacterCasts.ZD.pair_sum_bound` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.CountZPair`, lines 98–101)
Lean: `Formalism.HermitianLattices.Ext.PrimitiveCharacterCasts.ZD.pair_twist_bound` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.CountZTwist`, lines 32–73)
Lean: `Formalism.HermitianLattices.Ext.PrimitiveCharacterCasts.ZD.duality_inequality` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.CountZDual`, lines 9–124)
Lean: `Formalism.HermitianLattices.Ext.PrimitiveCharacterCasts.ZD.blockCountOf_statement` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.CountZBlock`, lines 23–166)
Lean: `Formalism.HermitianLattices.Ext.PrimitiveCharacterCasts.ZD.numeric_constants` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.CountZNum`, lines 10–134)
Lean: `Formalism.HermitianLattices.Ext.PrimitiveCharacterCasts.ZD.final_bound_of_assembly` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.CountZFinal`, lines 10–129)

□

Proof of 7.14

Proof. Under the Chinese remainder isomorphism

$$(\mathbf{Z}/fq\mathbf{Z})^\times \simeq (\mathbf{Z}/f\mathbf{Z})^\times \times (\mathbf{Z}/q\mathbf{Z})^\times,$$

take the product of ξ with the nontrivial quadratic character modulo q. A character of coprime product level has conductor equal to the product of the conductors of its two factors. Both factors here are primitive, so the conductor is fq. The quadratic factor also makes the product nonprincipal. Extending the characters by zero gives (18) on all integers.

For uniqueness, induce both characters to level $NN'L$. They agree on every unit of that ring by the assumed equality. Their induced characters at level NN' therefore agree as well. A character induced from both levels N and N' factors through $\gcd(N, N')$. Primitivity gives $N \le \gcd(N, N')$ and $N' \le \gcd(N, N')$. Hence $N = N'$, and injectivity of induction gives $\chi = \chi'$. □

Proof of 7.15

Proof. Write

$$x = Y^K, \quad V = x^{400}, \quad \theta = x^{-2}, \quad h = V\theta, \quad \delta = \frac{\alpha}{1600K}.$$

In particular $0 < \delta \le 1/800$. Introduce

$$\mathcal{S}_\chi(t) = \sum_{1 \le n \le t} \Lambda(n)\chi(n)\left(\frac{t}{n} - 1\right).$$

For every $0 < \delta \le 1/800$, there is $A_\delta \ge 0$ such that, for $Q \ge 2$, every $\chi \in \mathcal{Z}(Q)$ obeys

$$\left|\mathcal{S}_\chi(V + V\theta) - \mathcal{S}_\chi(V) - V\theta D_\chi\right| \le A_\delta \log Q\big((\log Q)\theta V^{1-1/400+\delta} + V^2/Q\big) \tag{19}$$

for $V \ge 1$ and $0 < \theta \le 1$.

Every $\chi \in \mathcal{Z}(Q)$ is a nonprincipal Dirichlet character modulo N with $1 \le N \le Q$ and $L(w, \chi) \ne 0$ on the rectangle $\operatorname{Re} w > 399/400$, $|\operatorname{Im} w| \le Q$; here χ need not be primitive. On $\operatorname{Re} s = 2$, the integrated Perron kernel for $\mathcal{S}_\chi(t)$ is

$$\frac{-L'(s,\chi)}{L(s,\chi)} \frac{t^s}{s(s-1)}.$$

Move the segment $|\operatorname{Im} s| \le Q/2$ to $\sigma_0 = 1 - 1/400 + \delta$. Because χ is nonprincipal, $L(s, \chi)$ is entire, so the only pole crossed is the simple pole of $(s-1)^{-1}$ at $s = 1$ with residue tD_χ, where $D_\chi = -L'(1,\chi)/L(1,\chi)$.

Suppose $\sigma_0 \le \operatorname{Re} s < 11/10$ and $|\operatorname{Im} s| \le Q - 1$. To bound $L'/L(s,\chi)$ without assuming χ primitive, note first that orthogonality modulo N gives $\left|\sum_{1 \le n \le u} \chi(n)\right| \le N$ for every $u \ge 1$, so partial summation of $\sum_{n \ge 1} \chi(n) n^{-z}$ on $\operatorname{Re} z \ge 1/2$ yields the convexity bound $|L(z,\chi)| \le 9N(1 + |z|)$ directly for $L(z,\chi)$ [33, Ch. 12]. Second, for $\operatorname{Re} z > 1$, complete multiplicativity of χ gives $L(z,\chi)^{-1} = \sum_{n=1}^{\infty} \mu(n)\chi(n)n^{-z}$ and $|L(z,\chi)|^{-1} \le \operatorname{Re} z/(\operatorname{Re} z - 1)$. Put $z_0 = 21/20 + i \operatorname{Im} s$. Then $|L(z_0,\chi)| \ge 1/21$, and every z in the closed disk $\overline{D}(z_0, 11/20) \subset \{\operatorname{Re} z \ge 1/2\}$ satisfies $1 + |z| \le 2(|\operatorname{Im} s| + 2)$, whence $|L(z,\chi)/L(z_0,\chi)| \le 378\, N(|\operatorname{Im} s| + 2)$. Since $\sigma_0 > 399/400 > 26/27$ and $L(s,\chi) \ne 0$, Landau's Borel–Carathéodory local zero-expansion lemma on the concentric disks $\overline{D}(z_0, 11/60) \subset \overline{D}(z_0, 11/20)$ [33, §12], [35, Lemmas 6.3 and 11.1] expresses $L'/L(s,\chi)$, up to an error $O\big(\log(N(|\operatorname{Im} s| + 2))\big) = O(\log Q)$, as $\sum_{\rho \in Z} m(\rho)/(s - \rho)$, where Z is the set of zeros of $L(z,\chi)$ in $\overline{D}(z_0, 11/60)$, with multiplicities $m(\rho)$ satisfying $\sum_{\rho \in Z} m(\rho) \ll \log(N(|\operatorname{Im} s| + 2)) \ll \log Q$ (as $N(|\operatorname{Im} s| + 2) \le Q(Q+1) \le Q^3$ for $Q \ge 2$). (Equivalently, if χ is induced by a primitive character χ^* mod q^* with $q^* \mid N$, then $L(z,\chi) = L(z,\chi^*)\prod_{p \mid N,\, p \nmid q^*}(1 - \chi^*(p)p^{-z})$; the extra Euler factors have all their zeros on $\operatorname{Re} z = 0$, disjoint from $\overline{D}(z_0, 11/20)$, and contribute $\sum_{p \mid N,\, p \nmid q^*} \chi^*(p)(\log p)/(p^s - \chi^*(p)) = O(\log(2N)) = O(\log Q)$ to $L'/L(s,\chi)$ on $\operatorname{Re} s \ge 1/2$ [33, Ch. 9].) Every $\rho \in Z$ satisfies $|\operatorname{Im} \rho| \le Q - 1 + 11/60 < Q$, so $\chi \in \mathcal{Z}(Q)$ gives $\operatorname{Re} \rho \le 399/400$ and $|s - \rho| \ge \operatorname{Re}(s - \rho) \ge \delta$. For $\operatorname{Re} s \ge 11/10$, the absolutely convergent Dirichlet series $-\sum_{n \ge 1} \Lambda(n)\chi(n)n^{-s}$ gives $|L'/L(s,\chi)| \le 1/(\operatorname{Re} s - 1) + O(1) = O(1) = O(\log Q)$. Thus

$$\left|\frac{L'}{L}(s,\chi)\right| \ll_\delta \log Q \quad \big(\operatorname{Re} s \ge \sigma_0,\ |\operatorname{Im} s| \le Q - 1\big).$$

For the difference of the two left-edge integrals, use

$$\left|(V + V\theta)^s - V^s\right| \ll \theta |s| V^{\sigma_0} \qquad (\operatorname{Re} s = \sigma_0).$$

After $|s|$ cancels, use $\int_{-Q/2}^{Q/2} |\sigma_0 - 1 + it|^{-1}\, dt \ll \log Q$, since $1 - \sigma_0 \geq 1/800$. The horizontal segments at $|\operatorname{Im} s| = Q/2 \leq Q - 1$, and the tails of the original line at real part 2, contribute $O_\delta((\log Q)V^2/Q)$. These estimates give (19).

Lean: `Formalism.HermitianLattices.Ext.BoxTail.tail_box_negLD` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.TailBox`, lines 35–119)

Lean: `Formalism.HermitianLattices.Ext.BoxTail.tail_contour` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.TailContour`, lines 14–135)

There is also a direct finite-sum estimate

$$\left| \mathcal{S}_\chi(V + h) - \mathcal{S}_\chi(V) - h \sum_{n \leq V} \frac{\Lambda(n)\chi(n)}{n} \right| \leq h(h + 1) \frac{\log(V + h)}{V}. \tag{20}$$

Indeed, terms with $n \leq V$ cancel exactly. There are at most $h + 1$ integers in $V < n \leq V + h$. Each corresponding term satisfies

$$\left| \Lambda(n)\chi(n) \left(\frac{V + h}{n} - 1 \right) \right| \leq h\, \frac{\log(V + h)}{V}.$$

Put $\Delta = \mathcal{S}_\chi(V + h) - \mathcal{S}_\chi(V)$ and $A_V = \sum_{n \leq V} \Lambda(n)\chi(n)/n$. Since $hT_\chi(V) = (\Delta - hA_V) - (\Delta - hD_\chi)$, the triangle inequality and (19)–(20), followed by division by h, give

$$|T_\chi(V)| \leq A_\delta (\log Q)^2 V^{-1/400+\delta} + A_\delta (\log Q) \frac{V}{Q\theta} + (h + 1) \frac{\log(V + h)}{V}. \tag{21}$$

Multiply by x. Since $V = x^{400}$, the first term becomes

$$A_\delta (\log Q)^2 x^{400\delta} \leq A_\delta Y^{\alpha/2} Y^{\alpha/4} = A_\delta Y^{3\alpha/4}.$$

The second becomes $A_\delta (\log Q) x^{403}/Q = o(1)$, uniformly because $Q \geq \exp(Y^{\alpha/6})$. The third is $O(x^{-1} \log x + x^{-399} \log x)$. Their sum is at most Y^α for sufficiently large Y. □

Proof of 7.16

Proof. For the first assertion, apply Lemma 7.15 with $K = 1$. It gives $|T_\chi(Y^{400})| \leq 1$, since $\alpha \leq 1$. Consequently

$$\left| \frac{L'}{L}(1, \chi) \right| \leq 1 + \sum_{n \leq Y^{400}} \frac{\Lambda(n)}{n} \leq C_{\mathrm{L}} \log Y$$

with an absolute constant.

For the explicit coefficient, there are absolute constants C_0, ℓ_0, with $\ell_0 \geq 1$, such that

$$\left| \frac{L'}{L}(1, \chi) \right| \leq 1000 \log\log Q + C_0 \qquad (\log Q \geq \ell_0,\ \chi \in \mathcal{Z}(Q)). \tag{24}$$

Put $\ell = \log Q$ and choose

$$x = \ell^{5/2}, \qquad V = x^{400} = \ell^{1000}, \qquad \theta = x^{-2}, \qquad \delta = 1/4000.$$

Use (21) with this fixed δ. Its first two terms are bounded respectively by constant multiples of

$$\ell^2 V^{-1/400+1/4000} = \ell^{-1/4}, \qquad \ell\, \frac{Vx^2}{Q} = \frac{\ell^{1006}}{e^\ell}.$$

The finite-difference term is bounded as well. Thus $|T_\chi(V)| \leq C_1$, for an absolute C_1 and all sufficiently large ℓ.

Mertens's upper bound gives

$$\sum_{n \leq V} \frac{\Lambda(n)}{n} \leq \log V + 2 \qquad (V \geq 1).$$

It follows that $|D_\chi| \le C_1 + \log V + 2 = 1000 \log \ell + C_1 + 2$, which proves (24).
Lean: Deduction.mertens_upper (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.mertensUpper, lines 103–112)

Finally,

$$\log\log Q \le \log A_1 + \frac{\varepsilon}{5}\log Y + 2\log\log Y.$$

The coefficient of $\log Y$ obtained from (24) is $200\varepsilon < 50/13$. Since $4 - 50/13 = 2/13 > 0$, the terms $2000\log\log Y + 1000\log A_1 + C_0$ are absorbed by the remaining multiple of $\log Y$. This gives (23). □

Proof of 7.17

Proof. Put $Z = (Y^K)^{400}$. For $0 < y \le Z$, splitting at $n = y$ gives

$$g_\chi(y) = -\sum_{n \le Z} \Lambda(n)\chi(n)\min\{1, y/n\} - yT_\chi(Z). \tag{25}$$

Indeed, for $n \le y$, $y/n - 1 = y/n - \min\{1, y/n\}$, whereas for $y < n \le Z$, the two terms on the right cancel.

The first sum in (25) is $S(Z, y, q, \kappa)$. Lemma 7.15 gives $|yT_\chi(Z)| \le |Y^K T_\chi(Z)| \le Y^\alpha$. For $Y \ge 16$, $\alpha \le 1/4$, and $y \ge Y$, we have $Y^\alpha \le \frac{1}{2}y^{1/2+\tau}$. The triangle inequality proves the desired bound for $g_\chi(y)$ at every point of the window. The absolute constant in (22) gives the logarithmic derivative bound. □

Proof of 7.18

Proof. Set $Q = FU$ and $Z = (Y^K)^{400}$. The inequalities $1 \le F \le \sqrt{U}$ give

$$Y^{\alpha/6} \le \log Q \le \tfrac{3}{2}\log U \le \tfrac{3}{2}Y^{\alpha/5}(\log Y)^2 \le Y^{\alpha/4}$$

for sufficiently large Y. Lemma 7.6 gives $|P| \ge U^{9/10} > 0$.

Define $B_1 \subseteq B$ to consist of primes admitting a primitive character of conductor fq, with $f \le F$ and $fq \le Q$, whose L-function has a zero in $\mathcal{R}(Q)$. By Lemma 7.13, $|B_1| \le C_{\mathrm{b}} Q^{1/4} \le C_{\mathrm{b}} U^{3/8}$. Since $3/8 < 9/10$, increasing Y_0 gives

$$|B_1| \le \tfrac{1}{2}\eta|P|. \tag{26}$$

For each $\xi \in \Xi$, let $B_2(\xi) \subseteq B$ consist of primes for which $|S(Z, y, q, \xi)| > \frac{1}{2}y^{1/2+\tau}$ at some $y \in [Y, Y^K]$. Theorem 7.10 supplies its mean-square hypothesis. Apply Theorem 7.11 with proportion $\eta/(2(R+1))$. It gives

$$|B_2(\xi)| \le \frac{\eta}{2(R+1)}|P|, \qquad \left|\bigcup_{\xi\in\Xi} B_2(\xi)\right| \le \tfrac{1}{2}\eta|P|. \tag{27}$$

We prove

$$B \subseteq B_1 \cup \bigcup_{\xi\in\Xi} B_2(\xi). \tag{28}$$

Take $q \in B$ outside the right-hand side, and let ξ, L witness its defining failure. With $f = \mathrm{cond}(\xi)$, we have $q > \sqrt{U} \ge F \ge f$, so $q \nmid f$. Also $q \equiv a_0 \pmod{m_0}$, where $8 \mid m_0$ and $(a_0, m_0) = 1$, so q is odd. Lemma 7.14 constructs a nonprincipal primitive χ of conductor fq with $\chi(n) = \xi(n)(n/q)$ for every n. Its conductor is at most FU. Since $q \notin B_1$, $L(s, \chi)$ has no zero in $\mathcal{R}(Q)$. Thus $\chi \in \mathcal{Z}(Q)$. Since $q \notin B_2(\xi)$, its finite sum obeys the required bound throughout the window. Lemma 7.17 gives its box.

Any primitive representative of the same function at level L agrees with χ on all integers coprime to L. The uniqueness assertion of Lemma 7.14 identifies it with χ. Thus every primitive representative at level L belongs to the box, contrary to the choice of ξ, L. This proves (28). Combine (26) and (27). □

Proof of 7.20

Proof. For each character ξ modulo N_0, choose its primitive representative ξ^*. Its conductor divides N_0, and $\xi^*(n) = \xi(n)$ whenever $(n, N_0) = 1$. The set Ξ of these representatives has at most $R = \#\{\text{Dirichlet characters modulo } N_0\}$ members.

Use $\alpha = \varepsilon$, $U = e^z$, $m_0 = 8$, $a_0 = 3$, and $F = N_0$ in Theorem 7.18. For large Y, we have $\frac{1}{4}(\log Y)^2 \log 8 \le z$, $Y^{\varepsilon/6} \le z \le z(\log Y)^2$, and $N_0 \le e^{z/2}$. Thus the progression is admitted and the conductor bound holds. The sole prime divisor of $m_0 = 8$ is $2 \le z$, so (13) holds.

If q fails the first selection condition, some character ξ modulo N_0 has a q-twist with a primitive representative at level $N_0 q$ outside $\mathcal{B}(N_0 e^z, C_1, \tau, Y, Y^K)$. On integers coprime to this level the values of ξ and ξ^* agree. The same representative therefore witnesses a failure for the member $\xi^* \in \Xi$ in Theorem 7.18. Apply Theorem 7.18 to this bad set. □

Proof of 7.21

Proof. Write $m_0 = 16\mathfrak{m}(z, q)$. The integer $\mathfrak{m}(z, q)$ is odd. By the Chinese remainder theorem there is a reduced residue class a_0 modulo m_0 specified by $a_0 \equiv 7 \pmod{16}$ and $a_0 \equiv -1 \pmod{\mathfrak{m}(z, q)}$. Hence

$$P_2(z, z', q) = P(m_0, a_0, e^{z'}). \tag{30}$$

Use the parameters

$$\alpha = \frac{6}{5}\varepsilon, \qquad U = e^{z'}, \qquad F = N_0 e^z \tag{31}$$

in Theorem 7.18.

The elementary primorial bound gives

$$\prod_{\substack{3 \le p \le z \\ p \text{ prime}}} p \le 4^{\lfloor z \rfloor_+}.$$

Since $q \le e^z$,

$$\log m_0 \le \log 16 + z + z \log 4 \le 4z \tag{32}$$

for large Y. It follows that $\frac{1}{4}(\log Y)^2 \log m_0 \le z(\log Y)^2 \le z' = \log U$. For the other two admission bounds, observe that $Y^{\alpha/6} = Y^{\varepsilon/5} = z$ and $Y^{\alpha/5} = z Y^{\varepsilon/25}$. The lower bound $z' \ge z(\log Y)^2$ gives $Y^{\alpha/6} \le \log U$. The upper bound on z', together with $K^2 \le Y^{\varepsilon/25}$, gives $\log U \le Y^{\alpha/5}(\log Y)^2$. Thus the progression in (30) is admitted.

Every prime factor of m_0 other than q is at most z for large Y. The small factors are at most $Y^{\alpha/5}$. For q, use $q > e^{z/2}$ and $z \ge 800K \log Y$ to obtain

$$q > (Y^K)^{400}. \tag{33}$$

This proves the prime-divisor condition (13).

For large Y, every $q \in P_1(z)$ exceeds N_0. For each character ξ modulo N_0 and each $a \in \{0, 1\}$, choose a primitive character $\xi^*_{q,a}$ representing $n \mapsto \xi(n)(n/q)^a$ on integers coprime to N_0. For $a = 0$, use the primitive representative of ξ. For $a = 1$, Lemma 7.14 gives the twist of that representative by q. In both cases the conductor is at most $N_0 q \le F$. The resulting family has at most $R = 2\#\{\text{Dirichlet characters modulo } N_0\}$ members, uniformly in q.

Finally, $\log F = \log N_0 + z \le z'/2$ for large Y, because $z' \ge z(\log Y)^2$. Hence $1 \le F \le \sqrt{U}$. The parameter $\alpha = 6\varepsilon/5$ is at most $1/4$.

The box height in Theorem 7.18 is $FU = N_0 e^{z+z'}$. If d fails the second selection condition for ξ, a, the same primitive character witnesses failure in Theorem 7.18 for $\xi^*_{q,a}$ at level $L = N_0 q d$, since the two twists agree on integers coprime to L. Apply Theorem 7.18 to this bad set. □

Proof of 7.22

Proof. Apply Lemmas 7.20 and 7.21, and take the larger of their constants. A larger constant preserves the logarithmic derivative bound because $\log Y \geq 0$. Restriction from $[Y, Y^K]$ to $[Y, X]$ preserves every box.

Let G_1 be the complement in $P_1(z)$ of the candidates failing the first box. For each $q \in G_1$, let $G_2(q)$ be the complement in its own candidate set $P_2(z, z', q)$ of the candidates failing the second box. The two bad-set estimates give the stated cardinality inequalities. The candidate sets are nonempty by the same lemmas. □

Proof of 7.23

Proof. Use Theorem 7.22 with $\eta = 1/2$. Its first good set has cardinality at least half the positive cardinality of $P_1(z)$, so it is nonempty. Choose q from it. Choose d from the second good set associated to q, which is nonempty by the same cardinality bound. Membership in $P_1(z)$ and $P_2(z, z', q)$ gives (34). Membership in the corresponding good sets gives both box bounds.

For large Y, we have $e^{z/2} > N_0$ and $z' \geq 2z$. Hence $q > N_0$ and $d > e^{z'/2} \geq e^z \geq q$. □

Proof of 7.25

Proof. Apply Theorem 7.23 with $N_0 = 8M$, $\tau = \varepsilon/2$, $K = A$, $X = Y^A$, and $z' = z_0'$. It gives primes q, d satisfying the window conditions and $8M < q < d$. Thus neither prime divides M, and $(M, qd) = 1$.

Let $f : \mathbf{N} \to \mathbf{C}$ agree, on integers coprime to $8M$, with a character ξ_8 modulo $8M$. Let χ be a primitive representative at level $2Mqd$ of $n \mapsto f(n)(n/q)^i(n/d)^j$, where $(i, j) \in \{0, 1\}^2 \setminus \{(0, 0)\}$. We claim that

$$\chi \in \mathcal{B}(H_j, C, \varepsilon/2, Y, Y^A) \tag{37}$$

holds, with the constant C from Theorem 7.23.

If $j = 1$, restrict the agreement defining χ to integers coprime to $8Mqd$. On these integers $f = \xi_8$, so χ represents the twist in the second selection condition with $a = i$. Its height is $8Me^{z+z_0'} = H_1$.

If $j = 0$, then $i = 1$. Choose the primitive representative ξ_8^* of ξ_8, with conductor $f_0 \mid 8M$. Since $q > 8M$, Lemma 7.14 gives a primitive character of conductor $f_0 q$ whose values are $\xi_8^*(n)(n/q)$. It agrees with χ on integers coprime to $2Mqd$. The uniqueness part of Lemma 7.14 therefore identifies the characters. In particular, χ represents $\xi_8(n)(n/q)$ at level $8Mq$. The first selection condition gives (37) at height $8Me^z = H_0$.

For the first family in Definition 7.24, induce the given character ξ modulo M to level $8M$. Induction preserves its values on integers coprime to $8M$. The claim supplies (37). In particular, χ is nonprincipal, so its centred sum is the function g_χ of Definition 7.1.

Lean: `QH.pairEx_g` (`Development.AutomorphicArithmetic.MeanSquareBounds.pairEx`, lines 61–77)

Its box yields $|g_\chi(y)| \leq y^{1/2+\varepsilon/2}$ for $Y \leq y \leq Y^A$.

For $j = 0, 1$, we have $\log H_j = \log(8M) + z + jz_0'$. For $Y \geq \exp(1 + \log(8M))$,

$$Y^{\varepsilon/6} \leq \log H_j \leq 3Y^{\varepsilon/5}(\log Y)^2.$$

By (37), $\chi \in \mathcal{Z}(H_j)$, so (23) applies with $Q = H_j$ and $A_1 = 3$. It gives $|L'/L(1, \chi)| \leq 4 \log Y \leq 4 \log y$. This proves (35).

Finally, let ρ be a real primitive character of conductor dividing $8M$. Induce it to level $8M$ and apply the same claim with $f = \rho$. By (37), $L(s, \chi) \neq 0$ throughout $\mathcal{R}(H_j)$, as required by (36). Take $Y_1 = \max\{Y_{\text{pair}}, Y_4, \exp(1 + \log(8M))\}$, where Y_{pair} is the threshold in Theorem 7.23 for the parameters used above and Y_4 is the threshold in (23) for $A_1 = 3$. □

Proofs for Chapter 8: Building the arithmetic host of a window

Proof of 8.1

Proof. The surjection $A^{(e)} \to H_e$ has kernel $\{1\} \times C_e$, where $C_e = \{b \in B : (1, b) \in A^{(e)}\}$. Thus $|A^{(e)}| = 2^e|C_e|$. Reduction gives $C_{e+1} \subset C_e$, so $|C_e|$ stabilizes to c. The projection onto B has kernel A_0 embedded in the cyclic group H_e, and $|B| = |\Xi|$.

By (8.2), the kernel of Art has both signs 1 and $\eta(a) = 1$, which gives the congruence and $\psi(a) = 1$. Primitivity of ψ gives $M \mid Q_S$ because ψ factors through Q_S. □

Proof of 8.3

Proof. Let e_0 exceed $\max(2, 4M, 64o_\psi)$ and the stabilization threshold of 8.1, and apply 8.1 at this e_0. Take E to be the fixed field of the kernel of (8.2) inside $\mathbf{Q}(\zeta_{Q_S qd})$. For each unit residue modulo Q_S, the Chinese remainder theorem realizes all four sign pairs, so the Artin map surjects onto $\{\pm 1\}^2 \times A$.

The third coordinate projects to $[a] \in H_e$ because $w(\pm 1)$ projects to 1. For $a \equiv 1 \bmod Q_S q$, the Artin image lies in $\{1, \epsilon_d\}$. For $a \equiv 1 \bmod Q_S d$ the third coordinate is $w((a/q))$ and the image squares to 1. For each unit a modulo $N = Q_S qd$, the automorphism $\zeta_N \mapsto \zeta_N^a$ restricts on E to the element with coordinates $\mathrm{Art}(a)$, as required in Definition 5.1. For $\ell = q, d$, the quadratic Gauss sum $\tau_\ell = \sum_{u \bmod \ell} (u/\ell)\zeta_\ell^u$ satisfies $\tau_\ell^2 = -\ell$ and $\sigma_a(\tau_\ell) = (a/\ell)\tau_\ell$ [35, Theorems 9.5 and 9.7]. The Artin kernel therefore fixes both sums, and the two signs cut out $\mathbf{Q}(\sqrt{-q})$ and $K = \mathbf{Q}(\sqrt{-d})$. Complex conjugation maps to $((-1,-1),1)$. These data give the raw host of 5.1.

The local calculation of 5.3 gives trivial inertia outside $Q_S qd$ and inertia generated by ϵ_d at d. Since ϵ_d acts nontrivially on K, the inertia at d in E/K is trivial. Thus every prime factor of $\mathfrak{D}_{E/K}$ lies above a rational prime dividing $Q_S q$. These primes split in K by 5.3. For each conjugate pair of primes of K above them, choose one member and put into $\mathfrak{b}$ every prime factor of $\mathfrak{D}_{E/K}$ above that member, with its multiplicity. Complex conjugation preserves the different and interchanges the two members of each pair. Every element of $\mathrm{Gal}(E/K)$ preserves the different and fixes each prime of K. Consequently

$$\mathfrak{D}_{E/K} = \mathfrak{b}\,\overline{\mathfrak{b}}, \qquad \mathfrak{b} + \overline{\mathfrak{b}} = \mathcal{O}_E, \qquad \sigma(\mathfrak{b}) = \mathfrak{b} \quad (\sigma \in \mathrm{Gal}(E/K)).$$

Fix a prime $\ell \mid Q_S$. If $2^{e+2} \mid Q_S/\ell$, minimality of Q_S gives a unit $1 \bmod Q_S/\ell$ with nontrivial joint image. Otherwise $\ell = 2$ and $v_2(Q_S) = e + 2$. The unit $1 + Q_S/2$ is $1 \bmod Q_S/2$ and $1 + 2^{e+1} \bmod 2^{e+2}$, so its H_e-coordinate is nontrivial. Lift the witness to a unit equal to $1 \bmod qd$. The lift is $1 \bmod Q_S qd/\ell$ and its Artin image is $((1,1), z)$ with $z \neq 1$. The bounds follow from $Q_S \le 2^{e+2}M \le 4Mn \le n^2$ and $o_\phi^{64} = o_\psi^{64} \le 2^{64 o_\psi} \le 2^e$. Every odd prime $\ell \mid Q_S$ divides M, so $4\ell \le B_0$.

For $\xi \in \Xi$ let ξ_A be its pullback to A. The character $\theta_\xi((s_q, s_d), z) = \xi_A(z\, w(s_q)^{-1})$ is trivial on A_0 and takes value 1 on ϵ_d. Substitution into (8.2) gives the primitive representative ξ^{prim}. On a unit equal to $1 \bmod Q_S d$ and nonsquare modulo q, the retained characters take value 1 and θ_q takes value -1. Apply 8.1 and 8.2 for the kernel and twist descriptions. □

Proof of 8.4

Proof. Let $\theta \in \mathcal{B} = (X_+ \setminus K_H(\mathcal{X})) \cup X_-$ and let χ be a primitive Dirichlet representative of θ. Since every element of $X_+ \cup X_-$ is trivial on A_0, Lemma 8.2 gives $\xi \in \Xi = \langle \psi \rangle$ and $i, j \in \{0, 1\}$ such that $\theta(\epsilon_d) = (-1)^j$ and

$$\theta(\mathrm{Art}(a)) = \xi(a) \left(\frac{a}{q}\right)^i \left(\frac{a}{d}\right)^j \qquad \big((a, Q_S qd) = 1\big).$$

We claim that $(i, j) \neq (0, 0)$. Suppose $(i, j) = (0, 0)$. If $\theta \in X_-$, then $\theta(\epsilon_d) = -1$, contradicting $\theta(\epsilon_d) = (-1)^0 = 1$. If $\theta \in X_+ \setminus K_H(\mathcal{X})$, then $\theta(\mathrm{Art}(a)) = \xi(a)$ for every integer a coprime to $Q_S qd$. Since $M \mid Q_S$, every integer coprime to $Q_S qd$ is coprime to M, so the primitive representative $\xi^{\mathrm{prim}} \in \mathcal{X}$ of ξ agrees with ξ and hence with $\theta \circ \mathrm{Art}$ on integers coprime to $Q_S qd$. Thus θ is represented by $\xi^{\mathrm{prim}} \in \mathcal{X}$, contradicting $\theta \notin K_H(\mathcal{X})$. Hence $(i, j) \neq (0, 0)$.

Let a be an integer with $(a, 2Mqd) = 1$. Then $(a, q) = (a, d) = 1$, and since every prime divisor of Q_S divides $2M$, we also have $(a, Q_S) = 1$, so $(a, Q_S qd) = 1$. Combining the representative identity $\chi(a) = \theta(\mathrm{Art}(a))$ with (8.5) yields

$$\chi(a) = \xi(a) \left(\frac{a}{q}\right)^i \left(\frac{a}{d}\right)^j = f_{\xi,i,j}(a) \qquad \big((a, 2Mqd) = 1\big).$$

Therefore χ is a primitive representative of $f_{\xi,i,j}$ at level $2Mqd$. Since (q,d) is a good pair for modulus M on $[Y,x]$ with exponent ε and $Y \le x$, (35) of Definition 7.24 applies at $y = x$ and gives

$$|g_\chi(x)| \le x^{1/2+\varepsilon/2} \le x^{1/2+\varepsilon}, \qquad \left|\frac{L'}{L}(1,\chi)\right| \le 4\log x,$$

using $x \ge 1$ and $\varepsilon > 0$ in the first inequality. □

Proof of 8.6

Proof. Set $P = 2 + 8\varepsilon$ and $\lambda = c^{1/P}$. For $v \ge 1$ take $a(v) = \lceil \log_2(v^P) \rceil$, and take $a(v) = 0$ for $v < 1$. Then $v^P \le 2^{a(v)} \le 2v^P$ for $v \ge 1$. Apply Definition 26.3 to this page function and exponent 401, obtaining $J \ge 1$ and X_0. Let Y_1 be the cutoff of 7.25 for this J and $M = \operatorname{cond}\psi$.

Given y, choose $X \ge X_0$ with $Y = X^{1/J} \ge \max(1, y, Y_1, \lambda^3, \lambda^{400})$. By 7.25 there is a good pair on $[Y, Y^J] = [Y, X]$. Its prime conditions at Y give the window pair required by 26.3. Keeping this pair fixed, choose $W \ge Y$ with $W^2 \le X$ as in that definition. Put $Z = \lambda W \ge y$.

For $Z \le x \le Z^{3/2}$, use $v = x/\lambda$ and $e = a(v)$. Lemma 8.5 gives $W \le v \le x \le W^2 \le X$ and the required bounds on $c2^e$. Thus $x^{1/J} \le Y \le x$, and restricting the good pair to $[Y,x]$ gives (8.8). Every field with the prescribed kernel at (e,q,d) satisfies the upstream bound at v. All its summands are nonnegative. The cutoff and budget inequalities of Lemma 8.5 therefore give (8.9) for the same field and all the local data of Definition 26.1. □

Proof of 8.8

Proof. Write $L_\chi(s) = \sum_{n\ge1} a_\chi(n) n^{-s}$ and $P_\chi(s) = \sum_{n\ge1} \widehat{\lambda}_\chi(n) n^{-s}$. For $\operatorname{Re} s > 1$ the convolution gives $L'_\chi = -P_\chi L_\chi$. Integration from $+\infty$, with $a_\chi(1) = 1$ and $|\widehat{\lambda}_\chi| \le 2\Lambda$, gives $|L_\chi(\sigma + it)| \ge \zeta(\sigma)^{-2}$. Put $r = \kappa|G|$. By (8.10) and character orthogonality, $E_\chi(y) = \sum_{n\le y} a_\chi(n) - \mathbf{1}_{\chi=1} ry$ satisfies $|E_\chi(y)| \le 10D\sqrt{y}$ for $y \ge 1$. Partial summation gives

$$L_\chi(s) = \mathbf{1}_{\chi=1}\frac{rs}{s-1} + s\int_1^\infty E_\chi(y) y^{-s-1}\,dy.$$

Hence F_χ extends to $\operatorname{Re} s > 1/2$ and $|F_\chi(s)| \le 300D(1+|s|)^2$ for $\operatorname{Re} s \ge 11/20$. On the disc of radius $7/10$ about $z_t = 5/4 + it$, the logarithm of the ratio of the maximum modulus to $|F_\chi(z_t)|$ is $O(\ell(t))$. Apply [35, Lemmas 6.1 and 6.3] to $z \mapsto F_\chi(z_t + 7z/10)$ with $r = 5/7$ and $R = 6/7$ in those lemmas. For $|s - z_t| \le 1/2$ away from the zeros,

$$\frac{F'_\chi}{F_\chi}(s) = \sum_{|\rho - z_t| \le 3/5} \frac{m_\rho}{s-\rho} + O(\ell(t)), \qquad \sum_{|\rho - z_t| \le 3/5} m_\rho \ll \ell(t), \tag{A.15}$$

where m_ρ is the multiplicity. Since $P_\chi(s) = \mathbf{1}_{\chi=1}/(s-1) - F'_\chi(s)/F_\chi(s)$ and all zeros have real part at most 1, each zero contributes a nonpositive real part to $P_\chi(s)$ when $\operatorname{Re} s > 1$.

Lean: `GlobalTheory.HkLocal.logDeriv_local (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.RayClassLFunctionLogDerivLocalFormula`, lines 21–156)

Write a zero as $1 - b + it$, with $0 \le b \le v \le 1/400$ and $v > 0$. At $\sigma = 1 + 6v$, the inequality $3 + 4\cos u + \cos 2u \ge 0$ gives $3\operatorname{Re} P_1(\sigma) + 4\operatorname{Re} P_\chi(\sigma + it) + \operatorname{Re} P_{\chi^2}(\sigma + 2it) \ge 0$. If $\chi^2 \ne 1$, both shifted pole terms vanish. If $|t| \ge 30v$, their real parts before weighting are at most $1/(156v)$. In either case the indicated zero and (A.15) give

$$4/(7v) \le 3/(6v) + 5/(156v) + O(\ell(t)).$$

For $\chi^2 = 1$ and $0 < |t| < 30v$, sum $P_\xi(\sigma)$ over $\xi \in \langle\chi\rangle$ at $\sigma = 1 + 60v$. Orthogonality makes the coefficients nonnegative. The conjugate pair contributes $2(60v+b)/((60v+b)^2 + t^2) \ge 120/(4621v)$ against one pole of size $1/(60v)$. Both comparisons give $1/v \ll \ell(t)$, which excludes these zeros for $v = c_0/\ell(t)$ and sufficiently small c_0.

Lean: GlobalTheory.Hk.ZeroFree.zero_free_region (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.RayClassLFunctionZeroFreeRegion, lines 286–311)

If the strip contains two real zeros counted with multiplicity, sum $P_\xi(\sigma)$ at $\sigma = 1 + 2c_0/\ell(0)$ over the subgroup generated by their characters. It has at most four elements and the sum has nonnegative coefficients. By (A.15) the zeros contribute at least $2\ell(0)/(3c_0)$ against one pole $\ell(0)/(2c_0)$ and an error $O(\ell(0))$. This is impossible for small c_0.

Lean: GlobalTheory.Hk.RealZero.main (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.RayClassLFunctionUniqueSimpleExceptionalZero, lines 135–258)

Write $H_\chi(t) = \sum_n a_\chi(n)e^{-nt}$. The partial summation formula gives $F_1(1) = r \geq D^{-1}$. For $n \geq 1$, nonnegativity and (8.10) give $a_g(n) \leq \kappa n + 10\sqrt{n} \leq 14n$. For a nontrivial real χ, (8.10) and this bound give

$$|H_1(t) - r/t| + |H_\chi(t)| \ll D^{11}\min(t, 1/t).$$

For $t \geq 1$, summing the coefficient bound against e^{-nt} gives $|H_\chi(t)| \ll De^{-t}$. Together with the bound at zero, this extends $\int_0^\infty H_\chi(t)t^{s-1}\,dt = \Gamma(s)F_\chi(s)$ from $\operatorname{Re} s > 1$ to $s = 1$. Hence $F_\chi(1) = \int_0^\infty H_\chi(t)\,dt$.

Put $J(y) = \int_0^\infty e^{-y/w-w}\,dw/w$. For $0 < y \leq 1$, expand the product and use $a_1 * a_\chi \geq 0$ and $(a_1 * a_\chi)(1) = 1$ to get

$$\begin{aligned} J(1) \leq J(y) &\leq \sum_{n\geq 1}(a_1 * a_\chi)(n)J(ny) \\ &= \int_0^\infty H_1(y/w)H_\chi(w)\,\frac{dw}{w} = \frac{r}{y}F_\chi(1) + O(D^{22}\sqrt{y}). \end{aligned}$$

The inequality $e^{-z} \ll z^{-3}$ gives $J(z) \ll z^{-3}$, so $\sum_n a_1(n)n^{-3} < \infty$ gives absolute convergence. In the error integral use $\min(y/w, w/y) \leq \sqrt{y/w}$ and the integrability of $\min(w, 1/w)w^{-3/2}$. Take $\sqrt{y} = \delta D^{-22}$ for a sufficiently small absolute $\delta > 0$. Since $r \leq 4D$, this gives $|F_\chi(1)| \gg D^{-45}$.

Lean: GlobalTheory.Hk.RealChar.norm_F_one_ge (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.NormRayClassFamilyFOneGeOfRealCharacter, lines 343–422)

The Cauchy integral formula with radius $1/10$ and the bound for F_χ give $|F_\chi'| \leq 18000D$ on $|s-1| \leq 1/4$. For either a trivial or a nontrivial real character, $|F_\chi(1) - F_\chi(\beta)| \leq 18000D(1-\beta)$ gives $1 - \beta \gg D^{-46}$.

Lean: GlobalTheory.Hk.LFam.real_zero_far_from_one (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsRayClassLFunctionRealZeroFarFromOne, lines 47–76)

□

Proof of 8.9

Proof. Put $w = \sqrt{\log u}$ and $\alpha = c_0/32$. For $\alpha w \leq 1$, Chebyshev's bound gives $|\Psi_\chi(u)| \leq 2\sum_{n\leq u}\Lambda(n) \ll u$ [35, Theorem 2.4], and any $\beta \in Z_\chi$ has $\beta > 1/2$. Otherwise take $T = e^{\alpha w}$, with $2 \leq T \leq u$ and $\ell(T) \leq 3w$. Truncated Perron inversion on $\operatorname{Re} s = 1 + 1/\log u$ [35, Theorem 5.2 and Corollary 5.3] costs $O(u\log^2 u/T + \log u)$. For $u/2 < n < 2u$, sum the kernel error $O(\log u)\min(1, u/(T|n-u|))$, with the minimum taken as 1 at $n = u$. For the other n, use $|\log(u/n)| \geq \log 2$ and $\sum_n \Lambda(n)n^{-1-1/\log u} \ll \log u$ by partial summation from Chebyshev's bound.

Lean: GlobalTheory.PerronCoeff.truncated_perron_of_le_two_vonMangoldt (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.TruncatedPerronOfLeTwoVonMangoldt, lines 9–157)

In $1 - c_0/(2\ell(t)) \leq \operatorname{Re} s \leq 3$ at height t, away from Z_χ, we have

$$\frac{F_\chi'}{F_\chi}(s) - \sum_{\beta\in Z_\chi}\frac{1}{s-\beta} = O_{c_0}(\ell(t)^2).$$

In (A.15) there are $O(\ell(t))$ zeros counted with multiplicity. For each nonexceptional zero in that sum, $\ell(\operatorname{Im}\rho) \leq 4\ell(t)/3$ gives a horizontal distance $\gg c_0/\ell(t)$ from s. If $|\beta - z_t| > 3/5$ and $|s - z_t| \leq 1/2$, then $|s-\beta| > 1/10$. For $\operatorname{Re} s \geq 7/4$ the absolutely convergent series for P_χ gives the bound directly.

Lean: GlobalTheory.Hk.LogDerivHalf.logDeriv_sub_exceptional_le (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.LogDerivSubExceptionalLe, lines 92–260)

Take $b = c_0/(8\ell(T))$. Shift the integral of $P_\chi(s)u^s/s$ to the line $\sigma = 1 - 2b$ or $1 - 4b$ at distance at least b from Z_χ. The residues give $\mathbf{1}_{\chi=1}u$ and $-u^\beta/\beta$ for $\beta > \sigma$. A zero to the left contributes at most $2u^{1-2b}$ in absolute value. On the new sides $|P_\chi| \ll_{c_0} \ell(T)^2 + 1/b$. The horizontal and vertical integrals cost $O_{c_0}(w^2u/T)$ and $O_{c_0}(Tw^2u^{1-2b})$, respectively. Since $2b\log u \geq c_0 w/12 \geq 2\log T$, the total error is $O_{c_0}(uw^4e^{-\alpha w}) = O_{c_0}(ue^{-\alpha w/2})$. □

Proof of 8.10

Proof. Use the c_0 of 8.8 in 8.9. Put $L = \sqrt{\log X}$, $U = X(1+h)$ and $\Delta = c_9 D^{-a}$. Choose $c = \min\{1, c_7/(2(a+2)), c_7c_9/(24C)\}$. Then $cL \geq \log D \geq 1$, so $X \geq \mathrm{e}$ and both endpoints satisfy $\log D \leq \sqrt{\log u}$. By orthogonality, $\sum_\chi \Psi_\chi(u) = |G| \sum_{1 \leq n \leq u} \lambda_0(n)$. Subtract (8.11) at the endpoints and write $W = \sum_{X < n \leq U} \lambda_0(n)$. Since $U \leq 2X$,

$$|G|W = hX - B + R, \qquad |R| \leq 3CDXe^{-c_7 L}.$$

Here $B = 0$ in the absence of an exceptional pair. If the pair is present,

$$B = \int_X^U t^{\beta-1}\,dt \leq hXe^{-\Delta \log X} \leq hX(1-\Delta/2)$$

since $0 < \Delta \leq 1$ and $\log X \geq 1$. The remainder satisfies

$$\frac{|R|}{hX\Delta} \leq \frac{3C}{c_9} e^{((a+2)c - c_7)L} \leq \frac{3C}{c_9} e^{-c_7 L/2} \leq \frac{1}{4}.$$

The last step uses $e^{c_7 L/2} \geq c_7 L/2 \geq c_7/(2c) \geq 12C/c_9$. Hence $|G|W \geq hX\Delta/4$. Use $|G| \leq D$ and take $A = a+1$, $\eta = c_9/4$. □

Proof of 8.11

Proof. Let $K = \mathbf{Q}(\sqrt{-d})$, G its ray class group modulo $N\mathcal{O}_K$, and $D = 4dN^2$. For ideals prime to N of norm n in class g, let $a_g(n)$ count the ideals and $\lambda_g(n)$ sum their ideal von Mangoldt weights. The kernel of the map to the ordinary class group is a quotient of $(\mathcal{O}_K/N\mathcal{O}_K)^\times$, so $|G| \leq h_K N^2$, where h_K is the class number. Minkowski's bound [41, Ch. I, §6, Exercise 3] gives an ideal of norm at most $\sqrt{d}$ in each ordinary class. At norm n there are at most $\tau(n) \leq n$ ideals by prime ideal factorization, where τ counts positive divisors. Thus $h_K \leq d$ and $|G| \leq D$.

Choose an integral ideal J prime to N in class $-g$. Ideals I in class g correspond to generators z of IJ with $z \equiv 1 \bmod N$. Such a generator is unique since $N \geq 3$ and $\mathcal{O}_K^\times = \{\pm 1\}$. The possible z form a translate of NJ in J. In a reduced basis of NJ, write

$$\frac{|z|^2}{\mathrm{N}J} = \alpha(m + \theta_1 + \ell(k+\theta_2))^2 + \gamma(k+\theta_2)^2 \qquad (m, k \in \mathbf{Z}),$$

where θ_1, θ_2 describe the translate. For the shortest basis vector w_1, $\alpha = |w_1|^2/\mathrm{N}J \geq N^2$ since $w_1/N \in J$. Choose the second vector with projection $|\ell| \leq 1/2$ on the first. Its length is at least $|w_1|$, hence $\gamma \geq 3\alpha/4$. The covolume gives $\alpha\gamma = dN^4/4$, so $\kappa = \pi/\sqrt{\alpha\gamma} = 2\pi/(\sqrt{d}\,N^2)$ lies in $[D^{-1}, 4]$. Each horizontal chord costs at most one in rounding. The sum of chord lengths differs from its integral by at most the total variation $4\sqrt{y/\alpha}$. The area is κy and the count error is at most $2\sqrt{y/\gamma} + 3 + 4\sqrt{y/\alpha} \leq 10\sqrt{y}$.

For the Gaussian sum use Poisson summation twice [35, Theorem D.3]. At scale s, the one-dimensional Gaussian sum divided by its integral differs from 1 by at most $2\sum_{j \geq 1} e^{-\pi^2 j^2/s} \leq s^2$, by $e^{-z} \leq 2/z^2$. First sum over m, uniformly in the shift $\theta_1 + \ell(k+\theta_2)$, then over k. The relative error is at most $(\alpha t)^2 + (\gamma t)^2 + (\alpha\gamma t^2)^2$. For $0 < t \leq 1$ the absolute error is at most $3\pi(\alpha\gamma)^2 t \leq D^{10} t$. This gives (8.10).

The counts give $a_g(0) = 0$ and $a_g(1) = \mathbf{1}_{g=0}$. Factorization gives the logarithmic convolution and $\sum_g \lambda_g(n) \leq 2\Lambda(n)$ since $[K : \mathbf{Q}] = 2$. For real χ, the coefficient at an ideal I prime to N in $L_1 L_\chi$ is $\prod_{\mathfrak{p}^e \| I} \sum_{j=0}^{e} \chi(\mathfrak{p})^j$. It is nonnegative and at least one on squares. Summing over ideals of norm n gives $(a_1 * a_\chi)(n) \geq 0$. The $a_1(n)$ ideals of norm n prime to N have distinct squares of norm n^2, each with coefficient at least one. Hence $(a_1 * a_\chi)(n^2) \geq a_1(n)$. By 8.10, $W = \sum_{X < n \leq U} \lambda_0(n) \geq \eta hX/D^A$, where $U = X(1+h)$.

Lean: `GlobalTheory.Hk.rayCountTheta` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.RayCountThetaFibreCount`, lines 53–121)

Lean: `GlobalTheory.Hk.exists_lFam_ray` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsLFamRay`, lines 73–128)

If p is prime and $\lambda_0(p) > 0$, an ideal of norm p has a generator $z = (u-v)/2 + v(1+\sqrt{-d})/2 \equiv 1 \bmod N$. Its norm and congruence give (8.12). Each such prime contributes at most $2\log U$, and the other positive weights lie on higher prime powers. Thus

$$W \le 2\log U\,|\mathcal{P}(d,N;X,U)| + 2\sum_{j\ge 2}\vartheta(U^{1/j}),$$

where $\vartheta(y) = \sum_{p\le y}\log p$. By Chebyshev's bound [35, Theorem 2.4], the last term is $O(\sqrt{U} + U^{1/3}\log U) = O(\sqrt{X})$. For $v = \sqrt{\log X}$,

$$hX/(D^A\sqrt{X}) \ge \exp(v^2/2 - (A+1)cv), \qquad v \ge \log 4/c.$$

Shrink c so that the higher powers cost at most $\eta hX/(2D^A)$. Since $\log U \le 2\log X$, take $c' = \eta/8$. □

Proof of 8.12

Proof. Take $V = (kb)^8$ and $D = 4dN^2$. Since $d \mid N$, $\log D \le \log 4 + 9x^{\varepsilon/5}(\log x)^2$ and $\log V \ge 8x^{\varepsilon}$. Thus $\log D = o(\sqrt{\log V})$ and $A\log D + \log\log V - \log c' \le (\log V)/10$ for large x. Apply 8.11 with $h = 1$ to obtain

$$|\mathcal{P}(d,N;V,2V)| \ge V^{9/10}. \tag{A.16}$$

Lean: `UpperTheory.T3.normPrimesN` (`Development.AutomorphicArithmetic.ShortNorms.normPrimesN`, lines 108–157)

By (8.1), $r \le x^5$ for large x. Fix $g_* \in [V,2V] \cap \mathbf{Z}$. For each j, four distinct exceptional prime powers would divide $g_*^j - 1$ and have product greater than g_*^j. Hence $|\mathcal{B}_r(g_*)| \le 3r+1$.

For composite ℓ, membership $\ell \in \mathcal{B}_r(g)$ forces $g = \ell$. For prime ℓ and $1 \le j \le r$, take the least a with $\ell^a > V^{j/4}$. For $g \in [V,2V]$, the inequality $g^j < \ell^{4v_\ell(g^j-1)}$ gives $v_\ell(g^j-1) \ge a$, hence $g^j \equiv 1 \bmod \ell^a$. For odd ℓ there are at most j roots by cyclicity. For $\ell = 2$, the cyclic subgroup $\langle 5\rangle$ has index at most two, so there are at most $2j$. Each residue class contains at most $1 + V/\ell^a \le 1 + V^{3/4}$ candidates. For each $\ell \in \mathcal{B}_r(g_*)$, sum over $1 \le j \le r$ and include $g = \ell$. At most

$$(3r+1)\bigl(1 + 2r^2(V^{3/4}+1)\bigr) \tag{A.17}$$

integers $g \in [V,2V]$ have $\mathcal{B}_r(g) \cap \mathcal{B}_r(g_*) \ne \varnothing$. Since $20r^3 < V^{3/20}$ for large x, (A.17) is less than $V^{9/10}$. Choose $g_0 \in \mathcal{P}(d,N;V,2V)$. By (A.16), some $g_1 \in \mathcal{P}(d,N;V,2V)$ lies outside the exclusion set for $g_* = g_0$. Disjointness gives $g_0 \ne g_1$.

Let $A \ge 1$ be the exponent of 8.7. Choose p, p', q_0 successively in the classes 1 modulo $15N$, $15Np$, and Npp'. They are pairwise distinct. Since $N \ge 100$,

$$p \le N^{2A}, \qquad p' \le N^{(2A+2)A}, \qquad q_0 \le N^{(1+2A+(2A+2)A)A}.$$

Take $C_8 = (1 + 2A + (2A+2)A)A$. For large x, $C_8\log N \le 3C_8x^{\varepsilon/5}(\log x)^2 < x^{\varepsilon}$. Thus $p, p', q_0 < b \le (kb)^8 \le g_i$.

Lean: `GlobalTheory.choice_count` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.DivLeRpowThreeQuarters`, lines 221–221)

□

Proof of 8.13

Proof. The primitive conductor N_χ divides $Q_S qd$. For a real character, the local component at an odd prime power is trivial on the principal units, which have odd order. At a power of 2 the component factors through modulus 8, since every unit congruent to 1 mod 8 is a square. The Chinese remainder theorem separates the components at q and d, each either trivial or Legendre. The remaining component has a primitive real representative ρ with $N_\rho \mid 8Q_{\mathrm{odd}}$. Thus on units modulo $Q_S qd$,

$$\chi(a) = \rho(a)\left(\frac{a}{q}\right)^i\left(\frac{a}{d}\right)^j \qquad (i,j \in \{0,1\}).$$

The product of these primitive characters at coprime conductors is primitive at $N_\rho q^i d^j$, so 7.14 gives $N_\chi = N_\rho q^i d^j$. If q or d divides N_χ, the twist is nonzero and $j = \mathbf{1}_{d \mid N_\chi}$. Every integer coprime to $2Mqd$ is coprime to $Q_S qd$ by the hypothesis on the prime divisors of Q_S. The twist identity therefore holds on the units used in Definition 7.24. Since $N_\rho \mid 8M$, the boxes in 7.24 apply. The prime sizes there give the conductor bound.

The nontrivial real host characters of conductor prime to qd have conductor at most $8\lfloor B \rfloor$. Nonvanishing at 1 gives this finite family a common zero-free interval to the left of 1 [35, Theorem 4.9]. Take the smaller of its width and $1/400$ to cover the other real host characters.

Let χ be any primitive host character and put $\lambda = \max(1, \log N_\chi)$. Combining the classical zero-free region with the preceding exclusion gives the following bound for each nontrivial zero ρ of $L(s,\chi)$ [35, §§ 6.1 and 11.1].

$$\operatorname{Re}\rho \le 1 - \frac{c}{\lambda + \log(|\operatorname{Im}\rho| + 3)}, \tag{A.18}$$

where $c > 0$ depends on B. By the functional equation, apply (A.18) to $1 - \rho$ for $\overline{\chi}$. For $|\operatorname{Im}\rho| \le 2$ this gives $|\rho| \gg_B 1/\lambda$. The zero count in a unit interval at height h is $O(\lambda + \log(|h| + 3))$ with multiplicity [35, Theorems 10.13 and 10.17]. The low zeros contribute $O_B(\lambda^2)$. Add the other unit-height counts with weights $1/j$ to obtain

$$\sum_{|\operatorname{Im}\rho| \le T} \frac{m_\rho}{|\rho|} \ll_B (\lambda + \log T)^2 \qquad (T \ge 1),$$

where m_ρ denotes multiplicity.

Put $\psi_0(\chi, u) = \sum_{n \le u} \Lambda(n)\chi(n) - \mathbf{1}_{\chi=1} u$ and

$$Z_\chi(T) = \{\rho : 0 < \operatorname{Re}\rho < 1,\ L(\rho,\chi) = 0,\ |\operatorname{Im}\rho| < T\}.$$

For $2 \le T \le u$ the truncated explicit formula gives

$$\left| \psi_0(\chi, u) + \sum_{\rho \in Z_\chi(T)} m_\rho \frac{u^\rho}{\rho} \right| \ll \frac{u \log^2(N_\chi u T)}{T} + \log(N_\chi u) + \sum_{\rho \in Z_\chi(1)} \frac{m_\rho}{|\rho|}.$$

For $u \ge \mathrm{e}$, put $v = \log u$, $T = \exp(\sqrt{v})$ and $D_v = \exp(-c'v/(\lambda + \sqrt{v}))$, with $c' = \min(c/3, 1)$. Then $2 \le T \le u$ and (A.18) gives $|u^\rho| \le u D_v$ for $\rho \in Z_\chi(T)$. The weighted zero count bounds the zero sum by $O_B(u(v+\lambda)^2 D_v)$ and the last term of the error by $O_B(\lambda^2)$. Since $T^{-1} \le D_v$ and $uD_v \ge 1$, we obtain

$$|\psi_0(\chi, u)| \ll_B u(v+\lambda)^2 D_v. \tag{A.19}$$

This bound implies absolute convergence of $I_\chi = \int_1^\infty \psi_0(\chi,u) u^{-2}\, du$. For real $s > 1$, partial summation gives

$$\int_1^\infty \psi_0(\chi,u) u^{-s-1}\, du = -\frac{L'(s,\chi)}{sL(s,\chi)} - \frac{\mathbf{1}_{\chi=1}}{s-1}.$$

As $s \downarrow 1$, dominated convergence and the Laurent expansion $\zeta(s) = (s-1)^{-1} + \gamma + O(s-1)$ give $I_\chi = -L'(1,\chi)/L(1,\chi)$ for $\chi \ne 1$ and $I_1 = -1 - \gamma$. For $y \ge 1$, termwise integration on $[1, y]$ gives

$$y \int_1^y \psi_0(\chi,u) u^{-2}\, du = \sum_{n \le y} \Lambda(n)\chi(n)(y/n - 1) - \mathbf{1}_{\chi=1} y \log y.$$

Subtract yI_χ and use 2.10 to obtain $G_\chi(y) = -y \int_y^\infty \psi_0(\chi,u) u^{-2}\, du$.

Since $N_\chi \mid Q_S qd$, we have $\lambda \le 1 + 2x^{\varepsilon/5}(\log x)^2$. For $v \ge V = \log y \ge x^{\varepsilon/3}/2$ and large x, this gives $\lambda \le v^{61/100}$. Thus $(v+\lambda)^2 D_v \le 4v^2 e^{-(c'/2)v^{39/100}}$. Substituting $u = \mathrm{e}^v$ in (A.19) bounds $|G_\chi(y)|/y$ by

$$4C e^{-(c'/4)V^{39/100}} \int_0^\infty v^2 e^{-(c'/4)v^{39/100}}\, dv$$

with C depending on B. The integral is finite and $V^{39/100} \gg x^{13\varepsilon/100}$, with $13/100 > 1/8$. This proves (8.14) with a threshold depending on B, ε.

Lean: `QH.psi_thin` (`Development.AutomorphicArithmetic.MeanSquareBounds.PsiThin`, lines 385–421)
Lean: `QH.g_integral` (`Development.AutomorphicArithmetic.MeanSquareBounds.GIntegral`, lines 229–233)
Lean: `QH.int_thin` (`Development.AutomorphicArithmetic.MeanSquareBounds.IntThin`, lines 169–262)

□

Proof of 8.14

Proof. Let $N \geq 1$ and $L \geq 2$. One sphere of constant squared norm in $\{0, \dots, d-1\}^S$ contains at least $d^S/(Sd^2)$ points. Map them to integers by $v \mapsto \sum_{\nu=0}^{S-1} v_\nu (4Ld)^\nu$. In a putative relation (8.16), each coefficient has absolute value less than half the base. Successive reduction modulo the base makes every coefficient zero. The three vectors would lie on a line, which meets the sphere in at most two points.

Choose the parameters so that $4Ls < N$ and the cardinality is at least

$$\frac{N}{L} \exp(-32\sqrt{\log N \log L}). \tag{A.20}$$

If this is at most 1, take $\{0\}$. Otherwise put $v = \log N$, $w = \log L$, $\sigma = \sqrt{vw}$, $S = \lfloor \sigma/w \rfloor$, $a = (v - w - 3)/S$ and $d = \lfloor e^a/(4L) \rfloor$. Then $S \geq 32$, $a \geq 7 + w$ and $e^a/(8L) \leq d \leq e^a/(4L)$. Since $N = e^{aS} L e^3$, we have $4L(4Ld)^S \leq N$. The sphere count gives (A.20) because $e^3 (8L)^S S e^{2a} \leq e^{32\sigma}$. This follows from $Sw \leq \sigma$, $a \leq 2\sigma$ and $3 + S(7 + w) + S + 2a \leq 32\sigma$.

Take $N = n_0$, $E = \exp(32\sqrt{\log n_0 \log L})$, and $Q = n_0/(LE)$. The bounds $\log n_0 = O(\log x)$ and $\log L = O(\log \log x)$ give $4C_o LE \leq x^\varepsilon$ eventually, hence $Q \geq 4x^{-\varepsilon} n$. Since $x^{-\varepsilon} n \geq 1$ and $E \geq 2$, deleting 0 leaves at least $Q - 1 \geq \max(x^{-\varepsilon} n, Q/E)$ points by (A.20). This gives (8.15) with $C_T = 64$. The remaining s are positive and $4Ls < n_0$, so $4L|T_2| \leq n_0$.

Lean: `Formalism.FrobeniusOrbits.hostThickExists` (`Formalism.FrobeniusOrbits.FrobeniusOrbitsProofs.ThickExists`, lines 181–334)

□

Proof of 8.15

Proof. Give $\{0,1\}^A$ the product of the Bernoulli measures with parameters $p_a = \delta_1$ on A_0 and $p_a = 3\delta_1/4$ elsewhere. Let T be the set of coordinates equal to 1. For each coset $C = \gamma + A_0$, put $Y_C = |T \cap C| - \mathbb{E}|T \cap C|$ and $D = \sqrt{2\delta_1 n}$. Then

$$\mu = \mathbb{E}|T| = \frac{3o_\phi + 1}{4o_\phi} \delta_1 n, \qquad \mathbb{E}|T \cap C| = \lambda_\gamma \mu,$$

$$\mathbb{E} \sum_C Y_C^2 = \sum_a p_a (1 - p_a) \leq D^2/2.$$

Condition on $t \in T$. The other indicators retain their product law with parameters in $[3\delta_1/4, \delta_1]$. For large x, $(M_0 + 1)\ell = O_\varepsilon(x^{\varepsilon/8} \log x) \leq x^{\varepsilon/4} \leq N_{\mathrm{orb}}$, so a run contains $M_0 + 1$ distinct full tiles. A tile away from t is marked with probability at least

$$q_* = ((1 - \delta_1)^u)^{3R_\varepsilon + 1} \left(\lfloor u/2 \rfloor \frac{3\delta_1}{4} (1 - \delta_1)^{u-1} \right)^{3R_\varepsilon}.$$

At least M_0 tiles in a run have disjoint coordinates away from t, so the run fails with probability at most $(1 - q_*)^{M_0}$. A short-window failure needs at least K_ε further selected points and a long-window failure needs at least $N_1 - 1$. The union bound over windows and runs gives conditional orbit failure probability at most

$$N_{\mathrm{orb}} \left[\binom{L_2}{K_\varepsilon} \delta_1^{K_\varepsilon} + (1 - q_*)^{M_0} + \sum_{j=N_1-1}^{G'} \binom{G'}{j} \delta_1^j (1 - \delta_1)^{G'-j} \right], \tag{A.21}$$

where the sum is the probability of at least $N_1 - 1$ successes in G' independent trials with success probability δ_1. The two window bounds use at most L_2 or G' distinct coordinates, each with conditional selection probability at most δ_1 away from t.

Write $\delta = \delta_1 \leq 1/100$. Since $u = \lceil 1/\delta \rceil$,

$$(1 - \delta)^u \geq \exp\left(-\frac{1 + \delta}{1 - \delta} \right) > \frac{3}{10}, \qquad \delta \lfloor u/2 \rfloor \geq \frac{1 - \delta}{2}.$$

An occupied-cell factor is at least $(3/4)(99/200)(3/10) > 1/10$. Thus $q_* \geq 10^{-(6R_\varepsilon + 1)}$ and $(1 - q_*)^{M_0} \leq x^{-(H_\varepsilon + 3)}$. For $a = N_1 - 1$, a union bound over a-element subsets gives the binomial tail bound $\binom{G'}{a} \delta_1^a \leq$

$(eG'\delta_1/a)^a$. The definition of N_1 gives $x^{-(H_\varepsilon+3)}$ as an upper bound for this term as well. Multiplying either bound by $N_{\mathrm{orb}} \le n \le 2x^{H_\varepsilon}$ gives $2x^{-3}$.

Put $C = \max(C_8, 2)$. Since every prime at most $Y^{\varepsilon/5}$ satisfies $(-d/p)_K = 1$, we have $p > Y^{\varepsilon/5} \ge x^{\varepsilon/(5C)}$ for inert-class primes. For $x \ge 16$, (8.1) gives $\log_p((kx^{26})^4) \le 620C/\varepsilon$, so $L_2 \le 2(620C/\varepsilon + 1)$. Since $\delta_1^{K_\varepsilon} \le x^{-(H_\varepsilon+2)}$, the first term of (A.21), including N_{orb}, is $O_{\varepsilon,C_8}(x^{-2})$. The conditional probability is thus at most $\beta = C_b x^{-2}$ for suitable C_b.

Let $B(T)$ be the left side of (8.18) with $TT = T$, and put $m_B = \beta\delta_1 n \sum_{p\in\mathcal{P}_s} \log p/p$. The conditional bound gives $\mathbb{E}B(T) \le m_B$. If $m_B > 0$, then $\mathbb{E}(\sum_C Y_C^2/D^2 + B(T)/(4m_B)) \le 3/4$. Some T has all $|Y_C| \le D$ and $B(T) \le 4m_B$. If $m_B = 0$, then $B(T) = 0$ almost surely and the second-moment bound gives the same choice. For this choice,

$$\big||T| - \mu\big| \le o_\phi D, \qquad ||T \cap C| - \lambda_\gamma |T|| \le (1 + \lambda_\gamma o_\phi)D.$$

By (8.1), $\delta_1 n \ge x^{2+63\varepsilon/8}$. Since o_ϕ is bounded, these inequalities give the size bounds and (8.17) for $TT = T$ at large x. The inequality $B(T) \le 4m_B$ gives (8.18) for this top set.

Lean: `UpperTheory.BernoulliSumsFrobeniusOrbits.not_good_le` (`Development.AutomorphicArithmetic.ThickFrames.NotGoodLe`, lines 84–138)

Lean: `UpperTheory.BernoulliSumsFrobeniusOrbits.select` (`Development.AutomorphicArithmetic.ThickFrames.select`, lines 70–174)

□

Proof of 8.16

Proof. By 7.24, the pair has the required prime ranges and $-d \equiv 1$ modulo q and every odd prime $p \le Y^{\varepsilon/5}$. The congruence $d \equiv 7 \pmod{16}$ gives Kronecker value 1 at 2. Put $z = Y^{\varepsilon/5}$. By (8.1), $n \le 2x^{9/4}$ and $\log Q_S \le 2C_0 + \frac{9}{4}\log x$. Using $\log x \le C_0 \log Y$, for large x we have $\log Y \ge 2$ and $z \ge 820C_0 \log Y$. Since $C_0 \ge J \ge 1$ and $\log Y \ge 2$ give $2C_0 \le C_0 \log Y$ and $(\log Y)^2 - 1 \ge 3$, it follows that $\log Q_S \le 2C_0 + \frac{9}{4}C_0 \log Y \le 4C_0 \log Y \le z((\log Y)^2 - 1)$. Add $\log q \le z$ and $\log d \le z(\log Y)^2$ for the pair-row bound.

For the pointwise bounds use the twists of the trivial character in 7.24 and $x^{1/2+\varepsilon/2} \le x^{1/2+\varepsilon}$. For the six quadratic twists use the base characters $1, \chi_8$ with conductors $f = 1, 8$. The product of the primitive factors at the coprime conductors is real and has conductor $fq^i d^j \le 8X_1(Y)X_2(Y)^j$. On units modulo $8qd$, quadratic reciprocity identifies $(-q/\cdot)$ and $(-d/\cdot)$ with the two sign characters. By 7.14, every primitive representative in either family agrees with the corresponding primitive twist. In Definition 7.24 at base modulus M, the first selection clause supplies the pointwise bounds at modulus $2Mqd$ and the second clause (for real primitive base characters of conductor dividing $8M$, here $f \in \{1, 8\}$) supplies the boxes, since integers coprime to $8Mqd$ coincide with integers coprime to $2Mqd$. Restrict the box to height $8X_1(Y)X_2(Y)^j$. The host datum of Definition 34.46 includes $q, d > 4tr + 1$. □

Proof of 8.17

Proof. Take the thickness constant C_{T_2} from Theorem 8.14, the exponent $C_1 \ge 1$ from Theorem 8.12 (using Theorems 8.7 and 8.11 for its progression-prime and short-interval norm-prime inputs), $c_6 = 1$ from Theorem 8.13, and $C_b > 0$ from Theorem 8.15. Set $C_8 = \max(C_0, C_1)$, and take x_0 to be the maximum of the scale thresholds of Theorems 8.12–8.15, Lemma 8.16, the threshold for $4\log x \le x^\varepsilon$, and the threshold for the prime-size lower bounds below.

Put $N = Q_S qd$ and $z = Y^{\varepsilon/5}$. By (8.1) and 31.1, we have $r \le x^5$ and $t \le x^{400}$ for large x. The lower bound $\log q > z/2$ from Definition 7.24, together with $z \ge x^{\varepsilon/(5C_0)}$ and $q \mid d + 1$, yields $d \ge 100$ and $q, d > 4tr+1$ for large x. To bound $\log N$ before choosing the final level primes, form an auxiliary choice record over H_0: take two primes greater than Q_S and two larger primes 1 mod 15 as the level set, set $\mathfrak{c}_K = \mathcal{O}_K$, $TT = T_2 = \varnothing$, $L_w = 2$, and $C_{T_2} = 0$. Admissibility of R for H_0 at splitting bound $B_0 = \lfloor z \rfloor$ yields an initial host D_0 extending H_0. Applying Lemma 8.16 to D_0 using the good-pair hypothesis for (q, d) on $[Y, x]$ gives $\log N \le 2z(\log Y)^2 \le 2x^{\varepsilon/5}(\log x)^2$. Since $d \equiv 7 \pmod{16}$, we have $d \equiv 3 \pmod 4$. Theorem 8.12 now supplies primes g_0, g_1, p, p', q_0 satisfying the size bounds, norm-prime conditions for (d, N), $\mathcal{B}_r$-disjointness, congruences $p \equiv p' \equiv 1 \pmod{15N}$ and $q_0 \equiv 1 \pmod N$, and upper bounds $p, p', q_0 \le N^{C_1} \le N^{C_8}$.

Set $L_w = 10\lceil 405 \log x\rceil + 2\lceil 30/\varepsilon\rceil + 22$, so $L_w \le (4084 + 60/\varepsilon)\log x$ for $\log x \ge 1$. Theorem 8.14 with $C_o = \max(C_0, 2)$ gives a thick set T_2 satisfying (8.15) and (8.16) at $L = L_w$. By (8.12) and $d \equiv 3 \pmod 4$, the norm primes satisfy (8.19) and $g_i \equiv 1 \pmod N$. All five primes g_0, g_1, p, p', q_0 are 1 mod N, hence coprime to N; they split completely in $\mathbf{Q}(\zeta_N)$, and therefore in E and K. Form the choice record with level set $\{g_0, g_1, p, p', q_0\}$, $\mathfrak{c}_K = \mathcal{O}_K$, $TT = \varnothing$, T_2, L_w, and C_{T_2}. Admissibility of R for H_0 at splitting bound $\lfloor z\rfloor \ge \lfloor x^{\varepsilon_z/5}\rfloor$ yields a host D extending H_0.

Apply Lemma 8.16 to D using the good-pair hypothesis (Definition 7.24) on $[Y, x]$. This supplies:

1. the size, splitting, and modulus conditions at Y;
2. the pointwise square-root bounds $|g_\chi(x)| \le x^{1/2+\varepsilon}$ and $|(L'/L)(1,\chi)| \le 4\log x$ (constant $C_5 = 4$) for every primitive representative χ at level $8qd$ of the three sign characters $(\cdot/q)$, $(\cdot/d)$, and $(\cdot/q)(\cdot/d)$ (obtained from (35) with the principal character $\xi = 1$ modulo M);
3. the reality, conductor, and zero-free box conditions in $\mathcal{R}(8X_1(Y)X_2(Y)^j)$ for the six quadratic twists (obtained from (36)).

Apply Theorem 8.15 to D to obtain a top set $TT \subset A$ satisfying the size bounds, the bad-orbit bound (8.18) with constant C_b, and the coset share bound (8.17) with $\rho = x^{-1+\varepsilon}$. Form H by replacing the top set of D with TT.

The divisibility $Q_S \mid \operatorname{lcm}(2^{e+2}, M)$ implies $Q_{\mathrm{odd}} \mid M$ and that every prime divisor of Q_S divides $2M$. By the first part of Theorem 8.13 applied to the good-pair boxes (36) of Definition 7.24, every primitive real host character χ with $q \mid N_\chi$ or $d \mid N_\chi$ satisfies $N_\chi \le H_{Q_{\mathrm{odd}}}(j)$ and $L(s,\chi) \ne 0$ on $\mathcal{R}(H_M(j))$, hence on $\mathcal{R}(H_{Q_{\mathrm{odd}}}(j))$ and in particular has no real zero greater than $399/400$. Since $Q_{\mathrm{odd}} \le M \le C_0$ and $\log N \le 2x^{\varepsilon/5}(\log x)^2$, the second part of Theorem 8.13 gives the thin-position bound (8.14) with $c_6 = 1$. The logarithmic-derivative bound of Lemma 8.16 for $\chi_d = (\cdot/d)$ gives $|\operatorname{Re}(L'/L)(1,\chi_d)| \le 4\log x \le x^\varepsilon$ for $x \ge x_0$. Since $\operatorname{lcm}(15, N) \mid 15N$, the congruences $p, p' \equiv 1 \pmod{\operatorname{lcm}(15,N)}$ hold as well. Thus H and q_0 satisfy all completed-host conditions with constants $C_8, 4, c_6, C_b$. □

Proof of 8.18

Proof. Choose ranks on ϕ by 38.3. Fix a choice record and B_0 as in admissibility. The field rows give the square root of $-q$ with its Galois action. The specified CM type consists of the embeddings taking the chosen square root of $-d$ to $i\sqrt{d}$. It has $2|A| > 0$ embeddings, so choose one. Since $K = \mathbf{Q}(\sqrt{-d})$ and an automorphism acts on $\sqrt{-d}$ by its d-sign, $\mathrm{Gal}(E/K)$ corresponds to the subgroup $\{((s_q, 1), z) : s_q \in \{\pm 1\},\ z \in A\}$ (identified with Φ under the coordinate map T of Lemma 5.2). For any arithmetic datum H_0 of Definition 5.1, Lemma 5.3 gives $I_p = \{0\}$ for $p \notin S \cup \{d\}$, $I_p \subseteq \Phi$ and $F_p \in \Phi$ for $p \in S$, and $I_d = \{0, \epsilon_d\}$. In particular, complex conjugation $c = ((1,1), 0)$ lies outside I_p for every rational prime p (since $c \notin \Phi$ and $c \ne \epsilon_d$), so E/E^+ is unramified at every finite place. Likewise $I_d \cap \Phi = \{0\}$ shows that E/K is unramified at d, while every rational prime $p \in S$ satisfies $I_p \subseteq \Phi$ and $F_p \in \Phi$ and therefore splits in K (as $d \equiv 7 \pmod{16}$ splits 2 and $(-d/p) = 1$ for odd $p \in S$). Choosing one prime of K from each conjugate pair above S (as in the proof of Theorem 8.3) yields the $\mathrm{Gal}(E/K)$-stable ideal $\mathfrak{b}$ with

$$\mathfrak{D}_{E/K} = \mathfrak{b}\,\overline{\mathfrak{b}}, \qquad \mathfrak{b} + \overline{\mathfrak{b}} = \mathcal{O}_E.$$

Adjoin these choices and set the splitting bound to B_0.

Lean: `HostParam.Seam.exists_host` (`Development.AutomorphicArithmetic.CoprimeRanks.SecondSetNoRelation`, lines 100–104)

□

Proof of 8.20

Proof. Fix $0 < \varepsilon < 1/52$. Take $c \ge 1$ and e_0 from Theorem 8.3. Apply Theorem 8.6 with the cohomological window hypothesis of Definition 26.3 and the unconditional good-pair existence theorem 7.25 to obtain $J \ge 1$ and the family of windows $[Z, Z^{3/2}]$. Choose $C_0 \ge \max(1, J, o_\psi, 8M)$. From the completion hypothesis (discharged by Theorem 8.17) at (ε, C_0) obtain C_{T_2}, C_8, c_6, C_b and threshold $x_{0,A}$. Take the rank-supply threshold $x_{0,B}$ and the threshold $x_{0,T}$ for $t \le x^{400}$ from 31.1. Increase $x_1 \ge \max(x_{0,A}, x_{0,B}, x_{0,T})$ so that $x > c2^{e_0}$ and $Y^{\varepsilon/5} \ge \max(5, 4M)$ whenever $x \ge x_1$ and $Y \ge x^{1/J}$.

Let $\mathcal{W}$ consist of the real x admitting Y, q, d, e as in (8.8) for which (8.9) holds for every field with the prescribed ψ-kernel at (e, q, d). By Theorem 8.6, $\mathcal{W}$ satisfies (8.20) with $\lambda = 3/2$. Fix $x \geq x_1$ in $\mathcal{W}$ and its datum (Y, q, d, e). Since (q, d) is a good pair for (M, ε) on $[Y, x]$ in the sense of Definition 7.24, it satisfies (8.3), $(M, qd) = 1$, and $(-d/p) = 1$ for every odd prime $p \leq Y^{\varepsilon/5}$. The lower bound $\log q > Y^{\varepsilon/5}/2 \geq 5/2$ gives $q \neq 3$, and $c2^e \geq x^{2+8\varepsilon} \geq x > c2^{e_0}$ gives $e \geq e_0$.

Apply Theorem 8.3 with $B_0 = \lfloor Y^{\varepsilon/5} \rfloor \geq 4M$ to obtain a frame ϕ with $n = c2^e$ and $o_\phi = o_\psi$, and an arithmetic datum H_0 on ϕ with primes q, d, satisfying the field conditions at B_0, (8.6), $M \mid Q_S$, (8.4), the twist representation of Lemma 8.2, and the retained-character and θ_q clauses of Definition 8.19. The rank-supply hypothesis gives ranks R on ϕ satisfying the rank conditions at $s = (x, \varepsilon, J)$ and admissible for H_0. Since $Q_S \leq 4Mn \leq C_0 n$ and $M \leq C_0$, the completion hypothesis at (ε, C_0), applied with this Y, produces a host H extending H_0 and a prime q_0 satisfying the completed-host conditions with constants $C_8, 4, c_6, C_b$ (including the pointwise bounds for $\chi_q, \chi_d, \chi_q\chi_d$ with $C_5 = 4$ from Lemma 8.16 and the real-character zero-free boxes and thin-position bound from Theorem 8.13) together with the top-set shares (8.17) at $\rho = x^{-1+\varepsilon}$.

From $Q_S \mid \operatorname{lcm}(2^{e+2}, M)$, every prime divisor of Q_S divides $2M$. Because $M \mid Q_S$, the twist representation of Lemma 8.2 holds on H, and (q, d) is a good pair for (M, ε) on $[Y, x]$, Theorem 8.4 gives $|g_\chi(x)| \leq x^{1/2+\varepsilon}$ and $|(L'/L)(1, \chi)| \leq 4 \log x$ for every $\theta \in \mathcal{B}_H(\mathcal{X})$ and every primitive representative χ of θ. Together with the frame index, retained-character, θ_q, and top-set share conditions, this establishes the arithmetic requirement of Definition 8.19 for $\mathcal{L}_\psi$ with $C = 4$.

Restriction from $\mathbf{Q}(\zeta_{Q_S qd})$ to E is the Artin map, so (8.4) shows that E has the prescribed ψ-kernel at (e, q, d). The window bound (8.9) therefore yields $V_E(\lfloor x^{400} \rfloor) \leq \sqrt{b}$. Since $t \leq x^{400}$, restricting the nonnegative sum in (8.7) gives $V_E(t) \leq \sqrt{b}$. Taking $C_5 = 4$ and $x_0 = x_1$ completes the proof of the first statement; applying Theorem 8.17 discharges the completion hypothesis. □

Proof of 8.21

Proof. Apply Theorem 8.20 with the rank supply of Theorem 8.18. In Theorem 8.20, the good pairs (q, d) on $[Y, x]$ are supplied unconditionally by Theorem 7.25 via Theorem 8.6, and the completion hypothesis is discharged by Theorem 8.17. The resulting completed hosts at $C_5 = 4$ inherit the pointwise square-root bounds for χ_q, χ_d, and $\chi_q\chi_d$ from Lemma 8.16, the real-character zero-free boxes and thin-position bound from Theorem 8.13, and the square-root bounds for every character in $\mathcal{B}_H(\mathcal{X})$ from Theorem 8.4. □

Proofs for Chapter 9: Background: the Weil representation and theta lifts

Proof of 9.1

Proof. Put $V = L \otimes \mathbf{R}$, $Q(x) = (x, x)/2$, and $k = d/2 + j$. The rank-zero case is immediate, so assume $d > 0$.

Claim 1. The series has the asserted transformation law on $\Gamma_0(N_L)$.

Let E be the Euler differential operator and Δ_Q the Laplacian in orthonormal coordinates for $(\ ,\ )$. The published theta transformation criterion applies to a function p satisfying

$$(E - \lambda)p = \frac{\Delta_Q p}{4\pi},$$

provided $p \exp(-2\pi Q)$, its derivatives of order at most two, and its products with polynomials of degree at most two belong to $L^1(V) \cap L^2(V)$. Its conclusion gives weight $d/2 + \lambda$, level equal to the level of Q, and the associated lattice character and half-integral weight multiplier for

$$\sum_{x \in L} y^{-\lambda/2} p(\sqrt{y}\, x) \exp(2\pi i z Q(x)), \qquad y = \operatorname{Im} z.$$

This is the unnumbered theorem of [44, pp. 20-01–20-02]. Take $p = P$ and $\lambda = j$. Homogeneity gives $EP = jP$, harmonicity gives $\Delta_Q P = 0$, and the required functions are Schwartz functions because Q is positive definite. Moreover $y^{-j/2} P(\sqrt{y}\, x) = P(x)$. The resulting series is exactly $\theta_{L,P}$.

Claim 2. In even rank the lattice character is the one displayed in the statement.

Choose a $\mathbf{Z}$-basis of L with Gram matrix A. The definition of N_L is equivalent to requiring that $N_L A^{-1}$ have integral entries and even diagonal. Thus it agrees with the level used in [45, p. 514]. With rank $d = 2r$, that paper's equation (16), at the zero residue class, gives weight $r + j$ and character

$$\chi(a) = \left(\frac{(-1)^r \det A}{a}\right).$$

Its series at that residue class is $N_L^j \theta_{L,P}$, by equation (10), so this scalar does not change the transformation law. This proves the asserted character formula.

Claim 3. The series is holomorphic at every cusp and vanishes there when $j > 0$.

Write $D = L^\vee/L$, and for $\mu \in D$ set

$$\Theta_\mu(z) = \sum_{x \in L+\mu} P(x) \exp(2\pi i z Q(x)).$$

All these series converge locally uniformly on the upper half-plane, together with their complex derivatives. Gaussian Fourier transformation, differentiated according to P, and Poisson summation give

$$\Theta_\mu(z+1) = \exp(2\pi i Q(\mu))\Theta_\mu(z),$$

$$\Theta_\mu(-1/z) = \frac{(-iz)^{d/2} z^j}{\sqrt{|D|}} \sum_{\nu \in D} \exp(-2\pi i(\mu,\nu))\Theta_\nu(z).$$

Here $(-iz)^{d/2}$ is the holomorphic branch positive when z is positive imaginary. The harmonicity of P removes the lower-degree terms in the differentiated Gaussian formula. Since the two displayed substitutions generate the modular group, the weight-k transform of $\Theta_0 = \theta_{L,P}$ at any cusp is a finite linear combination of the Θ_μ, with the corresponding square-root factor in odd rank. Positive definiteness makes every Fourier exponent nonnegative. The only possible zero exponent comes from $x = 0$, whose coefficient is $P(0)$, and this is zero for $j > 0$. Thus each transformed series is holomorphic at infinity, and its constant term vanishes when $j > 0$, as required. □

Proofs for Chapter 10: The local kernel and the differential operator

Proof of 10.7

Proof. Leibniz's rule and the chain rule preserve integrality of derivatives at zero under products and zero-constant substitutions. The derivatives of $\log(1+T)$ are $(-1)^{n-1}(n-1)!$ at order $n \geq 1$. Those of ℓ are integral since $(p^{2h})!$ is divisible by p^h. The same holds for ℓ^{-1} by differentiation of $\ell \circ \ell^{-1} = T$ and induction, since $\ell'(0) = 1$. Thus for $A \simeq 0$, the identity $A = \log(\exp(A^\ell)) \circ \ell^{-1}$ gives integral derivatives.

Differentiating the first additivity defect in X, X' and setting $X' = 0$ gives $\partial_X^2 b$. The other defect gives $\partial_Y^2 b$. For the remaining mixed coefficient, use the conjugation defect with η the Teichmüller lift of an element of $\mathbf{F}_{p^2} \setminus \mathbf{F}_p$. Then $(\eta - \sigma\eta)[XY]b \in W$ and $\eta - \sigma\eta$ is a unit. □

Proof of 10.11

Proof. Put $F = \exp \Psi$ and $\varphi Q = (\sigma Q)(X^p)$. By Legendre's formula [59, (1.7)], exponential and logarithm identify $p(X)W[[X]]$ with $1 + p(X)W[[X]]$, where (X) is the zero-constant ideal. The right side is therefore equivalent to $\varphi F/F^p \in 1 + p(X)W[[X]]$. Integral F gives this by reduction modulo p. Conversely, suppose it holds and the terms of F below degree $h \geq 1$ are integral. Divide F by their sum T to obtain $H = F/T$. Since $\varphi T/T^p$ lies in the same group, so does $\varphi H/H^p$. Its degree-h part is $-pH_h$, since $\varphi(H-1)$ starts in degree ph. Thus $F_h = H_h$ is integral. □

Proof of 10.13

Proof. Write $P \equiv Q$ if $P - Q$ has zero constant term and $\exp(P - Q)$ is integral over W. Set $\beta = b^{\ell}$ and $F(X,T) = \ell^{-1}(\ell(X) + \ell(T))$. The relation $\ell(T) - \ell(T^{p^2})/p = T$ gives $F \in \mathbf{Z}_p[[X,T]]$ by Hazewinkel's functional equation lemma with $q = p^2$ and $s_1 = p^{-1}$ [58, Lemma 2.3, (2.5)]. Let $\delta f = f(F(X,T)) - f(X) - f(T)$, with d_1, d_2 the corresponding defects in each variable. The pairing laws give $d_i\beta \equiv 0$ for $i = 1, 2$.

For a primitive (p^2-1)st root $\zeta \in W$, average $\beta(\zeta^k X, \zeta^{-pk} Y)$ over $0 \le k < p^2 - 1$ to obtain B. Since $\ell(\zeta T) = \zeta \ell(T)$, the conjugation law makes each summand equivalent to β. The criterion 10.11 allows division by $p^2 - 1$ in this average and gives $B \equiv \beta$. The equivariance $F(\zeta X, \zeta T) = \zeta F(X,T)$ gives $d_i B \equiv 0$. The average has support

$$[X^i Y^l]B \neq 0 \implies i, l \ge 1, \qquad i \equiv pl \pmod{p^2 - 1}. \tag{A.22}$$

For integral zero-constant P', D and a series C with integral derivative,

$$\big(C(P' + pD) - C(P')\big)/p \quad \text{is integral.} \tag{A.23}$$

Indeed, the D^k term from $[T^n]C$ has scalar $n[T^n]C\binom{n-1}{k-1}p^{k-1}/k$ for $k \ge 1$, and $v_p(k) \le k - 1$.

Put $h(T) = \ell(T^p)/p$. Set $P = F(X,T)^p$ and $P' = F(X^p, T^p)$. Then $P - P' \in p\mathbf{Z}_p[[X,T]]$. Since ℓ' is integral, (A.23) gives $\delta h = (\ell(P) - \ell(P'))/p$ integral. Also $\delta\ell = 0$. If $f(0) = 0$ and δf is integral, then

$$f = ah + b_0\ell + R, \qquad a \in W, \quad b_0 \in K_0, \quad R \in W[[T]]. \tag{A.24}$$

Subtract $[T]f$ times ℓ. If the remainder is integral below degree $n \ge 2$ and has coefficient u there, its mixed defect terms give $\binom{n}{i}u \in W$ for $0 < i < n$. The base-p digit expansion of $(1+X)^n$ modulo p gives a unit intermediate coefficient when n is different from every power of p. For $n = p^{k+1}$, the identity $\binom{p^{k+1}}{p^k} = p\binom{p^{k+1}-1}{p^k-1}$ gives $pu \in W$ by the same expansion. At degree p^{2j+1} subtract $p^{j+1}uh$, and at degree p^{2j+2} subtract $p^{j+1}u\ell$. The earlier coefficients change by integral amounts. The increments belong to $p^j W$, so the multipliers converge and the limiting remainder is integral.

Differentiate $\exp(d_1 B)$ in T at zero, where its value is one. Using $\partial_2 F(X,0) = 1/\ell'(X)$ gives $\partial_X B - \ell'(X)[X]B$ integral. Also $\delta([X]B)$ is integral by the X-linear term of $\exp(d_2 B)$. By (A.22), $[Y^{p^{2k}}][X]B = 0$ for all k. Thus in (A.24) for $[X]B$ one has $b_0/p^k \in W$ for all k, so $b_0 = 0$. Exchanging the variables gives $c, c' \in W$ with

$$\partial_X B - c\ell'(X)h(Y), \qquad \partial_Y B - c'\ell'(Y)h(X) \quad \text{integral.} \tag{A.25}$$

For any variable tuple $\mathbf{X}$, put $D_\sigma Q = ((\sigma Q)(\mathbf{X}^p) - pQ)/p$ and set

$$J = D_\sigma B - \sigma(c)h(X)\ell(Y) - \sigma(c')\ell(X)h(Y).$$

By 10.11, $D_\sigma(d_1 B)$ is integral. The commutator $d_1(D_\sigma B) - D_\sigma(d_1 B)$ equals $((\sigma B)(P, Y^p) - (\sigma B)(P', Y^p))/p$. Apply (A.23) to $(\sigma B)(U, Y) - \sigma(c)\ell(U)h(Y)$ in U. Modulo integral series the commutator is $\sigma(c)\delta h\, h(Y^p) = \sigma(c)\delta h\,(\ell(Y) - Y)$. Subtracting the first defects of the two products in J leaves an integral series since δh is integral and $\delta\ell = 0$. The second defect is integral by the exchanged calculation.

For $Q \in XYK_0[[X,Y]]$ with integral defects,

$$Q \in \gamma\ell(X)\ell(Y) + W[[X,Y]] \qquad (\gamma \in K_0). \tag{A.26}$$

Apply (A.24) to $[Y^m]Q$ and $[X^i]Q$. With h_i, ℓ_i the coefficients of h, ℓ, we obtain

$$A_m h_i + B_m \ell_i - A_i' h_m - B_i' \ell_m \in W, \qquad A_m, A_i' \in W.$$

Fix $i = p^{2k+1}$ and take $m = p^{2l}$. The first term has valuation at least $-k - 1$, so B_i'/p^l has the same bound for all l. Hence $B_i' = 0$. Fix m and vary k to get $A_m = 0$. Taking $i = p^{2k}$ gives $B_m = 0$ when $\ell_m = 0$. For $m = p^{2l}$, subtract p^{-l} times the comparison at $m = 1$. The valuation of $(B_m - p^{-l}B_1)/p^k$ is bounded below uniformly in k, so $B_m = p^{-l}B_1$. This proves (A.26).

The congruence in (A.22) is preserved by D_σ and holds for the two products in J. Thus $[X^{p^{2k}}Y^{p^{2l}}]J = 0$, and (A.26) gives $\gamma/p^{k+l} \in W$ for all k, l. Hence J is integral. For $m = p^{2l+1}$, the coefficient of XY^m in B is congruent to c/p^{l+1} by (A.25) and to $-\sigma(c')/p^{l+1}$ by integrality of J. Varying l gives $c = -\sigma(c')$.

Exchanging variables gives $c' = -\sigma(c)$ and $\sigma^2 c = c$. Reindexing gives $A(X^p, Y^p) = p(h(X)\ell(Y) - A(Y, X))$. Consequently $D_\sigma(B - G_c) = J$ and $\beta \equiv G_c$ by 10.11.

The coefficient of $T^{p^{2b}}$ in $\pi^{-1}\ell(\pi T)$ is $p^{\sum_{a<b} p^{2a}-b}$. The series lies in $T\mathbf{Z}[[T]]$ and reduces to $T + T^{p^2}$ modulo p. Its inverse μ therefore has integral coefficients and reduces to $\sum_{l\geq 0}(-1)^l T^{p^{2l}}$. The identity $\ell(\pi\mu(T)) = \pi T$ transports phase equivalences by substitution of $\pi\mu(L)$ in the logarithmic parameters. Also

$$G_c(\pi X, \pi Y) = \frac{\pi^{p+1}}{p}(cXY^p - \sigma(c)X^pY) + \pi^{p+1}R(X, Y), \qquad R \in (X, Y)R_\pi[[X, Y]]. \tag{A.27}$$

For an A term with $l \geq 1$, the exponent of p after factoring π^{p+1} is $\sum_{a<i}(p^{2a}-1)+\sum_{a<l}(p^{2a+1}-1)-1 \geq p-2$.

Reindexing the reduction of $\mu(L)$ gives

$$\mu(L^X_{r,a,b}) \equiv \sum_{s \text{ even}} \sigma^s(\omega_r^*)S_{s,a,b}, \qquad \mu(L^Y_{r,a,b}) \equiv \sum_{s \text{ odd}} \sigma^s(\omega_r)S_{s,a,b} \pmod p.$$

Trace duality makes the even conjugate matrices inverse transposes. Hence for s even and t odd, $\sum_r \sigma^s(\omega_r^*)\sigma^{t+1}(\omega_r) = \mathbf{1}_{s=t+1}$, with cyclic indices. Applying σ gives the other parity. Frobenius on the pth powers in (A.27) therefore gives H_u modulo p in the term with prefactor π^{p+1}/p. The resulting phase error belongs to $\pi^{p+1}(Z)R_\pi[[Z]]$. If $i! = p^a m_i$ with $p \nmid m_i$, Legendre's formula gives $(p-1)a \leq i$, so $\pi^{(p+1)i}/i! = m_i^{-1}\pi^{(p+1)i-(p^2-1)a} \in R_\pi$. Exponentiating the error gives an integral unit, whose product with the units from phase equivalence gives V. □

Proof of 10.14

Proof. If $V = \varnothing$, the generator phase in Definition 10.6 is a constant equivalent to the empty sum. By Definition 10.5 that constant is zero. The generator therefore equals its integral cofactor, so take $U_\varnothing = \Theta_A \in O_c$. Otherwise $p \geq 5$. In the coordinate tuple $\mathcal{L} = (L^X, L^Y)$, the integral cofactor becomes $U(\pi\mathcal{L}) = U^\ell(\pi\mu(\mathcal{L}))$. Put $q = \pi^{p+1}$. At a single place 10.13 gives the series as $V_0 \exp(-qH_u/p)$ with V_0 integral. Take $U_i = V_0(-q)^i \exp(qG_v)/i!$.

For the generator apply 10.13 to the pairing phase at each place. Put $\Delta = \sum_{v,e} b(X_{v,e}, Y_{v,e}) - \Phi_A$. The unit $E = \exp(\Delta^\ell)$ has coefficients in W. Since $\ell(\pi\mu(T)) = \pi T$, the factor $\exp(\Delta(\pi\mathcal{L})) = E(\pi\mu(\mathcal{L}))$ belongs to $1 + (Z)R'[[Z]]$. Absorb it into the integral cofactor V_A. The series is $V_A \prod_v \exp(-qH_{u_v}/p)$, so take $U_{\mathfrak{i}} = V_A \prod_v((-q)^{i_v}\exp(qG_v)/i_v!)$. The factorial bound in 10.13 gives integrality, and the zero constant terms of H'_v give the displayed degree bound. □

Proof of 10.16

Proof. For every product Q of flag minors of the same heights as H, $[H, Q] = [H, H]Q(1)$. Indeed a nonzero pairing requires equality of the degrees in each row. Row one must then occur in every nonempty minor. After deleting it, row two must occur in every remaining nonempty minor. Continuing gives the initial row sets in every column. In that case the ordering signs give both $Q(1)$ and Q/H. The result extends by linearity to the span of these products. Cauchy–Binet shows that this span is stable under row substitutions.

First take z to be an $s \times s$ matrix of independent variables. Reverse its columns to obtain R and work in the fraction field. For each h, factor R out of $C_h(Y, z)$. The last h columns left inside the determinant are the last h unit columns in reverse order. Thus, with $\epsilon = \prod_k (-1)^{h_k(h_k-1)/2}$,

$$\prod_k \det C_{h_k}(Y, z) = \epsilon(\det R)^c H(R^{-1}Y).$$

Substitution by R^{-1} in the first argument of the factorial pairing is adjoint to substitution by $R^{-\mathsf{t}}$ in the second. This follows by comparing coefficients in $\exp(\sum_{i,j} Y_{ij}Z_{ij})$. The normalized pairing is consequently $\epsilon(\det R)^c Q_I(R^{-\mathsf{t}})$. Multiplying $[E_{I_k} \mid z_{:,1:h_k}]$ by R^{-1} and expanding along the last h_k columns gives precisely its k-th factor, with sign $(-1)^{h_k(h_k-1)/2}$. Both sides are polynomials in the original entries. The identity therefore survives every specialization. Unused columns of z may be filled arbitrarily. □

Proof of 10.17

Proof. Choose basis matrices A_v of L_v. Lift to L'_v a basis of a complement to L''_v/pL'_v, then a basis of that subspace with lifts in L''_v. This gives a basis matrix C_v of L'_v. Multiplying its first $i = i_v$ columns by p gives a basis matrix B_v of L''_v. Thus $C_v = B_v D_i$ with $D_i = \mathrm{diag}(p^{-1}I_i, I_{N_{j+1}-i})$. Take $\phi = (A_v, B_v)_v$.

In one block (v, s) let A, B, C denote the embedded matrices, and recall from Definition 10.1 the shift $d_l = d_{l,v,s}$, partition $\kappa = \kappa_{l,v,s}$, highest-weight minor product H_κ, unframed Weyl lattice $\Delta_O(\kappa)$, and tuple Weyl weight lattice $H_l(L)$ transported by $Y \mapsto E^{-1}Y$ with scalar $(\det E)^{-d_l}$.

Lean: `Formalism.L2.PairingIndexBounds.Ent.delta_zero_eq_twisted_delta` (`Formalism.LocalToGlobal.PairingIndexBounds.RowActionCoefficients.AdmissibleInterpolation.Bridge`, lines 45–62)
Lean: `Formalism.L2.PairingIndexBounds.Ent.branching_source_identity` (`Formalism.LocalToGlobal.PairingIndexBounds.RowActionCoefficients.AdmissibleInterpolation.BranchingSourceIdentity`, lines 48–58)
Lean: `Formalism.L2.PairingIndexBounds.Ent.branching_target_identity` (`Formalism.LocalToGlobal.PairingIndexBounds.RowActionCoefficients.AdmissibleInterpolation.BranchingSourceIdentity`, lines 60–70)

Put $n = N_j$, $n' = N_{j+1}$, $w = w_{v,j+1}(s)$, and $c = \nu_{j+1,s,1}$. Write h_k for the c target column heights; they satisfy $h_k \le n$ since the final $t+1$ rows at level $j+1$ have length zero, and $\sum_{k=0}^{c-1} h_k = \sum_{a=1}^{n'} \nu_{j+1,s,a} = b$. For a square matrix M let $C_h(M, x)$ take its first $\dim M - h$ columns and then the first h columns of x. Write $x_{a,h} = \tau_{v,s,h,a}$ with matrix indices shifted to start at one, and let x_0 consist of the first n columns. Put $\mu = (\nu_{j+1,s,a})_{a=1}^{n}$ and $e = \mu_n$. Use the source polynomials

$$S^+(x, Y) = (\det x_0)^e H_{\mu - e\mathbf{1}}(x_0^{\mathsf{t}} Y), \qquad S^-(x, Y) = \prod_k \det C_{h_k}(Y, x).$$

The even source shape is $\mu - e\mathbf{1}$. The odd shape has heights $n - h_k$ and rows $c - \mu_{n+1-a}$. For off-diagonal indices $a \ne b$, an elementary row substitution $Y \mapsto EY$ with $E = I + TE_{a,b}$ has $\det E = 1$ and therefore transfers to $x \mapsto E^{\mathsf{t}}x$ in S^+ and $x \mapsto E^{-1}x$ in S^-. Put $d_a(\alpha) = \sum_h \alpha_{a,h}$. For diagonal indices $a = b$, the substitution $Y \mapsto (I + TE_{a,a})Y$ scales row a of Y by $1 + T$. The factor $(\det x_0)^e$ has degree e in each row of x and degree zero in Y. Thus every nonzero coefficient $[x^\alpha]S^+$ has degree $d_a(\alpha) - e$ in row a of Y, whereas $[x^\alpha]S^-$ has degree $c - d_a(\alpha)$. Extracting $[T^r]$ therefore multiplies these coefficients by $\binom{d_a(\alpha)-e}{r}$ and $\binom{c-d_a(\alpha)}{r}$, respectively. These are integer binomial coefficients, since the displayed degrees of nonzero coefficients are nonnegative. Thus the auxiliary coefficient spans are stable under divided powers for all a, b. Evaluation at the identity columns for S^+ and the reversed identity columns for S^- gives the corresponding highest-weight polynomial up to sign. Hence these coefficients span a module containing the Weyl generators. Under transport $Y \mapsto A^{-1}Y$, the free columns become $A^{-\mathsf{t}}x$ with scalar $(\det A)^e$ in the even case, and Ax with scalar $(\det A)^{-c}$ in the odd case.

Compute the symbol $R_G(d)$ of Definition 10.9 on S^+ and S^-, writing Y now for the $n' \times n'$ target variable matrix in the block. At an even position, substituting $D = (dY)_{:,1:n}$ and multiplying by $(\det D)^e$ combines with $(\det x_0)^e$ to supply the e height-n minors, so $N_- = 1$ in this block and $R_{S^+}(d) = H_{\nu_{j+1,s}}(z_0^{\mathsf{t}} Y)$ with $z = d^{\mathsf{t}}x$ and z_0 its first n' columns.

At an odd position, with $T = (\mathrm{adj}\, Y)_{\text{last } n \text{ rows}} d^{\mathsf{t}}$, the left inverse Π_j in Definition 10.9 evaluates the symbol of S^- via the column identity

$$\det T\, (\det Y)^{n'-h} \det C_h(\mathrm{adj}\, T, x) = (\det T)^{n-h} (\det Y)^{n'-1} \det C_h(Y, d^{\mathsf{t}}x).$$

Multiplying the first column matrix by T and the second by $\mathrm{adj}\, Y$ gives initial columns of scalar identity matrices, and expansion along those columns leaves the same minor in the last h rows and first h columns of $(\mathrm{adj}\, Y)d^{\mathsf{t}}x$. Put $u = \det Y$, $v = \det T$, $P = \prod_k \det C_{h_k}(\mathrm{adj}\, T, x)$ and $Q = \prod_k \det C_{h_k}(Y, d^{\mathsf{t}}x)$. Multiplying the column identities over k gives $v^c u^{cn'-b} P = v^{cn-b} u^{c(n'-1)} Q$. Since u and v are nonzero polynomials, in their localization this reads $Q = u^{c-b} v^{c-cn+b} P = (N_+/N_-)P$ in the block, so $N_+ P = N_- Q$ in the polynomial ring and every linear left inverse Π_j to multiplication by N_- gives Q on $N_+ P$, also coefficientwise in the source parameters.

Lean: `Formalism.L2.G.Pend.polDel` (`Formalism.LocalToGlobal.GroupLattices.PendingLemmas.DetYtMatNeZero`, lines 100–101)
Lean: `Formalism.L2.G.Pend.col_id` (`Formalism.LocalToGlobal.GroupLattices.PendingLemmas.CardFilterValLt`, lines 73–100)
Lean: `Formalism.L2.G.Pend.cols_id` (`Formalism.LocalToGlobal.GroupLattices.PendingLemmas.CardFilterValLt`, lines 102–112)

In Definition 10.9(3), the displayed symbols $H_{\nu_{j+1,s}}(z_0^{\mathsf{t}} Y)$ and $Q = \prod_k \det C_{h_k}(Y, d^{\mathsf{t}}x)$ already have homogeneous d-degree $\sum_k h_{v,s,k} = \sum_{a=1}^{n'} \nu_{j+1,s,a}$ in each block (v, s), so on the source generators and their

linear span the degree filter in $\mathcal{D}_j(G,f)$ is automatic. The arithmetic substitution B_ϕ on Z is $Z \mapsto A^{\mathsf{t}} Z B^{-\mathsf{t}}$ in an even block and $Z \mapsto A^{-1} Z B$ in an odd block; comparing coefficients in $\sum_{a,a'} d_{a,a'} Z_{a,a'}$ gives the transposed action $d \mapsto A d B^{-1}$ and $d \mapsto A^{-\mathsf{t}} d B^{\mathsf{t}}$, respectively, preserving each block degree.

For a target partition κ, use the pairing $[P,Q] = \sum_\alpha \alpha! [Y^\alpha] P [Y^\alpha] Q$ and put

$$C_\kappa = \sum_\alpha \alpha! \big([Y^\alpha] H_\kappa\big)^2 \in \mathbf{Z}_{>0}.$$

Use the image of this integer in the coefficient field. Row substitution by E is adjoint to substitution by E^{t}. For a product Q of initial-column minors with selected rows and shape κ, $[H_\kappa, Q]/C_\kappa = Q(I)$. Indeed, equality of row weights forces row 1 to appear in every column, row 2 in every column of height at least two, and so on. These tests equal H_κ with the row-order sign. Multilinearity gives the same identity for row substitutions in Q.

At an even block the normalized pairing of $H_\kappa(z_0^{\mathsf{t}} Y)$ with Q is therefore $Q(z_0)$. Apply Lemma 10.16 with an independent matrix of free columns before specializing it to $d^{\mathsf{t}} x$. At an odd block κ has rows $c - \nu_{j+1,s,n'+1-a}$. Reverse the first n' columns of z to form R. Factoring R from each $C_{h_k}(Y,z)$ and applying adjointness gives $(\det R)^c Q(R^{-\mathsf{t}})$ with the column reversal signs. If I_k lists the rows of the kth test minor, write E_{I_k} for its unit columns. The minor of $R^{-\mathsf{t}}$ on rows I_k and the first $n' - h_k$ columns, multiplied by $\det R$, is the determinant with first columns E_{I_k} and last h_k columns of R. This follows by multiplication by R^{-1} and expansion along the last columns. Undoing the reversal gives the determinant with columns E_{I_k} followed by the first h_k columns of z. Polynomiality extends the formula to singular R.

For the framed test $Q(C^{\mathsf{t}} Y)$, move C to the output by adjointness. After the source transport and the transposed frame action, the free columns become

$$C^{\mathsf{t}} (A d B^{-1})^{\mathsf{t}} A^{-\mathsf{t}} x = D_i d^{\mathsf{t}} x, \qquad C^{-1} (A^{-\mathsf{t}} d B^{\mathsf{t}})^{\mathsf{t}} A x = D_i^{-1} d^{\mathsf{t}} x.$$

At an odd block the substitution $Y \mapsto CY$ also contributes $(\det C)^c$. Repeated rows give zero tests. For the other tests let J_k be the selected rows in an even block and their complementary rows in an odd block. At an even position retain the order of the rows in the test minor. At an odd position take the complementary rows in increasing order and include the sign from expansion along the unit columns. Write q_k for the number of rows in J_k among the first i and $M = \sum_k \min(i, h_k)$. The minor D_k of $d^{\mathsf{t}} x$ on rows J_k and its first h_k columns is the transpose of the corresponding column determinant in 10.15. Since $q_k \le \min(i, h_k)$, the normalized pairings before weight twists are

$$(\det A)^e p^{-M} \prod_k p^{\min(i,h_k) - q_k} D_k \quad \text{and} \quad \pm (\det A)^{-c} (\det C)^c \prod_k p^{q_k} D_k$$

at even and odd positions, respectively.

Lean: `Formalism.L2.PairingIndexBounds.Ent.MiddleFrameLemmas.blkX` (`Formalism.LocalToGlobal.PairingIndexBounds.RowActionCoefficients.AdmissibleInterpolation.ProdItePow`, lines 372–414)

Let $\widetilde{y}$ be the asserted output on the universal transported source product, with its target twist $\prod_{v,s} (\det C)^{-d_1}$ divided out. Denote a product of framed tests by Q_I^C. First evaluate on $f_d = \exp(\sum_z d_z Z_z)$, where z indexes the coordinates. Contraction with f_d evaluates the symbol at d. Multiply the block identities, including the source twists $(\det A)^{-d_0}$. The factorial and target pairing both factor over disjoint blocks. This gives

$$\frac{[\widetilde{y}, Q_I^C]}{\prod_{v,s} C_\kappa} = \pm U \, \mathcal{D}_{P_{J,\mathrm{i}}}(f), \qquad U = \left(\prod_v p^{N'_{v,j+1}(i_v)} \right) D(L, L'') \prod_{v,s} (\det A)^{-d_0} (\det C)^{d_1} U_s, \tag{A.28}$$

for $f = f_d$, where J is the column choice given by the J_k, and $U_s = (\det A)^e p^{-M}$ or $(\det A)^{-c} (\det C)^c$ according to parity. For general $f = \sum_\alpha f_\alpha Z^\alpha$, apply the functional $d^\alpha \mapsto \alpha! f_\alpha$ on the finite block degrees of the identity, coefficientwise in τ. Indeed, for any matrix T, comparison of the two expansions of $\exp(d^{\mathsf{t}} T Z)$ gives $\alpha! [Z^\alpha] f(TZ) = \sum_\gamma \gamma! f_\gamma [d^\gamma] (T^{\mathsf{t}} d)^\alpha$. Thus this functional moves the transposed action back to the stated substitution on the series, with every factorial retained as in 10.18. On the left $f_d(B_\phi Z) = \exp(\sum_z (B_\phi^{\mathsf{t}} d)_z Z_z)$. The block degrees are preserved, so ordinary contraction recovers $\mathcal{D}_j(G, f(B_\phi Z))$ on the given generators. On the right it gives the indicated test. The symbolic outputs lie in the K_c-span of products of the target Weyl

spaces by their highest-weight formulas for invertible free-column matrices. The formulas are polynomial, so the outputs at singular matrices and the symbol coefficients also lie in that span. Hence contraction preserves it.

Lean: `Formalism.L2.PairingIndexBounds.Ent.xMid_of` (`Formalism.LocalToGlobal.PairingIndexBounds.RowActionCoefficients.AdmissibleInterpolation.ProdItePow`, lines 504–544)

Put $a = \operatorname{ord}_p \det A_v$ and $a' = \operatorname{ord}_p \det B_v$. Write $\lambda_s^{[i]} = \sum_{k=1}^{i} \lambda_{v,s,k}$. The embeddings preserve the determinant valuations since p maps to p and units and their inverses map to integral elements. Thus C has determinant valuation $a' - i$, and $D(L, L'')$ contributes $w(a' - a)$ at an even position and its negative at an odd position. At an even position $d_1 = -w$ and $e = d_0 + w$. Since $i \le r \le N_0$, counting boxes in the first i rows gives $M = \lambda_s^{[i]} + iw$. At an odd position negative reversal gives $d_1 = d_0$ and $c = w - d_0$. With the terms $\lambda_s^{[i]}$ and iw of $N'_{v,j+1}(i)$ included, the contributions to $v_p(U)$ are

$$\begin{aligned} \lambda_s^{[i]} + w(a' - a) - d_0 a + d_1(a' - i) + ea - M = 0, \\ iw - w(a' - a) - d_0 a + d_1(a' - i) - ca + c(a' - i) = 0. \end{aligned} \tag{A.29}$$

Lean: `Formalism.L2.PairingIndexBounds.Ent.xUnit_of` (`Formalism.LocalToGlobal.PairingIndexBounds.RowActionCoefficients.AdmissibleInterpolation.SumIccEqRange`, lines 233–308)

The standard bideterminants of the Schur lattice and its duality with the Weyl lattice are the straight-shape cases of [60, Theorem II.2.16 and Proposition II.4.1]. For a divided-power word u, $[uH_\kappa, Q]/C_\kappa = (u^{\mathrm{t}}Q)(I)$, where transposition reverses the word and transposes the substitutions. This is integral for integral Q. Conversely, restrict Q to lower unitriangular matrices. Its column degrees determine the diagonal exponents, so this restriction reads the lower triangular monomial coefficients. Produce the lower unitriangular matrix from I by row additions in decreasing order of source row index. Each source row is still a unit row at the time of addition. Thus every coefficient of the restriction is evaluation of a divided-power word at I.

Order exponents lexicographically by column and then by row. For a minor on increasing rows $i_1 < \cdots < i_h$, the leading monomial is $\prod_{r=1}^{h} Y_{i_r,r}$ with coefficient one. Its product over the columns of a standard tableau determines the tableau because each tableau row is ordered. These leading monomials are lower triangular. Successive subtraction of the leading terms therefore gives an integral Schur expansion whenever the normalized pairings are integral. The same argument with zero pairings gives nondegeneracy on the common characteristic-zero span. Hence the field dual of an integral Weyl basis for $[-,-]/C_\kappa$ consists of integral combinations of the tests.

Transport the Weyl basis by $Y \mapsto C^{-1}Y$ and the dual basis by $Y \mapsto C^{\mathrm{t}}Y$. Their products are dual under the product normalization. By (A.28) and (A.29), all pairings with framed tests have valuation at least g coefficientwise in τ. The source coefficient spans and the integral embedding of O give the same bound for every $G \in H_j(L)$. Each basis coordinate of $\widetilde{y}$ has this bound by integral duality. Restoring the target twists puts the basis vectors in $H_{j+1}(L')$ and gives the stated sum.

Lean: `Formalism.L2.PairingIndexBounds.Ent.xPair` (`Formalism.LocalToGlobal.PairingIndexBounds.RowActionCoefficients.AdmissibleInterpolation.NuTAntitone`, lines 36–99)

Lean: `Formalism.L2.PairingIndexBounds.Ent.dual_bases_holds` (`Formalism.LocalToGlobal.PairingIndexBounds.RowActionCoefficients.AdmissibleInterpolation.PolyKTorsionFree`, lines 44–124)

Lean: `Formalism.L2.PairingIndexBounds.Ent.tgt_bidet` (`Formalism.LocalToGlobal.PairingIndexBounds.RowActionCoefficients.AdmissibleInterpolation.TgtBidet`, lines 33–74)

□

Proofs for Chapter 11: Entry estimates: factorials against denominators

Proof of 11.3

Proof. For a monomial z^m, the contributions from multiplying by z_x on the series side and differentiating on the polynomial side both vanish when $m_x = 0$. When $m_x > 0$, their equality is

$$m!\,[z^{m-\mathbf{1}_x}]F = m_x(m - \mathbf{1}_x)!\,[z^{m-\mathbf{1}_x}]F.$$

By linearity, multiplication by L_k is adjoint to $\partial_{b_k} = \sum_x b_{k,x}\partial/\partial z_x$. Expanding the monomials of $P(z + \sum_k t_k b_k)$ gives

$$\prod_k \partial_{b_k}^{\beta_k} P = \left(\prod_k \beta_k!\right) D_b^{[\beta]} P.$$

The multinomial coefficients are integers, so this identity holds in every characteristic. Apply the adjoint relation repeatedly to obtain (11.5).

The expansion takes place over $\mathcal{O}_K$ when P and the directions are integral. For homogeneous P, every term has total (z,t)-degree d. If $|\beta| \le d$, every monomial in the coefficient of t^β has z-degree $d - |\beta|$. If $|\beta| > d$, the coefficient is zero. □

Proof of 11.4

Proof. If I_c repeats a label, $\Delta_c = 0$. For injective I_c, a direction at s_c with label $b \in \operatorname{im} I_c$ adds $t_k M_c \xi_k$ to the unique column labeled b. Every other direction leaves the matrix unchanged. The determinant has degree at most one in the parameters attached to any single column. If two directions have the same row vector, a term using both replacement columns vanishes by alternation.

The Hasse product formula is

$$D_b^{[\beta]} \prod_c \Delta_c = \sum_{\beta^{(1)}+\cdots+\beta^{(C)}=\beta} \prod_c D_b^{[\beta^{(c)}]} \Delta_c. \tag{A.30}$$

In a nonzero summand, directions from $\mathcal{K}_0$ go to factors at s_0. With a common row vector, each such factor receives at most one direction. With labels in J, it receives at most one per label in $\operatorname{im} I_c \cap J$. Summing the respective bounds over these factors proves (11.7) and (11.8). □

Proof of 11.5

Proof. If $E = 0$, then $D = 0$. Both families are empty and h has zero constant term, so $Z_{\ge p} = Y = 0$ suffices. Assume $E \ge 1$ and replace each $S_{s,a,b}$ by its sum over $0 \le l < E$. The omitted terms have degree at least p^{2E}. The linear substitution (11.10) preserves this degree bound, so the coefficients of h through degree D agree with those of the finite substitution.

Work modulo $p\mathcal{O}_K$, where Frobenius is additive. Since p is odd, the p-th power preserves the signs $(-1)^l$. For fixed v, a, b, the substituted product $S_{v,a,b}^p S_{v+1,a,b}$ becomes $(Z_1 + T_1)(Z_2 + T_2)$, where

$$\begin{aligned}
Z_1 &= \sum_{l<E} (-1)^l y_{v-2l,a,b}^{p^{2l+1}}, \\
T_1 &= \sum_{l<E} (-1)^l \left(\mathbf{1}_{v-2l\in T} \xi_{v-2l,a} t_{v-2l,b}\right)^{p^{2l+1}}, \\
Z_2 &= \sum_{l'<E} (-1)^{l'} y_{v+1-2l',a,b}^{p^{2l'}}, \\
T_2 &= \sum_{l'<E} (-1)^{l'} \left(\mathbf{1}_{v+1-2l'\in T} \xi_{v+1-2l',a} t_{v+1-2l',b}\right)^{p^{2l'}}.
\end{aligned}$$

Write $Z_2 = y_{v+1,a,b} + Z_2'$, where every monomial of Z_2' has remaining degree at least p^2. Then

$$(Z_1 + T_1)(Z_2 + T_2) = T_1 T_2 + T_1 y_{v+1,a,b} + Z_1(Z_2 + T_2) + T_1 Z_2'. \tag{A.31}$$

Multiply by u_v and sum over v, a, b. Expanding $T_1 T_2$ gives the first sum of (11.16), with the exponents and coefficients in (11.11) and (11.12). In the second term of (A.31), change variables from v to $s = v - 2l$. Summing over a gives

$$(-1)^l u_{s+2l}\, t_{s,b}^{p^{2l+1}} \sum_a \xi_{s,a}^{p^{2l+1}} y_{s+2l+1,a,b},$$

when $s \in T$, and zero otherwise. Summing over $s \in S$, $0 \le l < E$ and $0 \le b < r_1$ gives the sum indexed by $\kappa = (s, l, b)$.

Define $Z_{\ge p}$ by summing $u_v[Z_1(Z_2 + T_2) + T_1 Z_2']$ over v, a, b. Its monomials have y-degree at least p by the bounds for Z_1 and Z_2'. Subtract the two indexed sums and $Z_{\ge p}$ from the finite substitution and divide by p. The congruence modulo $p\mathcal{O}_K$ gives an integral quotient Y, which completes (11.16). □

Proof of 11.6

Proof. The polynomial $t^e Q_0$ is homogeneous of degree D. Thus (11.17) expresses its pairing with g as a finite sum of pairings with $p^{-i} h^i U_i$, $0 \le i \le D$. In these pairings we may use the displayed expression for h, because multiplication by power series preserves agreement through degree D.

Expand h^i by the multinomial formula. A summand chooses m_ϕ occurrences of each pure-top term, β_κ occurrences of each term with a linear remaining factor, i_2 occurrences of $Z_{\ge p}$, and i_3 occurrences of pY. Put

$$A = \sum_\phi m_\phi, \qquad B = \sum_\kappa \beta_\kappa, \qquad i = A + B + i_2 + i_3.$$

Its multinomial coefficient and the exponent of its extracted top monomial are

$$M = \frac{i!}{(\prod_\phi m_\phi!)(\prod_\kappa \beta_\kappa!) i_2! i_3!}, \qquad q = \sum_\phi m_\phi d_\phi^A + \sum_\kappa \beta_\kappa d_\kappa^B.$$

Use 11.3 to move the product of linear factors to the polynomial side. This contributes $\prod_\kappa \beta_\kappa!$ and replaces Q_0 by $D_b^{[\beta]} Q_0$, with t^e unchanged. The resulting pairing is a sum over $n \in \operatorname{supp} D_b^{[\beta]} Q_0$ with factorial factor $e!n!$. A nonzero coefficient requires $q \le e$.

Apart from integral scalar factors and a coefficient of the integral Hasse derivative, such a summand has the form

$$p^{-i+i_3} M \left(\prod_\kappa \beta_\kappa! \right) e! n! \, [t^{e-q} y^n] (Z_{\ge p}^{i_2} Y^{i_3} U_i). \tag{A.32}$$

If it is nonzero, every scalar factor selected with positive multiplicity is nonzero. The minimum remaining degree of $Z_{\ge p}^{i_2}$ gives $p i_2 \le |n|$.

The multinomial product is divisible by the factorial of the total number B of linear factors. Indeed,

$$M \prod_\kappa \beta_\kappa! = B! \, \frac{i!}{(\prod_\phi m_\phi!) B! i_2! i_3!},$$

where the quotient is an integer multinomial coefficient. Hence $|M \prod_\kappa \beta_\kappa!| \le |B!|$. The coefficient in (A.32) has norm at most one and $|p^{-i+i_3}| = p^{A+B+i_2}$. For each nonzero summand w this gives $|w| \le p^{A+B+i_2} |e!| \, |B!| \, |n!|$ with its own data. Choose a summand of maximal norm. The ultrametric inequality gives $|J| \le |w|$, so its data satisfy (11.20). □

Proof of 11.8

Proof. Put

$$c_p = \frac{1}{p-1}, \qquad A = \sum_s \alpha_s, \qquad L = \sum_s L_s, \qquad I = \sum_{s \in S} \sum_{l \in E} i_{s,l}.$$

By 11.7, monotonicity of ℓ_p, and (11.22),

$$\sum_b v_p(e_{s,b}!) \ge c_p \alpha_s - r_1 \ell_p(\alpha_s), \tag{A.33}$$

$$\sum_{a,b} v_p(a_{s,a,b}!) \ge c_p \sum_{a,b} a_{s,a,b} - r_0 r_1 \ell_p(L_s). \tag{A.34}$$

For the middle family,

$$\sum_{l\in E} \ell_p(i_{s,l}) \le \ell_p(\alpha_s)^2. \tag{A.35}$$

For $\alpha_s = 0$, all the $i_{s,l}$ vanish. A positive $i_{s,l}$ implies $p^{2l+1} \le \alpha_s$, hence $l < \ell_p(\alpha_s)$. There are at most $\ell_p(\alpha_s)$ such indices, and each $i_{s,l} \le \alpha_s$. Another application of 11.7 gives

$$\sum_{l\in E} v_p(i_{s,l}!) \ge c_p \sum_{l\in E} i_{s,l} - \ell_p(\alpha_s)^2.$$

Outside T, this inequality has zero on both sides. Summing the three bounds over s and using (11.25) gives

$$\mathcal{F} \ge c_p(A + L) - \delta. \tag{A.36}$$

Since $c_p - \rho_p = \gamma_p$, it follows that

$$\mathcal{F} - \rho_p(A + L) - (i_0 + I + i_2) \ge \gamma_p A - i_0 - I - \delta + \gamma_p L - i_2.$$

By (11.25) and $1/p \le \gamma_p$, we have $i_2 \le L/p \le \gamma_p L$. □

Proof of 11.9

Proof. Transposing the linear substitution (11.10) in the differential pairing gives

$$\mathcal{J} = \left\langle t^e Q_0(y), F\big(y + \mathbf{1}_T \xi t\big) \right\rangle. \tag{A.37}$$

A derivative in $t_{s,b}$ is carried by the chain rule to $\sum_a \xi_{s,a} \partial/\partial z_{s,a,b}$ when $s \in T$. Top exponents outside T vanish. A derivative in $y_{s,a,b}$ becomes $\partial/\partial z_{s,a,b}$. Iterating and evaluating at the origin gives (A.37).

The substituted F belongs to $\mathcal{C}_p(h)$. Choose E with $D < p^{2E}$ and apply 11.5. Each determinant contributes its height to the degree of Q_0, so Q_0 is homogeneous of degree $\sum_s L_s$. Apply 11.6 to (A.37).

For a nonzero entry, the resulting m, β, i_2, n satisfy (11.30) by the nonzero scalar conditions. The other conclusions give (11.31), (11.32), and

$$v_p(\mathcal{J}) \ge v_p(e!) + v_p(B!) + v_p(n!) - \left(\sum_{\phi} m_\phi + B + i_2 \right), \qquad B = \sum_{\kappa} \beta_\kappa. \tag{A.38}$$

Every integer has norm at most one. For an integer u with $p \nmid u$, a Bézout identity $au + bp = 1$ gives $|u| = 1$ since $|p| < 1$. Writing each factorial factor m in (A.38) as $p^k u$ with $p \nmid u$ gives $v_p(m) = k = \mathrm{ord}_p(m)$, where ord_p denotes the arithmetic valuation.

For each source s, set $i_{s,l} = \sum_{b<r_1} \beta_{(s,l,b)}$ for $l < E$ and $i_{s,l} = 0$ for $l \ge E$. The top budget gives $p^{2l+1} i_{s,l} \le \alpha_s$. Since $\sum_{s,l} i_{s,l} = B$,

$$\prod_{s,l} i_{s,l}! \;\mid\; B!, \qquad v_p(B!) \ge \sum_{s,l} v_p(i_{s,l}!).$$

A monomial y^n of $D_b^{[\beta]} Q_0$ comes from a monomial y^{n_0} of Q_0 with $n \le n_0$ and $|n| + B = |n_0|$. Every monomial of Q_0 has degree L_s in the variables at s, so

$$\sum_{a,b} n_{s,a,b} \le L_s, \qquad |n| + B = \sum_s L_s.$$

Also $p i_2 \le |n| \le \sum_s L_s$. Apply 11.8 with $a_{s,a,b} = n_{s,a,b}$, $i_0 = \sum_\phi m_\phi$, and the grouped array $i_{s,l}$. Its conclusion and (A.38) give (11.33). □

Proof of 11.10

Proof. The positive integer $a - a'$ satisfies $s + (a - a') = s'$, so minimality gives $d \le a - a' \le a$.

For $s \in T$, write $x_l = \alpha_s p^{-(2l+1)}$, with $d = 2l+1$. At an excluded distance, (11.35) gives $0 \le x_l - \mathrm{ch}_{s,l} - i_{s,l}$. At an unexcluded distance, (11.34) and (11.36) give

$$[x_l - C_{s+d}]_+ \le x_l - i_{s,l} = x_l - \mathrm{ch}_{s,l} - i_{s,l}.$$

The series of positive parts is bounded termwise by the convergent series $\sum_l x_l = \gamma_p \alpha_s$. The other two arrays have finite support. Summation therefore gives

$$\sum_{\substack{d \ge 1,\ d \text{ odd} \\ d \notin D_s}} [\alpha_s p^{-d} - C_{s+d}]_+ \le \gamma_p \alpha_s - \sum_l \mathrm{ch}_{s,l} - \sum_l i_{s,l}.$$

Sum over $s \in T$ and subtract δ. □

Proof of 11.11

Proof. The bound holds for $\mathcal{J} = 0$. For a nonzero entry take the multiplicities of 11.9. Set $i_{s,l} = \sum_b \beta_{(s,l,b)}$ for $l < E$ and $i_{s,l} = 0$ for $l \ge E$, as in that proof. For fixed s, l with $l < E$, the directions share the target $s + 2l + 1$ and the row vector $(\xi_{s,a}^{p^{2l+1}})_a$. By (11.32) and (11.7), $i_{s,l} \le C_{s+2l+1}$. The top budget gives $p^{2l+1} i_{s,l} \le \alpha_s$, so for every $l \ge 0$

$$i_{s,l} \le \min(\alpha_s p^{-(2l+1)}, C_{s+2l+1}). \tag{A.39}$$

Assign each occurrence of a pure-top term to an excluded distance. For $\phi = (v, l, l', a, b)$, put

$$s_1 = v - 2l, \qquad s_2 = v + 1 - 2l', \qquad a_1 = 2l + 1, \qquad a_2 = 2l'.$$

Then $s_1 + a_1 = s_2 + a_2$. The endpoints have opposite parity, and a_1, a_2 are unequal. Choose as source the endpoint with larger a_i, and as target the other endpoint. Let d_ϕ be the first positive distance from the source to the target. By 11.10, $d_\phi \le \max(a_1, a_2)$. The distance is odd. If $m_\phi > 0$, both endpoints belong to T, so $d_\phi \in D_{\mathrm{source}(\phi)}$. The pure-top monomial consumes at least p^{d_ϕ} units of top degree at its chosen source in column b.

For $d = 2l + 1$, let

$$\mathrm{ch}_{s,l} = \sum_{\substack{\phi:\ \mathrm{source}(\phi) = s \\ d_\phi = 2l+1}} m_\phi.$$

These counts have finite support, and

$$\sum_{s \in T, l} \mathrm{ch}_{s,l} = \sum_\phi m_\phi, \qquad \mathrm{ch}_{s,l} = 0 \quad \text{if } 2l + 1 \notin D_s.$$

For fixed s, l, sum the coordinatewise top budget (11.31) over the columns at s. The charged A-terms and the B-terms of distance $2l + 1$ consume at least $p^{2l+1}(\mathrm{ch}_{s,l} + i_{s,l})$ units of top degree. Consequently

$$\mathrm{ch}_{s,l} + i_{s,l} \le \alpha_s p^{-(2l+1)}. \tag{A.40}$$

Apply the scalar part of 11.10 with

$$V = \gamma_p \sum_{s \in T} \alpha_s - \sum_\phi m_\phi - \sum_\kappa \beta_\kappa - \delta.$$

By (A.40) and (A.39), this gives $V \ge g_A$. The common estimate (11.33) proves the theorem. □

Proof of 11.12

Proof. For a nonzero entry take the data of 11.9, with $i_{s,l}$ extended by zero as above, and put

$$A = \sum_{\phi} m_{\phi}, \qquad B_{\mathrm{ev}} = \sum_{s \in T_{\mathrm{ev}}} \sum_{l \geq 0} i_{s,l}, \qquad B_{\mathrm{odd}} = \sum_{s \in T_{\mathrm{odd}}} \sum_{l \geq 0} i_{s,l}.$$

Each pure-top term consumes at least one unit of top degree at its even endpoint. Each even-source linear-factor term also consumes at least one unit at an even position. Summing the top budget (11.31) over the even coordinates gives

$$A + B_{\mathrm{ev}} \leq A_{\mathrm{ev}}. \tag{A.41}$$

For odd sources, (A.39) gives

$$B_{\mathrm{odd}} \leq \sum_{s \in T_{\mathrm{odd}}} \sum_{l \geq 0} \min(\alpha_s p^{-(2l+1)}, C_{s+2l+1}).$$

Substitute both inequalities in the common lower bound

$$v_p(\mathcal{J}) \geq \gamma_p \sum_{s \in T} \alpha_s - A - B_{\mathrm{ev}} - B_{\mathrm{odd}} - \delta + \rho_p D.$$

This gives (11.39). The norm bound also holds for a zero entry. □

Proof of 11.13

Proof. For a nonzero entry take the multiplicities of 11.9. Let $\mathcal{P}$ consist of the even-source coordinates (s, b) with $b \notin J$ and the odd-source coordinates with $b \in J$. The endpoints of a pure-top term have opposite parity and share a column label, so exactly one belongs to $\mathcal{P}$. Each occurrence consumes at least one unit of top degree there, as does a B-term whose source coordinate belongs to $\mathcal{P}$. Summing the top budget over $\mathcal{P}$ gives

$$\sum_{\phi} m_{\phi} + \sum_{\kappa:\ (\mathrm{source}(\kappa), \mathrm{col}(\kappa)) \in \mathcal{P}} \beta_{\kappa} \leq \sum_{(s,b) \in \mathcal{P}} e_{s,b}. \tag{A.42}$$

The even-source B-terms left outside this payment have labels in J and odd targets. Group their directions by target. By the nonvanishing in (11.32), (11.8) applies at each such target with label set J. Summing the bounds gives

$$\sum_{\substack{\kappa:\ \mathrm{source}(\kappa)\ \mathrm{even} \\ \mathrm{col}(\kappa) \in J}} \beta_{\kappa} \leq \sum_{c:\ s_c\ \mathrm{odd}} \#(\mathrm{im}\, I_c \cap J).$$

Together with (A.42), this gives

$$\sum_{\phi} m_{\phi} + \sum_{\kappa:\ \mathrm{source}(\kappa)\ \mathrm{even}} \beta_{\kappa} \leq N_J. \tag{A.43}$$

For odd-source B-terms use (A.39). Add N_J to the common estimate (11.33) and use (A.43) to obtain

$$N_J + v_p(\mathcal{J}) \geq \gamma_p \sum_{s \in T} \alpha_s - \sum_{s \in T_{\mathrm{odd}}} \sum_{l \geq 0} \min(\alpha_s p^{-(2l+1)}, C_{s+2l+1}) - \delta + \rho_p D.$$

This is $v_p(p^{N_J}\mathcal{J}) \geq g_B + \rho_p D$. The norm bound also holds for a zero entry. □

Proof of 11.14

Proof. Let $F \in \mathcal{C}_p(H)$. For $A(\tau) \in K[\tau]$, put

$$\|A\|_{\mathrm{G}} = \max_{\eta} |[\tau^{\eta}]A|, \qquad \|0\|_{\mathrm{G}} = 0.$$

To prove multiplicativity, divide each of two nonzero polynomials by a coefficient of maximal norm. Each normalized polynomial is integral and has nonzero reduction modulo the maximal ideal of $\mathcal{O}_K$. Their reduced product is nonzero because the residue polynomial ring is a domain. The normalized product therefore has Gauss norm one. Rescaling proves multiplicativity.

The Gauss norm extends to the rational function field $K(\tau)$. Write $Y_{s,i,a}$ for its coordinate functions. For every $A \in K[\tau]$, evaluation at Y gives

$$|A(Y)| = \|A\|_{\mathrm{G}}. \tag{A.44}$$

Taking A constant gives the isometric inclusion of K. Taking $A = \tau_{s,i,a}$ gives $|Y_{s,i,a}| = 1$.

Evaluate the frame variables in $\mathcal{P}$ at Y. Separate the height-one factors at marked positions. Each has the form $\sum_{a<r_0} Y_{s,0,a} z_{s,a,b}$. Choose $\xi_{s,a} = Y_{s,0,a}$ and let $e_{s,b}$ count the separated factors at s with label b. If $r_0 = 0$, the vectors are empty and there are no height-one factors. Leave the remaining determinants in Q_0. Thus $\mathcal{P}(Y, z) = P_{\mathrm{ent}}(z)$ of (11.27), with integral frame and direction coordinates and counts (11.44) and (11.45). The isometric extension preserves the expansion-class identities for F.

Evaluation in the finite frame-jet sum gives $\mathcal{J}_F(Y) = \langle P_{\mathrm{ent}}, F\rangle$. By (A.44),

$$|[\tau^\eta]\mathcal{J}_F| \le \|\mathcal{J}_F\|_{\mathrm{G}} = |\langle P_{\mathrm{ent}}, F\rangle|. \tag{A.45}$$

Apply 11.11 and 11.12 to this entry to get

$$|[\tau^\eta]\mathcal{J}_F| \le p^{-(g_A+\rho_p D)}, \tag{A.46}$$

$$|[\tau^\eta]\mathcal{J}_F| \le p^{-(g_B-A_{\mathrm{ev}}+\rho_p D)}. \tag{A.47}$$

Lean: `Formalism.InvariantFactorPaths.G7.EGaussK.reduce` (`Formalism.ThetaLogCoordinates.UniformThetaSubstitution.FrobeniusLiftNorms.EGaussK`, lines 38–101)

For the prescribed-column bound, apply 11.13 with $J = J_i$. A separated height-one factor contributes 1 to N_{J_i} if its position is even and its label lies outside J_i, or if its position is odd and its label lies in J_i. Since $i \ge 1$, these are exactly its contributions to M_i. Each remaining determinant at an odd position contributes $q_{s,c}(i)$ to both. At an even position, a remaining determinant adds $\min(h_{s,c}, i) - q_{s,c}(i)$ to M_i. This is nonnegative because $q_{s,c}(i) \le h_{s,c}$ and $q_{s,c}(i) \le |J_i| \le i$. Hence

$$N_{J_i} \le M_i. \tag{A.48}$$

Since $|p| \le 1$, 11.13 and (A.45) give

$$|p^{M_i}[\tau^\eta]\mathcal{J}_F| \le p^{-(g_B+\rho_p D)}. \tag{A.49}$$

Take $F(z) = \Theta(\pi z)$. The product $\mathcal{P}(\tau, z)$ is homogeneous of z-degree D, since each determinant contributes its height. Substitution multiplies $[z^n]\Theta$ by $\pi^{|n|}$, and the frame pairing uses terms of degree D. Together with $\pi^{p^2-1} = p$ this gives

$$\mathcal{J}_{\Theta(\pi z)} = \pi^D \mathcal{J}_\Theta, \tag{A.50}$$

$$|\pi^D| = p^{-\rho_p D}. \tag{A.51}$$

Dividing each scaled bound by $|\pi^D| = p^{-\rho_p D}$ proves the three asserted estimates.

Lean: `Formalism.InvariantFactorPaths.G7.EGaussK.descend` (`Formalism.ThetaLogCoordinates.UniformThetaSubstitution.FrobeniusLiftNorms.EGaussK`, lines 132–144)

□

Proofs for Chapter 12: One lift on a lattice and the Hecke commutation

Proof of 12.1

Proof. Write $\nu_{s,a} = \nu_{j+1,s,a}$ for the target row lengths of Definition 10.3. The column heights $h_{s,c} = \#\{a : \nu_{s,a} > c\}$ are at most N_j, since the last $t+1$ rows vanish. At a marked position delete the $\alpha_{v,s} =$

$\nu_{s,1}-\nu_{s,2}$ height-one columns. The reduced diagram has $\nu_{s,2}$ columns there and $\nu_{s,1}$ elsewhere, with size $L_s=\sum_a \nu_{s,a}-\alpha_{v,s}$. This is the diagram of Theorem 11.14. The contraction sets variables outside v to zero, and the injection of local auxiliary variables into the full set identifies the local coefficients and gives zero off their support.

Claim 1. The presence of the local place v implies $e_O>0$. Indeed, $j_v: W_v\to O$ is finite by Definition 10.1, so $W_v/\ker j_v\subseteq O$ is integral. Since W_v is local with maximal ideal (p), the proper ideal $\ker j_v$ lies in (p). Lying-over gives a prime of O containing the image of p, so p is not a unit in O. If $e_O=0$, the association of p with $\varpi^{e_O}=1$ would make p a unit. This proves the claim.

Complete K_c for $|z|=p^{-v(z)}$. Equip the algebraic closure of the completion with the extending spectral norm and choose $\pi^{p^2-1}=p$ there. For the integral linear forms L in the local substitution, the inverse μ of $\pi^{-1}\ell(\pi T)$ from Theorem 10.13 gives $U(\pi L)=U^{\ell}(\pi\mu(L))$, an integral series. The scaled phase is $\exp(-\pi^{p+1}H_u/p)V$, where $H_u=\sum_{s,a,b}u_s S^{p}_{s,a,b}S_{s+1,a,b}$ is the formal series of Definition 10.12. The series V and the scalars $(-\pi^{p+1})^d/d!$ are integral. Thus the scaled product is the sum of $p^{-d}H_u^d$ times the integral series $(-\pi^{p+1})^dVU(\pi L)/d!$. Theorem 11.14 gives the norm bound p^{-A_v} after contraction in the original variables. □

Proof of 12.3

Proof. Definition 10.10 gives $\Theta=\exp(-\Phi)U_0$. For $\Delta=\Phi-\sum_v\Phi_v$, phase equivalence makes $\exp(\Delta^{\ell})$ an integral unit of constant term one, so take $U_1=\exp(-\Delta)U_0$. If there are no positions, the contraction is the integral constant coefficient. Otherwise expand $U_1^{\ell}=\sum_\beta a_\beta X^\beta$ and substitute the inverse $\mathfrak{e}=\ell^{-1}$ coordinatewise. Each $a_\beta\in O_c$, and the corresponding term of U_1 is $a_\beta\prod_v\mathfrak{e}(Y_v)^{\beta_v}$. Substitution and contraction factor over positions in this product. Only $|\beta|\le\deg_Z P_{I,\mathbf{i}}$ can contribute, since $\mathfrak{e}(0)=0$. The factor $\mathfrak{e}(Y_v)^{\beta_v}$ becomes $Y_v^{\beta_v}$ after logarithmic substitution. By (12.11) and Theorem 12.2, the contracted factor at v has coefficient bound q_v. Multiplication adds the bounds and finite addition preserves them. □

Proof of 12.4

Proof. For $f\in V_j$ and a target lattice pair L', Definition 10.10(3) gives over K_c the finite nonterminal expansion (10.1) for $j+1<m$ (with $\mathcal{R}_j$ containing one representative of each lattice orbit in each component) and the finite terminal expansion (10.2) for $j+1=m$ (with terminal index sets, weights w_x, and source pairs $L_{x,y}$ fixed for all f,L'), writing $\ker_{j,L,L'}=\mathcal{K}_{j,L,L'}$.

For each source L and each simultaneous lower neighbour L'' of L', Definition 10.10 supplies a frame from L to L'' in which the bound of Lemma 10.17 holds for every series and every real lower bound. Apply it to the framed theta series with source value $f(L)\in H_j(L)$, where $f\in\Lambda_j$. The value $\ker_{j,L,L''}(f(L))$, multiplied by $\prod_v p^{N'_{v,j+1}(i_v)}$, is a sum of elements of $H_{j+1}(L')$ with coefficients of bound g. For each class denominator w, $p^{d_G(j)}/w\in O_c$. Consequently the value at L' of the normalized class sum is

$$y=\sum_\alpha a_\alpha z_\alpha,\qquad z_\alpha\in H_{j+1}(L'),\qquad \nu_c(a_\alpha)\ge h=g-d_G(j). \tag{A.52}$$

In a nonzero K-linear relation $\sum b_\alpha z_\alpha=0$, choose $b_\gamma\neq0$ of least ϖ-order. Eliminating z_γ preserves the coefficient bound because the ratios b_α/b_γ lie in $O_c\cap K=\iota(O)$. Repeat until the vectors are independent. Since y is defined over K, an invertible coordinate minor gives $a_\alpha\in K$. For $e_O>0$, $\operatorname{ord}_\varpi(a_\alpha)\ge\lceil e_Oh\rceil$ by (12.3). For $e_O=0$ the same condition gives $a_\alpha\in\iota(O)$. Thus $y\in\iota(\varpi)^{\lceil e_Oh\rceil}H_{j+1}(L')$.

The lift sends V_j into V_{j+1} by admissibility. A level element gives an index-preserving bijection between the lower neighbours of a lattice and those of its translate. An upper endpoint of an edge with full lower endpoint is full. Thus the edge sums preserve equivariance and support, and the fibre bounds give (12.16). □

Proof of 12.6

Proof. Take $i_v=0$, $q_v=A_v$ by Theorem 12.1 and use Lemma 12.5 with the empty list. In Definition 11.1, for large odd distances d one has $\alpha_{v,s}p^{-d}<1$. The tail in each class modulo n'_v then vanishes if the capacity is positive and is geometric if the capacity is zero. Hence $A_v\in\mathbf{Q}$, so Theorem 12.3 gives the coefficient

bound. For $\mathbf{i} = \mathbf{0}$, Definition 10.10 gives the coefficient bound $g = 0$ at every lattice pair, frame and column choice. Lemma 12.4 then gives (12.19). □

Proof of 12.8

Proof. For the lift bound take the list $[v]$ in Lemma 12.5, with $i_v = i$, $q_v = B_v$ and $i_{v'} = 0$, $q_{v'} = A_{v'}$ elsewhere. For integrality choose an adapted basis on an index-i edge $A \to_i B$ at level b, so

$$A_v = gW_v^N, \qquad B_v = gDW_v^N, \qquad D = \operatorname{diag}(p^{-1}I_i, I_{N-i}), \qquad N = N_b. \tag{A.53}$$

At one embedding let κ^* be the actual weight, $e = \kappa_N^*$, and $\nu_a = \kappa_a^* - e$. The untwisted integral Weyl module is spanned by divided-power words on the highest-weight product of minors of shape ν. A divided power of order n in direction E_{ab} takes $[z^n]$ after $Y \mapsto (I + zE_{ab})Y$. It shifts the total degree in the first i rows by $n(\mathbf{1}_{b\le i} - \mathbf{1}_{a\le i})$, so each nonzero word is homogeneous in those rows. Row substitution expands each minor into minors of the same height. A height-h minor uses at most $\min(i,h)$ of the first i rows. Thus each nonzero word F has degree $d \le M_i = \sum_{a=1}^{i} \nu_a$ in the first i rows, so $p^{M_i}F(DY) = p^{M_i-d}F(Y)$.

Let G be the image of g at this embedding. A fibre element at A has the form $F = (\det G)^{-e}F_0$, where the polynomial $H(Y) = F_0(GY)$ lies in the untwisted module. The preceding degree bound puts $p^{M_i}H(DY) = p^{M_i}F_0(GDY)$ in the untwisted module, so $p^{M_i}F_0$ satisfies the row condition at B. Its determinant twist is $(\det(GD))^{-e}p^{M_i}F_0 = p^{M_i+ie}F$, where $M_i + ie = \sum_{a\le i}\kappa_a^*$. Tensoring over the embeddings gives $p^{\mathcal{A}_i(\kappa)}H_b(A) \subseteq H_b(B)$. Sum over the edges, with equivariance and support as in Lemma 12.4. For $b = j+1$, $i \le r$, the even terms use the initial entries. At an odd position, every entry of the row before negative reversal is at least $-w_{v,j+1}(s)$. Hence every entry of the actual, negatively reversed weight is at most $w_{v,j+1}(s)$, and the sum of its first i entries is at most $iw_{v,j+1}(s)$. Thus

$$\mathcal{A}_i(\kappa) \le \sum_{s\ \text{even}} \sum_{a=1}^{i} \lambda_{v,s,a} + i \sum_{s\ \text{odd}} w_{v,j+1}(s) = N'_{v,j+1}(i). \tag{A.54}$$

This proves integrality for the stated normalization. □

Proof of 12.10

Proof. In an abstract system the fourth item of Definition 10.10 gives the units and the additive maps with the stated quantifiers. For a constructed arithmetic system take the signed determinant lengths and the record of the two components and all entries outside v. Lemma 12.15 gives (12.26) for these data; its proof constructs the units a, a' depending only on the place and layer (before fixing the fibre), and (12.27) in Lemma 12.15 identifies the chart value with the value obtained from a representative. □

Proof of 12.11

Proof. For any three full lattices of the same rank, $\mathfrak{d}(U,W) = \mathfrak{d}(U,V)\mathfrak{d}(V,W)$, by taking determinants over each local ring. Hermitian duality sends the determinant ideal to its inverse conjugate. Thus the relative determinant of self-dual lattices has norm one, and its character value is nonzero.

The trace dual of $\mathfrak{b}^{-1}$ for the Hermitian trace pairing is $\mathfrak{D}_{E/K_0}^{-1}\bar{\mathfrak{b}} = \mathfrak{b}^{-1}$. In dual bases of A, B this calculation applies to each matrix entry. On the curve, division by the area makes an oriented period basis have alternating pairing one. Hence $P(A,B)$ is self-dual. On Q the operators W_y obey the additive group law. If $h \in P \cap Q$, then $W_{y+h}F = W_yW_hF = W_yF$ for a P-fixed section.

Use the complex move identities (17.3) and (17.2). Their hypotheses are self-duality and even ranks, as here. Their proof uses the normalized averages on complex period lattices before making the split-prime moment estimate. To see explicitly how the choice of ideal character enters, denote the Weil family's character by χ_*, its sections by $e^*_{A,B}$ and its determinant factor by $C_*(A,B)$. On the standard pair the theta space is a line and the Weil section is nonzero. Write the chosen standard section as a_0 times that section. Transport first the standard source to A, then the standard target to B. Linearity of both averages gives

$$e_{A,B} = a_0 \frac{C(A,B)}{C_*(A,B)} e^*_{A,B}.$$

The quotient makes sense even if $a_0 = 0$. In a source move $A \to A_1$ the ratio of the two C's formed with χ is $\chi(\mathfrak{d}(A, A_1))^{n'/2}$. In a target move $B \to B_1$ it is $\chi(\mathfrak{d}(B, B_1))^{-n/2}$. Dividing the corresponding move identities by the new C therefore gives $I_{P,Q}$ in both cases. Finally, a rational unitary change contributes a principal ideal generated by its determinant. On such generators both characters have value $\prod_{\tau\in\Psi} \tau(\det g)^{-1}$. Thus the ratio C/C_* is invariant under the changes at both ends. Applying the Weil family's pullback identity (17.1) gives the asserted formula for e. □

Proof of 12.13

Proof. An element of the determinant class group of the construction has a finite idèle representative. Strip off the finitely many uniformizer powers of that representative. The remaining integral idèle is in the kernel of the class character. At each stripped-off uniformizer the character is the unramified value of the defining Hecke character, which belongs to H by the choice of coefficient field. Their product with the respective integer exponents therefore belongs to $H^\times$. In particular the class character to power b_l at each chosen representative belongs to $H^\times$.

For $l = 0$ the power is zero. For $l > 0$ transport $\mathrm{diag}(x, 1, \dots, 1)$ from the standard Hermitian form to the level-l form. This gives a finite unitary point Y of determinant x. In the chosen component write $Y = \gamma y_C k$ with γ rational and k in the level. The determinant twist of this chart at Y is $\chi_I(x)^{b_l}$. Its value on k is one. Its value on γ is the inverse of the b_l-th power of the type norm of $\det\gamma$, whose factors are among the specified embeddings into H. Multiplication by the class value at y_C proves the assertion.
Lean: `GlobalTheory.TwV.Xc.ChiPow.tw_ch_rat (Development.AbelianVarietyTheory.Proofs.SatakeParameters.KernelSiteNormalisingCharPowValuesInCoeffField`, lines 79–99)

For the test, carry the lattice, its prescribed residue maps and its local frames by y, and do the same at the target using y'. A frame at a marked prime is a basis inducing the prescribed coordinates on its finite quotient. Lifts of a basis of the quotient give such frames. At primes of the word level take the indicator that Z maps the source lattice into the target lattice and preserves the prescribed residue map. Multiply it by the following factors at the type and twist primes. At a matrix step read Z in the two local frames and take the finite Fourier sum of the local test vector over its free matrix entries. At the penultimate step append the column in the specified target coset before evaluating the local vector. At the last step evaluate the test vector on the projected matrix and its finite Fourier factor on the omitted row against the specified source coset. Definition 12.12 gives the finite summation, its volume factor and the last-row Fourier character in the coordinates of these frames. In particular the coset belongs to the target in the second case and to the source in the third.

Two lifts of the same frame differ by a transformation acting trivially on the finite quotient on which the local vector is read. In the Fourier sum this transformation permutes the summation classes and preserves the additive character. In the appended column the difference lies in its defining congruence lattice. In the last row its pairing with the Fourier variable is integral. These observations prove independence of the lifts and the congruence assertion. They also show that equality of the two adelic points at the test places gives equality of the functions, including the residue-map indicator. The values are algebraic: the finite sums use values of the finite characters and rational volume factors. Their denominators are supported on the prescribed exceptional primes.

A rational change carries each decorated point and its frames to the changed point. Reading Z in those frames replaces it by $(g')^{-1}Zg$.

For a right level change (k, k'), distinguish the adelic elements from their relative frame matrices. At a test prime P, choose κ_P, κ'_P with

$$f_{yk,P} = f_{y,P}\kappa_P^{-1}, \qquad f_{y'k',P} = f_{y',P}(\kappa'_P)^{-1}.$$

Thus $X = f_{y',P}^{-1} Z f_{y,P}$ becomes $X_1 = \kappa'_P X \kappa_P^{-1}$. The level condition supplies lifts of these matrices to the enlarged spaces. Write J, J' for the original local levels. A source lift r belongs to hJh^{-1}, and a target lift r' belongs to $h'J'(h')^{-1}$. Hence the original vector is acted on by $h^{-1}rh \in J$ and $(h')^{-1}r'h' \in J'$. In particular,

$$(h')^{-1}(r'Dr)h = \left((h')^{-1}r'h'\right)\left((h')^{-1}Dh\right)\left(h^{-1}rh\right).$$

Apply the local vector's character law to these conjugated elements. In Definition 12.12 the scalar multiplying $\Phi((h')^{-1}Dh)$ is the same on both sides. The lifts have the determinants of their blocks on the original source

and target, and these determinants are local units, so their additional valuation exponents are zero. Change the free variables in the finite Fourier sum after refinement to common support and period depths. The Haar weight is preserved. If the character argument increases by d, the Fourier factor is multiplied by $\psi_P(-d)$; this is included in the induced level character, whose value on a lift r is the original vector character at $h^{-1}rh$ multiplied by $\psi_P(-\,\mathrm{n}_1(r))$, where $\mathrm{n}_1(r)$ is that additive shift. If c_P, c'_P are the relative-frame level-character values and μ_P is the local step character, the resulting local identity is

$$\Xi_P(\kappa'_P X \kappa_P^{-1})\frac{\mu_P(\det \kappa'_P)}{\mu_P(\det \kappa_P)} = \frac{c'_P}{c_P}\Xi_P(X).$$

The relative-frame construction gives $\prod_P c_P = \lambda_j(k)$, $\prod_P c'_P = \lambda_{j+1}(k')$, and $\prod_P \mu_P(\det \kappa_P) = \mu_{j+1}(\det k)$, with the corresponding target identity. The level-marking factor is unchanged. Multiplying the local identities gives the second displayed identity of the lemma.

At the appended-column step apply the same changes to the matrix and its marked target column. At the projected-row step the source change sends η to $\eta\kappa_P^{-1}$. For the old and new coset representatives y, y_1, the difference $\kappa_P^{-1}y_1 - y$ lies in $\mathcal{O}_P^a$ at a type prime and in $q^M\mathcal{O}_P^a$ at a twist prime. On the respective supports, η and $q^M\eta$ are integral, so $\psi_P(-\eta\kappa_P^{-1}y_1) = \psi_P(-\eta y)$. The target congruence preserves the supported test at its defining depth, and the upper block supplies the prescribed target character. These give the same identity for both coupled steps, including zero test values.

Lean: `GlobalTheory.TwV.levAt_mat` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.FiniteDatumLevelEquivarianceAtMatrixSteps`, lines 18–77)
Lean: `GlobalTheory.AdLevCpl.levAt_coupled` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.AdelicDatumLevelEquivarianceFirstCoupledStep`, lines 13–37)
Lean: `GlobalTheory.TwTop.levAt_top` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.LastStepAdelicDatumLevelEquivariance`, lines 42–115)

□

Proof of 12.14

Proof. Write M_i for the period module at pair i, identifying its elements with their analytic images. Let A_i have localizations $c^{-1}M_i$ on S, cM_i on $\bar{S}$, and M_i elsewhere. Interchange c and c^{-1} to define B_i. For the moment suppress i. Choose sets X, T, Y of representatives of

$$A/(A\cap M), \qquad (M\cap A)/(M\cap A\cap B), \qquad B/(B\cap M).$$

Write $B_H(x,y) = \operatorname{Im} H(x,y)$ for the alternating form, where $H(Z,x) = \sum_{\tau,a,b} Z_{\tau,ba}\overline{x_{\tau,ba}}/\operatorname{area}(\delta\mathfrak{c})$, and let $W_xF(Z) = \alpha(x)F(Z+x)\exp(-\pi H(Z,x) - \pi H(x,x)/2)$ be the theta translation with semicharacter $\alpha(x) = \exp(\pi i\sqrt{d_0}\operatorname{Re} H(x,x)/2)$ for $K_0 = \mathbf{Q}(\sqrt{-d_0})$. Fourier transformation in the middle quotient with counting measure of total mass one gives the operator

$$F \longmapsto \frac{1}{|T|}\sum_{x\in X,\ t\in T,\ y\in Y} \xi(t-x)\exp(2\pi i B_H(t,y))\, W_xW_yF.$$

Indeed the Fourier coefficient at y is the sum over t divided by $|T|$. Its inverse Fourier term acts by W_y, and translation of that test coset acts by W_x. The three quotients are finite because all their lattices are commensurable.

Here are the three changes of representatives on an M-fixed section. Adding $h \in M\cap A\cap B$ to t preserves ξ by its congruence at S, and $B_H(h,y)$ is integral since $h, y \in B$. Adding $h \in B\cap M$ to y preserves the character since $t, h \in M$, and $W_{y+h}F = W_yW_hF = W_yF$. If x is replaced by $x+h$ with $h \in A\cap M$, replace t by $t+h$ modulo the middle denominator. The argument $t-x$ stays fixed. The additional Fourier factor is $\exp(2\pi i B_H(h,y))$. Moving W_h past W_y gives its inverse, and then $W_hF = F$.

Fix a reference pair 0. At $P, \bar{P}$ choose exponents large enough to carry every source lattice into the reference source and conversely, and likewise for every target. The Chinese remainder theorem gives $g \in \mathcal{O}_E$ in these prime powers with $g \equiv 1 \bmod c$ and $g \equiv 1 \bmod Q$ at each $Q \in S$ prime to c. Put $b = g^2$. For a homomorphism z from the i-th source to the i-th target, $bz(v) = gz(gv)$ carries the reference source into the reference target. The same calculation in the opposite direction shows that b carries the respective period modules into one another. Localizing proves the same inclusions for A_i, B_i and their indicated intersections.

For each of the three quotients, denote the numerator by U_i and the denominator by V_i. The local descriptions give $cU_i \subseteq V_i$. Multiplication by b defines maps in both directions between the reference quotient and the i-th quotient. Their composite is multiplication by b^2, which acts as the identity since $b^2 - 1 \in c\mathcal{O}_E$. Thus the actual elements bx, bt, by obtained from one choice of reference representatives serve at every i. They have the same analytic images, the same character factors, and the same test values by Lemma 12.13. The resulting finite sum is one complex linear operator on the ambient function space. The representative-independence proved above identifies its restriction to each fixed section with the prescribed test there.

For the coefficient assertion, first consider the elliptic factor of the standard section. The discriminant normalization gives an algebraic Weierstrass equation: the CM j-invariant is algebraic, and the discriminant fixes the scale of g_2, g_3. The coordinate differential is the invariant differential of this equation. The reduced odd theta section with derivative one has algebraic Taylor coefficients, by [62, Corollary 2.2]. The divisor of this section is the origin, hence symmetric and defined over the coefficient field. Its translate by a rational period point has algebraic coefficients by the translation theorem for a symmetric divisor, after adjoining the torsion coordinates. An algebraic local equation for the translated divisor has algebraic Taylor coefficients in the invariant differential, so restoring this local equation also preserves algebraicity. This is the elliptic case of [62, Theorem 2.9]. Translating to the prescribed nonzero half-period and dividing by its nonzero value gives the normalized even section. Further torsion translations compose up to roots of unity.

For the one-entry matrix section use the fourth-power normalization in Lemma 17.14 with its initial rational isometry g_0. Its scalar κ_0 is algebraic by the defining root equations. The entries of the isometry in CM embedding coordinates are algebraic: the trace matrix, the change to a Hermitian orthogonal basis and the four-square matrices are over number fields. Translating the four factors by the same rational period point gives a finite sum of translates of the elliptic product. Commensurability makes each translating point torsion modulo the elliptic product lattice. The normalized averaging coefficients are inverse square roots of positive integers times roots of unity. It follows that the fourth tensor power of the translated one-entry section has algebraic coefficients. If $a_\gamma \neq 0$ is any coefficient of that section, the tensor coefficients at $(\gamma,\gamma,\gamma,\gamma)$ and $(\nu,\gamma,\gamma,\gamma)$ give a_γ^4 and $a_\nu a_\gamma^3$. Hence every a_ν is algebraic. The zero section satisfies the same assertion.

Tensor over the matrix entries and perform the two endpoint averages of Lemma 12.11. The determinant character values in those averages are algebraic. Indeed for a norm-one ideal $\mathfrak{a}$, principalize $\mathfrak{a}^h = (z)$; then $\mathfrak{a}^{2h} = (z/\bar{z})$, to which the principal-ideal character formula applies. Finally every x, y, t in the finite sum is a rational period point. Its translation phases are roots of unity, and the values of ξ are algebraic by Definition 12.12. The output is a finite algebraic combination of entire torsion translates with algebraic Taylor coefficients. Taylor expansion, followed coefficientwise by the prescribed embedding of $\overline{\mathbf{Q}}$, therefore carries every rational linear relation between these normalized outputs to the same relation on their series. □

Proof of 12.15

Proof. Write E/F for the host CM extension and $K_0 = \mathbf{Q}(\sqrt{-d_0})$ for its imaginary quadratic subfield. Write $K \hookrightarrow \mathbf{C}_p$ for the local coefficient embedding, compatible with $\lambda : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ on the global coefficient field. Fix a component at each endpoint. At level l let M_l, J_l be the base lattice and form, C_l the change to standard orthogonal coordinates, and a_* the level parameter. The point y of a tuple L has its component representative away from the moving places. At a moving split place its first matrix, in the chart basis, is a basis matrix of L_v. Its other matrix is determined by the unitary equation. With $\#$ denoting the J_l-dual, the kernel lattice and determinant idèle are

$$M(y) = E^{N_l} \cap y\widehat{M}_l, \qquad A(y) = C_l\big(M(y) + a_* M(y)^{\#}\big), \qquad d_y = \det y. \tag{A.55}$$

Indeed completion gives the corresponding translate of the base lattice, and taking the dual commutes with a unitary translate. The base sum in parentheses is the standard self-dual companion in orthogonal coordinates. Thus $A(y)$ is self-dual, and its first completion at v is the image of L_v under the chart matrix. At level zero use the initial chart and at positive levels the target chart of the preceding lift.

Lean: `GlobalTheory.pairOK_allSites` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.KernelSiteLatticeTuplePairsAdmissible`, lines 23–47)

Fix the place v and the layer j. Write $n = N_j$, $n' = N_{j+1}$, $2\eta = [\operatorname{Frac} W_v : \mathbf{Q}_p]$, and put

$$\Delta_w(l) = \sum_{s \text{ even}} w_{w,l}(s) - \sum_{s \text{ odd}} w_{w,l}(s), \qquad \Delta = \Delta_v(j+1). \tag{A.56}$$

Let π be the norm-one idèle equal to (p, p^{-1}) at the moving pair $P, \bar{P}$ and one elsewhere. Put $M_\pi = \lambda(\mu_{j+1}(\pi))$ and $C_\pi = \lambda(\chi_I(\pi))$. The lift $(p, 1)$ under $x \mapsto x/\bar{x}$ is supported above p. There the untwisted step character is unramified, since p differs from the type primes. Its value is the product of its uniformizer values to their respective orders. Those values and the value of the finite twist belong to the kernel coefficient field included in the global coefficient field. Hence $M_\pi \in K^\times$. By Lemma 12.13, $C_\pi^{b_j}$ and $C_\pi^{b_{j+1}}$ also belong to K if both levels have components.

Lean: `GlobalTheory.TwV.Xc.MuLL.mu_mem_LL` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.StepCharacterUniformizingIdeleMemCoeffField`, lines 91–175)

To compute the valuations, raise π to a positive power m_0 such that its first ideal factor to that power is principal, say (y_0). Then $\pi^{m_0} = zk$ with $z = y_0/\bar{y}_0$ and k an integral norm-one idèle. On the compact units the step character has finite image. On z it is $\prod_\tau \tau(z)^{e_\tau}$, where the filling gives $e_{w,s} = w_{w,j+1}(s) + b_j + n'/2$. The valuations at even and odd embeddings have opposite signs. The constant shift cancels because there are equally many of each. Only v contributes, with first valuation m_0. Thus $v_p(M_\pi) = \Delta$.

For a norm-one ideal $\mathfrak{a}$, write it as $\mathfrak{b}_1/\bar{\mathfrak{b}}_1$ by choosing one prime from each conjugate pair in its support. Principalizing a positive power of $\mathfrak{b}_1$ gives $\mathfrak{a}^h = (z_1)$, $z_1\bar{z}_1 = 1$. Hence $\chi(\mathfrak{a})^h = \prod_{\tau\in\Psi} \tau(z_1)^{-1}$. At an inert prime of K_0 a decomposition-group element inducing Frobenius on K_0 interchanges Ψ and its conjugate. It preserves the valuation and carries this product to its inverse. The product has valuation zero. This applies to the determinant-ideal value of π. Consequently

$$a = p^{-\Delta} M_\pi C_\pi^{b_j}, \qquad a' = p^{\Delta} M_\pi^{-1} C_\pi^{-b_{j+1}} \tag{A.57}$$

are units in the local coefficient field. Their definition involves neither component nor lattice. If either component set is empty take $a = a' = 1$ and all maps zero.

Lean: `GlobalTheory.TwV.Xc.lambdaCharacter_norm_eq_p_pow_of_pos` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.NormLambdaCharacterNormOneIdeleEqPPow`, lines 19–84)

Lean: `GlobalTheory.TwV.Xc.two_units` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.TwoUnitsAtPlaceUniformizingIdele`, lines 38–81)

Fix the two components and every entry away from v. This is the record ω. Consider a nonempty finite family of pairs in this fibre and designate one pair as 0. Denote its members' kernel lattices by A_i, B_i, and their source and target lengths by s_i, s_i'. In the fixed chart matrices the first completion of the matrix lattice is $D_i = G_i' \operatorname{Mat}_{n'\times n}(W_v) G_i^{-1}$, where G_i, G_i' are bases of the two tuple entries. Its indicator in these coordinates is precisely the containment test. The fixed ideal and chart factors cancel in relative volumes. The n columns see G_i' on the left and the n' rows see G_i^{-1} on the right. With $d_i = ns_i' - n's_i$, the length convention gives

$$\operatorname{vol}(D_i)/\operatorname{vol}(D_0) = p^{2\eta(d_i - d_0)}. \tag{A.58}$$

We now give the finite Fourier calculation that fixes the sign of this power. Put $P_i = P(A_i, B_i)$, $P_- = \bigcap_i P_i$, $P_+ = \sum_i P_i$. By self-duality $P_+^{\#} = P_-$. The alternating pairing on P_+/P_- is perfect. The only primary parts are the two over $P, \bar{P}$. Conjugation interchanges these primes, so each part is isotropic and the second is the character dual of the first. Write the first as V. The image of P_i in the two halves is $D_i \oplus D_i^\perp$, writing D_i also for its finite image. A local generator of the curve ideal identifies these first images with the preceding matrix lattice quotients.

On functions on V use $T_a f(t) = f(t+a)$ and $M_b f(t) = b(t) f(t)$. On sections the corresponding operations are

$$W_x F(Z) = \alpha(x) F(Z + x) \exp\bigl(-\pi H(Z, x) - \pi H(x, x)/2\bigr), \tag{A.59}$$

where $H(Z, x) = \sum_{\tau,a,b} Z_{\tau,ba}\overline{x_{\tau,ba}}/\operatorname{area}(\delta\mathfrak{c})$ and $\alpha(x) = \exp(\pi i\sqrt{d_0}\operatorname{Re} H(x, x)/2)$. Their commutator is $b(a) = \exp(-2\pi i \operatorname{Im} H(a, b))$ in period coordinates. To choose lifts in each half with its additive translation law, choose an integer of E in P coprime to its conjugate, taking $(\varrho) = P^h$ for a positive multiple h of the ideal class order. For sufficiently large J the two norm-one multiples $(\bar{\varrho}/\varrho)^{\pm 2J} P_0$ contain the respective local halves. Intersect these multiples with each P_i. The intersections sum to P_i: this can be tested locally, where

one of the two multipliers enlarges the required lattice and the two lattices agree away from the moving pair. The semicharacter law on each multiple is transported from P_0, since the multiplier preserves H.

Let $\theta_i = C(A_i, B_i)^{-1} e_{A_i,B_i}$. There is a linear map S sending $T_a M_b \mathbf{1}_{D_0}$ to $W_a W_b \theta_0$. Indeed shifts by D_0 and modulations by $D_0^{\perp}$ fix the generating vectors. Commuting a shift past a modulation gives the same scalar on each side. On each coset of D_0, the distinct characters of D_0 are independent and span the functions there. Distinct cosets have disjoint support. This accounts for all relations. Put $J_i = D_i \cap D_0$, $m_i = [D_i : J_i]$, $n_i = [D_0 : J_i]$. Sum first over the characters of D_0/J_i; this gives $n_i \mathbf{1}_{J_i}$ on D_0. Then sum over translations by D_i/J_i; this gives $n_i \mathbf{1}_{D_i}$. The normalized move divides by $\sqrt{m_i n_i}$, since the second-half index is n_i. Apply Lemma 12.11 first to the source and then to the target, including the intermediate pair in the family. The two square roots of relative volume multiply. Thus

$$S(\mathbf{1}_{D_i}) = \sqrt{\operatorname{vol}(D_i)/\operatorname{vol}(D_0)}\,\theta_i. \tag{A.60}$$

Lean: `Formalism.L2.PairingIndexBounds.Mv.MovesU_of` (`Formalism.LocalToGlobal.PairingIndexBounds.ModelLatticeMoves.ModelIndexPairings.MovesU`, lines 100–234)

The finite test applies one linear operator to these sections on the whole finite family. To see this, for each P_i enlarge its local lattice by c^{-1} at S and shrink it by c at $\bar{S}$, obtaining A; let B use the opposite changes. Choose representatives X, T, Y of $A/(A \cap P_i)$, $(P_i \cap A)/(P_i \cap A \cap B)$, $B/(B \cap P_i)$, respectively. The operator is

$$\mathcal{T}_\xi F = |T|^{-1} \sum_{x \in X,\, t \in T,\, y \in Y} \xi(t - x) e^{2\pi i \operatorname{Im} H(t,y)} W_x W_y F. \tag{A.61}$$

Use period preimages in the argument of ξ. Lemma 12.13 makes ξ invariant under $A \cap B$. Changing t by a period in the denominator of its quotient leaves the additive character unchanged. Adding $h \in B \cap P_i$ to y preserves the Fourier character because $t, h \in P_i$, and $W_{y+h} F = W_y W_h F = W_y F$ on a P_i-fixed section. For $x \mapsto x + h$, $h \in A \cap P_i$, reindex $t \mapsto t + h$; its extra Fourier factor cancels the commutator of W_h with W_y.

The moving primes are outside $S, \bar{S}$ and do not divide c. By the Chinese remainder theorem choose $g \equiv 1 \bmod c\mathcal{O}_E$ clearing the finitely many lattice comparisons in both directions. Then g^2 clears the Hom-lattice comparisons, one factor at either endpoint. These three finite quotients are killed by c. Multiplication by g^2 in the two directions induces inverse maps since $g^4 \equiv 1 \bmod c$. By Lemma 12.14, multiply each of the three reference sets by this same g^2 in the rational period module and then embed it into the analytic space. This gives three sets of actual points common to all the operators in (A.61). Lemma 12.13 gives the same ξ on them. If $H_i = \mathcal{T}_\xi e_{A_i,B_i}$, every finite rational relation $\sum_i q_i \mathbf{1}_{D_i} = 0$ consequently gives

$$\sum_i q_i\, p^{\eta(d_i - d_0)} C(A_i, B_i)^{-1} H_i = 0. \tag{A.62}$$

Lean: `GlobalTheory.KerCommon.kerTh_common_op` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.KernelFunctionsCommonLinearOperatorOffSplitPrime`, lines 128–174)
Lean: `GlobalTheory.KerRel.kerTh_relation_of_indicator` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.KernelThetaRelationOfIndicatorRelationSplit`, lines 76–121)

Lemma 12.14 gives algebraic Taylor coefficients for each H_i and algebraicity of the nonzero character factors $C(A_i, B_i)$. Also $p^{\eta(d_i - d_0)}$ is rational. Taking each Taylor coefficient of (A.62) therefore gives an equality in $\overline{\mathbf{Q}}$ to which the same embedding λ applies for every i.
Lean: `GlobalTheory.AlgTranslate.coefIn_translate` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.CoefInTranslate`, lines 53–85)
Lean: `GlobalTheory.kerTh_entire_algCoef` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.KerThEntireAlgCoef`, lines 22–40)

In the fixed coordinates the series is obtained by Taylor expansion and λ, followed by the chart substitution and restriction, from

$$p^{\sum_w \Delta_w (j+1)(s'_{i,w} - s_{i,w})} \frac{\mu_{j+1}(d_{y'_i})}{\mu_{j+1}(d_{y_i})} c(A_i)^{-b_j} c(B_i)^{b_{j+1}}\, C(A_i, B_i)^{-1} H_i. \tag{A.63}$$

The two chart matrices in the substitution are constant on the fibre. At level l, $c(A(y))$ equals $\chi_I(d_y)$ times a constant depending on the fixed rational change to standard coordinates. If the source and target lengths increase by h, h', the determinant idèles change by $\pi^{-h}, \pi^{-h'}$ times integral units supported at the moving pair. Both characters are trivial on those units. The ratio of the scalars preceding $C(A_i, B_i)^{-1} H_i$ is therefore

$$p^{\Delta(h' - h)} M_\pi^{h - h'} C_\pi^{b_j h - b_{j+1} h'} = a^h (a')^{h'}.$$

Set $k_i = a^{s_i}(a')^{s'_i}p^{-\eta d_i}$. Applying the common substitutions to (A.62) now gives $\sum_i q_i k_i^{-1}\Theta_i^F = 0$. Thus $\mathbf{1}_{D_i} \mapsto k_i^{-1}\Theta_i^F$ defines a rational linear map on the span of the fibre indicators. Extend it by zero on a vector-space complement. This is Ξ_ω in (12.26).

Finally let a source group element have chart matrices G_w, and write the changed point as gyk, with g rational unitary and k in the level. Lemma 12.13 gives the multiplier $\mu_{j+1}(\det k)\lambda_j(k)^{-1}$ and the substitution $Z \mapsto Zg$ in standard coordinates. The source idèle denominator in (A.63) cancels $\mu_{j+1}(\det k)$. The principal-ideal law cancels the constant $b_j + n'/2$ in the type at g. The remaining exponents are exactly the weights $w_{w,j+1}(s)$. The lengths change by $-\operatorname{ord}_w \det G_w$. Combining these factors yields (12.27), where $\chi_{\mathrm{lev}}(\gamma)$ is the local coefficient realization of $\lambda_j(k)$. For a stabilizer this formula gives the stabilizer identity. Applying it to any element taking the chosen representative to L recovers the original series at L. A lower neighbour of index i multiplies the basis determinant by a factor of order i, so its signed length decreases by i. The other lengths and ω are unchanged.

Lean: `GlobalTheory.XiGen.xiEx_theta_general` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.XiExThetaGeneral`, lines 86–303)
Lean: `GlobalTheory.TwV.site_law_xiIn` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.KernelSiteLawAndFixedCoordinateStabiliserLaw`, lines 20–48)

□

Proof of 12.17

Proof. For representatives A, B in a component the left contribution of their orbits is $\sum_{\gamma: A \to_h \gamma B} \phi(A, \gamma B)/(w(A)w(B))$, since each γB occurs $w(B)$ times. Inversion sends the finite indexing set to $\gamma^{-1}A \to_h B$. Simultaneous invariance identifies the summands. Summing over orbits and components gives (12.28).

Apply it coefficientwise to $\ker_{j,B,L'}(f(A))$. For each edge $A \to_h B$, the edge calculation in Theorem 12.8 gives $p^{\mathcal{A}_h} H_j(A) \subseteq H_j(B)$, so $f(A) \in KH_j(B)$. For a source polynomial G, put $\rho(\gamma)G = \chi_{\mathrm{lev}}(\gamma)(\gamma G)$, where γG is the level-group action on polynomials in Definition 10.10. Then $f(\gamma A) = \rho(\gamma)f(A)$ and Definition 10.10 gives $\ker_{j,\gamma B,L'}(\rho(\gamma)f(A)) = \ker_{j,B,L'}(f(A))$. Linearity sums the input at B to $(T^{(j)}_{v,h}f)(B)$. This proves (12.29) for (10.1). At the terminal step it is the terminal neighbour clause of Definition 10.10. For the arithmetic class data, y again has one value and x runs over the source representatives with their stabilizer orders. □

Proof of 12.18

Proof. If $xA \not\subseteq B$, both sides of (12.30) vanish. Otherwise let $d = \dim_{\mathbf{F}_q} \ker(\bar{x} : A/pA \to B/pB)$. The lower neighbours correspond to codimension-i subspaces of $(B/pB)/\operatorname{im}\bar{x}$ of dimension $d+g$. Multiplication by p identifies the upper neighbours in the ℓ-term with $(i-\ell)$-dimensional subspaces of $\ker\bar{x}$. Thus the count reduces to

$$\begin{bmatrix} d+g \\ i \end{bmatrix}_q = \sum_{\ell=0}^{\min(i,g)} q^{(i-\ell)(g-\ell)} \begin{bmatrix} g \\ \ell \end{bmatrix}_q \begin{bmatrix} d \\ i-\ell \end{bmatrix}_q . \tag{A.64}$$

To count i-dimensional subspaces of $D \oplus G$, with dimensions d, g, fix the intersection J with G and the image H in D. For $\dim J = \ell$, the subspace modulo J is the graph of a map $H \to G/J$. There are $q^{(i-\ell)(g-\ell)}$ such maps for each J, H.

For the lift take $g = t+1,\ N = N_j,\ \eta = \eta_v,\ \Delta = \Delta_v(j+1)$. A target lower edge of index i decreases s' by i and $d = Ns' - N's$ by Ni. A source upper edge of index h increases s by h and decreases d by $(N+g)h$. Other lengths and ω are fixed by Lemma 12.10. With b, G fixed in Lemma 12.16, the respective multipliers of $J_{L,L'}(b; G)$ are

$$C_A(i) = p^{\Delta i}(a')^{-i}p^{\eta N i}, \qquad C_C(h) = p^{\Delta h}a^h p^{\eta(N+g)h}. \tag{A.65}$$

Lean: `Formalism.MainNotions.FormalGroupSystem.Pf.kernel_target_identity` (`Formalism.SharpRestriction.GrowthBounds.MultisetValuation.IntegralSpanCoordinates`, lines 80–125)
Lean: `Formalism.MainNotions.FormalGroupSystem.Pf.kernel_source_identity` (`Formalism.SharpRestriction.GrowthBounds.MultisetValuation.IntegralSpanCoordinates`, lines 127–173)

Apply $J_{L,L'}(\,\cdot\,; G)$ at the original pair to (12.30) as an identity of indicator sums. For the target sum, multiply by $C_A(i)$ to obtain the actual kernel values. For each source sum, divide the actual kernel values

by $C_C(i-\ell)$ to obtain the evaluation at the original pair. The target neighbour sum of kernels is therefore the sum of the source neighbour sums with coefficients $C_A(i)C_C(i-\ell)^{-1}q^{(i-\ell)(g-\ell)}\begin{bmatrix} g \\ \ell \end{bmatrix}_q$.

Lean: `Formalism.MainNotions.FormalGroupSystem.Pf.ker_dn_sum` (`Formalism.SharpRestriction.GrowthBounds.MultisetValuation.CoupledSumOfEntries`, lines 38–63)

Use $G = f(L_{x,y})$ and sum over the class data with weights c_K/w_x. The target sum gives $T_{v,i}^{(j+1)}K_jf$. By Lemma 12.17, the source neighbour sums act on f to give $K_jT_{v,i-\ell}^{(j)}f$. Theorem 12.8 ensures that $T_{v,h}^{(j)}$ preserves V_j for $h \le N_j$, and the sum is empty for $h > N_j$. The resulting equality descends to K.

Lean: `Formalism.MainNotions.FormalGroupSystem.Pf.hk_point` (`Formalism.SharpRestriction.GrowthBounds.MultisetValuation.HeckeConditionOfAdm`, lines 15–73)

Put $Q = p^\eta$. Then

$$\begin{bmatrix} g \\ \ell \end{bmatrix}_{Q^2} = Q^{\ell(g-\ell)}e_\ell(Q^{g-1}, Q^{g-3}, \dots, Q^{1-g}). \tag{A.66}$$

Indeed the elementary generating polynomial at g is the one at $g-1$ with z replaced by Qz, times $1+Q^{1-g}z$. After the factor $Q^{\ell(g-\ell)}$, this gives the recursion with multipliers $Q^{2\ell}$ and 1 in (A.64) for dimensions $g-1$ and 1. Here $g = t+1$, so the arguments are $\beta_{v,a}$. In the normalized kernel coefficient the exponent of Q is

$$-i(N+g-i) + Ni - (N+g)(i-\ell) + 2(i-\ell)(g-\ell) + \ell(g-\ell) = -(i-\ell)(N-i+\ell).$$

This is the source normalization. The remaining factors are $((aa')^{-1})^i(ap^\Delta)^\ell e_\ell((\beta_{v,a})_a)$. Thus (12.31) holds with $u_{j+1} = (aa')^{-1}$ and $\mathfrak{a}_{j+1} = ap^\Delta$. Both sides are supported on lattice tuples. □

Proof of 12.20

Proof. Fix v and suppress it in subscripts. Take $u_l, \mathfrak{a}_l$ from Lemma 12.18. Set $U_0 = 1$ and $\mathfrak{m}_l = \mathfrak{a}_l/U_{l-1}$. Since the U_l are units, (12.33) follows. Set $\widehat{\mathcal{E}}_i^{(b)} = U_b^{-i}\mathcal{E}_i^{(b)}$ and $\widehat{\mathcal{E}}^{(b)}(z) = \sum_{i\ge0}\widehat{\mathcal{E}}_i^{(b)}z^i$. After normalization the scalar at i,ℓ in (12.31) is $U_{j+1}^{-i}u_{j+1}^i\mathfrak{a}_{j+1}^\ell U_j^{i-\ell} = \mathfrak{m}_{j+1}^\ell$. Thus the jth lift inserts $\prod_{a=0}^t(1+\mathfrak{m}_{j+1}\beta_a z)$ in the generating polynomial. Iterate and multiply by the reciprocal product $\sum_{\ell\ge0}(-1)^\ell h_\ell(\mathcal{F}_b)z^\ell$. The coefficient of z^r gives

$$\left(\sum_{\ell=0}^{r}(-1)^\ell h_\ell(\mathcal{F}_b)\widehat{\mathcal{E}}_{r-\ell}^{(b)}\right)K^{[b]} = K^{[b]}\mathcal{E}_r^{(0)}. \tag{A.67}$$

On V_b the index-zero operator is the identity, so $\widetilde{T}_i^{u,(b)} = p^{N_b'(i)+\eta i(N_b-i)}\widehat{\mathcal{E}}_i^{(b)}$ also holds at zero. Since $\ell(N_b-2r+\ell)+(r-\ell)(N_b-r+\ell) = r(N_b-r)$, the sum on the left of (A.67) multiplied by $p^{N_b'(r)+\eta r(N_b-r)}$ is Z_b. Inserting $\mathcal{E}_r^{(0)} = p^{-\eta r(k-r)-\mathcal{A}_v}y_v$ gives exponent

$$N_b'(r) - \mathcal{A}_v + \eta rb(t+1) = P_v^{\mathbf{Z}}(b).$$

For $1 \le l \le b$, Definition 10.2 gives $w_l(s) \ge w_1(s)$ at even s and $w_l(s) \le w_b(s)$ at odd s. Since $v_p(\beta_a) \ge -\eta t$, every argument of $h_\ell(\mathcal{F}_b)$ has valuation at least $\sum_{s\ \text{even}}w_1(s) - \sum_{s\ \text{odd}}w_b(s) - \eta t$. Each monomial has degree ℓ. Substituting this bound in (12.25) cancels the odd weight sum and gives

$$v_p(c_{b,\ell}) \ge \sum_{s\ \text{even}}\sum_{a=r-\ell+1}^{r}(\lambda_{s,a}+w_1(s)) + \eta\ell(N_b-2r+\ell-t). \tag{A.68}$$

The double sum is nonnegative by dominance and $\lambda_{s,k}+w_1(s) \ge 0$. Also $N_b-2r+\ell-t = (k-2r)+(b-1)t+b+\ell \ge 0$. At $\ell = r$, the term at $a=1$ at each marked even position is at least $\lambda_{s,1}-\lambda_{s,2} = \alpha_{v,s}$. This proves (12.35).

For $b = j+1$ and $\ell < r$, Theorem 12.8 gives the bound $B_v + R_j(v)$ on the Hecke term. Its coefficient is integral and U_b is a unit. At $\ell = r$ the Hecke term is the identity and the scalar adds at least E_v to the bound $B_v - E_v + R_j(v)$ of Theorem 12.7. All terms therefore lie in the lattice of exponent $B_v + R_j(v)$. □

Proofs for Chapter 13: Induction along the chain

Proof of 13.1

Proof. For the empty list, (13.2) is the identity. For $L = (v, L')$, pass the scalar from the induction hypothesis through $Z_{v,b}$ and apply (13.1) to $Y_{L'}F \in K\Lambda_0$. The two powers of p multiply to the power with exponent $\sum_{v\in L} P_v^{\mathbf{Z}}(b)$.

For the bound, expand the product of the $Z_{v,j+1}$. A summand is a scalar a times the ordered product of $\widetilde{T}_{v,i_v,j+1}$ on K_jF, where $0 \le i_v \le r$ on L and $i_v = 0$ elsewhere. The inverse powers of $U_v(j+1)$ are units. By the assumed coefficient bounds, each $v \in L$ with $i_v = 0$ contributes E_v to the bound for a, and the other factors are integral. Thus $\nu_K(a) \ge H$, where $H = \sum_{v\in L,\ i_v=0} E_v$.

Put $q'_v = q_v - E_v$ if $v \in L$ and $i_v = 0$, and $q'_v = q_v$ otherwise. For $v \in L$, the inequality $q_v \le B_v(j+1)$ gives $q'_v \le B_v(j+1) - E_v$ when $i_v = 0$ and $q'_v \le B_v(j+1)$ when $i_v > 0$. Outside L, use $\mathcal{L}_{v,j}(q_v)$. Thus the q'_v satisfy (12.11) of 12.3. Applying the real-exponent argument of Lemma 12.5 to the assumed coefficient estimate and coefficient-to-lattice implication gives the operator bound $\sum_v q'_v - d_j = \sum_v q_v - H - d_j$. Multiplication by a restores H, since

$$\lceil e(x+y)\rceil \le \lceil ex\rceil + \lceil ey\rceil.$$

The target lattice is closed under addition. □

Proof of 13.3

Proof. At $b = 0$ the assertion is $F \in \Lambda_0$. Suppose it holds at $b < m$ for every integral input. Put $C = C_b$ and let D be its complement in L. Write $H_b = \sum_{j<b} \beta_j$. Commutativity gives $Y_L^{b+1} = Y_C Y_L^b Y_D$. Since $Y_D F \in \Lambda_0$, the induction hypothesis gives $K^{[b]}W = \varpi^{\lceil eH_b\rceil}F'$ with $W = Y_L^b Y_D F$ and $F' \in \Lambda_b$. Each y_v preserves Λ_0, so $W \in \Lambda_0$.

Set $N = \sum_{v\in C} P_v^{\mathbf{Z}}(b+1)$. By 13.1 and linearity,

$$K^{[b+1]}Y_L^{b+1}F = p^{-N}Z_{C,b+1}K^{[b+1]}W = p^{-N}\varpi^{\lceil eH_b\rceil}Z_{C,b+1}K_bF'.$$

Use (13.3) with $q_v = G_v^+(b+1)$ on C and $q_v = G_v^0(b+1)$ elsewhere. Put $Q = \sum_v q_v - d_b$. Since p is a unit times ϖ^e, the right-hand side lies in

$$\varpi^{\lceil eH_b\rceil + \lceil eQ\rceil - eN}\Lambda_{b+1}.$$

The exponent is at least $\lceil e(H_b + Q - N)\rceil = \lceil e(H_b + \beta_b)\rceil$, which gives the bound at $b+1$. □

Proof of 13.4

Proof. Take a generator in 13.2 with $C \subseteq V$ and $F_0 \in \Lambda_0$. Enumerate C without repetition and apply 13.3 with

$$C_j = \{v \in C : G_v^0(j+1) \le G_v^+(j+1) - P_v^*(j+1)\}.$$

Let $g = \sum_{j=0}^{m-1}\beta_j$ be the resulting lower bound for $K^{[m]}Y_C^m F_0$. On C_j, the contribution to β_j is at least the maximum in (13.8) because $P_v \le P_v^*$. On $C \setminus C_j$ and outside V, both contributions equal $G_v^0(j+1)$. At each $v \in V \setminus C$, the excess in (13.8) over all layers is the sum of $[G_v^+(b) - P_v^*(b) - G_v^0(b)]_+$ over b. The bound on this sum gives

$$T \le g + \sum_{v\in V\setminus C} S_v. \tag{A.69}$$

Put $n = \sum_{v\in V\setminus C}\lceil eS_v\rceil$. Since $e \ge 0$, (A.69) gives $eT \le \lceil eg\rceil + n$. The right side is integral, hence $\lceil eT\rceil \le \lceil eg\rceil + n$. By 13.3 and linearity, the lifted generator lies in $\varpi^{\lceil eg\rceil+n}\Lambda_m \subseteq \varpi^{\lceil eT\rceil}\Lambda_m$. Linearity extends the inclusion to the O-span. □

Proof of 13.5

Proof. The summands of S are nonnegative. Dominance gives $\alpha_s \ge 0$. The lower bound on $w(1,s)$ and monotonicity in j give $\lambda_{s,i} + w(j,s) \ge 0$ for $1 \le i \le k_1+1$ and $1 \le j \le m$. Thus the capacities in 10.3 are nonnegative. The positive parts in 11.1 and the minima in 11.2 are nonnegative. With δ_j the digit loss of 10.4, this gives

$$g_A(j) \ge -\delta_j, \qquad g_B(j) \le \gamma_p \sum_{s\in T} \alpha_s - \delta_j.$$

The inner sum in the price formula of 12.19 has r terms, so

$$P(j) = \sum_{s \text{ odd}} \sum_{i=k_1+2-r}^{k_1+1} (w(j,s) + \lambda_{s,i}) + \frac{n' r j (t+1)}{2} \ge 0.$$

Hence $[g_B(j) - P(j) - g_A(j)]_+ \le \gamma_p \sum_{s\in T} \alpha_s$. Sum over $1 \le j \le m$. □

Proof of 13.8

Proof. For each $v \in \mathcal{P}$, Definition 10.1 supplies a DVR W_v with maximal ideal (p) and a finite ring map $j_v : W_v \to O$. Its kernel is contained in (p), and $W_v/\ker j_v \subseteq O$ is integral. The lying-over theorem of Cohen and Seidenberg says that every prime of the smaller ring in an integral extension is the contraction of a prime of the larger ring [63]. Applied to the image of (p) in $W_v/\ker j_v$, it gives a prime of O containing p. Thus p is a nonunit, and $(p) = (\varpi^e)$ forces $e > 0$.

The kernel field K_c contains $\operatorname{Frac} W(\overline{\mathbf{F}}_p)$, where W denotes Witt vectors, so $p \ne 0$ in K_c and in K. For $e > 0$, the map ι is injective, since a nonzero kernel would contain a power of ϖ and therefore kill p. The valuation ring $O_c \subseteq K_c$ contains the image of z or z^{-1} for every $z \in K^\times$. By Definition 10.6, its inverse image in K is $\iota(O)$. Thus each $z \in K^\times$ belongs to $\iota(O)$ or is the inverse of a nonzero element of $\iota(O)$, and $K = \operatorname{Frac}(O)$ via ι.

The cited results give the first nine assertions. In Theorem 12.20, the required inclusion $K_j(K\Lambda_j) \subseteq K\Lambda_{j+1}$ for $0 \le j < m$ is part of Definition 10.10.

By Definition 10.10, the input operators y_v commute pairwise on M_0, and each maps Λ_0 into itself. For $e > 0$, apply 13.4 with the sequences from 12.20. Use the joint coefficient estimate of 12.3 and the coefficient-to-lattice implication of Lemma 12.4.

If $e = 0$, then $\mathcal{P} = \varnothing$ and $I(S,\varnothing) = \Lambda_0$. Since $\lceil eh \rceil = 0$ for every h, the basic lift bound of 12.6 gives $K_j\Lambda_j \subseteq \Lambda_{j+1}$ at each step. Thus $K^{[m]}\Lambda_0 \subseteq \Lambda_m$, which is (13.7) at $e = 0$.

The local bounds of 12.1 and the external bound (13.7) give 13.6. Lemma 13.5 gives the asserted upper bound for the enhancement budget. Apply the system assertion at $p = \operatorname{char} k(w)$ to the given system at (u,w). □

Proofs for Chapter 14: Global Frobenius-orbit parameters

Proof of 14.7

Proof. Permission gives the inert class and $n_p = \operatorname{ord}(h)$. The map β_h is constant on each h-coset and takes its value there. Its fibre over t_0 is therefore the image of ι_{t_0} in Definition 14.5. Write $w_j(i)$ for the local shift, $c_{\mathrm{loc}} \le c$ for the local row bound, and $\lambda_{i,\ell}$ for the base weights, numbered from one. The filling identification of Definition 14.5 gives the branch rank r and

$$z_e(\iota(i)) = (0, \iota((i+e) \bmod n_p)), \qquad w_j(i) = B_j(0, \iota(i)). \tag{A.70}$$

Local marks correspond to $T(t_0)$ and odd marks to $T(t_0) \setminus T^V(t_0)$. By 12.19,

$$P_v(j) = \sum_{\substack{0 \le i < n_p \\ i \text{ odd}}} \left(r w_j(i) + \sum_{\ell=k-r+1}^{k} \lambda_{i,\ell} \right) + \frac{n_p r j (t+1)}{2}. \tag{A.71}$$

For $r = 1$, the marked summand at $s = \iota(i)$ is $B_j(0, s) - b_s$. At unmarked positions $w_j(i) \le w'' + \eta$ by 14.2 and $\lambda_{i,k} \le c$ by 10.2. Since $j(t+1)/2 \le N_m$, this gives the large-prime bound.

In (14.5), $x^\theta(\log x)\Lambda/k \ge 0$, so $4R \le x^{-2} \le 1$; together with (14.1) this gives (14.6). For $r = r_G < k$, the last r rows omit the first and each has weight at most c_{loc}. At a mark, $w_j(i) \le b_s + b'/x + \eta \le 2b' + \eta$ by (14.6) and $x \ge 2$. The unmarked shift is at most $w'' + \eta$. Thus (A.71) is at most $r_G(2N_v(t_0)b' + n_p M_{\mathrm{Pr}})$, which gives the small-prime bound. □

Proof of 14.12

Proof. Initial-column minors are invariant under the specified column operations by alternation. Row substitutions commute with these operations, so $D_\lambda \subseteq A_\lambda$.

Order monomials lexicographically by column and then by row, with earlier variables first. For a nonzero $P \in A_\lambda$, let m_{ij} be the exponents of its leading monomial. Homogeneity gives $\sum_i m_{ij} = \nu_j$. For adjacent columns $j, j+1$, write $x_i = Y_{i,j}$, $y_i = Y_{i,j+1}$. Retain the component with the leading monomial's total degree in each pair (x_i, y_i) and its degrees in each other variable. This component is invariant under $y \mapsto y + tx$, which preserves those degrees.

In that component successively take the coefficient of $x_a^K y_a^e$ in row a, where K is the largest exponent of x_a and $K + e$ is the fixed pair degree. At this step the polynomial has the form $\sum_{i=0}^{K} x_a^i y_a^{e+K-i} P_i$, with $P_K \ne 0$ and P_i independent of x_a, y_a. After $y \mapsto y + tx$, factor out $(y_a + tx_a)^e$. If the transformed polynomial has t-degree at most s, then $e \le s$. The remaining factor has t-degree at most $s - e$ and y_a-degree at most K. Evaluating it at $x_a = 1$, $y_a = -t$ gives the transformed P_K, of t-degree at most $s - e + K$. The initial bound is $s = 0$. The successive pairs (K, e) are $(m_{a,j}, m_{a,j+1})$, since the degrees outside the two columns are fixed. Iterating the degree bound gives

$$\sum_{i \le a} m_{i,j+1} \le \sum_{i<a} m_{i,j}.$$

Make a tableau of shape ν by placing m_{ij} copies of i in row j in increasing order. The inequality ensures strict increase down each column. For each tableau column take the minor on its indicated rows and the first matrix columns. Its diagonal term is leading with coefficient one, so the product has the same leading monomial as P. Subtract the leading coefficient of P times this product and induct on the leading monomial. This gives spanning over O, and the distinct leading monomials give independence.

The resulting basis is the straight-shape case of the standard basis of Akin, Buchsbaum and Weyman [60, Theorem II.2.16], with shape ν^{t} in their notation, where a single row gives an exterior power. The normalized pairing of 10.17 gives D_λ the same rank as this Schur lattice. Thus $V = FA_\lambda = FD_\lambda$.

Lean: `Formalism.GradedWeylDivisibility.leadInterlaced` (`Formalism.GradedWeylDivisibility.FinalC`, lines 36–36)
Lean: `Formalism.GradedWeylDivisibility.semistandard_basis` (`Formalism.GradedWeylDivisibility.FockSpaceTableaux.SSBasis`, lines 10–67)
Lean: `Formalism.L2.G.Model.sw_weyl_eq` (`Formalism.LocalToGlobal.GroupLattices.FockModels.SwRowGenMonomial`, lines 177–198)

For $i \ne j$ let e_{ij} be the first coefficient of the substitution adding row i to row j, and let e_{ii} be the degree operator in row i. The coefficient of order a is $e_{ij}^{(a)} = e_{ij}^a/a!$. These operators preserve A_λ by column invariance and D_λ by its generating words. Each tableau product uses l minors. A given row occurs at most once in each minor. Its vector of row degrees therefore has coordinates in $[0, l]$. Row weight ν occurs on the line Fh_λ. Row 0 must occur in every minor and row 1 in every minor of size at least two. Each subsequent row i must occur in every minor of size at least $i + 1$. Let ξ project to row weight ν and evaluate at the identity matrix. Then $\xi(A_\lambda) \subseteq O$ and $\xi(h_\lambda) = 1$.

Put $\mathscr{H} = \{T \in \mathrm{End}_F(V) : T(A_\lambda) \subseteq D_\lambda\}$ and $u(y) = \xi(y)h_\lambda$. Then $u \in \mathscr{H}$ has weight zero and trace one. On endomorphisms use $E_{ij}T = e_{ij}T - Te_{ij}$. The off-diagonal divided powers preserve $\mathscr{H}$ by

$$E_{ij}^{(a)}T = \sum_{b=0}^{a} (-1)^b e_{ij}^{(a-b)} T e_{ij}^{(b)}.$$

Endomorphism weights are differences of row weights of V, so each coordinate lies in

$$[-l, l]. \tag{A.72}$$

For a row index $0 \le c' < k$ and $S \subseteq \{0, \dots, c'-1\}$, define operators on $\mathrm{End}_F(V)$ by

$$W_0(S) = 1, \qquad W_{q+1}(S) = \sum_{i \in S} E_{c'i} W_q(S) E_{ic'}.$$

For $S_{c'} = \{0, \dots, c'-1\}$ put

$$P_{c'} = \sum_{q=0}^{l} b_q W_q(S_{c'}), \qquad b_q = \frac{(-1)^q c'!}{(q+c')! q!}.$$

Suppose z has weight zero and $E_{ab} z = 0$ for $a, b < c'$. Write $W_q = W_q(S_{c'})$. Each W_q preserves weight and commutes with E_{ab} for $a, b < c'$. In the recursion the commutators $[E_{ab}, E_{c'i}] = -\delta_{ia} E_{c'b}$ and $[E_{ab}, E_{ic'}] = \delta_{bi} E_{ac'}$ cancel by commutation with W_q at the preceding step.

For $j < c'$, let y be a product of s operators $E_{ic'}$ with $i < c'$ applied to z. Then

$$E_{c'c'} y = -sy, \qquad \sum_{i<c'} E_{ji} E_{ic'} y = (s + c') E_{jc'} y. \tag{A.73}$$

For the second identity at $s = 0$, commute E_{ji} to the right and use $E_{ji} z = 0$. Inserting $E_{bc'}$ adds one copy of $E_{jc'} E_{bc'} y$, since the column operators commute and $[E_{ji}, E_{bc'}] = \delta_{ib} E_{jc'}$. Commuting $E_{jc'}$ through the recursion gives

$$E_{jc'} W_q y = W_q E_{jc'} y + q(2s + c' + q) W_{q-1} E_{jc'} y, \tag{A.74}$$

with zero second term at $q = 0$. In the step from q to $q+1$, the recursion contributes $q(2s+2+c'+q)$. The commutator $[E_{jc'}, E_{c'i}] = E_{ji} - \delta_{ij} E_{c'c'}$ contributes $(s+c')+(s+1)$ by (A.73). The sum is $(q+1)(2s+c'+q+1)$.

At $s = 0$ the terms of $E_{jc'} P_{c'} z$ cancel because $b_{q+1}(q+1)(c'+q+1) = -b_q$. The terminal term $W_l E_{jc'} z$ vanishes by (A.72), since its rightmost $l+1$ column operators would give coordinate $-l-1$ at c'. For $v = P_{c'} z$, put $e = E_{jc'}$ and $f_- = E_{c'j}$. Then $ev = [e, f_-]v = 0$, and

$$e f_-^a v = -a(a-1) f_-^{a-1} v \qquad (a \ge 1).$$

If $f_- v \neq 0$, take the last nonzero $f_-^a v$, which exists by (A.72). The identity at $a+1$ gives a contradiction. Thus the new row operators also kill v, and its diagonal operators vanish by weight zero. For $q > 0$, each summand of $W_q z$ has trace zero by its outer commutator $E_{c'i}$. Hence $P_{c'}$ preserves trace.

Lean: `Formalism.MainNotions.G.GLWeightTheory.raising_casimirPower_commutator` (`Formalism.SharpRestriction.GrowthBounds.RootWeights.CommutatorTripleProductLeibniz`, lines 159–200)
Lean: `Formalism.MainNotions.G.GLWeightTheory.extremal_projector_properties` (`Formalism.SharpRestriction.GrowthBounds.RootWeights.CommutatorTripleProductLeibniz`, lines 259–302)

Put $d = \lfloor \log_p(l + k - 1) \rfloor$ and $C_{q,S} = c'! W_q(S)/((q+c')! q!)$. The operations $U \mapsto E_{c'i} U E_{ic'}$ commute for distinct $i < c'$. Expand the q-th power after adjoining $i \notin S$ and convert both outside powers to divided powers. This gives

$$C_{q,S\cup\{i\}} = \sum_{a=0}^{q} \binom{q+c'}{a}^{-1} E^{(a)}_{c'i} C_{q-a,S} E^{(a)}_{ic'}. \tag{A.75}$$

The scalar is $\binom{q}{a}(a!)^2(q-a+c')!(q-a)!/((q+c')!q!)$, which equals $\binom{q+c'}{a}^{-1}$. For $0 \le a \le N = q + c'$ and $0 \le q \le l$,

$$v_p\binom{N}{a} = \sum_{j \ge 1} \left(\left\lfloor \frac{N}{p^j} \right\rfloor - \left\lfloor \frac{a}{p^j} \right\rfloor - \left\lfloor \frac{N-a}{p^j} \right\rfloor \right) \le d. \tag{A.76}$$

Each term is zero or one and vanishes for $p^j > N$. The prime-to-p part of the binomial coefficient is a unit in O, so $p^d / \binom{N}{a} \in O$. For $S = \varnothing$, $C_{0,S} = 1$ and $C_{q,S} = 0$ for $q > 0$. Induction on $|S|$ in (A.75), using divided-power stability, gives

$$p^{|S|d} C_{q,S} \mathscr{H} \subseteq \mathscr{H} \quad (0 \le q \le l), \qquad p^{c'd} P_{c'} \mathscr{H} \subseteq \mathscr{H}. \tag{A.77}$$

Lean: `Formalism.MainNotions.G.GLWeightTheory.casimirPowerScaled_insert_dividedPower` (`Formalism.SharpRestriction.GrowthBounds.RootWeights.CasimirPowerScaledInsertDividedPower`, lines 86–98)
Lean: `Formalism.MainNotions.G.GLWeightTheory.extremalProjector_denominator_bound` (`Formalism.SharpRestriction.GrowthBounds.RootWeights.CasimirPowerScaledInsertDividedPower`, lines 217–234)

Apply $P_0, \dots, P_{k-1}$ to u in order. The result commutes with every e_{ij}. It acts by a scalar on Fh_λ and by the same scalar on its divided-power words. These span V, so trace one makes the result $1_V/\dim_F V$. By (A.77), multiplication by $p^{d\sum_{c'=0}^{k-1} c'}$ puts it in $\mathscr{H}$. Multiplying by $\dim_F V$ gives $p^{d_p(\lambda)}1_V \in \mathscr{H}$, which proves (14.15).

Lean: `Formalism.MainNotions.G.GLWeightTheory.restA_of_linf_integrality` (`Formalism.SharpRestriction.GrowthBounds.RootWeights.LinfAdjointLeEllWtHighest`, lines 51–87)
Lean: `Formalism.L2.G.Model.sandwich0` (`Formalism.LocalToGlobal.GroupLattices.FockModels.Sandwich`, lines 125–144)

For a frame matrix g, use $Y \mapsto g^{-1}Y$ and the twist $(\det g)^{-\lambda_{k-1}}$ as in 10.17. Both commute with scalar multiplication. The Weyl lattice is also independent of the frame. Over the DVR, integral invertible matrices are generated by elementary shears and diagonal units. The shears preserve D_λ by divided-power stability, and diagonal units preserve its row-homogeneous generators. Thus the framed inclusions apply to the intersections defining the two factors. Negative reversal at an odd position preserves l.

Use $R = 2n\delta_1 x^\varepsilon \Lambda/k$ as in 14.7. The parameter choice gives, for $s \in TT$,

$$0 < 4S \le \Lambda b'/k, \qquad 4R + C_\rho(\log x)\Lambda/k \le x^{-2}, \qquad b_s \le (1+R)b',$$

where $C_\rho > 0$. The first inequality gives $\Lambda > 0$, so the second gives $R \le x^{-2}/4 \le 1/4$. Hence at a top position the spread $a_s + b_s$ is bounded using (14.1) and $b_s \le 5b'/4$.

At an off-top embedding τ, let (u_i) and (v_i) be the upper and lower interlacing rows of its Harish–Chandra parameter. The rank shift gives $l = u_0 - u_{k-1} - (k-1)$. The cloud and extreme-coordinate clauses of Definition 14.11 give $u_0 - v_0 \le 2S$, $v_i - v_{i+1} = 4\Delta$ (telescoping to $v_0 - v_{k-2} = 4(k-2)\Delta < 4k\Delta = S$), and $u_{k-1} = v_{k-2} - 1/2$, hence $u_0 - u_{k-1} = (u_0 - v_0) + (v_0 - v_{k-2}) + (v_{k-2} - u_{k-1}) \le 2S + 4(k-2)\Delta + 1/2 < 3S + 1/2$. Thus

$$l \le \begin{cases} \frac{6}{5}a_* + \frac{5}{4}b', & \text{at a top position,} \\ 3S + \frac{1}{2} - (k-1), & \text{at an off-top position,} \end{cases} \qquad l \le 3a_*, \tag{A.78}$$

where $kS \le w'' \le b' \le a_*$ and $k \ge 5$. Since $\Delta \ge N_m m^{10}$ and $S = 4k\Delta$, we have $S \ge 1$ and $k \le kS \le w'' \le a_*$. Hence $l + k - 1 \le 4a_* - 1 \le a_*^2$, and

$$d_p(\lambda) \le \binom{k}{2} 2\log_p a_* \le k^2 \log_p a_*. \tag{A.79}$$

Lean: `Formalism.BasicNotions.ParameterChoice'.bT_le'` (`Formalism.LocalRingCardinality.ColumnWeightSpreadOffTop`, lines 85–102)
Lean: `Formalism.BasicNotions.ParameterChoice'.k_le_a` (`Formalism.LocalRingCardinality.ColumnWeightSpreadOffTop`, lines 77–81)
Lean: `Formalism.BasicNotions.columnWeight_spread_le` (`Formalism.LocalRingCardinality.ColumnWeightSpreadOffTop`, lines 182–194)
Lean: `Formalism.BasicNotions.gat_ellWt_lamAt_le` (`Formalism.LocalRingCardinality.ColumnWeightSpreadOffTop`, lines 241–246)

At a full lattice tuple the fibres are spanned by products in disjoint variables over the pairs (v, s). Apply (14.15) to each factor. By (A.79) the sum of the exponents over the f pairs is at most $fk^2 \log_p a_* \le e_p$. The remaining integral power of p preserves the Weyl fibre. The inclusion $D_\lambda \subseteq A_\lambda$ gives the reverse fibre inclusion. Both function lattices use the support and equivariance conditions of 10.2, so the fibre inclusions give (14.16) pointwise. □

Proof of 14.13

Proof. By 10.3, at a mark $\alpha_i = a_{\iota(i)} - \lambda_{i,2}$ and $C_i = w_j(i) + \lambda_{i,2}$. Elsewhere $\alpha_i = 0$ and $C_i = w_j(i) + \lambda_{i,1}$. These row weights have absolute value at most $c_{\mathrm{loc}} \le c$, and $w_j(i) \ge w_1(i) \ge c_{\mathrm{loc}}$ by 10.2. Thus (A.70) gives, with local indices modulo n_p,

$$\bar\alpha_{\iota(i)} \le \alpha_i \quad (i \text{ marked}), \qquad 0 \le C_{i+e} \le \bar C_{\iota(i),e} \quad (e \ge 0). \tag{A.80}$$

Even marks correspond to $T^V(t_0)$. The local exclusions D_i of 11.1 are the least positive distances to opposite-parity marks. The cycle bijection and (A.70) give

$$e \in D_i, \quad e \text{ odd} \quad \Longrightarrow \quad e \in \mathcal{E}_p(\iota(i)). \tag{A.81}$$

Put $A = 2a_*$ and $z = \log_p A = z_p$. The rank conditions give $m \le k$, $t \ge 1$, and $fk^2 \ge 3$, so $N_m \le k(t+2) \le 3kt \le t^3 fk^3$. By (14.5),

$$N_m \le b \le a_*, \qquad c \le 4w''/5, \qquad \eta \le w''/2. \tag{A.82}$$

At a mark of the represented first-sheet orbit the shift is at most $5b'/4 + b'/x + \eta \le 11b'/4$ by 14.2 and (14.6). At the other positions of that orbit it is at most $w'' + \eta \le 3b'/2$. After removing the single-box columns, each row has at most $w_j(i) + c_{\mathrm{loc}}$ boxes. By (14.1), (A.82), and $b' \le a_*$,

$$\alpha_i \le A, \qquad L_i \le N_j(w_j(i) + c_{\mathrm{loc}}) \le a_*(4a_*) = A^2, \qquad N_{j-1}, N_j \le N_m. \tag{A.83}$$

The digit lengths in 10.4 satisfy $\ell_p(\alpha_i) \le 1 + z \le 2z$ and $\ell_p(L_i) \le 1 + 2z \le 3z$. Hence

$$\delta_v \le n_p(2N_m z + 3N_m^2 z) + 4N_v(t_0)z^2 \le 5n_p N_m^2 z + 4N_v(t_0)z^2 \le \bar{\delta}(t_0). \tag{A.84}$$

By (A.80) and (14.2), the positive-part series are summable and each global summand is bounded by its local counterpart. By (A.81) each retained ordinary summand is retained locally. For the enhanced gain, the capacities in (A.80) permit (14.3). Compare its positive parts at odd marks with those of 11.2, and at even marks use $\gamma_p \bar{\alpha}_{\iota(i)} \le \gamma_p \alpha_i$. Subtract the errors using (A.84). Proposition 14.7 gives the price bound. □

Proof of 14.14

Proof. The logarithm bound gives $p > 2a_*$. The rank and parameter conditions give $m \ge 2$, $m + 3 \le k$, $N_m m^{10} \le \Delta$, and $S = 4k\Delta$. In particular,

$$S \ge 100, \qquad \kappa_j \le N_m \le S, \qquad kS \le w'' \le b'/x \le b' \le a_*/2. \tag{A.85}$$

The fillings are nonnegative by 14.2 and (14.1), so every capacity is at least $4S - \kappa_j \ge 3S$. Since $a_* \ge 2kS \ge 2c$ and $z_p \ge 0$,

$$0 \le \bar{\alpha}_s \le 6a_*/5 \quad (s \in T(t_0)), \qquad \bar{C}_{s,e} \ge 1, \qquad \bar{\delta}(t_0) \ge 0. \tag{A.86}$$

For $e \ge 1$, $\bar{\alpha}_s/p^e \le 6a_*/(5p) < 1$, so $g_0 = -\bar{\delta} \le 0$. The minima in 14.4 are nonnegative. As $a_* < p/2$ and $3p^2/5 \le p^2 - 1$ for $p \ge 2$,

$$g_U(t_0, j) \le \gamma_p \sum_{s \in T(t_0)} \bar{\alpha}_s \le N_v(t_0). \tag{A.87}$$

The coset has at most n_p elements by 14.1. At a mark $B_j(0,s) - b_s \ge b'/x - S \ge 0$, and $K_{\mathrm{Pr}}, M_{\mathrm{Pr}} \ge 1$. Thus the large-prime branch of (14.4) is at least $n_p \ge N_v(t_0)$. The small-prime branch is at least $N_v(t_0)$ by $r_G \ge 1$ and $2b' + c \ge 1$. This proves (14.18), and (14.9) gives zero counted gain. □

Proof of 14.16

Proof. At step $j-1$ and position v, the ordinary and zero bounds give the coefficient estimate of 10.19 with exponent $\max\{g_{A,v}(j), 0\}$, hence with every smaller exponent q.
Lean: `Formalism.MainNotions.FormalGroupSystem.lowAt_max` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 718–725)

Put $B = v(V')$. In 13.4 set the gain and price inputs G^+, P^* to zero off B. If $z_p \ge 1$, use $(G^+, P^*) = (g_U, \Pi_p)$ on B and the ordinary input $G^0 = [g_0]_+$ on $v(\mathcal{R}_p)$, with the corresponding orbit and layer arguments. By (14.17), these meet the gain and price bounds and $G^0 \le \max\{g_{A,v}, 0\}$. If $z_p < 1$, use $(G^+, P^*) = (\min\{g_{B,v}, P_v\}, P_v)$ on B and $G^0 = 0$ on $v(\mathcal{R}_p)$. The gain and price bounds hold, and the represented positions in (14.20) contribute zero by (14.18). Off $v(\mathcal{R}_p)$ take $G^0(v,j) = F_n(v,j)$ in both cases. This gives the required coefficient bound at every position.

Set $S(v) = \sum_{j=1}^{m} [G^+(v,j) - P^*(v,j) - G^0(v,j)]_+$. Injectivity on $\mathcal{R}_p$ makes $y_{V'}F$ a generator of $I(S,B)$ in 13.2, with all positions of B used. Theorem 13.4 gives the layer contribution $\max\{G^+ - P^*, G^0\}$ on B and G^0 elsewhere. On $v(\mathcal{R}_p)$, splitting $\mathcal{R}_p \setminus V'$ into $V_s \setminus V'$ and $\mathcal{R}_p \setminus V_s$ gives the three sums of (14.20). The complementary positions and the stabilizer subtraction give at least low by (14.19). Multiplication by $e_{\mathcal{O}}$ and taking ceilings preserve the bound. □

Proof of 14.17

Proof. Fix the twist, the comparison and the permitted place. The first clause gives a system on precisely this filling, with the chosen levels and chain. It gives coefficient bounds on every unrepresented position and their sum satisfies (14.19) after subtraction of the stabilizer losses. Apply 13.8 to this admissible system and its kernel. It supplies the ordinary coefficient bound and the conditional chain inequality. The zero bound at every position is part of the admissibility conditions of 10.10. The coefficient ring embeds in its fraction field. These are the local hypotheses of 14.16. The two numerical hypotheses in the statement cover its two ranges of p, so that theorem gives the asserted inequality. □

Proofs for Chapter 15: What each layer records

Proof of 15.2

Proof. By 14.1, the period is $\operatorname{ord}(h_p)$ off $\mathcal{S}$ and $2\operatorname{ord}(2h_p)$ on $\mathcal{S}$, where a return takes an even step. In the filling of 14.2 put $b' = xb/X'$. At a top position t, the weight is at least $b_t + b'/x$, with $b_t \geq b' > 0$. The lower bounds at thick and thin positions are b and $b'/(x|A|)$, respectively. Since $k \geq m + 3$, each orbit capacity $(k-1)w_j(z)$ is nonnegative. The feasible set is nonempty because it contains zero. Its defining inequalities are closed and $0 \leq x_{s,i} \leq p^{-i}d_s$ bounds each of its finitely many coordinates. The continuous total flow therefore attains its maximum, including when the index set is empty. Replacing z by $P_p^a z$ replaces d_s by d_{s+a}. The permutation $x'_{s,i} = x_{s+a,i}$ carries the old feasible set onto the new one. It preserves total flow, and the permutation of the capacities preserves their sum. □

Proof of 15.3

Proof. Write Lx for the vector of loads in 15.1 and set $|x| = \sum_{s,i} x_{s,i}$. Let M be the maximum at capacity d, attained by Lemma 15.2. Apply finite-dimensional linear programming duality to $Lx \leq d$, $-x \leq 0$, with objective $\mathbf{1}^{\mathsf{T}}x$. The primal is feasible and has finite maximum M. The dual has nonnegative multipliers $\tilde{y}, z$ with $L^{\mathsf{T}}\tilde{y} - z = \mathbf{1}$ and minimum $d \cdot \tilde{y} = M$ [70, Cor. 7.1g]. Eliminating z gives precisely $L^{\mathsf{T}}\tilde{y} \geq \mathbf{1}$. Set $y_s = \min(1, \tilde{y}_s)$. Each edge inequality involving a capped coordinate retains a summand at least one. Thus y is feasible, and $d \cdot y \leq M$ since $d \geq 0$. For any feasible x, y,

$$d \cdot y - |x| = \sum_s y_s(d_s - (Lx)_s) + \sum_{s,i} x_{s,i}(p^i y_s + y_{s+i} - 1) \geq 0.$$

This proves equality of the optima. At equality every summand vanishes. If $y_s \geq 2/p$, capacity at s is tight and $p^i y_s \geq 2$ on each outgoing edge. Hence $d_s = \sum_i x_{s-i,i}$. Summing over such vertices gives at most $|x|$, and summing all loads gives $p|x| \leq \sum_s d_s$.

For the comparison, take a dual minimizer y at W and set $S = \sum_s w_s$. Then $F(W) = \sum_s W_s(\gamma - y_s)$. With $(a)_+ = \max(a, 0)$, testing y at $W + e$ gives

$$F(W + e) \geq F(W) - \sum_s e_s(y_s - \gamma)_+ \geq F(W) - 8S/p.$$

Indeed, $(y_s - \gamma)_+ \leq 1/p$ where $y_s < 2/p$, since $\gamma \geq 1/p$. On the other vertices it is at most one, and the demand estimate gives a total w_s of at most S/p there. The capacities D are nonnegative, and every flow at D is feasible at $W + e$. Hence

$$F(W + e) \leq F(D) + 2\gamma \sum_{s \in T} w_s \leq F(D) + 3S/p,$$

since $2p^2 \leq 3(p^2 - 1)$. □

Proof of 15.4

Proof. Solve $L(g) = Z$ by total degree from

$$g_i = Z_i - \sum_{h\ge 1} p^{-h} g_{P^h i}^{p^{2h}}.$$

In each degree the sum uses finitely many h and smaller-degree coefficients. The identity linear part gives a two-sided inverse. Inductively the contribution at h has weight at least $p^{2h}\rho - h \ge \rho$, since $(p^{2h}-1)/(p^2-1) \ge h$.

Use the shift in each copy of the letters. For either coboundary $B = \Delta_X\beta, \Delta_Y\beta$, put $V = (\exp B) \circ L$. The integral series $V - 1$ has zero constant term, so $(V-1)\circ g = \exp B - 1$ has weight at least ρ in each term. Splitting a letter commutes with simultaneous torus scaling on its copies. Hence an exponent n with nonzero coefficient in $\exp B - 1$ satisfies

$$p^r - 1 \;\Big|\; \sum_i n_i (-1)^{s_i} p^{s_i},$$

where s_i is the letter of variable i. Because r is even, reduction modulo $p+1$ gives $(p+1) \mid |n|$. The coefficients of $\exp B$ lie in K_0, so the positive weights of $\exp B - 1$ lie in $q_*\mathbf{Z}$ and are at least q_*. In the logarithm of $\exp B$, the part of order $h \ge 1$ has weight at least $hq_* - v_p(h) \ge q_*$, since $h \ge p^{v_p(h)} \ge 1 + (p-1)v_p(h)$. Thus every term of B has weight at least q_*.

Split a parity block of a monomial cZ^n between two copies so that each has positive total degree. Assign degree k_i of each variable to the first copy. Its coboundary coefficient is $c\prod_i \binom{n_i}{k_i}$ since the copy exponents recover n and the subtracted series have zero split coefficient. If the block uses at least two variables, separating whole variables gives factor one. For a single exponent m with base-p digits m_h, use

$$(1+T)^m \equiv \prod_h (1+T^{p^h})^{m_h} \pmod p.$$

If m differs from every power of p, choose $m_h \ne 0$ with $p^h < m$. The factor $\binom{m}{p^h}$ has residue m_h. Both splits preserve degree and valuation, so the monomial has weight at least q_*.

In the remaining case each block consists of one variable to a power of p. The bidegree condition gives a power p^h with $h > 0$. Splitting off p^{h-1} costs one in valuation because $\binom{p^h}{p^{h-1}} = p\binom{p^h-1}{p^{h-1}-1}$ and the second factor is a unit by the digit expansion. This gives weight at least $q_* - 1$. For the pair on opposite-parity letters s, t, torus invariance gives $p^{s+u} \equiv p^{t+v} \pmod{p^r-1}$. The residues of $1, p, \dots, p^{r-1}$ are distinct and $p^r \equiv 1$, so $s + u \equiv t + v \pmod r$. In particular $u - v$ is odd. □

Proof of 15.5

Proof. Normalize $v_p(p) = 1$ and put $q_* = 1/(p-1)$. Let $g = \ell^{-1}$ in the matrix coordinates. Then

$$\Theta(\xi Z) = (U \circ g)(Z) \exp \Phi(Z), \qquad \Phi(Z) = \epsilon \sum_{a,b} \beta(Z_{\bullet,ab}).$$

Use the base weight $v_p(c) + \rho|n|$ of 15.4 for cZ^n.

Index rows and matrix entries from zero. Let $Y = (Y_{t,k,a})$ be independent variables and equip $\mathbf{C}_p(Y)$ with the Gauss norm of 11.14. Mark the letters of T and call all rows except their first rows ordinary. At target t take labels (s, j) with j a signed odd integer, $s = t - j \in T$, and

$$j > 0 \implies p^j \le A_s, \qquad j < 0 \implies p^{-j} \le |\nu_t|.$$

Keep distinct integers j even when they agree modulo r. Since $|j| \le \log_p a$, there are at most $2\lfloor \log_p a \rfloor + 1$ labels at t. Enumerate them from zero in increasing order of j. Label i uses row $e_i = i + \mathbf{1}_T(t) < N_0$ by the rank hypothesis.

Choose an integer $M \geq 0$ at least all the $|j|$ in the lists. Let $y_{t,k}$ have coordinates $Y_{t,k,a}$ except on marked first rows, where $y_{t,0,a} = Y_{t,0,a}^{p^M}$. At target t write (s_i, j_i) for label i. Put $L_i = p^{M+j_i}$ and set

$$\lambda_i(x) = \sum_a Y_{s_i,0,a}^{L_i} x_a, \qquad G_{ih} = \lambda_i(y_{t,e_h}).$$

Reduce the coefficients of $\det G$ modulo the maximal ideal of $\mathcal{O}_{\mathbf{C}_p}$, then specialize the target variables $Y_{t,k,a}$ to δ_{ka}. Since j_i is odd and r even, $s_i \neq t$, so the specialization gives $\det(Y_{s_i,0,e_h}^{L_i})_{i,h}$. A permutation monomial records the pair (s_i, L_i) chosen at each distinct coordinate e_h. These pairs are distinct, so the diagonal monomial has coefficient one. Thus $\det G$ has Gauss norm one, also for the empty matrix, and the adjugate gives an integral inverse.

Lean: `Formalism.DworkPowerSeriesWeights.labelsExist_statement` (`Formalism.GaussJetBounds.GaussJetProofs.Labels`, lines 57–127)

Lean: `Formalism.DworkPowerSeriesWeights.generic_minor_norm` (`Formalism.GaussJetBounds.GaussJetProofs.Generic`, lines 78–123)

Set $v_i = \sum_h (G^{-1})_{hi} y_{t,e_h}$, so $\lambda_j(v_i) = \delta_{ji}$. Keep each marked first row. Set $x_{t,e_i} = v_i$ and use $x_{t,k} = y_{t,k} - \sum_i \lambda_i(y_{t,k}) v_i$ on unreserved ordinary rows. Thus $\lambda_i(x_{t,k}) = \delta_{e_i k}$ on ordinary rows. Recover the reserved rows by $y_{t,e_h} = \sum_i G_{ih} v_i$ and the others by adding back the subtracted sums. Write these integral relations as $y_{t,h} = \sum_k C_{hk}^t x_{t,k}$. At a marked target $C_{00}^t = 1$ and $C_{0k}^t = C_{k0}^t = 0$ for $k > 0$.

Lean: `Formalism.DworkPowerSeriesWeights.frameA_letter` (`Formalism.GaussJetBounds.GaussJetProofs.FrameA`, lines 48–194)

Put $K_x(V) = \Theta(\xi Z(V))$ with $Z_{t,ab}(V) = \sum_k x_{t,k,a} V_{t,k,b}$. By the chain rule, direction $y_{t,h}$ in column b is represented by $\sum_k C_{hk}^t \partial_{V_{t,k,b}}$. With commuting differential symbols $D_{t,k,b}$ and a column choice I indexed from zero, a minor of height h has symbol

$$\det\left(\sum_k C_{ik}^t D_{t,k,I(j)}\right)_{i,j<h} = \sum_{k_0,\dots,k_{h-1}} \det(C_{i,k_j}^t)_{i,j<h} \prod_{j<h} D_{t,k_j,I(j)}.$$

Repeated k_j give equal columns in the coefficient determinant. At a marked letter each surviving minor uses row zero once by the form of C^t. A symbol exponent κ has degree $|\nu_t|$ at t. Each row is used at most once per diagram column. At a marked letter row zero has degree A_t, and ordinary rows occur in the B_t columns of height at least two. Thus the ordinary degree at t is D_t and each labelled row has degree at most c_t over all matrix columns.

Lean: `Formalism.DworkPowerSeriesWeights.op_prod_shape` (`Formalism.GaussJetBounds.GaussJetProofs.ThmAOp`, lines 244–297)

Write a_κ for the coefficient of D^κ in the transformed minor product. By 10.18, the evaluation of the scaled kernel at the frame $\tau = y$ is

$$\sum_\kappa a_\kappa \, \kappa! [V^\kappa] K_x(V), \qquad \kappa! = \prod_{t,k,b} \kappa_{t,k,b}!.$$

In a product of series, exponents contributing to $[V^\kappa]$ are at most κ coordinatewise. Integral linear substitution preserves base weight bounds by homogeneity. By 15.4, a phase term of base weight below q_* has weight at least $q_* - 1$ and takes the form $c Z_s^{p^u} Z_t^{p^v}$ in one matrix entry, with $t = s + u - v$ and $u - v$ odd. Let $\mathcal{O}$ be the Gauss valuation ring. The p-power map is a ring endomorphism of $(\mathcal{O}/p\mathcal{O})[V]$. Its u-fold iterate gives

$$\left(\sum_k d_k V_k\right)^{p^u} \equiv \sum_k d_k^{p^u} V_k^{p^u} \pmod{p\mathcal{O}[V]}$$

for $d_k \in \mathcal{O}$. A term that splits either power among several rows gains at least one in valuation and has base weight at least q_*. For an unsplit term on rows k, k' in a common matrix column, the sum over the original row coordinate a gives the multiplier $S = \sum_a x_{s,k,a}^{p^u} x_{t,k',a}^{p^v}$.

Orient a mixed pair with marked first row $(s, 0)$ and ordinary row (t, k'). Put $j = u - v$. The support bounds give $p^u \leq A_s$ and $p^v \leq |\nu_t|$, hence $p^j \leq A_s$ for $j > 0$ and $p^{-j} \leq |\nu_t|$ for $j < 0$. Thus (s, j) is a label i at t. Since $x_{s,0,a}^{p^u} = (Y_{s,0,a}^{L_i})^{p^v}$ by $M + u = M + j + v$, apply the same congruence to the integral elements $Y_{s,0,a}^{L_i} x_{t,k',a}$ and sum over a to obtain

$$S \equiv \lambda_i(x_{t,k'})^{p^v} \pmod{p\mathcal{O}}.$$

For $k' \neq e_i$ we have $\lambda_i(x_{t,k'}) = 0$, so $S \in p\mathcal{O}$. The extra unit of valuation gives base weight at least q_*.
Lean: `Formalism.DworkPowerSeriesWeights.pure_cross_term_weight` (`Formalism.GaussJetBounds.GaussJetProofs.PureCrossTermWeight`, lines 163–259)

Take a dual minimizer σ_t at capacities D_t by 15.3. For every positive odd e, its residue e_0 modulo r is odd with $1 \leq e_0 < r$, so $p^e\sigma_t + \sigma_{t+e} \geq p^{e_0}\sigma_t + \sigma_{t+e_0} \geq 1$. For marked s, let b_s be the first positive odd $e < h_s$ for which $A_s p^{-e} \leq c_{s+e}$, or h_s if there is none. Set $\delta_s = p^{-b_s}$ with $p^{-\infty} = 0$. Enlarge the weight of every variable on row (t,k) from ρ to $\rho + d_{t,k}$, where

$$d_{t,k} = \begin{cases} \delta_t, & k = 0,\ t \in T, \\ \sigma_t, & \text{ordinary row with no label}, \\ \max\{\sigma_t, 1 - p^j\delta_s\}, & \text{row labelled } (s,j),\ j > 0, \\ \max\{\sigma_t, p^j\}, & \text{row labelled } (s,j),\ j < 0. \end{cases}$$

The increments are nonnegative. For an unsplit pair on two ordinary rows orient the powers so that $e = u - v > 0$. The increase is at least $p^v(p^e\sigma_s + \sigma_{s+e}) \geq 1$. For two marked first rows use the same orientation. The target is marked, so $b_s \leq h_s \leq e$ and $p^u\delta_s \geq p^{u-e} \geq 1$. For a mixed pair with marked source, the congruence for S leaves the labelled partner to consider. For $j > 0$ the increase is at least $p^v(p^j\delta_s + 1 - p^j\delta_s) = p^v$. For $j < 0$ the partner gives at least $p^v p^j = p^u$. Thus every phase term of index $\leq \kappa$ has weight at least q_*.
Lean: `Formalism.DworkPowerSeriesWeights.dualRadiiInequalities` (`Formalism.GaussJetBounds.GaussJetProofs.OneLetterRadiusIdentity`, lines 124–206)

The phase has zero constant term. Terms of index $\leq \kappa$ in its h-th power have weight at least hq_*. Legendre's formula $v_p(h!) = \sum_{d \geq 1} \lfloor h/p^d \rfloor \leq hq_*$ then gives nonnegative exponential weight through κ [61, Ch. I, §2, Ex. 14]. Write $U = \sum_n u_n Z^n$. By 15.4, each term contributed by $u_n g^n$ has base weight at least $v_p(u_n) + \rho|n| \geq 0$. The integral linear substitution and the nonnegative row increments preserve this bound. Each coordinate of κ is at most a, so 11.7 gives digit loss at most $\log_p a + 1$ per factorial. The factorial-weighted coefficient has valuation at least

$$\sum_{t,k,b} \kappa_{t,k,b}(\gamma - d_{t,k}) - rN_0N_1(\log_p a + 1), \qquad \gamma = q_* - \rho.$$

Before the extra charges on labelled rows, the ordinary rows contribute $\sum_t D_t(\gamma - \sigma_t) = F(D)$ by 15.3. Fix a marked s and put $a_e = A_s p^{-e}$. The difference $\gamma A_s - H_s$ is

$$\sum_{e < h_s} \min(a_e, c_{s+e}) + \begin{cases} a_{h_s}, & h_s < \infty, \\ 0, & h_s = \infty. \end{cases}$$

A positive label e costs at most $c_{s+e}(1 - p^e\delta_s)_+$ above the dual charge. For $e < b_s$ this is bounded by $c_{s+e} = \min(a_e, c_{s+e})$. For $e \geq b_s$ it vanishes. The first-row charge $A_s\delta_s$ is zero at $b_s = \infty$. Otherwise it equals a_{b_s}, the minimum at $b_s < h_s$ or the endpoint at $b_s = h_s$. The remaining amount is at least H_s. A negative label $-e$ costs at most $p^{-e}c_{s-e}$ and satisfies $p^e \leq |\nu_{s-e}|$, so $c_{s-e} \leq M_s$. These labels cost at most γM_s in total.
Lean: `Formalism.DworkPowerSeriesWeights.rowRadiusAssembly` (`Formalism.GaussJetBounds.GaussJetProofs.CapacityLeReachCapacity`, lines 278–506)

The integral symbol coefficients preserve the valuation bound in the differential evaluation. The substitution of p^M-th powers for marked first-row variables preserves the Gauss norm because it sends distinct frame monomials to distinct monomials. The refined bound follows from $J_s \leq H_s$ term by term and 15.3.

For one letter use the variables $Y_{k,a}$ for the frame and substitute $Z_{ab} = \sum_k Y_{k,a}V_{k,b}$, with all V-variables of weight γ. Apply 15.4 at period two before identifying the letters. An exceptional pair has degree $p^u + p^v \geq p + 1$, so increasing the variable weight from ρ to γ adds at least $(p+1)(\gamma - \rho) = 1$. Substitution by an integral linear form and its unit multiple preserves degree and the resulting weight bound q_*. The factor $\epsilon/2$ has valuation zero. The same substitution bound gives $U \circ g$ nonnegative weight at γ. The exponential argument gives valuation at least $-\gamma|\kappa|$ before factorials. Since $|\kappa| = |\nu|$ and all N_0N_1 coordinates are at most a, 11.7 gives $(q_* - \gamma)|\nu| - N_0N_1(\log_p a + 1)$ after factorials. Here $q_* - \gamma = \rho$, and the Gauss norm bounds each frame coefficient.
Lean: `Formalism.DworkPowerSeriesWeights.oneLetterR` (`Formalism.GaussJetBounds.OneLetterRadius`, lines 23–25)

The minor product is homogeneous of degree equal to the box count, so scaling the kernel variables by ξ multiplies the evaluation by a unit by 10.18. □

Proof of 15.8

Proof. The long row is obtained from $z_i + e_l$ by subtracting $(k+m-2+t-2i)/2 + (p-(m-j))/2$. This is $p-1-i+(t+1)/2$. The short row uses the remaining $m-j$ entries with the sign displayed above. These rational expressions are integers since t is odd. In the construction of the row space the prescribed integer coordinates are the integers with these rational values, so the formulas apply to the row space used for the period.

The entries of z decrease by at least one. This follows within the v's by interlacing, at their last entry by $v_{k-2} > U_{k-1} > -b_0$, and within the b's by adding the two half-gap inequalities. Thus κ^{big} is weakly decreasing. Its last entry is $e_l - b_l - (t+1)/2 \geq 1$. When $j < m$, the negative of the first short entry is $b_j - e_l - (t+1)/2 \geq 1$. The increment in the negative short tuple is $b_{j+c+1} - b_{j+c} - 1 \geq 0$.

For the degree in a column, subtract $(N_l + N_{l+1} - 1 - 2i)/2$ from the source parameter plus e_l. The source parameter is U_i for $i < k$, and is $-e_q + (t-2d)/2$ for $i = k + q(t+1) + d$. Substitution gives the two formulas. The head decreases weakly because $U_i - U_{i+1} \geq 1$. Adding the upper half-gap at q to the lower half-gap at $q+1$ gives

$$e_{q+1} - e_q \geq t + 3.$$

Hence successive tail blocks decrease by at least two, and the last block, if present, has value at least two. The last head entry minus the first tail entry is $U_{k-1} + e_0 - 1 - t/2 \geq 1$, since $U_{k-1} \geq -b_0 + 1/2$. At $l = 0$ this same expression is the last head entry. This proves positivity, including the first layer where the tail is empty. □

Proof of 15.9

Proof. Use the integer base at each position in the weight construction of 33.32. The adjustment of the base to the unit character adds a residue in $[0, |\mu(E)|)$ at the designated thick position. The same adjusted base is used for all layers and words. The construction adds qh_0 for the q-th auxiliary coordinate and a further $h_0/2$ for the extension coordinate. At a mark the integer endpoints before addition of $1/2$ are $a_s + k/2 - 1$ and $-b_T(s) - k/2$. Elsewhere they are $4\delta(k-2) + 4k\delta$ and -1. This gives the asserted tuples.

For an interior coordinate the gaps to the two adjacent v's are $1/2 + \xi_i\delta$ and $4\delta - 1/2 - \xi_i\delta$, both positive. At a mark $a_s \geq a$, $b_T(s) \geq b'$, and eventually $S_c \leq a$ and $\lfloor b'/x \rfloor \geq k$. Indeed S_c, k, n grow at most polynomially by 33.27 and 38.3, whereas $b'/x = \exp(x^{\varepsilon} - x^{\varepsilon/3})$. These inequalities give $U_0 > v_0$ and $U_{k-1} > -B_\rho$. Off the marks $B_\rho \geq 1$ gives the latter inequality. Finally $t \leq x^{401}$ at large selected points by the rank formula of 38.3, and $c_h \geq 402$ in 33.32, so $h_0 \geq t + 3$. Substitution into 15.8 gives the tail degrees. □

Proofs for Chapter 16: From local lifts to bounds on whole columns

Proof of 16.3

Proof. The certificate proves the prime conditions, and the assigned comparison and model are independent of its choice.

Lean: `Formalism.BasicNotions.GIdx` (`Formalism.LocalRingCardinality.RhoDetPrimeEqMuTildeMulChiPow`, lines 68–69)

For $z \in V$, the tied comparison at a component j and full lattice tuple $\mathbf{L}$ uses a polynomial $Q_{z,j,\mathbf{L}}$ over K. To construct it, use basis matrices of $\mathbf{L}$ at p and the representative of j away from p. Write the resulting adelic point as $\gamma y_d \kappa$, with y_d a class representative, γ rational unitary and κ in the level group. Take the polynomial Q_d over K for z at d. Form the block matrix of γ through the specified embeddings $E \hookrightarrow K$ at the positions and their Frobenius translates, where E is the host field. Act by this matrix on Q_d and multiply by the element $r_\kappa \in K$ whose complex image is the level character $\chi(\kappa)$. Evaluation in the transported frame then gives $\chi(\kappa)$ times the value at d. Two presentations of the same tuple give adelic points related by a level element of character one, so the transformation law gives the same reading. The comparison applies i to the coefficients of $Q_{z,j,\mathbf{L}}$ and gives zero outside full lattice tuples.

Lean: `Formalism.L2.G.Cmp.IsVal` (`Formalism.LocalToGlobal.GroupLattices.ComparisonMaps.CmpLApply`, lines 18–25)
Lean: `Formalism.L2.G.Cmp.cmpC` (`Formalism.LocalToGlobal.GroupLattices.ComparisonMaps.CmpLApply`, lines 30–31)
Lean: `Formalism.L2.G.Cmp.CmpTied.valRat` (`Formalism.LocalToGlobal.GroupLattices.ComparisonMaps.NpDescFOfValRatF`, lines 18–21)

An edge $S \to L_v$ has $pL_v \subseteq S \subseteq L_v$ with quotient of residue dimension r. Its possibilities are finite by reduction modulo pL_v. The neighbour is

$$(y_v H)(j, \mathbf{L}) = p^{\mathcal{A}_v} \sum_{S \to L_v} H(j, \mathbf{L}[v \leftarrow S]), \tag{A.88}$$

where $\mathcal{A}_v$ sums the first r actual weights over the translates at v. This formula preserves coefficients in $i(K)$. To see integrality, choose a basis of L_v whose last $k-r$ vectors lift a basis of S/pL_v. With $\omega^2 = p$, the first r basis vectors of S are divided by ω at each of the two half-steps from S to L_v. For a weight λ, put $\nu_i = \lambda_i - \lambda_{k-1}$. A tableau product of this shape has degree $d \le \sum_{i<r} \nu_i$ in the first r rows, since a minor of height h uses at most $\min(h, r)$ of them. The Weyl generators lie in the same tableau lattice. In the enlarged frame a homogeneous generator changes by $\omega^{-(d+r\lambda_{k-1})}$, including its determinant twist. The normalization leaves the integral factor $\omega^{\sum_{i<r} \nu_i - d}$. In the tensor product, with H_S, H_a, H_{L_v} the successive fibres, this gives

$$\omega^{\mathcal{A}_v} H_S \subseteq H_a, \qquad \omega^{\mathcal{A}_v} H_a \subseteq H_{L_v}. \tag{A.89}$$

Level transport permutes edges and carries fibre lattices by unit character factors. Thus y_v preserves both M and the tableau lattice M_{na} of 14.12.

Lean: `Formalism.L2.G.Model.weyl_scale` (`Formalism.LocalToGlobal.GroupLattices.FockModels.OperatorsPreserveLattices`, lines 76–126)

Lean: `Formalism.L2.G.Model.operators_preserve_lattices` (`Formalism.LocalToGlobal.GroupLattices.FockModels.OperatorsPreserveLattices`, lines 334–343)

If $H = \sum_i t_i c(z_i)$ has coefficients in $i(K)$, choose a K-linear retraction $\pi : \mathcal{K} \to K$ of i. For any coefficient evaluation e, with $e(c(z_i)) = i(a_i)$, one has

$$e\left(c\left(\sum_i \pi(t_i) z_i\right)\right) = i\left(\sum_i \pi(t_i) a_i\right) = i\pi(e(H)) = e(H).$$

Evaluations separate functions. Apply this to $H = y_v c(z)$, which lies in $\mathcal{K}M$ and has rational coefficients by (A.88). Injectivity gives a unique K-linear y_v^K intertwining with y_v.

Lean: `Formalism.L2.G.Cmp.yvRat_of_valRat` (`Formalism.LocalToGlobal.GroupLattices.ComparisonMaps.YvRatOfValRat`, lines 269–279)

Replace y_v by y_v^K in Y_B to form Y_B^K. The module

$$C_w = \left(\sum_{B \subseteq W_p} Y_B^K (p^{n(B)} L_w) \right) \cap p^{b_0} L_w \cap V^r$$

extends to N_P. Indeed, $A \to \mathcal{O}$ is local by $i^{-1}(\mathcal{O}) = A$ and flat by torsion-freeness over a DVR. The first two modules in the intersection extend by intertwining and the input comparison. Flatness commutes with the kernels of the difference map for the first intersection and the map to V/V^r for the second. Inverting a uniformizer of A in $\mathcal{O}$ gives $\mathcal{K}$, and the basis independence identifies the resulting scalar extension with its image in $\mathscr{P}$. The same kernel argument gives $\mathcal{O}c(R_w) = M \cap V^{\mathrm{ram}}$.

For any finite free A-submodule of V, extend an A-basis of the submodule to a K-basis of V. Membership of a rational vector's image in the extended submodule forces its complementary coordinates to vanish and the others to lie in $i^{-1}(\mathcal{O}) = A$. Apply this to C_w, which is finite free as a submodule of $p^{b_0} L_w$, to obtain $J_w = C_w$. The upper inclusion follows from the second and third terms of the intersection. The lower inclusion follows from the empty subset's contribution $p^a L_w$.

Lean: `GlobalTheory.EX.OB.mem_span_of_semilinear` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.MemSpanOfSemilinear`, lines 16–77)

Lean: `Formalism.BasicNotions.FBC.span_inf_span_tower` (`Formalism.LocalRingCardinality.Lemmas`, lines 368–466)

□

Proof of 16.4

Proof. The half-step inclusions $\omega^{A_v} H_S \subseteq H_a$ and $\omega^{A_v} H_a \subseteq H_{L_v}$ (established in (A.89) inside the proof of Lemma 16.3) for the fibres of 14.10 give $H_L \subseteq R_a$ on outgoing edges and $H_L^\vee \subseteq R_a'$ on incoming edges.

The intersections give the first two pairs of inclusions. For $z = \omega^{-A_v} h \in R_a$ and a functional $\phi = \omega^{-A_v}\psi$ in the defining intersection for $P_L^\vee$, one has $\phi(p^{A_v} z) = \psi(h) \in \mathcal{O}$. This proves the last inclusion and gives integrality of each summand of y_v on $L^+_{B,v}$.

For the orbit function, every contributing edge is a transport of $S \to L'$. The terms with $L' \neq L_v$ lie in the target fibre multiplied by p^E. At a transport of $\mathbf{L}$, triviality of its stabilizer makes the transport of the specified edge unique. The remaining term is the orbit function of $p^{A_v}\xi$. The difference divided by p^E is equivariant and fibrewise integral. □

Proof of 16.5

Proof. If $W = \varnothing$, Definition 16.2 gives $N_P = R$, and both budgets are zero by 14.9. Suppose $W \neq \varnothing$. Set $D = D_p,\ q = p^D,\ S = \max_{v\in W} S_v$ and $s = m\sum_{v\in W}\kappa_v$. For $A = M, M_{\mathrm{na}}$ write

$$I(A) = \sum_{B\subseteq W} p^{\sum_{v\notin B} S_v}\Big(\prod_{v\in B} y_v^m\Big)A.$$

Here and below complements of subsets of positions are taken in W. The summand for $B = \varnothing$ shows that $I(A)$ has full rank in A.

Use the fibres of Lemma 16.4. At a target of local fibre rank d, choose a Smith basis b_j of P_L for the span of the tested vectors $p^{A_v}F_i$. Write its generators as $a_j b_j$, allowing $a_j = 0$. The forms and combinations in 14.10 give integral matrices with $C\,\mathrm{diag}(a_j)X = qI_{h_0}$, where $h_0 \geq (1-\kappa_v)d$. Discard the terms with $a_j \nmid q$. Each discarded entry belongs to $q\mathfrak{m}_{\mathcal{O}}$. The remaining product divided by q is therefore an integral matrix congruent to I_{h_0} in the residue field. Its rank over the fraction field is h_0. Thus at least h_0 indices satisfy $a_j \mid q$, and for each of them qb_j is an integral combination of the tested vectors.

For $\varnothing \neq B \subseteq W$, use these bases on B and the base fibre bases elsewhere. If a tensor b uses a passing index at $v \in B$, insert the integral combination for that index at its source edges. At every other edge, $F_i \in \omega^{c_*}R_{a'}$ and $p^{A_v}R_{a'} \subseteq P_{L'}$. Since $\omega^2 = p$ and $c_* \geq 4(D+1)$, the other contributions lie in $p^{D+1}L_B$. Trivial stabilizers allow Lemma 16.4 to be applied to each orbit function. With its notation this gives

$$q\delta_{i_0,\mathbf{L}}(b) \equiv y_v \sum_i u_i \delta_{i_0,\mathbf{L}[S_i]}(\xi_i) \pmod{p^{D+1}L_B}. \tag{A.90}$$

Here u_i are the combination coefficients and ξ_i has F_i in the replaced factor. Let C_B consist of the orbit functions of products failing at every $v \in B$. At each representative the number of such products is at most its fibre rank times $\prod_{v\in B}\kappa_v$. Summing over representatives gives

$$|C_B| \leq d_M \prod_{v\in B}\kappa_v. \tag{A.91}$$

Set $T_B = \sum_{v\in B} y_v L^+_{B,v} + \mathcal{O}C_B$, where $\mathcal{O}C_B$ denotes the span. All basis products give $qL_B \subseteq T_B + pqL_B$. The finitely generated module $(qL_B + T_B)/T_B$ equals its p-multiple. Nakayama's lemma gives $qL_B \subseteq T_B$, in particular

$$p^{D+1}L_B \subseteq T_B. \tag{A.92}$$

Order W, fix $0 \leq \beta < |W|$, and put $h = 2(D+1)$, $d'_v = 2q_v + h$, $K_v = md'_v$, $K_\beta = \sum_v K_v + 2\beta S$. A stage consists of $B \subseteq W$ of size $\beta+1$ and integers $0 \leq j_v < m$ for $v \in B$. Put $P = \{u < \max B : u \notin B\}$,

$$Y = \prod_{u\in P} y_u^m \prod_{v\in B} y_v^{j_v}, \qquad a = \sum_{u\in P} K_u + \sum_{v\in B}(q_v + j_v d'_v).$$

Let $C^{(\beta)}$ be the union of YC_B over these stages. Since $P_L \subseteq H_L$ and all the y_v preserve M_{na}, this is a subset of M_{na}. We prove

$$\omega^{K_\beta - a} Y L_B \subseteq I(M_{\mathrm{na}}) + \mathcal{O}C^{(\beta)}. \tag{A.93}$$

Use descending induction on $m|P| + \sum_{v\in B} j_v$, which is bounded by $m|W|$. At an active position the unused part of its allowance is

$$K_v - (q_v + j_v d'_v) = (m - 1 - j_v)d'_v + q_v + h.$$

Thus $K_\beta - a \geq h$. For $z \in L_B$, write $\omega^h z = \sum_{v\in B} y_v z_v + c$ using (A.92), with $z_v \in L^+_{B,v}$ and $c \in \mathcal{O}C_B$. Multiply by $\omega^{K_\beta - a - h}Y$. The term from c belongs to the claimed span. Fix one of the remaining terms.

If $j_v + 1 < m$, increase j_v by one. By (16.3), multiplication by ω^{2q_v} takes U into P_L, via H_L. Apply the induction hypothesis to $z' = \omega^{2q_v} z_v$. The new operator is $Y y_v$ and the new cost is $a' = a + 2q_v + h$. Thus $K_\beta - a' + 2q_v = K_\beta - a - h$, the exponent required for this term.

If $j_v = m - 1$ and there is a position after $\max B$, take the first such position w. Replace v by w in B, with $j_w = 0$. The completed set becomes $P \cup \{v\}$. At v, multiplication by ω^{q_v} takes U into H_L. At w, multiplication by ω^{q_w} takes H_L into P_L. Hence $z' = \omega^{q_v+q_w} z_v$ lies in the new stage lattice. Completing v uses its remaining $q_v + h$, and entering w uses q_w. The new cost is $a' = a + h + q_v + q_w$, so $K_\beta - a' + q_v + q_w = K_\beta - a - h$. Commutation gives the new operator $Y y_v$. In both transitions the induction index increases by exactly one.

Finally let $j_v = m - 1$ and $\max B = \max W$. Put $B' = B \setminus \{v\}$ and $n' = \sum_{u\in B'} S_u$. The vector $\prod_{u\in B'} y_u^{j_u} (\omega^{q_v} z_v)$ belongs to M_{na}. The summand of $I(M_{\mathrm{na}})$ indexed by $W \setminus B'$ therefore contains $\omega^{2n'+q_v} Y y_v z_v$. Since $n' \leq \beta S$, the $2\beta S$ in K_β and the remaining $q_v + h$ at v give $K_\beta - a - h \geq 2n' + q_v$. This proves the terminal case and finishes the induction.

Start with the first $\beta + 1$ positions, all at exponent zero. The product of the inclusions $\omega^{q_v} H_L \subseteq P_L$ pays its initial cost a. For fixed B there are $m^{\beta+1}$ exponent choices. Expanding $(\sum_v \kappa_v)^{\beta+1}$ counts each product on distinct indices $(\beta + 1)!$ times. Hence

$$|C^{(\beta)}| \leq d_M m^{\beta+1} \sum_{|B|=\beta+1} \prod_{v\in B} \kappa_v \leq d_M \frac{s^{\beta+1}}{(\beta+1)!}. \tag{A.94}$$

By 14.9, $K_\beta/2 + b_p = \tau_p + \beta S$. Multiply (A.93) at the initial stage by p^{b_p}. The sandwich (16.5) and linearity give an inclusion of $p^{\tau_p+\beta S} M$ in $I(M)$ plus a span in M with at most $|C^{(\beta)}|$ generators.

Write a_j for the uniformizer exponents of $I(M) \subseteq M$. The image of $p^{\tau_p+\beta S} M$ in $M/I(M)$ needs one generator for each $a_j > e(\tau_p + \beta S)$. A submodule of a module with b generators over a DVR needs at most b generators, by lifting to a submodule of $\mathcal{O}^b$. Thus (A.94) bounds this count. For $\beta \geq |W|$ it is zero by the empty summand of $I(M)$. Intersecting $I(M)$ with $p^{\tau_p} M$ changes the exponents to $\max(a_j, e\tau_p)$. For $S > 0$, sum the excess over successive intervals of width eS. It is at most $e d_M S \sum_{\beta\geq 0} s^{\beta+1} \leq 2e d_M S s$, since $s \leq 1/2$. For $S = 0$ the empty summand gives $I(M) = M$. Saturation gives $R \cap p^{\tau_p} M = p^{\tau_p} R$, and the inclusion of R induces an injection $R/N_P \hookrightarrow M/(I(M) \cap p^{\tau_p} M)$. Now $\mathrm{io}_p = 2Ss$ by 14.9, which gives both assertions. □

Proof of 16.7

Proof. The matching system gives the coefficient and terminal estimates of 13.8 and 13.4. At every position and every step $0 \leq i < m$, its differentiated coefficients satisfy the bound of 10.19 with exponent zero by the final sentence of item 7 of Definition 10.10. The filling and position map carry the tie and (16.11) to the twist. Apply 14.16 with $\mathcal{O} \hookrightarrow \mathcal{K}$, lower exponent $B^0_{u,w}(\sigma)$ and (16.10)–(16.12), and split the complementary sum at V. □

Proof of 16.8

Proof. For each $t \in W_p$, the ceiling gives

$$\sum_j H_{t,j} \leq \sum_j A_{t,j} + S_t. \tag{A.95}$$

Apply (16.13) with $V = W_p$, $V' = B$ and multiply by $p^{n(B)}$, which adds $n(B)$ to the local exponent. The bound suffices since

$$\sum_j g^c_{p,j} \leq n(B) + \sum_j \left(\sum_{t\in B} H_{t,j} + \sum_{t\in R_p\setminus B} A_{t,j} \right). \tag{A.96}$$

This follows from (A.95) on omitted representatives, and also holds when the gain vanishes. The bound extends to the span by $\mathcal{O}$-linearity. The filling, position map and ramified submodule in (16.1) are common. □

Proof of 16.9

Proof. First use frames with entries in the embedded coefficient field. Such frames exist simultaneously at the primes over p. Lift a basis of each localization by approximation in the host field. The condition of being a basis is open, since the change matrix must have integral entries and unit determinant. The Hermitian dual basis gives the companion frame. At a level-free place the marking conditions at the level and at the row primes impose no further change on these frames.

In one frame let D_ν be the integral span of the products of flag minors of shape ν, and take the tensor product over the types. Its integral dual under the normalized polynomial pairing is the Weyl lattice. Indeed, the semistandard products are an integral basis of D_ν, and the divided-power construction and the evaluation at the identity in 10.17 give the dual lattice. Consequently a functional lies in that Weyl lattice exactly when its values on all the flag-minor products are integral. The determinant power is part of the frame change on both sides of this test. In the source its removal and the inclusion of the full-height columns give the source polynomial of shape ν. In the target the same shape is read in rank N_{i+1}. Changing a lattice basis acts on these two integral lattices by inverse dual automorphisms. Thus the test gives membership in the prescribed lattice in any local basis.

Lean: `GlobalTheory.LatSCore.prSC_int_iff` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.FormsLatticeAtClassIffTwistedPolynomial`, lines 20–32)
Lean: `GlobalTheory.Mem.IntDualT.mem_locW_iff_prTC_le_one` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.MemLocalisedWeylLatticeIffTargetPairingIntegral`, lines 21–38)

We now give the rationality detail for the target test. A rational representative, evaluated on a rational matrix in a rational frame, has value in $\iota_w(K)$. Write the evaluation polynomial as a functional on the universal product of minors. Choose an $\iota_w(K)$-linear retraction $\pi : \mathbf{C}_p \to \iota_w(K)$, and replace the value of the functional on each monomial by its image under π. Extend the resulting assignment $\mathbf{C}_p$-linearly on monomials. The two functionals give the same evaluation polynomial on matrices over $\iota_w(K)$, since the universal product has integral coefficients. This field is infinite, so the evaluation polynomials agree identically. Their difference vanishes on each flag-minor test. To see this, polarize the columns of the universal product, or apply Cauchy–Binet with indeterminate columns and read the coefficients in the semistandard basis. The diagram fits in the rank, so all the columns used in this operation exist. The reading of the original functional therefore equals its image under π. In the adapted frames of the kernel one may further multiply the frame at each type by a local unit. Homogeneity multiplies each reading by products of powers of those units and of their determinants. Its absolute value is unchanged.

Lean: `GlobalTheory.KerVal.canM_ker` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.TargetPairingSizesInCoefficientFieldValueGroup`, lines 62–75)
Lean: `GlobalTheory.Pdv.KerScale.kerEntHyp_scB` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.TrueColumnChoicesKernelDatumAdaptedBases`, lines 160–212)

It remains to pass from values to forms. In representatives the rational space consists of the tuples of rational weight polynomials satisfying the transformation equations for forms. Indeed, transport from representatives determines a form, and two transports to the same point differ by a stabilizer. The equations at a representative are precisely invariance under that stabilizer, with its prescribed character and weight action. The section polynomial is determined by its evaluation in the frame model. Hence rational representatives for a tuple already in the complex form space give a rational form. The integral form lattice imposes the local Weyl conditions on these values. To localize this description, take a rational tuple satisfying the conditions at w. It has denominators at finitely many other places, since there are finitely many representatives and each value lattice is full. Choose an integer of $\mathcal{O}_K$ in sufficiently high powers of these other prime ideals and congruent to 1 modulo $\mathfrak{p}_w$. Its multiple satisfies all the integral conditions, and this integer is invertible in A. Conversely, division of an integral tuple by an element outside $\mathfrak{p}_w$ preserves the local conditions. Applying the frame test at every representative proves both equivalences. □

Proof of 16.10

Proof. Write K_i for the lift and h_c for the stabilizer order at a source class c. Let x be an integral source vector. By 16.9, its value at c is represented by a rational functional v_c with $|v_c(\Delta_J)|_p \le 1$ on every source product of minors Δ_J.

Expand each minor of $X_z Z_z$ by Cauchy–Binet before applying the differential operator. The resulting P_I lies in the span of the Δ_J. Fixed-shape straightening gives the semistandard products as an integral basis of this span of products over $\mathbf{Z}$ [64, Theorem 3.2(1)]. In a diagonal monomial order the leading monomial of a semistandard product has coefficient one. It records the multiset in every tableau row, which determines that row in its prescribed order. Thus the leading monomials are distinct. In decreasing order, subtract the leading coefficient times its basis product. The entry hypothesis bounds this coefficient by p^{-B}, and subtraction preserves the bound on the remaining polynomial since the basis products have integral coefficients. The argument in the product monomial order works for all types together. We obtain

$$P_I = \sum_J a_{IJ}\Delta_J, \qquad |a_{IJ}|_p \le p^{-B}.$$

By rationality of the pair map, its value from c to c' has a rational target representative $v_{cc'}$. Its I-reading is

$$t_{cc'I} = \epsilon_{cc'} v_c(P_I) = \epsilon_{cc'} \sum_J a_{IJ} v_c(\Delta_J).$$

This is the defining differential evaluation of the kernel in the two frames. The pair scalar $\epsilon_{cc'}$ has absolute value one. The ultrametric inequality gives $|t_{cc'I}|_p \le p^{-B}$. Choose a uniformizer $\pi_w \in K$ and set $a = \pi_w^{\lceil e_w B \rceil}$. By 16.9, a nonzero $t_{cc'I}$ has norm $p^{-v_w(b)/e_w}$ for some $b \in K^\times$. Hence $v_w(b) \ge \lceil e_w B \rceil$. This proves $|t_{cc'I}/\iota_w(a)|_p \le 1$, including zero readings.

Let n be a positive common multiple of the h_c, and put $k = v_p(n)$. The scalar $p^k/h_c = (n/h_c)/(n/p^k)$ lies in $\mathcal{O}_{K,(w)}$. At c' form the rational value

$$z_{c'} = \sum_c \frac{p^k}{a h_c} v_{cc'}.$$

Every reading is integral. These values evaluate to $\iota(p^k/a)K_i(x)$, since the class weight is $1/h_c$. The lift preserves the complex space of forms. The target implication of 16.9 gives a vector z in the localized target lattice with exactly this evaluation. The lift of x is represented by $(a/p^k)z$, and

$$v_w(a/p^k) = \lceil e_w B \rceil - e_w k \ge e_w(B-k).$$

Take $n = M_u$ by 15.12. When $i = 0$ and $\sigma = 1$, all $h_c = 1$, so take $n = 1$. These choices give (16.15). At μ_0 the stage isomorphisms identify the two lifts and carry the integral lattices onto each other. Extending those isomorphisms to $\mathcal{O}_{K,(w)}$ transports the same factor a/p^k. □

Proof of 16.12

Proof. For (1), setting the variables outside a block to zero eliminates the other phases because their constant terms vanish. For addition, expand the integral multiplier into monomials and factor each monomial by blocks. Inverse logarithmic substitution gives a product of kernels with integral monomial multipliers in logarithmic coordinates. Differential evaluation factors by 10.18. At each order the sum is finite, so the exponents add.

For (2) put $z_s = P^s r$, $w_s = w_j(z_s)$, $W_s = (k-1)w_s$. Let T be the marked indices and A_s, B_s the first two rows of ν_s. By (16.17), the refined cyclic estimate applies at size a_*^2 with base $F_{\mathrm{Orb},j}(p,r)$. Indeed $p \le a_*$, the rank inequality holds at N_i, and the flow capacities D_s of 15.5 are $W_s + e_s$ off T, $W_s - 2w_s + e_s$ on T, with $0 \le e_s \le 4w_s$.

At a signal prime, $r = (0, t_0)$ and $z_s = (s \bmod 2, t_0 + s h_P)$. As $n_P = 2\,\mathrm{ord}(2h_P)$, the marks correspond to $T(t_0)$ of 14.3. An odd positive distance e from $(0,t)$ ends on sheet one, so no positive odd distance

reaches another mark and none is excluded. For the row cutoff $c_{s+e} = \mathrm{cap}_{z_{s+e}}$ of 15.5, (16.17) gives $c_{s+e} \le B_j(z_{s+e}) + c_G = \bar{C}_{t,e}$ and $A_s - B_s \ge \bar{\alpha}_t$. Hence

$$[\bar{\alpha}_t p^{-e} - \bar{C}_{t,e}]_+ \le [(A_s - B_s)p^{-e} - c_{s+e}]_+ \le A_s p^{-e}.$$

Sum over odd $e > 0$ and the marks. The marked terms J_s of 15.5 cover $g_{0,r,j}$ because the digit subtraction is nonnegative and each $J_s \ge 0$. On the second inert sheet $g_{0,r,j} = 0$.

The reverse-partner maximum is at most $b + \eta + 1 + 3S_0$, and $C_6 \le c_6$. Thus the first two terms of (16.19) cover these losses. For $l = \log_p a_* \ge 1$, one has $C(2l+1) \le (1 + K_{\delta,1})l$ by (16.18). The rank and linear digit terms cover the error $Cn_P N_i N_{i+1}(2l+1)$ on the orbit, since their logarithms are at least l and $N_i N_{i+1} \le N_m^2$.

For (3) the quadratic phase pairs the two sheets at t with scalar $-Su_t$ of norm at most $p^{1/2}$. Its factorial-weighted coefficient for matching powers h, h at a matrix entry is $h!(-Su_t)^h$, with zero for unequal powers. Put $\Delta_t = \sum_{a,b} \partial_{(0,t),ab}\partial_{(1,t),ab}$. The contraction of order n_t has operator $(-Su_t)^{n_t}\Delta_t^{n_t}/n_t!$. The operator after removing the scalar is integral, by the multinomial expansion with divided derivatives on one side and ordinary derivatives on the other. For diagrams α, λ on the two sheets, with first rows a_1, b_1, at most $|\alpha| - a_1$ contractions use lower rows on the α side. A first-row contraction differentiates in that frame-row direction on the other sheet. Each of the b_1 minors is affine in this translation, since two columns replaced by the same vector give zero. At most $\min(a_1, b_1)$ first-row contractions survive. In particular $n_t \le |\alpha|$ off the top and $n_t \le |\alpha| - a_1 + \alpha_2 + b_1$ on the top. These are the contraction bounds of 15.12. The residual series has integral factorial-weighted coefficients, which proves (3).

For (4), (16.17) gives $|\nu_z| \ge (k-1)w_j(z)$ everywhere. On the top this uses both top inequalities. Thus the base term in B_C covers F_{inert} at $n_P = 1$. Since $C(2l+1) \le c_{\mathrm{dig}} l$, the digit term of $C^*_{j,p}$ covers the subtraction in B_C. Apply the one-letter bound at size a_*^2 and add by (1).

For (5) use optimal dual coordinates $0 \le y_s \le 1$ of 15.3 on each kind-BC orbit. With $W_s = (k-1)w_s$, these give

$$F(W) = \sum_s W_s(\gamma(p) - y_s), \qquad p^e y_s + y_{s+e} \ge 1 \quad (1 \le e < n_P,\ e \text{ odd}).$$

Setting $y'_s = \max(y_s, p^{-3})$ enforces all positive odd distances, since every omitted distance is at least three. This costs at most $p^{-3}\sum_s W_s$. These orbits have no top positions and $0 \le |\nu_s| - W_s \le 4w_s$. For $H = \{s : y_s \ge 2/p\}$, 15.3 gives $\sum_{s\in H} W_s \le p^{-1}\sum_s W_s$. Since $\gamma(p) \ge 1/p$,

$$\sum_s (|\nu_s| - W_s)[y'_s - \gamma(p)]_+ \le \frac{4}{p}\sum_{s\notin H} w_s + 4\sum_{s\in H} w_s \le \frac{8}{p}\sum_s w_s. \tag{A.97}$$

On kind A take $y'_s = 1/(p+1)$, which satisfies the odd-distance constraints and gives $\gamma(p) - y'_s \ge 0$.

Apply 15.4 at period $2n_P$ to the phases in (16.16). Give each letter at z weight $\rho(p) + y'_z$ in addition to the coefficient valuation. A term of base weight at least $q_* = 1/(p-1)$ keeps that bound. Each remaining term is an exceptional pair of base weight at least $q_* - 1$. With powers p^α, p^β, $\alpha > \beta$, and z carrying the larger power, its partner projects to $P^e z$ for $e = \alpha - \beta > 0$ odd. The weight increase $p^\beta(p^e y'_z + y'_{P^e z})$ is at least one, also if the positions coincide after projection. Identifying the letters preserves weights. The factors $\epsilon_r/2$ are units, so the resulting phase Φ has weight at least q_* and zero constant term.

Write $g = \ell^{-1}$. Each coordinate of g has at least the base weight $\rho(p)$ by 15.4, so $U \circ g$ has nonnegative weight. Also $v_p(n!) = \sum_{r\ge1}\lfloor n/p^r\rfloor \le nq_*$. Hence $\Theta(\xi Z) = (U\circ g)(Z)\exp\Phi(Z)$ has nonnegative weight. For $0 \le n \le a_*^2$, the factorial formula using base-p digits gives

$$v_p(n!) \ge nq_* - (2l+1), \qquad l = \log_p a_*, \tag{A.98}$$

since the digit sum of $n > 0$ is at most $(p-1)(1 + \log_p(a_*^2))$. For a differential index κ, homogeneity gives $\sum_{a,b}\kappa_{z,ab} = |\nu_z|$. Applying (A.98) to each coordinate and using $q_* - \rho(p) = \gamma(p)$ gives

$$v_p(\kappa![Z^\kappa]\Theta) \ge \sum_z |\nu_z|(\gamma(p) - y'_z) - fN_iN_{i+1}(2l+1).$$

Frame coefficients are integral combinations of these by 10.18. Sum the dual values and (A.97) over kind-BC orbits. Each orbit contributes its base value once, equivalently its position sum divided by n_P. Since

$2l+1 \le 3l$, the result is L_{odd}. The positive kind-A average in $I^{*}_{j,p}$ covers that part of F_{inert}. The cubic weight loss is in $X^{*}_{j,p}$, and $c_6 \ge 12$ and $K_{\delta,1} \ge 0$ (from 15.12 and (16.18)) cover the $8/p$ weight loss in L_{odd}, the nonnegative $R_{BC}(j,p)$ terms, and the $3fN_m^2 \log_p a_*$ loss via $X^{*}_{j,p}$. □

Proof of 16.14

Proof. In (i) the place is level-free. On the signal and even-order branches use the bound without a neighbour operator on kind A and 16.12(2) on kind BC. Kind A is the first sheet at even-order inert primes, and all signal orbits have kind BC. Add by 16.12(1). The full base sum is $F_j(w)$, and $I^{*}_{j,p}$ covers (16.19) and the positive part of the kind-A average. The entry exponent is at least

$$\sum_{t \in R_p} A_{t,j} + F_j(w) - I^{*}_{j,p}. \tag{A.99}$$

On the odd-order branch use 16.12(5), with the large-error constants of 15.12, to obtain $F_j(w) - I^{*}_{j,p}$. Both branches give, by (16.15) and the error identity, the step exponent

$$g_{0,S}(p,j) + F_j(w) - E^{\mathrm{ev}}_{u,j}(w) + \sigma \mathbf{1}_{\{i=0\}} \delta_{u,w}, \qquad j = i+1. \tag{A.100}$$

In (ii) use 16.12(4). In (iii) the host class gives $p = d$ and admissibility gives $g_0, g_1 \ne d$. The entry bound of 16.12(3) is at least $F_j(w) - D^{*}_j$ by the maximum over words in 15.12. Apply (16.15) in both cases and add the step bounds in (i)–(iii) by Lemma 16.13. In (iv) use the entry exponent zero of admissibility and (16.15), with $Q'(\mu_0) = \varnothing$ at the reference index. □

Proof of 16.18

Proof. For the arithmetic assertion let L^{loc}_j be the lattice in the local charts and $\jmath : K \hookrightarrow \mathcal{K}$ the coefficient embedding. At each $1 \le j \le m$ choose a finite set of full lattice tuples meeting every level-group orbit in each component, as in Definition 16.17. Choose frames at its representatives over the host field E by simultaneous local approximation of basis matrices. After clearing denominators this follows from the Chinese remainder theorem, since a sufficiently close matrix stays in the given right $\mathrm{GL}_k(W_v)$ coset, with W_v the local lattice ring at that position. The block embeddings and rational level characters give the stabilizer equations over K. In each rational frame take the invariant space for the stabilizer action with its character, and intersect with the framed integral lattice over $\mathcal{O}_{K,(w)}$. Clearing denominators expresses this intersection as a kernel in that lattice. Flat base change to $\mathcal{O}$ preserves the kernel. Evaluation at the finitely many representatives gives

$$L^{\mathrm{loc}}_j = \mathrm{span}_{\mathcal{O}}\{F \in L^{\mathrm{loc}}_j : F \text{ has rational readings}\}. \tag{A.101}$$

For an empty class set this is the zero module.

Lean: `Formalism.MainNotions.G.CmpJ.StepIn.desc` (`Formalism.SharpRestriction.GrowthBounds.ChartComparison.CmpJMemSpan`, lines 58–85)

Write V^{rat}_j for the rational space and $\Lambda^{\mathrm{rat}}_{j,(w)}$ for its localized lattice. At every positive level, evaluation in the rational embedding blocks gives a semilinear map c_j with rational polynomial readings. If $c_j(x) = 0$, these readings vanish at every class representative. The rational form therefore evaluates to zero in every frame, so injectivity of the form embedding gives $x = 0$.

Lean: `Formalism.MainNotions.G.CmpJ.stepsMu` (`Formalism.SharpRestriction.GrowthBounds.ChartComparison`, lines 128–147)

At the designated place w, the chart lattice test in Definition 16.17 applies. The normalizing factors for the determinant twist are units there, so integral local readings characterize membership in $\Lambda^{\mathrm{rat}}_{j,(w)}$. For a K-basis b_i, the images $c_j(b_i)$ are independent over $\mathcal{K}$. Indeed a nonzero relation, after applying to the reading equations a K-linear functional on $\mathcal{K}$ nonzero on one of its coefficients, would give a nonzero rational relation. The images lie in $\mathcal{K}L^{\mathrm{loc}}_j$, by scaling into the integral test. The dimension hypothesis gives $\dim_K V^{\mathrm{rat}}_j = \mathrm{rk}_{\mathcal{O}} L^{\mathrm{loc}}_j$ for $1 \le j \le m$, so the independent images span over $\mathcal{K}$. For $F = \sum_i a_i c_j(b_i)$ with rational readings, apply a K-linear retraction $\pi : \mathcal{K} \to K$ of $\jmath$ to each reading. Separation gives $F = c_j(\sum_i \pi(a_i) b_i)$. For integral F, the test puts this preimage in $\Lambda^{\mathrm{rat}}_{j,(w)}$. Together with (A.101) this yields

$$L^{\mathrm{loc}}_j = \mathrm{span}_{\mathcal{O}}\, c_j(\Lambda^{\mathrm{rat}}_{j,(w)}). \tag{A.102}$$

At level zero this equality is the input comparison.
Lean: Formalism.MainNotions.G.CmpJ.StepsMu.cmp_int (Formalism.SharpRestriction.GrowthBounds.ChartComparison.ComparisonZeroWeightBase, lines 201–204)

At $j = m$ take a basis of $\Lambda^{\mathrm{rat}}_{m,(w)}$. A bound s on $c_m(y)$ gives $v_{\mathcal{O}}(\jmath(a_i)) \geq \lceil v_{\mathcal{O}}(p)s \rceil$ on the nonzero coordinates a_i of y, by (A.102). Since $a_i^{e_w}/p^{v_w(a_i)}$ is a w-unit, its image is a unit and $e_w v_{\mathcal{O}}(\jmath(a_i)) = v_{\mathcal{O}}(p)v_w(a_i)$. Thus $v_w(a_i) \geq e_w s$. Taking a coordinate of least valuation as the scalar, or zero for $y = 0$, proves the arithmetic descent.
Lean: Formalism.MainNotions.G.CmpJ.CmpUpMu.val_of (Formalism.SharpRestriction.GrowthBounds.ChartComparison.ComparisonZeroWeightBase, lines 313–332)

For the general implication, start at $y_0 = e_0^{\mu} x$ and take rational representatives y_{j+1} of the successive lifts. Intertwining gives $c_{j+1}(y_{j+1}) = K_j^{\mathrm{loc}} c_j(y_j)$, so y_m represents the complex column and $c_m(y_m)$ is the local output. Apply the assumed terminal lattice bound to y_m. At μ_0 the inverse reference isomorphism at m carries this representative and the localized lattice to the reference chain.
Lean: Formalism.MainNotions.G.CmpJ.chainMu_cmp_K (Formalism.SharpRestriction.GrowthBounds.ChartComparison.ComparisonZeroWeightBase, lines 115–120)
Lean: GlobalTheory.Proof.LatticeValuationBound.exists_chain (Development.AbelianVarietyTheory.Proofs.SatakeParameters.LatticeValuationBound, lines 27–40)
Lean: GlobalTheory.Proof.ColumnWeightComparison.colGe0_of_colGeTw (Development.AbelianVarietyTheory.Proofs.SatakeParameters.ColumnWeightComparison, lines 25–35)

□

Proof of 16.19

Proof. The ordinary bound is 16.16, and the zero gain follows from 16.6. The local conclusion follows from 16.7–16.8. If $x \in \mathcal{N}_{u,w}$ through a permitted model, the common filling and position map carry the tie to the twist. Use (16.14) on the common lattice and descend by (16.27). The counted gain dominates the served gain by 16.6.

Otherwise $W_p = \varnothing$ and $x \in S^{\mathrm{ram}}_{u,w}$. Under (16.30) use (16.29), using the base-term identity from 14.15(3). In the remaining cases $p \leq X''$ is nonexceptional and signal or nonzero even-order inert. Formula (16.8) gives $g^{\mathrm{s}}_{p,j} = g_{0,S}(p,j)$. Apply (16.21). □

Proof of 16.20

Proof. Take $x \geq 3$ and put $U = mh\eta$. By (14.6), $b_T(t) \leq 5b'/4$. In the row formula of 15.12, interlacing bounds the initial word weights at indices $1, \dots, k-1$ by S_0, and at index zero off the top by $3S_0$, since the first base weight is at most $2S_0$ above the interlacing weight. Thus, for $w = w_j(\tau)$, the first k rows obey $\nu_i \leq S_0 + w$ for $1 \leq i < k$ and $\nu_0 \leq 3S_0 + w$ off the top. The later rows at $k + \ell h + a$, $0 \leq a < h$, $\ell \leq j-2$, equal $w_j(\tau) - w_{\ell+1}(\tau) - (j - \ell - 1)h$. The windows have width at most $\eta + 1$, and $h \geq 1$, so each such row is at most η and their sum is at most $2U$. With $A = \sum_i \nu_i$, this gives

$$A - \nu_0 + \nu_1 \leq k(S_0 + w) + 2U, \qquad A \leq (k+2)S_0 + kw + 2U \quad \text{off the top.} \tag{A.103}$$

The sheet-zero weight on TT is at most $b_T + b'/x + \eta \leq 23b'/12$. The first row on sheet one is at most $3S_0 + b + \eta + 1$ at thick positions and $3S_0 + z + \eta$ at thin positions. In (A.103) these give contraction bounds $23kb'/12 + 5z + 2b$ and $23kb'/12 + 6z$, respectively. They are at most $4kb' + 2b$ and $4kb'$ since $z \leq b'/3$ and $k \geq 5$. Off TT, sheet zero is thin and $c_{u,j,t} \leq (k+2)S_0 + 2kz + 2U \leq 4kb'$. Sum over $n = f/2$ positions to obtain (16.31). The stabilizer size assumption gives (16.32) for $C \geq C_D$.
Lean: PI.SzCt.UnitaryGroupMassFormula.ct_le (Formalism.PeriodIntegrals.UnitaryGroupMassFormula.SizeCount.SzCtOf, lines 9–83)

For the prime sum, using the family's constants $c_6, K_{\delta,1}, K_{\delta,2}$ from Definitions 14.2 and 15.12, put $c_1 = 12 + 5|c_6|$ and

$$\mathcal{R} = N_m^2(\log(8nN_m a_*) + \log Y) + |K_{\delta,1}| N_m^2 \log(2a_*) + 2|K_{\delta,2}|(\log(2a_*))^2. \tag{A.104}$$

Here $b = e^{x^{\varepsilon}}$, $a_* = xb$, $X' = e^{x^{\varepsilon/3}}$, $b' = a_*/X'$ and $Y = xnX'$. The size bounds give $n \leq x^3$, $N_m^2 \leq e^{x^{\varepsilon/3}}$, so $Y\mathcal{R} \leq C_\varepsilon x^{4+2\varepsilon} e^{2x^{\varepsilon/3}}$ for a family constant at large x. Consequently

$$b' \leq b, \qquad 0 \leq Y\mathcal{R} \leq b, \qquad \log Y \geq 1 \tag{A.105}$$

eventually. The weight windows then give $0 \le w_j(z') \le 5b$ at every position z', and $S_0 \le b$.

Subtract the kind-A positive part from $I^*_{j,p}$. By 15.12 the rest is $R_{BC}(j,p)$ of (15.1) plus $X^*_{j,p}$. The first two terms of (15.1) are bounded by $c_1 fb/p$, since $\gamma(p) \le 2/p$ and $\#Z_{\mathrm{top}}, \#Z_{BC} \le f$. After multiplication by $\log p$, the other terms are bounded by $f\mathcal{R}$ using $n_p \le 2n$, $p \le Y$, $N_i \le N_m$ and $1/\log p \le 2$. Hence

$$\log p\,(I^*_{j,p} - [\overline{F}_{A,j}(p)]_+) \le c_1 fb\,\frac{\log p}{p} + f\mathcal{R} + \log p\, X^*_{j,p}. \tag{A.106}$$

For primes supporting $X^*_{j,p}$, the inert shift $h_P \neq 0$ has odd order. Let 2^{e+2} be the modulus of the real cyclotomic factor in the host construction and put $q_e = 2^e$. The projection of the second Frobenius component to $(\mathbf{Z}/2^{e+2}\mathbf{Z})^\times/\{\pm 1\}$ is trivial because that quotient is a 2-group and the component has odd order. Arithmetic Frobenius acts on the cyclotomic quotient through the class of p, so $p \equiv \pm 1 \bmod 4q_e$. The lower bound on n gives $n \ge x^{2+\theta}$, and $n \le o\,2^e$ with $o \le x^\theta$ gives $q_e \ge x^2$. Also $k \le x^{9/2+10\varepsilon+\theta} \le x^5$. For every finite set S' of positive integers in the two classes,

$$\sum_{a \in S'} \frac{\log a}{a^3} \le \frac{\log q_e}{q_e^3}. \tag{A.107}$$

Indeed 1 contributes zero, and for $a = 4q_e r \pm 1$, $r \ge 1$, the bounds $\log a \le 2(a/q_e)\log q_e$ and $a \ge 3q_e r$ give $\log a/a^3 \le \log q_e/(4q_e^3 r^2)$. Sum both signs with $\sum_{r\ge 1} r^{-2} \le 2$. Using $k \le q_e^{5/2}$ and $\log q_e \le 2\sqrt{q_e}$, the cubic term of the weighted $X^*_{j,p}$ sum is at most $10fb$. Its logarithmic term is at most $|c_6|fb$ by $\#S \le Y$ and $Y N_m^2 \log a_* \le b$. Therefore

$$\sum_{p \in S} \log p\, X^*_{j,p} \le (10 + |c_6|) fb \log Y. \tag{A.108}$$

Rosser and Schoenfeld give $\sum_{p \le Y} \log(p)/p < \log Y$ for $Y > 1$ [34, (3.24)]. Nonnegativity gives the same upper bound for a subset of the primes. Sum (A.106) using $\#S\,\mathcal{R} \le b$ and (A.108). Take $C = \max\{C_D, c_1 + 1 + 10 + |c_6|, 0\}$ and x_0 above the size thresholds and x_D. □

Proof of 16.21

Proof. Combine the closed conditions by (16.34). For the numerical comparison take $\varepsilon_1 = 1$ and zero staged thresholds. The lower bound on k in 16.20 gives $k \ge x$ for $x \ge 1$, hence

$$r_G = 1 + \lceil 8(4 + 8\varepsilon)/\varepsilon \rceil < 32/\varepsilon + 66 \le k \tag{A.109}$$

for $x \ge x_n = \max\{x_1 + 1, 2, 32/\varepsilon + 66\}$. The built-in structural normalizations of every $\mathcal{F} \in \mathfrak{F}_\varepsilon$ give nonnegativity of all global constants K together with $K_{\delta,1} = 5$, $K_{\delta,2} = 4$, $K_{\mathrm{Pr}} = 1$ (which is why zero staged thresholds suffice in (16.34)), so the family sizes give the hypotheses of 14.13 and the parameter choice and nonnegative constants give those of 14.14.

Lean: `Formalism.MainNotions.G.rG_lt_k` (`Formalism.SharpRestriction.GrowthBounds.PlaneCountTheoremWithConstants`, lines 1798–1826)

Lean: `Formalism.MainNotions.G.numCmpF_fixed` (`Formalism.SharpRestriction.GrowthBounds.PlaneCountTheoremWithConstants`, lines 1828–1831)

Let x_b, x_c be the thresholds of the two clauses and C_s, x_a the choices of 16.20. For the splitting exponent $e_z > 0$ put

$$x_0 = \max\{x_a, x_b, x_c, x_n, 3^{5/e_z}\}. \tag{A.110}$$

For $x \ge x_0$ with $x > x_1$, the small-prime splitting condition applies to every odd prime up to $\lfloor x^{e_z/5} \rfloor \ge 3$. Tower divisibility gives $2 \mid Q_S$, so 2 belongs to the small-prime set. At 3 the splitting condition gives $(-d/3) = 1$. The host class rule excludes both primes from the inert and signal classes. At signal primes $n_P = 2\,\mathrm{ord}(2h_P)$, and at even-order inert primes $n_P = \mathrm{ord}(h_P)$, so in both cases $p \ge 5$ and n_P is even and at least two. The level primes g_0, g_1 split and d is the discriminant prime, so $g_0, g_1 \neq d$.

Form Y' and W from the first clause. Under the error hypotheses of 15.12 it supplies the matching systems and (16.10) on all its specified twists and places. The filling and representative map give (16.11) by 14.13, and 14.14 gives (16.12). Apply 16.7–16.8. The terminal hypothesis of 16.19 applies to every matching

admissible system and kernel, even at places with $p \in Q'(\mu)$. There the formal-group theorem of 13.8 gives the terminal estimate of 13.4.
Lean: Formalism.MainNotions.Pk.SysTw (Formalism.SharpRestriction.SmoothDesignExistence.SumOverPrimesHypotheses, lines 116–130)
Lean: Formalism.MainNotions.WeilJacquetShalikaLaw.local_theorem_of_statement (Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw, lines 371–374)

The second clause gives 16.10 at $\sigma = 1$, and 15.5 supplies the three letter estimates at $(C_6, C) = (11, 1)$. The built-in family normalizations $c_6 = 12$, $c_{\mathrm{dig}} = 3$, $K_{\delta,1} = 5$, $K_{\delta,2} = 4 \geq 0$ satisfy (16.18). Thus the orbit and kernel data satisfy Definition 16.11. The restriction and addition laws and the orbit estimates follow from 16.12. Apply 16.14–16.15 and then 16.19 to Y'. The filling and position map are common to Y and Y', as is the span of the rational ramified subspace. The base terms also agree. Thus the three conditions of 14.15 and membership in the enhanced input set of 16.2 carry over to Y'. The errors use the same global data. Hence (16.28) and (16.29), together with 16.20, give (16.35) with $C = C_s$ and threshold x_0.
Lean: Formalism.MainNotions.Pk.serv_re (Formalism.SharpRestriction.SmoothDesignExistence.SumOverPrimesHypotheses, lines 98–107)

For a prescribed selection, the first closed condition supplies the same levels, chains and comparisons. For x_c take the threshold assigned by the selection to the family $\mathcal{F} \in \mathscr{C}$. For $x \geq x_c$, $x > x_1$, $i \in I_x$ the selected member's kernel has the required properties. □

Proofs for Chapter 17: From columns to periods

Proof of 17.3

Proof. In the assumed reference and twisted evaluations, the integral interlacing map gives the lowered representative in local class coordinates. Reference frame changes multiply it by inverse finite-order character values in $\iota(K)$, which are roots of unity in K and hence local units at every finite place. The terminal lattice at one finite place spans the terminal K-space, so these representatives extend linearly. Injectivity of the point embedding identifies the extension with the local representatives at all places. Intersect the local point lattices.

Lift an $\mathcal{O}$-integral expression for H on the row generators to their rational lifts. At each allowed place w, the lifted row polynomial r_c lies in the w-local dual of the point lattice for the Weyl pairing, so $b_c = \langle x_c, r_c \rangle$ is integral at w. The Petersson class weight is $1/h_c$. The point stabilizer injects into the terminal stabilizer, whose order divides the chain denominator. The reference chain of word u has denominator Δ_u, so the common-denominator condition gives $h_c \mid \Delta_u$ on the twists as well. Thus $\Delta_u y = \sum_c (\Delta_u / h_c) b_c$ is integral at w. □

Proof of 17.4

Proof. Let e identify the complex input space with the ramified space of initial reference class functions. The rational ramified space is realized as a subspace of $V^0_{u,0}$. Its embedding in the complex input space sends ϕ to $e^{-1}(j^0_{u,0}(j\phi))$. Applying e cancels the inverse, so its initial reference class function is $j^0_{u,0}(j\phi)$.

For the reference tuple the period on a complex input ξ is defined by applying the reference column to $e\xi$, evaluating and lowering its output, and pairing against the inverse reference test identification applied to H. For a different tuple its definition applies the twisted column to $E_\mu e\xi$ and uses the inverse twisted test identification on H. Substitution of the preceding expression for ξ gives the displayed formula in each case. On an embedded rational terminal vector the same evaluation, lowering and test identification define $\mathcal{T}^\mu_u$ in 17.2. □

Proof of 17.5

Proof. By 16.1 the column bound supplies $a \in K$ and $z \in (\Lambda^\mu_{u,m})_{(w)}$ with

$$C^\mu E_\mu j^0_{u,0}(j\phi) = j^\mu_{u,m}(az), \qquad a = 0 \quad \text{or} \quad v_w(a) \geq e_w s.$$

Here and at the reference tuple we use the identifications of 17.4. Since H is admissible and $p_w \notin Q'(\mu)$, 17.2 applied to z, H gives $y_0 \in K$ with $\iota(y_0) = \mathcal{T}^\mu_u(z, H)$ and $v_w(y_0) \geq -e_w \delta_{u,w}$ if $y_0 \neq 0$. Evaluation and

lowering are linear, and the pairing with the fixed test is linear in the terminal variable. Thus 17.4 gives

$$P_u^{\mu}(\phi, H) = \mathcal{T}_u^{\mu}(az, H) = \iota(ay_0).$$

Take $y = ay_0$. When $y \neq 0$, add the valuations of its two factors. □

Proof of 17.9

Proof. The stabilizer condition on the design gives a cutoff x_s beyond which every prime divisor of every Δ_u is at most $2fN_m + 1$. The level condition gives a cutoff x_T beyond which $b^{m/2} \leq T_u$ for every word. Both cutoffs are uniform in the member index at x. Take $x_0 = \max(x_s, x_T)$. If $x \geq x_0$ and $x > x_1$, the member satisfies the size condition of Lemma 17.7. Hence $T_u > 2fN_m + 1$. Since T_u is prime, the stabilizer condition excludes $T_u \mid \Delta_u$. In the definition of closedness in 16.21, take the ε-cutoff to be 1 and all four size thresholds to be zero. The preceding choice of x_0 then works for each family in the design class. □

Proof of 17.10

Proof. Each vector of V^r has a nonzero integer multiple in L^0. Since $\mathcal{O}$ is Noetherian, $L^0 \cap V^r$ is finitely generated, so $M_{H,u}$ is full. Clearing denominators outside $\mathfrak{p}_w$ gives $j((M_{H,u})_{(w)}) = S_{u,w}^{\mathrm{ram}}$. At a selected permitted place 16.3 gives

$$p_w^{a+b}(M_{H,u})_{(w)} \subseteq B_w \subseteq (M_{H,u})_{(w)}$$

for the nonnegative integers a, b of that lemma. The first inclusion comes from the empty Hecke product. The second uses $N_P \subseteq M$ and the coordinate test.

At $w \mid T_u$, frame changes in a neighbor sum act by substitution in rational matrices, determinant powers and the rational level character. Thus U_v preserves rational coefficients. Rescaling markings permutes the neighbor incidences, so U_v preserves V^r, the common kernel of the diamond sums. For a marked vector $e \neq 0$, the down-up coefficient at $y \neq 0$ counts residue linear forms g with $g(e) = g(y) = 1$. There are T_u^{k-1} at $y = e$, none at the other points of its line and T_u^{k-2} off the line. On the common kernel the line sums and the unrestricted trace vanish. Thus $D_v U_v = T_u^{k-1} I$, where D_v is the down operator. It follows that U_v is invertible on V^r.

Lean: `Formalism.L2.GlobalTheoryTheory.UnramifiedTrace.du_formula_of_fibre_count` (`Formalism.LocalToGlobal.GlobalTheory.UnramifiedTrace.DuFormulaOfFibreCount`, lines 15–41)

Under the complex embedding, $N_{\lambda,u,w} = GS_{u,w}^{\mathrm{ram}}$ with $G = T_u^{a_0} U^m$ a rational automorphism of V^r. Clearing denominators of G and G^{-1} on generators of $L^0 \cap V^r$ gives an integer $n_w > 0$ with $n_w S_{u,w}^{\mathrm{ram}} \subseteq GS_{u,w}^{\mathrm{ram}} \subseteq n_w^{-1} S_{u,w}^{\mathrm{ram}}$. For an empty product take $G = 1$.

Lean: `GlobalTheory.LevelHecke.uop_rat` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.UpwardOperatorPreservesRationality`, lines 147–182)

Lean: `GlobalTheory.LevelCommensurable.commensurable` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.RamifiedFormsCommensurableOfUStableInjective`, lines 36–92)

The exceptional set S of selected permitted places and places above T_u is finite, since nonempty selection requires $p_w < X'$ by 14.8. At its places the preceding inclusions give

$$n_w(M_{H,u})_{(w)} \subseteq B_w \subseteq n_w^{-1}(M_{H,u})_{(w)}$$

for positive integers n_w. With $n = \prod_{w \in S} n_w$ this also gives $nM_{H,u} \subseteq N_u \subseteq n^{-1} M_{H,u}$. Each B_w in the display is full and finitely generated over $\mathcal{O}_{(w)}$. The $\mathcal{O}$-span of a set of its local generators is a full lattice in V_u with localization B_w. Use these lattices at the exceptional places and $M_{H,u}$ elsewhere in 18.4 to obtain the intersection and all its localizations.

Lean: `GlobalTheory.EX.Glue.exists_glue` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.MemLocSub`, lines 69–113)

Lean: `GlobalTheory.EX.Glue.inLoc_iInf` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.MemLocSub`, lines 51–67)

□

Proof of 17.12

Proof. The level-prime description of $(N_u)_{(w)}$ gives $j_u\phi \in N_{\lambda,u,w}$. The formula in 15.12, with m even and t odd, gives $n_0 = mT^*_{u,w} \in \mathbf{N}$. At μ_0, take $c = T_u^{n_0}$ in 17.20 and divide its terminal representative by c. On another used tuple the twisted bound gives exponent $-n_0$. Apply 17.5 at μ_w with $\delta_{u,w} = 0$. At $w \mid T_u$, the identities of 17.6 give zero raw gain and zero base term at each lift, with loss $T^*_{u,w}$. The screened gain also vanishes. Summing over the m lifts gives $S_{u,w} = -n_0$. □

Proof of 17.13

Proof. Each b_ν has norm at most one by continuity from $\mathbf{N}^r$. For a finite expansion $f = \sum_\nu a_\nu b_\nu$, its coefficient a_ν is $(\Delta_1^{\nu_1} \cdots \Delta_r^{\nu_r} f)(0)$, where $\Delta_j f(c) = f(c + e_j) - f(c)$. Hence $\|f\|_\infty = \max_\nu |a_\nu|_p$. These finite expansions are dense. In one variable this is Mahler's interpolation theorem [67]; it applies in particular to the $\mathbf{Q}_p$-valued indicators of residue disks. Products approximate the indicators of residue boxes in r variables. Locally constant functions, finite linear combinations of these indicators with coefficients in $\mathbf{C}_p$, approximate every continuous function by compactness. The rule $f \mapsto \sum_\nu a_\nu A_\nu$ now extends uniquely by completeness.

Choose k so large that on each box modulo p^k the oscillation of every g_i is less than η. Take its representative $c \in \{0, \dots, p^k - 1\}^r$ and put $d_c = \mu(\mathbf{1}_{c+p^k\mathbf{Z}_p^r})$. The assertion follows by applying μ to the difference between g_i and the corresponding step function. For $r = 0$ there is one box. □

Proof of 17.14

Proof. Use Lebesgue measure in H-orthonormal complex coordinates. The integral of $e^{-\pi H(z,z)}$ over the period space is one. A lattice self-dual for $B = \operatorname{Im} H$ has covolume one for this measure: in real orthonormal coordinates the matrix of B has determinant one. The trace dual of $\mathfrak{b}^{-1}$ is $\mathfrak{D}^{-1}_{E/K_d}\bar{\mathfrak{b}} = \mathfrak{b}^{-1}$; the area factor gives alternating pairing one on an oriented basis of Λ. Hence P_1 is self-dual.

Direct multiplication gives $U_x U_y = e^{-\pi i B(x,y)} U_{x+y}$. On each indicated lattice the semicharacter makes $W_x W_y = W_{x+y}$. For self-dual L, L' the commutator induces a perfect pairing of $L/(L \cap L')$ with $L'/(L \cap L')$. Summing it shows $I_{L',L} I_{L,L'} = 1$ on L-fixed sections. Change of variable in the Gaussian norm on a common period domain identifies the reverse average with the adjoint, for the norm with period-domain measure normalized to mass one. The averages are therefore isometries. The fixed holomorphic space for Λ is a line: division by its theta section gives an elliptic function with at most one simple pole, which the residue theorem excludes. Applying this in one variable at a time proves the product statement. The inverse averages transport this line to every commensurable self-dual period lattice.

Diagonalize the positive trace form over K_d. For a positive rational diagonal coefficient A/B_0, write AB_0 as the norm of an integral quaternion by the four-square theorem. Its left multiplication matrix divided by B_0 has transpose times itself equal to $(A/B_0)I_4$. This gives an isometry g_0 on four copies. Write $P_0 = g_0 P_1^4$, $P = \Lambda^{4f}$. At a split prime l choose one factor $\mathfrak{l}$, a generator π_l of $\mathfrak{l}^h$, with h the class number, and a root $r_l^{2h} = \pi_l/\bar{\pi}_l$. Let y_l be the determinant order at $\mathfrak{l}$ of a local unitary map taking P_0 to P. Such a map sends a basis on the first half to a basis there and acts by the inverse adjoint on the second half. The order is independent of the bases. Put $\kappa_0 = \prod_l r_l^{-y_l}$; the two lattices agree away from finitely many primes. The fourth-power normalization chooses Θ so that

$$\Theta^{\otimes 4}(z) = \kappa_0 (I_{P,P_0} \theta_\Lambda^{\otimes 4f})(g_0 z).$$

The fourfold line is the tensor product of four one-entry lines, so this chooses a fourth root.

At p the maps taking P_0 to P form an open coset. Choose a rational unitary u in it with no eigenvalue one. Indeed multiplying a local map by norm-one scalars arbitrarily close to one avoids eigenvalues $1, -1$. In the Cayley chart the resulting open set pulls back to an open set in the vector space of skew-Hermitian matrices, where rational points are dense. Set $g = ug_0$, $Q = uP_0$. Since $\det u$ has orders $y_p, -y_p$ at the factors over p, the defining equation of r_p, and the fact that r_l is a p-unit for $l \neq p$, give

$$|\jmath(\kappa_0)|_p^2 |\jmath(\det u)|_p = 1. \tag{A.111}$$

We compute the pullback factor. Let v be unitary on a period space, let L, vL be commensurable, and assume $A = 1 - v^{-1}$ invertible. Put $L_0 = L \cap vL$, $J = [L : L_0] = [vL : L_0]$, $D = \det A$, $P_A = A^{-1} - 1/2$. Pullback by v^{-1} and $I_{L,vL}$ carry the same theta line isomorphically to the vL-line. Write their ratio as c_v. Define

$$K(w,z) = \sum_{x \in L} W_x^{(w)} e^{\pi H(w,z)}, \qquad \phi(x) = \alpha(x) e^{-\pi H(P_A x, x)}.$$

For fixed z, $K(\cdot, z)$ belongs to the L-line by reindexing. The sum and the following integrals converge absolutely: on bounded sets of w, z the terms are bounded by a constant times $e^{-c\|x\|^2 + C\|x\|}$ with $c > 0$.

Apply the two maps in the first variable, set $w = z$, multiply by $e^{-\pi H(z,z)}$ and integrate over a fundamental domain for L_0. In the term of the average with representative $y \in vL$, the summand indexed by $x \in L$ is a constant times $e^{2\pi i B(y+x,z)}$. Since $B(y + x, L_0) \subseteq \mathbf{Z}$, its integral vanishes unless $y + x = 0$. Choose zero as the representative of the trivial coset. The surviving term has coefficient one. Thus the integral of the average is $J/\sqrt{J}$.

For the pullback the summand indexed by x is, by expanding the Hermitian form,

$$\alpha(x)\,(U_x^{(w)} e^{\pi H(w,z)})\big|_{w = v^{-1}z} e^{-\pi H(z,z)} = \phi(x) \exp\{-\pi H(A(z - A^{-1}x), z - A^{-1}x)\}.$$

For $h \in L_0$, $Ah = h - v^{-1}h \in L$. The semicharacter law and $\alpha(v^{-1}h) = \alpha(h)$ give $\phi(Ah) = 1$. The cross factor in $\phi(x + Ah)$ is $e^{-2\pi i B(A^{-1}Ah, x)} = 1$, using symmetry of the polarization on L. Thus the sum over x in one coset modulo AL_0 unfolds to the integral over the whole space. In an orthonormal eigenbasis of v, each eigenvalue λ of A has positive real part, and $2\pi \int_0^\infty e^{-\pi\lambda r^2} r\, dr = \lambda^{-1}$. The integral is D^{-1}. Since $[L : AL_0] = |D|^2 J$, comparison gives

$$c_v = \frac{|D|}{D} G, \qquad G = \frac{1}{\sqrt{|V|}} \sum_{x \in V} \phi(x), \qquad V = L/AL_0.$$

For completeness, $A^{-1} + (A^{-1})^* = 1$ makes P_A skew-adjoint. Substituting in the preceding expression for ϕ and using the semicharacter law on L gives polarization $e^{-2\pi i B(A^{-1}x, y)}$. This is trivial for every $y \in L$ exactly when $A^{-1}x \in L$. In that case $v^{-1}A^{-1}x = A^{-1}x - x \in L$, so $A^{-1}x \in L_0$. This proves nondegeneracy on V. Also $\phi(nx) = \phi(x)^{n^2}$, since $\alpha(x) = \pm 1$ on L. The finite quadratic-module Gauss formula gives $G^8 = 1$ [69, Proposition 2.11 and Lemmas 3.6–3.8].

There is also a factor from composing averages. For L_1, L_2, L_3 the composition is a scalar μ times I_{L_1,L_3}. Project the composition back to the L_1-line by the inverse average. Commutator orthogonality retains the pairs $z \in L_3,\ y \in L_2$ with $z + y \in L_1$. They form a finite group modulo $(t, h - t)$, $t \in L_3 \cap L_2,\ h \in L_2 \cap L_1$. Expanding $W_z W_y$ gives their phase

$$\alpha(z)\alpha(y)\alpha(z+y)^{-1} e^{-\pi i B(z,y)}.$$

The group laws on the three lattices make this expression independent of representatives; its polarization is $e^{2\pi i B(y,z')}$ and it is homogeneous quadratic. The averaging constants are positive. If its restriction to the radical were nontrivial, translation there would annihilate the sum. The composition is invertible, so the phase descends to the nondegenerate quotient. Its normalized sum has eighth power one by the same Gauss formula. Since the composition is an isometry, the positive factor relating μ to that normalized sum is one.

Apply this with $(L_1, L_2, L_3) = (P, P_0, Q)$ and $v = u$ acting on P_0. The scalar c relating the two normalized standard sections is $c_u \mu$. From $A^* = -uA$ in dimension $4f$ we get $\bar{D}/D = \det u$. Consequently

$$c^8 = (|D|/D)^8 = (\det u)^4.$$

In particular c is algebraic. Taking p-adic absolute values gives $|\jmath(c)|_p^2 = |\jmath(\det u)|_p$. Thus $e = \kappa_0 c$ has $|\jmath(e)|_p = 1$ by (A.111).

For the endpoint factors we need the sign of a repeated Gauss sum. For every integer $m_0 > 0$ there exist $\lambda, s \in \mathcal{O}_E$ with $\bar{s} = s$ and $\lambda\bar{\lambda} = -1 + m_0 s$. Indeed, by Definition 5.1 and Lemma 5.3, E/E^+ is unramified at every finite place. Since $d \equiv 7 \bmod 16$, the integer $w_0 = (1 + \sqrt{-d})/2 \in \mathcal{O}_{K_d} \subseteq \mathcal{O}_E$ satisfies $w_0(1 - w_0) = (1 + d)/4 \equiv 0 \bmod 2$ with $\bar{w}_0 = 1 - w_0$, so 2 splits in K_d and no prime ideal $P \mid 2$ of $\mathcal{O}_E$ satisfies $P = \bar{P}$. Partition the prime ideals dividing $m_0\mathcal{O}_E$ into conjugation orbits $\{P, \bar{P}\}$; since $m_0 \in \mathbf{Z}$, conjugate

primes P and $\bar{P}$ appear with the same exponent $k = \operatorname{ord}_P(m_0\mathcal{O}_E)$. For each two-element orbit $P \neq \bar{P}$ (which includes all $P \mid 2$), choose one representative P and prescribe $\lambda \equiv 1 \bmod P^k$ and $\lambda \equiv -1 \bmod \bar{P}^k$ (so $\lambda\bar{\lambda} \equiv -1$ modulo both P^k and $\bar{P}^k$); for each fixed prime $P = \bar{P}$ (hence $P \nmid 2$), finite unramifiedness of E/E^+ gives a nontrivial involution on the finite residue field $\mathcal{O}_E/P$ whose norm map is surjective, yielding $\lambda_0\bar{\lambda}_0 \equiv -1 \bmod P$, which lifts modulo P^k by Hensel's lemma since $2 \notin P$ and $\lambda_0 \notin P$ (and we prescribe $\lambda \equiv \lambda_0 \bmod P^k$). Combining all these pairwise-coprime prime-power congruences across all orbits once by the Chinese remainder theorem produces $\lambda \in \mathcal{O}_E$ with $\lambda\bar{\lambda} + 1 = m_0 s$ for some $s \in \mathcal{O}_E$, automatically satisfying $\bar{s} = s$.

Lean: `GlobalTheory.NegOneNorm.exists_norm_eq_neg_one` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.ExistsDvdMulInvolutionAddOne`, lines 81–127)

In one row take $m_0 = 4|V|$, with $\lambda, s \in \mathcal{O}_E$ satisfying $\bar{s} = s$ and $\lambda\bar{\lambda} = -1 + m_0 s$. Multiplication by λ, $\bar{\lambda}$, and s preserves both period lattices and commutes with the endpoint action and A, with $H(\lambda x, y) = H(x, \bar{\lambda} y)$ and $H(sx, y) = H(x, sy)$. Since $|V|$ annihilates $V = L/AL_0$, the identity $\bar{\lambda}\lambda x = -x + 4|V|sx$ induces -1 on V, so λ permutes V. Moreover, $H(\lambda x, \lambda x) = -H(x, x) + 4|V|H(sx, x)$ with $H(sx, x) \in \mathbf{R}$ and $\sqrt{d}\,\operatorname{Re} H(y, x) \in \mathbf{Z}$ on L, whence $\alpha(\lambda x) = \alpha(x)^{-1} = \overline{\alpha(x)}$. Likewise $H(P_A\lambda x, \lambda x) = -H(P_A x, x) + 4|V|H(P_A(sx), x)$; because $P_A = A^{-1} - 1/2$ is skew-adjoint and commutes with the self-adjoint s, the last term equals $4|V|i\big(B(A^{-1}(sx), x) - \frac{1}{2}B(sx, x)\big) \in 2i\mathbf{Z}$ (using $|V|sx \in AL_0$, so $A^{-1}(|V|sx) \in L_0 \subseteq L$, and $B(L, L) \subseteq \mathbf{Z}$). Thus $\phi(\lambda x) = \overline{\phi(x)}$. The sum G is therefore real, so $G^8 = 1$ gives $G^2 = 1$.

For a source change the b rows give identical one-row factors. On one row the determinant of the action on period coordinates is $\prod_\tau \tau(\overline{\det \gamma})$. The row dimension af is even, whence $c_v^2 = \bar{D}/D = \prod_\tau \tau(\overline{\det \gamma})$. Taking b rows gives the claimed exponent $b/2$. For a target change use a columns; the determinant on one column is $\prod_\tau \tau(\det \gamma')$. If the action has eigenvalue one, factor it by a rational norm-one scalar chosen so that neither factor has eigenvalue one. The triangle of averages in this factorization is a product of identical row triangles, or identical column triangles. On each triangle choose m_0 divisible by 4 times the orders of its finite phases and of its finite group. The same λ permutes that group and conjugates the displayed triangle phase (since $\alpha(\lambda z) = \overline{\alpha(z)}$ and $B(\lambda z, \lambda y) = -B(z, y) + m_0 B(sz, y) \equiv -B(z, y) \bmod 2\mathbf{Z}$). Its scalar is real of absolute value one. An even number of copies has scalar one, so the factors multiply also in this case. For zero rank the empty product equals one. □

Proof of 17.15

Proof. For commuting directional derivatives D_j in the scaled directions of the statement, the coefficient at Z^ν in $\prod_j \exp(D_j \log(1 + Z_j))F(0)$ is $A_\nu(F; v)$, after transport by $\jmath$. By 17.13 a bound of one on these coefficients gives a measure with transform

$$\int_{\mathbf{Z}_p^r} \prod_j (1 + Z_j)^{c_j}\, d\mu(c).$$

Expressing powers in falling factorials shows that the power moments give the iterated derivatives. Products of such transforms correspond to pushing product measures forward by addition. Linear substitutions in the directions with coefficients in $\mathbf{Z}_p$ correspond to pushforward by the transpose on the measure variables. Both operations preserve the upper bound one on the norm.

Apply 17.14 to the fourth tensor power. If $nt \in P_1$, $n > 0$, $p \nmid n$, put $v = g(t, t, t, t)$ and $J = [Q : P \cap Q]$. Commuting U_v past the terms of the average gives

$$(U_t\Theta)^{\otimes 4}(z) = \frac{e}{\sqrt{J}} \sum_{x \in Q/(P \cap Q)} \alpha(x) e^{-\pi i B(v, x)} \prod_{\ell=1}^{4f} (U_{v_\ell + x_\ell}\theta_\Lambda)((gz)_\ell).$$

Here $nJ(v_\ell + x_\ell) \in \Lambda$. The phases are roots of unity since $nv, x \in Q$, and $e/\sqrt{J}$ is a p-adic unit. Multiplication by J puts each transformed period direction in P, and $1/J \in \mathbf{Z}_p$. The elliptic hypothesis and the product and substitution formulas give integral coefficients for every list in the four blocks. Use independent variables in the blocks, with the same list and index ν in each. The coefficient at (ν, ν, ν, ν) is $A_\nu(U_t\Theta; v_\bullet)^4$. Hence the one-entry coefficients are integral. Taking products over the entries proves the assertion for the standard matrix vector.

Write $P(L,L') = \Lambda(L,L')$, let L_0, L_0' be standard, and form the vector $F_{L,L'}$ by averaging the standard vector first to $P(L,L_0')$, then to $P(L,L')$. Its normalization is

$$e_{L,L'} = \chi(\det(L_0,L))^{b/2}\chi(\det(L_0',L'))^{-a/2}F_{L,L'}.$$

The relative determinant here is the determinant ideal of one lattice in a basis of the other. The character is multiplicative and $\chi((z)) = \prod_\tau \tau(z)^{-1}$ for $z\bar{z} = 1$.

We record why lattice moves compose with this normalization. For moves with the other lattice standard, the triangle scalar is one by the paired Gauss calculation in 17.14. For moves in different variables consider the four period lattices modulo their common intersection. At every finite place where the lattices differ, the two target lattices admit the same two complementary isotropic halves. At a split place of E/E^+ take the two factors exchanged by conjugation. In the host of 5.1, the prime d satisfies $d \equiv 7 \pmod{16}$, so the defining polynomial $X^2 - X + (1+d)/4$ of $\mathcal{O}_{K_d}$ splits into two distinct linear factors modulo 2. Consequently 2 splits in K_d, and therefore every prime of E^+ above 2 splits in E/E^+. Every inert place of E/E^+ thus has odd residue characteristic. At an inert place the extension E_v/E_v^+ is unramified of odd residue characteristic, and the lattices are self-dual of even rank. Reduction modulo the maximal ideal gives a nondegenerate Hermitian space of even dimension over a finite quadratic extension, which is hyperbolic. A hyperbolic pair lifts integrally: lift an isotropic basis vector to a primitive vector, adjust its Hermitian norm to zero by Hensel's lemma (as the residue characteristic is odd and the Hermitian form is unimodular), choose a second vector pairing to one with the first, and subtract a multiple of the first using surjectivity of the trace map $\mathcal{O}_{E_v} \to \mathcal{O}_{E_v^+}$ for the unramified extension so that the second vector is also isotropic. Splitting off the hyperbolic plane inductively proves that the local space is hyperbolic. In odd residue characteristic, the two self-dual lattice norms are points of the Bruhat–Tits building of the unitary group, and a common apartment splits them simultaneously in a single Witt basis by the norm description of [65, §§2.8–2.12]. Projection onto one vector of each hyperbolic pair and its complementary projection preserve both lattices. For the finite quotients it suffices to use these projections modulo a common denominator carrying either source lattice into the other. The Chinese remainder theorem gives the projections to that precision, with $e^2 = e$ and $e^* = 1 - e$ on the quotients.

This reduces the finite Heisenberg calculation to a lattice A in one half and its annihilator $A^\perp$ in the other. In the function realization, translations by A fix $\mathbf{1}_A$, and characters from $A^\perp$ fix it. For a change $A \to D$, summation of the characters cuts $\mathbf{1}_A$ down to $\mathbf{1}_{A\cap D}$ with multiplicity $[A : A \cap D]$. Summation over $D/(A \cap D)$ then gives $\mathbf{1}_D$. The normalized average consequently takes $\mathbf{1}_A$ to the positive multiple $([A : A \cap D]/[D : A \cap D])^{1/2}\mathbf{1}_D$. The two paths round the square have the same factor by multiplicativity of relative volumes. On each isotropic half the Weyl operators multiply as translations: their Hermitian cross terms vanish on an exact half, and the congruence projections give the same operators on the finite quotients. Thus the square commutes. Interchanging successive source and target moves now transports the standard triangle identities to arbitrary endpoints.

Lean: `Formalism.L2.Ker.Cd.PairingIndexBounds.Eq.SquareSplitPairs.sqSplit_of` (`Formalism.LocalToGlobal.KernelLattices.ColumnData.ColumnDataEquations.SqLMain`, lines 21–139)

Lean: `Formalism.L2.Ker.Cd.PairingIndexBounds.Eq.SqG.eqP_car` (`Formalism.LocalToGlobal.KernelLattices.ColumnData.ColumnDataEquations.SplitPairOfGen`, lines 55–58)

Pullback preserves H, α and carries each summation quotient bijectively onto the moved quotient. The two factors computed in 17.14 are the changes of the character normalization on moving the standard endpoints. Together with the commuting-square identity above, this gives (17.1)–(17.3).

Finally move L, L' by rational unitary maps so that both agree with their standard lattices above p. The basis and dual basis at the split factors give the local maps, and Cayley approximation as in 17.14 gives the rational maps. For the moved pair both averaging indices are prime to p. Its relative determinant ideals have valuation zero above p. For any such norm-one ideal $\mathfrak{a}$, choose $h > 0$ with $\mathfrak{a}^h = (z)$. Then $\mathfrak{a}^{2h} = (z/\bar{z})$, so the principal-ideal character formula makes $\chi(\mathfrak{a})$ a p-adic unit. Expanding the two averages and the prime-to-p translation gives a finite sum of translates of the standard vector with integral coefficients. Each translating point has prime-to-p denominator, since the original shift and the two finite quotients do. The same denominator clearing puts the directions in the standard period lattice. The standard bound and integral substitution apply term by term. Equation (17.1) transports both translations and directions back to L, L'. The measure assertion follows from 17.13. □

Proof of 17.16

Proof. At the split prime the connected ordinary formal group has a multiplicative parameter. Normalize it as follows. On the connected p^n-torsion, the multiplicative coordinate and the pairing with the period 1 both give primitive characters into μ_{p^n}. For the pairing, the denominator-p^n periods modulo periods are the two cyclic split factors in $\mathcal{O}_{K_d}/p^n$. The alternating area pairing is perfect, and the projection of 1 onto the complementary factor is a generator. The congruence on $\bar{v}$ selects the connected factor. The powers relating the two characters are units modulo p^n, compatible under multiplication by p; their inverse limit is a unit in $\mathbf{Z}_p$. Changing the parameter by this unit power makes the coordinate of y exactly ζ_y. In this chart an elliptic function regular on the origin disk and with bounded expansion satisfies

$$\widehat{F(\,\cdot + y)} = S_{\zeta_y}\widehat{F}.$$

Indeed the parameter identifies the formal group law with multiplication of $1 + Z$. Substitution first gives the formula on its convergent coordinate functions, hence on rational functions wherever the denominator is nonzero, and then on their regular germs. We compute the extra factor for the theta section.

Put $f = \widehat{\theta}$. Both f and its inverse have integral coefficients. Here are the details of the coefficient bound. For a nonzero $\alpha \in \mathcal{O}_{K_d}$ put $m = \alpha\bar{\alpha}$ and

$$E_\alpha(z) = \theta(\alpha z)/\theta(z)^m.$$

The period multipliers cancel, so E_α is elliptic with $E_\alpha(0) = 1$. Its possible poles are at the nonzero half-period, and its possible zeros lie over that half-period under α. Neither meets the origin disk: reduction commutes with α, and reduction is injective on the prime-to-p torsion [68, Ch. VII, Proposition 3.1]. Both E_α and its reciprocal have bounded expansions. Explicitly, clear their poles by products of $X - X(P)$ in good Weierstrass coordinates. For such a pole P, $X(P)$ is integral, and in a formal parameter T, $T^2(X - X(P))$ is an integral unit series. The numerator is polynomial in the affine coordinates; multiplication by a power of T bounds its coefficients. Regularity removes the negative powers. The same argument works after the integral change to the multiplicative parameter. The Gauss norm on bounded series is multiplicative, by taking the supremum of the multiplicative Gauss norms on closed disks of radii tending to one. The two series are inverse with constant term one, so both have norm one. Choose $\alpha \equiv 1$ at the chosen prime and 0 at the other. Write $a = \jmath(\alpha)$ and $[a](Z) = (1 + Z)^a - 1$. In

$$f([a](Z)) = f(Z)^m \widehat{E}_\alpha(Z)$$

the coefficient of the unknown j-th coefficient of f, for $j > 0$, is $a^j - m$, a unit. Starting with $f(0) = 1$ proves integrality degree by degree. Inversion preserves it.

Lean: `GlobalTheory.MomOne.ThetaQuot.theta_quot` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.ThetaNormQuotientEllipticUnitSeries`, lines 71–140)
Lean: `GlobalTheory.MomOne.theta_series_integral` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.CmEvenThetaSeriesChartCoeffIntegral`, lines 15–76)

First take $t = 0$. Put $g = \widehat{W_y\theta}$, $h = S_{\zeta_y} f$, and $R = g/h$. The denominator is a unit series, since $S_{\zeta_y}(f^{-1})$ is its inverse. Also $g(0) \neq 0$: the point of y has p-power order, whereas a zero of θ represents $1/2$.

For every positive s_1 prime to p, choose $\alpha \in s_1\mathcal{O}_{K_d}$ congruent to 1 and 0 at the two primes modulo p^{n+1}, respectively. Then $m = \alpha\bar{\alpha} \geq 2$, $p \mid m$, $s_1 \mid m$, and $a = \jmath(\alpha)$ is a unit congruent to 1 mod p^n. We have $W_{\alpha y}\theta = W_y\theta$. $(\alpha - 1)y$ is a period: the chosen component of $\alpha - 1$ and the other component of v clear its denominator. Moreover $\overline{y(\alpha - 1)y}$ is integral, since $v\bar{v}$ is divisible by p^{2n}. In the principal lattice $H(z, w) = 2z\bar{w}/\sqrt{d}$, $\alpha(z) = e^{\pi i z\bar{z}}$; inserting the period in the composition law for W therefore gives factor one. In coordinates this last assertion uses $\operatorname{Re} h + \operatorname{Im} h/\sqrt{d} \in \mathbf{Z}$ for $h \in \mathcal{O}_{K_d} = \mathbf{Z} + \mathbf{Z}(1 + i\sqrt{d})/2$, with either sign of the imaginary term.

The two factors in the definition of W_y, including its constant semicharacter, give the following identity of germs:

$$(W_{\alpha y}\theta)(\alpha z) = (W_y\theta)(z)^m E_\alpha(z + y).$$

Indeed $H(\alpha z, \alpha y) = mH(z, y)$ and $\alpha(\alpha y) = \alpha(y)^m$, where in the latter equality $\alpha(\,\cdot\,)$ denotes the semicharacter. Thus

$$g([a](Z)) = g(Z)^m S_{\zeta_y}\widehat{E}_\alpha(Z).$$

The same equation holds for h. Apply S_{ζ_y} to the equation for f and use $\zeta_y^a = \zeta_y$, since $a \equiv 1 \bmod p^n$. Division gives $R([a](Z)) = R(Z)^m$. Its constant coefficient $c \neq 0$ satisfies $c^{m-1} = 1$. If $r_j Z^j$ is its first nonconstant term, comparison gives $a^j r_j = mc^{m-1} r_j$, impossible since a is a unit and $p \mid m$. Hence $R = c$. Take $s_1 = 1$ to obtain one exponent m_1, then $s_1 = m_1 - 1$ to obtain m_2 divisible by $m_1 - 1$. Now $c^{m_2} = c^{m_2 - 1} = 1$, so $c = 1$. This determines the scalar for the chosen section.

For the given t, write $F = U_t\theta/\theta$. The function $F(sz)$ is elliptic because st is a period. Its poles avoid the origin disk since s is a unit there and the denominator vanishes at the nonzero half-period. Apply elliptic translation to $F(sz)$ at y/s. The coordinate at y/s has s-th power ζ_y. Substitution by $[s]$ is invertible with inverse $[1/s]$, so cancellation of this substitution gives $\widehat{F(\cdot + y)} = S_{\zeta_y}\widehat{F}$. Multiply by the identity for $W_y\theta$, using $W_y(F\theta)(z) = F(z + y)W_y\theta(z)$. This proves the claim for $U_t\theta$, including when its value at the origin vanishes. □

Proof of 17.17

Proof. The host hypotheses give d prime, $d \equiv 7 \bmod 16$, and p odd split with $p \nmid N\mathfrak{c}$. A basis as stated exists. Choose bases locally at both primes, lift them simultaneously modulo those primes in M, and apply Nakayama's lemma. Independence follows by extension to either completion. Multiplication by a common prime-to-p denominator puts the dual basis in the dual module.

We specify the normalization and the norm bound. Write $h = N\mathfrak{c}$. The isogeny from the lattice $h\mathcal{O}_{K_d}$ to $\mathfrak{c}$ has degree h prime to p; it identifies integral invariant differentials up to a unit. The discriminant has weight twelve, so $|\jmath(\delta/\delta_0)|_p = 1$. For the principal chart of 17.16, $\Omega_0 = \omega\jmath(\delta_0)$ is a unit by the same weight-twelve formula on a good model. The ideal-curve period is $\Omega = h\Omega_0\jmath(\delta/\delta_0)$. Each differential generator is a unit at the chosen place. Indeed the split coordinate lattice is integral and contains a vector with unit coordinate at the specified type, by taking the corresponding coordinate of a split basis and its trace dual. Thus $|u_\tau|_p = 1$.
Lean: `GlobalTheory.MomN.PeriodData.laws_and_chart` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.LawsAndChart`, lines 12–32)
Lean: `GlobalTheory.MomDelta.delta_ratio` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.DiscOneScaleRatioIdealRingAlgebraicUnit`, lines 101–142)
Lean: `GlobalTheory.EigenGenerator.norm_generator_eq_one` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.NormPadicEmbeddingLeOneOfMemInverseHalfDifferent`, lines 202–221)

For the elliptic input to 17.15, pull θ_Λ back by $z \mapsto \delta h z$. The pulled-back Hermitian form is h times the principal form. Division by $\theta(z)^h$ gives an elliptic function of value one at zero. Its zeros and poles have order dividing $2h$ as points of the elliptic curve and avoid the origin disk. The denominator calculation of 17.16, applied also to its reciprocal, gives norm one. The ordinary expansion of the unshifted ideal-curve theta is therefore integral. Directions in $\delta\mathfrak{c}$ substitute c/h, $c \in \mathfrak{c}$, integral at the chosen split prime.

We prove that prime-to-p translation preserves the Gauss norm on any finite linear combination of translates of θ^h. Let $E/\mathcal{O}_K$ be the good Weierstrass model of the principal curve $\mathbf{C}/\mathcal{O}_{K_d}$ over a finite extension $K/\mathbf{Q}_p$ containing the relevant prime-to-p torsion points and coefficients, with smooth irreducible special fibre E_k and reduced origin $0_k \in E_k$. Let v_η be the discrete valuation on $K(E)$ at the generic point η of E_k, normalized to extend v_p on K, and put $|f|_\eta = p^{-v_\eta(f)}$. Let $f \in K(E)^\times$ be a rational function whose horizontal polar divisor avoids 0_k. Choose $c \in K^\times$ with $v_\eta(c^{-1}f) = 0$ and set $f_0 = c^{-1}f$. Because f_0 has neither vertical zero nor vertical pole along E_k and no horizontal pole through 0_k, it belongs to the local ring $\mathcal{O}_{E,0_k}$ and reduces to a nonzero rational function $\bar{f}_0 \in k(E_k)$ regular at 0_k. In the completed local ring $\widehat{\mathcal{O}}_{E,0_k} \simeq \mathcal{O}_K[[T]]$ the expansion $\widehat{f}_0(T) \in \mathcal{O}_K[[T]]$ reduces modulo $\mathfrak{m}_K$ to the formal expansion of $\bar{f}_0$ at the smooth point 0_k, which is nonzero, so its Gauss norm in T is one. After extension of scalars to $\mathcal{O}_{\mathbf{C}_p}$, the normalized multiplicative parameter Z is a power series in T with integral coefficients and unit linear coefficient, and so is its inverse; such a substitution preserves the Gauss norm. Thus $\|\widehat{f}_0\|_{\mathrm{Gauss}} = 1$ and hence

$$\|\widehat{f}\|_{\mathrm{Gauss}} = |c|_p = |f|_\eta.$$

Moreover, for every $P \in E(K)$, translation τ_P extends to an $\mathcal{O}_K$-automorphism of the abelian scheme E, so it fixes η and satisfies $|f \circ \tau_P|_\eta = |f|_\eta$. The same holds for negation $[-1]$.

First take t with $p \nmid s$, $st \in \mathcal{O}_{K_d}$, and $t \notin \frac{1}{2} + \mathcal{O}_{K_d}$, and define $F_t(z) = (U_t\theta)(sz)/\theta(sz)$. For any period $\lambda \in \mathcal{O}_{K_d}$, the Heisenberg commutation law gives $W_{s\lambda}(U_t\theta) = e^{-2\pi i B(\lambda, st)} U_t(W_{s\lambda}\theta) = U_t\theta$ since $st \in \mathcal{O}_{K_d}$, so numerator and denominator share the same period multiplier and F_t is elliptic. By the definition of U_t,

direct cancellation yields $F_t(z)F_{-t}(z+t/s)=1$, and evenness of θ gives $F_{-t}(-z)=F_t(z)$. Evaluating $|\cdot|_\eta$ on these two identities gives $1=|F_t|_\eta|F_{-t}(\cdot+t/s)|_\eta=|F_t|_\eta|F_{-t}|_\eta=|F_t|_\eta^2$, whence $|F_t|_\eta=1$.

Now let $h\geq 1$ and let $G=\sum_j c_jU_{t_j}(\theta^h)$ be a nonzero finite linear combination of prime-to-p translates of θ^h with coefficients in $\jmath^{-1}(K)$ (the case $G=0$ being immediate), of Hermitian form hH. For a prime-to-p shift t with $t\notin\frac{1}{2}+\mathcal{O}_{K_d}$, choose a positive integer s prime to p such that $st\in\mathcal{O}_{K_d}$ and $st_j\in\mathcal{O}_{K_d}$ for every j. Set

$$F(z)=\frac{G(sz)}{\theta(sz)^h},\qquad F^{(t)}(z)=\frac{(U_tG)(sz)}{\theta(sz)^h}.$$

Both F and $F^{(t)}$ are elliptic by the same period-multiplier check. Since G and U_tG are holomorphic, every pole of F and of $F^{(t)}$ is a zero of $\theta(sz)$, that is, a point with $sz\in\frac{1}{2}+\mathcal{O}_{K_d}$. Because p is odd and $p\nmid s$, these are nonzero prime-to-p torsion points, and reduction modulo p is injective on prime-to-p torsion [68, Ch. VII, Proposition 3.1]; hence no pole of F or $F^{(t)}$ reduces to 0_k. The Gauss-norm identification therefore applies to both F and $F^{(t)}$. Furthermore, expanding U_tG and F_t gives the identity in $K(E)$

$$F^{(t)}(z)=F(z+t/s)\,F_t(z)^h.$$

Taking $|\cdot|_\eta$ and using translation invariance and $|F_t|_\eta=1$ gives $|F^{(t)}|_\eta=|F|_\eta$. Since $\widehat{\theta}(Z)$ is an integral unit series by 17.16 and $[s](Z)=(1+Z)^s-1$ is an automorphism of $\mathcal{O}_K[[Z]]$ for $s\in\mathbf{Z}_p^\times$, multiplying by $\widehat{\theta}([s](Z))^h$ and inverting $[s]$ yields

$$\|\widehat{U_tG}\|_{\mathrm{Gauss}}=\|\widehat{F^{(t)}}\|_{\mathrm{Gauss}}=|F^{(t)}|_\eta=|F|_\eta=\|\widehat{F}\|_{\mathrm{Gauss}}=\|\widehat{G}\|_{\mathrm{Gauss}}.$$

If instead $t\in\frac{1}{2}+\mathcal{O}_{K_d}$, write $t=t_1+t_1$ with $t_1=t/2$. Because p is odd, t_1 has prime-to-p order and $t_1\notin\frac{1}{2}+\mathcal{O}_{K_d}$. Since $B(t_1,t_1)=0$, we have $U_tG=U_{t_1}(U_{t_1}G)$, and applying the preceding equality twice gives $\|\widehat{U_tG}\|_{\mathrm{Gauss}}=\|\widehat{G}\|_{\mathrm{Gauss}}$.

For the ideal curve use the finite theta space on the sublattice $\delta h\mathcal{O}_{K_d}$. Its sections in this frame are spanned by translates of θ^h of prime-to-p order. To see spanning, diagonalize one of the two cyclic translation groups of order h; translations in the other cyclic group permute its characters transitively by the perfect commutator. Each character line has dimension one by descent to a principal lattice. The invariant span of the translates of the nonzero section is therefore the entire space. Applying the preceding norm equality in that frame proves the prime-to-p bound for the ideal curve. Thus 17.15 applies to U_te, also for the companion lattices.

Lean: `GlobalTheory.MomOne.TranslateNorm.bd_translate` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.CpSeriesBoundedOneTorsionTranslateCmTheta`, lines 72–168)
Lean: `GlobalTheory.MomN.latInt_th0_of_split_prime` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.EvenThetaFunctionCmLatticeIntegralSplitPrime`, lines 23–142)
Lean: `GlobalTheory.MomN.latInt_opU_of_coprime` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.CmThetaTranslatesPrimeToQIntegral`, lines 202–236)

Choose an orthogonal frame in the rational period module whose squared lengths for the K_d-valued Hermitian form are positive integers a_j prime to p. Locally a basis and its dual on the two split factors give a vector of unit length. Lift such a vector, and project to its orthogonal complement by dividing by its unit length. This preserves the localized module. Repetition and clearing prime-to-p denominators give the frame. In this frame the section e is a finite linear combination $\sum_k c_k\prod_j U_{t_{kj}}(\theta^{a_j})$ of products of prime-to-p translates of θ^{a_j}, by applying the spanning argument just given in each factor. The coefficients c_k may be taken algebraic for the given algebraic jets: a basis chosen among the translates has an invertible minor of Taylor coefficients in algebraic scaled directions, and solving with that minor produces algebraic c_k. Applying U_t shifts the factors, and the composition phases are constants independent of y.

Each factor $U_{t_{kj}}(\theta^{a_j})$ has integral transform by the preceding norm equality, so the finite algebraic linear combination has bounded binomial moments in the orthogonal frame, with bound $\max_k|\jmath(c_k)|_p$ (no integrality of the coefficients c_k is needed at this step). By 17.13 this determines a bounded $\mathbf{C}_p$-valued measure μ on $\mathbf{Z}_p^r$ of some finite norm N. The change from the orthogonal frame to (β_j) is invertible over the chosen $\mathbf{Z}_p$, since both are bases locally and their trace-pairing denominators are units there. Push μ forward by the dual coordinate change to obtain a bounded measure μ_t, so that c is the coordinate vector of $x(c)$ in the basis (β_j). Let (β_j^*) be the algebraic trace-dual basis to (β_j) and choose a positive integer s prime to p with $sh\beta_j^*\in M$ for every j. In the analytic period corresponding to $sh\beta_j^*$, the normalized derivative of

Lemma 17.15 reads sc_j, since $sh\omega\jmath(\delta)/\Omega = s$. Thus the pushforward of μ_t by $c \mapsto sc$ has integral binomial moments by Lemma 17.15 and norm at most one by Lemma 17.13. Multiplication by s is an automorphism of $\mathbf{Z}_p^r$, so $\|\mu_t\| \le 1$.

Lean: `GlobalTheory.MomN.Bridge.isMom_dual` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.MomentMeasureNormLeOneIffLatticeIntegral`, lines 114–144)

We check that this change of frame also transports the Weyl operator. At torsion level p^n write an allowed shift as $y = N_0 z/p^n$, with $z \in M$ divisible by p^{2n} on the other half. For $y \in N_0 vM$ one can take $n = 1$, since pv is the square of the level factor that is a unit in the chosen component. Choose $\epsilon \in N_0\mathcal{O}_{K_d}$ congruent to $1, 0$ modulo the two prime powers p^{2n}. Reduce the chosen coordinates of $N_0 z$ in the basis β_j modulo p^n, obtaining integers b_j, and put $y' = \epsilon \sum b_j\beta_j/p^n$. The difference $l = y - y'$ lies in $N_0 M$ and is divisible by p^n on the other half locally. Indeed these assertions follow at the chosen half by the coordinate congruences, at the other half by divisibility by p^{2n} before division, and away from p by $N_0 \mid \epsilon$. The period l fixes $U_t e$, since $N_0 t$ is a period. The Hermitian cocycle of l, y' is integral: on the two factors the possible p^n denominator of y' pairs with the p^n divisibility of l, and its other component already contains p^n. Thus $W_y U_t e = W_{y'} U_t e$, and $[y, x] = [y', x]$ on periods.

Lean: `GlobalTheory.MomN.goodLift_eq` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.GoodLiftEqIdempotentLift`, lines 63–333)

Multiply this lift, if necessary, by a common multiple of the prime-to-p frame lengths and translate denominators congruent to 1 mod p^n. The same calculation leaves its operator and bracket unchanged. Each orthogonal coordinate now satisfies the hypotheses of 17.16, with an integer s clearing all the translates in that coordinate. The transform of each product is therefore shifted by its torsion coordinates ζ_j; the finite sum is shifted by the same substitution. On a transform,

$$\prod_j (\zeta_j(1 + Z_j))^{c_j} = \prod_j \zeta_j^{c_j \bmod p^n} \prod_j (1 + Z_j)^{c_j}.$$

Here the identity first holds for nonnegative integers and extends by continuity. Orthogonality gives the first factor in the new coordinates as $\jmath([y, x(c)])$: the Hermitian pairing is the sum of the coordinate pairings, and the dual change of variables preserves this sum. This identifies the translated measure as that character times μ_t.

Finally a directional derivative in the rational matrix direction Y reads the linear function $\Omega^{-1} \sum_j c_j \jmath(\mathrm{Tr}_{E/K_d} \sum_{kl} Y_{kl} \overline{(\beta_j)_{kl}})$, using the chosen embedding of K_d. Extension in the direction variable to individual type and matrix coordinates gives the entries of $x(c)^\dagger/\Omega$ in those embeddings. Multiplication by the differential generators and repetition for ν give $M_{x(c)}(\nu)$. Since $W_t = \alpha(t)U_t$, the one remaining scalar is $\jmath(\alpha(t))$, a root of unity independent of y, ν. This proves (17.4).

Lean: `GlobalTheory.MomOne.Translates.moments_of_translates` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.CmLatticeTranslationOperatorEqPhaseMulNegComp`, lines 288–368)

Lean: `GlobalTheory.MomT.jetMoments_of_unitPeriod` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.JetMomentsOfUnitPeriod`, lines 19–142)

□

Proof of 17.18

Proof. All index computations can be made on the primary factors. At $P \mid \varpi$, $M'_P \subseteq M_P$ imposes one residue equation per source basis vector. Thus M/I injects into the product of f copies of $\mathbf{F}_p^a$. At $\bar{P}$ the inclusion is reversed, and away from p the lattices agree. The perfect alternating pairing between the relative quotients of the two self-dual period lattices gives $[M' : I] = [M : I]$.

For $b \in I$, the condition $rb \in Z_0$ at $P \mid \varpi$ is $b \in \varpi M_P$, since $c_A g \bar{\varpi}$ is a unit there. At row primes it holds for every $b \in I$, by the factor g and integrality of b. Away from the primes over ϖ, the element ϖ is a unit. Thus the preimage of Z_0 in I is ϖM; the inclusion $\varpi M \subseteq I$ also follows directly at the primes over ϖ. Further, $g\varpi M \subseteq M \cap B$. Taking the indices in

$$g\varpi M \subseteq \varpi M \subseteq I \subseteq M$$

gives the asserted divisibility, since $[\varpi M : g\varpi M]$ is prime to p.

At primes away from p, every class of $B/(B \cap M)$ has a representative in pB. These representatives annihilate rI, because $prI \subseteq B$ and B is integral. At p the quotient is killed by p; its classes are represented

on the enlarged conjugate half. Pairing there with rI_P is pairing with I_P, since r is a unit at P. Its annihilator is the dual of $I_P = M'_P$, hence $M'_{\bar{P}}$ modulo $M_{\bar{P}}$. The containment $M'_{\bar{P}} \subseteq p^{-1}M_{\bar{P}} = B_{\bar{P}}$ follows by dualizing $pM_P \subseteq M'_P$. This identifies the p-part with M'/I. The Chinese remainder theorem realizes the local representatives by elements of $M' \cap B$; at primes away from p prescribe the zero class. The two primary parts have coprime orders, so their sum is direct. □

Proof of 17.19

Proof. Use representatives standard at the primes over ϖ. Write $R(x)$ for the requirement that at each such P, $xN_P \subseteq N'_P$ and $l'_P x = l_P$ on N_P. Set $\Xi = \mathbf{1}_R\Xi_0$, $H_S = \mathrm{Hom}_S(N, N')$, and $M = M(N, N')$. For a positive integer h let $B(h)$ have localizations hM_Q on S, $h^{-1}M_Q$ on $\bar{S}$, and M_Q elsewhere. Let $A(h)$ interchange the first two prescriptions. Write $B_0 = B(pg)$. The curve and half-different factors of M are units at the test primes and above p, by the prime exclusions in the constructed family.

A class $x \in (c_A pg)^{-1}H_S/(pgH_S)$ (in particular any test class $x \in c_A^{-1}H_S/(pgH_S)$) has a lift $t_x \in A(c_A pg)$ with $t_x - x \in pgH_S$, by the Chinese remainder theorem after clearing a common denominator. Its theta section is

$$\Theta_x = \frac{1}{n} W_{-t_x} \sum_{y \in B_0/(M \cap B_0)} W_y e_{N,N'}, \qquad n = [M \cap A(pg) : M \cap A(pg) \cap B_0]. \tag{A.112}$$

The finite Fourier transform of the indicator of zero on the quotient of order n is the constant $1/n$. Translation of the indicator gives W_{-t_x}. Two allowed lifts differ by an element of B_0; the integral period cocycle on the half-lattices and reindexing the sum give the same section. The natural map into $M/(M \cap B_0)$ gives $n \mid [M : M \cap B_0]$. On every contributing class $c_A g t_x \in M$: at $P \mid \varpi$ use integrality of x and its congruence with t_x; at the other primes use the defining localizations of $A(c_A pg)$.

Lean: `GlobalTheory.MomT.ThShape.thInd_shape` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ThetaMovedCosetIndicatorVacuumShape`, lines 284–395)

Write

$$c_\chi(N, N') = \chi(\det(L_0, N))^{b/2}\chi(\det(L'_0, N'))^{-a/2}$$

for the Weil-splitting normalization of 17.15, and put $\theta_{N,N'} = c_\chi(N, N')^{-1}e_{N,N'}$ for the unnormalized theta vector $F_{N,N'}$ of 17.15. Dividing (17.2) and (17.3) by c_χ cancels the relative character powers by multiplicativity of relative determinants and of χ, yielding

$$\theta_{N,N''} = I_{P(N,N'),P(N,N'')}\theta_{N,N'}, \qquad \theta_{N_1,N'} = I_{P(N,N'),P(N_1,N')}\theta_{N,N'}.$$

Set $\widetilde{\Theta}_{N,N',x} = c_\chi(N, N')^{-1}\Theta_x$, obtained from (A.112) by replacing $e_{N,N'}$ with $\theta_{N,N'}$. Let $J_\nu \in K$ be the rational jet sum determined by

$$\sigma_0(J_\nu) = \sum_x \sigma_0(\Xi(x))\,\partial_c^\nu(\widetilde{\Theta}_{N,N',x})(0) = c_\chi(N, N')^{-1}\sum_x \sigma_0(\Xi(x))\,\partial_c^\nu(\Theta_x)(0),$$

where $\sigma_0 : K \hookrightarrow \mathbf{C}$ is the complex embedding and the index ν lies in the rational-jet range stipulated in the statement. The point kernel $k(d, d')$ equals the rational jet operator attached to $(J_\nu)_\nu$, scaled by the step-character prefactor

$$\mathrm{scal}_i(N, \mathrm{fr}; N', \mathrm{fr}') = \mu_{\mathrm{inv},i}(N, \mathrm{fr})\,\mu_{\mathrm{inv},i}(N', \mathrm{fr}')^{-1},$$

where

$$\mu_{\mathrm{inv},i}(N, \mathrm{fr}) = \mu_{\mathrm{off},i}(N) \prod_{Q \in S_{\mathrm{row}}} m_{I,i}(Q, \det \mathrm{fr}_Q), \qquad \mu_{\mathrm{off},i}(N) = \prod_{\substack{v \notin S_{\mathrm{row}}|_{E^+} \\ v \text{ split in } E/E^+}} m_{U,i}(\mathrm{sel}(v))^{\mathrm{ord}_{\mathrm{sel}(v)} \det(L_0, N)}.$$

Here E^+ is the maximal totally real subfield of E, $S_{\mathrm{row}}|_{E^+}$ denotes the places of E^+ below S_{row} (above places v not split in E/E^+, self-duality of L_0 and N forces $\mathrm{ord}_Q \det(L_0, N) = 0$), $\mathrm{sel}(v)$ is the chosen prime of E over a split place v, $m_{U,i}(Q) \in K^\times$ is the unramified step-character value at $Q \notin S_{\mathrm{row}}$ (of infinity type $e_{i,\tau}$), and $m_{I,i}(Q, \cdot)$ is the local character at $Q \in S_{\mathrm{row}}$. At standard representatives above p, both $c_\chi(N, N')$ and $\mathrm{scal}_i(N, \mathrm{fr}; N', \mathrm{fr}')$ are units at w: the relative determinant ideals have order zero above p, every unramified

character value away from p is a w-unit, and the local frame factors at the row primes are w-units. We determine $(\iota_w J_\nu)$ on any finite set of rational indices.
Lean: GlobalTheory.ColPeriod.kerOf_thDistC (Development.AbelianVarietyTheory.Proofs.SatakeParameters.ValuationLeOneOfMemZmu, lines 198–220)
Lean: GlobalTheory.CharVal.charValAtT (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.CharacterValueAtT, lines 865–912)

At an unseen place put $v = \varpi/\bar{\varpi}$. Localizing at $S, \bar{S}$ and their complement gives

$$gvM \subseteq B_0, \qquad gv^{-1}B_0 \subseteq M, \qquad gpB_0 \subseteq M. \tag{A.113}$$

For example at $P \mid \varpi$, $v = p/\bar{\varpi}^2$ with $\bar{\varpi}$ a unit. The conjugate place reverses the factors, and $p, \varpi, \bar{\varpi}$ are units at the row primes.

Represent the image of pB_0 modulo M by R_2. Each $z - t_x$, $z \in R_2$, has denominator dividing $c_A g$ modulo M. Choose N_1 prime to p divisible by $c_A g$ and all the integers N_0 of 17.17 for these translates. Represent the image of $N_1 vM$ modulo M by R_1. In $B_0/(B_0 \cap M)$ these two images form a direct sum. The first is killed by g, the second by p because $pv = \varpi^2$, and $N_1 gB_0 \subseteq N_1 vM$ by (A.113). Bézout for $N_1 g, p$ proves that they generate. Multiplication by N_1 preserves the image of vM modulo M, and vM is self-dual since $v\bar{v} = 1$. The perfect alternating pairing on the two relative quotients gives

$$|R_1| = [vM : vM \cap M] = [M : vM \cap M].$$

For fixed z apply 17.17 with $t = z - t_x$ to every $y \in R_1$. In (A.112) commute the period operators to this order. The operators at z, y commute on B_0; moving the displacement $-t_x$ past y contributes $[y, -t_x]$. Thus at a matrix table point $b \in M$ the y-dependent phase is $[y, b - t_x]$. It is a character of the indicated finite image: $N_1(b - t_x) \in M$; a representative in $N_1 vM \cap M$ belongs to $N_1 M$, since $pvM \subseteq M$ and N_1 is prime to p. For a character nontrivial at a class y_0, translation by y_0 multiplies its sum by a scalar different from one, so its sum is zero. The trivial character contributes $|R_1|$. This is a single orthogonality test at each table point, common to all the jet coordinates. In the latter case $N_1(b - t_x)$ pairs integrally with vM, hence lies in vM. Locally at $P \mid \varpi$ it follows that $b - t_x \in pM_P$, so $R(b)$ holds. The denominator is absorbed because

$$n \mid [M : M \cap B_0] \mid [M : M \cap vM]\,[M \cap vM : M \cap vM \cap B_0].$$

The second index is prime to p, since its quotient is killed by g. Hence $|R_1|/n$ is integral at p. More explicitly, in using 17.13 take one approximation for all pairs (y, ν) for this fixed z. Orthogonality then applies to each approximate table point. The original tolerance may be made arbitrarily small compared with $|1/n|_p$. Summing the resulting finite approximations over z and the test classes proves the unseen row of the table below.
Lean: GlobalTheory.MomT.CosetBar.clo_class (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.RelIndexNeZeroOfSmul, lines 207–247)
Lean: GlobalTheory.MomT.CosetBar.moments_thInd (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.RelIndexNeZeroOfSmul, lines 251–457)

At a seen place put $D' = L_{d'}$, $N'' = D' + \varpi(D')^\vee$, $M'' = M(N, N'')$, and $I = M \cap M''$. At $P \mid \varpi$, $N''_P = \ker l'_P$; at the conjugate it is the dual of that kernel, and elsewhere it is N'. These descriptions give self-duality and the inclusions $\varpi N' \subseteq N''$, $\bar{\varpi} N'' \subseteq N'$. Write $F = fa$. For standard local source basis vectors e_j consider

$$M \longrightarrow \prod_{P \mid \varpi} k(P)^a, \qquad b \longmapsto \big(l'_P(be_j)\big)_{P,j}.$$

Its kernel is contained in I. The assertion at $P \mid \varpi$ follows by testing a basis; away from those primes $\varpi b \in M''$ and ϖ is invertible. The target has order p^F by 37.21. Taking the index of the kernel and then that of I proves

$$h_1 = [M : I] \mid p^F.$$

The alternating pairing gives a perfect duality between M/I and M''/I, since the annihilators of M and M'' are themselves. Thus $[M'' : I] = h_1 = p^\ell$, $0 \le \ell \le F$.
Lean: GlobalTheory.MomT.MomentCompanion.companion_gen (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.MomentCompanion, lines 7–57)
Lean: GlobalTheory.MomT.IdxBound.relIndex_per_dvd (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.RelIndexPerDvd, lines 91–117)

Put $r_0 = c_A g\bar{\varpi}$, $Z_0 = pgH_S$, and let J represent $r_0 I/(r_0 I \cap Z_0)$. Addition of its elements preserves the test, by the row periodicity and $I_P N_P \subseteq \ker l'_P$. For $b \in M$, $r_0 b \in Z_0$ if and only if $b \in \varpi M$. At $P \mid \varpi$ cancel the units $c_A, g, \bar{\varpi}$; at a row prime b is already integral and p is a unit. Division by ϖ preserves the other lattice conditions. Since $\varpi M \subseteq I$, this gives $|J| = [I : \varpi M]$. Also $g\varpi M \subseteq M \cap B_0$ by the local definitions. Lemma 17.18, applied with the row primes and the marked primes above ϖ, identifies the refinement quotient and gives

$$n \mid [M : g\varpi M] = h_1 |J| [\varpi M : g\varpi M], \qquad v_p(n) \le \ell + v_p(|J|). \tag{A.114}$$

Group the indicators in translates of J and use the lifts $t_x + j$. Each W_{-j} fixes $e_{N,N'}$. The sum of the characters produced by commuting it through W_y is $|J|$ on $B' = \{y \in B_0 : B(r_0 I, y) \subseteq \mathbf{Z}\}$ and zero elsewhere. By the annihilator assertion of Lemma 17.18 with $M' = M''$ and $B = B_0$, the surviving classes modulo M consist of h_1 classes of p-power order and the image of pB_0 of order prime to p. Thus the sum over the first factor is a complete companion-lattice average for each fixed element of the second. The first subgroup pairs integrally with $r_0 I$ by self-duality of M''. The second does so because $pr_0 I \subseteq B_0$ and B_0 is integral. Conversely, for $y \in B'$, use $r_0 \bar{\varpi} M'' \subseteq r_0 I$ and move scalars across the pairing to get $\varpi^2 c_A g y \in M''$. Also $g(\bar{\varpi}/\varpi)y \in M$ by (A.113), giving $\bar{\varpi} c_A g y \in M''$. Since $\varpi^2, \bar{\varpi}$ are coprime, $c_A g y \in M''$. Bézout for $c_A g, p$ gives the required decomposition. The images intersect trivially: the first is killed by p since $\bar{\varpi} M'' \subseteq M$, the second by g. Finally $g\varpi M'' \subseteq B_0$ and $\bar{\varpi} M'' \subseteq M$; Bézout for $g\varpi, \bar{\varpi}$ shows that $M'' \cap B_0$ represents all of M''/I.

By (17.2) the sum over M''/I contributes $\sqrt{h_1}\,\kappa e_{N,N''}$, where

$$\kappa = \chi(\det(N', N''))^{a/2} = \left(\prod_{\tau} \tau(\bar{\varpi}/\varpi) \right)^{a/2}.$$

The local determinant ideal has order one at $P \mid \varpi$, minus one at its conjugate, and zero elsewhere. At a seen embedding $|\jmath(\kappa)|_p^2 = p^F$.

For $z \in pB_0$ the remaining translate $t_1 = z - t_x$ satisfies $m_* := c_A g \varpi t_1 \in M''$. Write $A_1 c_A g \bar{\varpi} + B_1 \varpi = 1$ in $\mathcal{O}_E$ and put $t' = B_1 \varpi t_1$. Then $c_A g\, t' \in M''$. Use 17.17 for this t' and the companion pair, and write $A_2 N_0 + B_2 p = 1$. Direct substitution gives

$$t_1 = t' + N_0(\bar{\varpi}/\varpi) A_2 A_1 m_* + B_2 \bar{\varpi}^2 A_1 m_*.$$

The last term is a period and the middle term is an allowed displacement for the companion measure. The Heisenberg composition phases are roots of unity. The outer coefficient is therefore $b_z = (|J|/n)\sqrt{h_1}\,\kappa$ times a root of unity. By (A.114),

$$|\jmath(p^F b_z)|_p^2 = p^{-2F} \left| \frac{|J| h_1}{n} \right|_p^2 |h_1|_p^{-1} |\jmath(\kappa)|_p^2 \le p^{-2F+\ell+F} \le 1.$$

A period $b \in M''$ sends $N_{\bar{P}}$ into $N''_{\bar{P}}$; taking adjoints gives $b^\dagger(\ker l'_P) \subseteq N_P$. Applying 17.13 and summing the finite class contributions with their integral test coefficients proves that, on every finite set of rational indices, $(\iota_w J_\nu)_\nu$ lies in the closure of the integral span of

	table	support
seen	$(p^{-fa} M_b(\nu))_\nu$	$b^\dagger(\ker l'_P) \subseteq N_P \quad (P \mid \varpi)$
unseen	$(M_b(\nu))_\nu$	$R(b)$.

(A.115)

Lean: `GlobalTheory.MomT.UnitTr.moments_unit_translate` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.MomentsTranslateRationalPointUnitDenominator`, lines 71–159)
Lean: `GlobalTheory.MomT.CosetSum.moments_coset_sum` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.MomentsCosetSum`, lines 363–491)
Lean: `GlobalTheory.MomT.type_p` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.MemSuppBarOfNe`, lines 267–325)
Lean: `GlobalTheory.MomT.type_bar` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.MemSuppBarOfNe`, lines 248–265)

We pass from tables to the polynomial value lattice and record the determinant-twist compatibility across the step. Write $\lambda = \tilde{\lambda}_i$ and $\lambda' = \tilde{\lambda}_{i+1}$ for the source and target point weights in zero-based coordinates, where $\tilde{\lambda}_{j,\tau,r} = \lambda^u_{j,\tau,r} - js$ (matching 17.10 at $j = 0$). Put $\sigma_\tau = e_{i,\tau} - a/2$ and $\sigma'_\tau = e_{i,\tau} - b/2$, where $e_{i,\tau} \in \mathbf{Z}$ is the common chain parameter of the adjacent point weights. Both σ_τ and σ'_τ are integers because a, b are even. The point weights satisfy

$$\lambda'_{\tau,r} + \sigma_\tau = \lambda_{\tau,r} + \sigma'_\tau \quad (0 \le r < a), \qquad \lambda'_{\tau,r} = -\sigma_\tau \quad (a \le r < b).$$

Hence the target determinant shift (the last coordinate of λ'_τ) is $h'_\tau = \lambda'_{\tau,b-1} = -\sigma_\tau$, agreeing with the chain determinant shift $s_{i,\tau} = -h'_\tau = \sigma_\tau$ and with the zero-based last-coordinate convention $b_v = -\lambda^u_{0,\tau_w(v),k-1}$ of 17.10. Furthermore $\sigma_\tau - \sigma'_\tau = (b-a)/2 = s$, and writing $h_\tau = \lambda_{\tau,a-1}$ for the source determinant shift, dominance of λ'_τ gives

$$n_\tau := h_\tau + \sigma'_\tau = \lambda'_{\tau,a-1} + \sigma_\tau \ge \lambda'_{\tau,a} + \sigma_\tau = 0,$$

which equals the last coordinate of the step partition $\nu^u_{i+1,\tau}$.

Lean: `GlobalTheory.ColPeriod.chainWeights` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.SubHalfRankInt`, lines 17–69)
Lean: `GlobalTheory.JetPoint.detShift_eq` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsMulOfApprox`, lines 115–118)
Lean: `GlobalTheory.BarStd.sub_eq_half` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.AdmissibleAdjointRationalPointPadicMatrix`, lines 369–384)

On the polynomial part P of a source value first multiply by $\prod_\tau \det(X_\tau)^{n_\tau}$. Substitute the adjoint matrix in the first a columns in target coordinates. Equivalently, make this substitution with matrix indeterminates Y and write the result as $\sum_\nu Q_\nu(X)Y^\nu$. A table at b evaluates Y_τ at $u_\tau \iota_w(\tau(b^\dagger))$. All these matrices are integral in the indicated frames: at the seen half this is the first support condition, and at the unseen half b carries the source marking kernel into the target marking kernel, so its adjoint is integral on their duals.

Each Q_ν has the target polynomial weight $(\lambda'_{\tau,r} - h'_\tau)_{0 \le r < b}$: multiplying by $\det(X_\tau)^{n_\tau}$ adds n_τ full columns to the source diagram, and padding its minors to rank b produces the target diagram with its determinant shift h'_τ removed, since $(\lambda_{\tau,r} - h_\tau) + n_\tau = \lambda'_{\tau,r} + \sigma_\tau = \lambda'_{\tau,r} - h'_\tau$ for $0 \le r < a$ and $\lambda'_{\tau,r} - h'_\tau = 0$ for $a \le r < b$. Matrix substitution preserves the span of these minors by Cauchy–Binet. A linear functional annihilating the target span thus annihilates $\sum Q_\nu Y^\nu$ at every rational matrix Y, and consequently annihilates each coefficient because K is infinite. A source Weyl vector belongs to the dual Weyl lattice, so its polynomial part evaluated in a local source basis B_τ equals $\prod_\tau \det(B_\tau)^{-h_\tau}$ times a polynomial with integral coefficients (that is, has integral coefficients after scaling by $\prod_\tau \det(B_\tau)^{h_\tau}$). Multiplying by $\prod_\tau \det(X_\tau)^{n_\tau}$ therefore introduces the source determinant factor $\prod_\tau \det(B_\tau)^{n_\tau - h_\tau} = \prod_\tau \det(B_\tau)^{\sigma'_\tau}$. Fix a target monomial in a local target basis B'_τ. Write d_ν for its coefficient in the framed Q_ν, and let D be the finite set of indices occurring in the substitution. Equation (A.115) gives, simultaneously on D, sums $J'_\nu = \kappa \sum_j c_j M_{b_j}(\nu)$ approaching the embedded jets, with $|c_j|_p \le 1$, with all b_j in the indicated support, and with $\kappa = p^{-fa}$ or 1 on the respective side. After removing $\prod_\tau \det(B_\tau)^{\sigma'_\tau}$ and κ, the coefficient of each point evaluation is integral because the adjoint matrix maps the target frame into the source frame. Their integral combination has the same property. The error before division is bounded by $\max_{\nu \in D} |d_\nu|_p \max_{\nu \in D} |\iota_w J_\nu - J'_\nu|_p$. The divisor is fixed and nonzero, so closedness of the valuation ring gives the assertion for the actual coefficient. The output is rational, so its integral coefficients lie in $\mathcal{O}_{K,(w)}$. Restoring the target determinant twist $\prod_\tau \det(B'_\tau)^{h'_\tau} = \prod_\tau \det(B'_\tau)^{-\sigma_\tau}$ proves membership in the target dual Weyl lattice after removing κ times the total frame factor $\prod_\tau \det(B_\tau)^{\sigma'_\tau} \det(B'_\tau)^{-\sigma_\tau}$.

For every positive root $j < k$ of GL_b the pairing with $\lambda'_\tau + \rho$ is $\lambda'_{\tau,j} - \lambda'_{\tau,k} + k - j < p$ by the size hypothesis. Over an algebraic closure of the residue field, the Jantzen sum formula for the Weyl module has no terms: for each positive root the indexing integers satisfy $0 < mp < \langle \lambda'_\tau + \rho, \alpha^\vee \rangle < p$ [66, II, 8.19]. Its radical is zero, so the Weyl and induced modules of this highest weight are simple. The canonical map between them is nonzero: the highest bideterminant takes value one at the identity, also after reduction. It is therefore onto, and surjectivity descends to the residue field by faithful scalar extension. The integral cokernel is finitely generated and has zero reduction, so it vanishes by Nakayama's lemma. Apply this at every type, tensor the resulting equalities, and restore the determinant twists in the chosen frames.

Lean: `GlobalTheory.JetPoint.closureIntegral` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsMulOfApprox`, lines 313–398)
Lean: `GlobalTheory.KerT.localization_delAtr_eq_localization_nabAtr` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.LocPDelAtrEqLocPNabAtr`, lines 24–36)

Evaluate the combined frame factor $\prod_\tau \det(B_\tau)^{\sigma'_\tau} \det(B'_\tau)^{-\sigma_\tau}$ on each half. At the seen half the source basis is standard ($B_\tau = I_a$) and the target marking kernel has $\det(B'_\tau) = p$ up to a w-unit. Since $h'_\tau = -\sigma_\tau$, the frame factor together with the table scale p^{-fa} is $p^{\sum_\tau h'_\tau - fa}$ up to a w-unit. Multiplying by the w-unit prefactor $\mathrm{scal}_i(N, \mathrm{fr}; N', \mathrm{fr}')$ proves the seen inclusion. At the unseen half both dual marking frames have determinant of valuation $-e_w$, so the frame factor has valuation $e_w \sum_\tau(\sigma_\tau - \sigma'_\tau) = e_w fs$. Multiplying by the w-unit prefactor $\mathrm{scal}_i(N, \mathrm{fr}; N', \mathrm{fr}')$ proves the unseen inclusion. For arbitrary representatives, choose rational unitary changes $\gamma \in \mathrm{U}(E^a)$, $\gamma' \in \mathrm{U}(E^b)$ moving N, N' to standard representatives above p as in 17.14. The total kernel scalar separates into source and target factors

$$\mathrm{scal}_i(N, \mathrm{fr}; N', \mathrm{fr}')\, c_\chi(N, N')^{-1} = \rho_i(N, \mathrm{fr})\, \rho'_i(N', \mathrm{fr}'),$$

where

$$\rho_i(N, \mathrm{fr}) = \mu_{\mathrm{inv},i}(N, \mathrm{fr})\, \chi(\det(L_0, N))^{-b/2}, \qquad \rho'_i(N', \mathrm{fr}') = \mu_{\mathrm{inv},i}(N', \mathrm{fr}')^{-1}\, \chi(\det(L'_0, N'))^{a/2}.$$

By the infinity type $e_{i,\tau}$ of the step character (with principal-ideal law matching $\chi((z)) = \prod_\tau \tau(z)^{-1}$ of 17.15, so that $\mu_{\mathrm{inv},i}(\gamma N, \gamma \mathrm{fr})/\mu_{\mathrm{inv},i}(N, \mathrm{fr}) = \prod_\tau \tau(\det\gamma)^{-e_{i,\tau}}$), on a determinant change of order one at $P \mid \varpi$ and minus one at $\bar P$ the factor ρ_i changes by valuation σ'_τ times that of $\chi(P\bar P^{-1})$ and ρ'_i by $-\sigma_\tau$ times that of $\chi(P\bar P^{-1})$, where $\tau \in \Sigma$ is the unique type corresponding to P (at a seen place w) or $\bar P$ (at an unseen place w). Since $v_w(\chi(P\bar P^{-1}))$ has opposite sign to $v_w(\tau(\det\gamma))$, these changes cancel the determinant-twist exponents σ'_τ and $-\sigma_\tau = h'_\tau$ in the frame factor and in the GL-action on $V_\lambda, V_{\lambda'}$. Pullback carries the test to $x \mapsto \Xi_0(\gamma'^{-1} x \gamma)$; (17.1) transports the normalized theta section, and the integral frame actions transport the two value lattices equivariantly.

For the neighbor identity fix a rational map x and one $P \mid \varpi$. A lower source neighbor $d_0 \uparrow_P d$ has underlying lattice N_0 with $(N_0)_P = \ker l_P$, $(N_0)_{\bar P} = (\ker l_P)^\vee$, and $(N_0)_Q = N_Q$ for $Q \notin \{P, \bar P\}$. Its marking, if it contributes, is forced to be $z \mapsto l'_P(xz)$ on $\ker l_P$. If a target upper neighbor $d' \uparrow_P d'_1$ contributes, this functional takes value one at pz for $l_P(z) = 1$, by the defining marking of d'_1. It therefore gives a contributing lower source neighbor $d_0 \uparrow_P d$. Conversely, present the move from a contributing lower source neighbor $d_0 \uparrow_P d$ by adjoining $p^{-1}z_0$ to $(N_0)_P$, where $z_0 \in (N_0)_P$ has marking one on d_0. Then $xz_0 \in N'_P$ has marking one on d'. Adjoining $p^{-1}xz_0$ to N'_P gives the underlying lattice N'_1 of the target upper neighbor d'_1 (with $(N'_1)_{\bar P}$ its Hermitian dual and $(N'_1)_Q = N'_Q$ for $Q \notin \{P, \bar P\}$). One can use a global lift by clearing a denominator outside P and inverting it modulo P. Two possible lifts differ by pN'_P, so d'_1 is unique. Writing $\phi_T(d, d'; x)$ for the indicator of $R(x)$, this proves

$$\sum_{d' \uparrow_P d'_1} \phi_T(d, d'_1; x) = \sum_{d_0 \uparrow_P d} \phi_T(d_0, d'; x)$$

for every rational map x. Every x with a nonzero term on either side satisfies $x \in \mathrm{Hom}_P(N, N'_1) \cap \mathrm{Hom}_P(N_0, N') \subseteq p^{-1}\mathrm{Hom}_P(N, N')$. Off $P, \bar P$ the decorations and frames are fixed.

We relate the unnormalized theta sections on both sides to the common base pair (N, N'). Put $M = M(N, N')$, $M_0 = M(N_0, N')$, $M_1 = M(N, N'_1)$, and locally write B, B_0, B_1 for their level-pg half-lattices $B(pg)$ (and A_0, A_1 for the corresponding A-lattices of M_0, M_1). For any lattice pair (L, L') of ranks (a, b), the local index $[B(L, L')_{Q'} : M(L, L')_{Q'} \cap B(L, L')_{Q'}]$ is 1 for $Q' \notin \bar S$ and $[M(L, L')_{\bar Q} : pgM(L, L')_{\bar Q}] = [M(L, L')_Q : pgM(L, L')_Q]$ at $Q' = \bar Q$ for $Q \in S$; the product over $Q \in S$ depends only on a, b, S, pg and equals $n = [M \cap A(pg) : M \cap A(pg) \cap B]$. Hence $[B(L, L') : M(L, L') \cap B(L, L')] = n$, and the unnormalized class section built from $\theta_{L,L'}$ is

$$\widetilde\Theta_{L,L',x} = n^{-1/2} W_{-t_x} I_{M(L,L'),B(L,L')} \theta_{L,L'}.$$

For (N, N'_1), we have $\theta_{N,N'_1} = I_{M,M_1}\theta_{N,N'}$. The lattices M, M_1, B_1, B agree away from $S \cup \bar S$ and split along the complementary isotropic halves at S and $\bar S$, so the isotropic-half composition law of 17.15 gives $I_{M_1,B_1} I_{M,M_1} = I_{B,B_1} I_{M,B}$ with composition scalar 1. Locally at P, $B_P = pM_P \subseteq pM_{1,P} = B_{1,P}$ has index $[M_{1,P} : M_P] = [(N'_1)_P : N'_P]^a = p^a$; at $\bar P$, $B_{1,\bar P} = p^{-1}M_{1,\bar P} \subseteq p^{-1}M_{\bar P} = B_{\bar P}$; elsewhere B and B_1 agree. Thus $[B_1 : B \cap B_1] = p^a$, and by the Chinese remainder theorem each coset of $B_1/(B \cap B_1)$ has a representative $u \in B_1$ with $u_{\bar Q} \in c_A pgM_{1,\bar Q}$ for every $Q \in S$. For a test class $x \in c_A^{-1}\mathrm{Hom}_S(N, N'_1)$, choose a lift $t_x \in A_1(c_A pg)$ of x modulo the finer lattice $B_S = pg\,\mathrm{Hom}_S(N, N') \subseteq B_{1,S}$. Because $t_x \in A_1(c_A pg)$ and

$u \in B_1$ satisfies $u_{\bar{Q}} \in c_A pg M_{1,\bar{Q}}$ for all $Q \in S$, we have $\frac{\sqrt{d}}{2} H(t_x, u) \in \mathcal{O}_{K_d}$, whence $W_{-t_x} W_u = W_{-(t_x - u)}$ as in 17.16. Moreover, $x_P, u_P \in M_{1,P} \subseteq p^{-1} M_P = A(c_A pg)_P$ and $M_{1,\bar{P}} \subseteq M_{\bar{P}}$ (and likewise $M_{0,P} \subseteq p^{-1} M_P$ and $M_{0,\bar{P}} \subseteq M_{\bar{P}}$ below), so $t_x - u$ lies in $A(c_A pg)$; since $t_x - x \in B_S$, it lifts $x - u$ modulo B_S, which runs over the p^a cosets of B_S inside $x + B_{1,S}$. Therefore

$$\widetilde{\Theta}_{N, N_1', x} = p^{-a/2} \sum_{\substack{x' \equiv x \bmod B_{1,S} \\ x' \bmod B_S}} \widetilde{\Theta}_{N, N', x'}.$$

By the same calculation for (N_0, N'), we have $\theta_{N_0, N'} = I_{M, M_0} \theta_{N, N'}$, $I_{M_0, B_0} I_{M, M_0} = I_{B, B_0} I_{M, B}$, and $[B_0 : B \cap B_0] = [M_{0,P} : M_P] = [N_P : (N_0)_P]^b = p^b$, giving

$$\widetilde{\Theta}_{N_0, N', x} = p^{-b/2} \sum_{\substack{x' \equiv x \bmod B_{0,S} \\ x' \bmod B_S}} \widetilde{\Theta}_{N, N', x'}.$$

The functions $\phi_T(d, d_1'; \cdot)\Xi_0$ and $\phi_T(d_0, d'; \cdot)\Xi_0$ are invariant under $B_{1,S}$ and $B_{0,S}$ respectively: at P, $p \operatorname{Hom}_P(N, N_1')$ sends N_P into $p(N_1')_P \subseteq \ker l'_{1,P}$ and $p \operatorname{Hom}_P(N_0, N')$ sends $(N_0)_P$ into $pN'_P \subseteq \ker l'_P$, while at S_{row} the lattices are unchanged. Summing over the test quotients unfolds both sides onto the common quotient $p^{-1} c_A^{-1} \operatorname{Hom}_S(N, N')/B_S$:

$$\sum_{d' \uparrow_P d_1'} \sum_{x \bmod B_{1,S}} \phi_T(d, d_1'; x) \Xi_0(x) \widetilde{\Theta}_{N, N_1', x} = p^{-a/2} \sum_{x' \bmod B_S} \left(\sum_{d' \uparrow_P d_1'} \phi_T(d, d_1'; x') \right) \Xi_0(x') \widetilde{\Theta}_{N, N', x'},$$

$$\sum_{d_0 \uparrow_P d} \sum_{x \bmod B_{0,S}} \phi_T(d_0, d'; x) \Xi_0(x) \widetilde{\Theta}_{N_0, N', x} = p^{-b/2} \sum_{x' \bmod B_S} \left(\sum_{d_0 \uparrow_P d} \phi_T(d_0, d'; x') \right) \Xi_0(x') \widetilde{\Theta}_{N, N', x'}.$$

By the indicator equality above, the left-hand sum equals $p^{(b-a)/2} = p^s$ times the right-hand sum.

It remains to compare the step-character prefactors scal_i on the two sides. Since $P, \bar{P} \notin S_{\mathrm{row}}$ and all lattices and frames away from $P, \bar{P}$ are fixed, only the place $v_0 = P \cap \mathcal{O}_{E^+}$ changes in $\mu_{\mathrm{inv},i}$. Set $\varepsilon = 1$ if $\mathrm{sel}(v_0) = P$ and $\varepsilon = -1$ if $\mathrm{sel}(v_0) = \bar{P}$, and define the split step-character value $\mu_P = m_{U,i}(\mathrm{sel}(v_0))^{\varepsilon} \in K^{\times}$. Replacing N by N_0 decreases the P-localization by index p and increases the $\bar{P}$-localization by index p, so $\mathrm{ord}_{\mathrm{sel}(v_0)} \det(L_0, N_0) = \mathrm{ord}_{\mathrm{sel}(v_0)} \det(L_0, N) + \varepsilon$ and

$$\mu_{\mathrm{inv},i}(N_0, \mathrm{fr}) = \mu_P \, \mu_{\mathrm{inv},i}(N, \mathrm{fr}).$$

Replacing N' by N_1' increases the P-localization by index p and decreases the $\bar{P}$-localization by index p, so $\mathrm{ord}_{\mathrm{sel}(v_0)} \det(L_0', N_1') = \mathrm{ord}_{\mathrm{sel}(v_0)} \det(L_0', N') - \varepsilon$ and

$$\mu_{\mathrm{inv},i}(N_1', \mathrm{fr}')^{-1} = \mu_P \, \mu_{\mathrm{inv},i}(N', \mathrm{fr}')^{-1}.$$

Every summand on both sides therefore carries the same prefactor $\mu_P \, \mathrm{scal}_i(N, \mathrm{fr}; N', \mathrm{fr}')$, independent of the ranks a, b. Applying the linear rational jet operator from V_λ to $V_{\lambda'}$ and multiplying by this common scalar yields the neighbor identity.

Lean: `GlobalTheory.KerT.UK.sum_phiT` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.UkTermOfMoveStep`, lines 374–390)

Lean: `GlobalTheory.KerT.UK.ukTerm_of_moveStep` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.UkTermOfMoveStep`, lines 491–533)

□

Proof of 17.20

Proof. Work on the column construction defined by the parameter data of (33.1), the column and arithmetic datum of Definition 34.46, and the level-character datum of Definition 34.47, where the stage-i lifts are formed as reciprocal-stabilizer class sums with the rational jets and size bounds of 17.19, the selected initial weight signs satisfy $b_v \geq 0$ (giving $a_0 \geq 0$ below), and the level-zero identifications respect the localized base lattices

for all used twists μ with $p_w \notin Q'(\mu)$. Put $p = T_u$, $s = (t+1)/2$, and let V be the selected set of level indices from 17.10. Write $r = |V|$, K_i for the i-th lift, $C_0 = 1$, and $C_{i+1} = K_i C_i$. Let S_i be the complex image of the localized form lattice and U_i the ordered product of the upward operators at V on stage i.

The class stabilizers have orders prime to p. Indeed a stabilizer acts faithfully on E^{N_i}. An element of order p acting on this space as a rational vector space would have a primitive p-th root of unity as an eigenvalue. Its minimal polynomial would have degree at least $p-1$, whereas the rational dimension is $2fN_i \le 2fN_m < p-1$. Cauchy's theorem gives the assertion. Thus the class weights $1/|\operatorname{Stab}(c)|$ preserve the local bounds of 17.19.

At a seen place 17.19 puts the class sum, multiplied by $p^{fN_i - \sum_\tau h'_\tau}$, in the target value lattice with every marking kernel inserted. At an index in V, the marked kernel of an upper neighbor is the ordinary lattice at the lower endpoint. Summing over the upper neighbors therefore erases that index. At a remaining index choose a basis in which the marking reads the first coordinate modulo p; the kernel basis change is $\operatorname{diag}(p, 1, \dots, 1)$. Multiplying by $p^{\tilde\lambda_1}$ erases that index, where $\tilde\lambda_1$ is the first target point weight. Indeed if h' is its last weight, a generator of the polynomial part has degree $\beta_1 \le \tilde\lambda_1 - h'$ in the first row. Each of its $\tilde\lambda_1 - h'$ diagram columns uses that row at most once. This bound also holds on the Weyl generators obtained by integral row substitutions and coefficient extraction. The net power after the determinant twist is $\tilde\lambda_1 - h' - \beta_1 \ge 0$.

Lean: `GlobalTheory.LvT.pow_smul_mem_valB_erase` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.LevelPrimePowSmulMemValueLatticeErase`, lines 534–621)

Use one-based coordinates and write $\tilde\lambda_{j,\tau,k} = \lambda^u_{j,\tau,k} - js$. At stage $i+1$, $-h'_\tau = s_{i,\tau}$ is its chain determinant shift. The total exponent just paid is

$$fN_i + \sum_\tau s_{i,\tau} + \sum_{v \notin V} \tilde\lambda_{i+1,\tau_w(v),1}.$$

Each $s_{i,\tau}$ is bounded above by $W_{u,\tau}$ of 15.12 since it is minus the last point weight. For $v \notin V$, $\tau_w(v)$ is a non-top type, so the first point weight is also bounded above by $W_{u,\tau_w(v)}$. The map $v \mapsto \tau_w(v)$ is bijective. Hence we may use $d_0 = fN_m + 2\sum_\tau W_{u,\tau}$ at every stage. At an unseen place $V = \varnothing$; the factor p^{fs} supplied by 17.19 is integral, and we take $d_0 = 0$. The lift and the upward operators preserve the space of forms by their equivariant class-sum definitions. Testing at representatives now gives

$$p^{d_0} U_{i+1} K_i S_i \subseteq S_{i+1}. \tag{A.116}$$

For one level index the last identity of 17.19 passes to the class sum with its stabilizer weights. An incident-pair orbit through (d, e) has weight $[\operatorname{Stab}(d) : \operatorname{Stab}(d,e)]/|\operatorname{Stab}(d)| = 1/|\operatorname{Stab}(d,e)|$; counting from e gives the same weight. Thus $U_{i+1,v} K_i = p^s K_i U_{i,v}$. Move the factors of the ordered product through in order to obtain

$$U_{i+1} K_i = p^{sr} K_i U_i, \qquad U_i C_i = p^{sri} C_i U_0.$$

For $z \in S_0$ put $b_0 = 0$ and $b_{i+1} = b_i + d_0 + sr(i+1)$. If $p^{b_i} C_i U_0^i z \in S_i$, apply (A.116) and the second identity with $i+1$. This gives $p^{b_{i+1}} C_{i+1} U_0^{i+1} z \in S_{i+1}$. Hence $b_m = m d_0 + sr\, m(m+1)/2$. At a seen place $r = n_T$, and 15.12 gives $b_m = m T^*_{u,w}$. At an unseen place both sides are zero.

By 17.10 the input is $p^{a_0} U_0^m z$ after embedding, for a localized ramified vector with image $z \in S_0$. Here a_0 is minus m times the sum of the selected last initial weights; the weight signs give $a_0 \ge 0$. The scalar $c p^{a_0 - b_m}$ lies in $\mathcal{O}_{K,(w)}$ by the hypothesis on $v_w(c)$. Multiplying the terminal inclusion by this scalar proves the reference assertion.

At a used twist with $p_w \notin Q'(\mu)$, the level-zero identification preserves the localized base lattice. On corresponding decorated representatives the auxiliary decoration at stage zero is unique, so the frame action and class value are identified. Scaling markings commutes with this correspondence, which therefore carries the kernels defining the ramified subspace to one another. Upper neighbors correspond by the same residue vectors of marking one, so each upward sum, and then its ordered product and m-th power, intertwine. The twisted input thus has the same presentation with the same a_0 and the corresponding twisted z. Applying the iteration on the twisted chain puts p^{b_m} times its column in the localized terminal lattice. The scalar p^{-b_m} has valuation $-e_w m T^*_{u,w}$, giving exactly the bound of 16.1.

Lean: `GlobalTheory.LvT.ColSharp.chain_into` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.LevelChainSharpColumnValuationOfStability`, lines 91–114)

Lean: `GlobalTheory.LvT.TrueStep0.step0Tie_trueCols` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.TwistedChainInitialLatticeCompatibleTrueColumns`, lines 167–173)

□

Proof of 17.22

Proof. Work on the row construction defined by (33.1), Lemma 33.33, and Definition 34.46, using the common CM realization. Condition (1) supplies the rational normalized jets at every positive divisor of the product of the support denominator c_A and the test modulus c_B. Condition (2) couples the last two steps of the length-m chain ($m = m_2 + 2$) via the prescribed coset decoration. Fix $w \mid p = g_i \neq T_u$, a word and a used tuple. At step j use representatives standard above p with lattices N, N'. Let S be the summation prime set and S_i its primes over $\bar{\pi}_i$. Write Ξ as the word-level indicator times the local factors of Definition 12.12, using the marked coset at each coupled step. At a type prime the factor takes cyclotomic integer values, is supported on integral homomorphisms and depends there only on their reduction. At a twist prime its values are w-integral and its local period modulus divides c_B. A translation in the intersection displayed below therefore preserves every factor: at S_i it is zero modulo p, and away from S_i it is zero modulo the local test modulus, including the modulus of the word-level indicator. Thus the product Ξ has $|\Xi|_w \leq 1$ and is supported in $\mathrm{Hom}_{S_i}(N, N')$, and invariant under

$$p\,\mathrm{Hom}_{S_i}(N, N') \cap c_B\,\mathrm{Hom}_{S\setminus S_i}(N, N').$$

Here c_B is its period modulus with $p \mid c_B$, and c_A its support denominator. Write $c_A c_B = p^n c'$ with $p \nmid c'$. Both c_B and pc' divide $c_A c_B$ and their period groups preserve the test. At S_i both are divisible by p, and elsewhere in S the factor p is a unit and both moduli are divisible locally by c_B. Refining either sum to modulus $c_A c_B$ partitions the cosets with the same test value and preserves the normalized theta sum. Rationality and injectivity of ι give equality in K. At modulus pc', integral support on S_i and removal of the unit p^n elsewhere give $x \in (c')^{-1}\,\mathrm{Hom}_S(N, N')$ for a nonzero summand; because $|\Xi|_w \leq 1$, the triangle inequality reduces the w-integrality of the scaled jet sum for Ξ to the individual modulus-pc' terms at $x \in (c')^{-1}\,\mathrm{Hom}_S(N, N')$.

Write M for the split period lattice and put $d_0 = fN_{j-1}N_j$. At modulus $c = pc'$ with $\gcd(c', p) = 1$, the period lattice B satisfies $(pc')B \subseteq M$, so the finite quotient $B/(B \cap M)$ splits as the direct product of its c'-torsion and p-torsion subgroups with representative sets S_1 (satisfying $c'y \in M$ and $p \nmid |S_1|$) and S_2 (satisfying $py \in M$). The prime-to-p shifts t_k ($k \in S_1$, combining the lift of $-x$ with k) satisfy $c't_k \in M$ (taking $m_0 = c'$ prime to p), and their phase coefficients r_k have p-adic norm at most one. For the p-part, S_2 maps bijectively onto a representative set S_a for $(\bar{\pi}_i/\pi_i)M/\bar{\pi}_i M$; let S_b represent $M/\bar{\pi}_i M$. These dual quotients have cardinality $q = p^{d_0}$, with one residue coordinate per entry per type. Since $p \nmid |S_1|$, the normalized test function $\Psi = |S_1|^{-1}\mathbf{1}_0$ on S_b satisfies $|\iota_p(\Psi(b))|_p \leq 1$ for every $b \in S_b$. For $a \in S_a$ its Fourier coefficient is

$$\widehat{\Psi}(a) = q^{-1} \sum_{b \in S_b} \Psi(b) e^{-2\pi i B_H(b,a)} = (|S_1|q)^{-1}.$$

Write $G = \sum_{a \in S_a} \widehat{\Psi}(a) W_a e_{N,N'}$. Averaging over $S_1 \times S_2$ splits the test-class function at x as the sum $\sum_{k \in S_1} r_k U_{t_k} G$. In the following estimate $\iota_p : \mathbf{C} \simeq \mathbf{C}_p$ extends the split embedding at w, M_{fr} is the integral matrix of direction coefficients in the period frame, and $A_{\mathrm{fr}} = \iota_p(\bar{\pi}_i)M_{\mathrm{fr}}$. Denote the Taylor series of a function F, read through ι_p and evaluated in directions with coefficient matrix B, by $\mathcal{S}(F; B)$. For a finite tuple of variables Z, define the ring of Hurwitz-integral power series

$$\mathcal{O}_{\mathbf{C}_p}\langle\langle Z \rangle\rangle = \left\{ \sum_{\alpha} c_\alpha Z^\alpha \in \mathbf{C}_p[[Z]] : \alpha!\, c_\alpha \in \mathcal{O}_{\mathbf{C}_p} \text{ for all } \alpha \right\},$$

where $\alpha! = \prod_s \alpha_s!$. This subring contains $\mathcal{O}_{\mathbf{C}_p}[[Z]]$ and is closed under $\mathcal{O}_{\mathbf{C}_p}$-linear combinations, products (since the $Z^\gamma/\gamma!$-coefficient of a product of $\sum a_\alpha Z^\alpha/\alpha!$ and $\sum b_\beta Z^\beta/\beta!$ is $\sum_{\alpha+\beta=\gamma} \binom{\gamma}{\alpha} a_\alpha b_\beta$), and $\mathcal{O}_{\mathbf{C}_p}$-linear substitutions of the variables (since for $\ell_s \in \mathcal{O}_{\mathbf{C}_p}$ the multinomial expansion gives $(\sum_s \ell_s W_s)^n/n! = \sum_{|\beta|=n} (\prod_s \ell_s^{\beta_s}) W^\beta/\beta!$, so each $Z^\alpha/\alpha!$ maps into $\mathcal{O}_{\mathbf{C}_p}\langle\langle W \rangle\rangle$ and each degree receives finitely many terms).

The local estimate is

$$p^{d_0}\mathcal{S}\left(\sum_k r_k U_{t_k} G; A_{\mathrm{fr}}\right) \in \mathcal{O}_{\mathbf{C}_p}\langle\langle Z\rangle\rangle. \tag{A.117}$$

Here Z denotes the variables of the chosen directions.

Lean: `Formalism.L2.AlgTh.split_bound_of_xi_lemmas` (`Formalism.LocalToGlobal.AlgebraicTheory.AlgebraicTheoryProofs.XiConn`, lines 464–496)

We prove the étale translation estimate: if a shift t satisfies $N_0\pi_0 t \in M$ and $N_0 p t \in M$ for a positive integer N_0 with $p \nmid N_0$ and a CM integer $\pi_0 \in \mathcal{O}_{K_d}$ satisfying $\pi_0\bar{\pi}_0 = p$ and $|\iota_p(\pi_0)|_p = 1$, then $\mathcal{S}(U_t e_{N,N'}; B) \in \mathcal{O}_{\mathbf{C}_p}\langle\langle Z\rangle\rangle$ for every direction matrix B over $\mathbf{C}_p$ with entries of p-adic norm at most p^{-1}. Because $p = g_i$ is a fixed auxiliary prime of the host and $2fN_m + 1 < p$ (with $f \geq 1$ and $N_m \geq 2$), we have $p > 5$, $p \nmid d$, $p \nmid \mathrm{discr}(E)$, and p splits unramified in K_d.

First reduce from $U_t e_{N,N'}$ to one-variable elliptic theta translates. Write L_0, L_0' for the standard lattices of ranks N_{j-1}, N_j and M_{std} for the standard period lattice of (L_0, L_0'). Since N, N' agree with L_0, L_0' locally at p, there is a positive integer M_1 prime to p such that $M_1 N \subseteq L_0$, $M_1 L_0 \subseteq N$, $M_1 N' \subseteq L_0'$, and $M_1 L_0' \subseteq N'$; hence $M_1^2 M \subseteq M_{\mathrm{std}}$ and $M_1^2 M_{\mathrm{std}} \subseteq M$. As in the proof of 17.15, averaging $\Theta^{\otimes N_{j-1}N_j}$ from M_{std} to M (with prime-to-p indices and p-adic unit character factors) and applying U_t expresses $U_t e_{N,N'}$ as a finite linear combination of translates $U_{t+s_\rho}\Theta^{\otimes N_{j-1}N_j}$ with coefficients of p-adic norm at most 1, where each s_ρ satisfies $M_1^2 s_\rho \in M_{\mathrm{std}}$. Put $M_2 = M_1^2 N_0$, still prime to p. Then $M_2\pi_0(t+s_\rho) \in M_{\mathrm{std}}$ and $M_2 p(t+s_\rho) \in M_{\mathrm{std}}$ for every ρ, so each matrix entry $t_{\rho,i_1 i_2}$ ($1 \leq i_1 \leq N_{j-1}$, $1 \leq i_2 \leq N_j$) satisfies $M_2\pi_0 t_{\rho,i_1 i_2} \in P_1$ and $M_2 p t_{\rho,i_1 i_2} \in P_1$, where P_1 is the one-entry period lattice. Applying 17.14 to the fourth power of entry (i_1, i_2) yields

$$(U_{t_{\rho,i_1 i_2}}\Theta)^{\otimes 4}(z) = \frac{e}{\sqrt{J}} \sum_{x \in Q/(P\cap Q)} \alpha(x) e^{-\pi i B_H(v,x)} \prod_{\nu=1}^{4f} (U_{v_\nu + x_\nu}\theta_\Lambda)\big((gz)_\nu\big),$$

where $P = \Lambda^{4f}$, $Q = g(P_1^4)$, $J = [Q : P \cap Q]$ is prime to p, $|\iota_p(e)|_p = 1$, and $v = g(t_{\rho,i_1 i_2}, t_{\rho,i_1 i_2}, t_{\rho,i_1 i_2}, t_{\rho,i_1 i_2})$. Since g is $\mathbf{C}$-linear, it commutes with scalar multiplication by π_0, so $M_2\pi_0(v + x) \in Q$. Putting $N_1 = M_2 J$ (still prime to p) and using $JQ \subseteq P = \Lambda^{4f}$ gives $N_1\pi_0(v_\nu + x_\nu) \in \Lambda$ for every ν and x. Let G_g be the $4f \times 4f$ matrix of g in the standard complex coordinates of $\mathbf{C}^{4f}$. Because the auxiliary curve ideal is trivial, $P_1 = \{(\delta\tau(a))_{\tau\in\Sigma} : a \in \mathcal{O}_E\mathfrak{b}^{-1}\}$ and $P = (\delta\mathcal{O}_{K_d})^{4f}$. Canceling $\delta \neq 0$ from $JQ \subseteq P$ shows that for every $a = (a_1, a_2, a_3, a_4) \in \mathcal{O}_E^4 \subseteq (\mathcal{O}_E\mathfrak{b}^{-1})^4$ the vector $JG_g((\tau(a_q))_{q,\tau})$ has entries in $\varphi_0(\mathcal{O}_{K_d})$, where $\varphi_0 : K_d \hookrightarrow \mathbf{C}$ is the common restriction of all $\tau \in \Sigma$; hence $\iota_p(G_g((\tau(a_q))_{q,\tau})) \in \mathcal{O}_{\mathbf{C}_p}^{4f}$ because $p \nmid J$. Choose a $\mathbf{Z}$-basis $(e_1, \ldots, e_{2f})$ of $\mathcal{O}_E$. The $2f \times 2f$ matrix $V = (\iota_p(\sigma(e_\kappa)))_{\sigma\in\Sigma\cup\bar{\Sigma},\, 1\leq\kappa\leq 2f}$ has entries in $\mathcal{O}_{\mathbf{C}_p}$ and satisfies $(\det V)^2 = \mathrm{discr}(E)$; since $p \nmid \mathrm{discr}(E)$, we have $V \in \mathrm{GL}_{2f}(\mathcal{O}_{\mathbf{C}_p})$. Each standard basis vector of $\mathbf{C}_p^{4f}$ is therefore an $\mathcal{O}_{\mathbf{C}_p}$-linear combination of the embedded vectors $(\iota_p(\tau(e_\kappa))\delta_{q,q_0})_{q,\tau}$, so every entry of $\iota_p(G_g)$ lies in $\mathcal{O}_{\mathbf{C}_p}$. For the f-row submatrix $B_{i_1 i_2}$ of B at entry (i_1, i_2), the block-diagonal matrix $\widetilde{B} = \mathrm{diag}(B_{i_1 i_2}, B_{i_1 i_2}, B_{i_1 i_2}, B_{i_1 i_2})$ has entries of p-adic norm at most p^{-1}, so $\iota_p(G_g)\widetilde{B}$ has entries of p-adic norm at most $p^{-1} = |N_1 p|_p$ and factors as $(N_1 p)L$ with L having entries in $\mathcal{O}_{\mathbf{C}_p}$.

Next establish the one-variable Hurwitz estimate on the elliptic curve $E_\Lambda = \mathbf{C}/\Lambda$, where $\Lambda = \delta\mathcal{O}_{K_d}$ has the discriminant normalization $\delta^{12} = \Delta(\mathcal{O}_{K_d})$ of 21.22 (so $\Delta(\Lambda) = 1$) and $\theta = \theta_\Lambda$ is its normalized even theta section with $\theta(0) = 1$ and simple zero at the nonzero half-period $P_0 = \delta/2 \bmod \Lambda \in E_\Lambda[2] \setminus \{0\}$. Fix $\tau \in \mathbf{C}$ with $N_1\pi_0\tau \in \Lambda$, and put $\beta = N_1\pi_0 \in \mathcal{O}_{K_d}$ and $m_\beta = \beta\bar{\beta} = N_1^2 p$. Since E_Λ has complex multiplication by $\mathcal{O}_{K_d}$, its j-invariant $j(E_\Lambda) = 1728 g_2(\Lambda)^3$ is an algebraic integer. As $p \geq 5$, the integers 2 and 3 are units in $\mathbf{Z}_p$; from $1728 g_2(\Lambda)^3 = j(E_\Lambda)$ and $27 g_3(\Lambda)^2 = g_2(\Lambda)^3 - 1$, both $\iota_p(g_2(\Lambda))$ and $\iota_p(g_3(\Lambda))$ are algebraic and lie in $\mathcal{O}_{\mathbf{C}_p}$. Hence, after transporting coefficients via ι_p, the short Weierstrass equation

$$Y^2 = X^3 - \frac{\iota_p(g_2(\Lambda))}{4} X - \frac{\iota_p(g_3(\Lambda))}{4},$$

parametrized analytically over $\mathbf{C}$ by $(X, Y) = (\wp(z), \wp'(z)/2)$, has coefficients in the ring of integers $\mathcal{O}_F$ of a finite extension $F/\mathbf{Q}_p$ inside $\mathbf{C}_p$ and unit discriminant $\Delta(\Lambda) = 1$, defining a smooth proper elliptic curve $\mathcal{E}/\mathcal{O}_F$; via $(X, Y) = (\wp(z), \wp'(z)/2)$ and ι_p, we identify $E_\Lambda(\mathbf{C})$ with $\mathcal{E}(\mathbf{C}_p)$ and elliptic functions on E_Λ with rational functions on $\mathcal{E}_{\mathbf{C}_p}$. Write $T = -X/Y$ (corresponding analytically to $-2\wp(z)/\wp'(z)$) for the formal-group parameter at the origin, and for a rational function R on $\mathcal{E}_{\mathbf{C}_p}$ regular at 0, write $\widehat{R}(T) \in \mathbf{C}_p[[T]]$

for its formal T-expansion at 0. The invariant differential $\frac{dX}{2Y}$ equals dz analytically and $\omega(T)\,dT$ formally, with $\omega(T) \in 1 + T\mathcal{O}_F[[T]]$ [68, Ch. IV, §1], so $z = \log_{\widehat{E}}(T) = \int_0^T \omega(T)\,dT$ and its compositional inverse is $T(z) = \exp_{\widehat{E}}(z)$.

For any $\alpha \in \mathcal{O}_{K_d}$, multiplication by α acts on T by $[\alpha](T) = \exp_{\widehat{E}}(\iota_p(\alpha)\log_{\widehat{E}}(T))$. For every $n \in \mathbf{Z}$, the n-fold formal group sum $[n](T) = \exp_{\widehat{E}}(n\log_{\widehat{E}}(T))$ lies in $\mathcal{O}_F[[T]]$; since each coefficient of $\exp_{\widehat{E}}(a\log_{\widehat{E}}(T))$ is a polynomial in a and $\iota_p(\alpha) \in \mathbf{Z}_p$ (because p splits in K_d), density of $\mathbf{Z}$ in $\mathbf{Z}_p$ gives $[\alpha](T) \in \iota_p(\alpha)T + T^2\mathcal{O}_F[[T]]$. Because $\mathcal{E}/\mathcal{O}_F$ is the smooth Néron model of its generic fiber, $[\alpha]$ extends uniquely to an endomorphism of the group scheme $\mathcal{E}/\mathcal{O}_F$, so $T([\alpha](Q)) = [\alpha](T(Q))$ for every point Q in the origin residue disk $\widehat{E}(\mathfrak{m}_{\mathbf{C}_p})$. Since $|\iota_p(2\beta)|_p = |2N_1|_p|\iota_p(\pi_0)|_p = 1$, the ultrametric inequality gives $|[2\beta](t)|_p = |t|_p$ for all $t \in \mathfrak{m}_{\mathbf{C}_p}$, whence

$$E_\Lambda[2\beta] \cap \widehat{E}(\mathfrak{m}_{\mathbf{C}_p}) = \{0\}.$$

Enlarge the finite extension $F/\mathbf{Q}_p$ inside $\mathbf{C}_p$ so that all points of $E_\Lambda[2\beta]$ are rational over F. Consider the elliptic functions $E_\beta(z) = \theta(\beta z)/\theta(z)^{m_\beta}$ and $G_0(z) = E_\beta(z)^{-1} = \theta(z)^{m_\beta}/\theta(\beta z)$, which satisfy $E_\beta(0) = G_0(0) = 1$ (the period multipliers cancel because $H(\beta z, \beta l) = m_\beta H(z,l)$ and $\alpha(\beta l) = \alpha(l)^{m_\beta}$ for $l \in \Lambda$). On E_Λ,

$$\operatorname{div}(G_0) = m_\beta[P_0] - \sum_{Q \in [\beta]^{-1}(P_0)} [Q],$$

whose support is contained in $E_\Lambda[2\beta] \setminus \{0\}$ (as $[\beta](0) = 0 \neq P_0$ and $[2](P_0) = 0$), hence is disjoint from $\widehat{E}(\mathfrak{m}_{\mathbf{C}_p})$; both E_β and G_0 are defined over F. As in the proofs of 17.16 and 17.17, let π_F be a uniformizer of $\mathcal{O}_F$ and η_s the generic point of the smooth special fiber $\mathcal{E}_s$: if $R \in F(\mathcal{E})^\times$ is regular at 0 and its generic-fiber poles on $\mathcal{E}_F \setminus \{0\}$ are cleared by a monic polynomial $P(X) \in \mathcal{O}_F[X]$ of degree N_P (in particular whenever all poles lie in $\mathcal{E}(F) \setminus \widehat{E}(\mathfrak{m}_{\mathbf{C}_p})$, since $Q \notin \widehat{E}(\mathfrak{m}_{\mathbf{C}_p})$ implies $|X(Q)|_p \le 1$, so $X(Q) \in \mathcal{O}_F$), then $v_{\eta_s}(P(X)) = 0$ and $P(X)R = \pi_F^{v_{\eta_s}(R)}U$ with $U = A(X) + YB(X) \in \mathcal{O}_F[X,Y] \setminus \pi_F\mathcal{O}_F[X,Y]$ of pole order at most $2N_P$ at 0 (so $\deg A \le N_P$ and $2\deg B + 3 \le 2N_P$); since $T^2X(T), -T^3Y(T) \in 1 + T\mathcal{O}_F[[T]]$ [68, Ch. IV, §1], multiplying by T^{2N_P} gives $T^{2N_P}P(X(T)) \in \mathcal{O}_F[[T]]^\times$ and $T^{2N_P}U(X(T),Y(T)) \in \mathcal{O}_F[[T]] \setminus \pi_F\mathcal{O}_F[[T]]$, whence $\widehat{R}(T) \in \pi_F^{v_{\eta_s}(R)}(\mathcal{O}_F[[T]] \setminus \pi_F\mathcal{O}_F[[T]])$. Applying this to E_β and $G_0 = E_\beta^{-1}$, since $\widehat{E}_\beta(0) = \widehat{G}_0(0) = 1$ forces $v_{\eta_s}(E_\beta) \le 0$ and $v_{\eta_s}(G_0) \le 0$ while $v_{\eta_s}(E_\beta) + v_{\eta_s}(G_0) = 0$, we obtain $v_{\eta_s}(E_\beta) = v_{\eta_s}(G_0) = 0$ and $\widehat{E}_\beta(T), \widehat{G}_0(T) \in \mathcal{O}_F[[T]]^\times$. Write $f_0(T) \in \mathbf{C}_p[[T]]$ for the T-expansion of θ transported by ι_p. From $\theta(\beta z) = \theta(z)^{m_\beta}E_\beta(z)$ we obtain $f_0([\beta](T)) = f_0(T)^{m_\beta}\widehat{E}_\beta(T)$. Writing $f_0(T) = \sum_{k=0}^\infty a_k T^k$ with $a_0 = 1$ and comparing T^k-coefficients for $k \ge 1$ yields $(\iota_p(\beta)^k - m_\beta)a_k \in \mathcal{O}_{\mathbf{C}_p}$ by induction on k, because $[\beta](T) \in \iota_p(\beta)T + T^2\mathcal{O}_F[[T]]$ and $\widehat{E}_\beta(T) \in 1 + T\mathcal{O}_F[[T]]$. Since $|\iota_p(\beta)|_p = 1$ and $p \mid m_\beta$, the factor $\iota_p(\beta)^k - m_\beta$ is a unit in $\mathbf{Z}_p$, so $f_0(T) \in 1 + T\mathcal{O}_{\mathbf{C}_p}[[T]] \subseteq \mathcal{O}_{\mathbf{C}_p}[[T]]^\times$ and hence $f_0([\beta](T)) \in \mathcal{O}_{\mathbf{C}_p}[[T]]^\times$.

Now establish the Gauss-norm bound on the étale $N_1\pi_0$-torsion disk via multiplication by $\beta = N_1\pi_0$. Write $H_\tau(T) \in \mathbf{C}_p[[T]]$ for the T-expansion of $U_\tau\theta$ transported by ι_p, and define $G_\tau(z) = (U_\tau\theta)(z)^{m_\beta}/\theta(\beta z)$. Because $\beta\tau \in \Lambda$, the transformation law $W_{\beta\tau}\theta = \theta$ evaluated at βz, together with $H(\beta z, \beta\tau) = m_\beta H(z,\tau)$ and $H(\beta\tau,\beta\tau) = m_\beta H(\tau,\tau)$, gives

$$\theta(\beta z) = \alpha(\beta\tau)\,\theta(\beta z + \beta\tau)\exp\!\big(-\pi m_\beta H(z,\tau) - \tfrac{\pi}{2}m_\beta H(\tau,\tau)\big)\,.$$

Dividing $(U_\tau\theta)(z)^{m_\beta} = \theta(z+\tau)^{m_\beta}\exp(-\pi m_\beta H(z,\tau) - \frac{\pi}{2}m_\beta H(\tau,\tau))$ by $\theta(\beta z)$ cancels the exponential factors and yields

$$G_\tau(z) = \alpha(\beta\tau)^{-1}\frac{\theta(z+\tau)^{m_\beta}}{\theta(\beta(z+\tau))} = \alpha(\beta\tau)^{-1}G_0(z+\tau),$$

where $\alpha(\beta\tau) = \pm 1$. Put $P_\tau = \tau \bmod \Lambda \in E_\Lambda[\beta] \subseteq \mathcal{E}(\mathcal{O}_F)$. Translation by P_τ is an automorphism of the smooth group scheme $\mathcal{E}/\mathcal{O}_F$, so it preserves the valuation at η_s and gives $v_{\eta_s}(G_\tau) = v_{\eta_s}(G_0) = 0$. Since $[\beta](P_\tau) = 0$, the divisor of G_τ on E_Λ is

$$\operatorname{div}(G_\tau) = m_\beta[P_0 - P_\tau] - \sum_{Q \in [\beta]^{-1}(P_0)} [Q],$$

where $P_0 - P_\tau \in E_\Lambda[2\beta]$ because $[2\beta](P_0 - P_\tau) = [2\beta](P_0) - 2[\beta](P_\tau) = 0$. Passing to T-expansions gives $H_\tau(T)^{m_\beta} = \widehat{G}_\tau(T)\,f_0([\beta](T))$. We treat the two cases for P_τ:

- If $P_\tau \neq P_0$, then $P_0 - P_\tau \in E_\Lambda[2\beta] \setminus \{0\}$, so neither $P_0 - P_\tau$ nor any point of $[\beta]^{-1}(P_0)$ lies in $\widehat{E}(\mathfrak{m}_{\mathbf{C}_p})$. Thus both G_τ and G_τ^{-1} have valuation 0 at η_s, are regular on $\widehat{E}(\mathfrak{m}_{\mathbf{C}_p})$, and have all horizontal poles outside $\widehat{E}(\mathfrak{m}_{\mathbf{C}_p})$. Applying the generic-point criterion above to both G_τ and G_τ^{-1} (whose poles all lie in $E_\Lambda[2\beta] \setminus \{0\} \subseteq \mathcal{E}(F) \setminus \widehat{E}(\mathfrak{m}_{\mathbf{C}_p})$) shows that $\widehat{G}_\tau(T)$ and $\widehat{G}_\tau(T)^{-1}$ lie in $\mathcal{O}_F[[T]]$, so $\widehat{G}_\tau(T) \in \mathcal{O}_{\mathbf{C}_p}[[T]]^\times$. Set $R_\tau(T) = H_\tau(T)$ and $C(T) = \widehat{G}_\tau(T)\, f_0([\beta](T)) \in \mathcal{O}_{\mathbf{C}_p}[[T]]^\times$.
- If $P_\tau = P_0$, then $0 \notin [\beta]^{-1}(P_0)$, so G_τ has a zero of exact order m_β at 0 and poles at $[\beta]^{-1}(P_0)$, all outside $\widehat{E}(\mathfrak{m}_{\mathbf{C}_p})$. Consider the rational function $\widetilde{G}_\tau = T^{-m_\beta} G_\tau$ on $\mathcal{E}$. On $\mathcal{E}/\mathcal{O}_F$, the Weierstrass coordinates X, Y are nonzero on $\mathcal{E}_s$, so $v_{\eta_s}(T) = 0$ and hence $v_{\eta_s}(\widetilde{G}_\tau) = 0$. Inside $\widehat{E}(\mathfrak{m}_{\mathbf{C}_p})$, $T = -X/Y$ has a single simple zero at 0 and no other zeros or poles (since $0 < |T(Q)|_p < 1$ for $Q \in \widehat{E}(\mathfrak{m}_{\mathbf{C}_p}) \setminus \{0\}$). Therefore both $\widetilde{G}_\tau$ and $\widetilde{G}_\tau^{-1}$ have valuation 0 at η_s, are regular on $\widehat{E}(\mathfrak{m}_{\mathbf{C}_p})$, and have all horizontal poles outside $\widehat{E}(\mathfrak{m}_{\mathbf{C}_p})$. The poles of $\widetilde{G}_\tau = (-Y/X)^{m_\beta} G_\tau$ lie in $[\beta]^{-1}(P_0) \subseteq \mathcal{E}(F) \setminus \widehat{E}(\mathfrak{m}_{\mathbf{C}_p})$ and at $X = 0$ (cleared by $X^{m_\beta} \prod_{Q \in [\beta]^{-1}(P_0)} (X - X(Q)) \in \mathcal{O}_F[X]$), and the poles of $\widetilde{G}_\tau^{-1} = (-X/Y)^{m_\beta} G_\tau^{-1}$ lie in $E_\Lambda[2] \setminus \{0\} \subseteq \mathcal{E}(F) \setminus \widehat{E}(\mathfrak{m}_{\mathbf{C}_p})$ (cleared by $\prod_{Q \in E_\Lambda[2] \setminus \{0\}} (X - X(Q))^{m_\beta} \in \mathcal{O}_F[X]$); applying the generic-point criterion above to $\widetilde{G}_\tau$ and $\widetilde{G}_\tau^{-1}$ therefore gives $\widetilde{G}_\tau(T) \in \mathcal{O}_F[[T]]^\times \subseteq \mathcal{O}_{\mathbf{C}_p}[[T]]^\times$. Because θ has a simple zero at P_0 and $T(z) = z + O(z^5)$, the series $R_\tau(T) = H_\tau(T)/T$ lies in $\mathbf{C}_p[[T]]$ and satisfies $R_\tau(T)^{m_\beta} = C(T) := \widetilde{G}_\tau(T)\, f_0([\beta](T)) \in \mathcal{O}_{\mathbf{C}_p}[[T]]^\times$.

In both cases $R_\tau(T)^{m_\beta} = C(T) \in \mathcal{O}_{\mathbf{C}_p}[[T]]^\times$, and from $R_\tau(0)^{m_\beta} = C(0) \in \mathcal{O}_{\mathbf{C}_p}^\times$ we obtain $R_\tau(0) \in \mathcal{O}_{\mathbf{C}_p}^\times$.

We now pass from the formal-group parameter T to the scaled additive coordinate $z = N_1 p\, \tilde{z}$ and extract the m_β-th root in $\mathcal{O}_{\mathbf{C}_p}\langle\langle \tilde{z} \rangle\rangle$. Define the derivation $D = \omega(T)^{-1} \frac{d}{dT}$ on $\mathcal{O}_{\mathbf{C}_p}[[T]]$.

Lean: `Formalism.L2.Th1.A.D` (`Formalism.LocalToGlobal.TorsionIntegrality.WeierstrassTwoTorsion.CompInv`, lines 57–57)

Under $z = \log_{\widehat{E}}(T)$, D corresponds to $\frac{d}{dz}$. Because $\omega(T)^{-1} \in 1 + T\mathcal{O}_{\mathbf{C}_p}[[T]]$, induction on $k \geq 0$ shows $D^k(T) \in \mathcal{O}_{\mathbf{C}_p}[[T]]$, so $(D^k T)(0) \in \mathcal{O}_{\mathbf{C}_p}$. The compositional inverse $T(z) = \exp_{\widehat{E}}(z)$ therefore satisfies

$$T(z) = \sum_{k=1}^{\infty} \frac{(D^k T)(0)}{k!}\, z^k.$$

By Legendre's formula, for every $k \geq 1$ and $p \geq 5$,

$$v_p(k!) = \sum_{r \geq 1} \left\lfloor \frac{k}{p^r} \right\rfloor < \frac{k}{p-1} < k,$$

so $v_p(k!) \leq k - 1$ and $p^k/k! \in p\mathbf{Z}_p$. Substituting $z = N_1 p\, \tilde{z}$ yields

$$T(N_1 p\, \tilde{z}) = \sum_{k=1}^{\infty} (D^k T)(0)\, N_1^k \frac{p^k}{k!}\, \tilde{z}^k \in p\, \tilde{z}\, \mathcal{O}_{\mathbf{C}_p}[[\tilde{z}]].$$

Substituting $T = T(N_1 p\, \tilde{z})$ into $R_\tau(T)^{m_\beta} = C(T)$ gives

$$R_\tau\big(T(N_1 p\, \tilde{z})\big)^{m_\beta} = C(0)\big(1 + p\, y(\tilde{z})\big), \qquad y(\tilde{z}) \in \tilde{z}\, \mathcal{O}_{\mathbf{C}_p}[[\tilde{z}]].$$

Because $\mathbf{C}_p$ has characteristic 0 and $R_\tau(0) \in \mathcal{O}_{\mathbf{C}_p}^\times$ satisfies $R_\tau(0)^{m_\beta} = C(0)$, uniqueness of the formal m_β-th root with given nonzero constant term gives

$$R_\tau\big(T(N_1 p\, \tilde{z})\big) = R_\tau(0) \sum_{k=0}^{\infty} \left(k! \binom{1/m_\beta}{k} p^k \right) \frac{y(\tilde{z})^k}{k!}.$$

Since $m_\beta = N_1^2 p$ with $p \nmid N_1$, we have

$$k! \binom{1/m_\beta}{k} p^k = \prod_{s=0}^{k-1} \big(N_1^{-2} - sp\big) \in \mathbf{Z}_p \subseteq \mathcal{O}_{\mathbf{C}_p}$$

for every $k \geq 0$. Write $y(\tilde{z}) = \sum_{r \geq 1} b_r \tilde{z}^r$ with $b_r \in \mathcal{O}_{\mathbf{C}_p}$. By the multinomial expansion, for every $k \geq 0$ and $n \geq k$, the coefficient of $\tilde{z}^n$ in $y(\tilde{z})^k/k!$ multiplied by $n!$ equals

$$\sum_{\substack{(k_r)_{r\geq 1} \\ \sum k_r = k,\ \sum r k_r = n}} \frac{n!}{k!} \cdot \frac{k!}{\prod_{r\geq 1} k_r!} \prod_{r \geq 1} b_r^{k_r} \in \mathcal{O}_{\mathbf{C}_p},$$

because $n = \sum r k_r \geq \sum k_r = k$ makes both $n!/k!$ and $k!/\prod k_r!$ integers; for $n < k$ the coefficient vanishes. Hence $y(\tilde{z})^k/k! \in \mathcal{O}_{\mathbf{C}_p}\langle\langle \tilde{z} \rangle\rangle$ for all $k \geq 0$. Since each $\tilde{z}^n$-coefficient receives a finite sum over $0 \leq k \leq n$, it follows that $R_\tau(T(N_1 p\, \tilde{z})) \in \mathcal{O}_{\mathbf{C}_p}\langle\langle \tilde{z} \rangle\rangle$. When $P_\tau \neq P_0$, this gives $\iota_p((U_\tau \theta)(N_1 p\, \tilde{z})) = R_\tau(T(N_1 p\, \tilde{z})) \in \mathcal{O}_{\mathbf{C}_p}\langle\langle \tilde{z} \rangle\rangle$; when $P_\tau = P_0$, multiplying by $T(N_1 p\, \tilde{z}) \in \mathcal{O}_{\mathbf{C}_p}[[\tilde{z}]] \subseteq \mathcal{O}_{\mathbf{C}_p}\langle\langle \tilde{z} \rangle\rangle$ gives $\iota_p((U_\tau \theta)(N_1 p\, \tilde{z})) = T(N_1 p\, \tilde{z})\, R_\tau(T(N_1 p\, \tilde{z})) \in \mathcal{O}_{\mathbf{C}_p}\langle\langle \tilde{z} \rangle\rangle$.

To return to $U_t e_{N,N'}$, take four independent copies $Z^{(1)}, Z^{(2)}, Z^{(3)}, Z^{(4)}$ of the variable tuple Z and substitute the $\mathcal{O}_{\mathbf{C}_p}$-linear forms $(L(Z^{(1)}, Z^{(2)}, Z^{(3)}, Z^{(4)}))_\nu$ (from $\iota_p(G_g)\widetilde{B} = (N_1 p)L$) into the one-variable series $\iota_p((U_{v_\nu + x_\nu}\theta_\Lambda)(N_1 p\, \tilde{z}))$. Because $|\iota_p(e/\sqrt{J})|_p = 1$, $\alpha(x) = \pm 1$, and $e^{-\pi i B_H(v,x)}$ is a root of unity, closure of the Hurwitz ring under $\mathcal{O}_{\mathbf{C}_p}$-linear substitutions, products over $\nu = 1, \ldots, 4f$, and $\mathcal{O}_{\mathbf{C}_p}$-linear combinations over $x \in Q/(P \cap Q)$ yields

$$\prod_{q=1}^{4} \mathcal{S}(U_{t_{\rho,i_1 i_2}}\Theta; B_{i_1 i_2})(Z^{(q)}) = \mathcal{S}\Big((U_{t_{\rho,i_1 i_2}}\Theta)^{\otimes 4}; \widetilde{B}\Big) \in \mathcal{O}_{\mathbf{C}_p}\langle\langle Z^{(1)}, Z^{(2)}, Z^{(3)}, Z^{(4)} \rangle\rangle.$$

Write $\mathcal{S}(U_{t_{\rho,i_1 i_2}}\Theta; B_{i_1 i_2})(Z) = \sum_\alpha c_\alpha Z^\alpha$. Reading the multi-index $(\alpha, \alpha, \alpha, \alpha)$ gives $(\alpha, \alpha, \alpha, \alpha)!\, c_\alpha^4 = (\alpha!\, c_\alpha)^4 \in \mathcal{O}_{\mathbf{C}_p}$, hence $\alpha!\, c_\alpha \in \mathcal{O}_{\mathbf{C}_p}$ for every α, so $\mathcal{S}(U_{t_{\rho,i_1 i_2}}\Theta; B_{i_1 i_2}) \in \mathcal{O}_{\mathbf{C}_p}\langle\langle Z \rangle\rangle$. Taking products over all $N_{j-1}N_j$ matrix entries (i_1, i_2) and summing over ρ with coefficients of p-adic norm at most 1 proves $\mathcal{S}(U_t e_{N,N'}; B) \in \mathcal{O}_{\mathbf{C}_p}\langle\langle Z \rangle\rangle$, completing the étale translation estimate.

If $|\iota_p(\bar{\pi}_i)|_p < 1$, then $|\iota_p(\bar{\pi}_i)|_p = p^{-1}$ and $\iota_p(\pi_i)$ is a unit, since $\pi_i \bar{\pi}_i = p$ and the prime splits unramified in K_d. Thus A_{fr} has entries of norm at most p^{-1}. For $a \in S_a$ one has $\pi_i a \in M$ and $pa \in M$, so $m_0 \pi_i$ and $m_0 p$ kill $t_k + a$ modulo M. Expand $U_{t_k} W_a$ using the translation law. The phase is a root of unity and the coefficient $q r_k \widehat{\Psi}(a)$ has norm at most one. The étale estimate on the shifted series gives (A.117).

If $|\iota_p(\bar{\pi}_i)|_p = 1$, set $s = \pi_i/\bar{\pi}_i$, so $|\iota_p(s)|_p = p^{-1}$. The scalar change-of-frame formula gives

$$e_{N,N'}(sz) = c\, q^{-1/2} \sum_{a \in S_a} W_a e_{N,N'}(z), \qquad c^8 = (\bar{\pi}_i/\pi_i)^{4 d_0}.$$

It is the Cayley Gauss comparison of 17.14 for the scalar isometry s, with average denominator $\sqrt{q}$. Expand $\widehat{\Psi}$ and exchange the sums in a, b. Then commute W_b through the a-sum. The character produced in that commutation is $e^{-2\pi i B_H(b,a)}$, since W_b fixes $e_{N,N'}$ for $b \in S_b \subset M$. Transport this W_b by the isometry s. The resulting formula is

$$G(z) = D^{-1} \sum_{b \in S_b} \Psi(b)\alpha(b)(U_{sb} e_{N,N'})(sz), \qquad D = c\, q^{1/2}, \qquad D^8 = \bar{\pi}_i^{8 d_0}.$$

Thus D is a unit. After U_{t_k} the translates in this sum are $s(t_k + b)$, with the additional unit phase $e^{-\pi i B_H(s t_k, sb)}$. They are killed modulo M by $m_0 \bar{\pi}_i$ and $m_0 p$, since $\bar{\pi}_i s = \pi_i$ and $ps = \pi_i^2$. Apply the same étale estimate with the matrix $\iota_p(s) A_{\mathrm{fr}}$, whose entries have norm at most p^{-1}. In this direction the series in (A.117) lies in $\mathcal{O}_{\mathbf{C}_p}\langle\langle Z \rangle\rangle$ before multiplication by p^{d_0}.

In the diagonal differential frame A_{fr} gives diagonal entries $\iota_p(\bar{\pi}_i) c_{D,\tau}$, with the generators read through the comparison. Extracting the coefficient at multi-index α and multiplying by $\alpha!$ gives the jet with the factor $\prod_\tau \tau(\bar{\pi}_i)^{A_{u,j,\tau}}$ when the degree at τ is $A_{u,j,\tau}$. The normalizing character on the standard pair is a unit, and integrality of Ξ permits the weighted sum. Thus

$$d_j J_\alpha \in \mathcal{O}_{(w)}, \qquad d_j = p^{f N_{j-1} N_j} \prod_\tau \tau(\bar{\pi}_i)^{A_{u,j,\tau}}. \tag{A.118}$$

Lean: `Formalism.L2.AlgTh.jetAtSplit_of` (`Formalism.LocalToGlobal.AlgebraicTheory.AlgebraicTheoryProofs.CardTwoSidedQuotient`, lines 349–601)

On a tensor Weyl generator perform the polynomial substitution of 17.19 with its determinant factors and the matrix entries as indeterminates. Cauchy–Binet expands each minor in target minors whose column heights are the source column heights together with the full-height source columns added by the determinant power; by weight compatibility these are the column heights of the target partition κ_τ at τ, of total box count $|\kappa_\tau| = A_{u,j,\tau}$. Straightening by Plücker relations expresses each differential coefficient in the standard bideterminant basis of the target dual Weyl lattice $\nabla_{\mathcal{O}_{(w)}}(\kappa_\tau)$ [60, Theorem II.2.16]. To pass from $\nabla_{\mathcal{O}_{(w)}}(\kappa_\tau)$ to the target Weyl lattice $\Delta_{\mathcal{O}_{(w)}}(\kappa_\tau)$, recall from the normalized-pairing argument in the proof of Lemma 10.17, using the duality between Schur and Weyl functors in [60, Proposition II.4.1], that if (e_r) is an $\mathcal{O}_{(w)}$-basis of $\Delta_{\mathcal{O}_{(w)}}(\kappa_\tau)$ and (d_r) is its dual basis over K under the factorial pairing $[P,Q] = \sum_\alpha \alpha![Y^\alpha]P[Y^\alpha]Q$, then $(C_{\kappa_\tau} d_r)$ is an $\mathcal{O}_{(w)}$-basis of $\nabla_{\mathcal{O}_{(w)}}(\kappa_\tau)$, where $C_{\kappa_\tau} = [H_{\kappa_\tau}, H_{\kappa_\tau}]$. Every $Q \in \nabla_{\mathcal{O}_{(w)}}(\kappa_\tau)$ has coefficients in $\mathcal{O}_{(w)}$, so $[Q, C_{\kappa_\tau} d_r] \in \mathcal{O}_{(w)}$ and $C_{\kappa_\tau} Q = \sum_r [Q, C_{\kappa_\tau} d_r] e_r \in \Delta_{\mathcal{O}_{(w)}}(\kappa_\tau)$. By the repeated column-addition Capelli identity [80], $C_{\kappa_\tau} = \mathrm{Hook}(\kappa_\tau)$ is the product of the hook lengths of the diagram of κ_τ. Every hook of κ_τ is a subset of its $|\kappa_\tau| = A_{u,j,\tau} \le D_{u,j}$ boxes, and $2D_{u,j} + 2 < p$ gives $D_{u,j} + 1 < p$, so every hook length of κ_τ is positive and strictly less than p (likewise $\kappa_{\tau,1} - \kappa_{\tau,N_j} + N_j - 1 \le A_{u,j,\tau} + N_m - 1 \le D_{u,j} + N_m - 1 < p$ places κ_τ inside the fundamental p-alcove where Jantzen's sum formula gives $\Delta_{\mathcal{O}_{(w)}}(\kappa_\tau) = \nabla_{\mathcal{O}_{(w)}}(\kappa_\tau)$ [66, Part II, Proposition 8.19]). Thus $C_{\kappa_\tau} \in \mathcal{O}_{(w)}^\times$ and $\nabla_{\mathcal{O}_{(w)}}(\kappa_\tau) = \Delta_{\mathcal{O}_{(w)}}(\kappa_\tau)$, so each differential coefficient belongs to the target Weyl lattice $\Delta_{\mathcal{O}_{(w)}}(\kappa_\tau)$. Since the jet degree at τ is $A_{u,j,\tau}$, (A.118) gives integral jet multipliers at w after scaling by d_j. By linearity and equivariance the scaled operator preserves the local lattice condition.

Class weights are units by the stabilizer calculation of 17.20. Write $m = m_2 + 2$ for the chain length, separating the first $m_2 = m - 2$ kernels from the *last two kernels* at steps $m-2$ and $m-1$, which are coupled through a common *intermediate coset decoration* $Y = y + L_+$ belonging to the set $\mathcal{Y}(N)$ of admissible intermediate coset decorations on the step-$(m-1)$ lattice N. At each row prime $P \in S_{\mathrm{row}}$, write $q_P = |k(P)|$. The *distinguished generator primes* are the primes over g_0, g_1, where the local conditions on Y are $y \in q_P^{-1} N_P$ and $(L_+)_P = N_P$; at the other row primes $P \in S_{\mathrm{row}}$, the *row modulus exponent* M prescribes $y \in q_P^{-M} N_P$ and $(L_+)_P = q_P^M N_P$, with no extra condition outside S_{row}. Because the local integral jet datum and its w-valuation multiplier d_j depend only on the local lattice conditions at p, the bound for each of the last two kernels is uniform in $Y \in \mathcal{Y}(N)$; multiplying the kernel bounds along the first $m-2$ steps and across the coupled pair $(m-2, m-1)$ before summing over Y shows that $d_* \psi_c$ (with $d_* = \prod_{j=1}^m d_j$) lies in the w-local source Weyl lattice. The row tie lifts H via the dual lattice scaling $\widetilde{M}_L$ (placing each terminal row value H_c in the w-local Weyl-pairing dual of the target lattice), and the pairing tie gives the rational period y. In terminal class coordinates, with unscaled column values ψ_c and row values H_c, the class sum is

$$d_* y = \sum_c |\operatorname{Stab}(c)|^{-1} \langle \mathrm{br}(d_* \psi_c), H_c \rangle.$$

The interlacing weights have the same determinant shift, so integral branching sends the source Weyl lattice into the target Weyl lattice. The level factors introduced by branching are integral. Since $\pi_T \bar{\pi}_T = T_u \ne g_i$, both are units at w. The scaled column $d_* \psi_c$ lies in the source lattice, and H_c lies in the pairing dual of the target lattice. Every summand is w-integral. Hence, for $y \ne 0$,

$$\frac{v_w(y)}{e_w} \ge -\sum_j (\nu_{i,u,j,w} + f N_{j-1} N_j) \ge -\sum_j \nu_{i,u,j,w} - \mathcal{E}_G.$$

Lean: `GlobalTheory.ColPeriod.valuation_PtilE_le_one` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.NotDvdCardOfInjective`, lines 268–305)

For the budget put $A = f(N_m^2 + m^2\eta)$. By 15.10–15.12 the base terms cancel and $F_j - E_{u,j} = -\nu_{i,u,j,w} - c_g A - \delta_{u,w}$. Since $t \le N_m$, one has $\mathcal{E}_G \le 2mA$, so $m c_g A + m\delta_{u,w} \ge C_1 \mathcal{E}_G$. Substitution gives $B_{u,w} \le -\sum_j \nu_{i,u,j,w} - C_1 \mathcal{E}_G$.

For the family size assertion the middle threshold gives $c_g \ge 2$. Put $A' = kb \ge 10$, $N = N_m \ge 2$. The choice of fixed primes gives a size parameter Y with $(A')^8 \le Y \le g_i$ and $N^4 (Q_S q d)^{C_Y} \le Y^{1/4}$, with $C_Y > 0$ and $Q_S q d \ge 1$. Thus $N^{16} \le Y$. For $D_{u,j}$, the rows are decreasing, so use the zeroth entry of 15.12. It is at most the first input weight plus $w_j(\tau)$ minus $(2N_{j-1} + t)/2$. Write $A_* = xb$. The parameter choice gives $m(t+1)\eta \le w''$ and $kS_c \le w''$, where S_c is the cloud-coordinate scale of the member, with $w'' \le b'$. Since $m(t+1) \ge 2$ and $k \ge 5$, it follows that $2\eta \le b'$ and $5S_c \le b'$. At a top type the first input weight

is $a_T + (k-1)/2$ with $a_T \le 6A_*/5$, and $w_j \le b_T + b'/x + \eta$ with $b_T \le 2b'$. Using $b' \le A_*/2$ gives an entry at most $6A_*/5 + 7b'/2 \le 3A_*$. Off the top the parameter choice gives $u_0 - v_0 \le 2S_c$, and the first cloud coordinate satisfies $v_0 \le S_c$. Thus $u_0 \le 3S_c$. The thick and thin windows give $w_j \le b + \eta + 1$ (by Lemma 15.9, using $|\mu(E)| \le 8f^2 \le h_0/2$) and $w_j \le b' + \eta$, respectively. With the bounds on S_c, η and $1 \le b \le A_*$, the row entry is again at most $3A_*$. Summing over N_{j-1} coordinates and f types and using $x, f \le k$ gives $D_{u,j} \le 3(A')^2 N$.

Lean: `GlobalTheory.CoreHt.RowsRange.nuRow_zero_le_three_mul` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.NuRowZeroLeThreeMul`, lines 34–90)

Each size expression is at most $6(A')^2 N + 2$. For $N \le (A')^5$ this is at most $6(A')^7 + 2 < (A')^8$. For $N > (A')^5$ it is at most $6N^2 + 2 < N^{16}$. These bounds are uniform in the member index.

Lean: `GlobalTheory.AtG.KernelSizes.sizeAt_of_params` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.KernelDegreesSmallAgainstTypePrimes`, lines 115–181)

□

Proof of 17.24

Proof. By 14.8, $G_{p,j} \ge 0$. The screened gain vanishes on $\mathcal{D}$. By 15.16, the raw gain vanishes outside the finite support of the size data, and also when p_w lies in neither the inert nor the signal class. It also vanishes at g_0, g_1. In the inert and signal classes, the primes dividing $Q_S q d\, g_0 g_1$ are fixed primes. In the remaining cases the filling identity of 14.9 identifies the raw gain with the counted gain in (16.8). Outside $\mathcal{D}$ the inert case of odd order greater than one is excluded by hypothesis, so (16.7) identifies $G_{p_w,j}$ with the same counted gain. □

Proof of 17.25

Proof. The denominator 1 gives $\phi \in (N_u)_{(w)}$. For nonempty selection take the permitted model and use the equivalence to put $c_{u,w}(j_u \phi)$ in $N_{P,u,w}$. For empty selection the assumed inclusion puts ϕ in $(M_{H,u})_{(w)}$, so its image lies in $(\Lambda^0_{u,0})_{(w)} \cap V^r = S^{\mathrm{ram}}_{u,w}$. Proposition 17.10 supplies these local properties on the arithmetic member. In the first case the comparison image belongs to the enhanced lattice. In the second case the selection is empty and the initial vector belongs to the localized ramified lattice. Either condition admits the input to (16.28). By 17.24, the selected column has bound $s^{\mathrm{enh}}_{u,w} \ge S_{u,w} + \delta_{u,w}$. The common stabilizer denominator in 15.12 allows us to apply 17.5, which subtracts $\delta_{u,w}$ from this exponent. □

Proof of 17.29

Proof. Take the maximum of the given cutoffs and $3^{5/e_z}$, where J is the family scale exponent and $e_z = \varepsilon/J > 0$. To check the permitted-model hypothesis of 17.25, fix $p = p_w \ne T_u$ with $W_p \ne \varnothing$. By 14.8, p is an inert candidate and $\mathrm{ord}(h_p) \ge x^{\varepsilon/4}$. An odd order would divide o because the image in the cyclic 2-group would vanish. But $o \le x^{\varepsilon/100} < x^{\varepsilon/4}$. The inert class makes p prime to the conductor. At each $P \mid p$, a Frobenius has the same image in the abelian host group as the representative defining h_p. Its second component is the image of h_p under the frame embedding and has that even order. The inert class excludes 2. The host splitting condition applies to primes up to at least $\lfloor x^{e_z/5} \rfloor \ge 3$. The host splitting condition therefore forces 3 to split in K_d, incompatible with the inert class of a candidate. Host level primes are split. These facts and $p \ne T_u$ give the conditions of 16.2.

For an arithmetic member, 17.10 gives both local equivalences for the same N_u and its inclusion in $(M_{H,u})_{(w)}$ away from T_u. For a general member these are the corresponding clauses of its family data. The error identities 15.10–15.14 apply. Together with the agreement of the local fillings and the ramified-space and base-term identities of 16.19, they give the error and compatibility hypotheses of 17.28. Lemma 17.6 gives the required size identities at every level prime. An inert-class prime p with h_p of odd order greater than one meets the screening criterion of 15.14. Every level prime is split, so p is distinct from every T_u and belongs to the discarded set $\mathcal{D}$. With filling and period linearity apply 17.28.

For closedness use the finite conjunction rule of 36.10. Once the common threshold admits a family, the eighth input supplies the witness C_1 for that family. Use it in the common-cutoff argument to obtain both eventual assertions. □

Proof of 17.30

Proof. The design gives the logarithmic stabilizer bound of Definition 17.8 on the entire class. By 16.21 and the two closed selection hypotheses, the following assertion is closed there: for some $C \geq 0$, the size estimates of 16.20 and both column estimates of 16.19 at normalization 1 hold beyond one family cutoff. Lemma 17.9 gives closed eventual coprimality. Together with the four point and period hypotheses, the pair of sharp level-prime estimates and the common fixed-prime bound, these give the eight closed hypotheses of 17.29 on $\mathcal{D}_3$. Theorem 17.29 therefore gives closedness of the eventual rounded period bound. Taking the quantifiers in (16.34) in order gives ε_1 and then $\mathcal{K}$. The arithmetic inputs for the constructed family come from the closed rows of 33.77 and the fixed-prime row of 33.34. □

Proofs for Chapter 18: Gluing row lattices across Hecke characters

Proof of 18.6

Proof. For the sets U, TU fixed before the two coordinate systems in the statement, and each $w \in \mathrm{Tw}$, the chosen label a satisfies $\tau_a \in TU$, so $t_w = e^{-1}(\tau_a)$ is well defined and satisfies $\theta(t_w) = a$ and $\chi_{\theta(t_w)} = \xi_w$ by 18.2. Off Tw we have $t_w = t_0$.

The valuation of $a \in \mathcal{O}$ at the conjugate arithmetic place equals that of $\bar{a}$ at the original place. Thus the place correspondence respects conjugation and Tw is stable. Off Tw both types are t_0. On Tw we have $p_{\bar{w}} = p_w$ and $\psi_{\bar{w}} = \psi_w$, so $\xi_{\bar{w}} = \xi_w$. On local units above the selected auxiliary prime, ψ_w is trivial and ξ_w has order 15. The surjection $a \mapsto a_f$ and the embedding σ make the character map in 18.2 injective, with $\chi_{\theta(t_0)} = 1$. Hence $t_{\bar{w}} = t_w \neq t_0$ on Tw.

In either system, 18.5 gives $RR_{(z)} = (L_{t_z})_{(z)}$. For $z = w, \bar{w}$ the type t_z is t_w. □

Proof of 18.8

Proof. Fix $H \in M_t$. By the assumed integral decomposition at $\theta(t)$, write $J_t H = \sum_{y \in S} \sigma(b_y) y$ with $S \subseteq \widetilde{M}_{\theta(t)} \cap V_0(\theta(t))$ finite and $b_y \in \mathcal{O}$. For $x \in N$, choose the period representative $a_y \in \mathcal{V}_{w,s}$ for each $y \in S$. For the fixed column input $j_N E_u x$, let z be its image under the reference column chain followed by evaluation and lowering. Let e be the complex-linear identification from the period test space to the ambient row space. Let ℓ_z be the complex-linear functional on the period test space obtained by fixing z in the first argument of the terminal period pairing. The row functional is $\ell_z \circ e^{-1}$. The twisted chain has the same formula with its own z and e. Thus the period is complex-linear in the row in both cases. The bridge identity gives

$$\sigma(P_{u,t}(x, H)) = \sum_{y \in S} \sigma(b_y)\sigma(a_y) = \sigma\left(\sum_{y \in S} b_y a_y\right).$$

Injectivity of σ gives $P_{u,t}(x, H) = \sum_{y \in S} b_y a_y \in \mathcal{V}_{w,s}$. Since $\phi \mapsto P_{u,t}(\phi, H)$ is $\mathcal{O}$-linear, the bound extends to $\mathrm{span}_{\mathcal{O}} N$.

At w, take $t = t_w$. Apply the first part with $N = \mathcal{N}_u$ and $s = s_{u,w}$. For $x \in \mathcal{N}_u$, we have $E_u x \in N_u$. Since $\theta(t_w) = \mu_w$, Theorem 17.28 supplies the required period representatives. The span of $\mathcal{N}_u$ is $\mathcal{N}_u$. □

Proofs for Chapter 19: The pairing after division

Proof of 19.5

Proof. For $x \in (\iota\pi M)_{(w)}$, choose $t \in \mathcal{O} \setminus \mathfrak{p}_w$ and $F \in M$ with $tx = \iota\pi F$. Then $P(F) = \zeta\, t\, b(x)$. If $b(x) \neq 0$, this is nonzero, and $v_w(\zeta) = v_w(t) = 0$ gives $v_w(b(x)) = v_w(P(F)) \geq c$.

Take $x \in R$. The first assertion gives $v_w(b(x)) \geq 0$ at every finite place when $b(x) \neq 0$. The principal fractional ideal $(b(x))$ therefore has nonnegative exponent at every prime, so it is an integral ideal and $b(x) \in \mathcal{O}$. The value zero also belongs to $\mathcal{O}$. □

Proof of 19.7

Proof. Apply Theorem 19.6 with $R = R^c$, $B = B_i^c$, $M_w = M_{t_w}$, $\pi_w = A_w$, $\iota_w = \mathrm{id}$ and $P_w = P_{i,t_w}$.

Put $N = E_u^{-1}(N_u)$. Proposition 18.8 gives the bound on $N \times M_{t_w}$ with $\sigma_w = s_{u,w}$ viewed in $\mathbf{R}$. By Lemma 19.3, N_u^c is the shift of N by $\tau_w = s_{u,w} = \lceil \sigma_w \rceil$. The first assertion gives $B_u^c(N_u^c) \subset (R^c)^\vee$. For $\phi \in N_u'$ and $r \in RR$, take $x \in N_u^c$ and $y \in R^c$ mapping to ϕ and r. Then $B_u(\phi)(r) = B_u^c(x)(y) \in \mathcal{O}$. □

Proofs for Chapter 20: Slope change and the data for step 5

Proof of 20.9

Proof. Form P_u from the primes at most X'' and the four exceptional primes. Adjoin the prime divisors of M_u. Outside P_u the raw gain and loss vanish, so the gain after dropping primes vanishes as well. The size datum gives $F_j(w) = 0$ for $p_w > X''$ with $p_w \notin \{g_0, g_1, d\}$. Substitution in the net budget of 15.16 gives $S_{u,w} = 0$. By 20.6, 20.7 and $e_w \geq 1$, the error is at most $2\sum_{p \in P_u} \log p$.

Put $P(n) = \prod_{p \leq n} p$. For $k \geq 1$, the primes in $(k+1, 2k+1]$ divide $\binom{2k+1}{k} \leq 4^k$, since the binomial sum 2^{2k+1} contains two equal central terms. Thus $P(2k+1) \leq P(k+1)4^k$. Since $P(2k) = P(2k-1)$ for $k \geq 2$, induction from $n = 0, 1, 2$ gives $P(n) \leq 4^n$. Take $n = \lfloor X'' \rfloor$, use $\prod_{p \mid M_u} p \mid M_u$ and include the four exceptional logarithms. □

Proof of 20.12

Proof. Choose $a \in K^\times$ with $v(a) = E$ and put $B' = B \cap a^{-1}\Theta$. In an elementary-divisor basis of B let b_i be the exponents of Θ. Take V spanned by the basis vectors with $b_i > E$. By (20.1) and $(\Theta_1)_V \subseteq \Theta_V$,

$$\operatorname{length}(B/B') = \sum_i (b_i - E)_+ = [B_V : \Theta_V]_v - E \dim V \leq r. \tag{A.119}$$

The operator $H = \sum_{u \in U} \psi(u)^{-1}\rho(u)$ sends a stable lattice into its intersection with W and acts by h on W. For $g \notin U$, the specified u gives $\psi(u)H\rho(g)|_W = \psi(g^{-1}ug)H\rho(g)|_W$, so $H\rho(g)|_W = 0$. For left coset representatives g_i this yields

$$H\rho(g_i^{-1}g_j)|_W = h\delta_{ij}\,\mathrm{id}_W\,.$$

Thus the translates $\rho(g_i)W$ are in direct sum and $s \dim W \leq n$. Put $Y = B_W/B'_W$. Summing translates and applying the operators $H\rho(g_i^{-1})$ give maps $Y^s \to B/B' \to Y^s$ whose composite is multiplication by h. Both maps descend by stability of B, B'. Hence $s \operatorname{length}(hY) \leq r$. The quotient Y/hY is killed by h and has at most $\dim W$ generators. Since $B_Z/B'_Z \hookrightarrow Y$,

$$s[B_Z : B'_Z]_v \leq s[B_W : B'_W]_v \leq r + nv(h). \tag{A.120}$$

For $z \in B'_Z$, one has $az/\sigma \in (\Theta_K)_W$, so $haz/\sigma \in \Theta_0 \cap V_0$ is fixed by e. Thus $(ha/\sigma)B'_Z \subseteq e\Theta_0 \subseteq e\Theta_J$. Scaling adds $d_Z(E - v(\sigma) + v(h))$ to the index. Apply (A.120), (20.2) and $X_3 \subseteq A$. □

Proof of 20.16

Proof. The shift is full by 19.2. Apply 19.7 with $\tau_w \leq \lceil \sigma_w \rceil$ for the inclusion and 20.5 for the slope difference.

The positive-dimensional reference base quotient is spanned by its spectral subspaces, so $\mathcal{I} \neq \varnothing$. The coefficient bound of 21.17 and (20.4) give $\|K_i\|_v^2 \leq \exp(\sum_{j:i(j)=i} k_{j,v})$, with $K_i = 0$ when its coefficient bound is zero. For $B \neq 0$, 21.12 gives $\log \|B\|_v^2 \leq t_v$, where

$$t_v = h_P + \sum_j k_{j,v} - \log \kappa_v - \min_{\alpha \in \mathcal{I}} \sum_j \log \lambda_{j,\alpha,v}.$$

Applying the terminal bound $A \leq H_{P,u}$ and kernel mean bound $\langle s_{j,v} \rangle_v \leq H_{K,uj}$ of Lemma 21.17 to the ranks and weights in R gives $h_P \leq H_P$ and $\langle k_{j,v} \rangle_v \leq H_{B,j}$. By (20.3) and (20.4),

$$\min_{\alpha \in \mathcal{I}} \sum_j \log \lambda_{j,\alpha,v} \geq \sum_j l_{6,j} \geq \sum_j H_{X,j}.$$

Averaging gives $\bar{t} := \langle t_v \rangle_v \le H - \bar{\kappa}$. Take $c_v = \exp(t_v + H - \bar{\kappa} - \bar{t})$ for $B \neq 0$ and $c_v = \exp(H - \bar{\kappa})$ for $B = 0$.

By 21.28, $\mu_0 \le B_*$. The row budgets give $0 \le \delta \le 1/3$, so passage at $U = h_{\mathrm{r}} - \bar{\kappa}$ gives

$$\mu_C - \bar{\kappa} \le \mu_0 + L_{\mathrm{pass}}, \qquad L_{\mathrm{pass}} = 3\delta(\mu_0 + h_{\mathrm{r}} + 28\lambda_{\mathrm{r}}) + 28\lambda_{\mathrm{r}}.$$

By (20.6), $L_{\mathrm{pass}} \le 1 + 28\lambda_{\mathrm{r}} \le 1 + C_0 m \lambda_0$, since $m \ge 1$ and $\lambda_0 \ge 0$. The same budgets give $\mu_0 \le \sum_{j,\rho} \beta_j w_{j,\rho} \log \Omega + 2F_* + E_b$. Apply 21.26 with $\widetilde{\mu}_C = \mu_C - \bar{\kappa}$. □

Proof of 20.17

Proof. For the fixed scale write j for the arithmetic layer number in the comparison, and let p be the residue characteristic of $\iota^{-1}y$. At g_0, g_1, Lemma 15.8 identifies the rows as $\kappa^{\mathrm{big}} = a^+$ and $\kappa^{\mathrm{small}} = -a^-$. At each position use the auxiliary entries b_i, e_i of its weight record, numbered from zero, and put $t = t_{\mathrm{ch}}$. The last index of a^+ is $k + j - 2$, where the spectral parameter is $-b_{j-1}$. Substitution in the two rows gives

$$a^+_{\mathrm{last}} = e_{j-1} - b_{j-1} - (t+1)/2, \qquad a^-_0 = b_j - e_{j-1} - (t+1)/2 \quad (j < m).$$

The gaps $e_{j-1} - b_{j-1}$ and $b_j - e_{j-1}$ are at least $(t+3)/2$. The spectral parameter decreases by at least 1 at each step of the a^+ row, while its index term increases by 1. The a^- increment is $b_{j+c+1} - b_{j+c} - 1$ from entry c to entry $c+1$, nonnegative by the gaps between auxiliary entries. Thus a^+ is nonincreasing and a^- is nondecreasing. Every entry of either row is at least 1. After reindexing the embeddings, the sum of absolute weights is $\sum a^+ + \sum a^-$, and both base terms are minus half this Fock degree. At d both equal $-(k-1)/2$ times the sum of the minima of the two weights.

For the other primes, use the position map

$$(\beta, t') \longmapsto e_\tau^{-1}\big(\tau_{\beta+1}(t')\big), \qquad \beta \in \{0, 1\},$$

where e_τ is the bijection from embedding indices to the union of the two branches in 14.9. Each branch is injective. Their images partition the embedding indices under e_τ^{-1}. For an inert or signal prime within the cutoff, p is coprime to the conductor. By 14.1, the second component of the Artin symbol of p, expressed in the additive coordinates of the positions, gives the same Frobenius displacement h_p on both sides. Hence both sides have the same position sequence and period n_p. By 14.9, corresponding positions have equal weights, so the capacity vectors of 15.1 agree entry by entry. The fixed-class sums with denominator $p^2 - 1$ agree, as do the cyclic orbit values and averages. Both entries vanish in the remaining prime classes or above the cutoff outside the exceptional primes. These cases exhaust the formula for $F_j(w)$ in 15.10, proving (20.7).

Enlarge both place sums to $\mathrm{Tw} \cup \iota(S)$. At a place outside Tw, untwistedness gives $F_{0,j,y} = 0$ for every layer. At a place outside $\iota(S)$, arithmetic support gives $F_{a(j)}(\iota^{-1}y) = 0$. Thus (20.7) puts every place with nonzero base term in both sets. In particular the whole fibre of a type prime contributes whenever its Fock degree is positive. The place bijection preserves residue norms and ramification indices, and the weights in 21.3 are $2\ell_w e_w$, twice those in F_*. Sum (20.7) using the layer bijection. □

Proof of 20.18

Proof. Put $l = \log M$. In the ambient frames the change of mean slope from P_i lies in $[-2\log h_i, 0]$ by inclusion and 20.1. The same interval bounds the change of maximum slope, since multiplication by h_i sends submodules of P_i into $h_i P_i$. Frame scaling adds $\log h_i$. Hence

$$|\mu(R_i) - \mu(P_i)| \le l, \qquad |\mu_{\max}(R_i) - \mu_{\max}(P_i)| \le l. \tag{A.121}$$

Write $P_{i,s}$ for the cut in a nonzero ambient isotypic space. Scalar endomorphisms give absolute irreducibility by scalar extension of endomorphisms and semisimplicity. Over $\overline{K}$ the eigenvalues of $g \in G$ are roots of unity. Their sum is the K-valued trace $\chi_s(g)$, hence belongs to $\mathcal{O}$. The central projector formula [36, Section 2.6, Theorem 8] has numerator coefficients $\dim(s)\chi_s(g^{-1})$ and denominator M. Thus

$$MP_i \subseteq \bigoplus_s P_{i,s} \subseteq P_i, \tag{A.122}$$

with the sum orthogonal by unitarity.

On a copy of s, the translates of the base character line span the copy. Project onto the selected character space and choose n_s translates whose projections form a basis. The resulting projected translates A_i give the map $(x_i)_i \mapsto \sum_i A_i x_i$ from n_s copies of the base cut space to the selected cut space. It is an isomorphism on each copy of s and hence on the isotypic component. The reverse map projects the inverse translates onto the base character space. At infinity it is the adjoint, so it is also an isomorphism. Each operator uses at most one central projector and two subgroup averages. The subgroup characters have integral coefficients and the subgroup orders divide M. Multiplication by M^3 gives integral maps between $P_{0,s}^{\oplus n_s}$ and $P_{1,s}$ in both directions.

Each A_i has norm at most one, and at every infinite place

$$\left\| \sum_{i=1}^{n_s} A_i x_i \right\|^2 \le n_s \sum_{i=1}^{n_s} \|A_i x_i\|^2 \le n_s \sum_{i=1}^{n_s} \|x_i\|^2.$$

The adjoint has the same operator norm. The regular representation contains each irreducible with multiplicity its dimension [36, Section 2.4, Corollaries 1 and 2], so $n_s \le \dim(s) \le \sqrt{M}$. Either integral map f_s therefore satisfies

$$\log \|f_s\|_v^2 \le 6l + \log n_s \le \tfrac{13}{2} l. \tag{A.123}$$

For an injective lattice map with $\log \|f\|_v^2 \le c$ at every infinite place, the image slope is at least the source slope minus c. In rank r its real volume grows by at most $\exp(n_K r c/2)$. A same-rank target has slope at least that of the image by the index formula. Put $r_s = \mathrm{rk}_{\mathcal{O}} P_{0,s}$, $d_s = r_s \mu(P_{0,s})$, $R = \sum_s r_s > 0$ and $W = \sum_s n_s r_s$. Use the forward norm bound when adding the degrees of the orthogonal type cuts. In (A.122), the left inclusion and 20.1 give the upper bound for P_0, while the right inclusion gives the lower bound for P_1 with loss $13lW/2$, hence also $17lW/2$.

$$R\mu(P_0) \le \sum_s d_s + 2lR, \qquad W\mu(P_1) \ge \sum_s n_s d_s - \tfrac{17}{2} lW. \tag{A.124}$$

An orthogonal sum $\bigoplus_i Q_i$ has maximum slope $\max_i \mu_{\max}(Q_i)$ over its nonzero summands. Indeed, filter a submodule L by $L_j = L \cap \bigoplus_{i \le j} Q_i$. Projection embeds L_j/L_{j-1} in Q_j with norm at most one for the quotient metrics. Degree additivity and the slope inequality [75, Propositions 5.5(2) and 5.6] bound the slope of L by the maximum, which the summands also bound from below. Applying this to the targets of the reverse maps and using (A.123) gives $\mu_{\max}(P_{1,s}) \le \mu_{\max}(P_0) + 13l/2$. By (A.122) and 20.1, $\mu_{\max}(P_1) \le \mu_{\max}(P_0) + 17l/2$. Thus (A.121) gives $\mu_{\max}(R_1) \le U + 21l/2 \le U + 14\lambda$.

Set $A = U + l$ and $z_s = A r_s - d_s$. By (A.121) and $P_{0,s} \subseteq P_0$ these are nonnegative. By (A.124), $\sum_s z_s/R \le A - \mu(P_0) + 2l$. The multiplicity bounds (20.8) give

$$\frac{\sum_s n_s z_s}{W} \le \frac{1+\delta}{1-\delta} \frac{\sum_s z_s}{R} \le (1+3\delta) \frac{\sum_s z_s}{R}. \tag{A.125}$$

Substitute $\sum_s n_s d_s = AW - \sum_s n_s z_s$ in (A.124) and apply (A.125) and (A.121). This gives

$$-\mu(R_1) \le -\mu(R_0) + \tfrac{25}{2} l + 3\delta\big(U - \mu(R_0) + 4l\big).$$

The slope changes sign under duality by duality of the metrized determinant line [75, Definition 5.4]. The inequality $0 \le l \le \lambda$ gives the row dual bound. □

Proof of 20.19

Proof. By 15.16, $s_{u,w} = 0$ off S. Thus the product over all places in 19.2 is I_u of 19.1, and the coordinate shift is $I_u \mathcal{N}_u$. Its image under E_u is $N'_u = I_u N_u$. By 20.3, $I_u \neq 0$, so multiplying by I_u^{-1} gives $E_u(\mathcal{N}_u) = N_u$.

At each finite place w, apply the integral decomposition of 18.7 to $J_{t_w} F$ at the label μ_w. The period bound of 17.28 puts the value on each summand in $\mathcal{V}_{w,s_{u,w}}$ under the coefficient embedding. This module is closed under integral linear combinations, so the period identity of 18.8 gives $P_{u,t_w}(\phi, F) \in \mathcal{V}_{w,s_{u,w}}$ for $\phi \in \mathcal{N}_u$. Apply 20.16. The pairing identity transports the coordinate inclusion to $B_u(N'_u) \subseteq RR^\vee$. If $B_u \neq 0$, the isometries give $h(B_u) \le \langle \log c_v \rangle_v = H_u - \bar{\kappa} = h_u$. The dual row isometry gives $\mu_C = \mu(RR^\vee)$. Substitute (20.9) in $\mu_C - \bar{\kappa} + H_u - 2F_* \le R_{*,u}$. □

Proofs for Chapter 21: The lattice after division

Proof of 21.9

Proof. Let a diamond representative δ fix the class of y. Write $y = \gamma y \delta \kappa$ with γ rational and κ in the level group. The diamonds normalize the level. Their marked multipliers lie in $\mathbf{F}_T^\times$, so $(\delta\kappa)^H$ is in the level and γ^H is in the class stabilizer. That stabilizer is trivial by 37.21. At either g_i, the diamond is the identity locally and $y^{-1}\gamma y$ is in the local level. Its reduction lies in a unipotent g_i-group and the congruence kernel is pro-g_i. Thus the finite order of γ is a power of each of the two distinct primes g_0, g_1. Hence $\gamma = 1$ and $\delta = \kappa^{-1}$. Each marked multiplier of δ is one. The diamond action is free.
Lean: `Formalism.L2.GlobalTheoryTheory.LocalLatticeBases.S0.fixed_class_mem_level_general` (`Formalism.LocalToGlobal.GlobalTheory.GL6.DiagonalSlopes.EqOneOfCoprimeTorsionTwoPrimes`, lines 571–620)
Lean: `Formalism.L2.GlobalTheoryTheory.LocalLatticeBases.S0.diamond_free_on_classes` (`Formalism.LocalToGlobal.GlobalTheory.GL6.DiagonalSlopes.EqOneOfCoprimeTorsionTwoPrimes`, lines 670–698)

Diamond transport identifies the orthogonal coefficient slices on each orbit isometrically. The nonempty class set of 37.21 and the dominant weights of 21.8 give $d_M > 0$, and $H = T - 1 > 2f \geq 2$ then gives $d > 0$. The sum in one coordinate has kernel on $\mathbf{Z}^H$ with basis $e_i - e_H$, $i < H$, and Gram determinant H. The sum to $\mathbf{Z}$ splits by $1 \mapsto e_H$. Tensor the f splittings with the coefficient lattice. The common kernel has rank fraction $(1 - 1/H)^f$ and its coordinate Gram determinant is $H^{f(H-1)^{f-1}}$. The coefficient lattice contributes equally per rank to the two slopes. This gives the mean formula, while the induced metric gives the maximum-slope inequality.
Lean: `Formalism.L2.GlobalTheoryTheory.LocalLatticeBases.S0.free_rank_model_of_diamonds` (`Formalism.LocalToGlobal.GlobalTheory.GL6.DiagonalSlopes.GroupActionMulSingleEqDilation`, lines 287–300)

Claim 1. In the tied level-zero model, the local coefficient lattices S_0 and B_v satisfy

$$D_v S_0 \subseteq B_v, \qquad U_v B_v \subseteq S_0.$$

Both operators descend to the rational coefficient space. They preserve the common diamond kernel V^{ram}, and $U_v D_v = T^{k-1} I$ on that kernel.

Here S_0 is the level-zero column lattice localized at a seen place $w \mid T$. The lattice B_v is the local lower-neighbor lattice, formed with the frame of the kernel of the marked residue functional. The down and up sums use the transports of the compatible symmetric column system. Apply the level-zero integrality comparison to the split tied column system. Its away-from-place lattice condition follows from the tied column transports, and it gives the two inclusions using the finite-order character with values in the coefficient field. As both local lattices span their rational coefficient space, the inclusions give the rational descents.
Lean: `PI.IntegralAtBasePointLattice.integral_at_t0_lattice` (`Formalism.PeriodIntegrals.OrbitalBoundsConclusion.IntegralAtT0Lattice`, lines 6–7)
Lean: `Formalism.L2.GlobalTheoryTheory.LocalLatticeBases.rational_level_zero_of_integral` (`Formalism.LocalToGlobal.GlobalTheory.GL6.CongruenceNoKernelOfHeckeRamified`, lines 39–42)

At position v, the diamond transport commutes with the up and down sums by the same-place incidence matchings. At every other position v', the matching permutes the neighbors at v while keeping their transports compatible with the diamond at v'. Consequently both operators preserve every diamond-sum kernel on the column space. For $D_v U_v$, the incidence count for a marked vector $e \neq 0$ and endpoint $y \neq 0$ is the number of g with $g(e) = g(y) = 1$. It is T^{k-1} at e, zero at other points on its line and T^{k-2} off that line. In the unrestricted count only $g(y) = 1$ is prescribed. Let $\mathrm{Tr}_v = T^{-(k-1)} E_v U_v$, where E_v is the down sum over all nonzero residue functionals. The transported incidence identity is

$$D_v U_v = T^{k-1} I - T^{k-2} \mathrm{Nm}_v + T^{k-2} \mathrm{Tr}_v. \tag{A.126}$$

The trace is invariant under each diamond at v, so $\mathrm{Tr}_v \mathrm{Nm}_v = H \mathrm{Tr}_v$. Thus the last two terms vanish on V^{ram}. Since U_v, D_v restrict to endomorphisms of this same finite-dimensional space, $D_v U_v = T^{k-1} I$ gives $U_v D_v = T^{k-1} I$ there as well.
Lean: `Formalism.L2.GlobalTheoryTheory.UnramifiedTrace.du_formula_of_fibre_count` (`Formalism.LocalToGlobal.GlobalTheory.UnramifiedTrace.DuFormulaOfFibreCount`, lines 15–41)
Lean: `Formalism.L2.GlobalTheoryTheory.UnramifiedTrace.hecke_ramified_zero_all` (`Formalism.LocalToGlobal.GlobalTheory.UnramifiedTrace.HeckeRamifiedZeroAll`, lines 10–11)

Claim 2. At a seen level place the ordered round of scaled upward operators preserves the ramified lattice, with colength at most $9 e_w d_M n_T (a/k + b')$ per round.

Let $R = S_0 \cap V^{\mathrm{ram}}$ and $R_B = B_v \cap V^{\mathrm{ram}}$. At an active v write $\lambda = \lambda_{u,\tau_w(v)}$ and $b_v = -\lambda_k \geq 0$ in one-based coordinates. The second branch of the weight-lattice comparison gives $T^{b_v} S_0 \subseteq B_v$. In the adapted frame each torus weight β contributes $e_w(b_v + \beta_1)$ per vector. The mean of β_1 is $k^{-1}\sum_i \lambda_i$, by permutation symmetry and homogeneity of the coefficient representation. Thus

$$\mathrm{length}_w(B_v/T^{b_v}S_0) = e_w d_M \left(b_v + k^{-1} \sum_i \lambda_i \right). \tag{A.127}$$

Lean: Formalism.L2.GlobalTheoryTheory.LemB.cls_len_D (Formalism.LocalToGlobal.GlobalTheory.CoordinateLengths.PairingAdjointness, lines 81–136)
Lean: Formalism.L2.GlobalTheoryTheory.LemB.second_branch_all_members (Formalism.LocalToGlobal.GlobalTheory.CoordinateLengths.SecondBranchAllMembers, lines 6–7)

Claim 1 gives $T^{k-1}R \subseteq U_v R_B \subseteq R$. In particular $U_v R_B$ has finite colength at most $e_w(k-1)d_M$, since the rank of R is at most d_M. The other quotient $U_v R_B / T^{b_v} U_v R$ is an image of $R_B/T^{b_v}R$, which injects into $B_v/T^{b_v}S_0$ because $T^{b_v}S_0 \cap V^{\mathrm{ram}} = T^{b_v}R$. Adding the two lengths proves

$$\mathrm{length}_w(R/T^{b_v}U_v R) \leq e_w d_M (a_{\max}/k + 2W_{u,\tau_w(v)} + k). \tag{A.128}$$

Here $k^{-1}\sum_i \lambda_i + b_v \leq a_{\max}/k + 2W_{u,\tau_w(v)}$ uses the leading-entry bound and $b_v \leq W_{u,\tau_w(v)}$. The selected size bounds are

$$a_{\max} \leq 2a, \qquad k \leq N_m \leq b', \qquad b_v \leq 2b', \qquad W_{u,\tau_w(v)} \leq 3b'. \tag{A.129}$$

They bound the parenthesis in (A.128) by $2a/k + 7b' \leq 9(a/k + b')$ (since $a_{\max}/k \leq 2a/k$, $2W_{u,\tau_w(v)} \leq 6b'$, $k \leq b'$, and $a/k, b' \geq 0$).
Lean: Formalism.MainNotions.product_first_zeroth_parts.twist_le (Formalism.SharpRestriction.GlobalSchurBounds.FiniteModelOfAdmissibleTransports, lines 1706–1721)
Lean: Formalism.L2.GlobalTheoryTheory.LocalLatticeBases.hecke_ramified_lattice_stable_length (Formalism.LocalToGlobal.GlobalTheory.GL6.CongruenceNoKernelOfHeckeRamified, lines 44–73)

Take the product of $T^{b_v}U_v$ over the active positions in the level order. No commutation of these factors is needed. For injective operators A, B preserving R, $AR/ABR \simeq R/BR$, so the colengths add. The active positions inject into the top positions, of which there are n_T. The local lattice prescription uses m iterations of this ordered product. At an unseen place the active set is empty and the lattice is R. This proves that the prescribed lattice lies in R and its colength is at most $9e_w d_M m n_T (a/k + b')$.
Lean: Formalism.MainNotions.J.card_VV_le (Formalism.SharpRestriction.AdmissibleParameterJoins.DIVfromGLo, lines 185–201)
Lean: Formalism.L2.GlobalTheoryTheory.LocalLatticeBases.congruence_no_kernel_minimal (Formalism.LocalToGlobal.GlobalTheory.GL6.CongruenceNoKernelOfHeckeRamified, lines 75–158)

The local prescription identifies the localization of the input column lattice with that ordered image. The localization comparison for the original ramified lattice identifies its quotient length with the quotient length inside R, so $e_w \ell^{\mathrm{col}}_{uw}$ equals the just-computed colength. Its containment in the unshifted lattice proves the exponent-zero assertion. The level-size bound implies $T - 1 > 2f$. The rank formula and Bernoulli's inequality give $d/d_M \geq 1 - f/H \geq 1/2$ for $d > 0$. Multiply the colength bound by $2\ell_w/d$ and sum. By 20.7, $\sum_{w \in F} e_w \ell_w \leq \log T$, giving the stated factor 36.
Lean: Formalism.MainNotions.J.lenAtT (Formalism.SharpRestriction.AdmissibleParameterJoins.LenAtT, lines 8–9)
Lean: Formalism.MainNotions.J.division_rank_slopes_of_lines (Formalism.SharpRestriction.GlobalSchurBounds.LinkMin, lines 497–582)

□

Proof of 21.11

Proof. Put

$$Q_u = \sum_{w \in S} c_w \left(\ell^{\mathrm{col}}_{uw}/d - \tau_w \right), \qquad \eta_u = \frac{f \log(T_u - 1)}{T_u - 2}.$$

By Definition 21.5,

$$R_u = \sum_{w \in S} \frac{c_w}{e_w} \{ e_w \mathrm{sg}(u, w) \} \geq 0.$$

The rank identity and Bernoulli's inequality give $0 \le \delta \le 2f/T_u \le 1/2$, so $(1-\delta)^{-1} \le 2$. At a place outside S_T,

$$\frac{\ell^{\mathrm{col}}_{uw}}{d} - \tau_w \le \frac{\mathrm{io}_w + \delta\tau_w}{1-\delta}.$$

At a place in S_T, the term τ_w is zero. Using the definitions of IO and $\mathcal{T}$, nonnegativity of their summands, and $\mathcal{T} \le b^2$, we obtain

$$Q_u \le P_u + 4\mathrm{IO} + \frac{4fb^2}{T_u}.$$

The exact mean comparison of 21.9 is $\mu(M_{H,u}) = \mu(M_u) - \eta_u$. Together with 21.3 and 21.6, it gives

$$\begin{aligned}\mu(N'_u) &= \mu(M_{H,u}) + J_u - R_u - Q_u \\ &\ge \mu(M_u) + J_u - R_u - P_u - 4\mathrm{IO} - \frac{4fb^2}{T_u} - \eta_u \\ &\ge \mu(M_u) + J_u - R_u - P_u - 4\mathrm{IO} - 1.\end{aligned}$$

Apply the assumed bounds for P_u, R_u, and use $4\mathrm{IO} \le c_1\mathrm{IO}$. This is the required lower bound.

For the maximum slope, 21.7 and 21.9 give

$$\begin{aligned}\mu_{\max}(N'_u) &\le \mu_{\max}(M_{H,u}) + \sum_{w\in S} c_w(\sigma'_w - \tau_w) \\ &\le \mu_{\max}(M_u) + J_u - R_u \\ &\le \mu_{\max}(M_u) + J_u.\end{aligned}$$

Since $c_1 m B_0 \ge 0$, this implies the stated upper bound. □

Proof of 21.12

Proof. At μ_0 the comparison matrix is the identity in the image basis.
Lean: `Formalism.IndefR.Cx.CxData.rowMat_base` (`Formalism.LocalToGlobal.IndefiniteReal.CxLemmas`, lines 352–353)

At each infinite place v, the comparison yields (21.1). For $v \ne v_0$, let P_p be the projection onto Q_p along the other rational spectral summands, extended at v. Distinct types have disjoint joint Hecke spectra. The spherical operators commute with one another and with their adjoints in the quotient metric, so these summands are orthogonal. In the reference coordinates the constructed row metric is

$$\langle \iota z, \iota y\rangle_v = \sum_{p\in I} \left(\kappa_v \prod_{j=1}^{m} \lambda_{jpv}\right) \langle P_p z, P_p y\rangle_v.$$

The weights are positive, and the projections sum to the identity. For $z \in Q_p$ the only surviving term is the one indexed by p, with $\langle z, P_p y\rangle_v = \langle z, y\rangle_v$. This proves (21.1) also when a layer contributes the empty-part weight $e^{L_{6,j}}$ of 21.20: that very weight occurs in the displayed definition of the row metric.
Lean: `Formalism.L2.DIV.weightMet` (`Formalism.LocalToGlobal.Divisibility.Frames`, lines 49–52)
Lean: `Formalism.IndefR.Cx.ComplexDataConditionsBaseLevel.partsOrth` (`Formalism.LocalToGlobal.IndefiniteReal.PackageFromPrimitiveStatements`, lines 66–72)

At v_0, let P_b be the level subgroup for the side of the row under consideration. The row Gram matrix is the Petersson Gram matrix divided by $|P_b|$, and its undivided Petersson ratio on Q_p is $|P_b| c_{pv_0}$.
Lean: `Formalism.IndefR.Cx.rowXE_part` (`Formalism.LocalToGlobal.IndefiniteReal.CxLemmas`, lines 280–315)
Lean: `Formalism.IndefR.Cx.CxData.scal_rowsR` (`Formalism.LocalToGlobal.IndefiniteReal.CxCoreRG`, lines 60–87)
Lean: `Formalism.IndefR.Cx.CxData.colsOfR` (`Formalism.LocalToGlobal.IndefiniteReal.CxCoreRG`, lines 151–169)

Equation (21.1) makes Q_p a c_{pv}-eigenspace of $\iota^*\iota$. Thus the squared singular values of $\iota^\vee$ are at least $a_v = \min_p c_{pv}$. The dual $q^\vee$ is isometric, so

$$a_v \|B_u\|_v^2 \le \|(\iota q)^\vee B_u\|_v^2 \le \|P_u^{\mathrm{term}}\|_v^2 \prod_i \|K_i\|_v^2. \tag{A.130}$$

Take logarithms and average. For the selected realization apply 21.17 and 21.21. The constructed point assigns to each word the terminal allowance plus the sum of the kernel allowances, less the sum of the

spectral lower bounds and the embedding mean $[K:\mathbf{Q}]^{-1}\sum_{\sigma}\log\kappa_{[\sigma]}$. The family takes this assignment from the point at every index above its cutoff. Since each infinite place has exactly two embeddings, this last term equals $\langle\log\kappa_v\rangle_{\infty}$.

Lean: `GlobalTheory.EX.OB.LedPart.ht` (`Development.AbelianVarietyTheory.Definitions.GlobalTwisting.SetUnipotentTranslate`, lines 379–379)
Lean: `GlobalTheory.EX.OB.LedPart.pt` (`Development.AbelianVarietyTheory.Definitions.GlobalTwisting.SetUnipotentTranslate`, lines 381–401)
Lean: `GlobalTheory.EX.famOfKits` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsFamPOfKits`, lines 56–157)

□

Proof of 21.13

Proof. Since $A_i\le D_{0,\tau}$, one has $\sigma_i\in I_i$. For $v\in I_i$, the number $v-i$ lies strictly below σ_j-j when $j<i$, and strictly above it when $j>i$. Excluding σ_i ensures $c_\tau\neq 0$.

Let Q extract the coefficient of $\prod_i X_{N,i}^{B_i}$ with last-column degree zero and project to the σ-module. The unscaled map is $P_\tau=c_\tau\kappa_\tau Q$. In ξ, the required degree of $X_{N,n}$ forces the last-row terms of all Z_n to use column n. Descend in i to get $Q\xi=h_\sigma$. Row repetition makes ξ invariant under the upper unipotents of GL_n, and its row degrees $\sum_{j\ge i}A_j+\sum_{j>i}B_j$ give σ_i after restoring the shift. By multiplicity-one interlacing [79, §8.1], ξ spans the highest line of the σ-summand. In unit highest-vector frames $\|\xi\|^2=\kappa_\tau$, so adjunction and equivariance give $Q^*h_\sigma=\xi/\kappa_\tau$ and $P_\tau P_\tau^*=c_\tau^2\kappa_\tau I$. Tensoring yields

$$P_tP_t^*=\prod_{\tau}(t_\tau\bar t_\tau)^{-n_\tau}c_\tau^2\kappa_\tau I,\qquad P_t=\left(\prod_{\tau}t_\tau^{-n_\tau}\right)\bigotimes_{\tau}P_\tau. \tag{A.131}$$

The adjoint image attains the squared norm. At a coefficient embedding s_0, $t_\tau=s_0\tau(\bar\varpi)$ for the generator $\varpi\bar\varpi=T_u$ of 15.12. CM conjugation therefore gives the log squared norm

$$-\sum_{\tau}(\lambda_{\tau,1}-D_{0,\tau})\log T_u+\sum_{\tau}(2\log|c_\tau|+\log\kappa_\tau). \tag{A.132}$$

Lean: `GlobalTheory.BranchNorm.br_tr` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LastMapNorm`, lines 192–212)

For a class with stabilizer order h_c, the contribution is $h_c^{-1}b_\sigma(P_t\psi_c,F_c)$. The bilinear form is $b_\sigma(z,y)=z^{\mathsf T}G_\sigma y$ for a real symmetric positive Gram matrix. Cauchy–Schwarz on $\bar z,y$ bounds it by $\|z\|\|y\|$. Apply (A.131) and Cauchy–Schwarz in the class sum with weights h_c^{-1}. □

Proof of 21.14

Proof. Write J_V,J_W for the Gram matrices and $X^*=\bar X^{\,t}$. In the adelic model on r-by-n matrices the alternating form and the Heisenberg operators are

$$[X,Y]=\mathrm{Tr}_{E/F}\big(\delta\ \mathrm{tr}(J_V^{-1}Y^*J_WX)\big),\qquad D(Y)f(X)=\psi([X,Y]/2)f(X+Y).$$

Here $\bar\delta=-\delta\neq 0$ is the scalar in the Weil datum and ψ is its standard additive character of $\mathbb{A}_F/F$. Evaluation at zero in this model is the theta functional. For a rational matrix Y, the rational transformation law of a theta vector gives

$$D(Y)f(X)=\psi([X,Y])f(X).$$

Write ω for the Weil action. It satisfies $\omega(h,g)D(Y)=D(gYh^{-1})\omega(h,g)$.

Choose a nonzero rational isotropic column ξ in the first nonzero member of the flag defining the parabolic. Put

$$z(t)=1+t\xi\xi^*J_W,\qquad \bar t=-t,\qquad Z_\xi=\{z(t)\}.$$

Since $\xi^*J_W\xi=0$, multiplication in Z_ξ adds the parameters. Its rational quotient is a copy of $\mathbb{A}_F/F$. For a theta vector f and an adelic row a of length n, put

$$C_f(a)=\int_{[Z_\xi]}(\omega(1,z)f)(\xi a)\,dz,\qquad [Z_\xi]=Z_\xi(F)\backslash Z_\xi(\mathbb{A}_F).$$

Use Haar probability on this compact quotient. The integrand is periodic in a modulo E^n, since the alternating form vanishes on pairs $\xi a, \xi b$. It descends in z by rational invariance of theta evaluation and $z\xi = \xi$. These functions are continuous and bounded on the two compact quotients. We shall prove $C_f(a) = C_f(0)$.

Let $\mathcal{A}(f) = \int_{\mathbb{A}_E^n/E^n} f(\xi a)\, da$, with Haar probability. Translation of a gives $\mathcal{A}(D(\xi b)f) = \mathcal{A}(f)$. The characters $a \mapsto \psi([\xi a, Y])$, for rational Y, exhaust the characters of this quotient. Indeed $Y \mapsto m = \xi^* J_W Y$ is onto E^n, and the trace pairing with δJ_V^{-1} identifies these rows with the dual of $\mathbb{A}_E^n/E^n$. Thus, up to the choice of sign of the frequency, the Fourier coefficients of C_f are

$$\int_{[Z_\xi]} \mathcal{A}(D(Y)\omega(1,z)f)\, dz.$$

Suppose $m \neq 0$. The element $N = mJ_V^{-1}m^*$ belongs to $F^\times$, since J_V^{-1} is positive definite at a real place. Choose $s \in \mathbb{A}_F$ supported at a split finite place such that $\psi(\delta^2 Ns) \neq 1$. This is possible because the local component of the standard character is nontrivial. Set $t = \delta s$ and $b = tm$. Then

$$z(t)Y = Y + \xi b, \qquad [\xi b, Y]/2 = \tfrac{1}{2}\operatorname{Tr}_{E/F}(\delta t N) = \delta^2 N s.$$

We give the local action on $\mathcal{A}$. In the split Schrödinger model, $z(t)$ acts by the linear substitution $X \mapsto z(t)^{-1}X$ on one half of the matrix space, with a positive modulus factor c_t. The determinant character is one since $\det z(t) = 1$. The line operators from the opposite half act by the characters of the row $\xi^* J_W X$, using the perfect pairing with the coefficient space. A functional invariant under these operators depends only on the restriction of a local test to $\xi^* J_W X = 0$. To see this, take a locally constant compactly supported test which vanishes on that subspace. Its support has a finite partition into compact open sets on each of which some such character has constant value different from one. Invariance kills the restriction to each set. On the subspace itself $z(t)^{-1}X = X - t\xi(\xi^* J_W X) = X$. Applying this argument to the local test with the other factors fixed gives $\mathcal{A}(\omega(1, z(t))f) = c_t \mathcal{A}(f)$, with $c_t > 0$.

The Heisenberg multiplication law now gives, for every theta vector f',

$$\mathcal{A}(D(Y)\omega(1, z(t))f') = \psi([\xi b, Y]/2) c_t\, \mathcal{A}(D(Y)f').$$

Indeed one commutes $\omega(1, z(t))$ across $D(Y)$ using $z(t)^{-1}Y = Y - \xi b$, writes $D(Y - \xi b) = \psi([\xi b, Y]/2) D(-\xi b) D(Y)$, and uses the invariance of $\mathcal{A}$ under $D(-\xi b)$. The multiplier differs from one. Its absolute value is c_t, and if $c_t = 1$ its phase is $\psi(\delta^2 Ns) \neq 1$. Translation of z by $z(t)$ leaves the integral defining the Fourier coefficient unchanged. The coefficient therefore vanishes. When $m = 0$ the frequency is trivial. Fourier uniqueness on the compact additive quotient and continuity prove the asserted constancy. Since $z\xi = \xi$, covariance also gives

$$\int_{[Z_\xi]} (\omega(1,z) D(\xi a) f)(0)\, dz = C_f(0) \qquad \text{for every } a.$$

Lean: `GlobalTheory.TransvConst.exists_transv` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ConstantTermHeisOAlong`, lines 113–158)
Lean: `GlobalTheory.SplitTransvection.transvection_pos` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.SplitPlaceTransvectionActsPositivelyInvariantFor`, lines 251–285)
Lean: `GlobalTheory.TransvConst.exists_eigen` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ConstantTermHeisOAlong`, lines 229–241)
Lean: `GlobalTheory.TransvConst.constantTerm_heisO_along` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ConstantTermHeisOAlong`, lines 245–345)

Fix a translate (h, g) of the kernel. In the marked split frame take the local column of $g^{-1}\xi$ and divide by a coordinate of maximal absolute value. The resulting column v is integral and $\bar{v} \neq 0$ in the residue field k_w. Let $A_{\mathrm{full}} = J_V^{-1} X^* J_W \in \mathrm{Mat}_{n,r}(\mathcal{O})$ be the adjoint matrix of the local test variable. Because J_V, J_W are unimodular over $\mathcal{O}$ while the first r columns of X are invertible modulo the maximal ideal $\mathfrak{m}$, the conjugate-transpose X^* has full column rank r modulo $\mathfrak{m}$, and hence A_{full} also has full column rank r modulo $\mathfrak{m}$. Consequently $\bar{A}_{\mathrm{full}}\bar{v} \neq 0$ in k_w^n on the support. Fix a nontrivial additive character ψ_k of k_w. For each $\ell \in k_w^{n\vee}$ take the Heisenberg modulation with multiplier $\psi_k(\ell(\bar{A}_{\mathrm{full}}\bar{v}))$. These are line operators: lift ℓ to an integral dual row of length n and scale by the inverse of the coordinate used to make v primitive and

by the conductor of the residue pairing. The split trace pairing realizes precisely these multipliers. Their sum on the test is zero, since the sum of a nontrivial character on $k_w^{n\vee}$ is zero. Transporting the operators by (h, g) gives $D(\xi a_\ell)$ on the translated vector, with the coefficient rows transported by h^{-1}. These finite-place operators commute with the prescribed real polynomial operators. Taking the transvection integral of their sum gives $|k_w|^n C_f(0) = 0$ for this translated kernel.
Lean: `GlobalTheory.LevelSupport.exists_sum_zero` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LevelFockVectorHeisenbergAverageIsotropicEqZero`, lines 45–224)

The definite quotient is compact. We may integrate the vanishing transvection integral over it by Fubini. Finally, Z_ξ belongs to the unipotent radical N of the parabolic and centralizes it: each $u \in N$ fixes ξ, so $u(\xi\xi^* J_W)u^{-1} = \xi\xi^* J_W$. Average the constant term on $[N]$ also over $[Z_\xi]$. Translation invariance on $[N]$ gives the original constant term. Integrating first over $[Z_\xi]$ gives zero at every remaining translate. This proves cuspidality. □

Proof of 21.15

Proof. Write $Q = Q_w$. Let L be the lattice in the symplectic space of the dual pair and put $I = [gL : gL \cap L]$. In the lattice model the intertwiner on the vacuum is the sum of the Heisenberg translates indexed by $gL/(gL \cap L)$, divided by the signed square root of I prescribed by the splitting. In its pairing with the original vacuum only the zero coset survives. Indeed, for a representative $y \notin L$, self-duality supplies $a \in L$ whose commutator character with y differs from one. Moving the lattice operator for a across the pairing multiplies that summand by this character and leaves the vacuum pairing unchanged. The zero coset contributes the original squared norm. Thus division by the value at the identity removes exactly that norm.
Lean: `GlobalTheory.LatCoef.coef_of_local_vacuum` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.WeilLatticeIntertwinerCoefficientAtPlace`, lines 104–151)

At a split place choose one prime $P \mid w$ and use its GL_r chart. For Cartan exponents $d_1, \dots, d_r$, the two halves of L have moved indices $Q^{n\sum_i \max(d_i,0)}$ and $Q^{n\sum_i \max(-d_i,0)}$. The splitting uses the positive square root in this chart and multiplies by the local determinant character. At an inert place the exponents occur in opposite pairs, with central zero in odd rank. Put $D = \sum_i \max(d_i, 0)$. There $I = (Q^2)^{nD}$. The lattice splitting on a conjugation-stable simple residue quotient is the negative square root of its order, hence takes Q^2 to $-Q$. This is also the sign required by $\chi_P = (-1)^n$ on the uniformizer. Consequently the two coefficients on a Cartan representative a_d are

$$\phi_w^\circ(a_d) = \begin{cases} \chi_P^{\sum_i d_i} Q^{-n\sum_i |d_i|/2}, & w \text{ split}, \\ ((-1)^n Q^{-n})^D, & w \text{ inert}. \end{cases}$$

Both have value one on K_w.
Lean: `GlobalTheory.LamInert.lam_qV_inert` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.IndexNormalizationInertNormEqNeg`, lines 16–48)

We compare with the spherical doubling section normalized to one on the doubled hyperspecial. In the split chart its restriction to the graph has radial variable $X = Q^{-s-r/2}$ raised to $\sum_i |d_i|$, with the unramified determinant factor. In the inert chart the determinant of the change from the diagonal lattice to the integral graph frame has valuation D: in a hyperbolic Cartan pair of exponents $d, -d$ its integral graph basis contributes $|d|$, and a zero exponent contributes zero. The unramified inducing law on this frame gives $(\chi_P Q^{-(2s+r)})^D$. Taking $s = t/2$ gives the two preceding coefficients. This checks both the value at the identity and the restriction on every double coset.
Lean: `GlobalTheory.UGraph.sphericalDoublingSection_graph_diagonal_cartan` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.SphericalDoublingSectionGraphDiagonalEq`, lines 156–204)
Lean: `GlobalTheory.ZetaGL.zeta_sum_GL` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.UnramifiedZetaSumGLCosets`, lines 58–271)

Apply the unramified doubling calculation to the conjugate vacuum and the normalized zonal spherical function of F_1. The extension is split or unramified quadratic, both splitting characters are unramified off the marked places, the spherical section has the normalization just computed, and the two spherical vectors pair to one. With the inverse twist its denominator is

$$\prod_{a=0}^{r-1} L_w(2s + r - a, \chi_V^{-1}|_{F_w^\times} \eta_w^a),$$

where η_w is the quadratic character of E/F. Since $\chi_V|_{F_w^\times} = \eta_w^n$, the substitution $h = t + r - a = n - a$ gives d_w, including the sign at every inert place. The numerator is the standard base-change factor at $s + 1/2$. For both split and inert unramified places this is the identity in [150, §4.2.1, pp. 48–49, display immediately before (53)]; their (53) defines the denominator. Their local rank is our r, their unramified inducing character is χ_V^{-1}, and the section and vector pairing have value one in the normalizations above. In split coordinates the numerator uses the two multisets $\chi_P^{-1}B_P$ and $\chi_P B_P^{-1}$. In inert coordinates it uses $(-1)^n B_P$ and $NP = Q^2$.

On each double coset the sum of $R(g)F_1$ over right K_w-cosets equals the Hecke eigenvalue times F_1. Applying ℓ gives the same sum as for the normalized zonal function times $\ell(F_1)$, even when $\ell(F_1) = 0$. Conjugate the resulting identity. At a split place unitarity gives $\overline{B_P} = B_P^{-1} = B_{\bar P}$ as multisets, and $\chi_{\bar P} = \chi_P^{-1}$. At an inert place $\overline{B_P} = B_P$. Thus conjugation preserves the displayed local product, giving (21.2).

To justify evaluation at $s = t/2$, first use the trivial spherical representation and trivial inducing character. Group the positive Cartan terms by $\sum_i |d_i|$ in the split case and by $2D$ in the inert case. This gives a power series in Q^{-s} whose coefficients are nonnegative. Near zero the spherical integral converges and the cited formula identifies its Taylor series with the rational Euler quotient. For the trivial representation the standard Satake parameters have modulus at most $(NP)^{(r-1)/2}$. Thus every possible pole of the quotient satisfies $\operatorname{Re} s \le (r-2)/2$; the reciprocals of the normalizing factors are polynomials. The Taylor series consequently converges on a disk containing $Q^{-t/2}$ in its interior. Its value there is the sum of the absolute vacuum weights. It dominates the series with a unitary twist and a normalized unitary spherical function throughout a neighborhood of that point. The identity on the initial half-plane therefore extends to $s = t/2$ with the integral represented by its convergent coset sum. In the split case its Satake exponents have absolute value at most $(r-1)/2$ in base Q. In the inert case the base-change bound is the same in base Q^2. Since $t \ge r+5$, every Euler argument in the resulting denominator is at most Q^{-2} in modulus. There are at most $2r$ such arguments and r numerator factors of absolute value at most $1 + Q^{-2}$. Hence

$$1 \le \sum_{gK_w} |\phi_w^\circ(g)| \le (1 + Q^{-2})^r (1 - Q^{-2})^{-2r}.$$

Boundedness of $\ell(R(g)F_1)$ justifies the double-coset summation. The excess of this majorant over one is $O_r(Q^{-2})$, summable over the places of F.

Lean: `GlobalTheory.ZetaSumU.hasSum_zetaSum_inert` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.HasSumUnramifiedZetaSumUnitaryInert`, lines 139–220)
Lean: `GlobalTheory.FinZeta.AbsConv.local_place` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.LocalZetaSumsAbsolutelyConvergentUniform`, lines 81–115)

□

Proof of 21.16

Proof. Write $F = E^+$, $f = [F : \mathbf{Q}]$, $G = \mathrm{U}(W)$ and $H = \mathrm{U}(V)$, with $\dim_E W = r$ and $\dim_E V = n = r + t$. For an F-group J put $[J] = J(F)\backslash J(\mathbb{A}_F)$. Write S_f for the places below the marked primes. They are split, with distinct restrictions to F. Use the following product Haar measures, taking counting measure on rational points in forming the quotients.

On $G(F_w)$ give the lattice stabilizer volume one for $w \notin S_f$. For $w \in S_f$ give

$$K_w^1 = \ker\bigl(\mathrm{GL}_r(\mathcal{O}_w) \longrightarrow \mathrm{GL}_r(k_w)\bigr)$$

volume one. Thus $\mathrm{GL}_r(\mathcal{O}_w)$ has volume $|\mathrm{GL}_r(k_w)|$ at a marked place. At a real label ρ use $dg_\rho = V_r\, dg_{\rho,\mathrm{cd}}$, where $dg_{\rho,\mathrm{cd}}$ has the invariant density transferred from Haar probability on the compact dual $\mathrm{U}(r)$, and

$$V_r = \prod_{a=0}^{r-1} \frac{2\pi^{a+1}}{a!}.$$

Indeed the invariant density in independent skew-Hermitian coordinates has a factor two for each complex off-diagonal coordinate. The specified Lie algebra measure divides by these factors. Projection of the compact group to its last column then has the ordinary sphere density and fiber the same coordinate measure on

$\mathrm{U}(r-1)$. Multiplication of the sphere areas gives V_r. On H use probability at infinity and give the finite definite level volume one. On the additive symplectic space use rational covolume one, with the vacuum normalized to unit norm. At a marked split place the additive measure in the integral matrix chart gives $\mathrm{Mat}_{r,n}(\mathcal{O}_w)$ volume one. These conventions fix the quotient integrals and the metrics defining the adjoint.
Lean: `Formalism.IndefR.Lev.IsMeasLevW` (`Formalism.LocalToGlobal.IndefiniteReal.SubgroupCharExtensionOneApply`, lines 122–131)
Lean: `Formalism.IndefR.Links.RealDat.models_vW` (`Formalism.LocalToGlobal.IndefiniteReal.LinksChar`, lines 223–224)
Lean: `ReciprocityTheory.ChartRestriction.DS.volU` (`ReciprocityTheory.ReductiveGroups.DiscreteSeries.FormalDegree.VolUFactProdMulLe`, lines 14–14)
Lean: `GlobalTheory.LowRow.CompactDualScale.isMinf_eq_volU_smul_cd` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.StandardMeasureUnitaryPQEqVolumeSmulCompactDual`, lines 29–51)

Use inner products conjugate linear in the first variable. For a definite class form x, extend its values equivariantly to $x(h)$. Let $\Phi(v,b)$ be the reference Schwartz vector with lowest-type argument v and definite coefficient b of 37.24. Define the full lift on lowest-type vectors by

$$(Tx)(v)(g) = \int_{[H]} \Theta(h,g;\Phi(v,x(h)))\,dh. \tag{A.133}$$

The marked test has invertible leading r-by-r block, so Lemma 21.14 gives cuspidality at every translate. The level is principal and its character is trivial. At ρ, put $p = p_\rho$, $q = q_\rho$. In the r-by-n Fock array let M_s^+ be the minor in the first s rows and columns, and M_s^- the minor in rows $p+1,\dots,p+s$ and the last s columns. On the paired unit extremal vectors of 37.24 the polynomial is

$$\Delta_\rho = \prod_{s=1}^{p} (M_s^+)^{a_s^+ - a_{s+1}^+} \prod_{s=1}^{q} (M_s^-)^{a_s^- - a_{s+1}^-},$$

with parts numbered from one and a zero appended. Each mixed lowering operator $\sum_c \partial_{ac}\partial_{bc}$, $a \le p < b$, kills it: $p+q = r \le n$, so the two sets of columns are disjoint. Compact equivariance gives the lowering equations on the lowest type. These equations give the holomorphic lowest-weight module in the cuspidal realization. More explicitly, on words in raising operators the invariant pairing is determined by its value on the lowest type: move a raising operator to the opposite side as a lowering operator and commute it through the word. The commutators are compact operators and the last lowering operator kills the lowest vector. Induction gives the pairing on every pair of words, hence the same null space as in the prescribed unitary lowest-weight module. This gives the intertwiner on its irreducible quotient, and therefore the multiplicity vector in Y_σ with evaluation (A.133).
Lean: `GlobalTheory.LowRow.CuspLift.liftIsCuspC` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.AdelicLiftMemCuspFormMultiplicitySpace`, lines 22–49)
Lean: `GlobalTheory.LowRow.exists_isLift_lowC` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.AdelCOfInnerProductFormula`, lines 142–155)

Let d_ρ be the dimension of the lowest compact type, $N_K = \prod_\rho d_\rho$, and d_c the dimension of the definite coefficient space. Choose unit extremal vectors u_ρ, put $u = \bigotimes_\rho u_\rho$ and $F_z = z(u)$. Schur's lemma on the common discrete series gives a scalar Petersson pairing on the lowest type. Its trace is the multiplicity pairing, and evaluation at u gives

$$b(z,z') = \int_{[G]} \overline{F_z(g)} F_{z'}(g)\,dg = N_K^{-1}\langle z, z'\rangle. \tag{A.134}$$

In particular evaluation at u, as a function on $[G]$, is injective. Define Ψ_z by

$$\langle \Psi_z(h), b\rangle = \int_{[G]} \Theta(h,g;\Phi(u,b))\overline{F_z(g)}\,dg.$$

The transformation law of the Schwartz vector makes Ψ_z a definite class form. For representatives h_α with stabilizer orders s_α, the quotient integral of an invariant pairing on $[H]$ is its class sum with weights $1/s_\alpha$. Applying (A.134) to z, Tx therefore gives $T^*z = N_K \Psi_z$. With an orthonormal basis $b_1,\dots,b_{d_c}$ of the coefficient space,

$$\langle T^*z, T^*z'\rangle = N_K^2 \sum_{a=1}^{d_c} \int_{[H]} \langle \Psi_z(h), b_a\rangle \overline{\langle \Psi_{z'}(h), b_a\rangle}\,dh. \tag{A.135}$$

Lean: GlobalTheory.AdelicScalar.pet_eq (Development.AbelianVarietyTheory.Proofs.DivisorProofs.FormOfClassSums, lines 75–77)
Lean: GlobalTheory.LowRow.form_key_lowC (Development.AbelianVarietyTheory.Proofs.DivisorProofs.AdelCOfInnerProductFormula, lines 160–196)

Fix $z \in Y_\sigma(c)$; take z' arbitrary in Y_σ. Double the two theta kernels in (A.135). On the two restrictions of the doubled space the test is $\Phi_1(X_L)\overline{\Phi_2(-X_R)}$. The splitting contributes $\chi_V(\det g_2)$ on the second copy of G, which is the multiplier in the doubling integral with character χ_V^{-1}. The Siegel exponent is $s_0 + r/2 = n/2$, hence $s_0 = t/2$. The definite space has Witt index zero, the lattice frames are self-dual, and the doubled polynomial test is compact-finite. The restriction of χ_V is the quadratic character to power n. Since $n > 2r$, the definite identity of 34.16 replaces the theta integral over $[H]$ by $\mathrm{vol}[H]$ times the Eisenstein value. Here $s_0 > r/2$, in the half-plane of absolute unfolding. In the orbit decomposition each orbit with nonzero isotropic intersection integrates a proper parabolic constant term of a cusp form. The open orbit has representatives $(g, 1)$; its section there is the Weil matrix coefficient, and the other factor is the conjugate right-translation pairing of $F_z, F_{z'}$. Absolute convergence permits integration in this order.
Lean: GlobalTheory.Core.Rk.IP.innerProduct_convergent (Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.UnitaryThetaInnerProductFormulaConvergentRange, lines 36–155)

At infinity, the invariant-pairing calculation on raising operators gives

$$\langle \Phi_\rho, \omega_\rho(g)\Phi_\rho \rangle = \|\Phi_\rho\|^2 \langle u_\rho, \pi_\rho(g)u_\rho \rangle \tag{A.136}$$

for paired unit extremals, where π_ρ is the prescribed discrete series. The identity on the generated modules extends to their unitary completions. The commuting compact definite action makes the corresponding sesquilinear form on its irreducible coefficient space invariant, so its value on each b_a, b_a equals the value on the paired unit extremal. For a partition β supported on the first s rows let P_β be its product of principal minors. In the Fock metric the adjoint of multiplication by the s-row determinant is $\det(\partial)$. The determinant Capelli identity on this highest vector gives

$$\det(\partial)(\det(X)P_\beta) = \prod_{i=1}^{s} (\beta_i + 1 + s - i)\, P_\beta. \tag{A.137}$$

Indeed the upper row operators give determinants with a repeated row, and the diagonal row degrees after multiplication are $\beta_i + 1$. This is the column-determinant identity of [80] applied to the s-by-s corner. The same product is the quotient of the two hook products on adding a column of height s, by the factorial formula of 21.12. Adding columns in order of increasing height gives the squared norm of each minor product. Since the two corners use disjoint variables, their norms multiply. Thus

$$H_\rho := \|\Phi_\rho\|^2 = \mathrm{Hook}(a^+_{\sigma,\rho})\,\mathrm{Hook}(a^-_{\sigma,\rho}).$$

The real part of the open-orbit integral contributes $\prod_\rho H_\rho J_\rho$, where

$$J_\rho = \int_{\mathrm{U}(p_\rho,q_\rho)} |\langle u_\rho, \pi_\rho(g)u_\rho \rangle|^2 \, dg_\rho = V_r \prod_{a<b} \frac{b-a}{|\Lambda_{\rho,a} - \Lambda_{\rho,b}|}.$$

The second equality is Schur orthogonality and Harish-Chandra's discrete-series formal-degree formula in compact-dual measure [82]. Its hypotheses hold: the parameter Λ_ρ is strictly decreasing and $\Lambda_{\rho,a} - (r-1)/2$ is integral. These are the ordered parameters of the prescribed holomorphic discrete series; a unit vector has fourth power of its norm equal to one in the orthogonality formula.
Lean: GlobalTheory.FockCoefReal.fock_coef_real (Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.JointHarmonicsLowestKTypeCoefficientReal, lines 41–80)
Lean: GlobalTheory.LowRowFock.fock_delta (Development.AbelianVarietyTheory.Proofs.DivisorProofs.FockFormMinorProductEqProdHookProducts, lines 189–202)
Lean: GlobalTheory.LowRow.CompactDualCoef.inv_integral_coef_sq (Development.AbelianVarietyTheory.Proofs.DivisorProofs.InvIntegralSqDiscreteSeriesCoefficientCompactDua, lines 90–124)

At $w \in S_f$, write $Q = |k_w|$ and $X = (A \;\; B)$. At principal level the support is $A \equiv I_r,\ B \equiv 0$ modulo the maximal ideal. Its additive volume and the squared norm of the test are $m_w = Q^{-nr}$. If both X and $g^{-1}X$ meet the support, then $A, g^{-1}A \in K^1_w$, hence $g \in K^1_w$. On K^1_w the test has trivial total level character: the Weil determinant multiplier cancels the inverse splitting factor in the test. The cusp evaluation is fixed by

K_w^1. Thus the marked integral contributes $m_w \operatorname{vol}_G(K_w^1) = m_w$. More generally a residue subgroup H_w has inverse image of volume $|H_w|$ in this measure; the level-normalized adjoint divides by the residue subgroup order. Here that order is one.

On the conjugate split half the definite level at w reduces to matrices $h = \left(\begin{smallmatrix} I_r & C \\ 0 & D \end{smallmatrix}\right)$, with $D \in \mathrm{GL}_t(k_w)$. On the half used for the row coordinates X, its action is right multiplication by

$$h^{-\mathsf{t}} = \begin{pmatrix} I_r & 0 \\ -D^{-\mathsf{t}}C^{\mathsf{t}} & D^{-\mathsf{t}} \end{pmatrix}, \qquad (I_r\ 0)h^{-\mathsf{t}} = (I_r\ 0).$$

Thus it preserves the marked support. Reduction of the full lattice stabilizer surjects onto $\mathrm{GL}_n(k_w)$: a lift of an invertible residue matrix is integral invertible, and its inverse adjoint determines the other split component. Its index i_w satisfies

$$i_w m_w = \frac{|\mathrm{GL}_n(k_w)|}{Q^{rt+nr}|\mathrm{GL}_t(k_w)|} = \prod_{h=t+1}^{n} (1 - Q^{-h}) = d_w^{-1}. \tag{A.138}$$

The middle equality follows by counting ordered independent columns and using $n^2 - t^2 = rt + nr$. Let M be the mass of the full standard definite lattice genus, the sum of the reciprocal automorphism orders. Refining the level replaces a class with stabilizer A_0 by its orbits on the level cosets. The sum of their inverse stabilizer orders is the number of cosets divided by $|A_0|$. Consequently $\operatorname{vol}[H] = M \prod_{w \in S_f} i_w$.

Lean: `GlobalTheory.ThetaOrth.orth` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.ThetaTranslateOrthogonalOfSupportMovedOff`, lines 184–251)
Lean: `GlobalTheory.BaseScalar.adjLev_triv` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.AdelicScalarBase`, lines 60–65)
Lean: `GlobalTheory.LevelIndex.levelIndexNormC` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.ThetaLevelIndexMulSchwartzNormFormula`, lines 24–68)

At $w \notin S_f$ all nonzero evaluations of a vector in $Y_\sigma(c)$ have the Satake multisets of c. Indeed evaluation commutes with each finite sum of translates, and the Satake transform identifies a spherical character with its Weyl orbit. For the second evaluation take the functional $\ell(F) = \int_{[G]} F\overline{F_{z'}}$. Cauchy–Schwarz and right invariance bound it uniformly on translates of F_z. Lemma 21.15 applies also after translation at other places, since those translations commute. Its absolute majorant justifies passing from finite sets of places to the restricted product. Put

$$\mathcal{E} = \prod_{P:\, P|_F \notin S_f} \prod_{\beta \in B_P} (1 - \chi_P^{-1}\beta(NP)^{-(1+t)/2})^{-1}, \qquad d_O = \prod_{w \notin S_f} d_w, \qquad d_S = \prod_{w \in S_f} d_w.$$

The unmarked integral contributes $\mathcal{E}/d_O$. Combining it with (A.138) and using (A.134) once on the remaining pairing gives

$$\langle T^*z, T^*z' \rangle = M d_c N_K \left(\prod_\rho H_\rho J_\rho \right) \frac{\mathcal{E}}{d_O d_S} \langle z, z' \rangle. \tag{A.139}$$

One factor N_K survives from the two adjoints, and d_c counts the terms in (A.135). For $z' = z \neq 0$ the coefficient is real and nonnegative. Testing against arbitrary z' proves $TT^*z = l_c z$, with

$$l_c = M d_c \prod_\rho (d_\rho H_\rho J_\rho) \frac{|\mathcal{E}|}{d_O d_S}. \tag{A.140}$$

Lean: `GlobalTheory.FinZeta.finiteZetaL` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.FiniteZetaIntegralThetaLiftEqPartialLValue`, lines 177–231)
Lean: `GlobalTheory.LowRow.scalar_of_form_key_lowC` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.AdelCOfInnerProductFormula`, lines 214–242)

For the size estimate, the normalized spherical functions of the unitary constituents are bounded by one. The bounded spherical-function criterion [81, §4.7] puts the real Satake exponents in the convex hull of the Weyl translates of the half-modulus exponent. In the standard base-change coordinates this gives $|\beta| \le (NP)^{(r-1)/2}$; in the inert chart the radius is measured in $NP = Q_w^2$. As $t \ge r+5$, each Euler argument is at most $(NP)^{-2}$ in modulus. For a number field L, grouping the places over each rational prime gives

$\sum_P (NP)^{-2} \le [L:\mathbf{Q}] \sum_{a\ge 2} a^{-2} \le [L:\mathbf{Q}]$. From $\log|1-z| \le |z|$ and $-\log(1-y) \le 2|y|$ for real $|y| \le 1/4$, we obtain nonzero convergent products and

$$\log|\mathcal{E}| \ge -2rf, \qquad \log d_O,\ \log d_S \le 4rf. \tag{A.141}$$

The identity class gives $M \ge 1/(w_E^n n!)$, where $w_E = |\mu(E)|$. Indeed an integral automorphism of the standard definite lattice has entries of absolute value at most one at every embedding. A nonzero entry has absolute value one everywhere by its integral norm and is a root of unity: the monic polynomials of all its powers have bounded integral coefficients. The unit column lengths then give one nonzero entry in each column. Since $\varphi(w_E) \le 2f$ and the prime-power formula gives $\varphi(h)^2 \ge h/2$, we have $w_E \le 8f^2$. Hence

$$-\log M \le n\log 2 + \log(n!) + 4nf. \tag{A.142}$$

Lean: `GlobalTheory.LowRow.StdMass.massLower_of_stdV` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.UnitaryMassLowerBoundOfOrthonormalLatticeFrames`, lines 148–170)

The entries of Λ_ρ are the cloud entries and negatives of the auxiliary entries in decreasing order. Adding e_j puts the former in $[0,3b]$ and the latter in $[-b,b]$. Thus each factor $|\Lambda_{\rho,a} - \Lambda_{\rho,b}|$ in the numerator of J_ρ^{-1} is at most $4b$, and each denominator factor $b-a$, for the two coordinate indices $a<b$, is a positive integer. Also $V_r^{-1} \le r^{r^2}$ by bounding each $a! \le r^r$ in the volume product. In the selected size datum $b \ge 1$ and $L' = \log(b\,\mathrm{rd}(E) g_0 g_1 n)$, where $\mathrm{rd}(E) \ge 1$. The bounds $k \ge 5$ and $g_0, g_1 \ge (kb)^8$ give $L' \ge 5.5$, $\log n \le L'$ and $\log(4b) \le L'$. In particular,

$$J_\rho > 0, \qquad \log J_\rho^{-1} \le r^2 \log r + \tfrac{1}{2} r(r-1)\log(4b) \le 2 + 5r^2 L'. \tag{A.143}$$

As $2r < n$, $f \ge 1$ and $n \ge 8$, $n(\log 2 + \log n) + f(2 + 5r^2 L') \le 2fn^2 L'$. Equations (A.141)–(A.143), together with $d_c, d_\rho \ge 1$, give

$$l_c > 0, \qquad \log l_c \ge \sum_\rho \log H_\rho - 2fn^2 L' - 4nf - 10rf \ge \sum_\rho \log H_\rho - 8fn^2 L'.$$

The embedding permutes the real labels, preserving their signatures. The minus-hook factors are positive integers. Hence $l_c \ge e^{D_j}$.

Lean: `GlobalTheory.LowRow.lam_par_width` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LiftModelParameterDecreasingHalfIntegralWidth`, lines 118–162)

Lean: `GlobalTheory.LowRow.scalar_bound_lowC` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.AdelCOfInnerProductFormula`, lines 52–86)

The spherical Hecke operators commute and their adjoints belong to the same algebra by inversion of double cosets. Their joint eigenspaces therefore span Y_σ orthogonally. It follows that $TT^* \ge e^{D_j} I$. Let Γ be the finite level group, with action ρ_σ, and $\mathcal{X}_j$ its retained characters. The coefficients

$$e_g = |\Gamma|^{-1} \sum_{\chi \in \mathcal{X}_j} \chi(1)\chi(g^{-1})$$

give the orthogonal projector $e = \sum_g \sigma(e_g)\rho_\sigma(g)$ by character orthogonality. Applying it to (A.133) gives the required projected evaluation, so injectivity in (A.134) gives $S = eT$. For $y \in \operatorname{im} e$, $\langle y, SS^* y\rangle = \langle y, TT^* y\rangle \ge e^{D_j}\|y\|^2$. If $S^*Sx = \lambda x$ with $\lambda > 0$, then $Sx \ne 0$ is an eigenvector of SS^* in $\operatorname{im} e$ with the same eigenvalue. The spectral decomposition of S^*S on $(\ker S)^\perp$ gives the claimed source inequality.

Lean: `GlobalTheory.LowForm.parts_scalar_six` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.RationalLiftLowerBoundOffBasePlace`, lines 178–192)

Lean: `GlobalTheory.LowForm.lower_of_parts` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsFullLift`, lines 158–172)

□

Proof of 21.17

Proof. Put $U = S_c + m\eta$ and $I_\rho = \mathbf{1}_{TT}(\rho)$, and extend a_ρ by zero off TT. By (36.2), $D_{0,\rho} \le U + I_\rho b_\rho$, with $b_\rho \le 2b'$ at a mark. Since $m \ge 1$ and $\eta \ge 0$, $S_c + \eta \le U$. Hence $D_{1,\rho} \le 3D_{0,\rho} + S_c + \eta \le 4U + 6I_\rho b'$. At a mark,

$$-\lambda_\rho = -a_\rho + (N-k)/2 \le -a_\rho + (N-1)/2.$$

Elsewhere $-\lambda_\rho \le (N-1)/2 - S_c$. Adding $D_{0,\rho}$ and using $n_\rho = \lambda_\rho - D_{0,\rho}$ gives

$$D_{0,\rho} \le U + 2I_\rho b', \qquad D_{1,\rho} \le 4U + 6I_\rho b', \qquad -n_\rho \le -I_\rho a_\rho + (N-1)/2 + U + 2I_\rho b'. \tag{A.144}$$

Write $T = T_u$. Since $m \le N$ and $L \le L_a$, the table gives $0 \le \log T \le C_T N L_a$. Also $a = xe^L$, $\operatorname{rd} E \ge 1$ and $L_a \ge 1$, so $\log(2a) \le 2L_a$. Keep $-I_\rho a_\rho \log T$ at its exact value. The remaining exponent term is bounded by $C_T N L_a((N-1)/2 + U + 2I_\rho b')$. The other two entries of the terminal table satisfy

$$2c_\rho^* \le 4NL_a(4U + 6I_\rho b'), \qquad \xi_\rho^* \le N^2 L_a.$$

Sum over $|\mathcal{P}| \le f$ types, of which n_T are marked, and add $e_{\mathrm{ev}} \le C_{\mathrm{ev}} f N^2 L_a$. This gives

$$A + S_a \log T \le L_a\big((2C_T + 24)n_T b' N + (C_T + 16)fUN + (C_T/2 + 1 + C_{\mathrm{ev}})fN^2\big). \tag{A.145}$$

The first bound in (36.3) dominates each coefficient. The right side is therefore at most $H_{P,u} + S_a \log T$. In the selected table the evaluation allowance is zero: $\varpi\bar{\varpi} = T$ in 21.13, and (A.494) gives the squared norm bound in the stated frames. For a general table the same norm bound is (36.8).

For a layer j, take the mean in (36.4). By 21.22 the mean of $\omega_{\rho,v}$ is $\log \Omega$. The term $-n_{j,\rho} \log \Omega$ is therefore exactly $-\sum_i \nu_{uj\rho i} \log \Omega$. Together with the factorial terms this gives $\sum_\rho K_{uj\rho}$ with the normalization of 21.12. The constant and degree-log terms cost at most $fN^2(1 + 2c_6)L_a$. Since $p_j = |\mathcal{P}| g_j^0 \le fN^2$, the two terms multiplied by p_j cost at most $fN^2(C_\theta + c_5)L_a$.

For the remaining terms, insert $g_* \le C_8 L$ and $\log T \le C_T m L$ in (36.1). The term $2C_e f N^2 g_*$ costs at most $2C_e C_8 f N^2 L_a$. In the two corrections h_j, h_j', the unmultiplied L costs at most $2C_4 f N^2 L_a$. The terms containing $g_* + \log T$ are at most $2(C_2 + C_4)(C_T + C_8) f m^3 N^2 L$, since $m \ge 1$. Thus

$$\langle s_{jv} \rangle_\infty - \sum_\rho K_{uj\rho} \le fN^2\big((C_\theta + c_5 + 2c_6 + 1 + 2C_4 + 2C_e C_8)L_a + 2(C_2 + C_4)(C_T + C_8)m^3 L\big). \tag{A.146}$$

The last two bounds in (36.3) give $\langle s_{jv} \rangle_\infty \le H_{K,uj}$.

Finally, apply weighted Cauchy–Schwarz to the sum over input indices in (36.7). At stage i the sum of the weights is at most the number $a_i h_i^{\mathrm{cl}}$ of indices, since every weight is at most one. Summing the resulting squared estimates with the output weights costs at most $a_{i+1} h_{i+1}^{\mathrm{cl}}$. The two frame identities consequently give $\|K_i\|_v^2 \le a_i h_i^{\mathrm{cl}} a_{i+1} h_{i+1}^{\mathrm{cl}} t_{\max}(i, v)$. If the product vanishes, $K_i = 0$. Otherwise exponentiate (36.9). □

Proof of 21.18

Proof. The determinant ideal of L_a^h is $\mathfrak{c}_a/\overline{\mathfrak{c}_a}$, with $\mathfrak{c}_a$ integral and prime to its conjugate. Indeed, at a chosen split prime P write $e_1, \dots, e_n$ for the elementary-divisor exponents relative to L_0, with $\varpi^{e_i}\mathcal{O}_{E,P}$ as the convention. At $\bar{P}$ the exponents are the negatives, up to order, because the second matrix is the inverse adjoint. Put $s_P = \sum_i e_i$ and $Q_P = \mathrm{N}P$. Taking the positive part of the determinant divisor gives

$$\mathrm{N}\mathfrak{c}_a = \prod_P Q_P^{|s_P|}.$$

The contribution to $[L_a^h : L_a^h \cap L_0]$ at P is $Q_P^{\sum_i \max(-e_i, 0)}$, and at $\bar{P}$ it is $Q_P^{\sum_i \max(e_i, 0)}$. Outside the moving pairs the two lattices agree. Conjugation gives an additive isomorphism on the intersection quotients, so

$$I_a = \prod_P Q_P^{r \sum_i |e_i|}, \qquad Z_a = m_a I_a.$$

The second equality also follows directly by multiplying the indices in the tower from $N_a L_0$ to L_a.

We now give the normalization of the class coefficient in 37.24. Write $B(\mathfrak{c}_a)$ for the algebraic determinant value with its positive norm factor. The coefficient is

$$\epsilon_a = \frac{t_a}{B(\mathfrak{c}_a)}, \qquad t_a \in \mathbf{Q}_{>0}, \qquad t_a^2 I_a = \mathrm{N}\mathfrak{c}_a.$$

In particular the rational factor in this definition can be computed before choosing a coefficient embedding. The ranks in Theorem 38.3 give r odd. Since $|s_P| \equiv \sum_i |e_i| \pmod 2$, the unique positive choice is

$$t_a = \prod_P Q_P^{\left(|s_P| - r\sum_i |e_i|\right)/2}.$$

Lean: `Formalism.L2.Pj.EpIs` (`Formalism.LocalToGlobal.ProjectiveLattices.ThClass`, lines 106–114)
Lean: `GlobalTheory.ClassTieCompare.exists_epIs_iff` (`Development.AbelianVarietyTheory.Proofs.UnitaryExistence.ExistsFirstOfSecond`, lines 69–91)
Lean: `GlobalTheory.ClassScalarCompare.stdIndex_conjLat` (`Development.AbelianVarietyTheory.Proofs.UnitaryExistence.NotIsOfFinOrderUnitQuot`, lines 139–155)

We compute the image of $B(\mathfrak{c}_a)$. At the reference tuple the twist is trivial and the splitting character is $\mu_0\eta^{-1}$. The moving primes avoid the bad modulus, so the local characters there are unramified. On the two primes of a moving pair their product character has trivial restriction to the diagonal local field. Thus a factor with $s_P < 0$ on the chosen half can equally be evaluated with exponent $-s_P$ on the opposite half. The ideal-character value on $\mathfrak{c}_a$ is consequently $(\mu_0\eta^{-1})(d_a)$.

For the change of embedding, let π_P denote the idèle of a uniformizer on the chosen half, equal to one off P. The conjugate of $\mu_0(d_a)$ is specified by its coefficient-field value, since d_a has infinite part one. For η, the coefficient-field value b_P at π_P has base reading $\eta(\pi_P)/\sqrt{Q_P}$ and reading $\eta_\sigma(\pi_P)/\sqrt{Q_P}$ at σ. Hence the algebraic product

$$\prod_P b_P^{-s_P} Q_P^{(|s_P| - s_P)/2}$$

has reading $\eta_\sigma(d_a)^{-1}\prod_P Q_P^{|s_P|/2}$. Multiplying by the coefficient of $\mu_0(d_a)$ gives $B(\mathfrak{c}_a)$ at the base embedding, by its defining ideal-character formula. Injectivity at the base now identifies the algebraic elements. At $\widetilde{\sigma}$ this gives

$$\widetilde{\sigma}\big(B(\mathfrak{c}_a)\big) = \chi_{W,\sigma}(d_a)\sqrt{\mathrm{N}\mathfrak{c}_a}.$$

Since $\sqrt{\mathrm{N}\mathfrak{c}_a} = t_a\sqrt{I_a}$, substitution gives

$$\widetilde{\sigma}(\epsilon_a)\frac{Z_a}{m_a} = \frac{t_a I_a}{\chi_{W,\sigma}(d_a)\sqrt{\mathrm{N}\mathfrak{c}_a}} = \chi_{W,\sigma}(d_a)^{-1}\sqrt{I_a}.$$

Lean: `GlobalTheory.GalOne.ClassConj.coefficientField_reading_at_embedding_prod` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.EmbeddingConjugateOscillatorSixth`, lines 148–181)
Lean: `GlobalTheory.GalOne.ClassConj.exists_ideal_conj` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.EmbeddingConjugateOscillatorSixth`, lines 185–294)
Lean: `GlobalTheory.ClassScalarValue.zCard_eq` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.OscillatorModelValuesThird`, lines 293–298)
Lean: `GlobalTheory.ClassScalarValue.value_calc` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.OscillatorModelValuesThird`, lines 317–333)

□

Proof of 21.19

Proof. Write u_σ for the prescribed unit extremal tensor and $\mathrm{ev}_\sigma(y)(h) = y(u_\sigma)(h)$ for evaluation in the target multiplicity space. If f is an algebraic cusp section, denote its rational coordinate column by $[f]$. At the framed point belonging to h, write $R_\sigma(f,h)$ for the value of the section in its algebraic frame, $C(h)$ for the component, and $\mu_\sigma(h)$ for the adelic multiplier taking that framed value to the adelic section. If $c_\sigma(C)$ denotes the chosen component coefficient, the realization of $[f]$ is specified by

$$\mathrm{ev}_\sigma(J_\sigma[f])(h) = a_j c_\sigma(C(h))\mu_\sigma(h)R_\sigma(f,h), \qquad a_j = (2\pi\xi)^{-A_j/2}.$$

Here $\xi = (\sqrt{d}\,\mathrm{N}\mathfrak{c}_K)^{-1}$ and A_j is the total degree in 37.24. In particular, the function called $\varphi_j(c_j, f)$ in (37.32) acquires the factor a_j upon extremal evaluation of the rational column. This formula on the rational cusp frame extends by semilinearity to its coordinate space.
Lean: `Formalism.L2.Pj.CoreJ.phiJ` (`Formalism.LocalToGlobal.ProjectiveLattices.FamilyUniformisationBasePointOrthogonalBis`, lines 132–134)
Lean: `Formalism.IndefR.FWT.FWTie.evaluates` (`Formalism.LocalToGlobal.IndefiniteReal.FWExist`, lines 645–649)

Let Γ be the finite level group and let ρ_σ denote its action by right translation. Write $e_g \in K$ for the coefficients of the retained central idempotent in 21.16, so that $e_\sigma = \sum_{g\in\Gamma} \sigma(e_g)\rho_\sigma(g)$. Write $\delta : \Gamma \to K^\times$ for the character in the level action of 37.24. Choose definite class representatives. For $x \in X$, take the algebraic theta section of its list of values on those representatives. By 37.24 it has an expansion $f_x = \sum_\nu a_\nu f_\nu$ after base change from cusp sections, with a_ν read in K through the reference field identification. Use this expansion of the class section itself, before the central translation in the reference identity. Denote that translation by z_0 and the corresponding root of unity, pulled back to K, by ζ_0. Put

$$\theta x = \sum_{g\in\Gamma} e_g \delta(g)^{-1} \sum_\nu \zeta_0 a_\nu [\, g(z_0 f_\nu)\,]. \tag{A.147}$$

The level action preserves the rational cusp space. At the base embedding its determinant multiplier cancels $\delta(g)^{-1}$. By the reference class identity of 37.24, with phase one and with the factor a_j in the displayed evaluation formula, the inner sum then gives the unprojected theta function. The outer sum projects it. Hence $\mathrm{ev}_0(J_0\theta x) = \mathrm{ev}_0(S_0 j_0 x)$. Evaluation is injective by (A.134), and the realized cusp frame is independent. Thus the column in (A.147) is uniquely determined by x. Additivity and K-linearity follow by applying J_0 to the two sides of the corresponding identities, using its semilinearity and that of j_0.

Lean: `GlobalTheory.GalLift.Assembly.ratOf` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.GalLiftOfOneMap`, lines 127–130)

Lean: `GlobalTheory.ConjRatLiftFive.base_lift5` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ConjugatesOfRationalThetaLiftFromCharacterisatio`, lines 46–67)

Fix σ over the distinguished embedding of K_d. The map from K to the reference comparison field is an isomorphism onto that field, so composition of its inverse with σ gives an embedding $\widetilde{\sigma}$ of the entire chart. All subsequent evaluations of algebraic sections use this embedding.

First consider the finite test. Let M be the covering modulus, A the finite group of n-tuples of arguments of the level indicator φ, and $A^\vee$ its dual under the level pairing. At the reference tuple the twisting set is empty, so φ takes values in $\{0, 1\}$. For a primitive M-th root ζ_M, use the Fourier weights

$$w(q) = \sum_{b\in A} \varphi(b)\zeta_M^{\langle b,q\rangle}, \qquad q \in A^\vee.$$

Choose d_σ prime to M such that $\widetilde{\sigma}(\zeta_M) = \zeta_M^{d_\sigma}$ in the reference coordinates. Then $\widetilde{\sigma}(w(q)) = w(d_\sigma q)$. The sign ε converting the finite adelic Heisenberg action to the action on the theta chart is fixed by the choice of the two actions, hence is the same for every class. The required change of marking is multiplication by $n_\sigma = \varepsilon d_\sigma$. The base modulus divides M. At every marked prime the scalar matrix $n_\sigma I_r$ is therefore invertible. Under the level-group identification with the product of the marked general linear groups these matrices give an element $z \in Z(\Gamma)$. Its action on the marked half of the level lattice is multiplication by n_σ modulo the base level.

Lean: `GlobalTheory.GalOne.conj_coeff_eq_dilate_ofFive` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.CoefMapNormEquivariantBasisOfFive`, lines 2428–2454)

Lean: `GlobalTheory.GalCentral.central_lev_ofFive` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.CoefMapNormEquivariantBasisOfFive`, lines 2483–2505)

Fix a framed point h, let L be its self-dual target lattice, and let v be a unit vacuum for $L_0 \otimes L$. For a class a let v_a be the average of the Heisenberg translates of v by $L_a \otimes L$, weighted by the lattice character. This is the projection onto the vacuum line for $L_a \otimes L$. Its normalization can be seen in the split Schrödinger model. On one half write D_0, D_a for the old and new position lattices on the tensor space, $D_\cap = D_0 \cap D_a$, $m = [D_a : D_\cap]$, and $n' = [D_0 : D_\cap]$. Averaging the modulations for the new dual half restricts $\mathbf{1}_{D_0}$ to $D_\cap$. Averaging its translations over $D_a/D_\cap$ gives $m^{-1}\mathbf{1}_{D_a}$. The unitary change of variables sends the old indicator to $\sqrt{n'/m}\,\mathbf{1}_{D_a}$ times the inverse splitting character on the chosen determinant half. Thus its multiplier relative to the projection is the character times $\sqrt{mn'}$. The calculation applies to successive disjoint split pairs. Localizing L gives r free coordinates, so the product of the mn' is I_a of 21.18. Consequently the analytic change of vacuum has coefficient $\chi_{W,\sigma}(d_a)^{-1}\sqrt{I_a}$ against v_a. By 21.18 it is exactly the embedded class coefficient times Z_a/m_a.

Lean: `GlobalTheory.OneMapRef.SplitProj.step` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.SplitFinitePointMovesVacuumToProjection`, lines 115–209)

Lean: `GlobalTheory.T.splitFinitePoint_moves_vacuum_to_projection` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.SplitFinitePointMovesVacuumToProjectionStatement`, lines 16–17)

We give the marking calculation, since a change of a Heisenberg lift can introduce a scalar. For $q \in A^\vee$ choose a displacement $Y_a(q)$ for the signed marking. Row by row it is congruent to the marked representative divided by M, modulo L. The lifts are taken with the clearing condition that a sufficiently high power of the marked half-ideal carries them into $2N_aL$. Such lifts exist by the Chinese remainder theorem on the two coprime halves of the level. Equivalently, one prescribes the marking on its supporting half and divisibility by $2N_a$ off that half. Take N_a divisible by M and take a second clearing integer k_a, prime to M, carrying L_0 into L_a. These choices are possible because the moving places avoid the level.

We show that these displacements give the transported translates with their characters. Write H for the completed tensor lattice of the moved vacuum, X' for the displacement transported from the reference marking, and Y' for the displacement in the new marking, both in finite adelic coordinates with the same sign. It suffices to consider q with nonzero Fourier weight. Its coordinates are annihilated by an integer supported at the bad primes of the test, disjoint from the moving pairs. Indeed, the indicator is constant on cosets of the kernel of reduction to its test modulus, so summing a nontrivial character on that kernel gives zero. At the moving pairs q is therefore integral as a displacement. At the other places the two markings give the same coset modulo the original integral lattice. Clearing that lattice into the moved lattice gives

$$k_a(X' - Y') \in H, \qquad k_aMY' \in H.$$

Let $\mathfrak{b}$ be the marked half-ideal. For l large enough for both root systems, choose $s \in \mathfrak{b}^l$ and t with $\bar{t} \in \mathfrak{b}^l$ and $s + 2k_a^2 t = 1$. This is possible since the level is odd, k_a is prime to it and its two halves are coprime. By the clearing conditions on the old and new roots,

$$sX' = 2b_1, \quad sY' = 2b_2, \quad \bar{t}X' = 2Mb_3, \quad k_a\bar{t}Y' = 2Mb_4, \qquad b_i \in H.$$

Put $b = b_1 - b_2 + k_a t\, k_a(X' - Y')$. Then $b \in H$ and $X' - Y' = 2b$. If ω is the alternating form in the Heisenberg character, sesquilinearity gives

$$\omega(Y', b) = \omega(2b_2, b) + \omega(k_aMY',\, 2(k_ab_3 - b_4)).$$

Both terms belong to the kernel of the additive character by integrality on H. The doubled lattice $2H$ acts trivially on the moved vacuum. The Heisenberg multiplication law now gives identical actions for X' and Y'. This also proves independence of the admissible choices of roots. Replacing q by $d_\sigma q$ gives admissible roots for the marking multiplied by d_σ.

Lean: `GlobalTheory.OneMapRef.Term.heis_congr` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.OmegaFormSignIndependent`, lines 64–77)
Lean: `GlobalTheory.OneMapRef.Term.heis_root` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.OmegaFormSignIndependent`, lines 304–389)
Lean: `GlobalTheory.OneMapRef.Term.heis_root_zak` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.OmegaFormSignIndependent`, lines 412–459)

We next compare the polynomial applied to this finite vector. In the complex definite coefficient space at σ, consider the map obtained by pairing with the analytic extremal tensor, and the map obtained by embedding the algebraic polynomial coefficients and relabelling their variables. Put both maps into the same Fock space by applying the polynomials to the standard vacuum. Call the maps A and B, respectively. The relabelling keeps the column index. With $p = k + j - 1$, $q_- = m - j$, a row numbered i from zero goes to row $q_- + i$ on the positive side and $q_- - 1 - i$ on the conjugate side in the framed derivative coordinates. Thus the prescribed product of minors is the same polynomial in these coordinates for the two maps. The algebraic highest vector has norm one after embedding: its Gram value in the standard definite frame is the rational number one. On the analytic side the vector giving the minor polynomial has norm one by 37.24. Changing the definite orthonormal basis acts unitarily, including the vacuum character. We obtain unit vectors w_A, w_B with $Aw_A = Bw_B \neq 0$. Nonvanishing follows also by taking the leading monomial of the minor product in the polynomial realization of the Fock space.

We check that the comparison on these vectors holds on the whole coefficient space. For a rational unitary element γ, the analytic map intertwines its action with the oscillator action on the output. For the algebraic map, substituting the matrix of γ in the source columns of the minor polynomial gives the same oscillator action times a scalar: the scalar is the embedded coefficient-line multiplier divided by the vacuum character of the orthonormal change of basis. Thus, writing ρ for the coefficient action, there are an operator T_γ and a nonzero scalar b_γ with

$$A\rho(\gamma) = T_\gamma A, \qquad B\rho(\gamma) = b_\gamma T_\gamma B.$$

The rational unitary group is dense in the product of the compact real groups. Indeed the Cayley transform gives rational coordinates near the identity from the skew-adjoint matrices, and weak approximation there gives a dense subset of a neighborhood of the identity. The compact unitary groups are connected. Hence the irreducible coefficient representation remains irreducible on the rational group. Both maps are injective, since their kernels are invariant and they have a common nonzero value. Likewise $A^{-1}(\operatorname{im} B)$ is invariant and contains w_A. It is the full space, and equality of dimensions gives $\operatorname{im} A = \operatorname{im} B$. The automorphism $V = B^{-1}A$ satisfies

$$V\rho(\gamma)V^{-1} = b_\gamma^{-1}\rho(\gamma).$$

For every γ of nonzero trace, taking the trace gives $b_\gamma = 1$. These elements generate the rational group. To see this, the trace is a nonzero real analytic function on the connected compact group, since its value at the identity is the dimension. Its nonvanishing set is open dense. For any rational γ, density gives a rational β with both β and $\gamma\beta$ in this set. Schur's lemma gives $V = \alpha\,1$. As $\alpha w_A = w_B$, $|\alpha| = 1$. Put $p_\sigma = \alpha^{-1}$. This phase works for every class value and every input. Finally each derivative letter in the passage from Fock to Lie coordinates contributes $1/\sqrt{2\pi\xi}$. The coefficient polynomials are homogeneous of degree A_j, as follows by counting the row degrees of their generating minor product and applying linear substitutions. After multiplication by p_σ the analytic differential polynomial is therefore a_j times the embedded algebraic one.

Lean: `GlobalTheory.AdTh.PhaseMap.proportional` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.OscillatorModelExtensionFirst`, lines 31–75)
Lean: `GlobalTheory.GalOne.PhaseAt.io_eq_phase_smul_at_ofFive` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.CoefMapNormEquivariantBasisOfFive`, lines 827–995)
Lean: `GlobalTheory.OneMapRef.IoCmp.two_div_fockScale` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.OmegaFormRationality`, lines 78–79)
Lean: `GlobalTheory.GalOne.IoCmp.ioAt_of_conj_ofFive` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.CoefMapNormEquivariantBasisOfFive`, lines 1033–1142)

We can now write the two evaluations at h. Let $\mathcal{D}_h(P, v')$ mean that the theta series of the test vector v' is differentiated by P in the framed Lie directions and evaluated at zero. Denote the embedded, relabelled polynomial of x at class a by $P_{a,\sigma}$, the common representative level volume by V_{lev}, its stabilizer order at a by s_a, and Heisenberg translation by $D(Y)$. The common finite sum is

$$\mathcal{C}(h,x) = \sum_a \frac{V_{\mathrm{lev}}}{s_a}\frac{\widetilde{\sigma}(\epsilon_a)Z_a}{m_a|A^\vee|}\mathcal{D}_h\left(P_{a,\sigma}, \sum_{q\in A^\vee}\widetilde{\sigma}(w(q))D(Y_a(d_\sigma q))v_a\right).$$

For the analytic lift, finite Fourier inversion expresses the level test as the sum of translates with base weights $w(q)/|A^\vee|$. The equality $\widetilde{\sigma}(w(q)) = w(d_\sigma q)$ and the permutation $q \mapsto d_\sigma q$ give exactly the vector displayed above. The vacuum projection calculation supplies its class coefficient. Integration over each definite class contributes V_{lev}/s_a. On the algebraic side the central translation z pulls the marking back to n_σ times the original marking, so that the embedded Fourier weights and the roots $Y_a(d_\sigma q)$ occur together. These operations commute with evaluating a polynomial differential operator on a finite sum.

For the remaining frame scalars let β_h be the scalar expressing the product of the degree-one theta generators of the framed point in the theta function of v. Both are sections of the same principal theta lines, each of dimension one. Use the same generators on the covering level by pullback. Let $c'_\sigma(C)$ denote the component coefficient of the analytic vacuum in this framing after removing $\mu_\sigma(h)$. More explicitly, at one point of each component compare the transported unit vacuum with the theta generator normalized to have mean squared norm one. Their quotient, divided by the unitary multiplier $\mu_\sigma(h)$, gives $c'_\sigma(C)$, of modulus one. At another framed point of the component the real change of frame transports the Gaussian and its arguments together. The finite change between the two transported target lattices carries the self-dual vacuum by the quotient of their splitting multipliers. Dividing by μ_σ cancels that quotient. A rational change of representative fixes the theta series by change of summation variable. This gives the same $c'_\sigma(C)$ in every such chart. In particular the scalar of the analytic vacuum relative to v at h is $c'_\sigma(C(h))\mu_\sigma(h)\beta_h$.

Lean: `GlobalTheory.OneMapRef.FrameConst.exists_frConst` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.OmegaFormScalarExtension`, lines 246–315)
Lean: `GlobalTheory.GalOne.FrameConst.exists_frAtEmb_ofFive` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.CoefMapNormEquivariantBasisOfFive`, lines 516–535)

The covering section is constant at the reference tuple. At the distinguished embedding its value at a marked point is the prescribed root of unity times the twist character of a determinant root. The reference

twist is trivial. The marked charts cover the complex components, and evaluation on them identifies the section with the constant section of that value. This equality descends to the comparison field: scalar extension to $\mathbf{C}$ is injective on sections, and evaluation at the chosen coefficient-field point gives the constant in that field. If its base value has order m_0, its coefficient-field value has m_0-th power one by the same injectivity. Denote its reading under $\widetilde{\sigma}$ by c_0. Thus $|c_0| = 1$.

Write T_σ for the unprojected analytic lift. Substitution in the differentiated theta series and in the evaluation of the translated algebraic class section gives, respectively,

$$p_\sigma \operatorname{ev}_\sigma(T_\sigma j_\sigma x)(h) = c'_\sigma(C(h))\mu_\sigma(h)\beta_h\, a_j\, \mathcal{C}(h,x),$$
$$\sum_\nu \sigma(a_\nu) R_\sigma(z f_\nu, h) = c_0 \beta_h\, \mathcal{C}(h,x).$$

For the second equality the expansion of f_x is an equality of algebraic sections, so it can first be translated and then evaluated over $\widetilde{\sigma}$. Multiplying the first equality by c_0 and the second by $a_j c'_\sigma(C(h))\mu_\sigma(h)$ identifies the two sides even at a zero of the evaluated section.

Lean: `GlobalTheory.OneMapRefCompare.cover_const` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.OneMapTypingsEquiv`, lines 180–205)
Lean: `GlobalTheory.GalOne.adelic_value_ofFive` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.CoefMapNormEquivariantBasisOfFive`, lines 2533–2587)
Lean: `GlobalTheory.GalOne.geometric_value_ofFive` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.CoefMapNormEquivariantBasisOfFive`, lines 2591–2647)
Lean: `GlobalTheory.GalOne.identity_of_frConst_ofFive` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.CoefMapNormEquivariantBasisOfFive`, lines 2651–2731)

Put $u(h) = c'_\sigma(C(h))/c_\sigma(C(h))$. This function descends to the components of the quotient at principal level and has modulus one. Multiplication by u preserves the real representation and the principal level. It also preserves cuspidality: on the orbit of a unipotent radical the component is constant, by approximation of its finite adelic coordinates by rational ones and invariance under the real group. Multiplication by $\bar{u}$ is its inverse. The Petersson integral in (A.134) shows that it induces a unitary operator U_σ on the multiplicity space. Write $J' = U_\sigma J_\sigma$ and $\kappa = p_\sigma c_0$. The two evaluations just proved say

$$\kappa \operatorname{ev}_\sigma(T_\sigma j_\sigma x) = \sum_\nu \sigma(a_\nu) \operatorname{ev}_\sigma(J'[z f_\nu]).$$

In this frame the level law is

$$J'[gf] = \sigma(\delta(g))\, \rho_\sigma(g) J'[f].$$

Indeed pullback under g identifies the framed fibre at h with the fibre at hk_g, where k_g is a level lift. If their degree-one generators differ by λ, the section value of degree n changes by λ^n. Comparing the two vacuum readings gives $c'_\sigma(C(h)) = c'_\sigma(C(hk_g))\lambda^n$: the finite vacuum multiplier is $\mu_\sigma(k_g)$ and $\mu_\sigma(hk_g) = \mu_\sigma(h)\mu_\sigma(k_g)$. One can compare at a nonzero value of the generator since its mean squared norm is one. The generator factors thus cancel in the framed pullback. The remaining level coefficient is the embedded determinant character in 37.24. The common factor a_j multiplies both evaluations. Injectivity of extremal evaluation proves the level law on vectors.

Lean: `GlobalTheory.GalOne.Equivar.eqAt_of_frAtEmb_ofFive` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.CoefMapNormEquivariantBasisOfFive`, lines 1244–1322)
Lean: `GlobalTheory.CompPhase.exists_isCompPhase_five` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.CentralShiftPhaseConjReadingsFiveClauses`, lines 38–45)

Set $Z_x = \sum_\nu \sigma(a_\nu) J'[f_\nu]$. Apply e_σ to the evaluation identity. By 21.16 its analytic side is the evaluation of $\kappa S_\sigma j_\sigma x$. The level law and injectivity give the first equality below. Applying the same level law directly to (A.147) gives the second:

$$\kappa S_\sigma j_\sigma x = \sigma(\delta(z))\, e_\sigma \rho_\sigma(z) Z_x, \qquad J'\theta x = \sigma(\zeta_0 \delta(z_0))\, e_\sigma \rho_\sigma(z_0) Z_x.$$

Both translations commute with the projector. Multiplying the first equality by $\rho_\sigma(z_0 z^{-1})$ and substituting in the second proves (21.3) with

$$g_\sigma = \rho_\sigma(z_0 z^{-1}), \qquad \xi_\sigma = \sigma(\zeta_0)\, \frac{\sigma(\delta(z_0))}{\sigma(\delta(z))}\, p_\sigma c_0.$$

All four scalar factors are independent of x. The two determinant values have order dividing $|\Gamma|$, and ζ_0 has finite order. Every complex embedding sends these values to the unit circle. The norm computations for p_σ and c_0 therefore give $|\xi_\sigma| = 1$. Right translation preserves the quotient measure, so g_σ is isometric. At the distinguished embedding the defining base equality itself allows $U_0 = g_0 = 1$ and $\xi_0 = 1$.

For the metric comparison, the squared norm of $J_\sigma[f]$ is, by (A.134), the integral of $|c_\sigma \mu_\sigma R_\sigma(f)|^2$ times $a_j^2 \prod_\rho \dim \tau_\rho$, where τ_ρ is the lowest compact type. The core Petersson norm on the same section also includes the factor a_j on lengths. Thus on d' independent columns the squared Gram determinants on both sides of the core-to-realization comparison contain $a_j^{2d'}$. Their ratio is the measure and lowest-type ratio of 35.22. In the height calculation the rational target carries this realized metric, so the equality just proved gives $\|J_\sigma \theta x\|^2 = \|S_\sigma j_\sigma x\|^2$ with the metric used for the singular values in 21.16.

Lean: `Formalism.L2.Pj.l2C` (`Formalism.LocalToGlobal.ProjectiveLattices.L2DefiniteCuspForms`, lines 157–159)
Lean: `GlobalTheory.GalLift.Assembly.conj_abstract` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.GalLiftOfOneMap`, lines 84–102)
Lean: `GlobalTheory.ConjRatLiftFive.conj_lift5` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ConjugatesOfRationalThetaLiftFromCharacterisatio`, lines 71–138)

□

Proof of 21.20

Proof. We first describe how to find Z_{jp} over K. The Hecke identification in the primitive hypothesis of 36.7 has the following content. The base lift intertwines the operators on class forms with a commuting family on its target. In that family W_{jp} is the sum of the joint generalized eigenspaces with the prescribed layer type of p. A layer type prescribes a Hecke polynomial at each good split prime and a scalar for each spherical double coset at each good inert prime. Here the good primes are taken off the marked places of the reference tuple.

Decompose the rational class space into the simultaneous generalized eigenspaces of its split and inert operators. These operators commute and split over the coefficient field of the selected realization. The rational subspace of forms whose base images belong to $\operatorname{im} S_0^*$ is stable under them. Indeed, the kernel of the lift is stable under the intertwining operators, and their adjoints belong to the same family. Each component of Bx therefore still has base image in $\operatorname{im} S_0^*$. Intertwining sends each component into the target component with the same eigenvalues at the base embedding. On $\operatorname{im} S_0^* = (\ker S_0)^\perp$ the lift is injective. Thus $S_0 j_0 Bx \in W_{jp}$ precisely when Bx lies in the sum of the rational components with the prescribed restrictions. This gives Z_{jp} by taking an inverse image over K. Its basis can be found by linear equations: for an operator A and eigenvalue a, the generalized eigenspace is $\ker(A - aI)^{\dim_K X}$. In an intersection of such kernels one retains a new equation whenever it reduces the dimension. The resulting finite intersection gives the same space.

Lean: `Formalism.IndefR.Cx.HeckeIntertwiningLifts` (`Formalism.LocalToGlobal.IndefiniteReal.PrimitiveStatementsWithPassage`, lines 48–56)
Lean: `Formalism.IndefR.Cx.part_iff_lifts3` (`Formalism.LocalToGlobal.IndefiniteReal.LiftCoordinatesInGoodPartFamily`, lines 114–127)

Fix an embedding τ agreeing with the base embedding on K_d. Fix p before choosing x. The layer transfer assigns to p one list of split Hecke polynomials and inert double-coset values at the good places. By the reference Hecke identification of 36.7, every summand contributing a rational vector to Z_{jp} has this restriction. Thus it suffices to recover the spherical generator values from this one list. The recovery works on the entire sum of the rational components. At a split prime with residue cardinality q, the coefficient of degree h of the prescribed polynomial is $(-1)^h q^{h(h-1)/2} a_h$, where a_h is the eigenvalue of the h-th elementary double coset. Thus the polynomial determines every a_h. At an inert prime the type specifies the double-coset values individually. At the base embedding the spherical operators are normal: their adjoints are given by inversion in the commutative spherical algebra. Hence their generalized eigenspaces are eigenspaces. Each resulting equation on a class form is an equality of finite sums of translates in its rational coefficient frame. Applying τ gives the equation with eigenvalue $\tau(a_h)$, or the corresponding embedded inert value.

Lean: `GlobalTheory.ZPartEig.points_eig` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.ThetaLiftPointsPartEigenformsCoefficientField`, lines 198–207)
Lean: `GlobalTheory.ConjScalar.eigF_conj` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.SemilinearInjectiveOfIndep`, lines 53–60)

We show that the vectors $z = S_\tau j_\tau Bx$ with $x \in Z_{jp}$ have one target spherical system whenever one of them is nonzero. At a good place the test is the self-dual lattice vacuum, the splitting characters are unramified,

and the retained projector at the marked places commutes with these spherical operators. Matching the actions of two operators on the vacuum gives their intertwining relation on the lift: insert the finite coset sum in the theta integral and change the source integration variable by each representative. This replaces the source double coset by its inverse.

At an inert place the two lattices are unimodular over an unramified quadratic extension, and the standard additive character has conductor the integers of the fixed field. The residue characteristic is odd since the host decomposition condition excludes conjugation from the stabilizer of a prime above 2. The integral Heisenberg translations fix the vacuum and its translates span the lattice model. A target spherical operator applied to the vacuum is fixed by the target integral group. By 34.18 it is a finite linear combination of source translates of the vacuum. It is fixed by the source integral group as well, since the two actions commute. Averaging this combination over the source integral group gives a source spherical operator with the same value on the vacuum. Its inverse-double-coset action on $j_\tau Bx$ has the common source eigenvalue just computed. This determines the target eigenvalue for all x.

Lean: `GlobalTheory.InertLoc.exists_locPair` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.ExistsUnramifiedLocalPairOfInertWeilDatum`, lines 30–99)
Lean: `GlobalTheory.ConjScalar.LiftPart.loc_inert` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.ConjugateDatumLiftJointEigenspaceIntoOnePart`, lines 135–150)

At a split place write $n = r + t$ for the source rank and r for the target rank. Let q be the residue cardinality, and let s and w be the source and target splitting characters evaluated on the idèle which is a uniformizer at the chosen prime and one elsewhere. Write T_h for the target elementary double coset with h uniformizer entries, and α_k for the embedded source eigenvalue. The split vacuum calculation gives, for $0 \le k \le r$,

$$\alpha_k z = \sum_{h=0}^{k} A_{kh} T_h z, \qquad A_{kh} = q^{(r-h)(k-h)} \begin{bmatrix} t \\ k-h \end{bmatrix}_q (wq^{r/2})^{-k} (sq^{n/2})^{h}.$$

Here $\left[\begin{smallmatrix} t \\ i \end{smallmatrix}\right]_q$ counts the i-planes in the complementary residue space of dimension t, with value zero for $i > t$. For a fixed intersection of dimension h with the target block, choose the image of dimension $k-h$ in the complement. Its lifts are graphs of maps into the target block modulo the intersection, giving $q^{(r-h)(k-h)}$ possibilities. The two determinant multipliers in the split Weil action give the last two factors. Since $A_{kk} = (sq^{n/2}/(wq^{r/2}))^k \neq 0$, induction on k gives $T_k z = \beta_k z$ with β_k independent of x. The elementary cosets and the inverse determinant coset generate the split spherical algebra; the determinant coset is invertible. Together with the inert calculation this gives the desired target system c.

Lean: `GlobalTheory.Hk3.SplitMatch.match_split` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.ThetaVacuumHeckeIdentitySplitPlace`, lines 148–186)
Lean: `GlobalTheory.ConjScalar.LiftPart.loc_split` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.ConjugateDatumLiftJointEigenspaceIntoOnePart`, lines 154–188)

Put $u = j_\tau Bx$, and write $S_\tau = eT$ as in 21.16. If some $z \neq 0$, the space $Y_\tau(c)$ is nonzero. Lemma 21.16 gives $TT^* z = l_c z$, with $l_c > 0$ independent of x. Since $ez = z$,

$$S_\tau S_\tau^* S_\tau u = eTT^* z = l_c S_\tau u.$$

The vector $S_\tau^* S_\tau u - l_c u$ belongs to $\ker S_\tau$. It also belongs to its orthogonal complement by the hypothesis on u. It is therefore zero. For every $x' \in Z_j$, adjunction gives

$$\langle S_\tau j_\tau Bx, S_\tau j_\tau Bx' \rangle = l_c \langle j_\tau Bx, j_\tau Bx' \rangle.$$

If every z vanishes, the same orthogonality implies $j_\tau Bx = 0$ for every $x \in Z_{jp}$.

Lean: `GlobalTheory.ConjScalar.adjoint_apply_of_orthogonal_ker` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.AdjointApplyOfOrthogonalKer`, lines 17–29)

Use $\tau = \sigma$ if its restriction to K_d agrees with the base embedding. The two unitary maps and the common unit phase in (21.3) preserve the target inner product. The source inner product is the class sum with matrix $\sigma(H)$. Dividing both pairings by N therefore gives $\langle y_v(x), y_v(x') \rangle = l_c M_v(x, x')$. For the other restriction to K_d, take $\tau = \bar{\sigma}$. On the same two rational vectors the target pairing at σ is the conjugate of the pairing at $\bar{\sigma}$, by the conjugate evaluation in the reference realization of 36.6. The displayed class sum has the same conjugation law. Conjugating the equality preserves the real scalar l_c.

Lean: `GlobalTheory.ConjScalar.readings_scale_core` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.RationalLiftReadingsScaleClassSumEigenspaces`, lines 21–58)

Lean: GlobalTheory.ConjScalarCoord.liftMat_isRatLift_inner (Development.AbelianVarietyTheory.Proofs.TwistedOperators.LiftWeightComplexPointEqTwist, lines 68–78)

The diagonal $M_v(x,x)$ is the squared norm of $j_\tau Bx$ divided by N, for either choice of τ. If it vanishes on every member of a K-basis of Z_{jp}, semilinearity gives $j_\tau Bx = 0$ throughout Z_{jp}; both pairings then vanish against every x'. Otherwise the first basis member b with nonzero diagonal has $M_v(b,b) > 0$. Orthogonality to the kernel gives $S_\tau j_\tau Bb \neq 0$, so the computation with c applies. Evaluating at (b,b) identifies the quotient in the statement with l_c. It also proves uniqueness of the scalar in the paired row. The scalar characterization in the selected realization uses this unique value on an occurring part and the exponential of the fixed layer allowance on an empty part.

Lean: GlobalTheory.LowOff.partAt_iff (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LogLiftEigenvalueLowerBoundOtherPlaces, lines 38–42)

Lean: Formalism.IndefR.Links.RealDat3.IsScal (Formalism.LocalToGlobal.IndefiniteReal.ZeroPartEigenvalueConvention, lines 217–223)

□

Proof of 21.21

Proof. At the scale of 21.12 the sixth entry is $L_{6,j} = H_j^+ - C_{\mathrm{sz}} f n_V^2 L'$. The exponent of 21.16 is $D_j = H_j^+ - 8 f n_V^2 L'$. Since $C_{\mathrm{sz}} \geq 8$ and $L' \geq 0$, $L_{6,j} \leq D_j$. At the distinguished place take $0 \neq w \in W_{jp}$ if this space occurs. Its assigned scalar $\lambda > 0$ satisfies $S_0 S_0^* w = \lambda w$, so $\|S_0^* w\|^2 = \lambda \|w\|^2$. As $S_0^* w \perp \ker S_0$, the source estimate of 21.16, with the stipulated reference primes and sizes, gives

$$e^{D_j} \lambda \|w\|^2 \leq \|S_0 S_0^* w\|^2 = \lambda^2 \|w\|^2.$$

Thus $\lambda \geq e^{D_j}$. On an empty space the assigned value is $e^{L_{6,j}}$.

Lean: GlobalTheory.LowRow.l6m_le_log_lam3_base_place (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LogLiftEigenvalueLowerBoundBasePlace, lines 13–20)

Write $K_0 = \{a \in K : \tau(\bar{a}) = \overline{\tau(a)} \text{ for every } \tau : K \hookrightarrow \mathbf{C}\}$. Let H be the class Gram matrix in the chosen class basis, V the inclusion of the retained block, and $H_1 = V^* H V$. We check the metric hypotheses on the primitive reference row. Definition 20.14 gives H over K_0, positive at every infinite place, and group matrices over K_0. The character projector and the level-character average have entries over K_0: their character values are sums of roots of unity, on which every embedding commutes with conjugation. The image of their product therefore has a basis over K_0, obtained by taking independent columns. The block inclusion uses such a basis. At each τ, an injective minor of V stays nonzero, and

$$\tau(H_1) = \tau(V)^* \, \tau(H) \, \tau(V) > 0.$$

For the conjugate of a representative embedding this follows as well by conjugating its positive Gram matrix.

The quotient-frame assertion on this row can be checked before transporting its orthogonal section. The ordinary and inert operators of the primitive reference row have matrices over K_0, commute, split over K, and have their adjoints in the specified families for the class pairing. Hence their complex realizations are commuting normal operators at each embedding. If T' is the specified adjoint of T, and their eigenvalues on a common K-eigenvector are α, α', adjunction at the distinguished embedding gives $\alpha' = \bar{\alpha}$. At any other embedding it gives $\tau(\alpha') = \overline{\tau(\alpha)}$, since the embedded vector has positive norm. Thus $\alpha \in K_0$. The selected sum of joint eigenspaces in block coordinates consequently has a basis over K_0. Indeed each eigenspace is an intersection of kernels over K_0, and taking a sum or passing to the block coordinates preserves this property. Finitely many kernel equations suffice, by dimension. The reference coordinate basis is chosen over K_0 whenever such a basis exists.

In these coordinates the section matrix E_1 has entries in K_0. Put $G_1 = E_1^* H_1 E_1$. The orthogonal quotient projection satisfies $P = G_1^{-1} E_1^* H_1$, because subtracting $E_1 P x$ from x must give a vector orthogonal to each column of E_1. In particular $P E_1 = 1$. For any embedding put $A = \tau(H_1)$, $C = \tau(E_1)$ and $G' = \tau(G_1)$. Since the entries of H_1, E_1 lie in K_0, one has $G' = C^* A C$. The columns of C are independent by a nonzero minor, and $A > 0$, so $G' > 0$. The embedded quotient map is $G'^{-1} C^* A$; its kernel is the A-orthogonal complement of the columns of C. A lift of y is uniquely $Cy + z$ with z in this kernel, and its squared norm is $y^* G' y + z^* A z$. The infimum is therefore $y^* G' y$ at this embedding. Its orthonormal coordinate frame X_v therefore satisfies $(X_v^{-1})^* X_v^{-1} = \tau(G_1)$ at the representative of v. This proves the asserted frame compatibility for the selected

realization. The ordinary and inert rationality and adjoint conditions here are precisely the two conditions on the primitive Hecke action used with the reference eigenspace selection in 33.74.
Lean: Formalism.L2.DIV.BlockDat.HV_pos (Formalism.LocalToGlobal.Divisibility.RowLatLemmas, lines 38–39)
Lean: Formalism.IndefR.Cx.okOp_opAll (Formalism.LocalToGlobal.IndefiniteReal.MajAtEqZeroOfMemQpOfNe, lines 18–35)
Lean: Formalism.IndefR.Cx.proviso_of_lift_eigenvalues (Formalism.LocalToGlobal.IndefiniteReal.MajAtEqZeroOfMemQpOfNe, lines 45–54)
Lean: GlobalTheory.LowRow.baseLinks_of_prims (Development.AbelianVarietyTheory.Proofs.DivisorProofs.SplitPartBaseLinksOfPrimitiveRows, lines 15–28)

The transport can also be argued from the quotient-frame hypothesis directly. It applies in any of the rational coordinates in the statement. Write $\tau'(a) = \overline{\tau(\bar{a})}$, $C = \tau(E_1)$, $D = \tau'(E_1)$, $A = \tau(H_1)$, $G' = \tau(G_1)$, and $P' = \tau(P)$. The identities over K give

$$P'C = 1, \qquad D^*A = G'P', \qquad D^*AC = G'.$$

In particular P' is surjective. Since $A > 0$, $P'A^{-1}P'^* > 0$. Its inverse is the quotient Gram matrix at this embedding: the lift

$$y \longmapsto A^{-1}P'^*(P'A^{-1}P'^*)^{-1}y$$

is a right inverse and is A-orthogonal to $\ker P'$. Comparing with the quotient frame gives

$$G' = (P'A^{-1}P'^*)^{-1}.$$

For the other embedding representing the same place this identity follows by complex conjugation. In this calculation the adjoint is taken on the complex quotient map P'.

Taking the adjoint of $D^*A = G'P'$ gives $D = A^{-1}P'^*G'$, since A and G' are Hermitian. It follows that $P'D = 1$ and $D^*AD = G'$. Apply this result at τ'. The matrices of H_1, G_1 there are still A, G', since they have entries in K_0, and twisting τ' again gives τ. Thus $C^*AC = G'$. Together with $D^*AC = G'$ and its adjoint this gives

$$(C - D)^*A(C - D) = 0.$$

Positivity on each column implies $C = D$. This proves $\tau(E_1^*) = \tau(E_1)^*$ at every embedding. The identity $P = G_1^{-1}E_1^*H_1$ then gives $\tau(P^*) = \tau(P)^*$ as well. For the full inclusion $B = VE_1$ we obtain $\tau(B^*) = \tau(B)^*$.
Lean: Formalism.L2.DIV.BlockDat.XQ_quot (Formalism.LocalToGlobal.Divisibility.RowLatLemmas, lines 67–68)
Lean: GlobalTheory.QuotRational.entries_mem_cmMax (Development.AbelianVarietyTheory.Proofs.DivisorProofs.MapConjTransposeTwist, lines 70–112)
Lean: GlobalTheory.QuotRational.fE_mem_cmMax (Development.AbelianVarietyTheory.Proofs.DivisorProofs.MapConjTransposeTwist, lines 114–119)

Fix σ over the distinguished embedding of K_d. The complex class space is $\mathbf{C} \otimes_{K,\sigma} X$. Indeed, at each representative the allowed values are the image of the average of its finite stabilizer action, with the level character included. This rational idempotent extends to the corresponding complex idempotent, and the values at the representatives determine the class form. Now choose a K-basis of $\ker\theta$ and extend it to a basis of X. The images under θ of the additional basis vectors are independent. Their coordinate columns remain independent after embedding, by a nonzero minor, and the realized target frame is independent. Applying 21.19 to this basis proves that $\ker S_\sigma$ is spanned by $j_\sigma(\ker\theta)$, including when the rank is zero or full.
Lean: GlobalTheory.ConjSpan.span_conjForms (Development.AbelianVarietyTheory.Proofs.DivisorProofs.SpanEqTopOfTie, lines 126–133)
Lean: GlobalTheory.ker_le_span_image_ker_of_semilinear (Development.AbelianVarietyTheory.Proofs.DivisorProofs.KerLeSpanImageKerOfSemilinear, lines 6–55)

Write $C_\sigma(z, z') = \overline{\sigma(z)}^{\,t}\sigma(H)\sigma(z')$. This equals the realized class pairing by the stabilizer-weighted formula. For $u \in \ker\theta$, 21.19 gives $S_0 j_0 u = 0$. The reference layer subspace of rational class forms is the inverse image, under class realization, of the adjoint range of the phase-normalized lift. In block coordinates one takes its inverse image under the block inclusion V; the columns of E_1 are a basis of that inverse image. Thus $Bx = VE_1x$ belongs to the reference layer subspace for every x. To read its defining range in the base realization, multiplication of a lift by a unit phase multiplies its adjoint by the conjugate phase, and transport by the unitary or conjugate-unitary class identifications carries the adjoint range onto the transported adjoint range. Undoing these identifications gives $j_0Bx \in \operatorname{im} S_0^*$.
Lean: GlobalTheory.ConjScalarOff.jV_mem_range (Development.AbelianVarietyTheory.Proofs.TwistedOperators.LiftScalarIdentityOffFixedPlaceBaseLevel, lines 32–36)

Since $S_0 j_0 u = 0$, we have $0 = \langle j_0Bx, j_0u\rangle = C_{\sigma_0}(Bx, u) = \overline{\sigma_0(x)}^{\,t}\sigma_0(B)^*\sigma_0(H)\sigma_0(u)$ for every $x \in K^{d_j}$ and every $u \in \ker\theta$. Using $\sigma_0(B)^* = \sigma_0(B^*)$ and letting x run through the standard basis of K^{d_j} gives

$\sigma_0(B^*Hu) = 0$, whence $B^*Hu = 0$ over K by injectivity of σ_0. At any embedding $\sigma : K \hookrightarrow \mathbf{C}$ over the distinguished embedding of K_d, applying σ and $\sigma(B^*) = \sigma(B)^*$ yields $\langle j_\sigma Bx, j_\sigma u\rangle = C_\sigma(Bx, u) = \overline{\sigma(x)}^{\,t}\sigma(B)^*\sigma(H)\sigma(u) = \overline{\sigma(x)}^{\,t}\sigma(B^*Hu) = 0$ for all $u \in \ker\theta$. Because $\ker S_\sigma$ is spanned over $\mathbf{C}$ by $j_\sigma(\ker\theta)$, it follows that $j_\sigma Bx \perp \ker S_\sigma$. Applying Lemma 21.16 at σ, followed by the isometries of Lemma 21.19, gives

$$e^{D_j}\operatorname{Re} C_\sigma(Bx, Bx) \le \|S_\sigma j_\sigma Bx\|^2 = \|J_\sigma\theta Bx\|^2. \tag{A.148}$$

If σ induces the other embedding of K_d, then $\bar\sigma$ agrees with the distinguished embedding on K_d, so $j_{\bar\sigma}Bx \perp \ker S_{\bar\sigma}$ and (A.148) holds at $\bar\sigma$. The two target pairings on rational columns are conjugate by Definition 36.6, and the formula for C_σ gives the same conjugation law on the source. The diagonal inequality thus holds at σ.

Lean: `GlobalTheory.LowForm.lower_overK_six` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.RationalLiftLowerBoundOffBasePlace`, lines 208–236)

For $v \ne v_0$, divide by the order N of the reference level subgroup. In the notation of 21.20 the source form is $M_v = N^{-1}C_\sigma(B\,\cdot, B\,\cdot)$, and $y_v(x) = N^{-1/2}J_\sigma\theta Bx$. Hence

$$e^{D_j}\operatorname{Re} M_v(x, x) \le \|y_v(x)\|^2. \tag{A.149}$$

We have checked the kernel-orthogonality hypothesis of 21.20 at every required embedding. On an occurring part take its basis vector b with $M_v(b, b) > 0$. The paired-row identity gives $\|y_v(b)\|^2 = \lambda_{jpv}M_v(b, b)$, so division gives $\lambda_{jpv} \ge e^{D_j}$. On an empty part the specified scalar is $e^{L_{6,j}}$. Summing $L_{6,j} \le \log\lambda_{jpv}$ at fixed p, v and taking the minimum over the nonempty finite set I proves the second inequality. Replacing the zero-based layer index by $j-1$ identifies the plus-hook sum and its subtraction term with $\sum_j H_{X,j}$ in 21.26.

Lean: `GlobalTheory.LowForm.lowForm_of_emb_six` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.RationalLiftLowerBoundOffBasePlace`, lines 238–260)
Lean: `GlobalTheory.LowOff.l6m_le_log_lam3_other_places` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LogLiftEigenvalueLowerBoundOtherPlaces`, lines 46–53)
Lean: `GlobalTheory.LowMainTerm.sum_logHook_eq` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.HookLogSumSubSizeTermLeLowerBound`, lines 78–89)

□

Proof of 21.22

Proof. Logarithmic Chowla–Selberg gives the class mean of $\log(4\pi\operatorname{Im} z_C|\eta(z_C)|^4)$ as $\gamma + \log(2\pi/\sqrt{d}) - D_1$, with z_C the ideal-class points and η the Dedekind eta function [153]. In discriminant-one normalization the argument of the logarithm is area $/\pi$. Take a common extension containing the Hilbert class field of K_d. By the elliptic CM theorem of 1.19, embeddings fixing K_d traverse the ideal classes with equal multiplicities. Embeddings inducing its other embedding are obtained by conjugation, which preserves area. For a model of unit discriminant u, scaling by c with $c^{12} = u$ gives discriminant one and adds $\frac{1}{6}\log|\sigma u|$ to the log area. Its mean is zero by the product formula. Each embedding of M extends with the same multiplicity, so the resulting normalized mean descends to M.

Let $\mathfrak{b}\bar{\mathfrak{b}} = \mathfrak{D}_{E/K_d}$ for the host ideal $\mathfrak{b}$. The differential ideal is the extension of $\tau(\mathfrak{b}^{-1}\mathfrak{D}_{E/K_d})$. With $B = N_{E/\mathbf{Q}}(\mathfrak{b}^{-1}\mathfrak{D}_{E/K_d})$, one has $B^2 = N_{E/\mathbf{Q}}\mathfrak{D}_{E/K_d}$ and

$$\frac{\log|N_{M/\mathbf{Q}}(s_\tau)|}{[M:\mathbf{Q}]} = \frac{\log B}{[E:\mathbf{Q}]}.$$

The identity $|\operatorname{disc} E| = d^{[E:K_d]}B^2$ shows that subtracting this doubled log norm replaces $\sqrt{d}$ by $\operatorname{rd}E = 2v_E$ in the class mean. Multiplication by $-\sum_i \nu_{uj\rho i}$ gives the period term in 21.12. □

Proof of 21.23

Proof. By Lemma 15.9, $e_q = B_\rho + (q + 1/2)h_0$, where $h_0 > 0$ and $\eta = mh_0$. In particular

$$e_{m-1} = B_\rho + \eta - h_0/2.$$

At a mark, $B_\rho = b_s + \lfloor b'/x \rfloor \le 3b'$ by (14.5) and $x \ge 1$. At a thin position $B_\rho = \lceil w'' \rceil \le w'' + 1$. At a thick position $B_\rho = \lceil b \rceil + d_\rho$, with $0 \le d_\rho < |\mu(E)|$, including the designated position at which the weight is adjusted.

Let $h = |\mu(E)|$. Since $\mathbf{Q}(\zeta_h) \subseteq E$, Euler's totient satisfies $\varphi(h) \le [E : \mathbf{Q}] = 2f$. The prime-power formula gives $\varphi(h)^2/h \ge 1/2$. Indeed, the factor at p^a is $p^{a-2}(p-1)^2$, which is at least one except at 2^1, where it is $1/2$. Thus $h \le 8f^2$. By the rank bounds of 38.3 and the size threshold of 36.2, eventually $f \le x^3$ and $h_0 \ge x^{1300}$. Hence $h \le h_0/2$. The three upper bounds for e_{m-1} are consequently $3b' + \eta$, $w'' + \eta$, and $b + 1 + \eta$, respectively. By 15.9, $b'/b = x \exp(-x^{\varepsilon/3})$, $w''/b \le \exp(-x^{\varepsilon/3})$, and η grows at most polynomially in x at fixed family parameters. Since $b = \exp(x^\varepsilon)$, each bound is at most $2b$ eventually. □

Proof of 21.24

Proof. Take the integer spacing and h_0 of 15.9. By 38.3 the ranks and n are polynomially bounded, and by 15.9 so are S_c, η. For fixed $d > 0$, $d \log x = o(x^{\varepsilon/3})$ and $x^{\varepsilon/3} + d \log x = o(L)$. Thus X', b, and w'' eventually exceed each fixed power of x. This gives the four rank and η bounds with right side b. Since $N, mk \le X'$, the identities $b' = a/X'$, $w'' = b/(nX')$ give the other three product bounds $b'N, mkb' \le a$ and $Nw'' \le b$. The inequalities $kS_c, B \le w''$ follow also from 36.2.

The finite row group has $2f$ general linear factors of rank $k + m - 1$, over the residue fields at the two split type primes. Their orders are at most $g_i^{(k+m-1)^2}$, by counting all matrices. Hence $0 \le m \log M \le 2mf(k+m-1)^2 \log \max_i g_i$, which is polynomially bounded by 36.2. The formula for N_c is nonnegative.

By 36.2, $\log r_d, \log x, |D_1| \le L$. From 21.12,

$$|\log \Omega| \le \gamma + \log(2\pi) + \log r_d + L \le L_a$$

as soon as $x \ge 16$. Here $L_a = \log x + L + \log r_d$ and $r_d \ge 1$. Also $\log n_V \le L$ eventually, so $L_p \le (3 + 2C_{YL})L + 2$. For example $C_0 \ge \max\{28, C_1 + 1, (5 + 2C_{YL})C_h\}$ suffices for the constant bounds.

Since $s_j \le N \le w''$, all three lower endpoints of 14.2 give $w_{j\rho} \ge s_j$. At each position $w_{j\rho} = e_{j-1}(\rho)$ by 15.9. Lemma 21.23 gives the upper bound, including the root-of-unity adjustment at the designated thick position. Since $S_c \le b$ and $x \ge 3 + |C_c|$ eventually, $w_{j\rho} + S_w \le a$. At a mark the first box of 22.45 gives $a_s \le a + b$. The field E contains the quadratic field of discriminant $-d$, so the discriminant identity in a tower gives $r_d \ge \sqrt{d} \ge 2$. Consequently $a_s + w_{j\rho} \le a + 3b \le A_*$ for $x \ge 3$.

Positivity and interlacing are in 15.8. A tail block of 15.9 has $t+1$ copies of a number at most $(j-1-q)h_0 \le \eta$. This proves the stated sum bound, including $j = 1$. Off the marks the bottom endpoint is $-1/2$ and the shifted top endpoint is at most $2S_c$, by 15.9. At a mark the two box weights are positive, $v_{k-2} = 0$, and $v_0 \le S_c$. These give every endpoint condition with the specified S_w. □

Proof of 21.25

Proof. Write $w = w_{j\rho}$, $W = w - s_j$, $d_j = (j-1)t$, and $p = k + j - 1$. Both s_j and d_j belong to $[0, N]$, and W is a nonnegative integer. Remove the rank shifts from the two original parameters by setting

$$c_i = U_i - (k-1)/2 + i \quad (i < k), \qquad d'_i = v_i - (k-2)/2 + i \quad (i < k-1).$$

These are integers since k is even. By 15.8 the head of the column and the first $k-1$ hook rows are exactly

$$y_i := \nu_i = W + c_i, \qquad r_i := \alpha_i = W + d'_i + d_j.$$

The shifts differ by half a unit in the original coordinates. Hence $U_i > v_i > U_{i+1}$ gives $c_i \ge d'_i \ge c_{i+1}$.

We use the following factorial bound for these particular r_i:

$$\log H_{j\rho} \ge \sum_{i<p} \log(\alpha_i!) \ge \sum_{i<k-1} \varphi(r_i).$$

Indeed, the hook of a cell at column h, numbered from zero, has length at least $\alpha_i - h$. The product of these lower bounds along row i is $\alpha_i!$. To identify the cell product with the gamma quotient in 21.12, delete the first column of a diagram with p positive rows. The cell product and the gamma quotient both lose the

factor $\prod_{i<p}(\alpha_i+p-1-i)$. The pairwise differences in the denominator are unchanged. Removing a trailing zero row leaves the quotient unchanged, since the new denominator factors cancel the additional numerator factors. Induction from the empty diagram proves the identity. Finally $e^r \geq r^r/r!$ gives $\varphi(r) \leq \log(r!)$ for each positive integer r, with equality at zero under the convention $\varphi(0) = 0$. The remaining j factorials have nonnegative logs.

For nonnegative integers y, r with $y \leq A_*$ we shall use

$$\varphi(y) - \varphi(r) \leq |y - r| L_a.$$

For $y, r \geq 1$, integrate $\log s$ when $r \leq y$, and use monotonicity when $r > y$. For $r = 0 < y$, use $\varphi(y) \leq yL_a$. For $y = 0 < r$, use $-\varphi(r) \leq 1 \leq rL_a$. The tail degrees have sum at most $B \leq w'' \leq b \leq A_*$, so their entire contribution to the φ-sum is at most BL_a.

Suppose first that ρ is unmarked. The endpoint bounds and interlacing give $0 \leq c_{k-1} \leq c_i \leq c_0 \leq S_w$. Pair y_i with r_i for $0 \leq i < k-1$. Since $W + S_w \leq A_*$,

$$\sum_{i<k-1} \{\varphi(y_i) - \varphi(r_i)\} \leq L_a \sum_{i<k-1} (c_i - d'_i + d_j) \leq [S_w + (k-1)d_j]L_a.$$

The last inequality uses $\sum_{i<k-1}(c_i - d'_i) \leq c_0 - c_{k-1}$. The remaining head entry contributes at most $(W + S_w)L_a$. Thus

$$e_{j\rho} \leq [W + 2S_w + (k-1)d_j + B]L_a.$$

At a mark the endpoints are $c_0 = a_s$, $c_{k-1} = -b_s$. The endpoint conditions on v give $0 \leq d'_{k-2} \leq d'_0 \leq S_w$. Pair y_i with r_i for $1 \leq i \leq k-2$. Now $d'_{i-1} \geq c_i \geq d'_i$, so

$$\sum_{i=1}^{k-2} (c_i - d'_i) \leq d'_0 - d'_{k-2} \leq S_w.$$

The paired cost is at most $[S_w + (k-2)d_j]L_a$. The first head entry contributes at most $\varphi(a_s) + WL_a$, since $a_s + W \leq A_*$. The last is $W - b_s \in [0, W]$, and contributes at most WL_a. The unpaired hook term costs $-\varphi(r_0) \leq 1$. Hence

$$e_{j\rho} \leq [2W + S_w + (k-2)d_j + B]L_a + 1.$$

In both cases use $W \leq w$, $kd_j \leq N^2$, $2S_w \leq C_1 S_c$, $C_1 \geq 2$, and $1 \leq N^2 L_a$. These give the first claimed bound since $C_0 \geq C_1 + 1$.

For the degree sum write $b_{\mathrm{tail}} = \sum_{k \leq i < N_{j-1}} \nu_i$. Before replacing W by w, the discrepancy from $I_\rho a_\rho + kW$ is exactly

$$\begin{cases} \displaystyle\sum_{i<k} c_i + b_{\mathrm{tail}}, & I_\rho = 0, \\ \displaystyle -b_s + \sum_{i=1}^{k-2} c_i + b_{\mathrm{tail}}, & I_\rho = 1. \end{cases}$$

The first lies in $[0, kS_w + B]$, and the second in $[-W, (k-2)S_w + B]$, since $W - b_s \geq 0$. Restoring w subtracts ks_j. Therefore

$$|\Delta_{j\rho}| \leq W + kS_w + B + kN.$$

The desired bound follows from 21.24 and $I_\rho k b' \geq 0$. Finally $\varphi(0) = 0$ proves the assertion about extension of the tuple. □

Proof of 21.26

Proof. For each pair j, ρ, the row contribution, the subtracted hook and the term from the reference divisor have sum at most

$$I_\rho\{\varphi(a_\rho) - a_\rho \log \Omega\} + e_{j\rho} + \{(\beta_j - k)w_{j\rho} - \Delta_{j\rho}\} \log \Omega.$$

This follows by substituting the degree identity before estimating the term with $\log \Omega$, whose sign need not be fixed. For the arithmetic data the two remainders and the required bounds are those of 21.25. Each marked position occurs once in every layer. Adding the terminal term $-\sum_{\rho \in \mathcal{U}} a_\rho \log T$ gives exactly $\mathcal{D}(T)$.

Put $Z = n_T a L_a$, $V = mL(B_0 + N_c)$. We have $L \le L_a \le 3L$, $V \ge f m^2 b L^2$, $L \ge 1$. There are at most mf pairs. Per pair the error $e_{j\rho}$ is at most $5C_0 b L_a$, because $w_{j\rho} \le 2b$ and $S_c, n_*^2, m(t_c+1)\eta \le b$. Thus its total is at most $15C_0V$. The $\beta_j - k$ terms cost at most $2fm^2bL_a \le 6V$. In the degree discrepancy the unmarked part of the bound is at most $5C_0 b$ per pair, using $kn_*, kS_c \le b$. Its cost is at most $15C_0V$. The additional marked allowance costs at most $C_0 m n_T k b' L_a \le C_0 Z$.

The terminal error is at most $C_0 Z + 9C_0 V$, since $b' n_* \le a$ and $m\eta n_*, S_c n_*, n_*^2 \le b$. Here $S_c \le w''$ and $n_* w'' \le b$. The kernel and hook allowances are bounded by

$$mC_h f n_*^2 (L_a + m^3 L) \le 4C_0 V, \qquad mC_h f n_V^2 L_p \le 3C_0 V.$$

For the first use $n_*^2, m^3 n_*^2 L \le b$, and for the second use $n_V \le n_*$, $C_h L_p \le C_0 L_a$. Finally $E_b \le c_e V$ and $L_{\mathrm{pass}} \le 1 + C_0 m \lambda_0 \le 2C_0 V$. Adding all terms gives at most $2C_0 Z + 55 c_e V$ beyond $\mathcal{D}(T)$, which proves the displayed inequality. In the arithmetic application 21.28 and (20.6) give the required reference bound with $E_b = E_{l3} + L_G + \sum_j (E_{1,j} + E_{2,j}) + E_3$. □

Proof of 21.27

Proof. Fix $t = t_w = t_{\bar w}$ and take bases over the semilocal ring at $w, \bar w$. Let a be the determinant of the change from the L_{t_0} image basis to that of L_t. By elementary divisors $v_z(a) = [L_{t_0} : R_0]_z$ for $z = w, \bar w$. The corresponding layer determinants satisfy $v_z(b_j) = [M_j : A_{jt}]_z - [M_j : A_{j,t_0}]_z$. If h_{jt} is the layer Gram determinant, the inverse Gram gives $[T_{jt} : T_{jt}^\sharp]_w = -v_w(h_{jt})$.

Let $D_{\mathrm{row}}(t,v)$ and $D_j(t,v)$ be the squared determinant ratios for ι_t, θ_{jt}. For $t = t_0$ or $v = v_0$, the metric identities in 20.13 give factors $\kappa_v \prod_j \lambda_{jt\alpha v}$ and $\lambda_{jt\alpha v}$ on the row and layer blocks indexed by α. Each such block has the same dimension on the row and on each layer. Multiplying their factors with these dimensions as exponents gives

$$D_{\mathrm{row}}(t,v) = \kappa_v^{D_C} \prod_j D_j(t,v) \qquad (t = t_0 \text{ or } v = v_0),$$

where D_C is the row rank.

Lean: `Formalism.L2.Indef.IndefRows.detId` (`Formalism.LocalToGlobal.IndefiniteForms.IndefUse`, lines 196–205)

Divide the identities for t_w, t_0 at v_0. The common $\kappa_{v_0}^{D_C}$ cancels. The Gram multipliers for the image basis changes are $\bar a a$ and $\bar b_j b_j$. By injectivity of the embedding,

$$\bar a a \, G_{t_0,w} \prod_j h_{jt} = G_{t,w} \prod_j (\bar b_j b_j h_{j,t_0}).$$

Taking valuations gives the formula, since $v_w(\bar b) = v_{\bar w}(b)$, also at a self-conjugate place where the row index occurs twice. □

Proof of 21.28

Proof. By 21.27 and $\ell_w = \ell_{\bar w}$,

$$\mathrm{lcv}(R_0) - \mathrm{lcv}(L_{t_0}) = 2 \sum_w \ell_w [L_{t_0} : R_0]_w = \sum_w \ell_w (C_w(t_w) - C_w(t_0)).$$

For the base quotient map P and test Gram matrix H put $g = \det(PH^{-1}P^*)^{-1} \in K^\times$. At σ, write $d_\sigma = \det(P_\sigma H_\sigma^{-1} P_\sigma^*)^{-1}$, with complex adjoint. For $\sigma'(a) = \overline{\sigma(\bar a)}$ the entries of the test metric give $H_{\sigma'} = H_\sigma$. The two row frames have mixed Gram determinant $\sigma(g)^{-1}$. Cauchy–Schwarz on their exterior products yields $d_\sigma d_{\sigma'} \le |\sigma(g)|^2$. Average over the permutation of embeddings and use the product formula. A basis change of determinant c adds $v_w(c) + v_{\bar w}(c)$ to the local Gram valuation and the corresponding weighted term to the covolume. This gives $\mathrm{lcv}(\Lambda_{t_0}) \le \sum_w \ell_w v_w(G_{t_0,w})$.

The algebraic layer metrics and the product formula give $\mathrm{lcv}(T_{j,t_0}) = -\sum_w \ell_w [T_{j,t_0} : T_{j,t_0}^\sharp]_w$. Thus the layer determinant ratio satisfies

$$\langle \log D_j(t_0, v) \rangle_\infty = \mathrm{lcv}(M_j) + \sum_w \ell_w \big(2[M_j : A_{j,t_0}]_w + [T_{j,t_0} : T_{j,t_0}^\sharp]_w\big).$$

The row determinant identity at t_0 adds $D_C\bar\kappa_0$ and the layer log ratios to $\operatorname{lcv}(\Lambda_{t_0})$. Adding the index formula for R_0 cancels the contributions at t_0, so

$$\mu(R_0^\vee) \le \bar\kappa_0 + \sum_j \operatorname{lcv}(M_j)/D_C + D_C^{-1} \sum_{w\in S_y} \ell_w C_w(t_w)$$

for a support set S_y.

Write $i_{jw} = [M_j : A_{j,t_w}]_w + [M_j : A_{j,t_w}]_{\bar w}$ and $q_{jw} = [T_{j,t_w} : T^\sharp_{j,t_w}]_w$. The Gram term is at most L_G by (20.5). The combined layer contribution vanishes where the selected type t_w equals the base type t_0. On the places with $t_w \ne t_0$, 36.8 and (20.5) give

$$D_C^{-1}\sum_j i_{jw} \le \sum_j (2e_wF^0_{jw} + \operatorname{err}_{jw}) + 2e_w/p_w^2,$$
$$D_C^{-1} q_{jw} \le e_w J^0_{jw} + \eta_{jw}/D_C,$$
$$\sum_{w:\, t_w\ne t_0} \ell_w(\operatorname{err}_{jw} + e_wJ^0_{jw}) \le E_{1,j}, \qquad \sum_{w:\, t_w\ne t_0} \ell_w\eta_{jw}/D_C \le E_{2,j}.$$

Here F^0, J^0 are the base and pairing-index terms and err, η their allowances. Sum with $F_* = \sum_j \sum_{w:\, t_w\ne t_0} \ell_w e_w F^0_{jw}$.

In the construction let M be the level group order and $s > 0$ the integral denominator of 20.16. The adjoint of the quotient map embeds $\Lambda_t^\sharp$ in the test dual. Multiplication by Ms sends the test dual into the test lattice, hence $Ms\,\Lambda_t^\sharp \subseteq \Lambda_t$ on projecting back. In the semilocal basis $Ms\,G^{-1}$ is integral at w for the quotient Gram matrix G. Its determinant gives $v_w(G_{t,w}) \le D_C v_w(Ms)$, so $L_G = \lambda_0 + \log s$ suffices by 20.7 when $\log M \le \lambda_0$.

On a twisted layer let J_w be the least nonnegative containment exponent between the cut lattices in 36.8 and a_{jw} the length of the lift lattice modulo its cut. Containment and the two denominators give

$$q_{jw} \le a_{jw} + D_C(J_w + v_w(d_{j,t_w}) + v_w(s)), \qquad J_w \le e_wJ^0_{jw}.$$

Use 36.13 with $D_0 = D_C$ for $\sum_w \ell_w a_{jw}/D_C \le b_j$, with $b_j = b_N$ its cut allowance. Since $d_{j,t_w} \mid d_g$, 20.7 gives $E_{2,j} = b_j + \log d_g + \log s$. For E_3, the sum of $2\log p/p^2$ through 256 is less than 0.979. Partial summation with $\sum_{p\le y} \log p < 1.01624y$ [34, (3.32)] bounds the tail by $4.06496/256$, so $E_3 = 1$ suffices. □

Proof of 21.29

Proof. The passage cost is at most $1 + 28\lambda \le 1 + C_0 m\lambda_0$. Apply 21.26 with $\widetilde\mu_C = \mu_C - \bar\kappa$. The z_v of 21.12 have mean at most $H^* - \bar\kappa$. Add the difference to every z_v and exponentiate to get positive squared norm bounds of exact mean $\log H^* - \bar\kappa$. For $B_u = 0$ use the constant bound $\exp(H^* - \bar\kappa)$. Apply 20.16 with $(\delta, Q_{\text{pass}}, h_{\text{r}}, \lambda_{\text{r}}) = (\delta_0, Q_0, H_0, \lambda)$. In the arithmetic coordinates 18.8 allows $\sigma_w = s_{uw}$ from 21.5. □

Proof of 21.30

Proof. Under the row group isomorphism and coefficient embedding, a retained χ is $\prod_w \chi_w$. Each χ_w is the character induced from the prescribed cuspidal datum on its block parabolic. Reordering the local labels conjugates the subgroup character by an inner automorphism and preserves the normalized twisted trace. Thus 32.49 and 32.51 give the trace of the local subgroup character projector as

$$n_w = \sum_{j=0}^{k} c_j\gamma_q(k-j), \qquad \gamma_q(0) = 1, \quad \gamma_q(s) = q^s - q^{s-1}\ (s > 0),$$

where $q = q_w$ and c_j counts subsets of cuspidal blocks of total length j. A subset is determined by its union of positions, so $c_j \le \binom{r}{j}$. The empty subset contributes $q^k - q^{k-1}$. The other terms divided by q^k sum to at most $(1+q^{-1})^r - 1 \le 2r/q$. Hence

$$1 - q_w^{-1} \le n_w/q_w^k \le 1 + (2r-1)/q_w. \tag{A.150}$$

Over the $2f$ places, $q_w \geq g$ and $4fr/g \leq 1$ give a product between $1 - 2f/g$ and $\exp(4fr/g) \leq 1 + 8fr/g$. The row budget with $2F_* = \mathrm{Fo}$ from 20.17 gives $B_* \leq B$.

The sizes in 21.26, with $L_a \leq 4P$, give $A_3 \leq 16P^4 b$. For F_{jw} in 15.10 use the weight bound $W = 4P^5 b$. The type-prime and ramified terms are nonpositive. In the inert term, $1/(p^2-1) \leq 1/p$ and $p/(p^2-1) \leq 2/p$. The orbit length division counts each orbit once by 15.1. Each orbit costs at most $2(k-1)W/p$ per unit length, with at most P orbit labels and a factor two in the inert sum. Thus $F_{jw} \leq 16P^7 b/p$, with positive terms confined to $p \leq X''$. Group by 20.7 and apply the Mertens bound $\sum_{p \leq X''} (\log p)/p \leq 2\log(X''+1)$ from [34, (3.24)]. The cost per layer is at most $64P^7 b \log(X''+1)$. Since $m \leq P$, $\log(X''+1) \leq 4P$, this yields $\mathrm{Fo} \leq 256P^9 b$.

Lean: `Formalism.MainNotions.G.Joints.Lit.mertens_bound` (`Formalism.SharpRestriction.GrowthBounds.JointsLit`, lines 525–567)

For $n = |A|$, $T_2 \subseteq \{0, \dots, n-1\}$ gives

$$N_c = \frac{|T_2| k b L^2}{n^2} \leq P^3 b, \qquad B_0 = fmbL \leq P^3 b, \qquad L(B_0 + N_c) \leq 2P^4 b.$$

The level size identification is $M = \prod_w |\mathrm{GL}_r(k_w)|$ over the $2f$ selected residue fields, whose sizes are at most $2g_0$. Thus $\log M \leq 2fr^2 \log(2g_0)$. Use $2r^2 \leq b$ and the selected prime bound $\log(2g_0) \leq 2L^2$ to get $\log M \leq 2mL(B_0 + N_c) \leq 4P^5 b$.

Lean: `Formalism.L2.DIV.Arith4S.Nc_le_rec` (`Formalism.LocalToGlobal.Divisibility.EigenvalueSumLeClosedOf`, lines 1743–1785)

Lean: `Formalism.L2.DIV.Cons.log_card_le_unitE` (`Formalism.LocalToGlobal.Divisibility.DivisibilityConstruction.RBLam`, lines 62–114)

$B \leq (272 + 2c_e)P^9 b$. Since P is polynomially bounded in x, eventually $B \leq b^2$, $x^{C_H} \leq b$, $28 \log M \leq b^2$. This gives $Q_0 \leq 3b^2$. The prime size $g \geq (kb)^8$ gives $24frQ_0 \leq 72frb^2 \leq b^8 \leq g$ eventually. □

Proof of 21.32

Proof. Fix a member beyond the common cutoff and a word u. Use the bijections of positions and layers in 36.1. They carry the marks to $\{0\} \times TT$ and preserve a_t at every mark. Padding a column adds terms $\varphi(0) - 0 \log \Omega = 0$. Lemma 15.8 identifies its positive row partition with κ^{big}. Thus the terminal, kernel and hook entries in 21.26 are respectively $H_{P,u}, H_{K,uj}, L_{6,j}$ of 21.12. With $\bar\kappa = \langle \log \kappa_v \rangle_\infty$, the defining formula for h_u gives $H^* - \bar\kappa = h_u$. The fixed scale and the untwisted reference row give $2F_* = \mathrm{Fo}$ by 20.17.

Apply 21.17 to the definite tables and 21.21 to the reference spectrum. These give the transition bounds and the spectral subtraction in 21.12. The paired indices of 36.8, the cut estimate of 36.13 and the reference Gram bounds give the bound for μ_0 in 21.28. The numerical row bounds are 21.24 and 21.25; the numerical passage bounds on δ_0, Q_0 and λ are those of 21.30, while the passage inequality for $\mu(RR^\vee) - \bar\kappa$ follows from 20.18, 34.69 and 21.30. Theorem 21.29 therefore applies to the coordinate pairing B_u^{c} with $T = T_u$. Its image on the shifted coordinate lattice lies in the dual coordinate row lattice. Its squared operator norms are bounded by positive $c_{u,v}$ with

$$\langle \log c_{u,v} \rangle_\infty = H^* - \bar\kappa.$$

Its slope inequality, using the right hand side of 21.26, is

$$\mu_C + h_u - \mathrm{Fo} \leq \mathcal{D}(T_u) + (50C_0 + 10) n_T a L_a + 60 c_e m L(B_0 + N_c).$$

Here the row isometry of 20.19 identifies the two dual row slopes. Since $B_0, N_c \geq 0$, $x \geq 1$ and $L = x^{\varepsilon} \leq x^{e_{\mathrm{Div}}\varepsilon}$, the last two terms are at most E_{div}.

Let E_R, E_u be the rational row and column isometries. For arbitrary rational coordinate vectors ϕ, F one has

$$B_u(E_u\phi)(E_R F) = B_u^{\mathrm{c}}(\phi)(F).$$

Both isometries carry the specified lattices onto the actual lattices. Evaluating on lattice vectors gives $B_u(N_u') \subseteq RR^\vee$. After scalar extension to v, take the supremum of the absolute value over the two unit balls. Both isometries are onto, so $\|B_u\|_v = \|B_u^{\mathrm{c}}\|_v$. If $B_u \neq 0$, some matrix entry over K is nonzero at every embedding. Taking logarithms and averaging gives $h(B_u) \leq \langle \log c_{u,v} \rangle_\infty = h_u$.

The support assertion follows from 20.8 with the stabilizer denominator of the same word. The frame assertion follows from 21.33 applied with the chosen C_H to these full positive-rank lattices. Lemma 21.41

bounds the loss with constant c_E; Lemma 21.42 bounds the difference of the adjusted heights with constant $20 + 6/\varepsilon$. Both are at most c_{Div}. The fullness and complex period identifications in the hypotheses supply the remaining clauses of Definition 21.31. □

Proof of 21.33

Proof. Put $q_\kappa = \langle \log \kappa_v \rangle_\infty$. For the closed threshold datum $\mathcal{K}$ of 34.69 (conjoined with $C_{\mathrm{sz}} \geq 0$), set $a_{\mathrm{row}} = \mathcal{K}_{\mathrm{mid}}(c_{\mathrm{out}})$. The family meets $\mathcal{K}$ at height exponent a_{row}, since $a_{\mathrm{row}} \geq \mathcal{K}_{\mathrm{mid}}(c_{\mathrm{out}})$, the remaining middle constants already satisfy the same lower bound $\mathcal{K}_{\mathrm{mid}}(c_{\mathrm{out}})$, and the final-list threshold of 34.69 is evaluated with a_{row} in place of C_H. Hence 34.69 at exponent a_{row} gives $\mu_{\max}(R_0) \leq x^{a_{\mathrm{row}}} b - q_\kappa$ for the level-zero row lattice R_0. Applying 20.18 with $U = x^{a_{\mathrm{row}}} b - q_\kappa$ gives $\mu_{\max}(RR) + q_\kappa \leq x^{a_{\mathrm{row}}} b + 14\lambda_0$, with $\lambda_0 \leq 4b$. Take $F_0 = \max(a_{\mathrm{row}}, 1) + 1$, which bounds the right side by $x^{F_0} b$ for $x \geq 57$.

The rank windows give $P = 1 + f + k + m + t_{\mathrm{ch}} + N_m + L \leq x^M$ eventually. In the maximum-slope bound of 21.11, the $\mu_{\max}(M_u)$ term is at most $x^{4M+1} b$ eventually by 21.39. For Ga, the retained terms of 14.8–14.9 give $\sum_j \mathrm{gainD}_j(w) - \tau_w \leq 4 m n_T a / p_w$ through X'', with zero gain above X'' by 20.8. Grouping by 20.7 and applying [34, (3.24)] gives

$$\mathrm{Ga} \leq 16 m n_T a \log(X'' + 1) \leq 64 P^4 b$$

using $n_T x \leq 2k$ and $\log(X'' + 1) \leq 4P$. Add $\mathrm{Fo} \leq 256 P^9 b$ and $-\mathrm{Ee}_u \leq c_E m B_0$. Include $c_1 m B_0$, with $m B_0 \leq P^4 b$. The arithmetic part is eventually at most $x^{9M+2} b$.

For the height, (36.10) gives $S_a \log T_u \leq 2|C_{Tu}| P^4 b$. The terminal size coefficient before L_a is at most $4P^3 b$, hence $-H_{P,u} \leq (2|C_{Tu}| + 16|C_{\mathrm{sz}}|) P^9 b$. For the row entries, 36.1–36.5 give $|\nu| \leq M_\nu = 11 P^5 b$, $\log M_\nu \leq 16P$, $|\log \Omega| \leq 7P$. A single entry costs at most $M_\nu(23P + 1) + 1 \leq 300 P^6 b$ in the negative kernel sum. Thus $-K_{uj\rho} \leq 300 P^7 b$.

Fix a type ρ and a layer $1 \leq j \leq m$. The plus hook has $k + j - 1$ rows, so its last gamma argument is $1 + \kappa^{\mathrm{big}}_{j\rho, k+j-2}$. The last entry of the long row is $e_{j-1} - b_{j-1} - (t_{\mathrm{ch}} + 1)/2$. Hence the gamma argument is $e_{j-1} - b_{j-1} - (t_{\mathrm{ch}} - 1)/2 \geq 2$ by 15.7. Dominance gives the same lower bound on the earlier arguments, and their upper bound is $Y = 10 P^5 b$. Thus each factorial log is at most $Y \log Y \leq 160 P^6 b$. There are at most P numerator terms and the denominator factors are at least one, so $\log \mathrm{Hook}(a^+_{j\rho}) \leq 160 P^7 b$.

Summing over at most P^2 layer-type pairs and using $f N_m^2 L_a \leq 4P^4$, $f m^3 N_m^2 L \leq P^7$, $f n_V^2 L' \leq 6P^5$ per layer gives

$$-h_u - q_\kappa \leq (500 + 2|C_{Tu}| + 30|C_{\mathrm{sz}}|) P^9 b \leq x^{9M+1} b$$

eventually. Add the bounds for $\mu(N'_u)$, $-h_u - q_\kappa$ and $\mu_{\max}(RR) + q_\kappa$. The choice of C_H absorbs the fixed number of contributions for $x \geq 5$. Average with weights $d_u / \sum_{u'} d_{u'}$. □

Proof of 21.34

Proof. The bounds on $f = 2n, k, m$ follow from (8.1) since $9/2 + 10\varepsilon < 5$ and $m^2 L \leq 4x^{1+\varepsilon}$ eventually. The discriminant bound is 36.2; the quadratic subfield gives $2v_E \geq \sqrt{d}$, with d tending to infinity by 8.16. Apply 8.12 for the lower prime bound and 36.2 with $L \geq 1$ for the upper bound.

Use the exact coordinates of 15.9. Off the marks $U_0 - v_0 = S_c + 1/2$ and $U_{k-1} = -1/2$. At a mark the spread is $a_s + b_s + k - 1$, with $a \leq a_s \leq a + b$ and $b' \leq b_s \leq 2b'$ by 22.45 and 36.2. Here $b/a = 1/x$, $b'/a = 1/X'$, and k, S_c grow polynomially while b' exceeds every fixed power. These give all the endpoint inequalities, including the span bound in both cases. The base b_0 is at least $w'' - 1$ by the three base formulas. Thus $b_0/x \geq \max(S_c, h_0/2)$ eventually. Since $e_q = b_q + h_0/2$ and $b_q \geq b_0$, this proves the relative bound. Apply Lemma 21.23 to the same base and step size to obtain $e_{m-1} \leq 2b$. Finally $b_{q+1} - b_q = h_0$ and $e_q = b_{q+1} - h_0/2$. The node threshold in the proof of 36.2 gives $h_0 \geq x^{1300}$, while $r \leq x^5$, $f \leq x^3$, $L \leq x$ by 38.3. This gives the asserted gap. □

Proof of 21.35

Proof. The prime interval gives $\log T_u = \log X_u - 2 \log \Gamma_u/(fk) + \epsilon_u$ with $0 \leq \epsilon_u \leq \log(1 + 1/(fk)) \leq 1$. This loses at most 2 in the lower bound and adds at most 1 to the upper bound. The common denominator

cancels in differences, and $|\epsilon_u - \epsilon_{u'}| \le 1$. These errors fit in c_{Count}. In particular $-S_a \log T_u \le -S_a\lambda_5$, as used in 24.5.

For the family, take $C_0 = C_Y = \max(5, C_{YL} + 1)$, with C_{YL} the type-prime size constant. Take $x \ge 44^4$ with $x > x_1$ beyond the cutoff in 21.34. Apply that lemma to the same word and type, numbering the coordinates from one in the following calculation. Put $r = k + m - 1$, and let Δ be the spacing of the two choices of a word coordinate. The scalar conditions are $f, k \le x^{C_0}$, $m^2 L \le k$, $(kb)^8 \le g_i$, $\log g_i \le C_Y L$ and the bounds on m, k, v_E below. The consecutive auxiliary weights have gaps at least $r^{19} f^5 L/C_0$. Off the top the leading gap is between $4k\Delta$ and $8k\Delta$. On the top the leading coordinate is at most $2a$. The remaining coordinate conditions are given in the next four sentences. To obtain them, write the interior word coordinates using the two choices in 33.7; their common interlacing row has successive spacing 4Δ and lies in $[0, S_c]$, where $S_c = 4k\Delta$. The first and last coordinates agree with those of the reference word. On a marked type they equal $a_s + (k-1)/2$ and $-b_s - (k-1)/2$. The top window (14.1), $b_s \le 2b'$, $k \le b$, and $100b \le a$ give the asserted span and endpoint bounds there, using $S_c \le b' \le b$. Off the top the first gap is in $[S_c, 2S_c]$ and the last coordinate is $v_{k-1} - 1/2$, giving span at most $3S_c + 1/2 \le 2a$. The rank window of 33.3 gives $44 \le m \le x$ eventually. The prime window $Y \le g_i \le 2Y$ with $(kb)^8 \le Y$ and $\log Y \le C_{YL} L$ gives $\log g_i \le C_Y L$ since $L \ge 1$. The auxiliary-weight spacing follows from $b_{j+1,\rho} - b_{j\rho} \ge \eta/m$ and $r^{19} f^5 L \le \eta/m$ in the selected window.

Lean: `Formalism.MainNotions.GlobalTheoryTheory.JDB.db_cloud` (`Formalism.SharpRestriction.GlobalSchurBounds.JointsL`, lines 738–785)

Lean: `Formalism.MainNotions.GlobalTheoryTheory.JDB.dataB_of` (`Formalism.SharpRestriction.GlobalSchurBounds.JointsL`, lines 1085–1090)

Lean: `Formalism.MainNotions.GlobalTheoryTheory.JDB.dataA_fam` (`Formalism.SharpRestriction.GlobalSchurBounds.GlobalRankLatticeProofs.FamilyPartFiveOfExplicitConstants`, lines 303–315)

At each ρ the interlacing entries v_i have spacing $v_i - v_{i+1} = 4\Delta$, $\Delta \ge 1$, and lie in $[0, 4k\Delta]$. A middle shifted column entry is $v_{i-1} - 1/2$ or $v_{i-1} - 1/2 - \Delta$. At a nontop place the last entry is $v_{k-1} - 1/2$; at a top place it is at most $-4k\Delta$ and at least $-a$. The leading coordinate exceeds v_1 by at least $4k\Delta$, and the column span is at most $2a$. The row sizes include $44 \le m \le \min(k, x)$, $\log(2k) \le L$, $\pi \le v_E$, $\log(2v_E) \le L$. For the negative row entries $-b_{j\rho}$, use

$$1 \le b_{j\rho} < w_{j\rho} \le (1 + 1/x) b_{j\rho}, \qquad w_{j\rho} \le 2b.$$

The auxiliary entries are separated by the intervening weights, with $b_{1\rho}/x \ge 4k\Delta$.

Write z_a for the shifted column coordinates and $\Delta_\rho(u) = \prod_{a<b}(z_a - z_b)$. The first-entry gaps change by at most Δ and are at least $4k\Delta$. For the other pairs the change is at most 2Δ and the gaps are at least $3(b-a)\Delta$. By $|\log y - \log y'| \le |y - y'|/\min(y, y')$, the change in $\log \Delta_\rho$ is at most $1/4 + \frac{2}{3}k(1 + \log k)$. The Weyl denominator is common to the words, so $|\log X_u - \log X_{u'}| \le 2 \log k$.

Set $r = k + m - 1$, $F_n = \prod_{i=0}^{n-1} i!$. For the row parameter λ_ρ, 37.8 at $t = 1/3 \ge e^{-r}$ gives $d_{\Sigma,\rho} \ge (2/3)\Delta(\lambda_\rho)/F_r$, where $\Delta(\lambda_\rho)$ is its Vandermonde. With $C_3^+ = 8192$ from Lemma 37.8, the required gap is at most $9C_3^+ r^9 = 9 \cdot 8192\, r^9$, and the supply is $r^{19} f^5 L/C_0$. Thus $L \ge 9C_3^+ C_0$ suffices. The $2k-3$ column gaps incident to the first or last entry are at most $8a$; for interior pairs $z_i - z_l \le v_{i-1} - v_l$. Each cross gap at $-b_{j\rho}$ is at least $b_{j\rho}$; unused row gaps are at least one. It follows that

$$\Delta_\rho(u) \le (8a)^{2k-3} \Delta(v), \qquad \Delta(\lambda_\rho) \ge \Delta(v) \prod_j b_{j\rho}^{k-1}.$$

The first power costs at most $5kL$ once $2(\log 8 + 1 + \log x) \le L$. Also $k \log w_{j\rho} - (k-1) \log b_{j\rho} \le k/x + 2L$. With $m \le \min(k, x)$, this yields

$$S_{\mathrm{dim}} - k \sum_{\rho, j} \log w_{j\rho} \ge -f(1 + k + 7kL), \qquad S_{\mathrm{dim}} = \sum_\rho \log \frac{d_{\Sigma,\rho} F_r}{\Delta_\rho(u)}. \tag{A.151}$$

The volume identity 37.9 and adjacent-rank mass ratio 37.5 cancel the Weyl factorials. With $\ell_E = \log(v_E/\pi)$ their product is

$$\begin{aligned} fk \log X_u = f\left((m-1)k + \frac{m(m-1)}{2}\right)\ell_E - \frac{f(m-1)}{2}\log(2v_E) + S_L \\ + f\left(km + \frac{m(m-1)}{2}\right)\ell_g + fS_g + S_{\mathrm{dim}}, \end{aligned} \tag{A.152}$$

where $S_L = \sum_{n=k+1}^{r} \log L_\eta(E,n)$ and $S_g = \sum_{i=0}^{1}\sum_{n=k+1}^{r} \log(1-g_i^{-n})$. The Euler bounds give $|S_L| \le fm,\ -2 \le S_g \le 0$. Subtract $fk(\mathcal{W}+m\ell_g)$. By (A.151) the dimension term costs $f(1+k+7kL)$. The unmatched linear term in ℓ_E costs at most fkL, the discriminant term costs at most $f(m-1)L/2$, and the two Euler sums cost at most $fm+2f$. The remaining quadratic terms in ℓ_E, ℓ_g are nonnegative. Hence the difference is at least $-f(8kL+(m-1)L/2+m+3+k)$. Take the family cutoff so that $L \ge 2$. Since $m \le k$ and $k \ge 3$, $m+3+k \le 3k \le 3kL/2$; the displayed cost is at most $11fkL$.

For the upper bound, $v_i - v_l \le z_i - z_{l+1}$ gives $\Delta(v) \le \Delta_\rho(u)$, since the unused column gaps are at least one. The compact Weyl product in 37.8 has at most $2km$ factors left after the common-shift interlacing entries, each at most $H+2k$ for largest row gap $H \le 4e^L$. Its denominator F_pF_q is at least F_k in signature (p,q), and $pq \le km,\ r \le 2k$. Hence

$$\log \frac{d_{\Sigma,\rho}}{\dim V(u,\rho)} \le \log 4 + 2km\log(H+2k) + km\log(3pqrH) \le km(12+10L).$$

Using $\log(F_r/F_k) \le 2mkL$ gives $S_{\dim} \le fkm(12+12L)$. In (A.152), $\ell_E \le L,\ \ell_g \le 2C_YL$ give $\log X_u \le (15+3C_Y)mL$ for $L \ge 9,\ k \ge 5$.

Lean: `Formalism.MainNotions.StandardLattice.part_five_family_of_mass_volume` (`Formalism.SharpRestriction.GlobalSchurBounds.LinkMin`, lines 318–423)

The lower bound and $\ell_g \ge 16L,\ \mathcal{W} \ge 0$ give $\log T_u^\star \ge 16mL - 11L - 2 \ge mL$ eventually. For this choice of primes, Definition 34.19 and 38.3 give $m \ge x^{1/3},\ 1 \le fk \le x^{12},\ g_i \le b^9,\ \log(Q_S qd) \le 2x^{\varepsilon/5}(\log x)^2$. Lemma 37.11 applies with scale $T_u^\star$ and interval parameter fk, and its congruence yields the splitting of 37.21. The size bounds of Definition 14.6 give $\log\Gamma_u = C_\Gamma m^5 fL \le k/8$, so $\log\Gamma_u/(fk) \le 1$. Take $C \ge \max\{11, 16+3C_Y, 2\}$ in (21.4), and take x_0 above these cutoffs and the family cutoff x_1. □

Proof of 21.36

Proof. Fix ρ and spacing $\Delta > 0$. In the shifted coordinates the first entry is fixed. The middle entries are $v_i - 1/2 - \delta_i\Delta$ with $v_i - v_l = 4(l-i)\Delta,\ \delta_i \in \{0,1\}$. At a nontop type the last entry is in the list with bit zero. At a top type it is fixed at $-b_{\mathrm{end}}$, where $b_{\mathrm{end}} = b_v + (k-1)/2$ for b_v in 21.9. For $J = k-1$ regularly spaced entries at a nontop type, and $J = k-2$ at a top type, the ratio to the unperturbed Vandermonde is

$$R(\delta) = \prod_{i<l\le J}\left(1 + \frac{\delta_l - \delta_i}{4(l-i)}\right).$$

For $|z| \le 1/4$, $|\log(1+z) - z| \le 2z^2/3$, so the quadratic errors total at most $J\pi^2/144$. At distance d, the linear sum telescopes to two boundary sums with difference bounded by $\min(d, J-d)$. With $q = \lceil J/2\rceil$, four times the absolute linear contribution is at most

$$q + \sum_{d=q+1}^{J}(J/d - 1) \le 2q - J + J\log(J/q) \le 1 + 0.7J.$$

Thus $|\log R| \le (J+1)/4$. The first-to-middle gaps are at least $4k\Delta$ and change by at most Δ. Their total contribution is less than $1/3$. The first-to-last gap is fixed. At a top type the remaining edge changes cost at most $(k-2)\Delta/b_{\mathrm{end}}$. Since $b_{\mathrm{end}} \ge b_1/2,\ b_1 \ge C_\ell mk\Delta,\ C_\ell \ge 40,\ m \ge 2$, one has $4k\Delta/b_{\mathrm{end}} \le 1/10$. Combining the two word bounds for R with these edges gives

$$|\log(\Delta_\rho(u')/\Delta_\rho(u))| \le k/2 + 1 \le 3k/4 \qquad (k \ge 4). \tag{A.153}$$

Lean: `Formalism.FourierDecayAlt.P.vandermondeRatioHarmonic` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.VandermondeRatioHarmonic`, lines 379–446)

Lean: `Formalism.FourierDecayAlt.P.vandermondeWords` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.SumInvSqLeSub`, lines 738–749)

Put $g = g_0g_1,\ r = k+m-1$ and define $t_{u\rho} > 0$ by

$$t_{u\rho}^k = (v_E/\pi)^{(m-1)k+m(m-1)/2}(2v_E)^{-(m-1)/2}g^{km+m(m-1)/2}\frac{d_{\Sigma,\rho}F_r}{\Delta_\rho(u)}.$$

Then $t_{u\rho}/t_{u'\rho} \le e^{3/4}$. Write d_u^S for the level-zero dimension and $A_u = \prod_\rho (t_{u\rho}/T_u)^k$. The stabilizer condition of 37.21 follows from $g_i \ge (kb)^8$, so $d_u^S = (T_u^k - 1)^f d_u^\star$. Equation (A.152) gives

$$D^\star/d_u^S = E_u A_u, \qquad E_u = \frac{\prod_{j=k+1}^r L_\eta(E,j) \prod_{i=0}^1 \prod_{j=k+1}^r (1 - g_i^{-j})^f}{(1 - T_u^{-k})^f}.$$

For $j \ge k+1 \ge 6$, $\zeta(j) - 1 \le 2^{1-j}$ by integral comparison. The L-terms cost at most $2f/2^k$ in absolute log by 37.5. Each type-prime product and the denominator cost at most $2f/2^k$ by $-\log(1-y) \le 2y$. Hence $|\log E_u| \le z = f2^{3-k}$.

Lean: `Formalism.MainNotions.BridgeLemma.degree_ratio_product_formula` (`Formalism.SharpRestriction.CuspSpanNormBounds.BruhatCellsKronecker.DegreeRatioFormulaStatement`, lines 294–328)

The spectral row gives $D^\star/(C_0\Gamma^2) \le d_u^S \le C_0 D^\star/\Gamma^2$ with Γ common. Thus $A_{u'}/A_u \le e^{2z} C_0^2$, and

$$(T_u/T_{u'})^{kf} = \frac{A_{u'}}{A_u} \prod_\rho (t_{u\rho}/t_{u'\rho})^k \le e^{3kf/4 + 2z} C_0^2 \le (C_0 e^{7/4})^{kf}.$$

Here $2z \le kf$, $2 \le kf$. The Vandermonde bound follows from (A.153). □

Proof of 21.38

Proof. For the rank-k class lattice $A \subseteq A^\vee$ over E, the normalized degree is $-\log[A^\vee : A]/[E : \mathbf{Q}]$ in the Hermitian dual metric. The local factors give $[A^\vee : A] = T_u^{2f}$, so the slope is $-l/k$. Isometric scalar extension preserves it.

Let h_ν be the squared norm of the highest Fock vector and $c_\nu = h_\nu / \prod_i \nu_i!$. Write μ_ν for minus the log Gram determinant per rank of the standard Weyl lattice. In representation dimension d' the determinant character is $\det^{d' \sum_i \lambda_i/k}$ by permutation symmetry of weights. Tensor over ρ, then sum the orthogonal, equal-rank classes to get

$$\mu(M_u) = \sum_\rho [\mu_\nu - (d_\rho/k + t_\rho) l].$$

For an integral basis matrix V of the dual polynomial lattice in monomial coordinates, its Gram matrix is $V^{\mathsf T} \operatorname{diag}(m!/h_\nu) V$, where $m! = \prod_{a,i} m_{ai}!$ and $\sum_a m_{ai} = \nu_i$. The multinomial bound gives $m!/h_\nu \ge k^{-d_\rho}/c_\nu$. A nonzero maximal minor of V is an integer. Cauchy–Binet and duality give $\mu_\nu \ge -d_\rho \log k - \log c_\nu$. In the hook formula of 21.12 the denominator is an integer at least one. Dividing by $\prod_i \nu_i!$ leaves at most $k(k-1)/2$ factorial factors of size at most $s_\rho + k$.

For the maximum slope use $e = [E : \mathbf{Q}]$ and the trace metric on $A^\vee$ of Euclidean rank ek. For a nonzero Euclidean dual vector z, testing against $\mathcal{O}_E y$, $y \in A^\vee$, gives $h(z,y) \in \mathfrak{D}_E^{-1}$, with $\mathfrak{D}_E$ the absolute different. For $\delta \in \mathfrak{D}_E$, $\bar\delta z$ pairs integrally with A and lies in $A^\vee$. Testing against this vector gives $\mathfrak{D}_E^2 h(z,z) \subseteq \mathcal{O}_E$. Taking norms and applying the arithmetic-geometric mean to the positive conjugates gives $\operatorname{Tr} h(z,z) \ge e/\operatorname{rd}(E)^2$. Banaszczyk's transference bound $\lambda_n(\Lambda)\lambda_1(\Lambda^*) \le n$ for Euclidean rank n [78] gives ek real independent vectors in the trace lattice of squared length at most $ek^2(2v_E)^2$. Choose k independent E-vectors y_a among them.

Pair with y_a to form the substitution matrix on each column. Its Gram operator is bounded by its trace. The polynomial substitution in degree d_ρ therefore gives a squared norm multiplier at v at most $(\sum_a h_{v,\rho}(y_a, y_a))^{d_\rho}$ in the normalized monomial metric. The trace-length bounds and concavity of log give mean log at most $d_\rho \log(k^3(2v_E)^2)$ for this factor.

Multiply the polynomial coordinates by the determinant factor of exponent t_ρ in each column. The coordinates of the class lattice then lie in a common fractional ideal I of L. In the following calculation $N(I)$ is the absolute ideal norm over L, finite places are those of L, and the infinite average is $[L : \mathbf{Q}]^{-1} \sum_{\sigma : L \hookrightarrow \mathbf{C}}$. At each finite place its generator comes from the product of the local column determinant lattices to powers t_ρ. At an infinite place the squared inverse determinant factor of the column frames is a positive multiplier D_v. In the doubled slope convention the determinant degree of the ρ-column lattice is $k\,\mu(A) = -l$. Combining the ideal norm and the frame determinants therefore gives

$$-\frac{2 \log N(I)}{[L : \mathbf{Q}]} + \langle \log D_v \rangle_\infty \le -\sum_\rho t_\rho l.$$

In local column bases the ideal is the product of the determinants of the basis matrices to powers t_ρ, and at infinity D_v is the squared norm of the corresponding inverse frame determinant raised to these powers. The two log sums together are $\sum_\rho kt_\rho\mu(A)$ for the determinant lattices. This comparison uses the signed powers before passing to $|t_\rho|$.

Lean: `Formalism.L2.GlobalTheoryTheory.TraceIntegrality.classMapT_of` (`Formalism.LocalToGlobal.GlobalTheory.GL4.PacketOfFrameDegree`, lines 72–104)

For a rank-s sublattice choose a projection onto s coordinates whose determinant is nonzero on its L-span. At a finite place w, evaluate this determinant on a local lattice basis. Dividing each projected basis vector by a generator of I_w gives integral coordinates, so the original determinant has valuation at least $sv_w(I)$. At infinity Cauchy–Binet bounds the Gram determinant below by the squared modulus of that minor times the product of the monomial weight lower bounds. The product formula combines these estimates to bound the slope of the coordinate image by $\sum_\rho(d_\rho\log k+\log c_\nu)-2\log N(I)/[L:\mathbf{Q}]$. The squared norm bound for the substitution adds its mean log and $\langle\log D_v\rangle_\infty$ to the source slope bound. Apply the determinant comparison and $-t_\rho l\le|t_\rho|l$. Since $v_E=\mathrm{rd}(E)/2\ge 2$ by hypothesis, $4k^4v_E^2\le(kv_E)^4$. This gives the asserted bound on each class. Integral orthogonal projections, with their kernel-image rank filtration on a sublattice, give the same bound for the sum of classes. □

Proof of 21.39

Proof. The first assertion follows directly from the class-lattice slope $-l/k$ in the opening paragraph of 21.38. Beyond the cutoff of 21.34, one has $v_E\ge\pi>2$, so 21.38 applies. Put $C_n=\max(3,c_{\mathrm{Count}},C_{YL}+1)$, $c_L=3$ and $l_T=\log T_u\le c_{\mathrm{Count}}mL$. Increase the cutoff for $x^{14}\le X'$, $7\log x\le L$, $m\ge c_L+c_{\mathrm{Count}}+6C_n+9$. The size rows give $\log(xbv_E)\le c_LL$, $N_mm^4\le b'$. With $|D_1|\le L$ these give the numerical conditions with log argument c_LL.

At a top column the first entry is $\alpha_{u\rho}\in[a,2a]$ and the negative of the last is in $[b',2b')$. The interior entries are in $[0,S_c]$. Other columns have entries in $[0,C_nS_c]$. With $B=b'+S_c$, the decomposition of 21.38 gives

$$\sum_\rho d_\rho\le 2n_Ta+O_{C_n}(fkB),\quad \sum_\rho|t_\rho|\ll_{C_n}fB,\quad \sum_{\rho,i}\lambda_i\le\sum_{\rho\ \mathrm{top}}\alpha_{u\rho}+O_{C_n}(fkS_c).$$

Since $S_c\le b'\le b$, $k\le x^5$, $\log(s_\rho+k)\ll_{C_n}c_LL$. Put $S_1=\sum_{\rho\ \mathrm{top}}\alpha_{u\rho}l_T/k$. Substitution in 21.38 gives constants C_5,C_6 depending on C_n with

$$\mu(M_u)\ge -S_1-C_5(n_Ta\log k+fkB\log k+fS_cl_T+fk^2c_LL),$$
$$\mu_{\max}(M_u)\le C_6(n_Ta\log(kv_E)+fB(k\log(kv_E)+l_T)+fk^2c_LL).$$

Lean: `Formalism.MainNotions.flipTwoThree.nums_wts_of_fam` (`Formalism.SharpRestriction.GlobalSchurBounds.GlobalRankLatticeProofs.SevenLogLeLOfPowerBound`, lines 698–747)
Lean: `Formalism.MainNotions.flipTwoThree.numsL_of_fam` (`Formalism.SharpRestriction.GlobalSchurBounds.GlobalRankLatticeProofs.SlopesMFOfCountFam`, lines 17–23)

Here $B\le 2a/X'$, $f=O(x^3)$, $k\le x^5$, and $n_T\ge 1$, so both fkB and fk^2 are at most n_Ta eventually. With $m\le k$, $L\le L_a$, $\log k\le 5L_a$, $\log(kv_E)\le 6L_a$, the cost of S_1 in units of n_TaL_a is at most $2c_{\mathrm{Count}}m/k$. The terms multiplied by C_5,C_6 cost at most $10+c_{\mathrm{Count}}+c_L$ and $12+c_{\mathrm{Count}}+c_L$, respectively. Taking $c_{\mathrm{Slope}}=2c_{\mathrm{Count}}+\max(C_5,C_6,0)(12+c_{\mathrm{Count}}+c_L)$ suffices. □

Proof of 21.41

Proof. Write $v_{i,u}=v_{g_i}(D_{\Gamma,u})$. Since the loss is supported on S, the fixed-prime formula in 15.12, $2F_g(j)=-D_j^{\mathrm{Fock}}$ from 15.10, the weights of 20.7, and clause 2 of 21.40 give

$$\sum_{\substack{w\in S\\ p_w=g_i}}c_wE_{ujw}=\left(2v_{i,u}-D_j^{\mathrm{Fock}}+2H_g+A_{uj}\right)\log g_i.$$

Subtract $S_a\log g_i$. Since $v_{i,u},H_g\ge 0$, clauses 3 and 4 bound the absolute difference by C_gB_0. On the level fibre $E_{ujw}=E_{u,w}^T\ge 0$, so the absolute sum there is at most C_TB_0. On the other fibres group by rational prime. The stabilizer term costs at most $2\sum_p v_p(D_{\Gamma,u})\log p=2\log D_{\Gamma,u}$. The discriminant term costs at

most $2\log d\,|D_j^*|$, and the remaining terms cost at most $\sum_{p\in P} 2\log p\,|J_{j,p}^*|$ over eligible primes meeting S. Clauses 6–8 bound their sum by $(C_{\mathrm{stab}}+C_d+C_I)B_0$. Add the level cost for the off-type bound and the two fixed-prime differences for the first inequality. Sum over j for Ee_u. □

Proof of 21.42

Proof. Add $S_a \log T_u$ to the height formula in 21.12. The remaining terminal allowance is independent of the word. The hook, the two layer allowances and the mean metric correction are also common. Subtraction therefore gives the displayed identity.

By 15.8, every unpadded degree is at least one. By the coordinate formula there, changing the word changes a head degree by the change in the corresponding U_i, and all tail degrees remain fixed. The endpoints are common to the words. By 36.3, every interior U-coordinate lies in $[0, S_c]$, so at each interior head coordinate the two choices differ by at most S_c and the degrees are at most $w_{j\rho}+S_c$. Write $M = 6x^6 b$. For $1 \le y, y' \le M$, integrating the derivative $\log s - \log\Omega$ gives

$$|(\varphi(y) - y\log\Omega) - (\varphi(y') - y'\log\Omega)| \le |y-y'|(\log M + |\log\Omega|).$$

The stated bounds on D_1 and $\log \mathrm{rd}\, E$, with 21.12, give $|\log\Omega| \le 6 + 2L$. Since $\log x \le L/\varepsilon$ and $L \ge 1$, the last factor is at most $(14 + 6/\varepsilon)L$. There are at most k varying coordinates per layer and position. The resulting bound is $(14+6/\varepsilon)mfkS_cL \le (14+6/\varepsilon)B_0$. This implies the claimed inequality for each member and hence for each translated member with its own pair of words. □

Proofs for Chapter 22: Prime sums and net gain

Proof of 22.1

Proof. Group by rational prime. The model uses the full fiber on the geometric support, and $\sum_{w|p} e_w \log N\mathfrak{p}_w/[K:\mathbf{Q}] = \log p$ by $N\mathfrak{p}_w = p^{f_w}$ and $\sum_{w|p} e_w f_w = [K:\mathbf{Q}]$ [41, Chap. I, Prop. 8.2]. The values agree on the fiber by 14.9. Equation (14.1) gives $a_t \le B$ since a_t is integral. Partners have weight $\ge w'' \ge 1$ by 22.4 and 22.43, so all ordinary terms above B vanish.

For the cutoff, a positive term has $p^i < a_t/w_j(\rho) \le N$, using d in place of p for a ramified term. The ordinary primes in the arithmetic sum are those of the signal and inert classes, and their exponents are odd. At d, the coset $iF_d + I_d$ has one element outside Φ. In either case the partner $r_t(g) = c + g + ((0,0),t)$ has first sign bit zero and hence a unique matching place by 22.43. At an ordinary exponent i its position has A-coordinate $t + ih_p$ and the signal indicator as its remaining bit. Thus the model filling and the presentation read the same lift-$(j+1)$ weight there. The ramified terms read the weight at the matching place of the ramified coset. Substitution in the arithmetic sum gives the model sum.

Lean: `Formalism.MainNotions.Lnk.askIsF` (`Formalism.SharpRestriction.GlobalSchurBounds.FrameCodeOfAdditive`, lines 546–556)

□

Proof of 22.3

Proof. Geometric series give

$$A_{jtp} \le a_t p/(p^2-1) \le 2a/p, \qquad D_{jt} \le a_t \log d/(d-1) \le 2a\log d/d. \tag{A.154}$$

These give $E_{\mathrm{out}}, E_1, E_2, E_d$ in (22.3). Above X'' any contribution comes from the first power. Its contribution is at most $w''\vartheta$, and it vanishes for $p \ge X''(1+\vartheta)$. The intervening interval has at most $X''\vartheta + 1$ integers. This gives E_h. For $E_{\mathrm{fl}}, E_{\mathrm{cp}}$ use $|S| \le X''$, $|TT_p| \le n_T$, and $1/\log p \le 2$.

For $Y \ge 1$ one has

$$\Theta(Y) = \sum_{p\le Y} \log p \le Y\log 4, \qquad \sum_{p\le Y} \frac{\log p}{p} \le \log Y + \log 4. \tag{A.155}$$

The first bound follows by induction using $\binom{2r}{r} \le 2^{2r}$ and $\binom{2r+1}{r} \le 2^{2r}$ to count primes above r and $r+1$, respectively. For $n = \lfloor Y \rfloor$, multiply the second sum by n and use $\sum_{p \le n} \lfloor n/p \rfloor \log p + \Theta(n) \le \log(n!) + \Theta(n)$. The first bound gives E_{pr}. For E_{or} use the second bound, $p^{-L_1(p)} \le 1/(K_0 p)$ below p_0, and

$$\sum_{n \ge p_0} \frac{\log n}{n^2} \le \frac{2(\log(p_0 - 1) + 1)}{p_0 - 1}, \tag{A.156}$$

by integrating $(\log y)/y^2$ from $p_0 - 1$. □

Proof of 22.5

Proof. For odd $s > 1$ and $0 \le j \le e$,

$$h_e(s)^{2^j} = 1 \iff s \equiv \pm 1 \pmod{2^{e+2-j}}. \tag{A.157}$$

For $j \ge 1$, successive differences of squares give $v_2(s^{2^j} - 1) = v_2(s-1) + v_2(s+1) + j - 1$, and one of $s \pm 1$ has valuation one. For $j = 0$ use the quotient definition. With $r = \min(e, \lfloor \log_2 J_0 \rfloor)$ and $M = 2^{e+2-r}$, the small-order primes have the form $qM \pm 1$ with $q \ge 1$. Since $p \ge qM/2$, their sum is at most $4 \log Y_0 (1 + \log Y_0)/M$. Use $2^e \le J_0 M$.

Lean: `Formalism.RankGapBounds.LocalQuadraticParts.ordersSumY_of` (`Formalism.RankGapBounds.RankGapProofs.PmModulusFacts`, lines 252–287)

Lean: `Formalism.RankGapBounds.LocalQuadraticParts.powTwoPowEqOneIff_holds` (`Formalism.RankGapBounds.RankGapProofs.FourNotDvdOddSqAddOne`, lines 132–138)

In the band $2i \le \log(2X') \le 2 \log X'$, and taking i-th roots puts p in the interval $y_i[1-\rho, 1+\rho]$, where $y_i = (X')^{1/i}$. The interval has at most $2\rho y_i + 1$ integers. Each prime contributes at most $\log(2X') \min(2/y_i, 1/p_0)$. Summation proves the band bound. □

Proof of 22.7

Proof. For $m = 0$, use $\tau_p \le 0$ on S from (22.2) and (22.10) with $m = 0$. For $m > 0$ put $E = x^\nu$, $S_x = e^{x^{\varepsilon/4}}$, $B = e^{z/2}$, and $D_0 = 100 K_s E^4 x^4$. Thus

$$X' \ge D_0, \quad B \ge D_0 X' S_x, \quad b = B^2, \quad x \ge 100, \quad 0 \le \sigma_x \le 1. \tag{A.158}$$

In particular $\log x, \nu \log x \le u \le z$, $z^2 \le x$, and

$$\log a \le 2z, \qquad \log(2a) \le x. \tag{A.159}$$

The bounds for $E_{\mathrm{fl}}/Q, E_{\mathrm{cp}}/Q, E_{\mathrm{pr,low}}/Q$ from (22.7) are respectively

$$12 K_s E^4 x^3 X'/b, \quad 6 K_s E^2 x^2 X' S_x/b, \quad 3 K_s E^2 x^2/X'. \tag{A.160}$$

Each is at most σ_x. For the upper price part, $X' Q_2/b - C$ is at most the sum of $2K_s X' S_x/b$, $K_s z^2/x^2$, $K_s X' E x/b$, X'/B. Each is at most $\sigma_x/4$, so

$$E_{\mathrm{pr,high}}/Q \le 2C \log 4 + 3\sigma_x. \tag{A.161}$$

In E_{or} the three factors inside braces in (22.7) use $\mathcal{L}/k \le z^2/x^2$, $K_0^{-1} \le x^{-2}$, $\log X' + \log 4 \le 3z$, and a tail at most $4z/x^2$. This gives $E_{\mathrm{or}}/Q \le 20 K_s z^4/x$. In 22.5 use J_0, $\lfloor \log X' \rfloor \le z$, $\log(2^{e_2} \max(Y, X'')) \le 4z$, and $2^{e_2} \ge x^2$. Then

$$E_{\mathrm{ord}}/Q \le 64 K_s z^3/x, \qquad E_{\mathrm{band}}/Q \le 40 z^2/x. \tag{A.162}$$

The remaining four bounds are

$$E_{\mathrm{out}}/Q \le 4z/x, \quad E_2/Q \le 2x^{-8}, \quad E_h/Q \le 16 z/x, \quad E_d/Q \le 4z/x. \tag{A.163}$$

Here the first two use 22.6. For the last two use $\log(2X'') \le 4z$, $a\vartheta^2, w''\vartheta \le b/x$, and $\log d/d \le 2 \log x/x^2$. Add the ten bounds by σ_x to (A.161). The index bound follows from (22.11) and $amn_T x^{-9} = x^{-8} Q$. Finally $x^{4\varepsilon} = o(x)$, $\log x = o(u)$, and $u + x^{\varepsilon/4} = o(z)$. □

Proof of 22.9

Proof. With $d_I = \det H_I$, each minor is $d_I^{p^{z_j}} \det(T_{jli}^{p^{z_j-u_{j,l}}})$. If $d_{I_0} = 0$, the initial factor vanishes. Otherwise row-reduce the first block to I_r. On a competing plane a replacement minor is nonzero and its factor vanishes because $d^{\sum_j p^{z_j}(Q_j-1)} = d^{p^{z_b}-p^{z_0}} = 1$ for $d \in F^\times$. On the coordinate plane each replacement vanishes and the Δ_j have unit diagonal terms. Lift cyclic positions by a common multiple of n' when necessary. A Witt lift of L^h has order at least h on the competing planes. □

Proof of 22.11

Proof. In the numerator of $\binom{u+p^h Z}{i}$ at least $\lfloor i/p^\ell \rfloor$ factors have both coefficients in $p^\ell \mathbf{Z}_p$ for $1 \le \ell \le h$. Thus the polynomial is integral. In Smith coordinates at u, the Gauss norm on $|Z| \le p^{-E}$ of the determinant is at most $p^{-E\beta_u}$, by the row bounds. Multiplicativity of the Gauss norm and the roots ζ of the determinant give

$$E\beta_u \le v(b) + \sum_\zeta \min(v(u-\zeta), h+E).$$

At heights $\ell - 1 < y \le \ell$, the integers with $v(u-\zeta) \ge y$ lie in one residue class modulo p^ℓ. Integration to height h costs at most $H/(p-1)+h$ per root, and above h at most one point contributes, at cost E. Use $s \le dJ$. Removing the integral scalar subtracts its valuation from the finite diagonal valuations. □

Proof of 22.14

Proof. Write $P_k(b) = \binom{b+k-1}{k-1}$. The total count is $R = \prod_i P_k(b_i)$. Indeed, tensoring an irreducible with one symmetric power adds a horizontal strip with multiplicity one. Dualizing gives the removal rule. Restriction to the subgroup fixing one coordinate gives one copy of each interlacing row. These two dimension identities also follow by grouping semistandard tableaux, respectively by the horizontal strip in the Pieri rule and by the boxes occupied by the largest letter in the branching rule. Evaluating dimensions after the last restriction gives precisely the labels in 22.13. Negative entries are handled by a common determinant twist before restriction.

Keep two lists at each stage. The first consists of the degrees of the vector letters already used, in reverse chronological order, and the second is the corresponding list for the covectors. At a restriction step both lists stay fixed. For the current row form two decreasing lists u^+, u^- of its positive parts and of the absolute values of its negative parts. Extend lists by zeros. If B is the list of degrees of the corresponding sign, then for every $h \ge 0$

$$\sum_{j>h} u_j \le \sum_{j>h} B_j. \tag{A.164}$$

For example, on adding a strip of size b the sum of positive parts increases by at most b, and interlacing gives $u^+_{j+1,\mathrm{new}} \le u^+_{j,\mathrm{old}}$. These are the inequalities for $h = 0$ and $h \ge 1$ after b is prepended to B. Negative parts decrease coordinatewise. Reversing and negating gives the covector step. In a restriction both new lists are coordinatewise bounded by the old ones. This proves (A.164) by induction. It also proves that a deficit X in (A.164) persists, with h increased by one each time a subsequent signed letter of the corresponding sign is used.

For either sign let $B_1, \dots, B_v$ be its final list of degrees. Inequality (A.164) at $h = v$ makes all subsequent u_j zero. Put

$$F(y) = \sum_{d=1}^{q} \log(1 + y/d), \qquad T_h = \sum_{j=h}^{v} (u_j - B_j).$$

Concavity and summation by parts give

$$\sum_j \{F(u_j) - F(B_j)\} \le \sum_{h=1}^{v} \{F'(B_h) - F'(B_{h-1})\} T_h, \qquad F'(B_0) := 0. \tag{A.165}$$

Every $T_h \le 0$. Since $B_h \ge k \ge q$ and $B_{h-1} \ge 4B_h$ when $h > 1$,

$$F'(B_h) \ge q/(2B_h), \qquad F'(B_{h-1}) \le F'(B_h)/2.$$

Consequently a deficit X beginning at the entry B_h costs at least $qX/(4B_h)$ in (A.165).

For a failed positive- or negative-part test, the loss of mass creates a deficit of at least ξ_i at the start of the appropriate list. The list at that point is nonempty, since mass was lost. Its first degree is an earlier signed degree and is at most $4v_i/p$. Its position in the final list is the one specified by persistence in (A.164). The cost is at least $qL/(16k)$. For a failed test at coordinate r, omit the already restored positions from the row. In the positive degree list the corresponding entry is $b_{o(i)}$. Apply (A.164) to the tail strictly after that entry and then include the entry itself. The deficit is at least $b_{o(i)}/3$, so the cost is at least $q/12$.

For the final row there are at most t positive and t negative entries, by interlacing from zero. In its Weyl dimension formula reserve the middle $K - 2t = q$ zero coordinates. Pairing one exterior entry of absolute value y with these coordinates contributes at most $\exp F(y)$, since their distances from it are at least $1, \dots, q$ in increasing order. The remaining pairs lie among $2t$ coordinates. Each factor is at most $1 + 2at$, since the spread is at most the sum of all signed-letter degrees. Also $P_k(b) \geq \exp F(b)$. Applying (A.165) to both signs therefore bounds the number of labels on one discarded path by

$$R \exp\left(9t^2 \log(1 + 2at) - \min(qL/(16k), q/12)\right).$$

Here $\binom{2t}{2} \leq 9t^2$ suffices.

At each stage record the first t positive parts and the last t negative parts. These determine the row, including the shorter rows at restriction steps. Each recorded integer lies between 0 and $2at$. The initial row is fixed, so there are at most $(1 + 2at)^{2t^2}$ paths. Multiply the preceding bound by this number and use $q \geq k/2$. □

Proof of 22.16

Proof. Consider one coordinate, with power $P > 1$, digit d, and quotient b. Splitting the factorial into consecutive intervals of length P gives

$$\frac{(d + Pb)!}{d!\, b!\, (P!)^b} = \prod_{h=0}^{b-1} \prod_{j=1}^{P-1} \left(1 + \frac{P}{j} h\right) \prod_{j=1}^{d} \left(1 + \frac{P}{j} b\right).$$

Indeed the multiples $P, 2P, \dots, bP$ contribute $P^b b!$. For $1 \leq j < P$, $v_p(j) < v_p(P)$. Every displayed factor belongs to $1 + p\mathbf{Z}_p$. The factor for $P = 1$ is one.

Write $t_j = P/(pj) \in \mathbf{Z}_p$. Replace p by an indeterminate T in the factors $1 + pt_j h$ and $1 + pt_j b$, keeping t_j fixed. Fix a sign $\epsilon \in \{1, -1\}$. The coefficient of T^a in $\prod_{j<P}(1 + Tt_j h)^\epsilon$, with j starting at one, is a polynomial $V_a(h)$ of degree at most a with coefficients in $\mathbf{Z}_p$. For the negative sign use the formal geometric series. In particular $V_0 = 1$.

Let $H_e(b)$ be the coefficient of T^e in the product over $0 \leq h < b$. Then $H_0 = 1$, $H_e(0) = 0$ for $e > 0$, and

$$H_e(b + 1) - H_e(b) = \sum_{a=1}^{e} V_a(b) H_{e-a}(b).$$

By induction H_e is an integer-valued polynomial on $\mathbf{N}$ of degree at most $2e$. Indeed the right side has degree at most $2e - 1$, and summation from 0 to $b - 1$ takes $\binom{h}{j}$ to $\binom{b}{j+1}$. Integrality of its values also follows directly from the product. Multiplying by the last d factors preserves the degree bound $2e$ for the coefficient of T^e.

In the product over I, the indices contributing to T^e sum to e, so the total degree of that coefficient is at most $2e$, independently of $|I|$. At $T = p$ the terms with $e \geq c$ vanish modulo p^c. The remaining polynomial takes values in $\mathbf{Z}_p$ on $\mathbf{N}^I$. Its coefficients in the product binomial basis are its iterated forward differences at zero and belong to $\mathbf{Z}_p$. Its degree is at most $2(c - 1) \leq 4c$. □

Proof of 22.17

Proof. Fix $g' \in \mathrm{GL}_k(W(\mathbf{F}_{p^{n'}}))$ and put $H = (g')^{-1} g \in \mathrm{GL}_k(W(\mathbf{F}_{p^{n'}}))$ and $H_a = \sigma^a(H) = (g'_a)^{-1} g_a$. Write $u_1(a), \dots, u_k(a) \in O^k$ for the rows of H_a mapped into O, and $u_1^\vee(a), \dots, u_k^\vee(a) \in (O^k)^\vee$ for the rows of

$(H_a^{-1})^{\mathsf{t}}$. Let $\widetilde{W}_a \subseteq O^k$ be the rank-r submodule spanned by $u_1(a), \dots, u_r(a)$ and $\widetilde{W}_a^{\perp} \subseteq (O^k)^{\vee}$ the rank-K submodule spanned by $u_{r+1}^{\vee}(a), \dots, u_k^{\vee}(a)$. Since $H_a H_a^{-1} = I_k$, $\widetilde{W}_a^{\perp}$ annihilates $\widetilde{W}_a$ identically over O. Expanding $s_{g'}$ on the twisted rows gives

$$s_{g'}(v_a^{(g)}) = v_0(a) + \omega v_1(a), \qquad s_{g'}(w_a^{(g)}) = w_0(a) + \omega w_1(a),$$

where $v_0(a) = \sum_{b=1}^{r} X_{a,b}^{\mathrm{vec}} u_b(a) \in \widetilde{W}_a \otimes_O O[X]$, $v_1(a) = \sum_{b=r+1}^{k} X_{a,b}^{\mathrm{vec}} u_b(a)$, $w_0(a) = \sum_{b=r+1}^{k} X_{a,b}^{\mathrm{cov}} u_b^{\vee}(a) \in \widetilde{W}_a^{\perp} \otimes_O O[X]$, and $w_1(a) = \sum_{b=1}^{r} X_{a,b}^{\mathrm{cov}} u_b^{\vee}(a)$. All entries lie in $O[X]$.

We record two congruences in $O[X]$ for any integer $e \geq 1$. First, because $p = \omega^2$ and $p \geq 5$, every term of the binomial expansion of $(a+\omega b)^p - a^p$ is divisible by $p\omega = \omega^3$ or ω^p, hence by ω^2. Induction on e gives $(a+\omega b)^{p^e} \equiv a^{p^e} \pmod{\omega^2 O[X]}$. Second, for any finite sum in $O[X]$, $(\sum_\alpha c_\alpha)^{p^e} \equiv \sum_\alpha c_\alpha^{p^e} \pmod{pO[X]}$, and in $W(\mathbf{F}_{p^{n'}})$ one has $\sigma^e(c) \equiv c^{p^e} \pmod{p}$. Since p is odd, raising the Leibniz expansion of a minor to p^e preserves every permutation sign and yields

$$\big(\det((H_y)_{S,I})\big)^{p^e} \equiv \det\big(((H_y)_{S,I})^{[p^e]}\big) \equiv \det\big((H_{y+e})_{S,I}\big) \pmod{pO}$$

for every h-subsets $S, I \subseteq \{1, \dots, k\}$, and the same congruence holds with $(H_y^{-1})^{\mathsf{t}}$ in place of H_y.

For part (1), integrality $K_c(g; J) \in O[X]$ is immediate from the definition, and the telescoping product of powers after step i gives multidegree p^{z-x_i} at each used letter i. Suppose first that $c > 0$. For $0 \leq h \leq k$ and $a \in \mathbf{Z}$, let $M_{a,h} \subseteq \bigwedge^h (O[X]^k)$ be the $O[X]$-span of $u_S(a) = u_{s_1}(a) \wedge \cdots \wedge u_{s_h}(a)$ over subsets S of size h with $S \subseteq \{1, \dots, r\}$ if $h \leq r$ and $\{1, \dots, r\} \subseteq S$ if $h > r$. Writing any $A \in M_{y,h}$ as $\sum_S f_S u_S(y)$, the two congruences above show that for every $e \geq 1$ and $B \in \bigwedge^h (O[X]^k)$, the coordinatewise p^e-th power satisfies

$$(A + \omega B)^{[p^e]} \equiv A^{[p^e]} \equiv \widetilde{A} \pmod{\omega^2 O[X]}, \qquad \widetilde{A} = \sum_S f_S^{p^e} u_S(y+e) \in M_{y+e,h}.$$

If no letter is used in c, then $K_c(g; \varnothing) = 1$ and $\mathrm{oc}(c) = 0$, so (1a)–(1d) are immediate. Otherwise we prove by induction over the used letters $i_1 < \cdots < i_m$ ($m \geq 1$) in column c that after step i_t (at position $y_t = x_{i_t}$ and height h'_t), letting $d_t = \sum_{s \leq t,\ i_s \text{ costly}} p^{y_t - y_s}$, one has $s_{g'}(K^{(i_t+1)}) = \omega^{d_t} Q_t$ with $Q_t \in M_{y_t,h'_t} + \omega \bigwedge^{h'_t} (O[X]^k)$. Before the first used letter, $h_0 = 0 < r$ and $K^{(0)} = 1 \in M_{y,0}$; the first used step must rise from 0 to 1, is not costly, and gives $v_0(y_1) + \omega v_1(y_1) \in M_{y_1,1} + \omega O[X]^k$. At step i_t ($t \geq 2$) with $e = y_t - y_{t-1} \geq 0$, raising each coordinate of $s_{g'}(K^{(i_{t-1}+1)}) = \omega^{d_{t-1}} Q_{t-1}$ to p^e produces $\omega^{d_{t-1} p^e} Q_{t-1}^{[p^e]}$. Write ι_w for right contraction (matching the coordinate formula in the statement).

- If step i_t is a non-costly vector rise ($h'_{t-1} < r$), then $Q_{t-1}^{[p^e]} = \widetilde{A} + \omega C$ with $\widetilde{A} \in M_{y_t,h'_{t-1}}$, and right wedging with $v_0(y_t) + \omega v_1(y_t)$ yields $\widetilde{A} \wedge v_0(y_t) + \omega(\cdots)$ with $\widetilde{A} \wedge v_0(y_t) \in M_{y_t,h'_t}$.
- If step i_t is a costly vector rise ($h'_{t-1} \geq r$), the hypothesis gives $e \geq 1$, so $Q_{t-1}^{[p^e]} = \widetilde{A} + \omega^2 C$ with $\widetilde{A} \in M_{y_t,h'_{t-1}}$. Because every basis wedge in $M_{y_t,h'_{t-1}}$ already contains $u_1(y_t) \wedge \cdots \wedge u_r(y_t)$ and $v_0(y_t) \in \widetilde{W}_{y_t} \otimes_O O[X]$, one has $\widetilde{A} \wedge v_0(y_t) = 0$ identically in $O[X]$. Thus $(\widetilde{A} + \omega^2 C) \wedge (v_0(y_t) + \omega v_1(y_t)) = \omega(\widetilde{A} \wedge v_1(y_t) + \omega(\cdots))$, increasing the ω-order by 1 to $d_t = d_{t-1} p^e + 1$ with $\widetilde{A} \wedge v_1(y_t) \in M_{y_t,h'_t}$.
- If step i_t is a non-costly covector fall ($h'_{t-1} > r$), then $Q_{t-1}^{[p^e]} = \widetilde{A} + \omega C$ with $\widetilde{A} \in M_{y_t,h'_{t-1}}$. Since $w_0(y_t) \in \widetilde{W}_{y_t}^{\perp} \otimes_O O[X]$ annihilates $\widetilde{W}_{y_t}$, right contracting with $w_0(y_t) + \omega w_1(y_t)$ preserves the factor $u_1(y_t) \wedge \cdots \wedge u_r(y_t)$ in the leading term, giving $\iota_{w_0(y_t)}(\widetilde{A}) \in M_{y_t,h'_t}$ modulo ω.
- If step i_t is a costly covector fall ($h'_{t-1} \leq r$), the hypothesis gives $e \geq 1$, so $Q_{t-1}^{[p^e]} = \widetilde{A} + \omega^2 C$ with $\widetilde{A} \in M_{y_t,h'_{t-1}} = \bigwedge^{h'_{t-1}} (\widetilde{W}_{y_t} \otimes_O O[X])$. Because $w_0(y_t)$ annihilates $\widetilde{W}_{y_t}$, one has $\iota_{w_0(y_t)}(\widetilde{A}) = 0$ identically in $O[X]$. Right contracting with $w_0(y_t) + \omega w_1(y_t)$ gives $\omega(\iota_{w_1(y_t)}(\widetilde{A}) + \omega(\cdots))$, increasing the ω-order by 1 to $d_t = d_{t-1} p^e + 1$ with $\iota_{w_1(y_t)}(\widetilde{A}) \in M_{y_t,h'_t}$.

After the last used step $y_{\mathrm{last}} = y_m$, raising each coordinate of $\omega^{d_m} Q_m$ to $p^{z - y_{\mathrm{last}}}$ gives $\omega^{d_m p^{z-y_{\mathrm{last}}}} Q_m^{[p^{z-y_{\mathrm{last}}}]}$, where $d_m p^{z-y_{\mathrm{last}}} = \mathrm{oc}(c)$. This proves (1b) for $c > 0$. For $c \leq 0$, replacing $(O^k, \widetilde{W}_a, r)$ by $((O^k)^{\vee}, \widetilde{W}_a^{\perp}, K)$ and height h_t by dual height $k - h_t$ interchanges vector and covector steps and gives the identical induction.

To prove (1c), specialize the preceding induction to $g' = g$, where $H = I_k$, so $v_0(a), w_0(a)$ are the nonnormal coordinate tuples and $v_1(a), w_1(a)$ are the normal coordinate tuples. The four bullets show that at each used step i_t, the scaled reduction $\overline{Q}_t \in (O/\omega O)[X]$ is obtained from $\overline{Q}_{t-1}^{[p^e]}$ by applying the nonnormal update at non-costly steps and the normal update at costly steps. Because $O/\omega O$ has characteristic p and the coefficients of K_c^{sep} lie in $\mathbf{F}_p$, raising any polynomial in $\mathbf{F}_p[X] \subseteq (O/\omega O)[X]$ to the p^e-th power coincides with raising every variable entrywise to the p^e-th power. Induction over the used steps therefore gives $\overline{Q}_t = \mathrm{pow}_{y_t}(\overline{K_c^{\mathrm{sep},(\mathrm{i_t}+1)}})$, and the final raising by $p^{z-y_{\mathrm{last}}}$ gives $\overline{Q_m^{[p^{z-y_{\mathrm{last}}}]}} = \mathrm{pow}_z(\overline{K_c^{\mathrm{sep}}(J)})$. Multiplying by $\omega^{\mathrm{oc}(c)}$ proves (1c). For (1d), the strict-increase hypothesis forces all costly positions in c to be distinct integers in $[x_0, z]$, so $\mathrm{oc}(c) \le \sum_{m=0}^{z-x_0} p^m \le \frac{p}{p-1} p^{z-x_0} \le 2p^{z-x_0}$.

For part (2), each $x - y_a \ge 1$, so the congruences above applied to each row $s_{g'}(v_{y_a}^{(g)}) = v_0(y_a) + \omega v_1(y_a)$ give

$$s_{g'}(A_x(g)) \equiv T_x H_{x,[1,r]} \pmod{\omega^2 O[X]},$$

where $(T_x)_{a,b'} = (X_{y_a,b'}^{\mathrm{vec}})^{p^{x-y_a}}$ $(1 \le a, b' \le r)$ and $H_{x,[1,r]}$ consists of the first r rows of H_x. Modulo ω, the first r rows of $s_{g'}(B_{x,b}(g))$ and its $(r+1)$-st row $v_0(x)$ all lie in the r-dimensional subspace $\widetilde{W}_x \otimes (O/\omega O)[X]$, so they are linearly dependent and $s_{g'}(B_{x,b}(g)) \equiv 0 \pmod{\omega O[X]}$. Likewise, modulo ω every row of $s_{g'}(A_x(g))$ lies in $\widetilde{W}_x \otimes (O/\omega O)[X]$ while $s_{g'}(w_x^{(g)}) \equiv w_0(x) \in \widetilde{W}_x^{\perp} \otimes (O/\omega O)[X]$, so $s_{g'}(A_x(g)(w_x^{(g)})^{\mathrm{t}}) \equiv 0 \pmod{\omega O[X]}$, hence $s_{g'}(P_{x,b}(g)) \equiv 0 \pmod{\omega O[X]}$. Under s_g (where $H_x = I_k$), $s_g(M_x(g)) = T_x$ exactly, $s_g(D_x(g)) = \det T_x$, and for $c > r$ entry (a, c) of $s_g(A_x(g))$ is $\omega^{p^{x-y_a}}(X_{y_a,c}^{\mathrm{vec}})^{p^{x-y_a}}$. Expanding $s_g(B_{x,b}(g))$ along its last column gives $\omega s_g(D_x(g)) X_{x,b}^{\mathrm{vec}}$ plus terms of the form $\pm\omega^{p^{x-y_a}}(X_{y_a,b}^{\mathrm{vec}})^{p^{x-y_a}}$ times nonnormal cofactors; expanding $s_g(P_{x,b}(g))$ gives $\omega s_g(D_x(g)) X_{x,b}^{\mathrm{cov}}$ plus terms $\sum_{a,c>r}(\mathrm{adj}\, T_x)_{b,a}\omega^{p^{x-y_a}}(X_{y_a,c}^{\mathrm{vec}})^{p^{x-y_a}} X_{x,c}^{\mathrm{cov}}$. Since $p^{x-y_a} \ge p \ge 5$, both congruences hold modulo $\omega^2 O[X]$ and every error term has normal degree equal to its ω-order.

For part (3), the contraction $E_x = \sum_{c=1}^{k} X_{x,c}^{\mathrm{vec}} X_{x,c}^{\mathrm{cov}}$ is invariant under g_x and g_x', while $\mathrm{diag}(I_r, \omega I_K)\,\mathrm{diag}(\omega I_r, I_K) = \omega I_k$; hence $s_{g'}(E_x) = \omega \sum_{c=1}^{k} X_{x,c}^{\mathrm{vec}} X_{x,c}^{\mathrm{cov}}$ identically for every g'. Raising to $P = p^e$ gives the stated identity, and the multinomial congruence modulo $p = \omega^2$ gives the congruence modulo $\omega^{P+2} O[X]$.

For part (4), since $O/\omega O$ is a field, the ideal $\omega O[X]$ is prime in $O[X]$, so the Gauss valuation v_ω on $O[X]$ is additive on products. Because $s_{g'}$ is a ring homomorphism, $\mathrm{ord}_{g'}(FG) = \mathrm{ord}_{g'}(F) + \mathrm{ord}_{g'}(G)$ for all $F, G \in O[X]$. Multiplying the lower bounds from parts (1)–(3) with the order m of ω^m, order ≥ 0 of integral nonnormal factors, and order $\ge c_0$ of the localizer yields the stated bound. □

Proof of 22.18

Proof. Fix an incoming edge e, write $K = k - r$, $c' = c_* + o$, and normalize $v(p) = 1$, $v(\omega) = 1/2$. A source family on e has target tests for its primitive combinations and satisfies the bounds

$$\begin{aligned} &T = C_t N \left(C N_1^2 \log(a_0) \frac{a_0}{pk} + a_0 p^{-L_1} + k p^{G'} \right), \\ &t_S \le T + o, \quad t_T \le T + N c_t + o, \quad D_e \le C_D N k^2 \log_p a_0, \\ &n_e \ge (1 - C_r N r^2 k / Y)(1 - k c_t / a_0)^N \dim V \, p^{-n' r K}. \end{aligned} \tag{A.166}$$

It suffices to treat $CNr^2 k < Y$.

Lean: `Formalism.MainNotions.Pf14.EdgeStep` (`Formalism.SharpRestriction.GrowthBounds.ScaleEdges.ManyEdges`, lines 68–84)
Lean: `Formalism.MainNotions.Pf14.edgeStep_of` (`Formalism.SharpRestriction.GrowthBounds.ScaleEdges.InitialLevelMonoWeights`, lines 30–180)

Number the base-weight rows from 1. At a mark split off $d_s = \lambda_{s,1} - \lambda_{s,2}$. Let $B_k(d) = \binom{d+k-1}{k-1}$, $P = \prod_{s \in S} B_k(d_s)$, and B be the product of the residual dimensions. The Weyl formula [86, Lecture 24] gives at each mark the ratio

$$\prod_{j=2}^{k} \frac{d_s + j - 1}{d_s + \lambda_{s,2} - \lambda_{s,j} + j - 1} \ge 1 - k c_t / a_0.$$

Here $d_s \ge a_0$ and the residual spread is at most c_t. Negated reversal at odd marks preserves dimension. Hence $PB \ge (1 - k c_t / a_0)^N \dim V$.

Lean: `Formalism.MainNotions.Pf14.top_count` (`Formalism.SharpRestriction.GrowthBounds.ScaleEdges.RemainingWeightVectorCount`, lines 30–81)
Lean: `Formalism.MainNotions.TestSideCounting.card_all` (`Formalism.SharpRestriction.GrowthBounds.DualFockLinear.TestSideCounting`, lines 321–397)

Lift the marks periodically to $\mathbf{Z}$. In each marked tile take the middle r of its $3r$ even marks, $f_{j,0} < \cdots < f_{j,r-1} = z_j$. If there are J_f marked tiles in a period, extend by $f_{j+J_f,i} = f_{j,i} + n'$. Segment j contains the marks in $[f_{j-1,0}, z_j]$, with the r marks of frame $j-1$ as its left frame. The gaps in a marked tile and $L_2 < u_1$ separate its middle frame from the adjacent marks. Thus every letter after the left frame has r even predecessors in the segment. The oldest is more than L_1 steps back, since otherwise a window of length L_2 would contain those r marks. The full-tile condition bounds each segment length by G', including the segment crossing the incomplete tile, and each has at most $N_s = \min(N_1, 2N)$ letters.
Lean: `Formalism.MainNotions.Pf14.frames_of` (`Formalism.SharpRestriction.GrowthBounds.ScaleEdges.FramesOf`, lines 25–56)

At a letter use r plane and K complementary variables. Use plane variables on the left frame. The normal variables are the complementary ones at even letters and the plane ones at odd letters. The segment substitution raises each variable at x to $M_x = p^{z_j - x}$ at its position modulo n'. For a set of letters I, let v_I, l_I be the least and greatest exponents occurring under a generic conjugate of $\operatorname{diag}(T, 1, \dots, 1)$ on I, with inverse transpose on covectors. Use $+\infty, -\infty$ for zero. Set $Q_* = p^{G'}$, $\alpha_j = \mathbf{1}_{j=0} + rK(p^{z_{j+1}-z_j} - 1)$, and

$$m_* = (N_1 + 1)((2Q_* + 3)c' + kQ_* + c' \max_j \alpha_j + 3c').$$

For a non-left-frame letter x let y be its oldest of r even predecessors, $b_x = d_{y \bmod n'}/M_y$, and $m_x = m_* + \lfloor b_x/(k^2 Y) \rfloor$. Span products AB' of a plane and a complementary polynomial with $v_{<x}(A) \geq b_x/2$. At even x require $l_{\leq x}(B') + m_x \leq v_{<x}(A)$, and at odd x require $l_{<x}(B') + m_x \leq v_{\leq x}(A)$. Impose the normal degree bound

$$M_x^{-1} \left(C_{\mathcal{L}} N_s^2 \log(a_0) \frac{a_0}{pk} + \epsilon_x a_0 p^{-L_1} \right), \qquad \epsilon_x = 10 \text{ at even } x, \quad \epsilon_x = 1 \text{ at odd } x. \tag{A.167}$$

Here $C_{\mathcal{L}}$ is absolute. A symbol is a sum of products of the substituted segment polynomials and residual monomials with exponents below M_x at the non-left-frame occurrence. Its multidegree is what remains after reserving the fixed determinant factor below. The left frame uses no normal variables, and the residual monomial contributes at most kQ_* per row. Thus (A.167) gives the bound T.
Lean: `Formalism.MainNotions.Pf14.symbolSpaces` (`Formalism.SharpRestriction.GrowthBounds.ScaleEdges.SymbolRealisableBound`, lines 10–21)
Lean: `Formalism.MainNotions.SelectionLemmas.Pf.gen_bound` (`Formalism.SharpRestriction.GrowthBounds.SegmentDegree.StrictMonoIntExists`, lines 524–587)

For the count put

$$R_j = \left\lfloor \log_p \left\lfloor \frac{a_0}{8Yk^2 N_s p^{z_j - f_{j-1,0}}} \right\rfloor_+ \right\rfloor, \quad M_{j,x}^c = p^{z_j - x + R_j}, \quad Q_j = p^{z_{j+1} + R_{j+1} - z_j - R_j}.$$

Use zero for the rounded logarithm of zero. The shifted endpoints are the maxima of z_j and $f_{j-1,0} + \lfloor \log_p(a_0/(8Yk^2 N_s)) \rfloor$, so increase and have total increment n' per period. In particular $\prod_j Q_j = p^{n'}$. In addition to the fixed lifting factor in (A.169), reserve a determinant for each non-left-frame occurrence x. In the unpowered segment variables it is the determinant of the last r preceding vector letters on their plane coordinates, raised to $s_x = \lfloor b_x/(k^2 Y) \rfloor$. Apply the unshifted segment substitution to these determinants and multiply over occurrences. At a mark y their total degree is $\sum_{j,x} s_x \sum_v \mathbf{1}_{v \equiv y \bmod n'} p^{z_j - v}$, where v runs through those r predecessors in segment j. Let a'_y be the marked degree after subtracting both reserves. The second reserve is used when converting the coarse symbols back to the unshifted prefix spaces. Its reduction is a product of nonzero determinants. At each non-left-frame occurrence sum over coordinate residues $0 \leq u_c < M_{j,x}^c$ with $\sum_c u_c \leq a'_x$ and $\sum_c u_c \equiv a'_x \bmod M_{j,x}^c$. The remaining degree is $h = (a'_x - \sum_c u_c)/M_{j,x}^c$. At a right-frame slot choose $0 \leq e_{ji} < rQ_j$ in the class of h modulo Q_j. The left-frame degree of the next segment is $(h - e_{ji})/Q_j$.
Lean: `Formalism.MainNotions.SelectionLemmas.Pf.Dat.Qc` (`Formalism.SharpRestriction.GrowthBounds.SegmentDegree.Coarse`, lines 43–44)
Lean: `Formalism.MainNotions.SelectionLemmas.Pf.Dat.bfam` (`Formalism.SharpRestriction.GrowthBounds.SegmentDegree.RestCountOf`, lines 43–50)

For fixed digits use 22.13. Before the right frame assign degrees $b_i = h_i$ to the interior letters and $b_i = [B_w - m_0]_+$ to the left-frame letters. At the successive right-frame letters perform the r restriction steps. Equivalently, at the j-th such step adjoin j leading entries $b_* = \sum b_i$ to the restricted row. This gives a vector strip of uniquely determined degree. On the first r coordinates use the highest tableau and on the last K use the freely chosen label of 22.13. The path polynomials are products of the exterior columns described below, with monomial labels in the boundary coordinates. Keep all interior variables and all complementary variables. Number the right-frame rows and plane coordinates from 0 to $r-1$, and let $\mathscr{O}_j$ consist exactly of the plane variables $X_{f_{j,t},c}$ with $c \neq r-1-t$. Apply the substitution killing $\mathscr{O}_j$ to the path-polynomial factor, keeping its boundary monomial as a separate factor. Write q for a path together with its tableau and boundary labels, and $u(q)$ for its boundary exponent. The successive right-frame vector strips give

$$X^{u(q)} \operatorname{kill}_{\mathscr{O}_j}(\Psi_{\mathrm{pl}}(q)\Psi_{\mathrm{comp}}(q)) = \pm X^{u(q)+d(q)}\Psi_{\mathrm{pl}}^{\mathrm{cut}}(q)\Psi_{\mathrm{comp}}(q).$$

Here $d(q)$ counts the rises assigned to the surviving antidiagonal frame coordinates, and the cut plane path ends before the right frame. This formula follows column by column: at a right-frame rise the surviving coordinate wedges into its assigned position, contributing that variable and its permutation sign. The complementary path is unchanged. The two remaining factors use disjoint alphabets and no right-frame plane variable. In each alphabet the distinct labelled path polynomials are independent over the residue field. To see this, make a generic change of frame and extract a maximal terminal weight in a proposed relation. The flag-minor products for its standard tableaux have distinct diagonal leading monomials. For each tableau, the extreme monomial in the last letter of the canonical path polynomial records the absolute row increments of that strip, with coefficient equal up to sign to the preceding canonical polynomial. Removing letters successively recovers the path. Independence in the two alphabets gives independence of their distinct products. In each active plane column the complement of its cut tableau label records the right-frame rises, with coordinate c rising at step $r-1-c$. These sets recover all appended strips and hence $d(q)$. Subtracting $d(q)$ from the boundary exponent recovers $u(q)$. Thus the pairs consisting of the boundary monomial and the remaining path-basis element are distinct.

To pass back to the original polynomials, scale every variable of $\mathscr{O}_j$ by a new variable z. In a putative nonzero linear relation choose the least $\mathscr{O}_j$-degree of its boundary monomials. The coefficient of that power of z is a nonzero relation among precisely the specialized polynomials above, a contradiction. This argument works over the residue field and gives a unit maximal coefficient minor over O. On the left frame multiply by the determinant of the preceding frame to power $m_0 = kN_s + 1$ and by the compensating first-coordinate monomials of exponents $h_{\mathrm{left}} - B_w$, where

$$B_w = \left\lfloor a'_x / M^c_{j,x} \right\rfloor - (k + r + 1).$$

The left-frame degree bound is $h_{\mathrm{left}} \geq B_w$. These factors are common within the degree class and their reductions are nonzero, so independence is preserved. For products over segments, the exterior-column tensors are independent for fixed classes. Distinct residue arrays are recovered before quotienting by each $M^c_{j,x}$. On a shared frame the first r exponents of the right-frame polynomial are below Q_j. Their residues recover the right-frame part and the quotients recover the left-frame part. The other K coordinates of the shared frame belong to the right-frame part. This gives independence across the boundary choices.
Lean: `Formalism.MainNotions.SelectionLemmas.Pf.restInd_of` (`Formalism.SharpRestriction.GrowthBounds.SegmentDegree.ScaleSCoeffZero`, lines 162–188)

Apply 22.14 with t the number of segment letters, $v_i = a_0/M^c_{j,i}$, and $L = C_{\mathcal{L}} N_s^2 \log a_0$. The left frame and the last r letters before the right frame are vector letters by the choice of the middle marks of a marked tile. Earlier letters of the same parity are at least two positions back. The divided degrees before the right frame lie between $(2/5)a_0/M^c_{j,i}$ and $2a_0/M^c_{j,i}$: subtracting the reserved degree leaves at least $9a_0/10$ before division, and the digit sum and left-frame reserve after division cost at most $k+r+2+m_0$. The definition of R_j gives $a_0/M^c_{j,i} \geq 8Yk^2N_s$ when the rounded logarithm is positive. When it is zero, use $M^c_{j,i} \leq p^{G'}$ and the last size hypothesis instead. These bounds give $b_i \geq k$, the growth factor $p^2/5$, and the earlier-degree bound $4v_i/p$. Since $t \leq N_s$, enlarging C gives $K - 2t \geq k/2$. At a covector the plane degree removed is at most the total decrease of positive parts, since the first r entries are nonnegative. At a vector, interlacing bounds the increase of positive parts in the complementary row by the preceding plane entry at position r. This entry is at most $2b_{o(i)}$: interlace back across the intervening vector strips and sum vector degrees through the oldest one, using the geometric growth. The decrease in absolute negative parts is at most ξ_i. The retained

paths thus have complementary increase at most $2b_{o(i)}+\xi_i$ at a vector and plane decrease at most ξ_i at a covector, as required. Finally $1+2a_0t\le a_0^2$ and $\min(L,k)\ge 1088N_s^2\log a_0$. The exceptional fraction in 22.14 is at most $a_0^{-N_s^2}$. The unrestricted labelled count there supplies exactly the asserted product of binomials. For a shared slot the path determines a strip degree $d\le 2b_*$ by interlacing: after deletion of one leading coordinate the change in the shorter row sum lies between its former smallest and largest entries, both in $[-b_*,b_*]$. The reserve satisfies $4Yb_*\le Q_j$. Indeed, with $D=z_j-f_{j,0}$, summing the distinct powers from letters before the right frame gives $b_*\le 4a_0p^{-R_j-D-L_1-1}$. The definition of the next rounded logarithm gives $p^{R_{j+1}}>a_0/(8pYk^2N_sp^{z_{j+1}-f_{j,0}})$. Dividing by Q_j gives $4Yb_*/Q_j\le 128Y^2k^2N_sp^{-L_1}\le 1$. Apply 22.15 with $b=b_*$. The exponent array from that lemma is the boundary monomial label, and $e_{ji}=d+\sum v_i$. Its contribution is at least $Q_j^{r-1}(1-r/(2Y))$ per slot. Thus the comparison factor for a shared row, after restoring the divided scale and moving its left-frame count to the preceding segment, is

$$B_k([B_w-m_0]_+)\,Q_j^{r-1}(1-r/(2Y))(M_{j,x}^c)^{k-1}Q_j^K,$$

where B_w uses segment $j+1$. For an interior row of reduced degree $d=a_x'$ and scale $M=M_{j,x}^c$, the sum needed here is exactly

$$\sum_{\substack{0\le u_c<M,\ |u|\le d\\ |u|\equiv d\pmod M}} B_k((d-|u|)/M)=B_k(d).$$

Indeed, coordinatewise Euclidean division gives a bijection from the exponent vectors of degree d to the pairs counted on the left. At a shared slot only the count of residues is needed. There are M^{k-1}, since the last digit is determined by the other digits and the congruence, and $kM\le a_x'$ makes every such choice admissible. For fixed residue rows the allowed boundary totals are chosen separately at each right frame. The lower bound at the next left frame uses B_w, independent of the preceding choice of total. Thus summing the product over boundary totals gives the product of the segment sums. Summing next over the residue rows factors into the row sums above.

Here are the two dimension comparisons. Write d_x for the original degree. In the fixed factor (A.169), a determinant contributes degree at most Q_* in a given row. There are at most N predecessor determinants and a row occurs in at most one frame determinant. Since $\alpha_j\le 1+rkQ_*$, the charge of this factor is at most $Q_*N((2Q_*+3)c'+kQ_*)+Q_*(2+rkQ_*)c'$, which is at most $a_0/(kY)$ by the last size hypothesis, on enlarging the absolute constant. For the other reserve, a row at y among the predecessors contributes $s_xp^{z_j-y}\le 2a_0/(k^2Y)$, since the oldest predecessor is at or before y. There are at most N_s such occurrences per segment and at most two segments containing the row. Hence $a_x'\ge d_x-5a_0/(kY)$. In the case under consideration $Y>CNr^2k$, so in particular $a_x'\ge 9a_0/10$. Each factor of the binomial product for $B_k(a_x')/B_k(d_x)$ is at least $1-5/(kY)$, and their product is at least $1-5/Y$.

For a shared row put $M'=M_{j,x}^cQ_j$, its scale in the next segment. From the definition of the rounded logarithm, $M'\le\max(p^{G'},a_0/(8Yk^2N_s))=a_0/(8Yk^2N_s)$, using the last size hypothesis for the equality. Thus $a_x'/M'\ge(36/5)Yk^2N_s$. For $b=[B_w-m_0]_+$, the floor and the two subtractions give $a_x'-M'b\le(k+r+2+m_0)M'\le 4kN_sM'\le a_x'/(kY)$. For $1\le i<k$ it follows that $M'(b+i)/(a_x'+i)\ge 1-1/(kY)$. Multiplication gives $(M')^{k-1}B_k(b)\ge(1-1/Y)B_k(a_x')$. The scale restoration has $r-1+K=k-1$ at each of the r shared slots. Put $A_0=1-a_0^{-N_s^2}$. The exceptional fraction in the segment count of 22.14, with the parameters just checked, is at most $a_0^{-N_s^2}$ in each of the J_f segments. Removing Q_j^{rK} from each segment's scale factor and multiplying gives retained dimension at least

$$\frac{P}{p^{n'rK}}A_0^{J_f}\big((1-5/Y)(1-1/Y)(1-r/(2Y))\big)^N. \tag{A.168}$$

Indeed $N_s\ge 1$ and the size hypotheses give $a_0\ge 32Y\ge 1$, so $a_0^{N_s^2}\ge 32Y$ and $1-1/(32Y)\le A_0\le 1$.

Lean: `Formalism.MainNotions.SelectionLemmas.Pf.Dat.firstSumFactor` (`Formalism.SharpRestriction.GrowthBounds.SegmentDegree.FamilyDegreesAdmissible`, lines 43–43)

Lean: `Formalism.MainNotions.SelectionLemmas.Pf.first_sum_factor_bounds` (`Formalism.SharpRestriction.GrowthBounds.SegmentDegree.FamilyDegreesAdmissible`, lines 139–150)

Since $J_f\le N$, Bernoulli's inequality gives $A_0^{J_f}\ge 1-N/(32Y)$. For factors $1-u_i$ in $[0,1]$, induction gives $\prod_i(1-u_i)\ge 1-\sum_i u_i$. Applying this directly to all the factors in (A.168) bounds their product below by $1-N(1/32+5+1+r/2)/Y\ge 1-Nr^2k/Y$, since $k>r\ge 2$.

Lean: Formalism.MainNotions.SelectionLemmas.Pf.restSeg_of (Formalism.SharpRestriction.GrowthBounds.SegmentDegree.GoodPathWeight, lines 108–139)
Lean: Formalism.MainNotions.SelectionLemmas.Pf.secondPartClosed (Formalism.SharpRestriction.GrowthBounds.SegmentDegree, lines 33–36)
Lean: Formalism.MainNotions.SelectionLemmas.Pf.restSum_of (Formalism.SharpRestriction.GrowthBounds.SegmentDegree.SumTotalLoss, lines 101–249)
Lean: Formalism.MainNotions.SelectionLemmas.Pf.sum_total_loss (Formalism.SharpRestriction.GrowthBounds.SegmentDegree.SumTotalLoss, lines 11–54)

Choose a Witt matrix g for the edge. Write g_x for its x-th Frobenius conjugate. In adapted coordinates the edge substitution s_g scales the normal entries by ω. In the original coordinates its row substitutions are $X_x \operatorname{diag}(I_r, \omega I_K) g_x^{-1}$ at even letters and $X_x \operatorname{diag}(\omega I_r, I_K) g_x^{\mathrm{t}}$ at odd letters. Let Δ_j be the first r-column minor of frame j, with row powers $p^{z_j - x}$. Let D_x be the first r-column determinant of the r preceding even rows in reverse order, with row powers p^{x-y}. With $R_0 = \prod_j \Delta_j^{\alpha_j}$ the fixed factor is

$$U = R_0^{c'} \prod_j \Delta_j^{c'} \prod_{x \in S} D_x^{(2Q_* + 3)c' + kQ_*}. \tag{A.169}$$

It has normal degree zero and nonzero reduction, so multiplication by its reduction is injective. Choose symbols giving a basis of the reduced span in (A.168). Their coefficient matrix has a unit maximal minor. To express these symbols at the unshifted scale, take the ordinary p^{R_j}-th power of each coarse plane and complementary path polynomial in its segment variables. Write each digit as $u_c = d_c + M_x e_c$, with $0 \le d_c < M_x$ and $0 \le e_c < p^{R_j}$. Keep d_c in the residual monomial and multiply the powered factors by the plane and complementary parts of the monomial with exponents e_c, respectively. Multiply the plane factor by the reserved predecessor determinants. The powers multiply the prefix weights by p^{R_j}. The high digits cost at most $kN_s p^{R_j}$ in the prefix gap, so the left-frame reserve $(kN_s + 1)p^{R_j}$ leaves at least $p^{R_j} > m_*$ under the size hypotheses. The predecessor determinant supplies the further s_x. Expand the resulting plane and complementary factors over O in the rank-r and rank-K path bases of Definition 22.13 (constructed at power one as in Lemma 22.17(1)). Reduction modulo ω gives the counted symbols by Frobenius.
Lean: Formalism.MainNotions.SelectionLemmas.Pf.Dat.Aw (Formalism.SharpRestriction.GrowthBounds.SegmentDegree.ColumnWordReductionOfFour, lines 27–29)
Lean: Formalism.MainNotions.SelectionLemmas.Pf.Dat.Gw (Formalism.SharpRestriction.GrowthBounds.SegmentDegree.ColumnWordReductionOfFour, lines 31–33)
Lean: Formalism.MainNotions.SelectionLemmas.Pf.generator_modulo_uniformiser_of_adm (Formalism.SharpRestriction.GrowthBounds.SegmentDegree.DiscriminantNonzeroModUniformiser, lines 13–23)
Lean: Formalism.MainNotions.SelectionLemmas.Pf.restRank_of (Formalism.SharpRestriction.GrowthBounds.SegmentDegree.Coarse, lines 128–160)
Lean: Formalism.MainNotions.SelectionLemmas.Pf.restSel (Formalism.SharpRestriction.GrowthBounds.SegmentDegree.SelIsComplRestrictSupport, lines 106–144)
Lean: Formalism.MainNotions.Pf14.SPtrue (Formalism.SharpRestriction.GrowthBounds.ScaleEdges.SymbolRealisableBound, lines 43–46)

A column path on segment j starts at zero weight, with strictly increasing letter positions ending at z_j, and satisfies $N_s + r \le K$ by $N_1 + 3r \le k$. Add a horizontal strip at an even letter and remove one at an odd letter. At weight μ, column $c \in \mathbf{Z}$ has height $\#\{i : \mu_i \ge c\}$. Build the exterior column polynomial $K_c(g; J)$ over $O[X]$ using the g-twisted rows $v_x^{(g)} = X_x g_x$ (even x) and $w_x^{(g)} = X_x g_x^{-\mathrm{t}}$ (odd x) as in Lemma 22.17(1): at the first change use power one, at each subsequent change x raise the exterior coordinates K_I to the polynomial power p^{x-y} (where y is the preceding change), apply exterior multiplication for rising positive or falling nonpositive columns and contraction for the other changes (writing $u = v_x^{(g)}$ at even x and $u = w_x^{(g)}$ at odd x),

$$\sum_{b \in I} (-1)^{\#\{i \in I : i > b\}} u_b K_{I \setminus b}, \qquad \sum_{b \notin I} (-1)^{\#\{i \in I : i > b\}} u_b K_{I \cup b},$$

and after the last change at y raise to $p^{z_j - y}$. By Lemma 22.17(1), each $K_c(g; J)$ is integral of multidegree $p^{z_j - x}$ at every used letter x, satisfies $\operatorname{ord}_{g'}(K_c(g; J)) \ge \operatorname{oc}(c)$ for every $g' \in \mathrm{GL}_k(W(\mathbf{F}_{p^{n'}}))$, and agrees under s_g modulo $\omega^{\operatorname{oc}(c)+1} O[X]$ with $\omega^{\operatorname{oc}(c)}$ times the entrywise segment substitution of the separated column $K_c^{\mathrm{sep}}(J)$, where $\operatorname{oc}(c) = \sum_{x \text{ costly}} p^{z_j - x} \le 2Q_*$. The product of these entrywise-substituted leading exterior coordinates is the segment symbol.
Lean: Formalism.MainNotions.G.KKstate (Formalism.SharpRestriction.GrowthBounds.ObjPol, lines 538–551)
Lean: Formalism.MainNotions.G.symbolGeneratorData.key_column_leading_bound (Formalism.SharpRestriction.GrowthBounds.PathWeight.LetterWeightPos, lines 283–337)

At each mark $x \in S$ (and in particular at each non-left-frame letter of a segment), the r preceding even marks $y_1 > \cdots > y_r$ in reverse order define $D_x(g)$ (so $s_g(D_x(g)) = D_x$), the bordered minor $B_{x,b}(g)$ (at even x, $r < b \le k$), and the adjugate-pairing factor $P_{x,b}(g)$ (at odd x, $1 \le b \le r$) of Lemma 22.17(2). By Lemma 22.17(2), both $B_{x,b}(g)$ and $P_{x,b}(g)$ belong to $O[X]$, satisfy $\mathrm{ord}_{g'}(B_{x,b}(g)) \ge 1$ and $\mathrm{ord}_{g'}(P_{x,b}(g)) \ge 1$ for every $g' \in \mathrm{GL}_k(W(\mathbf{F}_{p^{n'}}))$, and satisfy $s_g(B_{x,b}(g)) \equiv \omega s_g(D_x(g))X_{x,b}$ and $s_g(P_{x,b}(g)) \equiv \omega s_g(D_x(g))X_{x,b}$ (mod $\omega^2 O[X]$), with every error term having normal degree equal to its ω-order (at least p). The reserve $D_x^{kQ_*}$ pays for the normal entries of the residual monomial, each lifted by its corresponding factor $B_{x,b}(g)$ or $P_{x,b}(g)$.

Lean: `Formalism.MainNotions.G.WittFrameLocaliserOrders.firstKeyMinor` (`Formalism.SharpRestriction.GrowthBounds.WittFrameLocaliserOrders`, lines 3153–3157)

Lean: `Formalism.MainNotions.G.WittFrameLocaliserOrders.pairingKey` (`Formalism.SharpRestriction.GrowthBounds.WittFrameLocaliserOrders`, lines 3159–3161)

Use $L^{c'}$ of 22.9 in place of $R_0^{c'}$, with unshifted frame powers. Define a term S at level h to be a product of an explicit power ω^m, leading exterior column factors of orders $a_c = \mathrm{oc}(c)$, q leading bordered or pairing factors $\omega s_g(D_x(g))X_{x,b}$, integral nonnormal polynomials (including the reserve factors of D_x and Δ_j), and $R_0^{c'}$, with $h = m + \sum_c a_c + q$. Its raw lift F^0 replaces each leading column by $K_c(g;J)$, each leading bordered or pairing factor by $B_{x,b}(g)$ or $P_{x,b}(g)$, each nonnormal variable by its g-twisted coordinate (which has order ≥ 0 under every $s_{g'}$), and $R_0^{c'}$ by $L^{c'}$ (which has order $\ge c'$ on every competing plane by Lemma 22.9). By Lemma 22.17(4), the raw lift F^0 satisfies $\mathrm{ord}_{g'}(F^0) \ge m + \sum_c a_c + q + c' = h + c'$ on every competing plane g'. For a normal component of degree b, the initial term has $m = 0$ and $h = \sum_c a_c + q = b$, where q counts the normal entries of the residual monomial. Keep with each term at level $b + d$ at least $(2Q_* + 3)c' - (2Q_* + 1)d$ copies of D_x at each row and $c' - d$ copies of each Δ_j, after accounting for the initial residual monomial. Expand $s_g(F^0) - S$. Terms of order at least $b + c'$ already meet the precision bound. Every other term has an additional rise $d' \ge 1$. For an exterior column $K_c(g;J)$ of order $a = \mathrm{oc}(c) \le 2Q_*$, every monomial of normal degree t in $s_g(K_c(g;J))$ has coefficient divisible by ω^t because $K_c(g;J) \in O[X]$, and after subtracting its leading part its coefficients are also divisible by ω^{a+1}. Write such a monomial as $\omega^z X^\nu H$ with H integral in the nonnormal coordinates, $|\nu| = t$, and $z = \max(t, a+1)$, absorbing the remaining coefficient into H. Its rise is $e = z - a \ge 1$, and $t \le a + e \le (2Q_* + 1)e$. For a bordered or pairing factor $B_{x,b}(g)$ or $P_{x,b}(g)$ of order 1, Lemma 22.17(2) gives rise $e = p^{x - y_a} - 1 \ge p - 1$ and normal degree $t = e + 1 \le 2e \le (2Q_* + 1)e$. For a localizer error of rise e, choosing replacement minors in the product of 22.9 loses one Δ_j for each replaced copy at frame j; each replacement minor contributes at least one to the order after substitution and each normal entry contributes one power of ω, so both the deficit of Δ_j and the normal degree are at most e.

Expand the product of the columns, bordered and pairing factors, and c' localizers, subtracting the product of their leading terms. Leave unchanged column, bordered, and pairing factors in their leading form. In each remaining summand the rises add. At total rise e there are at most $(2Q_* + 1)e$ exposed normal variables (with explicit ω-power at least the number of exposed variables) and at most e missing factors of each Δ_j. Pair each exposed $\omega X_{x,b}$ with one reserve copy of D_x to form a leading bordered or pairing factor $\omega s_g(D_x(g))X_{x,b}$, keeping any remaining explicit ω-power. Set aside $(2Q_* + 1)e$ copies of each D_x and e copies of each Δ_j (the latter restoring $R_0^{c'}$); unused copies multiply the nonnormal coefficient. The resulting child terms have level $h + e$, preserve every row multidegree, and retain the two prescribed reserves for level $h + e$.

Put $b_* = b + c'$ for this recursion. If $h \ge b_*$, the zero lift gives the required agreement modulo ω^{b_*}. Otherwise let F^0 be the raw lift of a term S at level h. The expansion just obtained gives $s_g F^0 - S = \sum_\nu S_\nu + R$, with $R \in \omega^{b_*} O[X]$ and $h < h_\nu < b_*$. By descending induction in h, choose lifts F_ν of the children to precision b_* and take $F = F^0 - \sum_\nu F_\nu$. Then $s_g F - S = R - \sum_\nu (s_g F_\nu - S_\nu)$ has the required order. By Lemma 22.17(4), F^0 has order at least $h + c' \ge b_*$ on every competing plane, and every subtracted lift F_ν has order at least b_* there by induction. All operations take place over O.

This also treats a target symbol $P_b = (ZU + p^{c'} Y')_b$ with Z in the symbol space and Y' integral. The term $\omega^b p^{c'} Y'_b$ already belongs to $\omega^{b+c'} O[X]$. Projecting to the row multidegree commutes with the substitution. The integral lift F_b therefore satisfies

$$s_g F_b - \omega^b P_b \in \omega^{b+c'} O[X], \qquad s_{g'} F_b \in \omega^{b+c'} O[X] \quad (g' \text{ on a competing edge}). \tag{A.170}$$

Sum $\omega^{-b} F_b$ for $b \le \lfloor T \rfloor$.

Lean: `Formalism.MainNotions.G.WittFrameLocaliserOrders.clean_symbol_order` (`Formalism.SharpRestriction.GrowthBounds.WittFrameLocaliserOrders`, lines 1247–1299)

Lean: `Formalism.MainNotions.G.symbolGeneratorData.core` (`Formalism.SharpRestriction.GrowthBounds.PathWeight.FixedSymbolFactorIntegral`, lines 148–207)
Lean: `Formalism.MainNotions.Pf14.exists_pol_family_of_mem_gen_of_vsDeg_le` (`Formalism.SharpRestriction.GrowthBounds.ScaleEdges.ExistsPolFamilyOfMemGenOfVsDegLe`, lines 140–252)
Lean: `Formalism.MainNotions.Pf14.flatSrc` (`Formalism.SharpRestriction.GrowthBounds.ScaleEdges.FlatSrc`, lines 60–111)
Lean: `Formalism.MainNotions.Pf14.edgeFamOf` (`Formalism.SharpRestriction.GrowthBounds.ScaleEdges.ScaleZpowMono`, lines 48–118)

Multiply the lifts by the following B residual tableaux. Number rows and columns from one, and let ν be the partition of the residual active weight after subtracting its last entry. At a vector mark take the standard tableaux whose leading monomial has exponent ν_1 at matrix entry $(1,1)$. At a covector mark require that the entire first matrix row have exponent zero. Off the marks take the full tableau basis. Thus the dimension factor at a mark is the Weyl dimension in rank $k-1$ of $(\lambda_{s,2},\dots,\lambda_{s,k})$. The additional order and depth per row are bounded by the sum of its first r residual entries minus r times its last entry, at most rc_t at marks and $2rc_b$ elsewhere. Reduce a proposed relation in the adapted coordinates. For each residual tableau tuple i, let Q_i be its coefficient polynomial in the flat letters, including the common factor U. These polynomials have the prescribed flat degree in each factor. The relation in the class variables substitutes the first-column entries for the flat letters at a vector mark, and the signed initial minors omitting one row for those letters at a covector mark. Expand each Q_i in monomials. We recover both the flat monomial and the residual tableau from the leading exponent of their product, using the column-then-row order of 14.12.

At a vector mark the residual first column has total degree ν_1, all at entry $(1,1)$. Subtract this known column from the first column of the product to recover the exponents of all the flat letters. At a covector mark write a_c for the exponent of the minor omitting row c. The leading term of that minor uses the diagonal entry (t,t) precisely when $t<c$, for $1\le t<k$. The residual tableau contributes zero on this diagonal. Indeed its entries omit the letter one, and strict increase down each tableau column makes the entry at depth t at least $t+1$. The diagonal exponents of the product therefore give the tails $\sum_{c>t} a_c$. Adjoining the prescribed total degree and the empty tail recovers every a_c by subtraction. Off the marks the flat degree is zero. At every factor, subtracting the recovered flat exponent leaves the leading exponent of the residual tableau, which determines its rows by 14.12. Different factors have disjoint matrix variables. All these leading coefficients are units, including the cofactor signs. The largest leading monomial of a nonzero term in the expanded relation consequently cannot cancel. Hence every Q_i vanishes over the residue field.
Lean: `Formalism.MainNotions.TestSideCounting.key_inj` (`Formalism.SharpRestriction.GrowthBounds.DualFockLinear.DgFalseDiag`, lines 168–236)
Lean: `Formalism.MainNotions.TestSideCounting.bigInd` (`Formalism.SharpRestriction.GrowthBounds.DualFockLinear.ThAllMonomial`, lines 154–157)

Cancel the common nonzero reduction of U and use the unit coefficient minor of the symbols. All coefficients in the original relation vanish modulo ω. This proves primitivity of a combination with a unit coefficient and the claimed source rank.
Lean: `Formalism.MainNotions.TestSideCounting.primStep_of` (`Formalism.SharpRestriction.GrowthBounds.DualFockLinear.TheoremIIClosedOfTestBound`, lines 13–135)

For the target side write $\lambda=\phi+\eta$, where ϕ is the split-off flat weight, zero off the marks. Factor each edge lattice into its marked and unmarked tensor factors. In the construction of $\Psi_\xi, L_\beta, \rho_\beta$ below, restrict all weights and lattices to the marked factors. Let Ψ_ξ identify functionals with vectors by the inverse normalized factorial form at weight ξ, namely the factorial pairing on each polynomial factor divided by its value $[H,H]$ on the product H of initial principal minors, as in Lemma 10.16. Write

$$k_\xi=\binom{k}{2}\sum_{s:l_s>0}\lfloor\log_p(l_s+k-1)\rfloor,\qquad l_s=\xi^{\mathrm{act}}_{s,1}-\xi^{\mathrm{act}}_{s,k}$$

on the marked factors. Transposing the divided raising operators shows that the image is the divided-power lattice. By 14.12, $p^{k_\xi}R^{\xi}_{e^\vee}\subseteq\Psi_\xi((R^\xi_e)')\subseteq R^{\xi}_{e^\vee}$ for the reversed dual edge $e^\vee$. Put $B_0=Nk^2\log_p a_0$. Since $l_s+k\le a_0^2$, $\kappa=k_\lambda+k_\phi\le 4B_0$.
Lean: `Formalism.InvariantFactorPaths.INST.Top.sandwichLocal` (`Formalism.ThetaLogCoordinates.SandwichModelDomains.FinalW`, lines 50–52)
Lean: `Formalism.L2.G.Model.testSide_of_core` (`Formalism.LocalToGlobal.GroupLattices.FockModels.TestSide`, lines 36–98)
Lean: `Formalism.MainNotions.TestSideCounting.sizeStep` (`Formalism.SharpRestriction.GrowthBounds.DualFockLinear.LogSizeBoundLe`, lines 94–166)

For $\beta\in R^{\eta}_{e^\vee}$, multiplication m_β defines $L_\beta=p^{k_\lambda}\Psi_\lambda^{-1}m_\beta\Psi_\phi$. Its transpose for the normalized edge pairings is ρ_β, given by $T(\rho_\beta x)=p^{A_\eta}(L_\beta T)(x)$. Choose an integral frame of the lattice of the reversed edge

at each translate, as in 14.10. For a dominant active weight ξ, remove the determinant twist ξ_k. In the remaining tableau the number of columns of height $h < k$ is $\xi_h - \xi_{h+1}$. Both this number and the removed twist are additive in $\lambda = \phi + \eta$, also after negated reversal. In a tableau product of minors for λ, assign $\phi_h - \phi_{h+1}$ columns of each height to the first factor and the rest to the second, using the active weights at that translate. Each subtableau retains its column order. Its product of minors is an integral section of the assigned weight after restoring its determinant twist. By the tableau basis of 14.12 this factors a spanning set. Taking products over translates gives the assertion for the tensor lattice. The edge factors are ω^{-A_ξ} in 14.10; the sums of the first r active entries satisfy $A_\lambda = A_\phi + A_\eta$, so this factorization gives the span on $R_{e^\vee}^{\lambda}$ itself.

For each flat factor y just used, $T = \Psi_\phi^{-1}(p^{k_\phi} y)$ belongs to the flat test lattice by the preceding inclusions. Then $L_\beta T = p^\kappa \Psi_\lambda^{-1}(y\beta)$. As $\Psi_\lambda(R'_e) \subseteq R_{e^\vee}^{\lambda}$, these tests span a lattice containing $p^\kappa R'_e$. If every $\rho_\beta x$ belonged to $\omega^{2\kappa+1} R_e^{\phi}$, its normalized edge pairing with every integral flat test would belong to $\omega^{2\kappa+1} O$. By the transpose identity and additivity of A_ξ, the same would hold for the normalized pairing of x with $p^\kappa R'_e$. Since $p^\kappa = \omega^{2\kappa}$, perfection would imply $x \in \omega R_e^{\lambda}$. For primitive x some integral component therefore has exact order $a \le 2\kappa$.

Lean: `Formalism.MainNotions.TestSideCounting.ontoAll` (`Formalism.SharpRestriction.GrowthBounds.DualFockLinear.RestCloudBOfTwo`, lines 23–80)
Lean: `Formalism.MainNotions.TestSideCounting.rhoAtG_of_pair` (`Formalism.SharpRestriction.GrowthBounds.DualFockLinear.RhoAtGOfPair`, lines 74–220)
Lean: `Formalism.MainNotions.TestSideCounting.rhoStep_of_top` (`Formalism.SharpRestriction.GrowthBounds.DualFockLinear.RestCloudBOfTop`, lines 115–132)

Use the exact symbol representatives for the primitive source combination x. For the component of exact order a selected above, write $\rho_\beta x = \omega^a y$. The flat polynomial G of this primitive y in the source edge coordinates has integral coefficients and a unit coefficient. The residual partition contributes words in $X_{s,a}\partial_{s,b}$ of length at most c_t per mark. This follows by differentiating the products of its at most c_t columns. A row substitution conjugates each first-order operator to a linear combination of such operators. The normal-degree bound becomes $\lfloor T \rfloor + N c_t$.

Lean: `Formalism.MainNotions.TestSideCounting.restND_of_adm` (`Formalism.SharpRestriction.GrowthBounds.DualFockLinear.RestNDOfAdm`, lines 14–80)

For a test polynomial Z, put $v_d = v(\prod_s d_s!)$ and $k_d = \sum_{s:d_s>0} k \lfloor \log_p d_s \rfloor$. The integral normalized pairing is

$$\frac{p^{k_d}}{\prod_s d_s!} \sum_\alpha \alpha! \, [X^\alpha] G \, [X^\alpha] Z. \tag{A.171}$$

The valuation of a multinomial coefficient in row s is at most $k \lfloor \log_p d_s \rfloor$, by successive binomial choices and 11.7. For one digit split $\alpha = u + Pb$, where P is a power of p and $0 \le u < P$, the unit $\gamma(u,b) = (u+Pb)!/(u!\, b!\, (P!)^b)$ has valuation zero by the same formula for factorial valuations. Separate the plane and normal variables. For a polynomial H let $v_{\mathrm{F}}(H) = \min_\alpha (v(H_\alpha) + v(\alpha!)/2)$. In the residue selection, the normal isotypic projections and the normal prefixes, the normal variables at every letter are its complementary coordinates. The normal degree used for lifting counts complementary coordinates at even letters and plane coordinates at odd letters. Fix a monomial of G with unit coefficient. At each physical row select its coordinate residues modulo the non-left-frame power M_x, divide out their monomial and replace X^{M_x} by X coordinatewise, and select the plane and normal quotient row degrees of this monomial. Write b_p, b_n for its plane and normal quotient exponent arrays. In the selected block let G_{b_p} be the coefficient at the plane exponent b_p, a polynomial in the normal variables whose coefficient at b_n is a unit. The joint normal isotypic projections on the normal prefixes sum to the identity, so one of them, e_{no}, satisfies $v_{\mathrm{F}}(e_{\mathrm{no}} G_{b_p}) \le v_{\mathrm{F}}(G_{b_p})$. Choose ν_{no} attaining $v_{\mathrm{F}}(e_{\mathrm{no}} G_{b_p})$, and let P_c^{raw} be the plane coefficient at ν_{no} of the block after e_{no} is applied to its normal part. If m minimizes $v_{\mathrm{F}}(P_c^{\mathrm{raw}})$, put $z = [X^m] P_c^{\mathrm{raw}} \, \omega^{v(m!)}$ and $P_c = z^{-1} P_c^{\mathrm{raw}}$. Then $v_{\mathrm{F}}(P_c) = 0$.

Lean: `Formalism.MainNotions.Proposition5Numerics.pick_core` (`Formalism.SharpRestriction.GrowthBounds.DiagonalSupport.PickCore`, lines 23–165)

Projection with denominators. On the multihomogeneous space of a signed alphabet, write $d_{\mathrm{tot}} = \sum_i d_i$ for the sum of the divided plane degrees. For a prefix S_0, the condition on the least signed weight for a conjugate of $\mathrm{diag}(T, 1, \dots, 1)$ selects the isotypic components with last highest-weight entry at least the prescribed threshold. The projection onto this space loses at most $e_C = r^2(1 + \log_p(2d_{\mathrm{tot}} + r))$ in v_{F}.

To see the denominator bound, choose a lattice basis b_i of the irreducible representation L_λ. The invariant projector P_{inv} on $\mathrm{Hom}(L_\lambda, V)$ gives

$$e_\lambda(v) = \dim L_\lambda \sum_i P_{\mathrm{inv}}(b_i^\vee \otimes v)(b_i).$$

On an equivariant copy of L_λ the invariant projection is normalized trace on endomorphisms. On a different isotypic summand the Hom space has zero invariants. This proves the formula. Use the lattice for v_{F}, whose monomial generators have coefficient scale $\omega^{-v(\alpha!)}$. It is preserved by divided root operators. For example, let c be the normalized coefficient for moving $h \le a$ powers from a coordinate of exponent a to one of exponent b. Since $v(\omega) = 1/2$,

$$2v(c) = 2v\binom{a}{h} + v\left(\frac{(a-h)!(b+h)!}{a!\,b!}\right) = v\binom{a}{h} + v\binom{b+h}{h} \ge 0.$$

Thus c is integral. For the Hom lattice take its intersection on the irreducible factor and the corresponding dual lattice. The invariant projection denominator is bounded by $\binom{r}{2}$ times the largest $\lfloor \log_p M \rfloor$ for the root strings, with $M \le 2d_{\mathrm{tot}} + r$. In the extremal-projector expansion each positive root contributes a divided raising and lowering product and the inverse of a binomial coefficient with upper argument at most M. Its valuation cost is at most $\lfloor \log_p M \rfloor$ by 11.7. Evaluation against the lattice basis and multiplication by $\dim L_\lambda$ introduce no further denominator. The sum of the selected e_λ has the same bound by the ultrametric inequality.

The factorial form is contravariant for transposition of the signed operators, so the isotypic projections are self-adjoint. For disjoint sets of letters the actions commute. For nested sets, the small-set projector commutes with the small-set action and with the action on the complement in the large set. Hence it preserves the large-set isotypic spaces and commutes with their projectors. Each product of h such projections therefore loses at most he_C and preserves the pairing against polynomials already in every selected subspace.
Lean: `Formalism.MainNotions.G.GLWeightTheory.isotypic_projector_trace_formula` (`Formalism.SharpRestriction.GrowthBounds.RootWeights.IsotypicProjectorTraceFormula`, lines 21–100)
Lean: `Formalism.MainNotions.G.GLWeightTheory.invariant_denominator_sharp_of` (`Formalism.SharpRestriction.GrowthBounds.RootWeights.InvariantProjectionExtremalFormula`, lines 92–102)
Lean: `Formalism.MainNotions.PlaneGram.PROJ` (`Formalism.SharpRestriction.GrowthBounds.MonomialOrderSegments.LiftBoundOfHypotheses`, lines 168–179)
Lean: `Formalism.MainNotions.PlaneGram.prodProj` (`Formalism.SharpRestriction.GrowthBounds.MonomialOrderSegments.ProjKeepsLetter`, lines 56–112)

Number the right-frame slots from 1 to r, and let I_j be the interior letters of segment j, excluding both frames. The link indexed by (j,m), $1 \le m < r$, is $I_j \cup \{(j,h) : m < h \le r\}$. The letter (j,h) here denotes the shared boundary letter in the original plane alphabet. The links are nested for fixed j and disjoint for different j. In the enlarged alphabet the corresponding set consists of the powered letters of the preceding frame, the ordinary letters of I_j, and the surplus letters of the indicated right-frame suffix. A chain prefix consists of all the preceding powered frame letters together with the indicated initial part of I_j. Put $D_l = J_f(r-1)e_C$, $n_0 = v(\prod_i d_i!)/2$, and $k' = \sum_i r(1 + \log_p(d_i + r))$. On the original plane alphabet fix the low digits d_{xa} of the selected class. In row x, let $M_x = p^{z_j - x}$ at its non-left-frame occurrence. Its factorial unit on plane exponent α is $u(\alpha) = \prod_{x,a} \gamma(d_{xa}, \alpha_{xa})$, with power M_x in each factor. For an exponent m minimizing the normalized P_c, put $t = v(m!)/2$, and let Y_{link} be the link projection of X^m. The preceding projector bound gives $v_{\mathrm{F}}(Y_{\mathrm{link}}) \ge t - D_l$. Apply Lemma 22.19 to the flat polynomial of y, with the selected normal projector and normal index. It gives $\langle P_c, Y_{\mathrm{link}} \rangle = [X^m] P_c\, m!$ of valuation t, and gives the chain bound on every boundary digit component of this same P_c. The transfer to the lifted source representatives uses the precision of (A.170) after the test has been constructed.

Take an integral binomial approximation g to u^{-1} modulo $p^{c'}$ from 22.16. Write D_g for its diagonal action. For all polynomials A, B, the factorial pairing satisfies $v(\langle A, B \rangle) \ge v_{\mathrm{F}}(A) + v_{\mathrm{F}}(B)$, term by term. Thus $\langle P_c, (D_u D_g - I) Y_{\mathrm{link}} \rangle$ has valuation at least $c' + t - D_l > t$. Put $a_{\mathrm{int}} = \lceil n_0 + D_l - t \rceil$. Since every monomial factorial in these row degrees divides $\prod_i d_i!$, the coefficients of $p^{a_{\mathrm{int}}} D_g Y_{\mathrm{link}}$ are integral. Its pairing against P_c with multiplier u has valuation $a_{\mathrm{int}} + t \le n_0 + D_l + 1$. A binomial of degree d in a coordinate exponent acts by $X^d \partial_X^{[d]}$. In any basis each summand of its conjugate lowers the least signed first-coordinate weight by at most d. Thus every link retains its threshold minus $6c'$, using $4c' \le 6c'$ in 22.16.

Let $T_0 = p^{\lceil n_0+D_l-t\rceil} D_g Y_{\mathrm{link}}$. It is integral and has every original link bound with loss $6c'$, as just proved. Lemma 22.20 gives an integral allowable preimage Z, with the same bounds on the enlarged links. Denote recombination by Φ. Thus $\Lambda(Z) = \langle P_c, D_u \Phi Z\rangle = \langle P_c, D_u T_0\rangle$ has valuation at most $n_0 + D_l + 1$. For an exponent array ζ on this alphabet, let $\bar\zeta$ be its array on the original letters. It keeps the interior exponents and assigns exponent $\rho + P_t\beta$ to a boundary variable with remainder exponent ρ and quotient exponent β. The separate right-frame surplus contributes zero to $\bar\zeta$. At a boundary t denote the remainder and quotient rows of ζ by ρ_t, β_t. The diagonal unit is

$$\Gamma(\zeta) = u(\bar\zeta) \prod_t \prod_{a=1}^{r} \gamma(\rho_{t,a}, \beta_{t,a}),$$

with u as above and with power P_t in the second product. On a fixed remainder class apply 22.16 once with the original row powers and digits, and once with the boundary powers and remainders. Both applications hold for either sign. Substitution of $\rho + P_t\beta$ into the first approximation preserves total degree, and the resulting polynomial is integer-valued, so its binomial coefficients in the free exponents are integral by forward differences. The class selection acts on the remainder letters alone. These letters are outside every chain prefix and enlarged link, so the selection commutes with their group actions. The remaining factorial constants depend on row degrees and remainder rows. On the allowed low remainder exponents the binomial expansions of these units and their inverses give integral multipliers g_+, g_- with $v(g_+ - \Gamma), v(g_- - \Gamma^{-1}) \ge c'$. A binomial in a quotient exponent is a divided diagonal derivative. There are two factors for each sign. The original factorial unit evaluated on the collapsed exponents and the product of quotient–remainder units each have a binomial approximation of total degree at most $6c'$. Their product costs $12c'$. The error still has valuation at least c', since both exact factors are units and both approximations are integral. A binomial in an affine exponent $u + Pv$ of degree d has an integral binomial expansion in v of degree at most d, by finite differences. On a fixed low-digit class each of the two diagonal multipliers therefore lowers a prefix order by at most $12c'$. In the enlarged alphabet the resulting chain bound is its original threshold minus its assigned slack and $12c'$, and the link bound is its original threshold minus $30c'$. For the quotient letters from the preceding frame the slack used before the projection is at least $rd_I/P + 13rc'$ for each boundary power P, where d_I is the total plane degree of that preceding interior. Summing the r quotient loads costs at most the assigned slack minus $12c'$, since $r \ge 1$.

On the enlarged alphabet, within each segment a prefix contains its left-frame powered letters and an initial part of its interior. These sets are nested there and disjoint across segments. They are also contained in, or disjoint from, the enlarged link sets of their segment. The product P_{pre} of their projections preserves the link conditions and commutes with diagonal derivatives on the remainder letters outside these prefixes. Write $D_m = n_c e_C$, where n_c counts the prefixes used. The operator is

$$\mathrm{diag}(g_-)\, P_{\mathrm{pre}}\, \mathrm{diag}(g_+).$$

The remainder degrees are preserved because the projections commute with those coordinate operators. Each prefix action preserves every letter degree, including the surplus right-frame degree. Hence the output still obeys the digit and row constraints. The projection imposes the chain thresholds, and the derivative slack preserves the required link thresholds after the last diagonal multiplication. Multiplying by $p^{\lceil D_m + k'/2\rceil}$ clears the remaining denominator in the plane test. To compare its value, take $\theta_{ji} = H_{ji} - s_i/8$ and $\sigma_{ji} = 3s_i/8$ in Lemma 22.21. We check the slack at a cut x_i of segment j. Put $D = z_{j-1} - f_{j-1,0}$ and $P = p^{z_j - z_{j-1}}$. Each interior position of segment $j-1$ is at distance at least $D + L_1 + 1$ from its endpoint and has divided degree at most $2a_0$ times the inverse power at that distance. Summing distinct distances gives $d_{I_{j-1}} \le 3a_0 p^{-D-L_1-1}$. The oldest even predecessor in segment j is at or after $f_{j-1,0}$, so $a_0 p^{-D}/P \le b_{x_i}$. The spacing bound and $Y \ge r$ imply $48rYk^2 \le p^{L_1+1}$, whence $rd_{I_{j-1}}/P \le s_i/8$, using $b_{x_i}/(Yk^2) \le 2s_i$. The last size bound and $b_{x_i} \ge a_0/p^{G'}$ give $s_i \ge 100r^2c'$ on enlarging the absolute constant. This proves the required slack. The row digit approximations above give the two unit congruences, and Lemma 22.19 gives the source chain bounds. Write $\Omega = D_{g_-} P_{\mathrm{pre}} D_{g_+}$. Lemma 22.21 now gives $v(\Lambda(\Omega Z) - \Lambda(Z)) \ge c' + n_0 - 2k' - D_m > n_0 + D_l + 1$, since $k' \ge 1$ and the size cutoff gives $c' > 4(D_m + D_l + 3k')$. Thus the two values have equal valuation. With $N_2 = \lceil D_m + k'/2\rceil$, the integral test $p^{N_2}\Omega Z$ has value of valuation at most $n_0 + D_l + 1 + N_2 \le n_0 + D_l + D_m + 3k'$. Here the denominator bound can also be read directly on each block: the label and surplus factorials are fixed, and the variation of the sum of the free letter factorial valuations is at most k' by the row multinomial

bound. Hence a loss D_m in v_{F} costs at most $D_m + k'/2$ in coefficient valuation. This is the plane contribution used in (A.172).

Lean: `Formalism.MainNotions.PlaneGram.TwistData.gaP` (`Formalism.SharpRestriction.GrowthBounds.MonomialOrderSegments.CollapseExponent`, lines 57–58)
Lean: `Formalism.MainNotions.PlaneGram.stepDEF_of` (`Formalism.SharpRestriction.GrowthBounds.MonomialOrderSegments.OmegaValue`, lines 108–119)
Lean: `Formalism.MainNotions.PlaneGram.plane_gram_theorem_of_five_lemmas` (`Formalism.SharpRestriction.GrowthBounds.MonomialOrderSegments.PlaneGramTheoremOfLemmas`, lines 13–16)

Put $Z_1 = p^{N_2}\Omega Z$. It is integral and allowable. By the chain and link bounds above, with losses $12c'$ and $30c'$, its least signed weight is at least $\theta_{ji} - \sigma_{ji} - 24c'$ on the enlarged chain prefix of every cut i of segment j, and at least $H_{ju} - 3s_u/8 - 60c'$ on the enlarged link of (j, m), where $u = l + r - m$. Write ϱ_{xc}, $0 \le \varrho_{xc} < M_x$, for the coordinate residues of the chosen unit monomial at the non-left-frame occurrences of the marks, so that its plane residues are the low digits d_{xa}, and X^{ϱ} for their monomial. Write d_x^{pl} and d_x^{no} for its plane and normal quotient row degrees. Thus $\sum_c \varrho_{xc} + M_x(d_x^{\mathrm{pl}} + d_x^{\mathrm{no}}) = d_{x \bmod n'}$.

Let D_{no} be the least nonnegative integer for which $p^{D_{\mathrm{no}}} e_{\mathrm{no}}$ preserves $v_{\mathrm{F}} \ge 0$ on polynomials in the normal variables of the row degrees d_x^{no}. It is at most the joint normal projection cost estimated below. Put $c_Y = p^{D_{\mathrm{no}}} \omega^{v(\prod_x d_x^{\mathrm{no}}!) - v(\nu_{\mathrm{no}}!)}$. Since $v_{\mathrm{F}}(\omega^{-v(\nu_{\mathrm{no}}!)} X^{\nu_{\mathrm{no}}}) = 0$ and every monomial factorial in these row degrees divides $\prod_x d_x^{\mathrm{no}}!$, the polynomial $c_Y e_{\mathrm{no}}(X^{\nu_{\mathrm{no}}})$ is integral. Write $u_{\mathrm{no}}(\alpha) = \prod_x \prod_{r<c\le k} \gamma_{M_x}(\varrho_{xc}, \alpha_{xc})$ for the normal digit unit. By 22.16, choose an integral binomial approximation g_{no} to u_{no}^{-1} modulo $p^{c'}$, of total binomial degree at most $4c'$, and put $N_{\mathrm{no}} = D_{g_{\mathrm{no}}}\big(c_Y e_{\mathrm{no}}(X^{\nu_{\mathrm{no}}})\big)$ and $N^{\mathrm{ex}} = D_{u_{\mathrm{no}}^{-1}}\big(c_Y e_{\mathrm{no}}(X^{\nu_{\mathrm{no}}})\big)$. Then N_{no} is integral of the row degrees d_x^{no}, and $N_{\mathrm{no}} - N^{\mathrm{ex}}$ has coefficients in $p^{c'}O$. On the normal prefix of length u of segment j, the polynomial $e_{\mathrm{no}}(X^{\nu_{\mathrm{no}}})$ lies in the isotypic space whose highest weight has first entry k_{ju}, so its greatest signed weight there is at most k_{ju}. A binomial of degree d in a coordinate exponent raises this weight by at most d. Hence N_{no} has greatest signed weight at most $k_{ju} + 4c'$ on that prefix.

Let Ξ be obtained from $X^{\varrho}\,\Phi(Z_1)\,N_{\mathrm{no}}$ by raising each variable of the last two factors at a non-left-frame occurrence x to its M_x-th power, and let Ξ^{ex} be obtained in the same way with N^{ex}. With z as above, put

$$C_{\varrho} = \prod_{x,c} \varrho_{xc}! \prod_x (M_x!)^{d_x^{\mathrm{pl}} + d_x^{\mathrm{no}}}, \qquad k_0 = C_{\varrho}\, c_Y\, \nu_{\mathrm{no}}!\, z.$$

Then $\sum_\alpha \alpha!\,[X^\alpha] G\,[X^\alpha]\Xi^{\mathrm{ex}} = k_0 \Lambda(Z_1)$. Indeed, on the selected digit block each exponent factorial is C_{ϱ} times the factorials of the plane and normal quotient exponents times the plane and normal digit units. The factor u_{no}^{-1} cancels the normal digit unit. Moving the self-adjoint projection e_{no} to the normal part of G and pairing with $X^{\nu_{\mathrm{no}}}$ gives $\nu_{\mathrm{no}}!$ times $P_c^{\mathrm{raw}} = zP_c$, and the plane digit unit gives D_u. Replacing Ξ^{ex} by Ξ changes this sum by an element of valuation at least $v_d + c' - k_d$, since G and $\Phi(Z_1)$ are integral, $N_{\mathrm{no}} - N^{\mathrm{ex}}$ has coefficients in $p^{c'}O$, and every monomial factorial in the flat degrees has valuation at least $v_d - k_d$. This k_0 is the scalar estimated in (A.172). For a polynomial χ on the letters of segment j, write $\chi^{[j]}$ for its segment substitution.

Claim 1. There is a finite decomposition $\Xi = \sum_q \Xi_q$ in which each Ξ_q has the multidegree of Ξ, all monomials of Ξ_q have one normal degree o_q, and $\Xi_q = X^{\varrho} \sum_i \prod_j \chi_{qij}^{[j]}$, the product running over the segments of a period. For fixed q and j there are degrees $d_{qj}(x)$ at the letters x of segment j, common to all i, such that each χ_{qij} is an integral polynomial on the letters of segment j with the following properties.

1. It is an O-combination of products AB' of a plane polynomial A on the letters of segment j and a complementary polynomial B' on its non-left-frame letters such that $v_{<x}(A) \ge b_x/2 - s_x$ at every non-left-frame letter x, with $l_{\le x}(B') + m_* \le v_{<x}(A)$ at even x and $l_{<x}(B') + m_* \le v_{\le x}(A)$ at odd x.
2. Its degree at each letter x of segment j is $d_{qj}(x)$, and $d_{qj}(x) \le \frac{101}{100} d_{x \bmod n'} p^{x - z_j}$ at every left-frame or interior letter x.

Condition (1) is the condition on the symbol generators above with $b_x/2$ lowered to $b_x/2 - s_x$ and m_x lowered to m_*.

Keep distinct variables for the occurrences of a mark in different segments. Place the interior letters and the label and surplus letters of frame j at their positions in segment j, and the quotient letters of frame $j-1$ at the left frame of segment j; the label and surplus letters of a slot become one variable. Write $\widetilde{Z}_1$ and $\widetilde{N}_{\mathrm{no}}$ for the images of Z_1 and N_{no}. On a set of segment letters, the least weight of $\widetilde{Z}_1$ is at least that of Z_1 on the inverse image. Let π send the variable at position x of segment j to the variable at $x \bmod n'$ raised

to $p^{z_j - x}$; on the letters of segment j it is the segment substitution. At a shared slot the left-frame position carries the right-frame power times the boundary power, so $\pi(\widetilde{Z}_1)$ is $\Phi(Z_1)$ with the powers M_x applied, and $\Xi = X^{\varrho}\,\pi(\widetilde{Z}_1)\,\pi(\widetilde{N}_{\mathrm{no}})$. Put $\tau_{ju} = H_{ju} - s_u/2 - 60c'$. For $u \le l$, the inverse image of the letters of segment j before x_u is the enlarged chain prefix of the cut u, and $\theta_{ju} - \sigma_{ju} - 24c' = H_{ju} - s_u/2 - 24c' \ge \tau_{ju}$. For $l < u < l+r$, with $u = l+r-m$, it is the enlarged link of (j,m) together with label letters. Label letters are vectors and do not lower the least weight, and $H_{ju} - 3s_u/8 - 60c' \ge \tau_{ju}$. Thus $\widetilde{Z}_1$ has least signed weight at least τ_{ju} on the letters of segment j before x_u, and at least $\tau_{j,u+1}$ on those up to an interior letter x_u. The normal factor $\widetilde{N}_{\mathrm{no}}$ has greatest signed weight at most $k_{ju} + 4c'$ on the letters before x_u and at most $k_{j,u+1} + 4c'$ on those up to x_u.

Write $\widetilde{Z}_1 = \sum_\delta Z_\delta$ by degree δ at every segment letter. These components keep the bounds, since the prefix actions preserve letter degrees, and $\Xi = \sum_\delta \Xi_\delta$ with $\Xi_\delta = X^{\varrho}\,\pi(Z_\delta)\,\pi(\widetilde{N}_{\mathrm{no}})$. Each Ξ_δ has the multidegree of Ξ, and it has one normal degree, since δ fixes the letter degrees of the plane factor and N_{no} has fixed row degrees. For a segment j, the integral polynomials on its letters of degree δ at each letter that satisfy the bounds above form a submodule X_j of a finite free O-module. It is saturated, since the bounds are linear conditions over the fraction field. As O is a discrete valuation ring, X_j is a direct summand, so there is an O-linear projection q_j onto X_j. The coefficients of Z_δ with respect to the monomials in the letters outside segment j keep the bounds on segment j, so applying q_j in the variables of segment j, coefficientwise in the other variables, fixes Z_δ. It sends a product of elements of $X_{j'}$ over earlier segments j', times a monomial in the remaining letters, to a sum of such products over these segments and j, times monomials in the remaining letters. Applying this for all segments in turn writes Z_δ as a finite sum of products over j of elements $A_j \in X_j$. The same argument writes $\widetilde{N}_{\mathrm{no}}$ as a finite sum of products over j of integral complementary polynomials B_j on the non-left-frame letters of segment j, of fixed degree at each letter and with the bounds above. The products A_jB_j, taken over pairs of terms of the two sums, are the factors χ_{qij} of $\Xi_q = \Xi_\delta$, and $d_{qj}(x)$ is the degree of δ plus the complementary degree at x.

As at the cuts above, $b_x \ge a_0/p^{G'}$ gives $s_x \ge 100r^2c' \ge 128c'$ at every non-left-frame letter x. Fix j and a non-left-frame letter $x = x_u$ of segment j. If x_u is even, then $H_{ju} \ge k_{j,u+1} + m_* + s_u$, so $\tau_{ju} \ge k_{j,u+1} + 4c' + m_*$. If x_u is odd, it is interior and $H_{j,u+1} \ge k_{ju} + m_* + s_{u+1}$, so $\tau_{j,u+1} \ge k_{ju} + 4c' + m_*$. In both cases $H_{ju} \ge b_x/2$ gives $\tau_{ju} \ge b_x/2 - s_x$. With the bounds of the preceding paragraphs, each product A_jB_j satisfies (1). For (2), the degree at an interior letter is a plane plus complementary row degree, at most $d_{x \bmod n'}/M_x$ by the residue selection. At a left-frame letter it is a quotient degree, at most the plane row degree of the shared slot divided by its boundary power. Both are at most $d_{x \bmod n'} p^{x - z_j}$.

Lean: Formalism.MainNotions.Proposition5Numerics.noAt (Formalism.SharpRestriction.GrowthBounds.DiagonalSupport.VBBMonomialOne, lines 28–107)
Lean: Formalism.MainNotions.Proposition5Numerics.value_exact (Formalism.SharpRestriction.GrowthBounds.DiagonalSupport.PickVal, lines 23–63)
Lean: Formalism.MainNotions.Proposition5Numerics.restValBdG (Formalism.SharpRestriction.GrowthBounds.DiagonalSupport.MembVB, lines 258–261)
Lean: Formalism.MainNotions.Proposition5Numerics.atl (Formalism.SharpRestriction.GrowthBounds.DiagonalSupport.ProjectionMemPartialSpan, lines 220–252)
Lean: Formalism.MainNotions.Proposition5Numerics.seg_dec (Formalism.SharpRestriction.GrowthBounds.DiagonalSupport.SumMulSumProd, lines 155–215)
Lean: Formalism.MainNotions.Proposition5Numerics.restDecG_of (Formalism.SharpRestriction.GrowthBounds.DiagonalSupport.DecIdentity, lines 116–139)
Lean: Formalism.MainNotions.Proposition5Numerics.XXb.cond_at (Formalism.SharpRestriction.GrowthBounds.DiagonalSupport.MembXXb, lines 59–100)
Lean: Formalism.MainNotions.Proposition5Numerics.restXXG_b (Formalism.SharpRestriction.GrowthBounds.DiagonalSupport.MembXXb, lines 108–141)
Lean: Formalism.MainNotions.Proposition5Numerics.restDdG_holds (Formalism.SharpRestriction.GrowthBounds.DiagonalSupport.MembDd, lines 227–429)
Lean: Formalism.MainNotions.Proposition5Numerics.membAt_core (Formalism.SharpRestriction.GrowthBounds.DiagonalSupport.ExistsMapOfCoefGe, lines 183–229)

Claim 2. Let $\Xi' = X^{\varrho} \sum_i \prod_j \chi_{ij}^{[j]}$, where X^{ϱ} is supported on the marks with exponents below M_x at the non-left-frame occurrences, the χ_{ij} satisfy (1) and (2) of Claim 1 with degrees $d_j(x)$ common to all i, and all monomials of Ξ' have one normal degree o_{sym}. Let g be a Witt matrix of an edge. Then Ξ' has an integral lift F, of the multidegree of Ξ' when Ξ' is multihomogeneous, satisfying (A.170) with Ξ', o_{sym} and F in place of P_b, b and F_b.

For a non-left-frame letter x of segment j, let D_x° be the determinant of the plane coordinates of its r even

predecessors in reverse order, and let Δ°_{j-1} be the determinant of the plane coordinates of the left frame, both without powers. Put $m_1 = (2Q_*+3)c' + kQ_* + c'$, $m_2 = c' \max_j \alpha_j + 2c'$, and $D_v^{(j)} = \big(\prod_x (D_x^\circ)^{m_1}\big)(\Delta^\circ_{j-1})^{m_2}$, with x running over the non-left-frame letters of segment j. Since χ_{ij} has the fixed letter degrees $d_j(x)$, it is an O-combination of products AB' as in (1) whose letter degrees add to $d_j(x)$. Fix such a product. Let x be a non-left-frame letter with oldest even predecessor y. The even letters of the segment before x other than its r even predecessors are left-frame or interior letters at least two places before y. By (2) and $d_s \le 2a_0$, their total plane degree t_x satisfies $t_x \le \frac{101}{100}\, 2a_0 p^{y-z_j} \sum_{m\ge1} p^{-2m} \le b_x/80$, since $b_x \ge a_0 p^{y-z_j}$ and $p \ge 13$. The size hypotheses and $b_x \ge a_0/p^{G'}$ give $m_* \le s_x \le b_x/100$ on enlarging the absolute constant. Hence $v_{<x}(A) \ge b_x/2 - s_x \ge t_x + m_*$. At the first non-left-frame letter the same bound gives least signed weight at least m_* on the left frame, which contains no other vector letters. A segment has at most N_1 letters, so the exponents in $D_v^{(j)}$ add up to at most $N_1 m_1 + m_2 \le m_*$.

We use the following divisibility over the fraction field of O. Let $\mathcal{F}$ be multihomogeneous in the plane coordinates of signed letters, let I be a set of letters containing r distinct vector letters $y_1, \dots, y_r$, let t_{oth} be the total degree of the other vector letters of I, and let $\theta_I > t_{\mathrm{oth}}$ be an integer with $v_I(\mathcal{F}) \ge \theta_I$. Let $\mathcal{D}$ be the determinant of the matrix $\mathcal{M}$ of the plane coordinates of $y_1, \dots, y_r$. Then $\mathcal{D}^{\theta_I - t_{\mathrm{oth}}}$ divides $\mathcal{F}$. Indeed, let $c_{\mathcal{M}}$ be the cofactor of the entry of $\mathcal{M}$ in the row of y_r and the weighted column. Over the polynomial ring localized at $c_{\mathcal{M}}$, the column of the adjugate of $\mathcal{M}$ belonging to y_r has the unit $c_{\mathcal{M}}$ in the weighted position, so it is the weighted column of a change of basis. Write $\mathcal{F}$ in the new coordinates. By specializing the generic change of basis, every monomial has signed weight at least θ_I. Substituting back, the weighted coordinates of $y_1, \dots, y_{r-1}$ become 0 and that of y_r becomes $\mathcal{D}$. A monomial not using the weighted coordinates of $y_1, \dots, y_{r-1}$ has signed weight at most t_{oth} plus its exponent at the weighted coordinate of y_r, since the other vector letters of I contribute at most t_{oth} and covector letters contribute nonpositive weight, so this exponent is at least $\theta_I - t_{\mathrm{oth}}$. Hence $\mathcal{D}^{\theta_I - t_{\mathrm{oth}}}$ divides $\mathcal{F}$ after localization, and since $c_{\mathcal{M}}$ and $\mathcal{D}$ are coprime it divides $\mathcal{F}$. For the loss, specialize the generic frame to the identity and the rows $y_1, \dots, y_r$ to the standard basis vectors in their order. The determinant scaled on a set I' then becomes T if the row carrying the weighted basis vector lies in I', and 1 otherwise. Specialization cannot create a lower exponent, so $v_{I'}(\mathcal{D}) \le 1$ for every set I'. Least weights add on nonzero products, so $\mathcal{F} = \mathcal{D}^n \mathcal{F}'$ gives $v_{I'}(\mathcal{F}) \le v_{I'}(\mathcal{F}') + n$ for every I'. The determinant has unit coefficients, so by Gauss's lemma the quotient of an integral polynomial is integral, and its row degrees are those of $\mathcal{F}$ lowered by n at $y_1, \dots, y_r$.

Apply this to A successively at every non-left-frame letter x, with I the letters before x, its r even predecessors and exponent m_1, and at the left frame, with I the left frame and exponent m_2. At each step take θ_I to be the current degree of the other vector letters of I plus the exponent. Earlier divisions lower each least weight by at most the exponents already used and do not raise the other degrees, and the exponents add up to at most m_*, so every step applies. This gives $A = D_v^{(j)} A'$ with A' integral and every least signed weight of A' at most m_* below that of A. Hence $l_{\le x}(B') \le v_{<x}(A')$ at even x and $l_{<x}(B') \le v_{\le x}(A')$ at odd x.

Expand A' in the integral rank-r path basis with standard terminal tableaux of the proof of Lemma 22.20. The argument there, applied to the letters of segment j in order, shows that the integral polynomials of the given letter degrees whose least signed weight on the letters up to each x is at least a given integer are exactly the integral combinations of the path polynomials whose last row entry after x is at least that integer. The greatest first-coordinate weight of a column is $\mathbf{1}_{h(c)>0}$ for $c > 0$ and $-\mathbf{1}_{h(c)=0}$ for $c \le 0$, and these sum to the first row entry. So the same argument with greatest weights, in rank K, shows that B' is an integral combination of complementary path polynomials whose first row entry after each x is at most the greatest signed weight of B' on the letters up to x. By the inequalities just obtained, $A'B'$ is an integral combination of products $P_\nu P_\kappa$ of a plane path polynomial with rows ν and a complementary path polynomial with rows κ such that, at every non-left-frame letter, κ_1 after the letter is at most ν_r before it at an even letter, and κ_1 before the letter is at most ν_r after it at an odd letter. On the left frame $\kappa = 0$ and ν has nonnegative entries, since these letters are vectors and B' does not use them. These are the interlacing conditions of Definition 22.13 between the two blocks of the concatenated rows. The segment has at most N_1 letters and $N_1 + r \le K$, so the concatenated rows form a path of rank k.

For a column c of the joined path, let I'_c and I''_c be the terminal index sets of column c in P_ν and P_κ, and put $J_c = I'_c \cup \{r + a : a \in I''_c\}$. In the separated recursion of Lemma 22.17(1) for column c, a non-costly step uses nonnormal variables and a costly step uses normal variables. Hence a step changes a plane index

if it is a vector step from height below r or a covector step from height at most r, and a complementary index otherwise. For $c > 0$, a plane index changes while the complementary index set is empty, and a complementary index changes while the plane index set is full. In the latter case every plane index is smaller than the changing index, so the right exterior sign agrees with the sign in the complementary block. For $c \le 0$, at most N_1 row entries can be negative, since each covector letter adds at most one negative row. Thus $h(c) \ge k - N_1 > r$ throughout. The exterior index records the $k - h(c)$ missing indices and lies entirely in the complementary block, so its updates also have the complementary block signs. Therefore $K_c^{\mathrm{sep}}(J_c)$ is the product of column c of P_ν with terminal index I_c' and column c of P_κ with terminal index I_c'', and $\prod_c K_c^{\mathrm{sep}}(J_c) = P_\nu P_\kappa$. After segment substitution each costly step at a letter x contributes $p^{z_j - x}$ to the normal degree, so this product has normal degree $o_j' = \sum_c \mathrm{oc}(c)$. By Lemma 22.17(1), $\omega^{o_j'}$ times its segment substitution is the product over c of the leading parts of the column polynomials $K_c(g; J_c)$. Hence $\chi_{ij} = D_v^{(j)} \chi_{ij}'$, where χ_{ij}' is an integral combination of such products $P_\nu P_\kappa$.

Entrywise Frobenius modulo p gives $(D_x^\circ)^{[j]} = D_x^{M_x} + pE_x$ and $(\Delta_{j-1}^\circ)^{[j]} = \Delta_{j-1}^{p^{z_j - z_{j-1}}} + pE_j'$, with E_x, E_j' integral in the plane coordinates at the even marks, which are nonnormal. For integral D, E, a power $P \ge 1$ of p and $n \ge c'$, the terms of $(D^P + pE)^n$ with at least c' factors pE lie in $p^{c'} O[X]$, and the others contain at least $n - c'$ factors D^P. Hence $(D^P + pE)^n = D^{n-c'} \mathcal{H} + p^{c'} \mathcal{Y}$ with $\mathcal{H}, \mathcal{Y}$ integral in the same variables. Apply this with $n = m_1$ and $n = m_2$, where $m_1 - c' = (2Q_* + 3)c' + kQ_*$ and $m_2 - c' = c'(\max_j \alpha_j + 1) \ge c'(\alpha_{j-1} + 1)$. Every mark occurs once as a non-left-frame letter in a period, and the frames $j - 1$ run over a period. Put $V_{\mathrm{res}} = \prod_{x \in S} D_x^{kQ_*}$ and $U_0 = R_0^{c'} \prod_j \Delta_j^{c'} \prod_{x \in S} D_x^{(2Q_* + 3)c'}$, so that $U = U_0 V_{\mathrm{res}}$ for U in (A.169). Multiplying over the segments of a period and putting the excess powers of the Δ_j into the integral factor gives $\prod_j (D_v^{(j)})^{[j]} = U_0 V_{\mathrm{res}} \Upsilon + p^{c'} Y''$, with Υ integral of normal degree zero in the variables at the marks and Y'' integral.

Put $S_\chi = \sum_i \prod_j \chi_{ij}'^{[j]}$. Then $\Xi' = U_0 X^{\varrho} V_{\mathrm{res}} S_\chi \Upsilon + p^{c'} X^{\varrho} S_\chi Y''$. Take the components of normal degree o_{sym}. The second term gives an element of $p^{c'} O[X]$. Expand $\omega^{o_{\mathrm{sym}}}$ times the first component into products of $\omega^q X^{\varrho}$, $\omega^{o'}$ times products over j of segment substitutions of the path products above, V_{res}, Υ and U_0, where q counts the normal entries of X^{ϱ} with multiplicity, $o' = \sum_j o_j'$ for the chosen segment terms, and $q + o' = o_{\mathrm{sym}}$. At each mark these entries number less than $kM_x \le kQ_*$, so each normal entry $\omega X_{x,b}$ can be paired with its own copy of D_x from V_{res} to form a leading bordered or pairing factor $\omega s_g(D_x(g)) X_{x,b}$ of Lemma 22.17(2). The path products give leading exterior column factors. The unused part of V_{res}, the nonnormal entries of X^{ϱ} and Υ are integral nonnormal factors, and U_0 supplies $R_0^{c'}$ and the reserves of $(2Q_* + 3)c'$ copies of each D_x and c' copies of each Δ_j. So every product is a term at level o_{sym} of the recursion preceding (A.170), with its initial reserves, and that recursion lifts it. The second term, multiplied by $\omega^{o_{\mathrm{sym}}}$, lies in $\omega^{o_{\mathrm{sym}} + c'} O[X]$. The sum of the lifts, projected to the multidegree of Ξ' when Ξ' is multihomogeneous, is the asserted F.

Lean: `Formalism.MainNotions.TestSideCounting.one_sided_lemma4_of_interfaces` (`Formalism.SharpRestriction.GrowthBounds.DualFockLinear.OneSidedLemma4OfAdmittedTheoremII`, lines 189–238)
Lean: `Formalism.MainNotions.TestSideCounting.restGeomU` (`Formalism.SharpRestriction.GrowthBounds.DualFockLinear.OneSidedLemma4OfAdmittedTheoremII`, lines 64–148)
Lean: `Formalism.MainNotions.G.WittFrameLocaliserOrders.numeric_symbol_bound_real` (`Formalism.SharpRestriction.GrowthBounds.WittFrameLocaliserOrders`, lines 5556–5572)
Lean: `Formalism.MainNotions.TestSideCounting.restDivMU` (`Formalism.SharpRestriction.GrowthBounds.DualFockLinear.OneSidedLemma4OfAdmittedTheoremII`, lines 150–187)
Lean: `Formalism.MainNotions.G.symbolGeneratorData.dv_segment` (`Formalism.SharpRestriction.GrowthBounds.PathWeight.DvDegOtherMono`, lines 418–555)
Lean: `Formalism.MainNotions.G.symbolGeneratorData.dv_div_one` (`Formalism.SharpRestriction.GrowthBounds.PathWeight.DvDegOtherMono`, lines 130–161)
Lean: `Formalism.MainNotions.G.GLWeightTheory.support_coprimality_lemma_holds` (`Formalism.SharpRestriction.GrowthBounds.GLWeightTheory`, lines 102–102)
Lean: `Formalism.MainNotions.G.GLWeightTheory.height_bound_of_weights` (`Formalism.SharpRestriction.GrowthBounds.RootWeights.FrameChangePreservesMultidegree`, lines 184–250)
Lean: `Formalism.MainNotions.G.GLWeightTheory.valuation_bound_det_power_factor` (`Formalism.SharpRestriction.GrowthBounds.RootWeights.ValuationBoundDetPowerFactor`, lines 51–71)
Lean: `Formalism.MainNotions.G.GLWeightTheory.mul_valS_detA_le` (`Formalism.SharpRestriction.GrowthBounds.RootWeights.IdentityEvaluationGenericMatrix`, lines 122–175)
Lean: `Formalism.MainNotions.G.symbolGeneratorData.generator_pair_conditions_of_admitted` (`Formalism.SharpRestriction.GrowthBounds.PathWeight.CoeUntopDLeAlt`, lines 136–222)
Lean: `Formalism.MainNotions.LemmaBAlt.prod_plTan` (`Formalism.SharpRestriction.GrowthBounds.FrobeniusPaths.JointTypeLatticeSaturated`, lines 180–188)

Lean: `Formalism.MainNotions.G.symbolGeneratorData.generator_path_homogeneous_holds` (`Formalism.SharpRestriction.GrowthBounds.PathWeight.LetterWeightPos`, lines 368–404)
Lean: `Formalism.MainNotions.G.symbolGeneratorData.restPh` (`Formalism.SharpRestriction.GrowthBounds.PathWeight.SymbolGeneratorData`, lines 223–293)
Lean: `Formalism.MainNotions.G.symbolGeneratorData.restMon` (`Formalism.SharpRestriction.GrowthBounds.PathWeight.LeadingDepthOneMemClosure`, lines 226–228)
Lean: `Formalism.MainNotions.G.symbolGeneratorData.core` (`Formalism.SharpRestriction.GrowthBounds.PathWeight.FixedSymbolFactorIntegral`, lines 148–207)

Write $b = \log_p a_0$. There are at most N original plane letters, each of degree at most $2a_0$. Both the number of chain cuts and the number of links are at most N: a segment with l interior letters contributes $l+1$ cuts and $r-1$ links, and the interiors together with one frame per segment use each mark once. Since $p \le a_0$, $4N \le a_0$ and $r \le k \le a_0$, the logarithmic cost $1 + \log_p(2\sum d_i + r)$ is at most $4b$. This gives $D_m, D_l, k' \le 4B_0$; summing with coefficients $1, 1, 3$ gives $20B_0$. For the normal selection, the joint normal projection costs at most $2Nk^2(3b)$: there are at most $2N$ cuts and each root-string upper argument is at most a_0^3. The two further costs are half the valuations of the plane and normal row multinomials. They are at most $Nr(3b)/2$ and $NK(3b)/2$, by 11.7. Their sum with the projection cost is at most $9B_0$. More explicitly, the coefficient of P_c^{raw} at b_p is the coefficient of $e_{\mathrm{no}} G_{b_p}$ at ν_{no}. The minimality of m, the choice of ν_{no} and $v_{\mathrm{F}}(e_{\mathrm{no}} G_{b_p}) \le v_{\mathrm{F}}(G_{b_p}) \le \frac{1}{2} v(b_n!)$ therefore give

$$v(z) + \tfrac{1}{2} v(\nu_{\mathrm{no}}!) \le \tfrac{1}{2} v(b_p!) + \tfrac{1}{2} v(b_n!).$$

The digit units are units and the monomial factorial divides $\prod_s d_s!$, so the row factorial identity gives $v(C_\varrho) + v(b_p!) + v(b_n!) \le v_d$. With $k_0 = C_\varrho c_Y \nu_{\mathrm{no}}! z$ and the definition of c_Y,

$$v(k_0) \le v_d - n_0 + D_{\mathrm{no}} + \tfrac{1}{2} v\Big(\frac{\prod_x d_x^{\mathrm{pl}}!}{b_p!}\Big) + \tfrac{1}{2} v\Big(\frac{\prod_x d_x^{\mathrm{no}}!}{b_n!}\Big),$$

and the last three terms are the costs just estimated. Hence

$$D_m + D_l + 3k' \le 20B_0, \qquad k_d \le 20B_0, \qquad v(k_0) \le v_d - n_0 + 9B_0. \tag{A.172}$$

The last scalar restores the factorials of the normal variables and the normalization of P_c. The main term before the common factorial normalization has valuation at most $v_d + 29B_0$. Replacing Ξ^{ex} by Ξ changes the main term by an element of valuation at least $v_d + c' - k_d$, as shown above. Choose C so that $c' > 4(D_m + D_l + 3k')$ and $c' > (9 + 3 \cdot 20)B_0$. The main term determines the valuation in (A.171), which is at most $49B_0$. Use coordinates in which the target lattice has the standard basis, and let B_e be the basis matrix of the source lattice of e determined by the Witt matrix originally chosen for e and used in computing G, so that the columns of $B_e \operatorname{diag}(p^{-1} I_r, I_K)$ form a basis of the target lattice. Let $g^\vee$ be the Witt matrix with image $B_e \operatorname{diag}(p^{-1} I_r, I_K)$; it is invertible over the Witt ring. Then $(g^\vee)^{-\mathrm{t}} = B_e^{-\mathrm{t}} \operatorname{diag}(pI_r, I_K)$ is a basis matrix of the reversed dual edge $e^\vee$, and in these two frames the normalized edge pairing of the source with a target vector is (A.171) applied to their substituted flat polynomials. For each term Ξ_q of Claim 1 with $o_q \le \lfloor T \rfloor + Nc_t$, Claim 2 with $g^\vee$ gives an integral lift F_q of the multidegree of Ξ_q with $s_{g^\vee} F_q - \omega^{o_q} \Xi_q \in \omega^{o_q + c'} O[X]$ and $s_{g'} F_q \in \omega^{o_q + c'} O[X]$ for every g' on a competing plane of $e^\vee$. The normalized dual identification turns $\omega^{-o_q} F_q$ into a target functional of depth at most o_q, takes its competing-plane conditions to the conditions on competing incoming edges, and makes its pairing with the source the normalized pairing of G with Ξ_q up to the error of the lift. Sum these functionals over q. The terms with $o_q > \lfloor T \rfloor + Nc_t$ pair to zero with G, whose monomials have normal degree at most $\lfloor T \rfloor + Nc_t$, and are omitted. The multinomial bound puts the resulting pairing error in $\omega^{c'} O$. We also have $c' > 2\lceil 49B_0 \rceil$, so the lifted flat test has valuation at most $D_F = \lceil 49B_0 \rceil$, depth at most $T + Nc_t$, and order c' on competing incoming edges.

Lean: `Formalism.MainNotions.Proposition5Numerics.value_core` (`Formalism.SharpRestriction.GrowthBounds.DiagonalSupport.Proposition5OfThreeParts`, lines 61–123)
Lean: `Formalism.MainNotions.Proposition5Numerics.sizeStep'` (`Formalism.SharpRestriction.GrowthBounds.DiagonalSupport.SizeStepAlt`, lines 11–144)
Lean: `Formalism.MainNotions.Proposition5Numerics.numStep_of` (`Formalism.SharpRestriction.GrowthBounds.DiagonalSupport.LeOfOneLeMul`, lines 182–230)
Lean: `Formalism.MainNotions.TestSideCounting.restFlat_of` (`Formalism.SharpRestriction.GrowthBounds.DualFockLinear.FockSumRight`, lines 47–139)

For the full lattice, tensor the marked pullbacks by L_β with the dual basis of the unmarked edge lattice, using its normalized perfect pairing. A primitive tensor has a primitive marked coefficient in this basis. The unmarked factors cost at most $2rn'c_b$ in the order and depth allowance and preserve κ. The factor ω^a leaves the valuation at most $\kappa + D_F \le 54B_0$. The residual factors cost at most o in depth and order, so $c' - o = c_*$. A source approximation error in $\omega^{c_*}O$ preserves the value bound since $c_* > 2D_e$.
Lean: `Formalism.MainNotions.TestSideCounting.cloudBAt_of_rho` (`Formalism.SharpRestriction.GrowthBounds.DualFockLinear.CloudBAtOfRho`, lines 87–298)
Lean: `Formalism.MainNotions.TestSideCounting.test_bound_of_two_halves` (`Formalism.SharpRestriction.GrowthBounds.DualFockLinear.TestSideExplicitIdentification`, lines 149–199)

For $q = p^{n'}$ use the q^{rK} incoming edges of 14.10 given by the kernels of $(u, v) \mapsto \bar{u} + A\bar{v}$, $A \in M_{r,K}(\mathbf{F}_q)$. The residue vectors $(-Ae_j, e_j)$ recover A, so the edges are distinct. A lattice basis at the target and its Frobenius conjugates transport the construction to this family. The size bound gives $Nkc_tY \le a_0$, hence $(1 - kc_t/a_0)^N \ge 1 - 1/Y$. Summing (A.166) and applying 22.10 proves the result with C enlarged to absorb $C_t, C_D, C_r + 1$.
Lean: `Formalism.MainNotions.Pf14.manyEdges` (`Formalism.SharpRestriction.GrowthBounds.ScaleEdges.ColSpanMap`, lines 166–211)
Lean: `Formalism.MainNotions.Pf14.prvE_of_gl_operators_form_representation` (`Formalism.SharpRestriction.GrowthBounds.ScaleEdges.TheoremIIOfEdgeBoundConstants`, lines 9–74)

□

Proof of 22.19

Proof. Take one generating product of the segment symbols $A_jB'_j$, its low monomial and one monomial of U. All calculations in this paragraph precede the residual differential operators. In each non-left-frame occurrence the substituted segment exponents are multiples of M_x. The left-frame occurrence of a shared row uses M_xP_t. Thus the first residue selection either kills the entire product or leaves a plane factor times a normal factor. On the plane side the result is a fixed monomial X^b times the product of the renamed A_j, with each left-frame variable raised to its boundary power. The exponent b comes from the quotient of the low monomial times the fixed determinant monomial. It vanishes on interior plane covectors: the low exponents were below M_x, and the determinants use even plane rows. On the normal side there are no left-frame variables. Selecting the two row multidegrees preserves this factorization. The normal projection followed by coefficient extraction now multiplies the plane factor by a scalar.

Suppose that scalar is nonzero. On each normal prefix its type occurs in the product of the B'_j after row-degree selection. Only B'_j uses the normal letters of that prefix. Selection of row degrees commutes with the prefix action. Consequently k_{ju} is at most the greatest signed weight of B'_j on that prefix. Indeed, in an occurring irreducible a highest vector attains the first highest-weight entry, and a bound on the greatest weight in every frame holds throughout the module generated by the polynomial. The inequalities defining $A_jB'_j$ therefore give $v_{<x_u}(A_j) \ge H_{ju}$. For the term involving $u - 1$, the cut after x_{u-1} is the cut before x_u. An intervening odd letter adds no even predecessor, so the oldest even predecessor and the reserve s_u are the same at these two letters.
Lean: `Formalism.MainNotions.PairMembershipBlocks.restN` (`Formalism.SharpRestriction.GrowthBounds.PlaneCoefficients.CompDEqHomComp`, lines 17–34)
Lean: `Formalism.MainNotions.PairMembershipBlocks.restTh` (`Formalism.SharpRestriction.GrowthBounds.PlaneCoefficients.ValSCongrSupp`, lines 151–194)

For a chain cut, keep separate variables for the left and right occurrences of each shared row. Use a quotient letter at the left occurrence and a temporary label letter at the right occurrence. On this alphabet the cut acts on exactly the original prefix of A_j. The other segment factors use disjoint letters there. Projecting to the prescribed degrees after recombination commutes with this action on the separated alphabet: the degrees tested are sums of whole letter degrees with coefficients 1 or P_t. Expand next in the temporary label monomials. Extracting any one of their coefficients preserves the cut since no temporary label lies in it. If such a monomial has exponent h_t at a shared row, adding b_t selects the residue of $h_t + b_t$ and multiplies the quotient component by the monomial of exponent $\lfloor (h_t + b_t)/P_t \rfloor$. The other new factor on the cut is the interior part of X^b. Both have nonnegative least weight there, since their variables on the cut are vector variables. Summing the terms with the prescribed residues proves the chain bound H_{ju} for every digit component of the undifferentiated product.
Lean: `Formalism.MainNotions.TraceSpectra.restChain_of` (`Formalism.SharpRestriction.GrowthBounds.AevalValuation.MemSupportedOf`, lines 280–334)

For a link, decompose A_j by its individual letter degrees δ_y. This preserves its prefix bound. Removing the left frame from the action can lower its least weight by at most $\sum_{y \text{ in left frame}} \delta_y$. To see this, apply the inverse twist on those rows to the twisted prefix polynomial. Its entries on a vector of degree δ_y have powers at least $-\delta_y$. The removed rows are now outside the link, so their power substitutions preserve the resulting lower bound. The other segment factors involve, on this link, at most shared vector letters. Their least weights there are nonnegative.

Write d_y^{pl} for the selected physical plane degree, and b_y^0 for the row degree of X^b. In a nonzero component,

$$b_y^0 + P_y\delta_y \le d_y^{\mathrm{pl}} \qquad (y \text{ in the left frame}).$$

Indeed the remaining factors have nonnegative degree in that row. The total loss is consequently at most

$$L_j = \sum_{y \text{ in left frame}} (d_y^{\mathrm{pl}} - b_y^0)_+ / P_y.$$

Put $g = z_j - z_{j-1}$. In the first digit split the nonfixed degree at row y, before division by $p^{z_{j-1}-y}$, is at most $2a_0$. Each summand in L_j is therefore at most $2a_0 p^{-g-(z_{j-1}-y)}$. The distances $z_{j-1} - y$ are distinct nonnegative integers. As $p \ge 3$, summation gives $L_j \le 3a_0 p^{-g}$. At a right-frame cut the oldest of the preceding r even letters lies at least $L_1 + 1$ places beyond z_{j-1}: the first r interior letters after that frame are even, with the gap furnished by the marked tile. Thus $a_0 p^{-g} \le b_{x_u} p^{-L_1-1}$. The spacing hypothesis gives $48Yk^2 \le p^{L_1+1}$. Also $b_{x_u} \ge a_0/p^{G'}$, so the last size hypothesis gives $b_{x_u}/(Yk^2) \ge 1$, whence $b_{x_u}/(Yk^2) \le 2s_u$. It follows that $L_j \le s_u/8$. This proves the link bound $H_{ju} - s_u/8$ before differentiation.

Lean: `Formalism.MainNotions.PairMembershipBlocks.restLink_of` (`Formalism.SharpRestriction.GrowthBounds.PlaneCoefficients.SupportedOfSupport`, lines 77–199)

Lean: `Formalism.MainNotions.PairMembershipBlocks.arithCondition_of_one_le` (`Formalism.SharpRestriction.GrowthBounds.PlaneCoefficients.FortyEightBoundArith`, lines 51–214)

We propagate these two bounds through a residual word. Keep the assertions simultaneously for every first digit block, every normal type and every extracted coefficient. Charge one unit to cut u for a differentiation or multiplication at row s if that physical row occurs in the segment at or before x_u, and zero otherwise. The following digit calculation gives this charge at both levels of digit selection. In one coordinate write a polynomial as $\sum_{h=0}^{P-1} X^h f_h(X^P)$, with quotient variable V. Multiplication takes f_h to the next label, multiplying by V on a carry. Differentiation on a term with $h > 0$ gives $(h + PV\partial_V)f_h$ at label $h-1$. At $h = 0$ it gives $P\partial_V f_0$ at label $P-1$. The Euler operator $X\partial_X$ acts on a fixed label by $h + PV\partial_V$. These formulas include $P = 1$. A carry or a borrow changes the input quotient degree by one in exactly the affected letter, so the simultaneous assertion supplies the requisite input block.

In any frame, multiplication or differentiation shifts the least signed weight down by at most one on a cut containing its letter, and by zero outside it. A conjugate of $V_a\partial_{V_a}$ is a sum of $V_b\partial_{V_c}$; the two coordinate weights have difference at least -1 for either sign of the letter. Thus the Euler term also costs at most one. This proves the charge for plane operators, on the original links and, by the same digit calculation, on every quotient prefix. A quotient letter from the preceding frame is charged at its left occurrence.

For a normal operator, insert the finite sum of all joint normal projections before acting. On a prefix containing its letter, a multiplication, differentiation or Euler term raises the greatest signed weight by at most one, by the same coordinate calculation. Outside the prefix it preserves that bound. An input type with first entry k can therefore contribute to an output type with first entry k' only if $k' \le k+1$ on a touched prefix and $k' \le k$ on the others. Each active normal entry in the maximum defining H_{ju} uses a prefix ending at or before x_u. The change in that maximum is at most the assigned charge. After projecting, each output coefficient is a scalar linear combination of plane coefficients of the contributing input types. Least-weight bounds pass to such combinations. This proves the induction step for normal operators as well.

Lean: `Formalism.MainNotions.PairMembershipBlocks.oneOpStep_of` (`Formalism.SharpRestriction.GrowthBounds.PlaneCoefficients.RestOneOpOf`, lines 106–131)

Lean: `Formalism.MainNotions.LetterSubstitution.noOpStep` (`Formalism.SharpRestriction.GrowthBounds.MonomialSupport.Lam1SPderivAndXMulLeAddIndicator`, lines 38–140)

Each $X_{s,a}\partial_{s,b}$ incurs at most two charges. In the segment setup above, the entire segment, including both frames, lies in a window of length G', so it has at most N_1 physical rows; there are at most c_t occurrences per marked row in a residual word. For $C \ge 16$ the last size bound in the segment setup gives

$$16N_1c_tYk^2p^{G'} \le a_0,$$

using $N_1 \le rN$ and $c' \ge rNc_t$. Hence $2N_1c_t \le s_u/8$. Subtracting this from the chain bound gives the first assertion. Subtracting it from the link bound gives at least $H_{ju} - s_u/4$, which implies the second assertion.

To apply the induction, write the primitive combination in the exact symbol representatives as $\sum_{i,b} c_{ib}x_{ib}$, with i the flat symbol and b the residual tableau. For each summand the residual contraction gives a linear combination of the words just considered applied to symbol i. Changing to the edge coordinates conjugates a row matrix unit to a linear combination of matrix units in the same row. The two bounds are linear subspace conditions over the fraction field. They therefore hold for the sum with coefficients c_{ib}, for $\omega^{-a}\rho_\beta \sum c_{ib}x_{ib}$ when the component has exact order a, and after the nonzero scalar division defining P_c. In particular the second assertion puts P_c in the image of every link projector. Moving the self-adjoint projectors across the factorial form now gives the stated coefficient identity. At a minimizing monomial $v([X^{m_0}]P_c) = -v(m_0!)/2$, which gives its valuation.

Lean: `Formalism.MainNotions.TestSideCounting.cloudBAt_of_rho` (`Formalism.SharpRestriction.GrowthBounds.DualFockLinear.CloudBAtOfRho`, lines 87–298)
Lean: `Formalism.MainNotions.TestSideWords.memAt_of_wordConditionAt` (`Formalism.SharpRestriction.GrowthBounds.TwistedSubstitution.MemRed`, lines 40–64)

□

Proof of 22.20

Proof. Order a segment's interior first and append its right frame in chronological order $r, \dots, 1$. Thus each of its original links is an initial segment ending before the last letter. We give the integral basis argument used to separate the variables.

For this ordered word use paths of horizontal strips as in Definition 22.13, in rank r, with a standard terminal tableau and no restriction steps. Form a polynomial for each path and tableau by the exterior multiplication and contraction formulas of Lemma 22.17(1), with power one throughout. At a cut let μ be the path row and $h(c) = \#\{a : \mu_a \ge c\}$ its height in column c. Under the prefix action the state of a positive column is in $\bigwedge^{h(c)} V$; a nonpositive column uses $\bigwedge^{r-h(c)} V^\vee$. Later operations are linear in that state with coefficients in the suffix variables. The least possible first-coordinate weights of these two exterior powers are respectively

$$\mathbf{1}_{h(c)=r}, \qquad -\mathbf{1}_{h(c)<r}.$$

The sum over columns is μ_r, by counting integer columns on either side of zero. Each path polynomial consequently satisfies every lower bound read from its path rows.

These polynomials form an integral basis. Here are the independence and count arguments. They work over every residue field. Under a generic matrix change of frame all weights at the end of a path are dominated by its terminal row μ: in each exterior power the sum of the largest a coordinate weights is attained by the initial index set, or by its complementary final set for a dual column. Attaining μ forces that choice in every column. By exterior change of basis its coefficient is the canonical path polynomial with these index sets times the product of the corresponding flag minors of the generic matrix. For a negative column the complementary-minor identity changes the inverse minor to a flag minor divided by the determinant, up to sign. At fixed μ the determinant power is common. The flag products for distinct standard terminal tableaux are independent: under row-by-row lexicographic order on the entries of the generic matrix, the diagonal monomial is the unique leading monomial of each flag minor and hence of each flag product, with leading coefficient one, and its exponents give the entries in each tableau row, whose weak ordering recovers the row.

The canonical path polynomials at fixed μ are independent even over the field of the generic matrix entries. Inspect the last letter first, ordering its coordinate exponents lexicographically from r down to 1 for a vector and from 1 up to r for a covector. If the preceding row is λ, the leading coefficient in this letter is the preceding canonical polynomial, up to sign, at the monomial with exponents $|\mu_a - \lambda_a|$. Indeed a changing column ends on an initial set or its complementary final set. In the exterior sum the extreme admissible coordinate is the index of the changed row. Erasing it in a wedge or adjoining it in a contraction leaves the preceding canonical index. The term has coefficient 1 or -1. Multiplication over columns gives the indicated exponents. Since the direction of the strip is fixed, these exponents recover λ. Induction from the empty word proves independence. In a relation among all the labelled paths take a maximal terminal row in dominance order and extract that weight after the generic change of frame. Only this row contributes. Independence

of the canonical polynomials, followed by independence of the flag products, kills all its coefficients. This proves independence over the residue field.

Lean: Formalism.InvariantFactorPaths.GelfandTsetlinBases.pathProductLinearIndependent (Formalism.ThetaLogCoordinates.JacobiPathWeights.JacobiMinorProofs.IsWeightedHomogeneousZeroStripPathWord, lines 752–890)
Lean: Formalism.InvariantFactorPaths.GelfandTsetlinBases.Top.psiPath_basis (Formalism.ThetaLogCoordinates.JacobiPathWeights.PathBasisTheorem, lines 23–26)

By the horizontal-strip dimension identity used in Lemma 22.14, summing terminal tableau counts gives the product of the dimensions of the symmetric powers of the individual letters. For a dual letter the identity follows by dualizing, and shifting all entries of a row by the same integer tensors with a determinant character of dimension one. This count equals the number of ordinary monomials of the prescribed multidegree. The square coefficient matrix is therefore invertible modulo the maximal ideal and has unit determinant over the coefficient DVR.

At one cut the paths with $\mu_r \geq \theta$ span the required subspace over the fraction field. They lie in it by the exterior-power computation. Their number is its dimension: Pieri decomposition on the prefix gives one copy along each prefix path, the specified types are precisely those with $\mu_r \geq \theta$, and continuing along unrestricted suffix letters multiplies the dimension by their symmetric-power dimensions. Since the full family is an integral basis, an integral vector satisfying the cut has zero coefficients at all other paths and integral coefficients at the retained ones. Applying this to each cut in the same basis gives the simultaneous assertion. The alphabets comprising an interior and its right frame are disjoint across segments, so their tensor product gives the corresponding expansion for T.

Initially write every boundary variable as R_t. In each column that uses at least one interior letter of its segment, replace a boundary occurrence at $h > 1$ by $S_{j,h}$. Keep slot 1 on $R_{j,1}$. Leave columns without interior occurrences on the label letters. Each change recombines to the original column. A column is multilinear in its used letters, since a horizontal strip changes each column at most once per letter. There are at most d_{I_j} columns using the interior. Hence each surplus degree is at most d_{I_j}.

Consider the cut for m on the interior and surplus letters $h > m$. For a changed column this action pulls back to the original prefix action, so the column retains its exterior-power lower bound. An unchanged column is independent of the letters in the new cut. Its old lower bound is at most zero: a positive column using no interior letter could have received at most $r-m < r$ vector occurrences by the cut, and a nonpositive column has lower bound at most zero at every height. The column product therefore retains the full path lower bound at every cut.

Finally expand in all R-monomials. Their coefficients separately satisfy the bounds on the interiors and surplus letters. Replace R_t^h coordinatewise by $R_t^{h \bmod P_t} W_t^{\lfloor h/P_t \rfloor}$. This preserves integrality and recombination. In the enlarged link the additional W's are vectors and contribute nonnegative least weight in every frame. This proves the assertion after summing the expansion. □

Proof of 22.21

Proof. Fix an allowable exponent ζ. Its block fixes all letter degrees and all individual coordinates on the label and surplus letters. Denote its surplus exponent array by M, extended by zero on the interiors of the original alphabet. Recombination has exponent $\alpha + M$. Prefix projections preserve the block: they preserve each letter degree and act only on interior and quotient letters. Thus the proof can be made separately on each block.

Let $\partial^{[M]}$ be the product of the divided coordinate derivatives, and set $A = D_{u^{-1}} \partial^{[M]} D_u F$. The coefficient tested at α is

$$A_\alpha = u(\alpha)^{-1} u(\alpha + M) F_{\alpha+M} \prod_i \binom{\alpha_i + M_i}{M_i}.$$

Write $C_B = \prod_t (P_t!)^{|\beta_t|}$, where $|\beta_t|$ is the fixed quotient degree of slot t. Factoring the boundary factorials coordinatewise gives

$$\alpha!\, M! = C_B\, \zeta!\, \Gamma(\zeta)/u(\alpha), \qquad \Lambda(X^\zeta) = C_B\, \zeta!\, \Gamma(\zeta) A_\alpha.$$

In particular C_B is constant on the block.

Lean: Formalism.MainNotions.PlaneGram.TwistData.coeff_GM (Formalism.SharpRestriction.GrowthBounds.MonomialOrderSegments.SubstitutedExponentApplyLabel, lines 108–123)
Lean: Formalism.MainNotions.PlaneGram.TwistData.Lam_monomial_blk (Formalism.SharpRestriction.GrowthBounds.MonomialOrderSegments.SubstitutedExponentApplyLabel, lines 141–158)

We compute the chain orders of the digit component of A at ρ. Every coefficient of F has valuation at least $-n_0$, because its monomial factorial divides the product of the row factorials. Perform the calculation modulo coefficients of valuation at least $c' - n_0$. Both D_u and $D_{u^{-1}}$ may then be replaced by their integral binomial approximations of degree at most $6c'$ from Lemma 22.16. Its hypotheses hold for the first row split, whose digits lie between zero and $M_x - 1$, with $p \geq 13$. On any fixed boundary class, substitution of $\rho_t + P_t\beta_t$ preserves polynomial degree and integer-valuedness. The expansion in binomials of the free exponents is integral by forward differences. A factor $\binom{\beta_a}{h}$ acts by $W_a^h \partial_{W_a}^{[h]}$. In any frame its least signed weight shift is at least $-h$, since the coordinate weight differences are at least -1. Each of the two approximations therefore costs at most $6c'$ at every cut. All exact units, approximate multipliers and divided derivatives in this computation preserve coefficient lower bounds.

For the intermediate derivative consider a single shared row with power P, input label d, and derivative multi-index m. In one coordinate define

$$a_\ell = [T^m](1+T)^d\big((1+T)^P - 1\big)^\ell.$$

The binomial theorem gives

$$\binom{d+Pb}{m} = \sum_{\ell=0}^{m} a_\ell \binom{b}{\ell}.$$

Here $a_\ell = 0$ if $d + P\ell < m$. Since $(1+T)^P - 1 = T^P + pH(T)$ with $H \in \mathbf{Z}[T]$, at most $\lfloor m/P \rfloor$ of the ℓ factors can contribute T^P. Hence $v(a_\ell) \geq (\ell - \lfloor m/P \rfloor)_+$. Multiply these expansions over the coordinates of the row. Terms with $\sum_a (\ell_a - \lfloor m_a/P \rfloor)_+ \geq c'$ contribute to the discarded coefficient error. Every remaining nonzero term has

$$|\ell| \leq |m|/P + c', \qquad d_a + P\ell_a - m_a \geq 0.$$

On the quotient polynomial its operator is a scalar times

$$W^{\lfloor (d+P\ell-m)/P \rfloor} \partial_W^{[\ell]}.$$

Indeed on W^b its coefficient is $\prod_a \binom{b_a}{\ell_a}$, its output quotient is $b - \ell + \lfloor (d + P\ell - m)/P \rfloor$, and its output label is $d - m \bmod P$. The multiplier monomial uses a vector letter and has nonnegative least weight. Thus the loss is at most $|m|/P + c'$ on a cut containing this quotient letter and zero on other cuts. This calculation treats every input and output label, including borrows.

Lean: `Formalism.MainNotions.G.GLWeightTheory.derivative_digit_expansion` (`Formalism.SharpRestriction.GrowthBounds.RootWeights.BinomialExpansionCoeffZeroOfDeg`, lines 110–163)

The quotient letters on a chain cut are exactly those of the preceding frame. By allowability each $|M_t| \leq d_{I_{j-1}}$ there. The hypothesis on σ_{ji} gives

$$12c' + \sum_{t \text{ in frame } j-1} (|M_t|/P_t + c') \leq \sigma_{ji} + (12 - 12r)c' \leq \sigma_{ji}.$$

It follows that in the digit component of A the retained polynomial Q satisfies every bound $\theta_{ji} - \sigma_{ji}$; each coefficient of the difference has valuation at least $c' - n_0$. Multiply Q by the fixed label and surplus monomial and select the block letter degrees. Call the result Q_B. Both operations commute with all the prefix actions. Consequently $P_{\mathrm{pre}} Q_B = Q_B$.

For every ζ' in this block, with original collapsed exponent α', put

$$e_{\zeta'} = g_-(\zeta')\Gamma(\zeta')A_{\alpha'} - [X^{\zeta'}]Q_B.$$

The coefficient formula gives $v(A_{\alpha'}) \geq -n_0$, and $g_-\Gamma \equiv 1 \bmod p^{c'}$. Hence $v(e_{\zeta'}) \geq c' - n_0$. For W supported on the block we obtain the exact equality

$$\Lambda(D_{g_-}W) = C_B\langle Q_B, W\rangle + C_B \sum_{\zeta'} W_{\zeta'} e_{\zeta'}\, \zeta'!.$$

The first terms agree for $W = P_{\mathrm{pre}} X^\zeta$ and $W = X^\zeta$, by self-adjointness.

For either choice of W, the projector bound, also valid for the identity with loss zero, gives

$$v(W_{\zeta'}) \geq \tfrac{1}{2}v(\zeta!) - D_m - \tfrac{1}{2}v(\zeta'!).$$

We record the factorials needed to use this estimate. For any original exponent b in the prescribed degrees, $v(b!) \geq 2n_0 - k'$. In a row of degree d_i, successively choosing its coordinate exponents uses at most r binomial coefficients with upper argument at most d_i. Each costs at most $\lfloor \log_p d_i \rfloor$ for $d_i > 0$, by the carry formula from Legendre's identity in Lemma 11.7. A zero row costs zero. Splitting $b = \alpha' + M$ costs at most another k' by the same bound on each coordinate binomial. The block factorial identity therefore gives

$$v(C_B) + v(\zeta'!) = v(\alpha'!) + v(M!) \geq 2n_0 - 2k'.$$

It also holds for ζ. Average these two inequalities and add the bounds for $W_{\zeta'}$ and $e_{\zeta'}$. Each summand of the error has valuation at least $c' + n_0 - 2k' - D_m$. Subtraction proves this bound for the difference of the two evaluations with D_{g_-}.

Finally $D_{g_+}X^{\zeta} = g_+(\zeta)X^{\zeta}$. The additional error is $(g_+(\zeta)g_-(\zeta) - 1)\Lambda(X^{\zeta})$. Its first factor has valuation at least c'. The second has valuation at least $\frac{1}{2}v((\alpha + M)!) \geq n_0 - k'/2$, by $v_{\mathrm{F}}(F) = 0$ and the unit property of u. This proves the claimed precision on a monomial. Taking the finite linear combination with the integral coefficients of Z proves the result. □

Proof of 22.23

Proof. Use the invariant projection on a space of linear maps, first extracting its zero-weight part. For lattices L_1, L_2 preserved by divided roots, the lattice of maps from L_1 to L_2 is preserved by the divided adjoint action. Its j-th operator sends T to

$$\sum_{b=0}^{j} (-1)^b E_{uv}^{(j-b)} T E_{uv}^{(b)}.$$

Weight-zero extraction is integral when the lattices are graded. In the applications below all weights of the space of maps have entries in $[-l, l]$, since both its arguments have weights from A. Thus the row recursion (A.74) terminates at $q = l$. By (A.75), adjoining one of the j earlier indices in row j, numbered from zero, costs at most $\lfloor \log_p(l + h - 1) \rfloor$. Summing j gives d, as in (A.77). These identities use the matrix-unit commutators and the weight interval, so they apply to each of the spaces of maps used here. Write $I(T)$ for the resulting invariant map. On endomorphisms of an irreducible module U it is $\operatorname{tr}(T)1_U / \dim U$, since the recursion preserves trace. It commutes with composition by intertwiners, by the same recursion.

Choose one irreducible copy $U \subset A$ of type λ, and a basis b_j of $U \cap L$, with dual basis $b_j^{\vee}$. For $v \in A$ form

$$(\dim U) \sum_j I\big(b_j^{\vee} \otimes v\big)(b_j).$$

On a copy of U this gives v by the trace formula. On any other irreducible type it is zero since there is no nonzero intertwiner from U to that type. Complete reducibility applies over $\mathbf{Q}_p$ to the split semisimple Lie algebra with the center acting diagonally. Thus this is the type projector. For $v \in L$, the maps inside the sum take $U \cap L$ into L. The denominator calculation gives the first assertion, including sums of types.

On A_λ take the map extracting weight λ. It takes $L \cap A_\lambda$ into Δ_λ, since vectors of this weight in that component are highest. Apply the same recursion to maps from $L \cap A_\lambda$ to Δ_λ. Both lattices are divided-root stable and weight graded. On each irreducible copy the map being averaged has trace one. Its average is the inclusion divided by $\dim U$. Multiplication by $p^d \dim U$ gives the second inclusion. In rank one both assertions follow directly from integral weight extraction. □

Proof of 22.24

Proof. A cap on a group of coordinates says that the first-coordinate weight of every translate is at most its threshold. Indeed, the orbit span is the smallest stable subspace containing the polynomial. In a representation of the general linear group the largest first-coordinate weight of an irreducible summand is the first entry of its highest weight.

Restriction commutes with the actions on the proper prefixes and with the action of $\mathfrak{gl}(I)$ on the whole prefix. To see the cap on the first m letters, differentiate a polynomial of last degree d in W. The span of all its first derivatives, as the polynomial varies in a full cap, stays in that cap. The derivatives transform among themselves and each preserves or lowers first-coordinate weight. Differentiating d times shows that every coefficient in W satisfies the preceding cap, since the full action at last degree zero is the preceding action. This proves $RY_d \subseteq X_d$. With no finite full cap each element of X_d lifts to itself. The same holds in degree zero.

Let C be the least finite full cap. Write $A_{ab} = E^m_{ab}$ for the action on the earlier letters and $D_W = \sum_b W_b \partial_{W_b}$. For $J = I$ or $J = \{1, \dots, r\}$, and $a \in J$, put

$$U^J_a = W_a(C - D_W) - \sum_{b \in J} W_b A_{ab}.$$

Restriction intertwines the full U_a with U^I_a for $a \in I$. It also intertwines ∂_{W_a} and the matrix units with both indices in I.

The full U_a sends Y_j into Y_{j+1}. At an earlier cut, A_{ab} is the sum of the action at that cut and the action on the remaining earlier letters, both of which preserve its cap. For the full cut put $J_{ab} = A_{ab} + W_a \partial_{W_b}$. Then

$$U_a = CW_a - \sum_b W_b J_{ab}, \qquad [J_{uv}, U_a] = \delta_{va} U_u.$$

Consequently the span of the $U_a Y_j$ is stable under the full group. Only U_1 can raise first-coordinate weight. On the weight-C boundary, $J_{11} = C$ and $J_{1b} = 0$ for $b > 1$, since the latter operators raise that weight. Thus U_1 vanishes on the boundary. The span satisfies the full cap.

We prove surjectivity by induction on j. Write $d' > 0$ for the new degree. Derivatives ∂_{W_a}, $a \in I$, take $X_{d'}$ into $X_{d'-1}$ by the derivative argument above for the smaller group. By induction and the intertwining formula,

$$\Omega X_{d'} \subseteq RY_{d'}, \qquad \Omega = \sum_{a \in I} U^I_a \partial_{W_a} = (C - d' + 1)d' - \sum_{a,b \in I} W_b \partial_{W_a} A_{ab}.$$

We now give the kernel calculation. The operator commutes with $\mathfrak{gl}(I)$, since the two indices in its sum are contracted. A nonzero kernel on $X_{d'}$ therefore contains a highest vector $H = \sum_\beta W^\beta v_\beta$ of weight θ with $\theta_1 \le C$. Choose a nonzero coefficient maximizing $\sum_{a \in I} a\beta_a$, and put $\lambda_a = \theta_a - \beta_a$. For $a < b$, the equation $(A_{ab} + W_a \partial_{W_b})H = 0$ gives

$$A_{ab} v_\gamma = -(\gamma_b + 1) v_{\gamma - \epsilon_a + \epsilon_b}$$

when $\gamma_a > 0$, and gives zero when $\gamma_a = 0$. This implies $\lambda_a \ge \theta_b$. Indeed, with $k = \beta_b$, start at $\gamma = \beta + k(\epsilon_a - \epsilon_b)$. Applying A^k_{ab} gives a nonzero multiple of v_β, and one further application gives zero by maximality. In the finite-dimensional $\mathfrak{sl}_2$ for a, b, the resulting highest vector has weight $\lambda_a - \lambda_b$ and lies k steps up a root string. Hence $\lambda_a - \lambda_b \ge k$.

At W^β, the diagonal summands $W_a \partial_{W_a} A_{aa} H$ contribute $\sum_a \beta_a \lambda_a v_\beta$. For $a > b$ the required coefficient of H has larger $\sum a\beta_a$ and vanishes. For $a < b$ the highest-vector equation gives $-(\beta_a + 1)\beta_b v_\beta$. Thus the coefficient of ΩH is v_β times

$$(C - d' + 1)d' - \sum_a \beta_a \lambda_a + \sum_{a<b} (\beta_a + 1)\beta_b.$$

Successive use of $\lambda_a \ge \lambda_{a+1} + \beta_{a+1}$ and $C \ge \lambda_1 + \beta_1$ gives $C - \lambda_a \ge \sum_{j \le a} \beta_j$. Substitution and $\sum_a \beta_a = d'$ bound the displayed integer below by $d' + \sum_{a<b} \beta_b > 0$. This contradicts $\Omega H = 0$. Hence Ω is injective on $X_{d'}$. Its image lies in $RY_{d'} \subseteq X_{d'}$, and finite dimension gives equality.

Lean: `Formalism.InvariantFactorPaths.L4.omega_injective_holds` (`Formalism.ThetaLogCoordinates.FockResolutionOnto.FockMonomialProofs.OmegaInjectiveHolds`, lines 673–726)

We now keep the thresholds fixed and vary the last degree over all j. Let d_0 be the sum of the earlier degrees. If $C \ge S$, the full cap is automatic at the prescribed degree d and an integral element lifts to itself. Suppose $C < S$. Set

$$X = \bigoplus_{j \ge 0} X_j, \qquad L_X = X \cap L, \qquad L_R = \bigoplus_{j \ge 0} R(Y_j \cap L).$$

The preceding kernel calculation also bounds the possible degrees. On a highest vector in X_j, its last nonzero leading coefficient as chosen above satisfies $\lambda_{r-1}+j\le C$. Since λ_{r-1} is an earlier-letter weight, it is at least $-d_0$. Thus $X_j=0$ for $j>C+d_0$.

On X take matrix indices $0,1,\dots,r-1$ and use the operators

$$E_{00}=C-D_W,\quad E_{0b}=\partial_{W_b},\quad E_{a0}=U_a^I,\quad E_{ab}=A_{ab}+W_a\partial_{W_b}\quad (a,b\in I).$$

They preserve X by rational surjectivity and the intertwining just proved. They satisfy the matrix-unit relations. In particular $[\partial_{W_b},U_a^I]=\delta_{ab}(C-D_W)-E_{ab}$, the U_a^I commute by expansion using $[A_{ab},A_{ij}]=\delta_{bi}A_{aj}-\delta_{ja}A_{ib}$, and $[E_{ij},U_a^I]=\delta_{ja}U_i^I$. The weights are read off on monomials. Their zeroth entry is $C-j$, and the others are earlier signed weights plus the corresponding W-exponents. Each lies in $[-d_0,C+2d_0]$ on the nonzero spaces. For the zeroth entry this uses $C\ge -d_0$ whenever $X\ne 0$. The coordinate differences are at most $C+3d_0\le 4S$.

Both L_X and L_R are stable under the divided roots. For the derivatives this follows from the binomial formula for a divided derivative. For E_{ab}, $a\ne b$ in I, it follows by taking coefficients of an elementary shear. For U_a^J one can check all divided powers at once. Put $q=1+tW_a$. Multiply by q^C and make the following substitutions in a polynomial:

$$W\longmapsto W/q,\qquad V_b\longmapsto V_b-tW_bV_a/q\ (b\in J),\qquad Z_a\longmapsto Z_a+t\sum_{b\in J}W_bZ_b.$$

Here V runs through the earlier vectors, Z through the earlier covectors, and their undisplayed coordinates stay fixed. These substitutions compose by addition of t, including the prefactor, and their derivative at zero is U_a^J. The coefficient of t^h is therefore $(U_a^J)^h/h!$. All coefficients are integral, also for negative integral C, since q has constant term one. Taking J full and then putting $W_r=0$ gives the formula for $J=I$. Applying a full divided power to an integral lift thus gives an integral lift of the restricted divided power. Weight extraction on L_X is coefficient extraction by the monomial weights.

Lean: `Formalism.InvariantFactorPaths.L4.raising_divided_integral` (`Formalism.ThetaLogCoordinates.FockResolutionOnto.FockMonomialProofs.DerivativeDividedIntegral`, lines 434–437)
Lean: `Formalism.InvariantFactorPaths.L4.restrictedLattice_divided_power` (`Formalism.ThetaLogCoordinates.FockResolutionOnto.FockMonomialProofs.RestrictedLatticeDividedPower`, lines 52–72)

Apply 22.23 to X,L_X with $l=4S$. For $x\in L_X$, the component x_λ of each type satisfies $p^b x_\lambda\in L_X$. A second application puts $p^{2b}x_\lambda$ in the divided-root span of integral highest vectors for the displayed action. Each such vector is annihilated by every $E_{0b}=\partial_{W_b}$. In characteristic zero it is independent of W, so belongs to $X_0\cap L$ and lifts to itself. Divided-root stability puts the whole span in L_R. Sum over λ and take the last-degree-d component. This gives the asserted integral lift.

Lean: `Formalism.InvariantFactorPaths.L4.sat_main` (`Formalism.ThetaLogCoordinates.FockResolutionOnto.FockMonomialProofs.RestrictedSaturationHolds`, lines 49–87)

□

Proof of 22.25

Proof. Write $x=W_r$. For a cut containing W, multiplication by x^e raises the maximum first-coordinate weight over all translates by exactly e. To see this, take a generic frame. The largest weight of the translated polynomial is attained on a nonempty Zariski open set, and the coefficient of W_1 in the translated x is nonzero on another such set. On their intersection the top weight terms multiply to a nonzero polynomial. For an earlier cut x is fixed. Division by x^e therefore gives exactly the stated shifts, preserves coefficient integrality, and subtracts $e\epsilon_r$ from the weight. It commutes with the raising operators of $\mathfrak{gl}_{r-1}$.

After division, 22.24 describes the rational image before the weight and highest-vector conditions. Full weight extraction commutes with restriction. On the fixed weight, projection onto highest vectors of the smaller group is projection onto the isotypic component whose highest weight equals the restriction of that weight. If the restricted weight is not dominant, the projection is zero. This description follows because the common kernel of the raising operators in an irreducible module is its highest line. Restriction is equivariant for the smaller group, and the prefix conditions are stable under it. These projections therefore commute with restriction and preserve the caps. This proves $\mathbf{Q}_pG_e=x^eH$.

For an integral $y\in H$, first lift $p^{2e_2}y$ by 22.24. The total degree after division is $S-e\le S$. Extracting the specified full weight from the lift is integral. For the smaller-group type projection use 22.23. In any

coordinate the weights of the signed polynomial space lie between minus the sum of covector degrees and the sum of vector degrees, an interval of width at most S. The coefficient lattice is weight graded and stable under divided elementary shears. Thus p^{e_1} clears the projection. Multiplying the result by x^e gives an element of $M_{\geq e}$ with initial part $p^{e_1+2e_2}x^e y$.

For the partner projection, first keep just the monomials with x-exponent e and strip off x^e. On the resulting space impose the full-group type caps for prefix lengths $0, \dots, m$, combining coincident caps by their minimum. At length m include the bound inherited from each full cut. On all letters impose the smaller-group type cap, extract the full weight, and impose the smaller-group highest-vector projection. These projections commute. A type projector for an earlier full group commutes with its matrix units and with the matrix units on the remaining letters, hence with the later group action. The smaller group preserves the exponent-zero condition on x. The factorial pairing interchanges E_{ab} and E_{ba}, so distinct isotypic components are orthogonal, and weight extraction is orthogonal on monomials. Restoring x^e multiplies the factorial form on the retained monomials by a common scalar. The resulting map is consequently the required orthogonal projection.

At lengths zero and one there is a single full-group type: respectively the trivial type and a symmetric power of the standard module or its dual. Thus the two projections are integral. Lengths $2, \dots, m$ cost at most $(m-1)e_1$ altogether by 22.23. The smaller-group cap and the highest-vector projection cost at most e_1 each. This gives $D = (m+1)e_1$. For $m = 0$ the specified full weight singles out at most one monomial in W. The caps and the highest-vector equations either retain that line or kill it. A retained monomial is its own integral lift and coefficient extraction is its orthogonal projector.

Lean: `Formalism.InvariantFactorPaths.L4.highest_vector_projection_holds` (`Formalism.ThetaLogCoordinates.FockResolutionOnto.FockMonomialProofs.HighestVectorProjectionPos`, lines 252–514)

□

Proof of 22.26

Proof. Let u be primitive in G_e. Select a coefficient $a = u_\alpha$ of least valuation. Then $a^{-1}u \in V_e \cap P_{\mathbf{Z}_p}$. If $v_p(a) > c$, the equality $u = (a/p^c)(p^c a^{-1}u)$ contradicts primitivity. The vector $p^{c+D}y_\alpha$ belongs to G_e, and

$$v_p([u, p^{c+D}y_\alpha]_0) = c + D + v_p(a) - \sum_i v_p\binom{d_i}{\alpha_{i1}, \dots, \alpha_{ir}} \leq D + 2c.$$

Apply this to every source basis vector in a Smith reduction of the Gram matrix over $\mathbf{Q}_p$ by unimodular integral row and column operations. A zero diagonal would annihilate that row. A diagonal of valuation greater than $D + 2c$ would take every integral partner to a value of greater valuation. Hence every diagonal is nonzero with the stated bound, and taking their product proves the assertion. □

Proof of 22.27

Proof. Each $M_{\geq e+1}$ is saturated in $M_{\geq e}$, as it is the kernel of coefficient extraction. Starting at $e = o$ and extending bases downwards gives the asserted basis. In particular the g_b of order e form an integral basis of G_e.

First take monomial inputs with coefficient one. The two substitutions have integer coefficients. In their expansions write the two powers of η as $n - S + v, n - S + v'$, before applying contractions. The power of W_ρ in each term is at most v, respectively v'. For the term with l contractions move E^l across the form. If $a, a' \leq l$ contractions on the two sides use ∂_{Y_ρ}, matching monomials require $v - a = v' - a' = b \geq 0$. The n-dependent factors, apart from constants, are

$$\binom{o+n+r+z-1-l}{o-l}(n-S+v)_{\underline{a}}(n-S+v')_{\underline{a'}}\frac{(n-S+b)!}{(n-S)!},$$

where $x_{\underline{j}} = x(x-1)\cdots(x-j+1)$. The displayed expression is polynomial of degree at most $o-l+a+a'+b = o-l+a'+v \leq o+S$. The binomial in front follows by cancelling $\binom{o+n+r+z-1}{l}$ against the prefactor. This proves polynomiality over $\mathbf{Q}$ on monomials and hence over $\mathbf{Q}_p$ on the given basis.

We compute the top term first on real monomials, where the factorial form is positive. On bidegree (o, n) the spaces $E^j \ker F$, with the kernel taken in degree $(o-j, n-j)$, are mutually orthogonal and span. Indeed

F is adjoint to E, so the orthogonal complement of E times the preceding bidegree is $\ker F$. Iterate this equality. From $[F,E] = r + \deg_W + \deg_Y$, the eigenvalue of T_z on $E^j \ker F$ is

$$\sum_l (-1)^l \binom{j}{l} \frac{\binom{o+n+r-j-1}{l}}{\binom{o+n+r+z-1}{l}} = \frac{(z+j)_{\underline{j}}}{(o+n+r+z-1)_{\underline{j}}}.$$

The last identity is inclusion-exclusion for choosing j elements from a set of size $o+n+r+z-1$ while avoiding a specified subset of size $o+n+r-j-1$.

For a monomial of exact order e, its substituted polynomial is $E^e V_n$, where $V_n = \eta^{n-S} V$, V is independent of n, and V contains no W_ρ. The largest possible η-degree in V is $S-e$. Its coefficient there is the original monomial divided by W_ρ^e. Thus the squared norm of V_n, divided by $(n-S)!\, n^{S-e}$, tends to the factorial of this latter monomial. The same normalized squared norm of FV_n stays bounded: the W_ρ-derivative vanishes and every other Y-derivative acts on a bounded degree. On every nonharmonic component of V_n the positive eigenvalue of EF is at least $n-o$. Its total normalized squared norm is consequently $O(1/n)$.

On harmonic level j of the reduced bidegree, the eigenvalue of $F^e T_z E^e$ is

$$(j+e)_{\underline{e}} (o+n+r-e-j-1)_{\underline{e}} \frac{(z+j+e)_{\underline{j+e}}}{(o+n+r+z-1)_{\underline{j+e}}}.$$

It tends to $(e!)^2 \binom{z+e}{e}$ for $j=0$ and is $O(n^{-j})$ for $j>0$. The preceding norm estimate and $\binom{o+n+r+z-1}{o}/n^o \longrightarrow 1/o!$ now give, for equal orders e, the coefficient at degree $S+o-e$:

$$\frac{p^u e! \binom{z+e}{e}}{o!\, S!} [g_b, g_{b'}]_0.$$

For distinct monomials the corresponding factorial pairing is zero. More generally, in the norm defined by T_z, a monomial of order e, after this normalization, has all its limiting mass in harmonic level e of the full bidegree. For two different orders, orthogonality and Cauchy–Schwarz therefore make the mixed degree strictly less than $\big((S+o-e)+(S+o-e')\big)/2$. These are coefficient identities and vanishings of rational polynomials for pairs of monomials. Bilinearity transfers them to arbitrary $\mathbf{Q}_p$-coefficients. In a basis vector of order e, higher-order monomials do not affect the coefficient just calculated.

In the determinant expansion the sum of the half-degree bounds along any permutation equals s_A. Equality requires preservation of each order block. The coefficient at s_A is therefore the product of the determinants of the displayed blocks. It is nonzero under the hypothesis. □

Proof of 22.28

Proof. Evaluation at $Y_\rho = 1$, $Y_c = 0$ for $c \neq \rho$ recovers f. The upper root operators with both indices different from ρ act on $f[n]$ by acting on f before substitution. Those with second index ρ annihilate the substituted generators and Y_ρ, by the vector and covector substitution formulas. Hence $f[n]$ is a nonzero highest vector of weight $\lambda = w - n\epsilon_\rho$. Put $a = n' - n$. The vector $Y_\rho^a f[n]$ equals $f[n']$ and has weight $\lambda' = \lambda - a\epsilon_\rho$.

In the space of products of the prefix module with degree-a polynomials in Y, the largest possible weight is λ', with a one-dimensional weight space. Indeed each factor has a one-dimensional highest weight space, and every other weight is lower in the dominance order. Let $\mathcal{C} = \sum_{i,j} E_{ij} E_{ji}$ for the prefix including Y. On the irreducible of highest weight ν its scalar is $c(\nu) = \sum_j \nu_j(\nu_j + r + 1 - 2j)$, by applying the commutators to a highest vector. If $\nu < \lambda'$ has the same total weight, summation by parts against the strictly decreasing sequence $\lambda'_j + \nu_j + r + 1 - 2j$ gives $c(\nu) < c(\lambda')$. The polynomial $P(\mathcal{C}) = \prod_{\nu<\lambda'} (\mathcal{C} - c(\nu))$, with one factor for each other irreducible type present, maps into the module generated by $f[n']$ and is nonzero on its highest line.

Use the map $v \mapsto P(\mathcal{C})(Y_\rho^a v)$. It commutes with all raising operators and shifts every weight by $-a\epsilon_\rho$. A nonzero kernel on the irreducible prefix module would contain a highest vector: take a maximal weight in the kernel. The unique highest line maps nontrivially. Extend the injection coefficientwise on the disjoint suffix monomials. For a cut before Y, the new highest vector satisfies its threshold by the substitution, which is equivariant on those letters, and the prefix action preserves that cut space. For a cut including Y,

multiplication by a covector polynomial preserves an upper first-coordinate bound: every weight in its orbit has first coordinate at most zero. The prefix Casimir preserves such a cut space because it commutes with the action on the whole larger prefix and preserves its weight spaces. The image thus lies in $Z_{n'}$. □

Proof of 22.29

Proof. Use polynomial representatives as in 14.12, subtracting the last entry from every entry before forming the column degrees. Put

$$\beta_j = c + \lambda_{j+1}, \quad \alpha_j = Q + \beta_j, \quad N = Q + 2c, \quad \alpha'_j = N - \alpha_{h+1-j} \qquad (1 \le j \le h).$$

The middle coefficient spaces have shapes α, α'. We need a pairing ℓ with

$$\ell(gv, gt) = \det(g)^N \ell(v, t).$$

We now give its integral normalization. Put $e = \alpha_h$, $\nu = \alpha - e\mathbf{1}$. Divide the first input by $\det^e$. For the second, on a generic h-square matrix Z substitute $Z_{ij} \mapsto \operatorname{adj}(Z)_{h+1-j,i}$, then divide by $\det(Z)^{|\alpha'|-(N-e)}$. Jacobi's complementary minor identity sends an initial minor of height d to the complementary initial minor times $\pm\det^{d-1}$, with the indicated reversal of columns. The full product has $N-e$ columns, so the displayed power divides it over O. Complementing the columns again recovers every generator of shape ν, up to sign and the prescribed full determinant columns. Thus both maps onto shape ν are integral surjections. Under g in the second input the complement substitution transforms by $\det(g)^{N-e}$ and inverse transpose. The first input contributes $\det(g)^e$.

On shape ν take the contravariant form normalized to have value one on the highest vector. Its dual of the dual Weyl lattice is the Weyl lattice: transposing a divided root operator interchanges its two indices, and evaluation on the highest line is the integral highest coefficient. Equivalently, the pairings with all divided-power words are the coefficients of the successive elementary row substitutions. The integral dual Weyl lattice consists of the sections with all those coefficients integral, by the tableau basis in 14.12. For $h \ge 5$, put $d = \binom{h}{2}\lfloor \log_p(\nu_1 + h - 1)\rfloor$. The inclusions in 14.12, with $\nu_1 \le 2c$, make the form integral after multiplication by p^d, and its image contains p^d times the dual lattice. For $1 \le h \le 4$, use the same nondegenerate contravariant form over the fraction field. The fraction-field highest-weight module is irreducible; the radical is invariant under transposed root operators, and the highest-vector value is one, so the form is nondegenerate. In bases of the two finite free lattices its matrix has finitely many entries, so multiplication by a sufficiently large power of p makes every entry integral. Complementation retains the determinant covariance.

Apply Smith reduction to this integral pairing. Its nonzero diagonal entries s_b have nonnegative normalized p-valuations. Set $k_b = \lceil v_p(s_b) \rceil$ and multiply the second Smith vector by p^{k_b}/s_b, which is integral. The resulting full-rank pairs satisfy $\ell(u_b, u'_a) = \delta_{ab} p^{k_b}$. When $h \ge 5$, the inclusion of p^d times the dual gives $v_p(s_b) \le d$, hence $k_b \le d$; for $h \le 4$ no numerical bound is needed. Lift the pairs through the two integral surjections.

Lean: `Formalism.MainNotions.flipOneThree.Pf.tauP` (`Formalism.SharpRestriction.GrowthBounds.HermitianTensor.TauP`, lines 303–482)

Lean: `Formalism.MainNotions.flipOneThree.Pf.pb_smith_lam` (`Formalism.SharpRestriction.GrowthBounds.HermitianTensor.PbSmithLam`, lines 27–69)

On a generic k-square matrix X, let $D(Z) = \operatorname{diag}(1, Z, 1)$. Define a bilinear map on sections by

$$\mathcal{B}(v, t)(X) = \ell_{Z,Z'}\big(v(XD(Z)), t(XD(Z'))\big).$$

The subscript means to pair coefficients of the two polynomials in X and convolve their X-exponents. The coefficients in Z, Z' have the middle column degrees of v, t, respectively. Addition of an earlier middle column to a later one preserves these coefficients by the defining column invariance. They are therefore integral elements of the two middle lattices whenever v, t are integral. Left multiplication on X commutes with both substitutions. The determinant covariance of ℓ makes the middle degrees of the output all equal to N. Adding the first column to a middle column amounts, in either input, to adding multiples of its first column to its middle columns. The last column has degree zero. These checks give the defining column invariance and degrees of the upper flat space when v, t have weights $\lambda, \delta^\vee$, with $\delta^\vee_j = -\delta_{k+1-j}$.

Lean: Formalism.MainNotions.flipOneThree.Pf.lamOf (Formalism.SharpRestriction.GrowthBounds.HermitianTensor.CoeffSumMonomialLinear, lines 53–55)
Lean: Formalism.MainNotions.flipOneThree.Pf.lamMemP (Formalism.SharpRestriction.GrowthBounds.HermitianTensor.SubCoeffP, lines 10–57)

Write $u_b = \det^{Q} v_b$, which is possible integrally since every tableau of shape α has Q full columns available for removal. For each middle initial minor in v_b adjoin the first row and column, shifting the middle row indices up by one. If the product has d_0 columns, multiply by $X_{11}^{2c-d_0}$. This exponent is nonnegative since $\beta_1 \le 2c$. Extend linearly and denote the lift by s_b. On $X = D(Z)$ it restricts to $v_b(Z)$. The same construction on u'_a, whose first row has length at most $2c$, gives t_a restricting to u'_a. These have weights $\delta, \delta^\vee$; their first-row degrees are $2c$ and their last-row degrees zero. Set $U_b F = F s_b$, $L_a v = \mathcal{B}(v, t_a)$.
Lean: Formalism.MainNotions.flipOneThree.Pf.liftP (Formalism.SharpRestriction.GrowthBounds.HermitianTensor.InitialMinorLiftFirstEntry, lines 284–315)

For flat F, $F(XD(Z)) = \det(Z)^{Q} F(X)$: the height-one columns see the first column, and each height-$(k-1)$ minor sees all the middle columns. For the fixed lifts, $\mathcal{B}(F s_b, t_a) = F H_{ab}$ has the same multiplier for every height-one weight with middle determinant weight Q. Compute it on the auxiliary vector $F = \Delta_{k-1}^{Q}$ of weight $[0, Q]$, where Δ_{k-1} is the initial principal $(k-1)$-minor. The output has the maximum possible degree in the first row and degree zero in the last. In the two-column-height tableau description of a flat space this line is spanned by the highest tableau. Indeed all height-one factors must use row one, and all height-$(k-1)$ factors must omit row k. Evaluation at $X = I$ gives its scalar $\ell(\det^{Q} v_b, u'_a) = \delta_{ab} p^{k_b}$. Cancel the nonzero initial minor power. The remaining highest tableau is the one of flat weight $[2c, 2c]$, denoted by z_0. Hence $iF = z_0 F$ has the asserted composition identity on every F.

In an adapted frame, the normalized z_0 is primitive since its tableau polynomial has a unit diagonal coefficient. If $z_0 F$ belongs to the integral upper lattice, choose the least $j \ge 0$ with $\omega^j F$ integral in the lower frame. For $j > 0$, reduction of $z_0(\omega^j F)$ is zero. Both factors have nonzero reduction as polynomials, a contradiction. In using coefficientwise reduction here, the integral tableau lattices are saturated in the polynomial coefficient lattices by their unit leading monomials. Undoing the common determinant twists gives saturation in the selected edge lattice. The resulting retraction is required in this one lattice. □

Proof of 22.30

Proof. Fix a tie i, and let j be the length of its prefix including the covector. The highest vectors on this prefix are the polynomials $f[n_i]$ of 22.27. Indeed, evaluation at the last coordinate covector recovers f. Conversely, on the chart $Y_r \neq 0$, the upper elementary substitutions with indices a, r take a covector to $Y_r e_r^\vee$. Invariance under those substitutions and homogeneity in Y recover exactly the formula in 22.27. Evaluation preserves integrality, and the inverse formula is integral since n_i is at least the preceding degree. A cap on the earlier letters is preserved by each simultaneous change of their frame, and evaluation recovers the same cap. Thus this is also an identification of the integral highest-vector lattices, weight by weight. The new highest weight is $w - n_i \epsilon_r$.

Use left and right integral Smith bases, with substituted highest vectors h_a, h'_a, and diagonal entries d_a. Write B_{F} for the unnormalized factorial pairing on the j letters. With o the marked vector degree and S the preceding total degree, the common scalar is

$$c_i = \frac{p^{\lceil \kappa \rceil} \binom{o+n_i+r+z_i-1}{o}}{(n_i - S)!\, S! \prod_{\nu<j} d_\nu!},$$

where d_ν here denotes a letter degree. Thus $c_i B_{\mathrm{F}}(\Phi_i h_a, h'_b) = \delta_{ab} d_a$ within each weight. Also $c_i B_{\mathrm{F}}(h_a, h_b)$ is integral. In a monomial pairing the normalized factorial denominators are row multinomials. By the factorial formula of 11.7 their total valuation is at most $jr\lfloor \log_p X \rfloor$: at each power $p^\alpha \le X$ the difference of floors in one row is at most $r-1$. Replacing $n_i!$ by $(n_i - S)!S!$ contributes the integer $\binom{n_i}{S}$. The other binomial in c_i is integral.

Define $\pi_i h_a = h_a$ if $v_p(d_a) \le E$, and zero otherwise. On highest vectors define a second map by

$$A_i h_b = \sum_{\substack{a:\ \mathrm{wt}(h_a) = \mathrm{wt}(h_b) \\ v_p(d_a) \le E}} \frac{p^{\lfloor E \rfloor}}{d_a}\, c_i B_{\mathrm{F}}(h_a, h_b)\, h'_a.$$

These coefficients are integral. Extend both maps equivariantly on the rational prefix space satisfying its caps. This extension exists because in each isotypic component the highest vectors are its multiplicity space. Precompose with the commuting type projections for the caps on the prefix, and act coefficientwise on the suffix monomials. On Y these preliminary projections fix every prefix coefficient. Each extended map preserves Y: it takes its values in the earlier cap spaces, and for any later cut it commutes with that cut's group action and preserves weights. In particular it carries the largest stable space with the stated weight bound into itself.

There is a uniform denominator for A_i. Projection onto the prefix cap space costs at most ja_0, by 22.23 applied to the distinct positive prefix lengths. It commutes with the action on the j letters, so takes a highest vector to a highest vector. The map just defined takes integral highest vectors in the cap space to integral polynomials. On an arbitrary integral prefix coefficient, project first onto each type, at cost a_0, and apply the divided-root generation assertion of 22.23, at cost another a_0. The map is equivariant, and divided roots preserve integral coefficients. This proves

$$p^{K_i} A_i M \subseteq M, \qquad K_i = (j+2)a_0 \le \kappa.$$

All applications have the required weight bound since a coordinate on any prefix ranges from minus its covector degree to its vector degree.

The Smith identities on highest vectors give

$$B_0(\Phi_i v, A_i v') = p^{\lfloor E \rfloor} B_0(\pi_i v, v')$$

for v satisfying the caps through tie i and $v' \in Y$. To extend the identity, move each root operator on one argument to its transpose on the other. The two forms are contravariant and agree on the highest multiplicity spaces, so agree on the modules they generate. Distinct suffix monomials are orthogonal and matching monomials multiply both sides by the same factorial. Also π_i commutes with every later Φ_j, since the later pair is disjoint from its prefix.

In Y, the codimension of the fixed space of π_i is $\sum_w b_{iw} z_{iw}$. To see this, split the rational prefix module into the copies generated by the h_a. Projection onto any one copy extends coefficientwise and preserves the later caps by equivariance, as above. Its image on Y is the intersection of those caps with the product of the copy and the suffix polynomial space. Copies of the same type have equal image dimension: the isomorphism carrying one highest vector to the other and its inverse both extend coefficientwise and preserve every later cap. Their common dimension is z_{iw}. This argument also gives $\dim Y = \sum_w (\operatorname{rank} M_w) z_{iw}$ for each fixed tie.

Let W be the intersection in Y of all these fixed spaces. Its codimension is at most the sum of their codimensions. The lattice $W \cap M$ is saturated in M. A primitive v in it has a unit polynomial coefficient, since division by p preserves every rational condition. Pair v with the monomial of that coefficient. The normalized factorial on this monomial is an inverse product of multinomials, so the pairing has valuation at most zero. Project the monomial onto all the caps and multiply by p^{K_0}, $K_0 = (L+1)a_0$. This yields $y \in M$ with $v_p B_0(v, y) \le K_0$. The projections are self-adjoint and fix v.

Take $y' = (p^{K_1} A_1) \cdots (p^{K_t} A_t) y \in M$. Applying the pairing identity in order $1, \dots, t$ gives

$$B(v, y') = p^{\sum_i (K_i + \lfloor E \rfloor)} B_0(v, y).$$

At step i, the later tie operators preserve the earlier prefix caps and commute with π_i, which fixes v. Hence the valuation is at most $\lfloor t(E+\kappa) + \kappa \rfloor$. If the number of Smith entries above this threshold exceeded $\operatorname{codim}_Y W$, their rational left Smith span would meet W nontrivially. Scale an element of the intersection to be primitive in M. Every partner in M would then have pairing of valuation above the threshold, contrary to the construction. This proves the bound. □

Proof of 22.31

Proof. Let (u_a) represent the residue line over $\mathbf{F}_{p^{n'}}$. Modulo the uniformizer, the edge substitution at s_j sends X_{ja} to $u_a^{p^{s_j}}$ times the surviving coordinate of that row. For $u \neq 0$,

$$\prod_j (u^{p^{s_j}})^{h_j} = \prod_j \frac{u^{p^{s_j + G_j}}}{u^{p^{s_j}}} = 1,$$

since $u^{p^{n'}} = u$ at the wraparound. If $u_0 = 0$, the first factor of R vanishes. Otherwise a competing line has $u_a \neq 0$ for some $a > 0$. Its a-th difference vanishes by the product identity. At the distinguished line the nonzero coordinate is normalized to $u_0 = 1$, which gives the indicated monomial with coefficient one. Taking the c-th power gives the divisibility and the exact multidegrees. □

Proof of 22.32

Proof. There are at most k digits in D_i, so $D_i \le kQ$. Since successive odd distances differ by at least two,

$$P_i C_i \le a_1/p^{d_i} + 2b' \sum_{\nu<i} p^{d_\nu - d_i} \le a_1/p^{d_i} + 2b'/p.$$

In the first range insert this inequality in the bound on e_i, using $P_i H_i \le b_i \le 2b'$. The excess of $P_i e_i$ over the minimum is at most

$$8b'/p^2 + L'a_1/(pk) + 2L'b'/(pk).$$

In the other range the minimum is at least a_1/p^{d_i}. The excess is at most $2b'/p + 2L'b'/k$. Thus in either range an upper bound is $4b'/p + L'a_1/(pk) + 4L'b'/k$.

Finally $C_s \le a_1/p^2 + 2sb'/p$, since $G \ge 2$ and $P_i \ge p$. The terminal contribution is at most

$$a_1/p^2 + 2sb'/p + 2sL'a_1/(pk) + 8sa_1/p^3 + 2L'b'/k.$$

Adding the s interior bounds and their digits proves the inequality coefficient by coefficient, using $p \ge 5$. □

Proof of 22.33

Proof. By the polynomial calculation in 22.27, each entry has degree at most $S + o \le 2S$. Evaluate it at $n = S, S+1, \dots, S+2S$. The two input polynomials $f_b[n]$ and $f_{b'}[n]$ have integral coefficients, since the substitutions in 22.27 have integral coefficients and $n \ge S$.

For the term of contraction order l, set $D = o + n + r + z - 1$. Cancellation gives

$$\frac{\binom{D}{o}}{\binom{D}{l}} = \frac{\binom{D-l}{o-l}}{\binom{o}{l}}.$$

The denominator costs at most ℓ_X in valuation, by 11.7 with $l \le o \le S$. The size of D does not enter this estimate.

On a monomial with tied exponents $\alpha, \beta \in \mathbf{N}^r$, the contribution of a multi-index a with $|a| = l$ to $F^l/(l!)^2$ is zero unless $a \le \alpha, \beta$. Otherwise its coefficient is

$$\frac{\prod_{h=1}^{r} \binom{\alpha_h}{a_h}\binom{\beta_h}{a_h}}{\binom{l}{a_1,\dots,a_r}}.$$

This follows by expanding the power of the sum of the commuting contractions and writing each falling factorial as $a_h!$ times a binomial. The denominator is a product of at most $r-1$ binomials with upper argument at most l. Its cost is at most $r\ell_X$. Multiplication by E^l has integral coefficients and restores the row degrees.

There are now $m+2$ rows, of degrees $d_1, \dots, d_{m+1}, n$. On two matching monomials the factorial pairing divided by $n! \prod_i d_i!$ is the product of the inverses of their row multinomials. Every row degree is at most $3S \le pX$. By successive binomial choices and 11.7, each row costs at most $r\ell_X$. Replacing $n!$ by $(n-S)!S!$ multiplies the value by $\binom{n}{S}$. The total denominator cost is at most $((m+2)r + r + 1)\ell_X$, so the scaled entry is integral at every evaluation point.

If $a(n)$ is this entry, its coefficient of $\binom{n-S}{j}$ is $\Delta^j a(S)$. This is an integral linear combination of the first $j+1$ values just computed. Finally

$$\binom{n-S}{j} = \sum_{h=0}^{j} \binom{-S}{j-h}\binom{n}{h},$$

and $\binom{-S}{j-h} \in \mathbf{Z}$. □

Proof of 22.34

Proof. For one variable x, the operator of index n takes the coefficient of t^n after $x \mapsto (1+t)x$. On the reduced word y this is $x^n \partial_x^{[n]}$, where the divided derivative has monomial coefficient $\binom{\mu_x}{n}$. At a prefix containing that letter, every conjugate of the operator is a sum of products of n coordinate multiplications and n derivatives on the same letter. The signed first-coordinate weight increases by at most n. Indeed on a vector letter each multiplication has weight zero or one and each derivative has weight zero or minus one. On a covector letter the signs reverse. A prefix omitting the letter commutes with the operation. Applying this argument in every basis gives the stable prefix bounds.

If x occurs in an internal tie polynomial, write $H_h = A + B$, where A is its unique summand containing x. The coefficient of t^l in the transformed $H_h^{j_h}$ is

$$\binom{j_h}{l} H_h^{j_h - l} A^l.$$

For an ordinary coordinate $A = W_{ha} Y_{ha}$. Multiplying the reduced word by A^l adds l to the two reduced degrees and uses at most l in any prefix bound. Under any prefix change of basis each vector-covector product has greatest first-coordinate weight at most one, including when only its vector occurs in the prefix. For a scalar coordinate $A = Z_h S'_h = H_h - E_h$. Expanding A^l gives terms with $d \le l$ factors of E_h, level $j_h - d$, and cost at most d by the same argument. The factor $Z_h^{z_h}$ contributes integral binomial coefficients. The other ties use disjoint variables. In extracting t^n, the orders falling on all factors sum to n. This proves the one-coordinate assertion, and composition in the different coordinates proves the displayed inclusion.

For saturation, express a polynomial in the field span as $\sum_{j,\alpha} (\prod_h H_h^{j_h} Z_h^{z_h}) y_{j,\alpha} \xi^\alpha$, collecting equal indices. Set all $Z_h = 1$, then substitute $S'_h - E_h$ for S'_h at every tie. These substitutions preserve integral coefficients and take H_h to S'_h, also at the terminal tie where E_h involves ξ. The coefficient of $\prod_h (S'_h)^{j_h} \xi^\alpha$ is $y_{j,\alpha}$. Hence all these reduced polynomials are integral. They already satisfy the prefix and degree conditions in the field span, so the displayed expression belongs to $L(b)$. □

Proof of 22.35

Proof. Put $r = k - 1$, $\ell = 1 + \log_p(8a_1 + r)$, $Q_p = p^{G'}$, and $q'_p = p^{n'}$. The number of edges of rank one into a target is $E_q = ((q'_p)^k - 1)/(q'_p - 1)$, by counting nonzero residue forms modulo scalars. At a source count residue lines. For flat weight $(A_s, 0, \dots, 0, -B_s)$ at the marks and zero elsewhere, let $f(A,B)$ be the least nonnegative loss allowing $(1 - f(A,B)) \dim V / E_q$ pairs on each edge at depth t_*, order c_4, and common diagonal valuation at most D_1. Index the budgets by the gaps between successive even marks. Set

$$\kappa_0 = 2Nk \log_p a_1, \quad D_j = C_A Y^2 N_1^2 k^2 \ell + 2\kappa_0, \quad D_0 = \sum_j D_j + 4\kappa_0, \quad u_0 = \lceil (k+1)\ell \rceil, \quad D_1 = D_0 + 4N u_0,$$

$$t_0(A,B) = J_{\mathrm{odd}}(A,B) + C_t N (a_1/p^2 + b'/p + L' a_1/(pk) + L' b'/k + kQ_p + Q_p c_4),$$
$$t_* = \lceil t_0(A,B) + N J_0 \rceil_+. \tag{A.173}$$

Here $J_{\mathrm{odd}}(A,B)$ uses the A-weight at the left even endpoint and the B-weights at interior odd marks.
Lean: `Formalism.MainNotions.flipOneThree.Pf.Dat.kF` (`Formalism.SharpRestriction.GrowthBounds.ColumnSpanDensity.RhoLowerBoundLargeConductor`, lines 138–140)

On the tensor of the two full symmetric powers at each mark let $g = \kappa_P + \sum_j \kappa_{T,j}$, where $\kappa_{T,j}$ is the fraction of rank in the direct sum of the kept block lattices on segment j lost by retaining Smith factors of valuation at most D_j. Use loss zero for rank zero. Define

$$\kappa_P = C_\kappa N \left(e^{-L'/8} + \frac{kQ_p(m_0 + N_1 k)}{b'} + \frac{1}{Y} + \frac{kE_0}{a_1} \right), \tag{A.174}$$

where $m_0 = C_m k N_1 Q_p c_4 + 12 c_4$ and

$$E_0 = C_{E1}(a_1/p^2 + b'/p + L' N_1 a_1/(pk) + L' b'/k) + C_{E2} k^2 N_1 Q_p^2 c_4.$$

The constants here are absolute. Equation (22.18) gives $e^{-L'/8} \le 1/(NY)$, $k^3 N_1 Q_p^2 c_4/b' \le 2N_1/(HNY)$, and $k^2 N_1 Q_p/b' \le 2N_1/(HNY)$. The other contributions of kE_0/a_1 are bounded by a constant times $2k/p^2 + L'(N_1+1)/p$, which is $O(N_1/Y)$. Thus $\kappa_P \le C_{\mathrm{mrg}} N N_1/Y$ (with $C_{\mathrm{mrg}} \le C_s$).
Lean: `Formalism.MainNotions.flipOneThree.Pf.Dat.kx` (`Formalism.SharpRestriction.GrowthBounds.ColumnSpanDensity.RhoLowerBoundLargeConductor`, lines 142–144)
Lean: `Formalism.MainNotions.G.kappaMarginBound` (`Formalism.SharpRestriction.GrowthBounds.ObjSeg`, lines 434–435)

Consider one gap of length G, with inner distances d_i for $0 \le i < s$, weights a_i, b_i, and right endpoint weights a_+, b_+. Set $P_i = p^{G-d_i}$ and $\sigma = \#\{i : p^{d_i} < a_1/b'\}$. In each row of k exponents fix digits $u_{ic}, v_{ic} \in [0, P_i)$ with

$$H_i = (b_i - \sum_c u_{ic})/P_i \in \mathbf{N}, \qquad K_i = (a_i - \sum_c v_{ic})/P_i \in \mathbf{N}.$$

The blocks in this digit class have nonnegative integer parameters o_i, w_i, h_i, o_+. Their signed capacity is given by

$$C_0' = \left\lfloor \frac{a_1 - E_0}{p^G} \right\rfloor \in \mathbf{Z}, \quad C_0^{\mathrm{cap}} = C_0', \quad C_{i+1}^{\mathrm{cap}} = C_i^{\mathrm{cap}} + H_i - e_i, \qquad e_i = \begin{cases} w_i + h_i & i < \sigma, \\ o_i & i \ge \sigma. \end{cases}$$

Write $C_i^{\mathrm{cap}} = C_i$ in the segment formulas. For $i < \sigma$ keep $o_i = 0,\ w_i \le H_i,\ h_i \le K_i$ and

$$w_i + h_i \le (1 + 4/p^2) H_i + L' C_i/k, \qquad h_i \le 2L' C_0'/k.$$

For $i \ge \sigma$ keep $w_i = h_i = 0,\ o_i \le H_i$ and $o_i \le C_i + L' H_i/k$. At the terminal letter keep $o_+ \le C_s + (L'/k + 4/p^2) \sum_i 2a_1/P_i + L' b_+/k$. The r-variable letters alternate between vectors and covectors, starting with a vector. The degrees are $(w_i, K_i - h_i)$ for $i < \sigma$ and (o_i, K_i) for $i \ge \sigma$, followed by o_+. Adjoin a covector ξ of degree b_+.
Lean: `Formalism.MainNotions.G.segData` (`Formalism.SharpRestriction.GrowthBounds.ObjSeg`, lines 319–330)
Lean: `Formalism.MainNotions.G.Kept` (`Formalism.SharpRestriction.GrowthBounds.ObjSeg`, lines 161–172)

Internal ties pair the vector and covector at $i \ge \sigma$ with shift $z_i = H_i - o_i$. The terminal tie pairs the final vector and ξ with shift $z_+ = a_+ - o_+$. A level chooses a count j_i at each internal tie and j' at the terminal tie, at most the lesser degree in its pair, and subtracts it from both degrees there. Let Y_{pol} be the subspace on the first $2s+1$ reduced letters cut out by the prefix bounds $\lfloor C_i - m_0 \rfloor$ through the vector at $i \ge \sigma$ and $\lfloor C_s - m_0 \rfloor$ on the full prefix. At each prefix take the largest stable subspace in the allowed signed first-coordinate weight span. It is obtained by intersecting successively with inverse images under the prefix operators $\sum_{\mathrm{vec}} X_{la}\partial_{lb} - \sum_{\mathrm{covec}} X_{lb}\partial_{la}$. The level lattice is the sum of $Y_{\mathbf{Z}}\xi^\mu$ over $|\mu| = b_+ - j'$, where $Y_{\mathbf{Z}} = Y_{\mathrm{pol}} \cap \mathbf{Z}_p[X]$.
Lean: `Formalism.MainNotions.G.blockData` (`Formalism.SharpRestriction.GrowthBounds.ObjSeg`, lines 207–209)

For a tied pair w, e, let E multiply by $\sum_c w_c e_c$ and $F = \sum_c \partial_{w_c}\partial_{e_c}$ be its adjoint for $\langle X^\alpha, X^\beta \rangle = \delta_{\alpha\beta}\alpha!$. At reduced degrees o, n and shift z put

$$\Phi_z = \sum_{l=0}^{\min(o,n)} \frac{(-1)^l E^l F^l}{(l!)^2 \binom{o+n+r+z-1}{l}}. \tag{A.175}$$

With a fresh scalar pair Z, S' for the tie put $\widehat{E} = E + ZS'$, $\widehat{F} = F + \partial_Z \partial_{S'}$. The extended harmonic projector on $Z^z y$ has this formula with hats. Indeed $\widehat{F}^l(Z^z y) = Z^z F^l y$ and $[\widehat{F}, \widehat{E}] = r + 1 + \deg_{(w,Z)} + \deg_{(e,S')}$ cancels consecutive terms under $\widehat{F}$. Specializing $Z = 1, S' = 0$ gives Φ_z. Let T compose the internal operators and d_F be the product of the reduced degree factorials, including that of ξ. The preterminal form is $B(u, v) = \langle Tu, v \rangle / d_F$. The internal ties are disjoint, so the operators commute and B is symmetric.
Lean: `Formalism.GM.tiePhi` (`Formalism.MahlerMeasure.LevelSum`, lines 21–25)
Lean: `Formalism.GM.M.tieP` (`Formalism.MahlerMeasure.TieVars`, lines 18–22)

For a lattice with a form write g_D for the greatest number of integral pairs with matrix $p^D I$ and b_D for the complementary rank. For integral pairs x_a, y_a in $Y_{\mathbf{Z}}$ and $m = b_+ - j'$, use $x_a \xi^\mu, (m!/\mu!) y_a \xi^\mu$ for every $|\mu| = m$. These are orthogonal across distinct μ and preserve the diagonal scalar. If $d_\xi = \binom{m+r-1}{r-1}$ and $r_l = d_\xi \dim Y_{\mathrm{pol}}$, the bad rank before the terminal operation is at most $d_\xi b_D(Y_{\mathbf{Z}})$.
Lean: `Formalism.MainNotions.G.SymmetricFormBlocks.plusAW_of_blockOK` (`Formalism.SharpRestriction.GrowthBounds.MonomialFactorials.PlusAWICondition`, lines 63–145)

Internal Gram average. Fix digits, block, and level. The allowed a_i form one progression of step P_i per coordinate, so the active covector degrees $n_i = K_i - j_i$ range over a product of integer intervals of lengths Y_i. The block and level indices are constant there. Indeed $kP_i \le b'$, $b_i \le 5b'/2$, and $a_i \ge a_1 > 25b'$ imply $H_i \le K_i$. For $i < \sigma$, $P_i h_i \le a_1/2$ by the upper bound on h_i and $2L'/k \le 1/2$. For $i \ge \sigma$ use $j_i \le o_i \le H_i$. The terminal index bound uses o_+, b_+ alone.
Lean: `Formalism.MainNotions.G.SymmetricFormBlocks.idx_sub` (`Formalism.SharpRestriction.GrowthBounds.MonomialFactorials.DigitsOKOfCompat`, lines 88–187)

Let $K^{(i)}$ replace K_i by its largest compatible value and let S_i be the degree of the prefix through its vector letter at the upper endpoint of the grid. If n_{0i} is the least reduced covector degree, then

$$P_i n_{0i} \ge a_1 - kP_i - \tfrac{5}{2}b'.$$

Here the digit sum is at most kP_i, and $P_i j_i \le b_i$. The prefix and window bounds are

$$P_i S_i \le \tfrac{5}{2}b' + (2s+1)2a_1/p, \qquad P_i Y_i \ge a_1/(2kY).$$

For the first bound use H_i for the current vector and the smaller divided degrees for earlier letters, since $P_i/P_h \le 1/p$ for $h < i$. The second follows by counting the integers of the residue class in the upper weight interval and allowing one incomplete step at each end. As $kP_i \le b'$, $2s+1 \le k$, $p \ge k^2Y^2$, and $kY^2b' \le a_1/2$, these bounds give $S_i \le n_{0i}$ and $S_i/Y_i \le 7/Y$. At the upper endpoint, $K_i \le 2a_1/P_i$, $o_i, w_i \le 5b'/(2P_i)$, and $o_+ \le C_s + (L'/k + 4/p^2)\sum_i 2a_1/P_i + L'b_+/k$. Inserting the signed capacity and using $P_i \ge p$ bounds the total reduced degree by $8a_1 + r$. For each internal tie the pair degree plus $r + z_i + 1$ is at most $(2a_1 + 5b'/2)/P_i + r + 1 \le 8a_1 + r$. Also $Y_i \le 2a_1 + 1$ and $2S_i + 1 \le 8a_1 + r$ by the prefix bound. These inequalities hold throughout the grid by monotonicity of the degrees.
Lean: `Formalism.CountingSumsFrames.TheoremMV7` (`Formalism.PolynomialDimension.ChainOrderLetters`, lines 19–36)
Lean: `Formalism.MainNotions.G.SymmetricFormBlocks.lower_bound_initial_multiplicity` (`Formalism.SharpRestriction.GrowthBounds.MonomialFactorials.LowerBoundInitialMultiplicity`, lines 23–74)

Put $\kappa_S = 3(s+2)r^2\ell$, let t be the number of internal ties, and set

$$D_{1r} = t(Y + C_M)\kappa_S + C_M\kappa_S, \qquad D' = \lfloor D_{1r} \rfloor$$

with $C_M = 1$. At tie i use the lattice in 22.22 on the prefix ending immediately before the varying covector. The weight w is a full torus weight and is highest for the group on the first $r-1$ coordinates. Fix all other degrees. This prefix lattice is independent of n_i. Put $S = \sum d_j \le S_i$, $o = d_{m+1}$, $z = z_i$, and $u = \lceil \kappa_S \rceil$. The matrix for the window is the matrix of 22.27, including its binomial multiplier and its $(n_i - S)!$ normalization. Apply 22.25 to this prefix, whose last letter is a vector. For $m > 0$ it gives $c = e_1 + 2e_2$ and $D = (m+1)e_1$ in 22.26, with e_1, e_2 as in 22.25. Take for each partner the orthogonal projection of its monomial. For $m = 0$ take $c = D = 0$. With these bounds, 22.26 gives $v_p(\det H_e) \le d_e\kappa_4$, where $\kappa_4 = (s+4)r^2\ell$: both logarithms are at most ℓ since $2S + 1 \le 8a_1 + r$, $p \ge 3$, and $m + 7 \le 2(s+4)$. For the scalar $\beta_e = p^u e!\binom{z+e}{e}/(o!S!)$ in 22.27, the factorial formula in 11.7 gives

$$v_p(\beta_e) + \frac{S + o - e}{p-1} = u + v_p\binom{z+e}{e} + \frac{s_p(S) + s_p(o) - s_p(e)}{p-1} \le \kappa_S + 3\ell,$$

where s_p denotes digit sum. Indeed $S, o, z + e \le 8a_1 + r$, the binomial costs at most $\lfloor \log_p(8a_1 + r) \rfloor$, and the ceiling of κ_S costs less than one. Multiplying along the order blocks proves $v_p(b_A) + s_A/(p-1) \le d(\kappa_S + 3\ell + \kappa_4) \le 2d\kappa_S$. In particular the leading determinant is nonzero. Apply 22.33 with $X = 8a_1 + r$. Here $2S + 1 \le 2S_i + 1 \le X$. There are at most $2s + 1$ reduced letters before the terminal covector, so $m \le 2s$ and $((m+2)r + r + 1)\ell \le 3(s+2)r^2\ell = \kappa_S \le u$. Thus the matrix has integral binomial coefficients. In 22.11 take $H = Y_i$, $J = 2S_i$, and $h = 1 + \lfloor \log_p X \rfloor$. The grid bounds give $H, J + 1 \le X < p^h$, and $h \le \ell$. With the leading-coefficient bound just proved, the Smith window estimate at threshold $Y\kappa_S$ thus gives the coefficient

$$\frac{2S_i}{Y_i} + \frac{2\kappa_S + 2S_i\ell/Y_i}{Y\kappa_S} \le \frac{30 + C_M}{Y}.$$

Apply 22.30 at each grid point with $L = 2s + 1$, $X = \lfloor 8a_1 + r \rfloor$, $E = Y\kappa_S$, and $\kappa = \kappa_S$. The caps end at the marked vectors or the final letter. The preceding degree bounds give $n_i \ge n_{0i} \ge S_i$. The total degree

is at most $8a_1 + r$, so $S_{\mathrm{tot}} + r - 1 \le p(8a_1 + r)$ and the exponent a_0 in the lemma is at most $r^2\ell/2$. Both denominator inequalities follow from $\kappa_S = 3(s+2)r^2\ell$. The threshold in the conclusion is exactly D'. On a fibre in coordinate i fix the same prefix basis and the same nonzero representative for each nonzero weight space. By 22.28, its suffix dimension is at most its value at the top of the fibre. Multiply the single-variable count by that value and sum over weights. The dimension identity in the proof of 22.30 gives $\dim Y_{\mathrm{pol}}(K^{(i)})$ at this endpoint. The step in n_i is one because the step in a_i is P_i. Summing over the other coordinates gives

$$\sum_{K\in\mathcal{C}} b_{D'}(Y_{\mathbf{Z}}(K)) \le \frac{30 + C_M}{Y} \sum_{\text{active } i} \sum_{K\in\mathcal{C}} \dim Y_{\mathrm{pol}}(K^{(i)}) \tag{A.176}$$

for the digit progression $\mathcal{C}$.

Lean: `Formalism.InvariantFactorPaths.MC.theoremMXB_of_point` (`Formalism.ThetaLogCoordinates.PartnerMeanCoordinates.PartnerMeanProofs.TheoremMXBOfPoint`, lines 7–230)
Lean: `Formalism.InvariantFactorPaths.MC.theoremMV7` (`Formalism.ThetaLogCoordinates.PartnerMeanCoordinates.PartnerMeanProofs.GramHypothesisOfLogBound`, lines 188–250)
Lean: `Formalism.InvariantFactorPaths.MC.Top.theoremMV7` (`Formalism.ThetaLogCoordinates.PartnerMeanCoordinates.LatticeTheoremEps28Valid`, lines 7–7)
Lean: `Formalism.MainNotions.G.CountingStatementReduction.alternative_mean_value_theorem` (`Formalism.SharpRestriction.GrowthBounds.CountingStatementReduction`, lines 19–22)
Lean: `Formalism.MainNotions.G.SymmetricFormBlocks.restTM_closed` (`Formalism.SharpRestriction.GrowthBounds.MonomialFactorials.EpsilonBoundMConstant`, lines 228–236)

Terminal Gram average. Diagonalize the symmetric preterminal form over $\mathbf{Z}_p$. A smallest-valuation off-diagonal entry can be put on the diagonal by summing the two basis vectors, since 2 is a unit. Subtract integral multiples of the resulting vector to clear its row and repeat. Diagonal entries of valuation $\le D'$ give $g_{D'}$, by Smith reduction. For the terminal reduced degrees o, n put $J = \min(o, n)$, $q_0 = o + n + r - 1$. Multiplying the terminal form by $\sigma_0 = p^{x_0}\binom{q_0+z}{J}$ gives the matrix entries

$$\frac{p^{x_0}}{d_F} \sum_{\lambda=0}^{J} (-1)^\lambda \binom{q_0 + z - \lambda}{J - \lambda} \frac{\langle E^\lambda F^\lambda T u, v\rangle}{(\lambda!)^2 \binom{J}{\lambda}}. \tag{A.177}$$

This uses $\binom{x}{J}\binom{J}{\lambda} = \binom{x}{\lambda}\binom{x-\lambda}{J-\lambda}$. With $n_L = 2s+1$, $a_{\mathrm{big}} = (3s+2)a_1$, $\ell_p = 1 + \lfloor \log_p(8a_{\mathrm{big}} + r)\rfloor$, take $x_0 = ((t+1)(r+1)+(n_L+1)r+2(t+1))\ell_p$. In divided contractions the coordinate binomial factors are integral and the multinomial denominator costs at most $(r-1)\ell_p$. The tie and terminal binomial denominators cost at most ℓ_p each. Comparing coordinate factorials to row factorials costs at most $((n_L+1)r+2(t+1))\ell_p$. These are the two contributions to x_0. Vandermonde's identity writes (A.177) integrally in $\binom{z}{i}$, with coefficient $p^{x_0}B$ at $i = J$.

Lean: `Formalism.MainNotions.G.GramWindowAlt.phi_pair` (`Formalism.SharpRestriction.GrowthBounds.FockGramWindow.PhiPair`, lines 18–46)
Lean: `Formalism.MainNotions.G.GramWindowAlt.winLvl` (`Formalism.SharpRestriction.GrowthBounds.FockGramWindow.WinLvl`, lines 77–97)

On the preceding good directions, the leading determinant coefficient is the product of the scaled preceding entries divided by $(J!)^d$ in rank d. Since $J/(p-1) - v_p(J!) \le \ell_p$, apply 22.11 with $L \le x_0 + D' + \ell_p$ and $E_* = \lfloor YD_{1r}\rfloor + 1$. The terminal window has $H_a \ge a_1/(kY)$, while $J \le 5b'/2$. Since $k \ge 2N_1 + 3 \ge 5$ and $a_1 \ge a > pb' \ge k^2Y^2b'$, we have $H_a > 5Yb'$ and hence $J/H_a \le 1/(2Y)$. With $h = \lceil \log_p(a_1+1)\rceil$, $H_a, J+1 \le p^h$, and $x_0 + \ell_p, h \le \kappa_S \le D_{1r}$. The fraction discarded on the good directions is at most $1/(2Y) + (2 + 1/(2Y))/Y \le 5/Y$. For a retained entry d_i, scale its partner by $p^{E_*-1}\sigma_0/d_i$, which is integral and gives the terminal diagonal p^{E_*-1}. Thus

$$\frac{1}{H_a} \sum_{a_+} b_{\lfloor YD_{1r}\rfloor}(L, l, a_+) \le d_\xi b_{D'}(Y_{\mathbf{Z}}) + 5r_l/Y. \tag{A.178}$$

The first term bounds the entire rank if the preceding good span has rank zero.

Lean: `Formalism.MainNotions.G.GramWindowAlt.restWinNum_proved` (`Formalism.SharpRestriction.GrowthBounds.FockGramWindow.InitialPointNumericBound`, lines 183–235)
Lean: `Formalism.MainNotions.G.GramWindowAlt.bad_le_smith` (`Formalism.SharpRestriction.GrowthBounds.FockGramWindow.GramWindow2Proved`, lines 31–133)
Lean: `Formalism.MainNotions.G.SymmetricFormBlocks.restPlus_of` (`Formalism.SharpRestriction.GrowthBounds.MonomialFactorials.RestPlusOf`, lines 53–60)

From levels to block lattices. At each level l, the ordinary generators are $\prod_h \widehat{E}_h^{j_h} Z_h^{z_h} u$ with u in the level lattice, including the terminal tie. Define $\Theta_l u$ by harmonically projecting each pair after multiplication by $Z_h^{z_h}$ and before applying $\widehat{E}_h^{j_h}$. With $L_p = \lfloor \log_p(8a_{\mathrm{big}} + r) \rfloor$ and $U = (t+1)\lceil (r+1)(1+\log_p(8a_{\mathrm{big}}+r)) \rceil$, $p^U \Theta_l u$ belongs to the integral span of ordinary generators. The contraction coefficient for $F^{h'}/(h'!)^2$ at $|\nu| = h'$ is

$$\frac{\prod_c \binom{\alpha_c}{\nu_c}\binom{\beta_c}{\nu_c}}{\binom{h'}{\nu_1,\dots,\nu_r}}.$$

The denominator costs at most $(r-1)L_p$ and the projector binomial at most L_p. A contraction lowers the degrees and increases the level count after the compensating multiplication. At a prefix containing both letters it commutes with the action. At the prefix ending at the vector letter each derivative lowers its signed first-coordinate weight or leaves it fixed. The derivative span is stable by the commutator with $X_a \partial_b$. Thus the new generators satisfy the prefix constraints of their levels.

Lean: `Formalism.MainNotions.G.SymmetricFormBlocks.restLL_proved` (`Formalism.SharpRestriction.GrowthBounds.MonomialFactorials.BlockValuationZpOfValuationBound`, lines 190–194)

The images of different levels are orthogonal. Move a power of $\widehat{E}$ at a differing tie across the pairing as $\widehat{F}$ until a contraction meets a harmonic factor. At level l the factorial pairing on lifted vectors is the level form times

$$a_l = d_F \prod_h (j_h!)^2 \binom{o_h + n_h + r + z_h + j_h}{j_h} z_h!,$$

where o_h, n_h are reduced degrees. This follows by iterating the commutator on the harmonic space and evaluating the scalar coordinates. The factorials are nonzero. Put $M = \max_l v(a_l)$, $b_l^0 = \lfloor Y D_{1r} \rfloor$. For level pairs u_i, v_i of diagonal $p^{b_l^0}$ use $p^U \Theta_l v_i$ and $(p^M/a_l) p^U \Theta_l u_i$. Their common factorial diagonal is p^e with $e = b_l^0 + 2U + M$.

Lean: `Formalism.MainNotions.G.SymmetricFormBlocks.Th_fock` (`Formalism.SharpRestriction.GrowthBounds.MonomialFactorials.GoodCountMem`, lines 101–131)

Lean: `Formalism.MainNotions.G.SymmetricFormBlocks.Th_orth` (`Formalism.SharpRestriction.GrowthBounds.MonomialFactorials.GoodCountMem`, lines 133–150)

The field span of ordinary generators has dimension at most the sum of level ranks. Conversely each Θ_l is injective on the level space. Multiplication by $\prod_h \widehat{E}_h^{j_h}$ is injective in the polynomial ring. Remove these factors and set $Z_h = 1, S_h' = 0$ at each tie. By (A.175) the specialized map is the product of the operators Φ_{z_h}. For one pair of reduced degrees o, n, decompose into the harmonic levels $E^a \ker F$, $0 \le a \le \min(o,n)$, as in 22.27. On this level the eigenvalue is $\binom{z+a}{a}/\binom{o+n+r+z-1}{a}$. For $a = 0$ it is one. For $a > 0$ both binomial coefficients are positive integers since $z \ge 0$, $r \ge 1$ and $o, n \ge a$. This computation over the rationals remains valid after extension to $\mathbf{Q}_p$. Each specialized operator is invertible, including at the terminal tie, and the operators act on disjoint pairs. A zero image before specialization therefore implies zero input. The distinct level images are independent by the distinct harmonic degrees in the disjoint tied pairs. They lie in the block span by the denominator bound above. This gives the level-dimension sum bound by the block rank.

Lean: `Formalism.MainNotions.G.SymmetricFormBlocks.blkZ_gen` (`Formalism.SharpRestriction.GrowthBounds.MonomialFactorials.DegFactNeZero`, lines 58–126)

Put $s_b = \prod_{i<\sigma} (H_i - w_i)!\, h_i!$. The block lattice in this span is defined coefficientwise by $2v(c_\mu) + v(\mu!) + v(s_b) \ge 0$, with form $s_b \langle\ ,\ \rangle$. For a monomial of an ordinary generator,

$$M - v(\mu!) \le ((2s+2)r + 3(t+1))L_p. \tag{A.179}$$

Assign the two scalar coordinates to their original letters. The total degrees per letter are fixed at every level. The factorials in a_l, except its $t+1$ binomials, divide the product of these total-degree factorials. Comparing to $\mu!$ costs at most L_p per coordinate. Use $L_p \le \ell$ and $r \ge 4$ to get $e \le b_l^0 + L_* + v(\mu!)$ with $L_* = 10(s+1)(r+1)\ell + \kappa_0$.

Lean: `Formalism.MainNotions.G.SymmetricFormBlocks.val_gen` (`Formalism.SharpRestriction.GrowthBounds.MonomialFactorials.FinPosEqCast`, lines 199–237)

With $f_0 = \lfloor b_l^0 + L_* \rfloor$ scale both families by $\omega^{f_0 - e - v(s_b)}$. They belong to the block lattice and have a diagonal of valuation f_0. Saturating the first family preserves rank. A primitive vector in its saturation has some coordinate of valuation ≤ 0 in the original family, so the opposite vector detects it at valuation $\le f_0$. Smith reduction gives at least that many good directions in the block. Increasing C_A makes $f_0 \le D_j$ since $Y D_{1r} = O(Y^2 (s+2)^2 r^2 \ell)$.

Lean: Formalism.MainNotions.G.SymmetricFormBlocks.block_dual_family_bound (Formalism.SharpRestriction.GrowthBounds.MonomialFactorials.BlockBoundOfDualFamilies, lines 105–213)

For a digit class the full dimension in the k-variable rows is

$$m_{\mathcal{C}}(K) = P(C_0')P(b_+)\prod_i P(H_i)P(K_i).$$

The sum of all level ranks over the kept blocks in this class is at most $m_{\mathcal{C}}(K)$. First use the independent level maps into each block. Then embed the block spaces into the full digit-class space. Their first-coordinate degrees and contracted-pair degrees distinguish the block labels. The prefix capacities are the bounds on these first-coordinate degrees in the successive interlacing steps. Counting without those bounds gives precisely the displayed product, since each full row is a symmetric power on k variables.

Changing one coordinate from $x = K_i$ to its class maximum y changes one factor. The residue sum is at most $kP_i \le b'$, so $xP_i \ge a_1 - b' \ge 3a_1/4$. The enlarged a-window gives $r(y-x)P_i \le 3(3a_1/4)$ by its width $a/(kY) + J_b + c_b$ and the size inequalities. Hence $x \ge 1$, $r(y-x) \le 3x$, and

$$\frac{P(y)}{P(x)} = \prod_{j=1}^{r}\frac{y+j}{x+j} \le (y/x)^r \le e^3.$$

Multiply (A.176) by d_ξ and sum (A.178) over levels and classes. The internal loss is at most $(30+C_M)se^3/Y$ times the summed class dimension and the terminal loss at most $5/Y$ times that dimension. The comparison with the full segment rank costs at most $2e^3$ in the weighted count. It follows by summing the class dimensions over digits and using the kept-block comparison below, with the degree ratio bound for variation inside each progression. Thus each segment contributes at most $C_A(s+1)N_1/Y$, for $C_A \ge 2((30+C_M)e^3+5)e^3$. For fixed B, the product average restricts to the uniform average of the segment's interior A-weights and its right endpoint. Sum using $\sum_j(s_j+1) \le N$ and add κ_P. This proves (22.19).

Lean: Formalism.MainNotions.G.CountingStatementReduction.mCl (Formalism.SharpRestriction.GrowthBounds.CountingStatementReduction, lines 2946–2950)
Lean: Formalism.MainNotions.G.SymmetricFormBlocks.claim2 (Formalism.SharpRestriction.GrowthBounds.MonomialFactorials.SegmentWeightedCountingBoundOfAllRemainingBounds, lines 40–54)
Lean: Formalism.MainNotions.G.SymmetricFormBlocks.ratioCondition_of_segFacts (Formalism.SharpRestriction.GrowthBounds.MonomialFactorials.RatioConditionOfSegFacts, lines 74–210)
Lean: Formalism.MainNotions.G.SymmetricFormBlocks.segment_weighted_counting_bound_of_all_remaining_bounds (Formalism.SharpRestriction.GrowthBounds.MonomialFactorials.SegmentWeightedCountingBoundOfAllRemainingBounds, lines 134–173)
Lean: Formalism.MainNotions.G.GramWindowAlt.theorem_A_closed_form_holds (Formalism.SharpRestriction.GrowthBounds.FockGramWindow, lines 24–25)
Lean: Formalism.MainNotions.flipOneThree.Pf.segment_mean_kappa_bound (Formalism.SharpRestriction.GrowthBounds.ColumnSpanDensity.LocalHypothesisOfGlobal, lines 143–177)

Boundary rank. Let r_j be the total rank of the kept block lattices on segment j. The ranks satisfy

$$\prod_j r_j \ge (1-\kappa_P)\prod_{s\in S}P(A_s)P(B_s)/E_q.$$

In a fixed digit class count interlacing paths of the k-row chain starting at a single row of degree $C_0' - \lceil m_0\rceil$. At the end retain the r-row terminal weight, with multiplicity its Weyl dimension. To compare with the r-row blocks, strip the first row at each step. If the first-row length before a vector-covector pair is c_i and afterwards c_{i+1}, the removed degree is $c_i + H_i - c_{i+1}$. In the early pairs split it into the increase of the sum of the remaining rows and the part removed on the covector step. These are w_i, h_i. In the late pairs the removed degree gives o_i with the residual contraction count recorded at its tie. At the terminal pair use the first-row shift and the terminal contraction count. The remaining rows, these counts, and the boundary row recover the first row by its degree at every step. More explicitly, adjoin a terminal index $0 \le j' \le \lfloor L'b_+/k\rfloor$. Write $|t| = \sum_a t_a$ for the sum of a row and $t^- = (t_2,\dots,t_k)$ for its tail in this paragraph. At an inner pair let t, v, t' be the rows before the vector, after the vector, and after the covector. Set $w = |v^-| - |t^-|$, $h = v_1 - t_1'$. Interlacing shows $0 \le w \le H_i$, $0 \le h \le K_i$. The two tail degrees are $w, K_i - h$, and $w + h = t_1 + H_i - t_1'$. At an early pair the block records $(w_i, h_i) = (w, h)$. At a late pair it records $o_i = w + h$ and its level is $j_i = h$, so the reduced degrees are again $w, K_i - h$. If t is the last full row and λ the terminal interlacing row, set

$d = |\lambda| - |t^-| \geq 0$, $o_+ = d + j'$. Adjoining λ to the tail chain gives the last vector strip of degree d. The output includes a tableau of shape λ with its determinant twist, and a monomial of degree $b_+ - j'$ in ξ. These are precisely the two multiplicities. From the reduced path recover each full row by appending its missing first entry, equal to the prescribed signed degree sum to that point minus the tail sum. The final row gives λ, and the terminal level gives j'. This proves injectivity. Along the path $C_i = c_i + \lceil m_0 \rceil$, by the capacity recursion and its initial value. The largest entry of the tail after the vector is $v_2 \leq t_1 = c_i = \lfloor C_i - m_0 \rfloor$. At the last prefix $\lambda_1 \leq c_s$. These are the prefix caps of the reduced path.

For the unrestricted path count, Pieri interlacing counts the product of the symmetric-power dimensions at its letters. For the paired-chain count put $A = C'_0 - \lceil m_0 \rceil$ and use the signed degrees $A, H_0, K_0, \ldots, H_{s-1}, K_{s-1}$, with signs $+, +, -, \ldots, +, -$. After the last signed step choose an interlacing r-row λ, weighted by $\dim V_\lambda$. Number the inner pairs from zero. Let c_i be the first coordinate just before pair i, c_s the last first coordinate, R_i the second coordinate just after the vector of pair i, $u_i = c_i + H_i - c_{i+1} - R_i$, and $n_i = -\lambda_{r-s+i+1}$ in one-based row indices. The two rejection tests are

$$u_i + 4(H_i + r)/p^2 > L' \max(c_i, H_i)/k, \qquad n_i < K_i(1 - L'/k).$$

We check that a pair passing both tests gives a kept block. Write $a_c = C'_0$, $\theta = L'/k$ and $A = a_c - \lceil m_0 \rceil$. The digit bounds give $(b' - kP_i)/P_i \leq H_i \leq 2b'/P_i$. Successive odd distances differ by at least two, so $\sum_{h<i} H_h \leq 4H_i/p^2$, using $kQ_p \leq b'/4$ and the geometric series. The size bounds give E_0/a_1, $(m_0 + 1)Q_p/a_1$, $kQ_p/b' \leq \theta/2$ on increasing the fixed thresholds. Indeed the first four terms of E_0/a_1 are bounded by $C_{E1}(2/p^2 + \theta(N_1 + 1)/p)$, using $b'/a_1 < 1/p$. For the remaining terms use $Q_p^4 c_4/b' \leq 2/(Hk^3NY)$, $m_0 = (C_m k N_1 Q_p + 12)c_4$, $p \geq k^2$, and $HN\mathcal{L}^2 \leq k$. In particular $\lceil m_0 \rceil / a_c \leq 2\theta$. At an early position $p^{d_i} b' < a(1 - \rho_{\mathrm{loc}})$, and $b_i/b' \leq 1 + \Delta/b' + 3\mathcal{L}/k$. At a late position $p^{d_i} b' > a(1 + \rho_{\mathrm{loc}})$. Using $a_1 \leq a + 1$, $1/a \leq \theta$, and $\rho_{\mathrm{loc}} \geq HN_1\theta + 4\Delta/b'$, these give $H_i \leq (1 - 10N_1\theta)a_c$ early and $a_c \leq (1 - 10N_1\theta)H_i$ late, for sufficiently large H.

Write $O_i = w + h$ in the preceding tail construction. The second entry before a vector is at most the total of the preceding vector degrees $\sum_{h<i} H_h$, since the initial second entry is zero and covector steps decrease entries. Hence $R_i \leq w + 4H_i/p^2$. Also $R_i \leq c_i$ by interlacing and $c_i \leq A + \sum_{h<i} H_h$. Early, this gives $c_i \leq 2a_c$. At each preceding early step the first acceptance test gives $O_i \leq H_i + 4H_i/p^2 + 2\theta a_c$. The recursion then gives $c_i \geq A - i(4/p^2 + 2\theta)a_c \geq (1 - 6N_1\theta)a_c \geq H_i$, using $4/p^2 \leq \theta$. The maximum in the test is therefore c_i. We obtain $O_i \leq (1 + 4/p^2)H_i + \theta c_i$ and $h = O_i - w \leq \theta c_i \leq 2\theta a_c$. Late, $c_i \leq a_c + 4H_i/p^2 \leq (1 - \theta)H_i$. Thus $O_i \leq c_i + \theta H_i \leq H_i$. These are the two inner block tests. For the terminal test the increase in positive parts from the last tail to λ is at most c_s, by telescoping the interlacing inequalities. The decrease in absolute negative parts is at most $\sum_i K_i - \sum_i n_i$, since the covectors introduced at most $\sum_i K_i$ negative boxes. The second acceptance test bounds this by $\theta \sum_i K_i$. Hence $o_+ \leq c_s + \theta \sum_i K_i + \theta b_+$, which is within the stated terminal bound since $K_i \leq 2a_1/P_i$. To estimate the rejected weight put $C = c_i + r$, $B = (H_i + r)(1 + 4/p^2)$, $R = R_i + r$, $v = u_i + 4(H_i + r)/p^2$. Then

$$R(c_{i+1} + r) = CB - (C - R)(B - R) - Rv.$$

The interlacing bounds give $0 \leq R \leq \min(C, B)$, $v \geq 0$, $\lambda_1 \leq c_s$, and $0 \leq n_i \leq K_i(1 + 4/p^2)$. For $\tau = L'/(4k) \leq 1/16$, failure gives $v > 2\tau \max(C, B)$: here $H_i, K_i, A \geq k^3$, $p \geq k^2$, and $\max(C, B) \leq 2\max(c_i, H_i)$. If $R \geq \min(C, B)/2$, then $Rv \geq \tau CB$. Otherwise $(C - R)(B - R) \geq CB/4$. A failed step therefore costs a factor $1 - \tau$ in the product. A failed second test gives the same factor in the bound for $n_i + r$ after the additive r is absorbed. Indeed, with $\eta = k^{-2}$, $r/K_i \leq \eta$, and $1 - L'/k + \eta \leq (1 - \tau)e^{2\eta}$ for $L' \geq 1$. At all other steps $(H_i + r)(1 + 4/p^2) \leq e^{2\eta} H_i$ and $n_i + r \leq e^{2\eta} K_i$. Multiply the inequalities for $R(c_{i+1} + r)$; the intermediate $c_i + r$ cancel. Using $A + r \leq e^\eta A$ and raising the result to power $r - 1 = k - 2$ gives the loss $\exp(2 - L'/5)$, since $(k - 2)(4s + 1)/k^2 \leq 2$ and $(k - 2)/(4k) \geq 1/5$ for $k \geq 10$. To pass to Weyl dimensions use $y^{r-1}/(r - 1)! \leq \binom{y+r-1}{r-1} \leq (y + r)^{r-1}/(r - 1)!$. There are at most $(8a_1)^{4(N_1+1)^2}$ unlabelled pairs: each intermediate row and the last interlacing row is determined by its first $s + 1$ and last s entries, since all intervening ones vanish by interlacing. Let $D_r(y) = \binom{y+r-1}{r-1}$. The Weyl product for a decreasing row λ is at most $\prod_a D_r(|\lambda_a|)$. In its factor with indices $a < b$, bound $1 + (\lambda_a - \lambda_b)/(b - a)$ by $(1 + \max(\lambda_a, 0)/(b - a))(1 + \max(-\lambda_b, 0)/(b - a))$, then multiply. Repeated interlacing bounds the positive entries after the first by $R_{s-1}, \ldots, R_0$ in order: a covector step decreases each entry, and at a vector step each entry after the first is at most the preceding entry of the previous row. The last s entries have absolute

values $n_0, \dots, n_{s-1}$, and all intervening entries vanish. The same interlacing in the last $s-1-i$ covector steps bounds n_i by $\sum_{h \le i} K_h \le K_i(1+4/p^2)$, by the geometric series for the divided covector degrees. Thus the product just estimated bounds the rejected Weyl dimension by $e^{2-L'/5} D_r(A) \prod_i D_r(H_i) D_r(K_i)$. Each $D_r(y) \le P(y)$, so the ratio is relative to the unrestricted k-row dimension. Thus the logarithm of the total exceptional fraction is at most $2 - L'/5 + 64 N_1^2 \log a_1 \le -L'/8$ for $C_0 \ge 880$, using $k \le a_1$, $a_1 \ge 9$, and $\log a_1 \ge 1$. The terminal cutoff retains at least $1 - e^{-L'/3}$. For example its count sums the homogeneous dimensions on r variables for degrees $b_+ - j'$ with $j' \le \lfloor L' b_+ / k \rfloor$; the binomial sum gives $P(b_+) - P(b_+ - \lfloor L' b_+ / k \rfloor - 1)$. The reserve at the start retains at least $1 - r\lceil m_0 \rceil / C_0'$ by the same dimension ratio. Summing over the digits restores $P(a_i)P(b_i)$, since each monomial has a unique quotient and remainder row. For a shared even mark the starting degree of the next segment is $\lfloor (a_1 - E_0)/p^G \rfloor$, compared with the undivided A-weight in the preceding endpoint. The binomial product compares these dimensions with scale p^{-Gr} and losses bounded by a constant times $kQ_p(m_0 + N_1 k)/b' + 1/Y + kE_0/a_1$. Here the first term allows the reserved and rounded degrees, the second allows the width $a_1/(kY)$, and the last allows the subtraction of E_0. Each boundary row is used once in this comparison. Multiplication over segments uses $\sum_j G_j = n'$ and $p^{-n'r} E_q = \sum_{i=0}^{r} p^{-n'i} \ge 1$. The sum of the segment losses is at most κ_P by (A.174). If that sum is at least one the desired lower bound follows from nonnegativity.

Lean: `GlobalTheory.EX.RangeOne.tensor_of_pairLift` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.TensorOfPairLift`, lines 583–595)

Paired lifting. For a tuple β of kept blocks, map each segment into the symbol coordinates. Insert its common low digits and raise the divided coordinates to the powers P_i. Insert the prescribed scalar coordinates of its ties. Here are the two symbol maps on a segment block. Assign each ordinary coarse coordinate to the corresponding nonzero coordinate of its physical row. At an internal or terminal tie assign Z to coordinate zero of the vector row and S' to coordinate zero of the covector row, where the vector row here is the physical row carrying the coarse vector and its fixed shift. Thus Z supplies the shift $H_i - o_i$, or $a_+ - o_+$ at the end. At an early pair the unused coordinate-zero slots carry the fixed quotients $H_i - w_i, h_i$. Insert the prescribed digits, using power P_i at interior rows and power one and digit zero at the endpoint. Denote the resulting fine exponent of a coarse monomial X^μ by $a(\mu)$.

For a physical row x let d_x be its fine degree and e_x the sum of the coarse exponents in its assigned variable slots, excluding fixed quotients. This sum is constant on the block. Indeed $W_a Y_a$ and ZS' have the same degree in each physical row, and lowering a pair of reduced degrees by one is compensated by one tie factor. Put $s_b = \prod_{i<\sigma}(H_i - w_i)! h_i!$, $N_b = s_b \prod_x e_x!$, and $t_b = \omega^{v_p(N_b)}$. Write $U_b(\mu)$ in this paragraph for the product of the digit units in the variable slots. By 22.16 with $c = c_4$, choose $g \equiv U_b^{-1} \pmod{p^{c_4}}$ of total binomial degree at most $4c_4$. Coordinates of power one contribute one, so the indices used by g are all interior coordinates, including the internal scalar coordinates. The maps before substitution are $u \mapsto t_b D_g u$ and $v \mapsto t_b v$, respectively.

For a block-lattice coefficient u_μ, the defining inequality gives $2v(t_b u_\mu) \ge v(N_b) - v(s_b \mu!) \ge 0$, since $\mu! \mid \prod_x e_x!$. Thus $t_b u$ has integral coefficients and lies in the field span of the ordinary generators. Apply saturation in 22.34, then its derivative bound with $b_J = \lfloor C_i - m_0(C_m) \rfloor$ at each prescribed cut. It puts $t_b D_g u$ in the integral span with reserve $m_0(C_m) - 4c_4$, which is at least $m_0(13)$. The second map has the full original reserve. This proves the two simultaneous symbol-membership assertions used below. We carry out the rank-one lift explicitly using Lemma 22.17 with plane rank 1 and complementary rank $r = k-1$, numbering the k coordinates from 0 to r so that coordinate 0 is the distinguished line. Fix a Witt matrix $g \in \mathrm{GL}_k(W(\mathbf{F}_{p^{n'}}))$ for the edge e. At each mark x, the twisted vector row $v_x^{(g)} = X_x^{\mathrm{vec}} g_x$ has nonnormal coordinate 0 and normal coordinates $1, \dots, r$, while the twisted covector row $w_x^{(g)} = X_x^{\mathrm{cov}} g_x^{-\mathrm{t}}$ has normal coordinate 0 and nonnormal coordinates $1, \dots, r$. For the depth bound, apply 22.32 with $Q = Q_p$ and a_S the weight at the left even endpoint: $J_b + c_b \le 3\mathcal{L} b'/k$ (since $L' \le 2\mathcal{L}$) gives $b' \le b_i, b_+ \le b'(1+\rho_{\mathrm{loc}}) < 2b'$, and $1/a < \rho_{\mathrm{loc}}$ by (22.18) for large H. If $p^{d_i} < a_1/b'$, the excluded odd-power interval forces $p^{d_i} < (a/b')(1-\rho_{\mathrm{loc}})$, so $p^{d_i} b_i \le a_S$; otherwise $a_1/p^{d_i} \le b' \le b_i$. Summing 22.32 over segments using $\sum_j (s_j^{\mathrm{int}} + 1) = N$ (where s_j^{int} counts interior marks of segment j) bounds the initial block normal degree h_0 by t_0 for $C_t \ge 30$.

For the one-sided lift put $K = 13kN_1Q_p$, $m_0(v) = vkN_1Q_pc_4 + 12c_4$, and $m_0' = m_0(13) = 12c_4 + c_4K \in \mathbf{N}$; taking $C_m \ge 17$ gives $m_0(C_m) - 4c_4 \ge m_0'$. Because the segment symbol maps have disjoint alphabets (each containing its interior pairs and its right endpoint), at every even start s_j (the right endpoint of segment $j-1$)

the preceding terminal shift $Z_{+,j-1}^{a_{s_j}-o_{+,j-1}} = (X_{s_j,0}^{\mathrm{vec}})^{a_{s_j}-o_{+,j-1}}$ has exponent $a_{s_j} - o_{+,j-1} \geq a_1 - E_0 \geq C'_{0,j}p^{G_j}$ (for $C_{E1} \geq 3$, using $\sum_i P_i^{-1} \leq p/(p^2-1)$, $G_{j-1} \geq 2$, and $b_{+,j-1} < 2b'$ in the kept-block bound for $o_{+,j-1}$), supplying $(C'_{0,j} - m'_0)p^{G_j}$ powers of $X_{s_j,0}^{\mathrm{vec}}$ for segment j's column starts, $m'_0 p^{G_j}$ reserve powers of $X_{s_j,0}^{\mathrm{vec}}$ at s_j, and $(a_{s_j} - o_{+,j-1}) - C'_{0,j}p^{G_j} \geq 0$ leftover powers. On each segment j (from s_j to $s_{j+1} = s_j + G_j$, with interior marks $x_i = s_j + d_i$, $0 \leq i < s$, and powers $P_i = p^{G_j - d_i}$), expand each reduced polynomial y from either symbol map (which has integral coefficients and satisfies the prefix bounds $\lfloor C_i - m'_0 \rfloor = C_i - m'_0$ at $i \geq \sigma$ and at s) in the integral rank-r exterior-column path basis from the proof of Lemma 22.20 (applied symmetrically to upper bounds $\kappa_1 \leq C_i - m'_0$, since the greatest first-coordinate weight of a column is $\mathbf{1}_{h(c)>0}$ for $c > 0$ and $-\mathbf{1}_{h(c)=0}$ for $c \leq 0$, summing to κ_1). At each early mark x_i ($i < \sigma$), lift the h_i fixed covector quotient units by h_i two-letter columns $P_{x_i,0}(g)^{P_i}$ on $(s_j^{\mathrm{vec}}, x_i^{\mathrm{cov}})$ (positions $s_j < x_i$, order $P_i \leq Q_p$, leading term $\omega^{P_i} X_{s_j,0}^{p^{G_j}} (X_{x_i,0}^{\mathrm{cov}})^{P_i}$ modulo $\omega^{P_i+1}O[X]$ by Lemma 22.17(1)). Since $h_l \leq 2\theta C'_0$ gives $\sum_{l<\sigma} h_l \leq 2N_1\theta C'_0$, the early inequality $\sum_{h \leq i} w_h \leq (1+4/p^2)H_i \leq (1-9N_1\theta)C'_0 \leq C'_0 - m'_0 - \sum_{l<\sigma} h_l$ (from $H_i \leq (1-10N_1\theta)C'_0$ and $m'_0/C'_0 \leq 2\theta$) together with the prefix bounds $C_i - m'_0$ at $i \geq \sigma$ and at s ensures that every complementary path κ occurring in y joins with the nondecreasing rank-one plane path ν (starting at $C'_0 - m'_0 - \sum_{l<\sigma} h_l$ at s_j^{vec}, gaining $H_i - w_i$ for $i < \sigma$ or $H_i - o_i = z_i$ for $i \geq \sigma$ at x_i^{vec}, and changing by 0 at every x_i^{cov} and at s_{j+1}^{vec}, so $\nu = C_i - m'_0$ before x_i^{vec} for $i \geq \sigma$ and at s_{j+1}^{vec}) to give a rank-k path μ on the $2s + 2 \leq 2N_1 + 2 \leq r$ letters $(s_j^{\mathrm{vec}}; x_0^{\mathrm{vec}}, x_0^{\mathrm{cov}}, \ldots, x_{s-1}^{\mathrm{vec}}, x_{s-1}^{\mathrm{cov}}; s_{j+1}^{\mathrm{vec}})$. The only equal-position letter pairs are $(x_i^{\mathrm{vec}}, x_i^{\mathrm{cov}})$: every step at x_i^{vec} follows letters strictly before x_i, while at x_i^{cov} the plane entry $\nu \geq 0$ is unchanged, so any column $c \in \mathbf{Z}$ used at x_i^{cov} still has $\mu_0 = \nu \geq c$ after x_i^{cov} (for $c > 0$ because $c \leq \kappa_1^{\mathrm{before}} \leq \nu$, and for $c \leq 0$ because $\nu \geq 0 \geq c$), hence height ≥ 1 after x_i^{cov} and height $\geq 2 > 1$ before it; thus x_i^{cov} is never costly. Lemma 22.17(1) therefore applies to every active column c of μ with $z = s_{j+1}$: $K_c(g; J) \in O[X]$ has degree p^{G_j}, P_i, or 1 at each used letter at s_j, x_i, or s_{j+1} (and 0 at unused letters), satisfies $\mathrm{ord}_{g'}(K_c(g;J)) \geq \mathrm{oc}(c)$ for every $g' \in \mathrm{GL}_k(W(\mathbf{F}_{p^{n'}}))$, agrees modulo $\omega^{\mathrm{oc}(c)+1}O[X]$ under s_g with $\omega^{\mathrm{oc}(c)}$ times the entrywise-powered separated column, and has $\mathrm{oc}(c) \leq 2p^{G_j} \leq 2Q_p$; taking terminal index $J = \{0\} \cup J_\kappa$ for $c > 0$ and $J = J_\kappa$ for $c \leq 0$ (where $J_\kappa \subseteq \{1, \ldots, r\}$ is the terminal index in κ) gives $K_c^{\mathrm{sep}}(\{0\} \cup J_\kappa) = X_{a_0,0}^{\mathrm{vec}} K_c^\kappa(J_\kappa)$ for $c > 0$ (with $a_0 \in \{s_j, x_i\}$ the step where ν crosses c) and $K_c^{\mathrm{sep}}(J_\kappa) = K_c^\kappa(J_\kappa)$ for $c \leq 0$, with no sign change since $0 < 1, \ldots, r$ and coordinate 0 vanishes in all subsequent updates.

Lift the remaining factors of each generator on segment j as follows. Each internal tie $\widehat{E}_i^{j_i}$ ($i \geq \sigma$) lifts as the product of j_i tie factors $E_{x_i}^{P_i}$, and the terminal tie $\widehat{E}_+^{j'}$ lifts as j' factors $E_{s_{j+1}}$; by Lemma 22.17(3) (with $P = P_i$ and $P = 1$), each $E_{x_i}^{P_i}$ has exact order P_i under every $s_{g'}$ and agrees modulo $\omega^{P_i+2}O[X]$ with $\omega^{P_i}\,\mathrm{pow}_{s_{j+1}}(\widehat{E}_i)$ under s_g, while each $E_{s_{j+1}}$ has exact order 1 and zero error under s_g. At each interior mark x_i, lift each normal digit $u_{ic} < P_i$ ($1 \leq c \leq r$) by $B_{x_i,c}(g)^{u_{ic}}$ and $v_{i,0} < P_i$ by $P_{x_i,0}(g)^{v_{i,0}}$, where $D_{x_i}(g) = (v_{s_j,0}^{(g)})^{p^{d_i}}$, $B_{x,c}(g)$ is the 2×2 minor on columns $\{0, c\}$, and $P_{x,0}(g) = \sum_{a=0}^{r} (v_{s_j,a}^{(g)})^{p^{x-s_j}} w_{x,a}^{(g)}$ are the rank-one bordered and pairing factors of Lemma 22.17(2). Each has order at least 1 under every $s_{g'}$ and leading term $\omega X_{s_j,0}^{p^{d_i}} X_{x_i,c}^{\mathrm{vec}}$ or $\omega X_{s_j,0}^{p^{d_i}} X_{x_i,0}^{\mathrm{cov}}$ modulo $\omega^2 O[X]$ under s_g. Across all k normal coordinates and $s \leq N_1$ interior marks, lifting the initial normal digits consumes at most $kN_1 p^{G_j}$ degrees of $X_{s_j,0}$ (at most kN_1 units of p^{G_j}). Since the early fixed vector quotients $(X_{x_i,0}^{\mathrm{vec}})^{P_i(H_i - w_i)}$ ($i < \sigma$) and the shifts $Z_i^{z_i}$ ($i \geq \sigma$) are already supplied by the plane steps of ν at x_i^{vec}, and $Z_{+,j-1}^{a_{s_j}-o_{+,j-1}}$ supplies segment j's $C'_{0,j}p^{G_j}$ start and reserve powers together with the $(a_{s_j} - o_{+,j-1}) - C'_{0,j}p^{G_j}$ leftover powers of $X_{s_j,0}$, across all segments we lift $X_{s_{j_0},0}^{c_4} \prod_j X_{s_j,0}^{c_4(k-1)(p^{G_j}-1)}$ (carved from the reserve powers of $X_{s_j,0}$) by the rank-one localizer $R(g)^{c_4}$ of 22.31 in the twisted rows $v_{s_j}^{(g)}$, which has order at least c_4 on every competing line, leading term $X_{s_{j_0},0}^{c_4} \prod_j X_{s_j,0}^{c_4(k-1)(p^{G_j}-1)}$ under s_g, and degree $c_4((k-1)(p^{G_j}-1) + \mathbf{1}_{j=j_0}) \leq kc_4 p^{G_j}$ at start s_j (at most kc_4 units of p^{G_j}), while the remaining nonnormal digits, the terminal monomial ξ^α, and all remaining reserve or leftover powers of $X_{s_j,0}$ lift by substituting $v_{x,0}^{(g)}$ and $w_{x,c}^{(g)}$ ($c \geq 1$), which are integral of order ≥ 0 under every $s_{g'}$ and exact under s_g. Together the localizer and initial normal digits consume at most $kc_4 + kN_1 \leq 2kN_1Q_pc_4$ units of p^{G_j} at start s_j, leaving at least $11kN_1Q_pc_4 + 12c_4$ units of p^{G_j} in $X_{s_j,0}$ for the recursion.

We now lift any term S at level $h = h_0 + d$ ($0 \leq d < c_4$) by descending induction in h to precision $h_* = h_0 + c_4$, maintaining at least $11kN_1Q_p(c_4 - d) + 12c_4$ reserve units of p^{G_j} in $X_{s_j,0}$ at every start s_j

beyond the localizer degree. Here h is the sum of the explicit ω-power, column orders $\mathrm{oc}(c)$ (and P_i for the early two-letter columns), individual tie orders P_i and 1, and number of bordered or pairing factors. Let F^0 be the raw lift of S. By Lemma 22.17(4), F^0 satisfies $\mathrm{ord}_{g'}(F^0) \geq h + c_4 \geq h_*$ on every competing line g'. Expand $s_g(F^0) - S$, leaving unchanged column, tie, and bordered or pairing factors in their leading form, and discard terms in $\omega^{h_*}O[X]$. In every remaining summand, rises add across factors: an exterior column error of rise $e_{\mathrm{col}} \geq 1$ has normal degree $t \leq \mathrm{oc}(c) + e_{\mathrm{col}} \leq (2Q_p+1)e_{\mathrm{col}}$; an internal tie factor error of rise $e_{\mathrm{tie}} \geq 2$ has normal degree $P_i \leq Q_p \leq Q_p e_{\mathrm{tie}}$; a bordered or pairing error of rise $e_{\mathrm{bor}} = p^{x-s_j} - 1 \geq 1$ has normal degree $e_{\mathrm{bor}} + 1 \leq 2e_{\mathrm{bor}}$; and a localizer error of rise $e_{\mathrm{loc}} \geq 1$ has normal degree e_{loc} and loses at most e_{loc} powers of $X_{s_j,0}$ at each start s_j. Thus a child summand of total rise $e \geq 1$ has at most $(2Q_p+1)e$ exposed normal variables (with available ω-power at least the number of exposed variables) and loses at most $(2Q_p+1)e$ powers of $X_{s_j,0}$ at each start s_j. Pair each exposed normal variable ($X^{\mathrm{vec}}_{x,b}$ for $1 \leq b \leq r$, or $X^{\mathrm{cov}}_{x,0}$) at $x \in (s_j, s_{j+1}]$ (using the preceding start s_{j-1} when $x = s_j$) with $s_g(D_x(g)) = (X^{\mathrm{vec}}_{s_j,0})^{p^{x-s_j}}$ from the reserve powers of $X^{\mathrm{vec}}_{s_j,0}$ to form a leading factor $\omega(X^{\mathrm{vec}}_{s_j,0})^{p^{x-s_j}} X^{\mathrm{vec}}_{x,b}$ or $\omega(X^{\mathrm{vec}}_{s_j,0})^{p^{x-s_j}} X^{\mathrm{cov}}_{x,0}$ (preserving every row multidegree, lifted at the next step by $B_{x,b}(g)$ or $P_{x,0}(g)$), and restore the localizer leading monomial $X^{c_4}_{s_{j_0},0} \prod_j X^{c_4(k-1)(p^{G_j}-1)}_{s_j,0}$. Together these consume at most $(2Q_p+1)e$ units of p^{G_j} for the bordered factors plus at most $(2Q_p+1)e$ powers of $X_{s_j,0}$ to restore lost start degrees, altogether at most $(4Q_p+2)e \leq 11kN_1Q_pe$ units of p^{G_j} at each start s_j. Subtracting the inductively constructed lifts F_ν of all child terms gives an integral lift $F = F^0 - \sum_\nu F_\nu$ of the exact multidegree satisfying $s_g(F) - S \in \omega^{h_0+c_4}O[X]$ and $s_{g'}(F) \in \omega^{h_0+c_4}O[X]$ on every competing line g'. Applying this at $d = 0$ to $S = \omega^{h_0}S_{\mathrm{sym}}$ (for the assembled symbol S_{sym}) and dividing F by ω^{h_0} yields an integral vector in the distinguished edge coordinates agreeing with S_{sym} modulo ω^{c_4} and having order at least c_4 on every competing edge of the same source. Apply the identical construction on the reversed dual edge $e^\vee$, at depth $t_1 = \lfloor t_0(A,B) \rfloor_+$. The normalized dual identification turns the second lift into a target functional and takes its competing-source conditions to the competing-incoming conditions.

A fine exponent determines its block. Reduction modulo P_i gives both digit rows, and hence H_i, K_i. At an early pair the two coordinate-zero quotients are $H_i - w_i, h_i$. After removing these fixed quotients and all the digits, division by the row powers recovers every variable exponent, since distinct variable slots go to distinct coordinates. At a late pair recover o_i as the sum of the ordinary vector exponents plus the exponent of S'_i. Each factor $\widehat{E}_i$ contributes one to this sum, and the fixed shift $Z_i^{z_i}$ contributes zero. The same sum at the terminal tie recovers o_+. Thus distinct blocks have disjoint fine supports and their factorial pairing vanishes exactly.

Fix a segment block and use its row degrees and maps above. Put $L_j = \sum_x \lfloor \log_p(\max(1, d_x)) \rfloor$ and $z_j = p^{kL_j} / \prod_x d_x!$. There is a positive rational number C_b, constant on the block, such that $a(\mu)! = C_b\, \mu!\, U_b(\mu)$. Indeed a variable slot of digit d and power P contributes $d!(P!)^{\mu_v}$ after its coarse factorial and digit unit are removed. The sum of the exponents of $P!$ in a physical row is e_x, and every unused slot has a fixed fine exponent. Set $\kappa_j = z_j p^{v(N_b)} C_b / s_b$. For two block coefficients the quantity $s_b \mu! u_\mu v_\mu$ is integral, by adding their two lattice inequalities. The factorial identity and $gU_b \equiv 1 \pmod{p^{c_4}}$ give the normalized block pairing $\kappa_j(s_b\langle u, v\rangle + \omega^{c_4} y)$ for some $y \in O$.

To bound κ_j, take the level-zero exponent μ_0 placing each ordinary letter degree on its first ordinary coordinate, each shift on Z, and zero on S'. This exponent has the required row degrees, whether or not its monomial meets the prefix cuts. Let $M_f = \prod_x d_x!/a(\mu_0)!$ and $M_c = \prod_x e_x!/\mu_0!$. Since the digit factor is a unit, $v(\kappa_j) = kL_j - v(M_f) + v(M_c)$. Each fine multinomial has at most k slots. Each coarse multinomial at μ_0 has at most two nonzero slots and $e_x \leq d_x$. By 11.7, $0 \leq v(M_f) \leq (k-1)L_j$ and $0 \leq v(M_c) \leq L_j$. Hence $0 \neq \kappa_j \in O$ and $v(\kappa_j) \leq (k+1)L_j$.

The segment alphabets are disjoint: each contains its interior pairs and its right endpoint. The factorial form and its row normalization therefore multiply. Set $\kappa_\beta = \prod_j \kappa_j$. There are two rows per mark and each degree is at most a_1^2, so $\sum_j L_j \leq 4N \log_p a_1$ and $v(\kappa_\beta) \leq 4\kappa_0$. Finally write the distinguished substitutions of the two lifts as P, Q, with integral symbols S, T. The difference between the normalized pairings is the sum of the pairings of $P - S, T$ and $P, Q - T$. Each row normalization is integral on integral polynomials because its monomial denominator is a multinomial of valuation at most $(k-1)\lfloor \log_p(\max(1, d_x)) \rfloor$. Both

differences therefore contribute elements of $\omega^{c_4}O$. Under the normalized dual-edge identification this gives

$$[F_{\beta,u}, G_{\beta',v}] \equiv \mathbf{1}_{\{\beta=\beta'\}}\kappa_\beta \prod_j \langle u_j, v_j\rangle_j \pmod{\omega^{c_4}O}, \qquad 0 \neq \kappa_\beta \in O, \quad v(\kappa_\beta) \leq 4\kappa_0. \tag{A.180}$$

Lean: Formalism.MainNotions.ColumnSystems.Pf.srcHalf1_of (Formalism.SharpRestriction.GrowthBounds.CoarseMultiplicity.PairedLiftAtOfIdentity, lines 85–161)
Lean: Formalism.MainNotions.ColumnSystems.Pf.paired_lift_at_of_identity (Formalism.SharpRestriction.GrowthBounds.CoarseMultiplicity.PairedLiftAtOfIdentity, lines 163–234)
Lean: Formalism.MainNotions.ColumnSystems.Pf.paired_lifts_of_identity (Formalism.SharpRestriction.GrowthBounds.CoarseMultiplicity.PairedLiftAtOfIdentity, lines 236–242)
Lean: Formalism.MainNotions.ColumnSystems.Pf.restForm_of_oplus (Formalism.SharpRestriction.GrowthBounds.CoarseMultiplicity.BFFinal, lines 26–27)

Take the retained Smith pairs on each segment. The matrix on their lifted tuples is diagonal modulo ω^{c_4}, with diagonal valuations at most D_0. Since $4D_0 < c_4$, division of each column by its diagonal gives an integral matrix congruent to the identity modulo ω. Let z be a diagonal of greatest valuation. After multiplying the columns by the integral ratios of z to their diagonals, use the inverse unit matrix to obtain exact matrix zI. The number n of pairs is the product of the retained segment ranks. The boundary rank comparison and $\prod_j(1-\kappa_{T,j}) \geq 1 - \sum_j \kappa_{T,j}$ give

$$n \geq (1 - g(A,B)) \prod_s P(A_s)P(B_s)/E_q. \tag{A.181}$$

For $g > 1$ this follows from $n \geq 0$.
Lean: Formalism.MainNotions.ColumnSystems.Pf.liftVecAt_of (Formalism.SharpRestriction.GrowthBounds.BinomialRatio.PairLift, lines 102–152)
Lean: Formalism.MainNotions.ColumnSystems.Pf.factors_of_perturb (Formalism.SharpRestriction.GrowthBounds.BinomialRatio.FactorsOfPerturb, lines 72–143)
Lean: Formalism.MainNotions.ColumnSystems.Pf.extended_statement_of_parts (Formalism.SharpRestriction.GrowthBounds.BinomialRatio.ExtendedStatementOfParts, lines 93–151)

Passing to harmonic summands. At a mark let $h = \sum_i X_iY_i$ on bidegree (a,b) in two rows of k variables. Exchange the roles of the rows at odd marks. The quotient by h in this bidegree is the flat dual Weyl lattice. Map X_i to entry i of the first column of a matrix and Y_i to the signed minor on its first $k-1$ columns omitting row i. Cofactor expansion gives the relation. Straightening gives the standard tableau basis with columns of heights one and $k-1$, together with its ordinary integral coefficient image.
Lean: Formalism.MainNotions.flipOneThree.Pf.PsiXY (Formalism.SharpRestriction.GrowthBounds.ColumnSpanDensity.DeltaPsiH, lines 12–14)

Let $E^{[l]}$ multiply by $h^l/l!$, and $F^{[l]} = (\sum_i \partial_{X_i}\partial_{Y_i})^l/l!$. The harmonic and level projectors are

$$H_{a,b} = \sum_{l=0}^{\min(a,b)} \frac{(-1)^l E^{[l]}F^{[l]}}{\binom{a+b+k-2}{l}}, \qquad P_j = \frac{E^{[j]}H_{a-j,b-j}F^{[j]}}{\binom{a+b+k-j-1}{j}}. \tag{A.182}$$

The commutator $[F,h] = k + \deg_X + \deg_Y$ gives $FH_{a,b} = 0$, $H_{a,b} \equiv 1 \bmod h$. On a harmonic v of bidegree $(a-i,b-i)$ it gives $F(h^iv) = i(a+b+k-i-1)h^{i-1}v$. Hence $P_j(h^iv) = \delta_{ij}h^iv$. Reducing modulo h and iterating proves $\sum_j P_j = 1$ and gives the harmonic dimension $d(a,b)$.

With $L = \lfloor \log_p(a+b+k) \rfloor$, the two divided operators preserve the lattice with basis $X^\alpha Y^\beta/\alpha!$. The projectors cost at most $2L$ there by the binomial bounds. The inclusions between that lattice times $a!$ and the ordinary lattice cost at most $(k-1)L$ by the multinomial bound. Thus $p^{(k+1)L}$ clears the projectors in ordinary coordinates, including in every dual edge basis since h, F are invariant. Use $i_j[f] = h^jH_{a-j,b-j}f$ and $r_jx = [P_jx/h^j]$. Division by h^j on polynomial multiples preserves integrality because its leading coefficient is a unit. The tensor maps satisfy $\sum_j i_jr_j = 1$. By $a+b+1 \leq 8a_1$, their denominators in both directions are at most p^u for $u = Nu_0$. The summand dimension is $d_j^{\mathrm{harm}} = \prod_s d(A_s - j_s, B_s - j_s)$.
Lean: Formalism.MainNotions.flipOneThree.Pf.oneTop_abs (Formalism.SharpRestriction.GrowthBounds.ColumnSpanDensity.HarProjEqZeroIff, lines 87–224)

Let $A_H = \sum_{s \text{ even}} A_s + \sum_{s \text{ odd}} B_s$, $A_j = A_1$ on the summand, and $d_j = A_H - A_j \geq 0$. Use $r'_j = \omega^{d_j}r_j$, $i'_j = \omega^{-d_j}i_j$. For the tensor pairs x_a, y_b use in the summand $x_{ja} = p^u r'_j x_a$ and $y_{jb} = p^{2u+d_j}y_b \circ i'_j$. A

lattice exponent $-A_H$, or $-A_H + c_4$ on a competing edge, becomes $-A_j$, or $-A_j + c_4$, respectively. Indeed $p^{d_j}\omega^{-d_j} = \omega^{d_j}$ on the dual side. Depth t_1 persists because $d_j \geq 0$.

Lean: `Formalism.MainNotions.flipOneThree.Pf.lattice_maps_of_functor_comparison` (`Formalism.SharpRestriction.GrowthBounds.ColumnSpanDensity.LatticeMapsOfFunctorComparison`, lines 100–219)

For $M_j = (y_{jb}(p^{A_j}x_{ja}))_{b,a}$, cancellation of the ω-powers and $A_j + d_j = A_H$ give $\sum_j M_j = p^{3u}zI_n$. Apply 22.8. It gives n_j pairs in each summand, $\sum_j n_j \geq n$, with diagonal valuation at most $D_0 + 3u \leq D_1$.

Lean: `Formalism.MainNotions.flipOneThree.Pf.rho_sum_bound_of_maps_splitting` (`Formalism.SharpRestriction.GrowthBounds.ColumnSpanDensity.RhoSumBoundOfMapsSplitting`, lines 88–210)

Transport these pairs to the E_q outgoing edges of one source. After division by block diagonals their matrix is the identity modulo ω since $2D_1 < c_4$. Hence $E_q n_j \leq d_j^{\mathrm{harm}}$. If all $j_s \leq J_b$, each term $\min(B_y, A_x/p^d)$ in J_{odd} decreases under the shift by at most J_0, also across the cyclic boundary. Hence $t_1 \leq t_*(A-j, B-j)$ and $E_q n_j \leq (1 - f(A-j, B-j))d_j^{\mathrm{harm}}$. Since $\sum_j d_j^{\mathrm{harm}} = \prod_s P(A_s)P(B_s)$ by telescoping, (A.181) gives the pointwise inequality of 22.12.

Lean: `Formalism.MainNotions.flipOneThree.Pf.keyIneq_of` (`Formalism.SharpRestriction.GrowthBounds.ColumnSpanDensity.FactorizationBoundMonoLattices`, lines 169–285)

Lean: `Formalism.MainNotions.flipOneThree.Pf.rho_upper_bound_of_edge_count` (`Formalism.SharpRestriction.GrowthBounds.ColumnSpanDensity.RhoUpperBoundOfEdgeCount`, lines 106–205)

Middle rows. Write $\eta = 12Nkc_b/b'$. When the lower shift leaves the box, rank zero suffices. Otherwise let V_-, V_+ have the flat extremes $(A_s - c_b, -B_s + c_b)$ and $(A_s + c_b, -B_s - c_b)$, with respective marked dimensions d_-, d_+. Let d_{mid} be the product of the middle-row dimensions in the general dominant weight λ. The Weyl formula gives

$$d_{\mathrm{mid}}d_- \leq \dim V_{\mathrm{marked}} \leq E_c d_{\mathrm{mid}} d_-, \qquad d_+ \leq E_c d_-, \qquad E_c = (1 + 6c_b/b')^{(2k-3)N}.$$

At each mark the varying root factors are the $2k-3$ factors meeting an extreme row. Each varies by at most $6c_b/b'$ of its lower value. Since $E_c \leq e^{\eta}$, $\eta \leq 1/9$, we have $E_c \leq 2$ and $(1-\eta)E_c \leq 1$.

Apply 22.29 at each mark with $c = c_b$, $P = A_s - c_b$, $Q = B_s - c_b$. On the lower-shift domain both degrees are nonnegative. The size hypotheses with the stated large constants give $k \geq 10$, $2c_b \leq 8a_1$. At odd marks use reversed dual coordinates. Tensor the maps, indexing a middle direction by a tuple of local Smith indices. An off-diagonal tuple has a zero factor, and the diagonal exponents add. The maps on the marked lattices are $i : V_- \to V_+$, $U_b : V_- \to V_\lambda$, and $L_a : V_\lambda \to V_+$. Here i multiplies by the extra extreme columns, and the U_b insert the middle tableau indexed by b. Use the corresponding dual middle basis for L_a, clearing its divided-power denominator. In column coordinates the paired middle basis contracts to

$$L_a U_b = \delta_{ab} p^{k_b} i, \qquad 0 \leq k_b \leq k_{\mathrm{mid}} = N\lceil r^2 \ell \rceil.$$

For the frame calculation let g carry the standard frame to the selected frame and let $D_t = \mathrm{diag}(t^{-1}, 1, \ldots, 1)$. An inserted section of determinant twist $-b$ and first-row degree $a + b$ is transported as $\det(g)^b g^{-1} v$, with matrix actions on the rows. Checking t^a times this section in the gD_t frame gives the scalar $(\det(g)t^{-1})^{-b} t^a \det(g)^b t^{-(a+b)} = 1$. For the two lifted middle factors take $a = b = c_b$; for the multiplier of i take $a = b = 2c_b$. Use $t = 1, \omega, \omega^2$ at source, selected edge and target. Hence the multipliers for i, U_b are integral at the source and in the normalized selected edge; the source lattice is contained in every normalized outgoing lattice for these dominant residual weights. Multiplication therefore preserves the competing-edge order as well as the source depth. Saturation at the selected edge follows from 22.29, giving an integral retraction there. For a target functional g_+ take $p^d g_+ \circ L_a$, where $d = A_1(V_+) - A_1(V_\lambda) = Nc_b$. The residual factor for L_a is integral at the selected normalized edge, and multiplying it by ω^{2d} puts it in the target vertex lattice. This vertex lattice lies in each normalized incoming residual lattice: in an adapted frame every weight of that factor is between its two extremes, which are $c_b, -c_b$ at each mark. Bilinearity in the common frame now gives target depth t, selected-edge integrality, and competing-edge order $c_2 - 2d$ from an upper test of order c_2. Finally $[v, p^d g_+ \circ L_a]_\lambda = [L_a v, g_+]_+$ by the definition of the normalized pairing.

Take n_-, n_+ pairs with scalar diagonals z_-, z_+ at the lower and upper flat weights. Push the lower source family by i and pull its test family back by the retraction, with the normalization powers, to the upper edge lattice. Let M be the matrix pairing the pushed family with the upper test family. In Smith bases retain the h entries dividing $z_- z_+$, and scale them to that scalar. The $s = n_- - h$ other rows have $M_{\mathrm{bad}} \in z_- z_+ \mathfrak{m}$. Pair these s source directions, together with the n_+ upper source directions, against the retracted tests and

the upper tests. The block matrix has diagonals $z_- I_s, z_+ I$ and off-diagonal blocks M_{bad}, B with B integral. Eliminating the first diagonal gives a Schur complement $z_+ I - B M_{\mathrm{bad}}/z_-$, which is z_+ times a unit matrix. Multiplication by $z_- z_+$ clears the inverse of the block matrix. The $s+n_+$ resulting directions have the upper source conditions and tests in the distinguished upper edge dual. On the E_q outgoing edges their normalized matrix reduces to the identity because $4D_1 < c_4$. Thus $s + n_+ \le d_+/E_q$ and hence $h \ge n_- + n_+ - d_+/E_q$. Apply every U_b to each retained source and the corresponding L_a to its test. The displayed composition identity, with the additional factor $p^{k_{\mathrm{mid}}-k_a}$ on the test, gives $d_{\mathrm{mid}} h$ pairs of valuation at most $2D_1 + k_{\mathrm{mid}}$.
Lean: `Formalism.MainNotions.flipOneThree.Pf.cloud_count` (`Formalism.SharpRestriction.GrowthBounds.ColumnSpanDensity.CloudCount`, lines 30–186)
Lean: `Formalism.MainNotions.flipOneThree.Pf.cloud_prop_of_maps_retraction_basis` (`Formalism.SharpRestriction.GrowthBounds.ColumnSpanDensity.CloudPropOfMapsRetractionBasis`, lines 36–54)
Lean: `Formalism.MainNotions.flipOneThree.Pf.latRetract` (`Formalism.SharpRestriction.GrowthBounds.ColumnSpanDensity.CloudPropOfMapsBasis`, lines 8–101)

The Weyl inequalities now give loss at most $f_- + 2f(A + c_b, B + c_b) + \eta$. The denominator is at most D by (22.18) and (A.173), since $N_1^2 \le k$ and there are at most N segments. Off-mark factors cost at most $2n'c_b$ in depth and order. The marked pullbacks leave order $c_4 - 2Nc_b - 2n'c_b = c_*$. The upper shift in J_{odd} costs at most Nc_b. Substitution in $t_*(A + c_b, B + c_b) + 2n'c_b$ gives (22.20) after enlarging C. Apply 22.10.
Lean: `Formalism.MainNotions.flipOneThree.Pf.midLP_of` (`Formalism.SharpRestriction.GrowthBounds.ColumnSpanDensity.LatticeCountPrimeMonoLevel`, lines 183–276)

□

Proof of 22.36

Proof. Write $N = |T|$, $n' = n_p$, and let T_{ev} be the even marks. For $s \in T$ put

$$M_s = \sum_{\substack{h \ge 1 \\ h \text{ odd}}} \min\{a_s/p^h, W_j(z_h(s))\}, \qquad \gamma_p = \frac{p}{p^2 - 1}.$$

By 22.4 the capacities W_j are at least $w'' \ge 1$ at these scales. Both series are bounded termwise by the geometric series with terms a_s/p^h. The identity $[u - w]_+ + \min(u, w) = u$ gives $A_s + M_s = \gamma_p a_s$.

The numerator and capacity in 14.4 are $\bar\alpha_s = a_s - 2c$ and $\bar C_{s,h} = W_j(z_h(s)) - \kappa_j + c$. The size conditions of 14.6 give $0 \le \kappa_j \le N_m \le S_c$ and $2c \le a_s$ eventually. Thus $0 \le \bar C_{s,h} \le W_j(z_h(s)) + c$. At an odd mark, replacing $a_s, W_j(z_h(s))$ by $\bar\alpha_s, \bar C_{s,h}$ decreases any positive-part summand by at most $2c/p^h + c \le 3c$. A decrease requires a positive original summand, hence $p^h < a_s \le 2a$ by (14.1). There are at most z such positive integers h. At an even mark the corresponding decrease in $A_s + M_s$ is $2c\gamma_p \le 3cz$, since $p < X' = a/b' \le a$. Summing the two estimates and using the definition of g_U gives

$$\sum_{s \in T} A_s + \sum_{s \in T_{\mathrm{ev}}} M_s \le g_U(t_0, j) + \bar\delta(t_0) + 3Ncz. \tag{A.183}$$

Lean: `Formalism.MainNotions.AuxBlock8.Pf.St.gU_lower` (`Formalism.SharpRestriction.GrowthBounds.LogarithmicSums.PartnerCountOfCentral`, lines 308–343)

Put $G_1 = C_{31} L/\delta_1$, $U = 40n'\delta_1$, $N_1 = N_{\mathrm{win}}$, $q_p = p^{G'}$, and $z_a = \log_p a$. The good gaps cover the cycle, so $N \ge 1$ and $n' \le NG'$. In (14.7), $u_1\delta_1 \le 2$ and $M_0 = O_\varepsilon(L)$, giving $G'\delta_1 \le C_{31}L$ and $N_1 \le C_N L$ for suitable constants. Thus

$$N \ge 1, \qquad n' \le NG_1, \qquad U \le 40NC_{31}L, \qquad N_1 \le C_N L.$$

All coefficients below are the nonnegative family coefficients of 14.7 and 14.10. We use one common bound

$$Y_0 = \exp(x^{23\varepsilon/48}).$$

For sufficiently large x, each of $G_1, U, n', N_1, k, N_m, L, z_a, x^{13}, c, \eta, q_p$ and each fixed coefficient is at most Y_0, uniformly in $p < X'$. Indeed $n' \le n$, and 14.6 bounds n, k, m, t by fixed powers of x. Also $z_a \le \log_2 a$, $c = 16k\Delta$, $\log\Delta \le x^{\varepsilon/4}/2$, and $\log\eta \le x^{\varepsilon/4}$. Finally (14.7) gives $G' = O_\varepsilon(x^{\varepsilon/8}L)$. Thus

$$\log q_p \le G' x^{\varepsilon/3} = O_\varepsilon(x^{22\varepsilon/48}L) \le x^{23\varepsilon/48}$$

eventually. Each fixed power of Y_0 is $o(b')$, since $\log b' = L + x^{\varepsilon} - x^{\varepsilon/3}$.

First suppose $p < p_0$. Write $r = r_G$ and put

$$M = w'' + \eta + c + N_m, \qquad D' = K_{\mathrm{DII}}(N_1^3 + r^5 + k^2)z_a.$$

We have $kq_p, D' \le b'$ by the preceding bounds. Also $G_1 M \le b'$ eventually. Indeed $G_1 w''/b' = C_{31}L/(xn\delta_1) \to 0$, and $G_1(\eta + c + N_m) \le 3Y_0^2 = o(b')$. In particular $n'M \le Nb'$. With $c_0 = 2b' + 2c \le 3b'$, the small-prime formulas of 14.7 and 14.10 now give

$$\begin{aligned} P_{v,j} &\le 6rNb', \\ \lceil t_v \rceil_+ &\le K_{\mathrm{tII}} rN(H_- + 4b') + 2rNb' + 1, \\ D_p &\le 40NC_{31}Lb' + 2rNb' + 120rNC_{31}Lb' + 1. \end{aligned}$$

In the last line the three terms before rounding bound UD', $2rn'c$, and rUc_0, respectively. Since $N, r, b', L \ge 1$, adding these lines and the unit in (22.21) gives

$$P_{v,j} + \lceil t_v \rceil_+ + D_p + 1 \le N(C_* r b' L + C_* L H_-)$$

as soon as $C_* \ge K_{\mathrm{tII}} r$ and $C_* \ge 13 + 4K_{\mathrm{tII}} + 160C_{31}$. Now use (A.183) and $M_s \ge 0$.

Suppose next that $p_0 \le p < X'$. By 22.39 the cycle has even length and contains an even mark. For each odd mark y, let h be the positive backward distance to the preceding even mark s. Then h is odd and $z_h(s) = (0, y)$ in orbit coordinates. The pair (s, h) determines y, so different odd marks use different summands of $\sum_{s \in T_{\mathrm{ev}}} M_s$. By 14.2,

$$\min(b_y, a_s/p^h) \le \min(W_j(0,y), a_s/p^h), \qquad B_j(0,y) - b_y \le b'/x + \eta.$$

The first inequality bounds the contribution of y to J_{odd} in t_v. The second bounds its contribution to the marked part of $P_{v,j}$. Their sum over the odd marks is at most $\sum_{s \in T_{\mathrm{ev}}} M_s + N(b'/x + \eta)$.

Lean: `Formalism.MainNotions.AuxBlock8.Pf.St.kinkHH` (`Formalism.SharpRestriction.GrowthBounds.LogarithmicSums.PartnerCountOfCentral`, lines 369–438)

To bound the remaining terms, put

$$E = K_{\mathrm{DI}}(x^{13})^2(N_1^5 + k^3)z_a.$$

By 14.10, $D_p = D_I \le UE + 1$ and

$$c_6 = 8(D_I + 1) + K_2 U k z_a + 4Uc + 2n'c.$$

In the price, $K_{\mathrm{Pr}} n' w'' \le NC_* w'' L/\delta_1$ for $C_* \ge K_{\mathrm{Pr}} C_{31}$. In the depth, $K_1 N N_1 H_+ \le NC_* L H_+$ for $C_* \ge K_1 C_N$. After dividing by N, the sum of the remaining costs is at most

$$K_{\mathrm{Pr}} G_1(\eta + c + N_m) + K_2(q_p c_6 + k q_p + k z_a + c) + 2G_1 c + 40C_{31}LE + 3.$$

Here $n'/N \le G_1$ and $U/N \le 40C_{31}L$. The final 3 covers the two ceiling errors and the unit in the asserted inequality, since $N \ge 1$.

Write $Y = Y_0$. Replacing the nonnegative factors by their upper bounds gives

$$E \le Y^4(Y^5 + Y^3), \qquad D_I \le d(Y) := Y^5(Y^5 + Y^3) + 1, \qquad c_6 \le h(Y) := 8(d(Y) + 1) + Y^4 + 6Y^2.$$

Consequently the preceding cost per mark is at most

$$\begin{aligned} &3Y^3 + Y\{Yh(Y) + 2Y^2 + Y\} + 2Y^2 + 40Y^2Y^4(Y^5 + Y^3) + 3 \\ &\quad = 8Y^{12} + 40Y^{11} + 8Y^{10} + 40Y^9 + Y^6 + 6Y^4 + 5Y^3 + 19Y^2 + 3. \end{aligned}$$

The coefficients sum to 130, so for $Y \ge 130$ this is at most $130Y^{12} \le Y^{13}$. Since $13x^{23\varepsilon/48} \le x^{\varepsilon/2}$ eventually, $Y^{13} \le b_0$. This gives coefficient one for b_0. Combining the price and depth bounds yields

$$P_{v,j} + \lceil t_v \rceil_+ + D_p + 1 \le \sum_{s \in T_{\mathrm{ev}}} M_s + N\big(b'/x + \eta + C_* w'' L/\delta_1 + b_0 + C_* L H_+\big).$$

Substitution in (A.183) proves (22.21), since $3cz \le C_*(c + \eta)z$.

Lean: `Formalism.MainNotions.AuxBlock8.Pf.St.B.hboB_arith` (`Formalism.SharpRestriction.GrowthBounds.LogarithmicSums.StarNumB`, lines 98–129)

Lean: `Formalism.MainNotions.AuxBlock8.Pf.St.B.facts_ev` (`Formalism.SharpRestriction.GrowthBounds.LogarithmicSums.StarNumB`, lines 159–355)

□

Proof of 22.37

Proof. The duties of 33.6 identify the cyclic coordinates of a selected representative with those of the twist. The rows are dominant and their unspecified entries have absolute value at most $\lfloor c \rfloor_+$, where $c = 4S_c$. Take the componentwise maximum of the thresholds of 22.38 and the constant middle and last thresholds C of 22.40. Below p_0 apply that lemma. Above or at p_0 use the attained minimum of 14.10. The coefficient-ring map of the design preserves the Frobenius frames and the chosen square root of p. The tableau generators in each frame therefore extend to generators of the corresponding lattice over the model ring O, including the intermediate edge lattice. Source vectors and integral tests extend by scalars, and their pairing matrix extends to the same scalar identity over O. This preserves the rank and the inclusions defining the test.

Below p_0, 14.8 bounds the share by $C_{\mathrm{frac}} N_v r_G^2 k / x^{26}$. Since r_G is fixed and $k \le x^6$ eventually by 14.6, this is at most $N_v x^{-12}$. Above or at p_0, 22.41 gives the mean bound $C_m N_v N_{\mathrm{win}} / x^{13}$, and 22.39 gives $C_m N_{\mathrm{win}} \le x$ eventually. The selection rule (14.8) includes all good representatives of a candidate with $\Sigma_p \le x^{-\varepsilon}$.
Lean: `Formalism.L2.G.Model.lpAt_dom` (`Formalism.LocalToGlobal.GroupLattices.FockModels.LPBase`, lines 256–384)
Lean: `Formalism.MainNotions.G.Joints.linear_programming_input_of_hypotheses` (`Formalism.SharpRestriction.GrowthBounds.Joints`, lines 2040–2127)

Fix u, w with $p = p(w) \ne T_u$. For nonempty selection, 17.29 gives a permitted place, and the orbit identification gives distinct positions for the selected representatives. The definite level stabilizers are finite. Each embeds into pro-groups at two distinct primes by the two unipotent level conditions, so each has order one. The neighbor sums at distinct positions commute and are integral by 16.3. Put $E_p = \lceil C_e f k^2 \log_p a \rceil_+$. By 14.12 the level-zero lattices satisfy $p^{E_p} M_{\mathrm{na}} \subseteq M \subseteq M_{\mathrm{na}}$. The intersection $R = M \cap V^{\mathrm{ram}}$ is saturated in M. We have $m \ge 2$ and $m \sum_{v \in V_p} \kappa_v = \Sigma_p \le x^{-\varepsilon} \le 1/2$ at the selected scales. These are the lattice and share hypotheses of 16.5.
Lean: `Formalism.L2.G.gatEF_of_tied` (`Formalism.LocalToGlobal.GroupLattices.LinkGConcreteNoKer`, lines 20–26)
Lean: `GlobalTheory.Mem.Top.LayOfCore.ledOK` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ExistsLayerParametersAdmissible`, lines 65–76)

Write $A = \mathcal{O}_{K,(w)}$ and $e_O = v_O(p)$, with v_O normalized by a uniformizer of O. Under the initial inclusion and the comparison of 16.3 and 17.10, the extensions of $(M_{H,u})_{(w)}$ and $(N_u)_{(w)}$ are R and the enhanced lattice N_P. Indeed localization commutes with intersection, and $(N_u)_{(w)} \subseteq (M_{H,u})_{(w)}$ off T_u. If π is a uniformizer of A, then p is a unit times π^{e_w}, so $v_O(\pi) = e_O / e_w$. Flat extension carries each composition factor of the A-quotient to $O/\pi O$. Hence

$$\mathrm{length}_O(R/N_P) = \frac{e_O}{e_w} \mathrm{length}_A((M_{H,u})_{(w)}/(N_u)_{(w)}) = e_O \ell_{u,w}.$$

Dividing the bound of 16.5 by e_O gives $\ell_{u,w} \le d_M(\tau_w + \mathrm{io}_w)$. For containment, put $b_\tau = \lceil \tau_p \rceil_+$. By (16.1) the image of $p^{-b_\tau} x'$ lies in M for every $x' \in (N_u)_{(w)}$. The coordinate test of 16.3 puts $p^{-b_\tau} x'$ in $(M_{H,u})_{(w)}$, and $e_w b_\tau \ge \lfloor e_w \tau_w \rfloor_+$ by 14.9. For empty selection the two lattices coincide by 17.10, and both budgets vanish by 14.9.
Lean: `Formalism.BasicNotions.LatCmp.NP_eq_span` (`Formalism.LocalRingCardinality.Lemmas`, lines 990–1025)
Lean: `Formalism.BasicNotions.LatCmp.length_quotient_of_nonempty` (`Formalism.LocalRingCardinality.Lemmas`, lines 1061–1116)
Lean: `Formalism.MainNotions.G.nuServed_of` (`Formalism.SharpRestriction.GrowthBounds.PlaneCountTheoremWithConstants`, lines 664–708)

For the upper bounds on gains, $\gamma_p = p/(p^2-1) \le 4/(3p)$ and $0 \le \bar{\alpha}_s \le 6a/5$ by (14.1) and 14.6. Both $g_0(v,j)$ and $g_U(v,j)$ are at most $\gamma_p \sum_{s \in T(v)} \bar{\alpha}_s$ by 14.3 and 14.4. Here $T(v)$ denotes the fibre of the representative of v. The subtracted minima and the logarithmic error are nonnegative. The price in 14.7 is also nonnegative, since at a mark $B_j(0,s) - b_s \ge b'/x - \kappa_j \ge 0$ by 14.2 and $N_m \le S_c \le w'' \le b'/x$. Each maximum in (14.9) is therefore between zero and $8N_v a/(5p)$. Summing over the disjoint fibres gives the asserted bound $4n_T a/p$. For S_v, 14.9 takes the ceiling of a sum bounded by $8mN_v a/(5p)$. If $mN_v = 0$ this ceiling is zero. Otherwise selection implies $p < X'$ and $a/p \ge b' \ge 500$ eventually, so $8mN_v a/(5p) + 1 \le 2mN_v a/p$.
Lean: `Formalism.MainNotions.AuxBlock8.Pf.orbit_max_le` (`Formalism.SharpRestriction.GrowthBounds.LogarithmicSums.NotSingularOfCentral`, lines 469–482)
Lean: `Formalism.MainNotions.G.svBound_of` (`Formalism.SharpRestriction.GrowthBounds.PlaneCountTheoremWithConstants`, lines 409–545)

Fix a signal or inert-class prime $p \le X''$. Let $g_{p,j}^{\mathrm{count}}$ be the counted gain of 14.8. We first bound the total logarithmic error. Each fibre of β_{Δ_p} in H has $\mathrm{ord}(\Delta_p)$ elements by 14.1. Thus $\sum_{v \in \mathcal{R}_p} n_p$ is n in the inert

class and $2n$ in the signal class. Both are at most f. Also $\sum_{v\in\mathcal{R}_p} N_v = n_T \le f$. Write $z_a = \log_p a$. Since $p \le X'' = a/w'' \le a$ and $a \ge 2$, we have $z_a \ge 1$ and $\log_p(2a) \le 2z_a$. By 14.2,

$$\sum_{v\in\mathcal{R}_p} \bar{\delta}(v) \le 2K_{\delta,1} f N_m^2 z_a + 4K_{\delta,2} f z_a^2.$$

Moreover $E_p \le (C_e+1) f k^2 z_a$ by 14.10. Take C_* in 22.36 also at least $2K_{\delta,1} + C_e + 1$ and $4K_{\delta,2}$. Since $k \le N_m$, (22.4) with $N_0 = N_m$ gives, for every final coefficient $C \ge C_*$,

$$E_p + \sum_{v\in\mathcal{R}_p} \bar{\delta}(v) \le F_p.$$

Lean: `Formalism.MainNotions.AuxBlock8.Pf.sum_dl_add_ep_le` (`Formalism.SharpRestriction.GrowthBounds.LogarithmicSums.NotSingularOfCentral`, lines 326–349)

Suppose $V_p \neq \varnothing$. By 14.8 this is an inert-class candidate with $p < X'$ and $\operatorname{ord}(h_p) \ge x^{\varepsilon/4} > 1$. Thus the maximum branch of (14.9) applies. At a selected orbit its contribution is

$$\max\{g_U(v,j) - P_{v,j},\, g_0(v,j),\, 0\} \ge g_U(v,j) - P_{v,j},$$

where $P_{v,j}$ is the global price defined in 22.36. Each unselected orbit contributes $[g_0(v,j)]_+ \ge 0$. Hence

$$\sum_{v\in V_p} \big(g_U(v,j) - P_{v,j}\big) \le g_{p,j}^{\mathrm{count}}.$$

By 14.9, $TT_p = \coprod_{v\in V_p} T(v)$. Put $d_v = \lceil t_v \rceil_+ + D_p + 1$. Summing (22.21) over that partition, for the fixed lift j, gives

$$\sum_{s\in TT_p} (A_{jsp} - \ell_p^{\mathrm{int}}) \le g_{p,j}^{\mathrm{count}} + \sum_{v\in V_p} \bar{\delta}(v) - \sum_{v\in V_p} d_v.$$

Here $A_{jsp} = A_s$: the partner in (22.5) has weight $W_j(z_h(s))$ by 22.4 and 14.9. All terms beyond the finite arithmetic cutoff vanish, since $p^h > a_s$ gives $a_s/p^h < 1 \le W_j(z_h(s))$.

There are m lifts. The depth costs sum to $m\sum_{v\in V_p} d_v = \tau_p - E_p$ by 14.9. Subtracting mF_p after summing over the lifts gives

$$\begin{aligned}\sum_{j=1}^{m} \sum_{s\in TT_p} (A_{jsp} - \ell_p^{\mathrm{int}}) - mF_p &\le \sum_{j=1}^{m} g_{p,j}^{\mathrm{count}} - \tau_p + E_p + m\sum_{v\in V_p} \bar{\delta}(v) - mF_p \\ &\le \sum_{j=1}^{m} g_{p,j}^{\mathrm{count}} - \tau_p.\end{aligned}$$

The last line uses $m \ge 1$, $E_p \ge 0$, and $\sum_{v\in V_p} \bar{\delta}(v) + E_p \le F_p$.

Lean: `Formalism.MainNotions.AuxBlock8.Pf.sum_TTP_served` (`Formalism.SharpRestriction.GrowthBounds.LogarithmicSums.NotSingularOfCentral`, lines 601–608)

Lean: `Formalism.MainNotions.AuxBlock8.Pf.gain_bound_at_of_star` (`Formalism.SharpRestriction.GrowthBounds.LogarithmicSums.NotSingularOfCentral`, lines 619–728)

Take $C = 4C_* + 1$ in (22.4) and (22.12). At a candidate, $C(S_c + k + \eta) \ge C_*(c + \eta)$. For $p < p_0 = k^2 x^{26}$ we have $p k^2 x^{78} \le (k x^{26})^4$. Taking base-$p$ logarithms and ceilings gives $L_1(p) \le L_1^{\mathrm{loc}}$, so $a p^{-L_1^{\mathrm{loc}}} \le a p^{-L_1(p)}$. In the other branch,

$$b'/x + \eta + C_* w'' L/\delta_1 + b_0 \le C\{b'/x + \eta + S_c + (w'' + t)L/\delta_1\} + b_0.$$

These comparisons and $\Lambda = \mathcal{L}$ from 14.9 give $\ell_p^{\mathrm{int}} \le \ell_p$ with the stated ℓ_p.

It remains to consider $V_p = \varnothing$. Then $\tau_p = 0$. At an inert-class prime $p < X'$, TT_p is empty by 14.9, so nonnegativity of F_p and of the counted gain proves the inequality. In the other cases $TT_p = TT$. At a signal prime every odd partner is on sheet one, so none of the odd exponents is excluded in 14.3. At $p \ge X'$, for $h \ge 1$ we have

$$a_s/p^h \le \big(a + a/(k x^{13})\big)/X' \le b' + b'/x.$$

Every excluded partner is a mark, where $W_j \ge b' + b'/x$ by 14.2. Its summand in A_{jsp} is therefore zero. In particular, if p is inert-class with $h_p = 0$, every odd partner is the mark $(0, s)$, and both the prime sum and the counted gain vanish. In the remaining cases apply the positive-part comparison in the proof of 22.36 on the unexcluded odd exponents. For each orbit it gives

$$\sum_{s \in T(v)} (A_{jsp} - 3c \log_p(2a)) \le g_0(v, j) + \bar{\delta}(v) \le [g_0(v, j)]_+ + \bar{\delta}(v).$$

Sum over all orbits. The positive parts sum to $g^{\mathrm{count}}_{p,j}$ by (14.9), and the errors sum to at most F_p. Since $\ell_p \ge C_p \ge 3c \log_p(2a)$, summing over the m lifts proves the same inequality with the stated losses.

Finally 14.9 identifies the counted gain with the gain on the supported fibre. At a served prime 15.14 leaves this gain in the sum, so $g_{jp} = g^{\mathrm{count}}_{p,j}$. Taking $r_p = F_p$ gives (22.2).

Lean: `Formalism.MainNotions.G.ask_eq_askP` (`Formalism.SharpRestriction.GrowthBounds.PlaneCountTheoremWithConstants`, lines 223–244)
Lean: `Formalism.MainNotions.G.lo_le_loG` (`Formalism.SharpRestriction.GrowthBounds.PlaneCountTheoremWithConstants`, lines 102–205)
Lean: `Formalism.L2.GlobalTheoryTheory.Loss.loss_bound_at` (`Formalism.LocalToGlobal.GlobalTheory.PowFourLtCeilThresholdPrime`, lines 766–816)

□

Proof of 22.38

Proof. Take T_0, T_2, C_* from 22.35. Fix $h_G = 2 + 8\varepsilon$, $r_G = \lceil 8(h_G + 2)/\varepsilon \rceil + 1$, as in 14.8. In (14.7) we have $u_1 \delta_1 \le 2$ and $M_0 \le (h_G + 3) 10^{6 r_G + 1} \log x + 1$. Thus for $\log x \ge 1$ that formula gives $N_{\mathrm{win}} \le A_\varepsilon \log x$, where one may take

$$A_\varepsilon = 2 + \max\{12 \exp(1)(6 r_G + 1)\big((h_G + 3) 10^{6 r_G + 1} + 6\big), (h_G + 3)/\log 2\}.$$

Require every middle entry to be at least $\max(1, T_0)$. For a concatenated outer and middle list ℓ, take the last threshold to be

$$\max\big(\{A_\varepsilon\} \cup \{\, 2T_2(s),\, 2C_*(s) : s \in \ell,\ s \ge T_0 \,\}\big).$$

The pre-threshold can be zero. The entry C_0 belongs to the middle list at CH, so the last threshold bounds $2T_2(C_0)$ and $2C_*(C_0)$. Applied to the depth, denominator, band and two loss coefficients of 22.35, these are exactly the lower bounds needed in (22.22). Applied to C_N, the same test gives the window bound. All these choices precede the choice of a family. For a family passing the tests, the cutoff $x \ge \max(\exp(1), x_1 + 1)$ ensures that the scale in the displayed window formula is the indexed scale x.

Lean: `GlobalTheory.EX.RangeOne.rangeOne_abs` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.TensorOfPairLift`, lines 637–640)

□

Proof of 22.39

Proof. Use $k \le x^6$, $p \ge 2$, $u_1 = \lceil x^{\varepsilon/8} \rceil$ in L_2 and (14.7) for $G' = O_\varepsilon(x^{\varepsilon/8} \log x)$ and the stated bound on N_1. An odd orbit order would have trivial image in the 2-group and divide $o \le x^{\varepsilon/100}$, contrary to the candidate condition $n' \ge x^{\varepsilon/4}$. The run condition on full tiles in the good orbit gives even gaps at most G', so $n' \le N G'$ and there are at least two gaps by $3G' \le n'$. Cover the cycle with $\lceil n'/G' \rceil$ windows and use $N_1 \le 30 G' \delta_1$ for the upper bound on N. □

Proof of 22.40

Proof. Use 22.18 with $Y = x^{26}$, $a_0 = a - c$, $c_b = \lfloor c \rfloor_+$, $c_t = \lfloor c_0 \rfloor_+$, $D = D_{II}$, $c_* = 8(D + 1)$, and $c = 4S_c$. By 22.39, $N \ge x^{\varepsilon/8}/(C_{31} \log x)$, so $N_1^2 \le rN$. Let C_{rank} be the family coefficient in the rank requirement $C_{\mathrm{rank}} N_1^2 \log a \le k$. Since $k \ge x^{9/2}$, both this requirement and $N_1 + 3r \le k$ hold eventually. The host split-prime threshold forces candidate $p \ge 13$. Since $N \le x^3$, $p < p_0 \le k^2 x^{26} + 1$,

$$256 k^2 N Y^2 \le k^2 x^{78} \le (k x^{26})^4 / p \le p^{L_1^{\mathrm{loc}} - 1}.$$

The family thresholds bound C above by each of the following coefficients: the share coefficient C_{frac}, the depth coefficient in t_{II}, the coefficient of $N_1^2 \log a$ in $\mathcal{L}$, the denominator coefficient K_{DII}, the coefficient of $k^3 r^2 N p^{5G'}(c_* + rNc_0 + 2rn'c)Y$ in the weight-size requirement, and C_{rank}. Use $N \le 40n'\delta_1$ in D_{II}. Since $D = O(x^{16}b')$ and $\log(p^{G'}) = O_\varepsilon(x^{\varepsilon/8}\log^2 x)$, $k^3 r^2 N p^{5G'}(c_* + rNc_0 + 2rn'c)Y = o(a)$ by $a/b' = X'$. At a mark, $a_0 + c_b \le a \le a_s$. By (14.1), $a_s \le 6a/5 \le 2a_0 - c_b$ eventually, since $c = o(a)$. By (14.6), $0 \le b_s \le 5b'/4$, so the integer $b_s + 2c_b$ is at most $\lfloor 2b' + 2c \rfloor = c_t$. Put $z = \log_p a$ and $U = 40n'\delta_1$. The denominator hypothesis of 22.18 follows from

$$CNk^2 \log_p a_0 \le K_{\mathrm{DII}} U(N_1^3 + r^5 + k^2)z \le D_{II}.$$

For the depth, $CN_1^2 \log a_0 \le \mathcal{L}$, $a_0 \le a$, $c_t \le c_0$, and $c_b \le c$. The expression in (22.17) is therefore at most

$$K_{\mathrm{tII}} rN\{\mathcal{L}a/(pk) + ap^{-L_1^{\mathrm{loc}}} + c_0 + kq_p\} + 2rn'c = t_{II}, \qquad L_1^{\mathrm{loc}} = \lceil \log_p((kx^{26})^4) \rceil \text{ as in 22.36.}$$

Thus the integer depth $\lceil t_{II} \rceil_+$ is admissible. The witness has rank at least $(1 - \min\{1, CNr^2k/Y\}) \dim V$, which is at least the required rank because $C \le C_{\mathrm{frac}}$.

Lean: `Formalism.MainNotions.G.Joints.thirteen_le` (`Formalism.SharpRestriction.GrowthBounds.SizesAtRecord`, lines 163–183)

□

Proof of 22.41

Proof. Use 22.39 and $k^2Y^2 \le p < X'$. Take $\Delta = b_{\mathrm{top}}(x) - b'$, $c_b = \lfloor 4S_c \rfloor_+$, $Y_b = x^\varepsilon/20$, $D = D_I$, and take C_0 to be the family coefficient in $\mathcal{L} = C_0 N_1^2 \log a$, as in 14.8. By Lemma 22.38, the family thresholds satisfy (22.22). Use $N \le x^3$, $k \ge x^{9/2}$ and $\mathcal{L} = O_\varepsilon(x^\varepsilon \log^2 x)$ for $2N_1 + 2 \le k - 1$, $2Y_b N\mathcal{L} < k$, $HN\mathcal{L}^2 \le k$. The width of I_b is at least $Y_b N\mathcal{L}b'/k$ by $N \le 40n'\delta_1$, $n' \le n$. By 14.8, $\Delta \le \rho_B b'/4$ and the local band half-width is at most $\rho_B = o(1)$. The candidate band exclusion thus applies. The remaining elementary size inequalities use $X' = e^{x^{\varepsilon/3}}$ and $D_I \le x^{49}$, with $HNY^2(N_1^5 + k^3)\log_p a \le D_I$. In particular

$$Hk^3 N q_p^4 [8(D_I + 1) + (N + n')c_b]Y = o(b').$$

The logarithm of the left side is bounded by $O(\log x) + O(x^{23\varepsilon/48}) + O(x^{\varepsilon/4})$ from the bounds on q_p, c_b, while $\log b' \sim x^\varepsilon$. Both integer windows are nonempty eventually. This gives (22.18).

With the notation of 22.35, let $m' = k - 1$. For $0 \le i \le J_b$ and $u \ge a_1$, $P(u)/P(u+i) \ge 1 - m'J_0/a_1$. The binomial product gives

$$\frac{d(u,v)}{P(u)P(v+i)} \ge \frac{P(v)}{P(v+i)} - \frac{P(v)}{P(v+i+1)}.$$

For $v \in I_b + \tau$, $\tau = 0, c_b$, one has $v + J_b + 1 + m' \le 2b'$ eventually, so $P(v)/P(v + J_b + 1) \le e^{-L'/3}$. Summing gives the density in 22.12 with $\delta = (1 - m'J_0/a_1)(1 - e^{-L'/3})$. For the interval sizes w_a, w_b,

$$N(J_b + c_b)/w_b \le 6/Y_b, \qquad N(J_b + c_b)/w_a \le 6N\mathcal{L}Y/X', \qquad Nm'J_0/a_1 \le NL'/X'.$$

These tend to zero, as does $Ne^{-L'/3}$. Thus the cardinality inequality of 22.12 holds eventually. The downward boundary probability is at most $Nc_b(1/w_a + 1/w_b)$. This and $12Nkc_b/b'$ are at most $1/Y$ eventually, by the exponential size of b'. Therefore

$$\mathbb{E}_0(f_- + 2f(A + c_b, B + c_b) + 12Nkc_b/b') \le 6(C_A + C_s)NN_1/Y + 2/Y. \tag{A.184}$$

Teichmüller expansions identify the source with $W(\mathbf{F}_{p^{n'}})$. The coefficient DVR has unramified subring with residue field $\mathbf{F}_{p^{2n'}}$, and separating even and odd powers of ω in its expansion gives the full ring by adjoining $\omega^2 = p$. Extend the smaller residue-field identification to the larger one and carry ω to its counterpart. This transports the lattices and pairings, so the displayed majorant holds for every local model and dominant weight.

Lean: `Formalism.MainNotions.flipOneThree.Pf.restOIso` (`Formalism.SharpRestriction.GrowthBounds.ColumnSpanDensity.RestOIso`, lines 8–12)

Lean: `Formalism.MainNotions.flipOneThree.Pf.theorem_I_of_intermediate_bounds` (`Formalism.SharpRestriction.GrowthBounds.ColumnSpanDensity.TheoremIOfIntermediateBounds`, lines 28–77)

Let $C = C_*(C_0)$ be the coefficient of (22.20), put $U = 40n'\delta_1$, and write $z = \log_p a$. Since $p < X' = a/b'$, the sum of the four terms in the first parenthesis of that depth bound is at most $2(a/p^2 + \mathcal{L}a/(pk))$. Its contribution is bounded by the corresponding term of t_I because $2C \le K_1$ by (22.22). In the second parenthesis, $N \le U$ by 22.39, $C \le K_2$, and $c_b \le c = 4S_c$. Thus

$$c_* + CNkz + 4Nc_b + 2n'c_b \le 8(D_I + 1) + K_2Ukz + 4Uc + 2n'c = c_6.$$

Multiplication by $q_p = p^{G'}$ preserves this inequality. The terms $kq_p + kz + c_b$ are at most $kq_p + kz + c$, the outer coefficient CN is at most K_2N, and the last term $2n'c_b$ is at most $2n'c$. Each odd mark uses the same least positive distance back to an even mark in both formulas, by the cyclic order in 22.39. This gives the required depth $\lceil t_I \rceil_+$ with denominator D_I and order $8(D_I + 1)$. For each pair of weights the transported fraction is now admissible for every model and dominant weight in 14.10, so it bounds the least passing fraction from above. Average this pointwise inequality over the original box. Since $NN_1 \ge 1$, (A.184) gives the assertion with $C_m = 6(C_A + C_s) + 2$, or any larger family constant.

Lean: `Formalism.MainNotions.G.Joints.tI_le_tIf` (`Formalism.SharpRestriction.GrowthBounds.Joints`, lines 1161–1266)

□

Proof of 22.42

Proof. In applying 22.7, use the integer cutoff $\widetilde{p}_0 = \lceil k^2x^{26} \rceil$ for its parameter p_0. For every prime p, $p < k^2x^{26}$ if and only if $p < \widetilde{p}_0$, so the prime classes and local models agree. The family sizes give $k^2x^{26} \le x^{36}$; for $x \ge 3$, $\widetilde{p}_0 \le x^{36} + 1 \le x^{40}$. Increase the fixed exponent ν to at least 40. The remaining family sizes give (22.13) with $C = \max(C_G, 0)$, and eventually $b_0 \le b^{1/2}$. Increase the threshold for (22.14), $k^2x^{26} \le X'$, and $n_T > 0$.

Lean: `Formalism.L2.GlobalTheoryTheory.Loss.windows_ev` (`Formalism.LocalToGlobal.GlobalTheory.PowFourLtCeilThresholdPrime`, lines 1555–1595)

Use $K_0 = k^2x^{78}$, $L_1(p) = 1 + \lceil \log_p K_0 \rceil$, and $\vartheta = (kx^{13})^{-1}$. Lemma 22.4 gives the ordinary side conditions. At d the exponents are positive and the weights nonnegative.

Lean: `Formalism.L2.GlobalTheoryTheory.Loss.side_at` (`Formalism.LocalToGlobal.GlobalTheory.PowFourLtCeilThresholdPrime`, lines 1199–1209)

Use $\rho = \rho_B$, $Y = X'(1 + C_r\rho_B)$ from 15.14. The word primes split. By the discard criterion of 15.14, a prime $p \in P_s$ is discarded if $p > X''$, or if it is of inert class and either $h_p \ne 0$ has odd order or $p < Y$ satisfies the small-order, band, or excess-share test of 15.14. At each such prime the gain and deduction vanish. Served primes satisfy $p \le X''$.

Lean: `Formalism.L2.GlobalTheoryTheory.Loss.nonserved_zero` (`Formalism.LocalToGlobal.GlobalTheory.PowFourLtCeilThresholdPrime`, lines 687–698)

Use 22.37 for the served-prime comparisons. In the small-order branch of P_1 the order of $[p]$ divides $\mathrm{ord}(h_p) < x^{\varepsilon/4}$. In particular an inert-class $h_p \ne 0$ of odd order has trivial 2-group image. The band branch satisfies the second alternative of 22.5, so that lemma applies with $Y_0 = \max(Y, X'')$. The remaining discarded candidates have $\Sigma_p > \theta$.

Signal primes and primes $p \ge X'$ have $TT_p = TT$. Every other served prime is an inert candidate below X' with eligible share. Its TT_p consists of the positions on good representatives. Every omitted position therefore lies on a bad orbit, even when there are no good representatives. This gives the required R_{out}. If V_p is nonempty, $\mathrm{io}_p = 2\Sigma_p \max_v S_v$ by 14.9. Apply $S_v \le 2mn_Ta/p$. If V_p is empty, the index vanishes. Hence (22.10) holds.

Lean: `Formalism.L2.GlobalTheoryTheory.Loss.unserved_bad` (`Formalism.LocalToGlobal.GlobalTheory.PowFourLtCeilThresholdPrime`, lines 722–764)

Lean: `Formalism.L2.GlobalTheoryTheory.Loss7.loss_setting_family` (`Formalism.LocalToGlobal.GlobalTheory.LossRequirementsOfSmallShares`, lines 41–80)

The index is supported on full candidate fibers below X'. The fiber identity of 22.1 gives $\mathrm{IO}_i = \sum_{p \in \mathcal{P}_c} \log p \, \mathrm{io}_{i,p}$.

Lean: `Formalism.L2.GlobalTheoryTheory.Loss.IO_at` (`Formalism.LocalToGlobal.GlobalTheory.PowFourLtCeilThresholdPrime`, lines 253–268)

By 22.43, $\lceil 2a \rceil$ is an eligible cutoff. For every eligible cutoff, 22.1 identifies the ordinary weights and the ramified coset.

Lean: `Formalism.MainNotions.Lnk.wPart_eq` (`Formalism.SharpRestriction.GlobalSchurBounds.FrameCodeOfAdditive`, lines 464–477)

Lean: Formalism.MainNotions.Lnk.reads_w (Formalism.SharpRestriction.GlobalSchurBounds.FrameCodeOfAdditive, lines 223–234)
Lean: Formalism.MainNotions.ZetaZeroFreeRegion.Br.isInert_iff (Formalism.SharpRestriction.GlobalSchurBounds.RamificationTwoAndConjugationAtQAndD, lines 371–381)
Lean: Formalism.MainNotions.Lnk.askIsF (Formalism.SharpRestriction.GlobalSchurBounds.FrameCodeOfAdditive, lines 546–556)

Apply 22.7 with $c_{\mathrm{Loss}} = 3K_s + 13$. □

Proof of 22.43

Proof. The bijection e_τ labels the indices by the host subgroup Ψ. The position $T(e_\tau(\tau))$ of 5.2 has first sign bit zero and recovers the host element from its signs and A-coordinate. Its image has size $f = 2n = |\Phi|$. The bounds of 14.2 give $\sqrt{b} \le w'' \le b'$, $t_0\eta_{\mathrm{sz}} \le w''/x$, and $t_0 \ge 1$. At a mark, $b' \le b_s \le (1+v)b'$ with $v = 2n\delta_1 x^{\varepsilon}\mathcal{L}/k$ and $4v \le x^{-2}$. Its weight interval $[b_s + b'/x, b_s + b'/x + \eta_{\mathrm{sz}}]$ is contained in $[1, 2b']$, as is the thin interval $[w'', w'' + \eta_{\mathrm{sz}}]$. The thick labels $((0,1), s_2 + u_0)$ have the indicated thick interval. □

Proof of 22.45

Proof. Take $\varepsilon_0 = 1/53$. For each $0 < \varepsilon \le \varepsilon_0$ choose the numerical threshold $\mathcal{K}_W(\varepsilon)$ of 22.38. Now fix any family and any real CH passing this threshold. Lemmas 22.39 and 22.41 apply to this family. Take the maximum of their point cutoffs and the family starting scale plus one. The averaging below is over trial weights in the box of this fixed point. Eventually the two interval widths are at least one by $k \le x^6$, $a = xe^{x^{\varepsilon}}$, $b' = ae^{-x^{\varepsilon/3}}$. Give the integer box its uniform law. Below p_0, the family bound $C_{\mathrm{frac}} r_G^2 k \le x^{14}$ gives $\kappa_v \le N_v x^{-12}$ pointwise by 14.8. Above p_0, 22.41 and 22.39 give $\mathbb{E}\kappa_v \le C_m N_v N_1 / x^{13} \le N_v x^{-12}$.
Lean: GlobalTheory.EX.RangeOne.rangeOne_abs (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.TensorOfPairLift, lines 637–640)
Lean: Formalism.MainNotions.G.Joints.type_one_hypothesis_of_segments (Formalism.SharpRestriction.GrowthBounds.Joints, lines 1283–1325)
Lean: Formalism.MainNotions.AuxBlock8.Pf.restHypI_ev (Formalism.SharpRestriction.GrowthBounds.LogarithmicSums.MeanNum, lines 416–660)

The orbit coordinate map is injective on a period, so its induced law is uniform. The good orbits are disjoint, so $\mathbb{E}\Sigma_p \le m n_T x^{-12}$. For $u = \log X'$ put $I_p = \mathbf{1}_{\{\Sigma_p > x^{-\varepsilon}\}}$ and $Z = \sum_{p \in \mathcal{P}_c} (\log p/p)(I_p + \Sigma_p)$. Using $I_p \le x^{\varepsilon}\Sigma_p$ and (A.155) gives

$$4\mathbb{E}Z \le 4(x^{\varepsilon}+1) m n_T x^{-12} u(2+u) \le 48 x^{-19/2 + (9+2/100+2/3)\varepsilon} \le x^{-9}$$

eventually, by $m \le x^{1/2+\varepsilon/100}$ and $n_T \le 2x^{2+8\varepsilon+\varepsilon/100}$. Choose a pair with $Z \le \mathbb{E}Z$ and use $R_W \le 4Z$. □

Proof of 22.46

Proof. At a window point beyond the thresholds choose a model $i_0 \in I(x)$ and carry its weights along the top bijection to $\mathcal{T}(x)$. Extend by zero. This pair satisfies (22.27) and is realized by i_0. Every realization satisfies the share bound by hypothesis. For the constructed family, Proposition 33.26 gives $R_W(i) \le x^{-9}$ at the model's own weights for every $x > x_1$ and every $i \in I(x)$.
Lean: Formalism.MainNotions.FamCore.coverU_hyp_self (Formalism.SharpRestriction.FamCore, lines 207–212)

□

Proofs for Chapter 23: A slope bound for families of maps

Proof of 23.4

Proof. Put $T = \operatorname{tr} Q_Z$ and $a_i = \|P_i - I\|_{\mathrm{HS}}$. For an orthogonal projection E and an operator M on the same space, Cauchy–Schwarz gives

$$|\langle E, M\rangle_{\mathrm{HS}}| \le \|E\|_{\mathrm{HS}} \|M\|_{\mathrm{HS}}.$$

Since $E = E^* = E^2$, this yields

$$|\operatorname{tr}(EM)| \le \sqrt{\operatorname{rank} E}\, \|M\|_{\mathrm{HS}}. \tag{A.185}$$

With $E = I$, $M = P_i - I$, the same inner product inequality for the sequences $(\sqrt{d_i})_i$ and $(a_i)_i$ and (23.1) give

$$T = \sum_i \operatorname{tr} P_i \geq R - \sum_i \sqrt{d_i}\, a_i \geq R - \sqrt{R \sum_i a_i^2} \geq (1 - \sqrt{e})R \geq R/2 > 0. \tag{A.186}$$

Let Π_W be the orthogonal projection onto $W = (\sum_i \widetilde{B}_i V_i')^{\perp}$. Put $T_W = \operatorname{tr}(\Pi_W Q_Z)$. Let E_i project onto $(V_i')^{\perp}$, of dimension k_i. Since $\Pi_W \widetilde{B}_i$ vanishes on V_i', we have $\Pi_W \widetilde{B}_i = \Pi_W \widetilde{B}_i E_i$ and

$$\operatorname{tr}(\Pi_W c_i \widetilde{B}_i \widetilde{B}_i^*) = c_i \|\Pi_W \widetilde{B}_i E_i\|_{\mathrm{HS}}^2 \leq c_i \|\widetilde{B}_i E_i\|_{\mathrm{HS}}^2 = \operatorname{tr}(E_i P_i E_i).$$

By (A.185),

$$\operatorname{tr}(E_i P_i E_i) = k_i + \operatorname{tr}(E_i(P_i - I)) \leq k_i + \sqrt{k_i}\, a_i.$$

The subspaces satisfy $\sum_i k_i \leq zR$. Apply Cauchy–Schwarz to $(\sqrt{k_i})_i$ and $(a_i)_i$ and use (23.1) to obtain

$$T_W \leq \sum_i k_i + \sqrt{\sum_i k_i} \sqrt{\sum_i a_i^2} \leq R(z + \sqrt{ze}). \tag{A.187}$$

By (23.2), the operator $A = Q_Z - (T/D_Z)1$ satisfies

$$\|A\|_{\mathrm{HS}}^2 = \operatorname{tr}(Q_Z^2) - T^2/D_Z \leq \eta T^2/D_Z.$$

Put $q = \dim_{\mathbf{C}} W/D_Z$. Apply (A.185) to Π_W, A to get $T_W/T \geq q - \sqrt{\eta q}$. By (A.186) and (A.187),

$$q - \sqrt{\eta q} \leq 2(z + \sqrt{ze}).$$

For $z \geq 1/4$, use $q \leq 1$. For $0 \leq z < 1/4$, use $\sqrt{\eta q} \leq (\eta + q)/2$ and $\sqrt{ze} \leq \sqrt{e}$ to obtain $q \leq \eta + 4z + 4\sqrt{e}$. □

Proof of 23.7

Proof. By (A.186), $D_Z > 0$ and hence $D_C > 0$. For $\tau \geq 0$, define

$$\theta_i(\tau) = \frac{s - \tau + h_i - c_K}{2}, \qquad N_i^\tau = (N_i)_{\geq \theta_i(\tau)}, \qquad x_{ij} = 2\lambda_j(N_i) + c_K - h_i \quad (1 \leq j \leq n_K d_i).$$

A rank-zero source contributes an empty sum. By 23.2, the sum of the adjusted slopes is $n_K \sum_i d_i(\mu(N_i) - h_i) \geq n_K R s$ and each is at most $s + U$. The count in 23.3, divided by n_K using 23.2, gives

$$\sum_i (d_i - r(N_i^\tau)) \leq R z_U(\tau). \tag{A.188}$$

Let $S^\tau = (C^\vee)_{\geq (s - \tau - c_K)/2}$. For a nonzero image $B_i N_i^\tau$, 23.2 and 23.1 give

$$\mu_{\min}(B_i N_i^\tau) \geq \mu_{\min}(N_i^\tau) - h(B_i) \geq s - \tau.$$

By 23.2 the image lies in S^τ. Thus $M^\tau = \sum_i B_i N_i^\tau \subseteq S^\tau$. Apply 23.5 with (A.188) and use $D_Z \leq D_C/(1-\delta)$ to obtain

$$D_C - r(S^\tau) \leq D_C - r(M^\tau) \leq D_C w(\tau), \qquad w(\tau) = \frac{\eta + 4 z_U(\tau) + 4\sqrt{e}}{1 - \delta}. \tag{A.189}$$

By 23.2, the adjusted target slopes $y_j = 2\lambda_j(C^\vee) + c_K$ have mean $\mu(C^\vee)$ and are at least $s - H$. By (A.189), at most $n_K D_C w(\tau)$ lie below $s - \tau$. The function w is continuous on $[0, H]$, including when $U = 0$. Apply 23.6 and integrate z_U to get

$$s - \mu(C^\vee) \leq \int_0^H w(\tau)\, d\tau = E_{\mathrm{fr}}(U, H; \delta, \eta, e).$$

The second inequality follows since $\log \# I \geq 0$. □

Proof of 23.8

Proof. By (23.8),

$$\mathcal{R} \geq \#\mathcal{W}\,\frac{D_*}{C_0\Gamma^2} \geq \frac{D_*}{\sigma}. \tag{A.190}$$

Put $a = \sqrt{e_0}$, $t = \sqrt{d_e}$ and

$$Q_f = \sum_u c_u A_u A_u^*, \qquad Q_Z = \sum_u c_u A_u J_u J_u^* A_u^*.$$

Complete the columns of each J_u to an orthonormal basis. In these bases $J_u^* M J_v$ is a submatrix of M, so its squared Hilbert–Schmidt norm is bounded by that of M. Apply this to $c_u \widetilde{B}_u^* \widetilde{B}_u - I = J_u^*(K_{uu} - I)J_u$ to get a source defect sum at most $e_0\mathcal{R}$ by (23.5). By (23.6), $R \geq (1 - d_e)\mathcal{R}$, so the sum is at most $2e_0 R$. By (23.7), $2e_0 \leq \sigma^2/2 \leq 1/800$.

By (23.5), $\|K_{uu}\|_{\mathrm{HS}} \leq (1 + a)\sqrt{f_u}$. Expand the trace and use (23.4) to obtain

$$\operatorname{tr}(Q_f^2) = \sum_{u,v} \|K_{uv}\|_{\mathrm{HS}}^2 \leq (1 + \eta_0)\frac{\mathcal{R}^2}{D_*} + (1 + a)^2\mathcal{R}.$$

Here $1 + \eta_0 \geq 0$. By (A.190),

$$\operatorname{tr}(Q_f^2) \leq \frac{\mathcal{R}^2}{D_*}\big(1 + \eta_0 + (1 + a)^2\sigma\big). \tag{A.191}$$

The restricted cross matrices are $J_u^* K_{uv} J_v$, so contraction in each pair gives $\operatorname{tr}(Q_Z^2) \leq \operatorname{tr}(Q_f^2)$.

By (23.5) and (A.185), $\operatorname{tr} Q_f = \sum_u \operatorname{tr} K_{uu} \geq (1 - a)\mathcal{R}$. For the projection $E_u = I - J_u J_u^*$, of rank $f_u - d_u$, the same estimates give

$$\begin{aligned} \operatorname{tr} Q_f - \operatorname{tr} Q_Z &= \sum_u \operatorname{tr}(K_{uu} E_u) \\ &\leq \sum_u (f_u - d_u) + a \sum_u \sqrt{f_u(f_u - d_u)} \\ &\leq (d_e + a\sqrt{d_e})\mathcal{R} \end{aligned}$$

by Cauchy–Schwarz for $(\sqrt{f_u})_u$ and $(\sqrt{f_u - d_u})_u$ and (23.6). Consequently

$$\operatorname{tr} Q_Z \geq \beta\mathcal{R}, \qquad \beta = 1 - a - t^2 - ta \geq 57/80, \tag{A.192}$$

since $a \leq 1/40$ and $t \leq 1/2$.

For the coefficient in (A.191), we have

$$(1 + \eta_1)\big(1 + \eta_0 + (1 + a)^2\sigma\big) \leq (1 + 5\sigma)(1 - a)^2.$$

Indeed $1 + \eta_1 \geq 0$ by (23.6). The left side increases with η_0, η_1, a, and the right side decreases with a. At $\eta_0 = \eta_1 = a = \sigma/2$ the difference of right and left is

$$\sigma\left(2 - \frac{13}{2}\sigma + \frac{1}{2}\sigma^2 - \frac{1}{8}\sigma^3\right) \geq 0 \qquad (0 < \sigma \leq 1/20).$$

Put $P = 1 + 5\sigma + 3t$ and $B = 1 - a$. Write $v = t(t + a)$. Then $P \leq 11/4$ and $t + a \leq 21/40$, so $2P(t + a) \leq 231/80 < 117/40 \leq 3B$. Hence

$$P\beta^2 - (1 + 5\sigma)B^2 = tB\big(3B - 2P(t + a)\big) + Pv^2 \geq 0.$$

Combining (23.6), (A.191) and (A.192) gives

$$D_Z \operatorname{tr}(Q_Z^2) \leq P\beta^2\mathcal{R}^2 \leq P(\operatorname{tr} Q_Z)^2.$$

Thus (23.1)–(23.2) hold with the stated constants. The definition of s gives the mean hypothesis of 23.7 with equality. By 23.2, $U \geq 0$, and the definition of U gives the required upper bound on each adjusted maximum slope. Apply 23.7. □

Proof of 23.9

Proof. For each word u let K_2 be the compact open section level, $F_{N,u}$ a fundamental domain for the unipotent lattice, H_u the compact real stabilizer and V_{π_u} its nonzero coefficient space. With the Haar measures of the selected realization the unfolding constant is

$$\alpha_u = \frac{\operatorname{vol}(K_2)\operatorname{vol}(F_{N,u})\operatorname{vol}(H_u)}{\dim_{\mathbf{C}} V_{\pi_u}} > 0.$$

The compact open subgroup has positive finite Haar measure, the lattice has positive finite covolume, and the compact group has positive finite total measure. Thus every block can be divided by $\sqrt{\alpha_u\alpha_v}$.
Lean: `Formalism.L2.SG.PoincareInHH.PoincareSetting.c0X` (`Formalism.LocalToGlobal.SiegelLatticeTheory.HaarTransversals.PoincareInHH`, lines 257–258)
Lean: `Formalism.L2.SG.TermOp.IdBlocks.idDiagBlocks_of` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.IdBlocks`, lines 761–820)

For $u \neq v$, 21.36 gives both orders of the level and Vandermonde comparisons with $A = 3e^{7/4}$. By Lemma 33.8, these comparisons and the source conditions imply all the counting conditions of Definition 33.7 for the pair (u, v) with this value of A. With the size thresholds of 33.78, the ties identify the column dimensions with f_u, f_v and the main term with $D_* = D^\star$. Apply 33.113 to the full normalized Gram operator. Its right hand side is $(1 + b^{-6})\Delta_{u,v}$, where the dimension and main-term identifications just made give $\Delta_{u,v} = f_u f_v / D_*$. This is (23.4) with $\eta_0 = b^{-6}$. The bound is uniform because the differing coordinate used in 33.111 can be chosen for every pair of distinct words and the threshold in 33.113 is common to those pairs.

Fix u and use $r_0 = (m-2)/2$. For a term in the unfolding of 33.78, let p_ℓ be the order-ℓ leading pairing minor of the two flags, and let T_u be the level prime. The Gram invariant is

$$G = p_{r_0+1}\overline{p_{r_0+1}} - T_u^2 p_{r_0}\overline{p_{r_0}},$$

where the bar is CM conjugation. The full normalized block is $K_0 + K''$, with K_0 the sum over the identity cell and the $G \neq 0$ terms, and K'' the sum over the remaining $G = 0$ terms.

Let E/F_{real} be the host CM extension. For the fixed flag F and the transported flag F', write $\mathfrak{n}_F, \mathfrak{n}_{F'}$ for their unipotent Lie algebras and $\ell_F, \ell_{F'}$ for the simple-root functionals of 33.18. Relevance means

$$\operatorname{Tr}_{E/F_{\mathrm{real}}} \ell_F(Y) = \operatorname{Tr}_{E/F_{\mathrm{real}}} \ell_{F'}(Y) \qquad (Y \in \mathfrak{n}_F \cap \mathfrak{n}_{F'}).$$

If the equality fails at Y, set $\Delta = (\ell_F - \ell_{F'})(Y) + \overline{(\ell_F - \ell_{F'})(Y)} \neq 0$ and $s = g_0 g_1/(2f\Delta)$. The real image of $\exp(sY)$ belongs to both unipotent groups. The character ratio at this element is

$$\exp\left(\frac{2\pi i}{g_0 g_1}\sum_{\tau} \tau(s\Delta)\right) = \exp(\pi i) = -1,$$

where τ runs over the f embeddings of the real field. The finite support and the global character identity make the ratio trivial on the arithmetic subgroup of the intersection used in the orbital integral. In the quotient average over the intersection, translation by $\exp(sY)$ preserves the measure and multiplies the character integral by -1. Thus the orbital integral vanishes.

Let h_ρ represent a finite column class ρ with row lattice Λ_ρ. Let x_u be the fixed level translate and K_2 the subgroup supporting the finite section. For a term t, write t_f for the finite component of its rational representative. The term meets (ρ, ρ') exactly when its ordered row lattices are $(\Lambda_\rho, \Lambda_{\rho'})$ and

$$(h_\rho x_u)^{-1} t_f (h_{\rho'} x_u) \in K_2. \tag{A.193}$$

In the identity cell the rational stabilizer element is $n\eta$, with n in the unipotent radical and η in the rational Levi. Write U_u^{lev} for the column level, the Levi subgroup cut out by membership in K_2 after conjugation by x_u. The level decomposition projects (A.193) to $h_\rho^{-1}\eta_f h_{\rho'} \in U_u^{\mathrm{lev}}$. Thus the two representatives define the same double coset and $\rho = \rho'$. Neatness of the column level forces $\eta = 1$. The finite condition then implies that n belongs to the unipotent stabilizer of the row lattice. The ratio of any two such representatives lies in this stabilizer, so they represent the same term in the unipotent quotient. The unit representative supplies a meeting term. The finite support makes every nonmeeting contribution zero.

Lean: Formalism.L2.SG.TermOp.IdBlocks.Meets (Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.IdBlocks, lines 717–720)
Lean: Formalism.L2.SG.Pf.IdMeet.idMeetKeyC_holds (Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.IdMeetKeyPf, lines 136–177)
Lean: Formalism.L2.SG.Pf.IdMeet.idMeetOneC_of (Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.IdMeetOnePf, lines 61–89)

In the meeting block put $k_\rho = (h_\rho x_u)^{-1} n_f (h_\rho x_u)$. The level decomposition gives $\chi_2(k_\rho) = \psi_f(n_f)$, where χ_2 is the level character and ψ_f is the finite unipotent character. By unitarity, $\overline{\psi_\infty(n_\infty^{-1})} = \psi_\infty(n_\infty)$. The global character identity on rational points gives

$$\chi_2(k_\rho)\overline{\psi_\infty(n_\infty^{-1})} = \psi_f(n_f)\psi_\infty(n_\infty) = 1.$$

For h in the finite adelic Levi, let $F(h)$ be the covolume of its real unipotent lattice $\Lambda(h)$. Passing to $\Lambda(g) \cap \Lambda(gh)$ multiplies each covolume by the index of this intersection in the corresponding lattice. The indices are the relative indices in the finite unipotent level subgroups. These subgroups for g and gh are simultaneous conjugates by g of the subgroups for 1 and h. Conjugation preserves both indices, so

$$\frac{F(gh)}{F(g)} = \frac{F(h)}{F(1)}.$$

Right multiplication by the column level leaves the unipotent subgroup unchanged. Left multiplication by a rational Levi representative conjugates the real lattice by its compact component, which preserves the unipotent Haar measure. Thus F is constant on column double cosets and has finite range. The positive character $F(h)/F(1)$ is one, since the powers of each of its values also belong to its finite range. Every meeting block therefore has the same unipotent volume $\mathrm{vol}_N = F(1)$.

Let π_u be the irreducible coefficient representation and vol_H the compact-group Haar volume. The real averaging operator after phase cancellation commutes with π_u and is scalar. For the coefficient vector W, $\mathrm{tr}(W \otimes W^*) = \|W\|^2$. With the quotient measure ν the normalization is $\int \|W\|^2 \, d\nu = 1$. The Weil integration formula thus gives trace $F(1)\,\mathrm{vol}_H$. Multiplication by the finite volume $\mathrm{vol}(K_2)$ gives the scalar $c_{0,X} = \mathrm{vol}(K_2)\,\mathrm{vol}_N\,\mathrm{vol}_H \,/\, \dim \pi_u$ of the unnormalized block. The level and unfolding ties give the normalized identity block $\delta_{\rho\rho'} I_{\pi_u}$. Consequently $K_0 - I = K_{\mathrm{gen}} + K_{\mathrm{ng}}$, where K_{gen} and K_{ng} are the generic and nongeneric nonzero-G sums.

Lean: Formalism.L2.SG.Pf.IdMeet.idMeetPhaseC_of (Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.IdMeetPhasePf, lines 14–41)
Lean: Formalism.L2.SG.TermOp.IdBlocks.idDiagBlocks_of (Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.IdBlocks, lines 761–820)

The adjacent weight gaps in 33.7 imply the pair gaps of 33.18. The same source conditions give $(kb)^8 \le g_i \le b^{C_g}$. They also give $T_u \le b^{C_T m}$ and $(T_u, g_0 g_1) = 1$. In the normal form of 33.18, let X, X' be the spans of the isotropic vectors of the two frames over E. That proposition gives $p_j \neq 0$ at $j = r_0 - \dim_E(X \cap X')$, with $1 \le j \le r_0$ for each relevant term of K''.

For a rank-j label δ, use the leading minors $\delta_\ell = p_\ell$ and the quantities $q_\ell, H_1, \mathfrak{d}, D_\delta, S_\delta$ of 33.18. Here $\mathfrak{d}$ is the extended different of the real subfield. Write N for the absolute ideal norm and $\mathrm{rd}(E)$ for the root discriminant. Since E/F_{real} is unramified at finite places, $N(\mathfrak{d}) = \mathrm{rd}(E)^{2f}$. With τ in the CM type,

$$0 < D_\delta = \prod_\tau \big(|\tau(\delta_j)|\,\mathrm{rd}(E)^{-2q_j}\big)^2 \le (2H_1)^{4r_0 f} S_\delta^2.$$

Let $A(j,\delta)$ and $M(j,\delta)$ be the amplitude and counting factors of 33.18, with the constants C_s, C_p there. The bound for one vector in (A.294) supplies $D_\delta^{-(k-1-j)/2} S_\delta^{-C_s k}$. The sum of relative stabilizer weights supplies $D_\delta^{k/2} S_\delta^{C_p k}$. Thus the label factor of AM satisfies

$$D_\delta^{(j+1)/2} S_\delta^{(C_p - C_s)k} \le (2H_1)^{2r_0 f(j+1)} S_\delta^{-3r_0}. \tag{A.194}$$

Since $S_\delta \ge 1$, it suffices that $(C_p - C_s)k + j + 1 \le -3r_0$. This follows from $C_s \ge C_p + 4$ and the bounds $j \le r_0$ and $k \ge r_0 + 3$.

The pairing minors are algebraic integers because the relevant vectors of both frames lie in the integral row lattice. Put $R_\ell = \max(1, \mathrm{rd}(E)^{2q_\ell}(2H_1)^{2r_0})$. For a finite set of values ξ of one minor,

$$\sum_\xi \prod_\tau \max(1, |\tau(\xi)|/R_\ell)^{-3} \le (64R_\ell)^{2f}. \tag{A.195}$$

Partition the real and imaginary coordinates of the embedding vector into intervals of length $1/2$. A box contains at most one ξ, since two distinct integers in the box would have $0 < |N_{E/\mathbf{Q}}(\xi - \xi')| < 1$. The weight in one complex coordinate is bounded by the product of the weights $\max(1, |z|/R_\ell)^{-3/2}$ of its two real coordinates z. In each real coordinate, the sum of the suprema over the half-unit intervals is at most

$$2 + 4\int_0^\infty \max(1, t/R_\ell)^{-3/2}\,dt = 2 + 12R_\ell \le 64R_\ell.$$

There are at most 2^j choices for the nonzero minor positions, and their values determine the label. Since $j \le r_0$, $S_\delta^{-3r_0}$ is bounded by the product of the weights in (A.195) over the nonzero positions. Thus for every finite set $\mathcal{D}_j$ of occurring rank-j labels,

$$\sum_{\delta\in\mathcal{D}_j} S_\delta^{-3r_0} \le 2^j \prod_{\ell=1}^{j} (64R_\ell)^{2f} \le \exp(C_2 m^2 f L r_0). \tag{A.196}$$

For the last inequality, use $q_\ell \le r_0^2$ and the fact that H_1 is a fixed power of b^m. By 33.18, $\log \mathrm{rd}(E) \le L/C_\ell$. Thus $\log(64R_\ell) = O(m^2 L)$ with a uniform constant, and C_2 can be chosen uniformly.

By (A.194) and (A.196), the product AM satisfies

$$\sum_{\delta\in\mathcal{D}_j} A(j,\delta)M(j,\delta) \le a_j, \qquad a_j = \exp(Cm^6 fL)(C/k)^{j(k-1-j)f}. \tag{A.197}$$

Here $C \ge 1$ enlarges the constants of 33.18. The height factors there, $H(C)^{Cr_0^4}$ and $H(C_h)^{C_h r_0^4}$ with $H(c) = b^{cm^2 f}$, have logarithms $O(m^6 fL)$ before the enlargement. The extra $2H_1$ factor and the label-sum bound have the same upper order.

Let v_t be the orbital vector of term t. For a set of terms S, use the operator $\mathrm{part}(S)$ of 33.78 and put

$$w_i(S) = \sum_{\substack{t\in S\\ \iota(\mathrm{row}(t))=i}} v_t,$$

where ι gives the finite-type index of a row. For finite S, expand the Hilbert–Schmidt square as a sum of inner products over the column blocks. In each pair of terms, the finite character coefficients pair to the entry G_{ij} of the finite-type Gram matrix, where i, j are their finite-type indices. The remaining factors pair to the orbital-vector inner product. Grouping the expansion by i, j gives

$$\|\,\mathrm{part}(S)\|_{\mathrm{HS}}^2 = \mathrm{Re}\sum_{i,j} G_{ij}\langle w_i(S), w_j(S)\rangle.$$

Absolute summability gives this identity for the full cell. Apply 33.112 to bound it by $P_g\Gamma_1\sum_i \|w_i(S)\|^2$, where $P_g = \prod_{i=0}^{1}(1 + 3/g_i)^f$ and Γ_1 is the finite-type normalization. Within a fixed type i, each v_t is supported on its ordered row-lattice pair, so different rows give zero cross terms.

In the singular cell, bound the same-row inner products by products of norms. By 33.18 and (A.197), the sum of these norm products over any finite set of pairs of ranks j, j' is at most $\mathrm{mass}_u \dim(\pi_u) a_j a_{j'}$. Here mass_u is the sum of the inverse row-lattice stabilizer orders. If n_Λ is the number of column classes over a row lattice Λ with stabilizer order a_Λ, the finite level gives at least Γ_1 distinct right cosets over Λ. At most a_Λ of these can lie in one double coset, since the row stabilizer identifies the right cosets over Λ. Thus $\Gamma_1/a_\Lambda \le n_\Lambda$. Neatness gives a contribution of $\dim \pi_u$ from each column class, so

$$\Gamma_1\,\mathrm{mass}_u \dim \pi_u \le \dim \pi_u \sum_\Lambda n_\Lambda = f_u.$$

Since $f \le k$ and $g_i \ge (kb)^8$, $P_g \le 2$ for large x. Sum over the two ranks and take the supremum of the finite nonnegative pair subsums. Absolute convergence gives

$$\|K''\|_{\mathrm{HS}}^2 \le 2f_u \left(\sum_{j=1}^{r_0} a_j\right)^2 \le f_u e^{-kf\log k}. \tag{A.198}$$

For the last inequality, $j(k-1-j) \geq k-2$ in this range, so for large k the sum is at most $k\exp(Cm^6fL)(C/k)^{(k-2)f}$. With $C_\ell \geq 8C$ and $C_\ell m^6 L \leq k\log k$, the squared height factor is at most $\exp(kf\log k/4)$. For large k the squared power is at most $\exp(-3kf\log k/2)$, and $2k^2 \leq \exp(kf\log k/4)$.

For (u,u), the level and Vandermonde factors agree, so 33.8 gives the counting hypotheses with $A=1$. Put $T = m^5 fL$ and $\Delta_{uu} = f_u^2/D^\star$. Set $\ell = e^{-kf\log k/3}$. In a same-row pair from a nonzero-G cell, $G(t)\neq 0$ gives linear independence of the $2r_0+2$ vectors of the fixed frame and the frame of t. The pair space lies in their orthogonal complement and has dimension at most $(k+2r_0+1)-(2r_0+2) = k-1$. Thus the pair has corank one or corank at least two.

Apply the norm expansion and 33.112 to the two nonzero-G cells with the signed same-row inner products. The higher-corank and nonscalar loop bounds of 33.106, together with the signed corank-one estimates of 33.99, give

$$\|K_{\mathrm{gen}}\|_{\mathrm{HS}}^2 \leq P_g(e^{C_B T}+2\ell)\Delta_{uu},$$
$$\|K_{\mathrm{ng}}\|_{\mathrm{HS}}^2 \leq P_g(\ell + 2^{-(k-2r_0-5)}e^{C_4 T})\Delta_{uu}.$$

Using $\ell \leq 1$ and $k \geq 2r_0+5$, their sum is at most $P_g(e^{C_B T}+3+e^{C_4 T})\Delta_{uu}$. For large x, $10P_g \leq e^T$. Since $C_B, C_4 \geq 0$ and $C_\Gamma \geq 1+\max(C_B,C_4)$, twice the sum is at most $\Gamma\Delta_{uu}$. Hence $\|K_0 - I\|_{\mathrm{HS}}^2 \leq \Gamma\Delta_{uu}$ by the squared triangle bound $\|M+N\|_{\mathrm{HS}}^2 \leq 2(\|M\|_{\mathrm{HS}}^2 + \|N\|_{\mathrm{HS}}^2)$. Apply this bound to $K_0 - I$ and K''. With (A.198) and the column-size bound $\Gamma f_u/D^\star \leq C_0/\Gamma$ of 33.7, the full squared defect is at most $e_0 f_u$.

In 21.32 let V be the common self-isometry of the retained row space and U_u the source isometries. Write $a_u \neq 0$ for the scalars. Let $\jmath$ denote the antilinear isometric embedding of the retained row space into the full target. For an orthonormal basis $(q_a)_a$ of the retained rows, extend $(\jmath(Vq_a))_a$ to an orthonormal basis of the full target. Let $(e_l)_l$ and $(e'_j)_j$ be orthonormal bases of the unrestricted and restricted sources, respectively, and let ι_u be the isometric source inclusion. Set $(J_u)_{lj} = \langle e_l, \iota_u U_u e'_j\rangle$. Parseval gives

$$(J_u^* J_u)_{jj'} = \langle \iota_u U_u e'_j, \iota_u U_u e'_{j'}\rangle = \delta_{jj'}.$$

Let P_u^{per} be the unrestricted period operator and A_u its Bessel matrix. The retained entries of $A_u J_u$ are $\langle \jmath(Vq_a), P_u^{\mathrm{per}}\iota_u U_u e'_j\rangle$, with the inner product linear in the second argument. The antilinearity of $\jmath$ makes these pairings linear in q_a. If P_E, P_u are the changes from orthonormal to arithmetic coordinates before applying the isometries, the Bessel identity therefore gives the retained block as

$$P_E^{\mathsf{T}} B_{u,v}(a_u^{-1}P_u).$$

Conjugate all matrices for the conjugate embedding. The target change is common, and the invertible source change includes a_u^{-1}.

With $c_u = \alpha_u^{-1}$, the normalized matrices are $\sqrt{c_u c_v}A_u^* A_v$, up to entrywise conjugation, which preserves the squared norms.

In 37.13 take the entire type set at either distinguished prime. Its cylinder contains every label pair and its dimension is $D_{\mathrm{all}} = D_Z$. By (32.3), its idempotent is the full level idempotent e. For the selected realization, the row main term $D^\star$ equals the spectral main term $D_A^\star(e)$ of the full level idempotent. Thus (37.16) gives

$$D_A^\star(e) = D^\star = d_\infty \operatorname{vol}(\mathscr{F})\operatorname{Re} e(1) > 0,$$

with the degree factor d_∞ and fundamental domain $\mathscr{F}$ of 37.14. By the cylinder estimate of 37.13 with $c_{\mathrm{Ic}} = 1/16$ from (A.557),

$$D_Z - D^\star \leq e^{-rf\log r/16}D^\star = \eta_1 D^\star.$$

The inclusion in 37.13 identifies the retained rows with the indices good at both primes. Every omitted index belongs to at least one of the two bad cylinders. For these cylinders 37.13 supplies the dimension and relative-count conditions. Equation (32.3) gives the main term for a finite set S of bad tuples at the distinguished prime p_j,

$$D_j(S) = D^\star \sum_{t\in S}\prod_{v\mid p_j} w_{j,v}(t_v),$$

where $w_{j,v}$ is the local weight of this row system at v. With the cuspidal parametrization of Definition 32.6, Theorem 32.53 gives the local laws for this row system; with the family size bounds at $C_s = 6$, Theorem 32.62 (as in Theorem 32.63) gives (23.9) for $\delta = 1 - D_{\mathrm{rows}}/D_Z$.

In the level model of 36.5, the commensurability and ideal twist in 21.6 preserve the ramified column rank. By 21.9, $d_u = f_u(1 - 1/(T_u - 1))^f$. By 37.10, $T_u \geq b^{m/2} \geq b \geq 3$ for large x. For $0 \leq p \leq 1$,

$$1 - (1-p)^f = p \sum_{j=0}^{f-1} (1-p)^j \leq fp. \tag{A.199}$$

At $p = 1/(T_u - 1)$, (A.199) gives $f_u - d_u \leq f_u f/(T_u - 1) \leq 2ff_u/b$. Summing over u proves the rank-loss bound. □

Proof of 23.10

Proof. Take x above the common cutoffs and $20^{1/C_H}$. By 23.8, $\eta = \bar{\eta}$ and $e = 2e_0 \leq \bar{e}$. The numerical hypothesis gives

$$(\bar{\eta} + 4\sqrt{\bar{e}})x^{C_H} = 5 + (3\sqrt{d_e} + 4\sqrt{e_B})x^{C_H} \leq 6.$$

Arakelov slope duality [77, §1.3, (2)] and the height clause of 21.32 give

$$s - \mu_{\min}(RR^\vee) = s + \mu_{\max}(RR) \leq x^{C_H} b$$

at each translation. Apply 23.8 with this H and use $(\eta + 4\sqrt{e})H \leq 6b$ in (23.3). □

Proof of 23.11

Proof. Take the maximum of the class-number and column cutoffs, 3, and the family starting scale plus one. The type primes split in the real host field by the host splitting conditions of 8.17. The selected norm-prime congruence of 37.11 gives $T_u \equiv 1 \bmod Q_S qd$, hence splitting in the cyclotomic host and its real subfield. The type-prime size bound of 8.12 gives $g_0 \geq (kb)^8$. Since $b \geq 2$ and $1 \leq f \leq k$, it follows that $g_0 - 1 > 2fk$. The freeness argument of 37.21 applies to every level-zero class.

Let c run through those classes, and choose a representative y_c. By the compatible symmetric column system from the bundle selection (item 2) of 33.6, a lowest form is an equivariant function with coefficient space $\bigotimes_\rho V(u,\rho)$, where ρ runs over the archimedean embeddings of the CM type. Its values at y_c determine it. Conversely prescribe a coefficient vector at each y_c and extend by the rational action and the level character. Two expressions for the same adelic point differ by a stabilizer at y_c; this stabilizer is trivial. Evaluation therefore identifies the lowest form space with one copy of the coefficient space for each class. The archimedean weights here are the column weights: passing from the shifted coordinates to the Weyl dimension product subtracts a common shift, which leaves all its differences unchanged. Hence

$$f_u = \#\mathrm{cls}_u(0) \prod_\rho \dim V(u,\rho).$$

Formula (37.31) of 37.21 now gives the asserted rank.

Lean: `Formalism.MainNotions.Ext.Lw.finrank_lowest_forms_space` (`Formalism.SharpRestriction.GlobalSchurBounds.GlobalRankLatticeProofs.UnitTraceSumBounds.RankConditionsOfLayerTwo`, lines 230–256)
Lean: `Formalism.MainNotions.Ext.Lw.card_ssIdx_eq` (`Formalism.SharpRestriction.GlobalSchurBounds.GlobalRankLatticeProofs.UnitTraceSumBounds.RankConditionsOfLayerTwo`, lines 258–282)
Lean: `Formalism.MainNotions.Ext.Lw.rankTiesF_of_C2S_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ClosedRankTiesFC2S`, lines 12–19)

Put $N = fk$. The lower endpoint of the word window is $T_u^\star = (D^\star/(d_u^\star \Gamma^2))^{1/N}$, and the selected prime lies between $T_u^\star$ and $(1 + 1/N)T_u^\star$. Substitution gives

$$\frac{f_u \Gamma^2}{D^\star} = \left(\frac{T_u}{T_u^\star}\right)^N (1 - T_u^{-k})^f. \tag{A.200}$$

The first factor lies in $[1, (1+1/N)^N] \subseteq [1, \exp(1)]$. The second is at most one and at least $1 - fT_u^{-k} \geq 1 - k2^{-k} \geq 1/2$, since $T_u \geq 2$ and $k \geq 3$. These bounds are uniform in the word.

Lean: `Formalism.MainNotions.LinkB.rank_formula_of_part_four` (`Formalism.SharpRestriction.GlobalSchurBounds.TopDensityNGeSixteen`, lines 102–103)
Lean: `Formalism.MainNotions.proviso_e` (`Formalism.SharpRestriction.GlobalSchurBounds.MainSup`, lines 196–245)

□

Proof of 23.12

Proof. Fix a sufficiently small positive ε. In 33.32 take the maximum of the requested threshold and the thresholds for the class-number row, 23.9 and 21.33, with the outer-list substitution prescribed in 33.32. Thus the outer constants are fixed before the common middle value $C_F = C_H = C_\Gamma \geq 1$ is chosen, and the final thresholds are evaluated on the resulting lists. The bundle selection (item 2) of 33.6 supplies the compatible symmetric column system on these same points. *Claim 1.* Beyond a common cutoff the constructed family satisfies $\mathcal{T} \leq b^2$ at each member.

The containment exponents are those of the selected family. At a finite place w, their weighted contribution is $2e_w\ell_w\tau_w = c_w\tau_w$. The selected cut-size bounds and the elementary estimate

$$\sum_{\substack{p<Y\\ p \text{ prime}}} \frac{\log p}{p} \leq 2\log Y \qquad (Y \geq 1)$$

give a family constant C, a cutoff and a function $\delta(x) \to 0$ such that, beyond this cutoff and for every index $i \in I_x$,

$$\mathcal{T} \leq Cmn_T a \log X' + b^{\delta(x)}.$$

Lean: `Formalism.BasicNotions.SizeData₂.sz` (`Formalism.LocalRingCardinality.DataParams₂`, lines 205–206)
Lean: `Formalism.L2.GlobalTheoryTheory.Places.mertens` (`Formalism.LocalToGlobal.GlobalTheory.PlacesP`, lines 441–481)
Lean: `Formalism.MainNotions.G.TopChoiceBounds.bt_Ta_at` (`Formalism.SharpRestriction.GrowthBounds.TopChoiceBounds`, lines 3939–3995)
Lean: `Formalism.MainNotions.G.TopChoiceBounds.stmtFTa_proved` (`Formalism.SharpRestriction.GrowthBounds.TopChoiceBounds`, lines 4018–4089)
Lean: `Formalism.MainNotions.G.TopChoiceBounds.family_exponent_sizes_hold` (`Formalism.SharpRestriction.GrowthBounds.TopChoiceBounds`, lines 4455–4501)
Lean: `Formalism.MainNotions.G.TopChoiceBounds.tau_sum_family_of_mertens` (`Formalism.SharpRestriction.GrowthBounds.TopChoiceBounds`, lines 4503–4504)

Increase the common cutoff so that $x \geq 5$, $x^9 \leq b$, $4|C| + 1 \leq x$ and $\delta(x) \leq 1$. For these x, the member size bounds give $m, n_T \leq 2x^3$ and $0 \leq \log X' \leq x$. Since $a = xb$ and $b \geq 1$, it follows that

$$\mathcal{T} \leq (4|C|x^8 + 1)b \leq (4|C| + 1)x^8 b \leq x^9 b \leq b^2.$$

This proves the claim.
Lean: `Formalism.L2.GlobalTheoryTheory.Places.sizes_at` (`Formalism.LocalToGlobal.GlobalTheory.PlacesP`, lines 539–588)
Lean: `Formalism.L2.GlobalTheoryTheory.Places.ta_le` (`Formalism.LocalToGlobal.GlobalTheory.PlacesP`, lines 590–632)
Lean: `Formalism.L2.GlobalTheoryTheory.Places.placesF_of` (`Formalism.LocalToGlobal.GlobalTheory.PlacesP`, lines 638–745)

Let $L = x^\varepsilon$ and $\Gamma = \exp(C_\Gamma m^5 f L)$. By (8.1) and $f = 2n$,

$$m^5 f L \leq 128 x^{9/2+9\varepsilon}, \qquad k \geq x^{9/2+10\varepsilon}.$$

At fixed parameters, $\log(3\Gamma^2 x^{C_H}) = o(k)$ uniformly at the constructed points. Since $k \to \infty$, this logarithm is eventually at most $(k-3)\log 2$. Exponentiating gives $3\Gamma^2 x^{C_H} \leq 2^{k-3}$. The right side is an integer, hence also bounds the ceiling. This proves the logarithmic bound required in 25.2.
Lean: `GlobalTheory.EX.szRow_card_words_le` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.SzRowCardWordsLe`, lines 75–104)

The code in 33.32 injects exactly that many indices into the binary words of length $k-2$ and uses the same indices at each translation. Lemma 23.11 gives the rank bound on this family. Since $[1/2, \exp(1)] \subset [1/3, 3]$, the word count and rank bound give both inequalities in (23.8) with $C_0 = 3$.

For the family satisfying the additional hypotheses of 23.9, put $y = x^{\varepsilon/8}$ and $E = e^{-y}$. Then $L = y^8$. The displayed rank bounds give $k \geq 8\log\Gamma$ eventually. With $r = k + m - 1 \geq k$, the formulas of 23.9 yield

$$\eta_0, \eta_1 \leq E, \qquad e_0 = 2C_0/\Gamma + 2e^{-kf\log k} \leq (2C_0 + 2)e^{-C_\Gamma y^8} \leq E^2.$$

Its source rank-loss bound $2f/b$ is at most E^2 eventually, since f grows polynomially and $b = e^{y^8}$. Take $d_e = E^2$, $e_B = 2E^2$. Eventually $Ex^{C_H} \leq 1/18$, so (23.7) holds, $d_e, e_B \leq 1/4$, and

$$(3\sqrt{d_e} + 4\sqrt{e_B})x^{C_H} \leq (3 + 4\sqrt{2})/18 < 1.$$

These are the common bounds required in 23.10. □

Proofs for Chapter 24: The pointwise bound and its eight error terms

Proof of 24.1

Proof. Take e_K from the size bounds and $c_{\mathrm{Count}}, c_{\mathrm{Slope}}, c_1, c_{\mathrm{Loss}} \ge 0$ from the family estimates. Use a common constant C_d for 21.32, 21.41 and 21.42. The budgets are nonnegative, so take

$$c_{\mathrm{Div}} = c_E = c_H = C_d^+ = \max(C_d, 0), \qquad K = C_d^+ + c_{\mathrm{Count}} + c_{\mathrm{Slope}} + c_1 + c_{\mathrm{Loss}} + 6.$$

Take x_0 above the common thresholds and at least $\max(x_1 + 1, 3)$, where x_1 is the family threshold. At $x \ge x_0$, choose i_0, a_v, b_v, i_1 by the loss hypothesis. The covering in 22.44 from i_1 supplies a model $i(u_0)$ at each $u_0 \in A$ realising the same pair. Use the frame and global data of i_0.

At fixed x the family fixes $\mathrm{rd}(E)$ and d. Thus $v = \mathrm{rd}(E)/2$ and the quadratic logarithmic derivative LpL are common, so $\log \Omega$ is common. The shared scale a gives the same $L_a = \log(a\,\mathrm{rd}(E))$, and the family fixes g_0, g_1 and hence ℓ_g. The common ranks, scales and thick-label count give the same B_0, N_c. Identify each top set with TT by the integer labels. The weight at label t is the same value of a_v in each selected model. The sum in the division estimate of 21.32 therefore becomes

$$\sum_{t \in TT} \left[m(a_t \log a_t - a_t - a_t \log \Omega) - a_t \log T_u \right] = S_{\mathrm{div}} - S_a \log T_u.$$

Its error becomes E_{div}, which gives H2. The triangle inequality sums the m layer bounds of 21.41 to give H3. The remaining estimates H4–H7 follow from 21.42, 21.35, 21.39 and 21.11.

Put $\varepsilon_z = \varepsilon / J_{\mathcal{F}} > 0$, with $J_{\mathcal{F}} \ge 1$ the family scale parameter. By 32.62, $0 \le \delta_{i(u_0)} \le \bar{\delta} = \exp(-x^{\varepsilon_z/7}) < 1/2$. Use this same $\delta_{i(u_0)}$ in the frame inequality, with $\eta_{i(u_0)}, e_{i(u_0)}$. By 21.32 and target duality,

$$s - \mu_{\min}(RR^{\vee}) = s + \mu_{\max}(RR) \le x^{C_H} b.$$

Thus $\mathcal{H} = x^{C_H} b > 0$ gives H9–H11. Apply Lemma 22.43 with the scale signs and lattice hypotheses to obtain H1.

Admissibility gives $a \le a_t \le a + a/(kx^{13}) \le 2a$ and $0 \le a_t/b - x \le 1/(kx^{12}) \le 1$, since $k \ge 1$ and $x \ge 3$. Lemma 22.43 gives the thick and other node bounds and $w_{u_0 j \rho} \ge 1$ on Φ. Hence $a_t / w_{u_0 j \rho} \le N$.

The family fixes q, d, Q_S and the Artin map on units in the common frame. By 5.1–5.2, reduction to the prime-to-p part of $Q_S q d$ gives the same inertia subgroup and Frobenius coset as for the datum D_{ar} of i_0. Use the top labels and the place bijection of 22.43, which identifies the weights at lift $j+1$ with $w_{u_0 j}$ on Φ. Extend $w_{u_0 j}$ by 1 off Φ. Then

$$\mathrm{ASK}(u_0) = \sum_{t \in TT} \sum_{j=0}^{m-1} \mathrm{ASK}(D_{\mathrm{ar}}, N, a_t, t, w_{u_0 j}).$$

In 5.4 each contributing $g \notin \Phi$ has its partner ρ in Φ, for ordinary primes and for d. A positive summand requires $p^i < a_t / w_{u_0 j \rho} \le N$, which implies $p, i \le N$. The common-weight loss hypothesis gives H8.

The size and rank budgets, with $f = 2n$ and $|S_2| = n_2$, give

$$L = x^{\varepsilon}, \qquad B_0 = 2nmbx^{\varepsilon}, \qquad N_c = \frac{n_2 k b x^{\varepsilon}}{n(nx^{-\varepsilon})}.$$

By (24.5), $n_T \ge 1$. The seven coefficients in 25.2 lie in $[0, K]$, and (24.4) gives

$$(\bar{\eta} + 4\sqrt{\bar{e}}) x^{C_H} b \le 6b \le K x^{\varepsilon} n_T a.$$

We have $b' X' = a = xb$. Enlarge x_0 so that x^{ε} exceeds both $\log x$ and $200(1+K)^2(1+c_{HH})(1+e_K)$, as required in 25.2.

Apply 5.3 to the base host with $S = \{q\} \cup \{p : p \text{ prime},\ p \mid Q_S\}$ for the local conditions of 25.5 and 25.6. Apply Lemma 24.2 to D_{ar} using the hypotheses at i_0. Use Euler's constant γ and $\mathrm{LpL} = \mathrm{Re}\,\ell(\chi_C)$. The character data and the displayed sum for ASK then give 25.3 via the top and place bijections. Lemma 22.44 gives $e_L = 0$ for its family. □

Proof of 24.2

Proof. Pull ξ back by the change of coordinates T of 5.2. Both the model representative χ_ξ and the primitive reduction of the pulled-back Artin character evaluate ξ on Frobenius at primes unramified in the host field E. Induce both to the product of their conductors and $|\operatorname{disc} E|$. Dirichlet's theorem [89, §4] gives a prime in each unit class at this modulus. The induced characters agree, so their primitive reductions agree. By Lemma 6.6, $\chi_\xi(n) = \prod_{p|n} V_\xi(p^{v_p(n)})$ for every $n \geq 1$.

For $\xi \neq \mathbf{1}$, set $\ell(\xi) = L'(1,\chi_\xi)/L(1,\chi_\xi)$. Choose $\ell(\mathbf{1}) = 0$, since Definition 5.4 sets $G_{\mathbf{1}} = G_z$ independently of this value. Any primitive character with the prescribed values equals χ_ξ by Lemma 7.14. For n coprime to $Q_S qd$, the equality at unramified primes gives $\chi_\xi(n) = \xi(T(\mathrm{Art}(n)))$ by multiplicativity. This is the membership condition in 37.4. By 6.6, $G_\xi = g_{\chi_\xi}$, with $g_1 = G_z$ for the primitive trivial character. This gives the thin-character bound.

On integers coprime to $Q_S qd$, 8.3 and 5.1 give the Artin signs (n/q) and (n/d). Since $2 \mid Q_S$ and $q, d \equiv 3 \pmod 4$, quadratic reciprocity [90, Chap. 5] gives

$$\left(\frac{-p}{n}\right) = (-1)^{(n-1)/2}\left(\frac{p}{n}\right) = \left(\frac{n}{p}\right) \qquad (p = q, d).$$

The product sign has the product of these values. For each sign let χ^* be the primitive reduction of its character modulo $8qd$. On the units modulo $(8qd)(Q_S qd)$ the reciprocity identity gives $\chi^* = \chi_\xi$, so Lemma 7.14 identifies the primitive characters. Thus (24.7) bounds $G_\xi(x)$ and $\ell(\xi)$. The d-sign cuts out $\mathbf{Q}(\sqrt{-d})$ by 5.1, so its representative is the Kronecker character χ_d of discriminant $-d$. The constant $D_1 = \mathrm{Re}(L'/L)(1,\chi_d)$ in 21.22 gives $\mathrm{LpL} = D_1 = \mathrm{Re}\,\ell(\chi_C)$.

For the hosts of Theorem 8.20, let M be the target conductor and fix $K = J_{\mathrm{win}}$ from Theorem 8.6. For sufficiently large Y, Theorem 7.25 selects q, d on $[Y, Y^K]$. In the proof of Theorem 8.6, the same pair determines the window containing x, with estimates restricted to $[Y, x]$.

Put $z = Y^{\varepsilon/5}$, $z' = z(\log Y)^2$. In the two stages of 7.20 and 7.21, the candidate p lies in $(\sqrt{U_{\mathrm{p}}}, U_{\mathrm{p}}]$, with $(U_{\mathrm{p}}, F) = (e^z, 8M)$ and $(e^{z'}, 8Me^z)$, respectively, where F bounds the conductors of the primitive bases. Apply Theorem 7.18 in those progressions with $\tau = \varepsilon/2$ and discarded proportion $1/2$. The maximal estimate of Theorem 7.11 discards at most a quarter of each candidate set after summing over the bases. Lemmas 7.6 and 7.13 bound the proportion discarded for zeros by $C_{\mathrm{b}} U_{\mathrm{p}}^{3/8-9/10} \leq 1/4$ for large Y, with $C_{\mathrm{b}} \geq 0$. For each surviving prime, Lemma 7.17 gives the window bound.

Lean: `QH.stage` (`Development.AutomorphicArithmetic.MeanSquareBounds.StagePow`, lines 41–170)

Choose q from the surviving first set, then d from the surviving second set for that q. The first stage uses the primitive reductions of characters ξ_8 modulo $8M$ as bases. The second uses the primitive reductions of $\xi_8(n)(n/q)^i$ at level $8Mq$, $i = 0, 1$. Thus the first stage controls $(i, j) = (1, 0)$ and the second controls $(0, 1), (1, 1)$, where i, j are the exponents of the q- and d-factors. A character ξ modulo M lifts to ξ_8 by reduction modulo M. On integers coprime to $8Mqd$ the corresponding twists have the same values, so Lemma 7.14 transfers the stage estimates to the primitive representatives at level $2Mqd$.

For each stage $H_j = FU_{\mathrm{p}} = 8Me^{z+jz'}$. A primitive base κ has conductor $f_\kappa \leq F$, and the odd candidate prime p exceeds F. Lemma 7.14 gives a nonprincipal primitive twist of conductor $f_\kappa p \leq H_j$. The exclusion in Lemma 7.13 gives nonvanishing for $\mathrm{Re}\, s > 399/400$ and $|\mathrm{Im}\, s| \leq H_j$. Since $\log(8M) \leq z$ and $\log Y \geq 1$ for large Y,

$$Y^{\varepsilon/6} \leq \log H_j \leq 3Y^{\varepsilon/5}(\log Y)^2.$$

Here $0 < \varepsilon < 1/52$, so $200\varepsilon < 50/13 < 4$. In the proof of (23) the remaining $2000 \log\log Y + O(1)$ is eventually at most $(4 - 200\varepsilon)\log Y$. Thus the refined assertion of Lemma 7.16 applies with $A_1 = 3$ and gives $|(L'/L)(1,\chi)| \leq 4 \log Y$.

Lean: `QH.pairEx` (`Development.AutomorphicArithmetic.MeanSquareBounds.pairEx`, lines 112–169)

Take the principal character modulo M in Definition 7.24. If $(a, 2Mqd) = 1$, then a is a unit modulo M and $8qd$. The principal character equals 1 there, and reciprocity gives $\chi^*(a) = (a/q)^i(a/d)^j$ for the exponent pair of the sign. Hence χ^* is a primitive representative at level $2Mqd$. Evaluating at $Y \leq x \leq Y^K$ gives the bounds of 8.16

$$|g_{\chi^*}(x)| \leq x^{1/2+\varepsilon/2} \leq x^{1/2+\varepsilon}, \qquad |(L'/L)(1,\chi^*)| \leq 4\log Y \leq 4 \log x.$$

Lean: `qdq_core` (`Development.AutomorphicArithmetic.CharacterBoxes.PairRows`, lines 264–297)

Lean: qdq_of_pair (Development.AutomorphicArithmetic.CharacterBoxes.PairRows, lines 299–321)

□

Proof of 24.5

Proof. By H7, $\mu_u - h_u \ge \mu_{M,u} + \mathrm{Ga} - \mathrm{Ee}_u + \mathrm{Fo} - h_u - \mathrm{Lo}_u$. By H2 and H5, since $S_a \ge 0$,

$$\mathrm{Fo} - h_u \ge \mu_C - S_{\mathrm{div}} + S_a \log T_u - E_{\mathrm{div}} \ge \mu_C - S_{\mathrm{div}} + S_a \lambda_5(u_0) - E_{\mathrm{div}}.$$

Use $\mathrm{Ee}_u \le m(S_a \ell_g + c_E B_0)$ and $\mu_{M,u} \ge -c_{\mathrm{Slope}} n_T a L_a (1 + m/k)$ from H3 and H6. This gives $\mu_u - h_u \ge \mathrm{Ga} + \mu_C - K_5(u_0) - \mathrm{Lo}_u$. Take the d-weighted mean. □

Proof of 24.7

Proof. Subtract the heights in the upper bound of H7 for u and in the lower bound for u'. Their difference cancels $\mathrm{Ga} + \mathrm{Fo}$. By H6 and 24.6,

$$\begin{aligned}(\mu_u^+ - h_u) - (\mu_{u'} - h_{u'}) \le \mu_{M,u}^+ - \mu_{M,u'} + (-\mathrm{Ee}_u - h_u) - (-\mathrm{Ee}_{u'} - h_{u'}) \\ + c_1 m B_0 + \mathrm{Lo}_{u'} \le K_3 + \mathrm{Lo}_{u'}.\end{aligned}$$

Average over u' with weights $d_{u'}$ and maximise over u. For the lower bound, each $\mu_u - h_u \le \mu_u^+ - h_u$ by H1, so their weighted mean is at most $\max_u(\mu_u^+ - h_u)$. □

Proof of 24.9

Proof. The setting and the bounds on height and frame parameters give H9. With 24.5 this gives

$$\mathrm{Ga} + \mu_C - K_5 - J \le s \le \mu_C + \log|W(u_0)| + \frac{4\vartheta(U,\mathcal{H}) + (\eta + 4\sqrt{e})\mathcal{H}}{1-\delta}.$$

Cancel μ_C and use $K_5(u_0) = \mathrm{Main}(u_0) + K_0$ by 24.4. By 24.7 and H10, $0 \le U \le K_3 + J$ and $0 < \mathcal{H} \le Xb$. Thus 24.8 and H11 give

$$4\vartheta(U,\mathcal{H}) \le 4\log(1+X)(K_3 + J + b), \qquad (\eta + 4\sqrt{e})\mathcal{H} \le (\bar{\eta} + 4\sqrt{\bar{e}})Xb.$$

The numerator is nonnegative and $1/(1-\delta) \le 1/(1-\bar{\delta})$. □

Proofs for Chapter 25: Four terms and the square-root bound

Proof of 25.1

Proof. Put $r = m^2/k$, $A_1 = n_T a$, $Y_0 = y_D y_L y$, and

$$E = A_1(1+r) + m(B_0 + N_c), \qquad P = Y_0 E.$$

We have $P \ge E \ge A_1 \ge a \ge 1$ and $Y_0 \ge y, y_D, y_L$. In R_1, use $S_a(1+r) \le 2E$ and $L_a \le 2Y_0$. In R_4 and R_6, use $m/k \le 1$ and $mb \le a$, respectively. The definitions give

$$\begin{aligned}R_1 \le 4KP, \quad R_2 \le 3KP, \quad R_3 \le KP, \\ R_4 \le 4KP, \quad R_5 \le P, \qquad R_6 \le KP.\end{aligned}$$

Since $b' \le a/k$, we have $n_T m^2(a/k + b') \le 2A_1 r \le 2E$. The middle term of J_0 is therefore at most $4K^2P$. The other terms satisfy $c_1 P_0 + c_1 X'' + 1 \le (K^2 + K + 1)P$, so

$$J_0 \le (5K^2 + K + 1)P. \tag{A.201}$$

The slope term in $\bar{U}$ is at most $6KP$. The first two summands of V_0 contribute at most $5KP$ together, and $S_a c_{\mathrm{Count}} \ell_k \le 2KgP$. Also $c_1 m B_0 \le KP$ and $b \le P$. Together with (A.201), these give

$$\bar{U} + b \le uP, \qquad u = 5K^2 + 13K + 2Kg + 2.$$

Since $\tau \le KyA_1 \le KP$,

$$R_8 \le d_I\{4hy\,uP + KP\} \le 2\{4hy\,uP + KP\}.$$

Using $y \ge 1$ gives

$$\sum_i R_i \le (5K^2 + 14K + 2)P + 2\{4hy\,uP + KP\} \le (5K^2 + 16K + 2 + 8hu)yP.$$

Since $h, g \ge 1$,

$$\begin{aligned} 5K^2 + 16K + 2 + 8hu &\le hg(45K^2 + 136K + 18) \\ &\le 200(1+K)^2 hg \le y. \end{aligned}$$

Thus $\sum_i R_i \le y^2 P = y_D y_L y^3 E$. □

Proof of 25.2

Proof. Apply Theorem 25.1 with

$$\begin{gathered} y = x^{\varepsilon}, \quad y_D = x^{e_{\mathrm{Div}}\varepsilon}, \quad y_L = x^{e_{\mathrm{Loss}}\varepsilon}, \quad g = 1 + e_K, \quad h = 1 + c_{HH}, \\ \ell_W = \log \#W(u_0), \quad \ell_k = \log k, \quad \ell_H = \log(1 + x^{c_{HH}}), \\ d_I = (1-\bar{\delta})^{-1}, \qquad \tau = (\bar{\eta} + 4\sqrt{\bar{e}})x^{c_{HH}} b. \end{gathered}$$

The signs of the exponents give $y, y_D, y_L, h, g \ge 1$. Since $\varepsilon \le 1$, we have $m \le x$, so $mb \le xb \le a$. Also $m \le xm \le k$, $1 \le b \le xb \le a$, and $0 \le S_a \le 2n_T a$ by summation. The logarithm bounds are

$$\begin{gathered} \log \#W(u_0) \le (k-3)\log 2 \le k \le a, \\ 0 \le \log k \le e_K \log x \le (1 + e_K)y, \\ 0 \le \log(1 + x^{c_{HH}}) \le \log 2 + c_{HH} \log x \le (1 + c_{HH})y. \end{gathered}$$

The last line uses $1 + x^{c_{HH}} \le 2x^{c_{HH}}$, $\log 2 \le 1$, and $\log x \le y$. Also $0 < d_I \le 2$ by the bound on $\bar{\delta}$. The terms $R_1, \dots, R_8$ sum to $E_8(u_0)$ by Definition 24.11, and $y_D y_L y^3 = x^{C_{\mathrm{err}}\varepsilon}$. □

Proof of 25.11

Proof. Since $x^{\varepsilon} x^{-\varepsilon} = 1$, the left side equals

$$(mb)\frac{n_2 k}{n^2} x^{2\varepsilon}.$$

Since $k \ge 0$, we have $n_2 k \le nk$. Apply the upper bound on k and then the density bound to obtain

$$\frac{n_2 k}{n^2} x^{2\varepsilon} \le n x^{1/2+\varepsilon} \le 2n_T x^{1/2+9\varepsilon/8} \le 2n_T Y.$$

Multiply by $mb \ge 0$. For $Q > 0$, the normalized term is

$$\frac{m}{Q}\,\frac{n_2 k b x^{\varepsilon}}{n(nx^{-\varepsilon})} = \frac{n_2}{n_T}\,\frac{k}{n^2} x^{2\varepsilon}.$$

□

Proof of 25.12

Proof. The lower bounds imply $n_T, m, k > 0$, hence $Q > 0$. Put $P = x^{C\varepsilon}$, $X = x^{1/2+\varepsilon}$, and $Y = x^{1/2+13\varepsilon/8}$. Then $P, Y \geq 1$ and $X \leq Y$. Lemmas 25.8–25.11 give

$$\begin{aligned}\mathcal{B}_x(a_E) &= n_T a_E + n_T a_E \frac{m^2}{k} + mB_* + mN_* \\ &\leq (2+2+4+2)YQ = 10YQ.\end{aligned}$$

Thus $E/Q \leq 10PY$. Multiply the assumed inequality by $2/Q$ to obtain

$$\delta_2(G_x - 4X) \leq 20PY + \frac{4}{x} \leq 22PY,$$

where $x \geq 3$ and $PY \geq 1$. Since $0 < \delta_2^{-1} \leq x^{\varepsilon}$ and $X \leq PYx^{\varepsilon}$, we get

$$G_x \leq 4X + 22PYx^{\varepsilon} \leq 26x^{1/2+(C+21/8)\varepsilon} \leq x^{1/2+(C+3)\varepsilon}.$$

The last inequality uses $26 \leq x^{3\varepsilon/8}$. □

Proofs for Chapter 26: The window bound and its inputs

Proof of 26.4

Proof. For the first sentence, Lemma 39.51 supplies weak unitary descent in ranks two through four with the split set containing the places over the chosen prime and with spherical data outside it; together with the four classical input groups h_N of Definition 26.30 and the published results in Remark 26.31, these are the hypotheses of Proposition 26.39, which gives Definition 26.3 with exponent 401. The second sentence then follows immediately from Proposition 26.32, which assembles h_N from the representation assertion $\mathcal{R}$ of Theorem 39.2 and the base-change transfer of Lemma 26.22. □

Proof of 26.5

Proof. Regroup the diagonal of the Hecke kernel on the finite double coset space by rational conjugacy. The regular support condition makes each term separable. The trace on an automorphic summand is d_λ times the trace on its finite fixed vectors. Compact support bounds the coefficient denominators, and the unit-circle condition bounds the conjugates of the coefficients by the binomial coefficients in rank n. These bounds leave finitely many polynomials.

Exclude the test places and primes dividing coefficient denominators. Also exclude primes dividing the determinants or discriminants. At the remaining places the order generated by the regular norm is finite étale over the local integer ring. Its factors are unramified integer rings, and every full stable lattice is a product of their fractional ideals. At a split place the centralizer translates any such lattice to the standard one. At an inert place use the trace description of the forms and lattices proved in 26.10. On a field fixed by the involution, the trace dual of the integer ring is itself. Self-duality for coefficient c thus requires the ideal exponent j to satisfy $2j = -v(c)$. This leaves one contributing norm class, since unit norms are surjective. On exchanged factors the two ideal exponents have prescribed sum, and the norm-one group acts transitively on pairs with that sum. The same lattice conditions apply to an integral twisted representative by the unit coefficient construction in the proof of 26.10. Thus in each case the stable unit integral has one contributing integral conjugacy orbit. The stabilizer is the integral centralizer, so its contribution is one under the volume conventions. Classes without an integral orbit contribute zero.

For a fixed elliptic P, the algebra $E[X]/P$ has involution given by conjugation on E and inversion of the generator. On each field factor an invariant form is $\operatorname{Tr}(c\,x\bar{y})$ with c fixed by the involution and specified modulo norms. Indeed, pairing with 1 defines the trace functional and the adjoint relation determines the form. Let u_j be the product of the local determinant signs on factor j. Quadratic norm reciprocity realizes the prescribed signs with positive infinity type exactly when all $u_j = 1$. The orthogonal sum is then locally isometric to the standard form, hence globally isometric to it by the local-global theorem for Hermitian

forms. The same theorem on each factor gives uniqueness of the rational tuple [96]. Summing the sign over $P_1 \mid P$ gives $\prod_j (1+u_j)$, so the surviving tuples acquire multiplicity $2^{r(P)}$.

To compare masses, take a matrix N over E with characteristic polynomial P. Conjugate inversion preserves P, and separability implies that $(N^{-1})^*$ and N are conjugate, where * denotes conjugate transpose. Thus there is an invertible X with $NXN^* = X$. Put $\epsilon = (-1)^{n-1}$ and choose $c \in E^\times$ with $c/\bar{c}$ outside the spectrum of $-\epsilon N X^* X^{-1}$. The possible ratios form an infinite set. Then

$$M = cX + \epsilon \bar{c} N X^*, \qquad N M^* = \epsilon M, \qquad \delta = M J_n^t.$$

The choice of c makes M invertible. Since $J_n^t = \epsilon J_n$ and $J_n J_n^t = 1$, $\delta \theta_J(\delta) = \epsilon M (M^*)^{-1} = N$. The twisted centralizer consists of x with $xMx^* = M$. Such x commute with N, and the adjoint on $E[N]$ sends scalars to their conjugates and N to N^{-1}. Thus the centralizer is the same norm-one torus as for a unitary representative. This identifies rational points and maximal compact subgroups, so the unitary mass is $2^{r(P)}\kappa(P)$. For coprime P_1, P_2 the tori and their mass factors split. The geometric sum can therefore be written over ordered factorizations $P = P_1 P_2$ with summand

$$\kappa(P_1)\kappa(P_2) h_\lambda(P) \prod_{w \in \Sigma} O_w^{P_1}(P; f_w).$$

Lean: `GlobalTheory.PreStabU.fixedP` (`Development.WeilBound.CoreStatements.AutomorphicKernels.IsobaricBaseChange.IntegralCoefficientBaseChange.UnitaryBaseChangeCharacters.UnitaryBaseChangeLinks.UnitaryGeometricSideSeparableTest`, lines 357–435)

Inverting the two root multisets gives $R/\bar{R} = (-1)^{ab} D$ for the global resultant. At a split pair Δ_w is the product of $\mu_u(R)|R|_u^{1/2}$ over the pair, by the restriction of μ to the base. At an inert place this expression is the stated factor. Write R_v for the image of R at the chosen embedding over v. The product formulas give

$$\prod_{w \in \Sigma} \Delta_w \prod_{v \mid \infty} (R_v/|R_v|)^{e_v} |R_v| = 1. \tag{A.202}$$

Expand the Weyl numerator at v along the first a rows. The minor in columns $j_0 < \cdots < j_{a-1}$ has exponents $\lambda_{v,j_i} + n - 1 - j_i$, hence highest weights $\lambda_{v,j_i} + b + i - j_i$ after division by the block denominator. The complementary minor has the formula with a in place of b. Put $h_v = (e_v + 1)/2$. The cross denominator in the conjugated character gives

$$\frac{(R_v/|R_v|)^{e_v}|R_v|}{\overline{R}_v} = ((-1)^{ab} D_v)^{h_v}.$$

This factor subtracts $b h_v$ and $a h_v$ from the respective highest weights in the conjugated weight characters. In infinitesimal parameters the shifts are $-b e_v/2$ and $-a e_v/2$. The local sign is $(-1)^{\sum_{j \in I_v} j + a(a-1)/2 + ab h_v}$. Multiply over v and use (26.2) and (A.202). An empty block gives the full minor with sign $+1$.

Lean: `UpperTheory.ConvolutionTheory.G2.termId` (`Development.AutomorphicArithmetic.Foundations.Convolution.WeightRankUnits.TermId`, lines 24–41)

Lean: `UpperTheory.ConvolutionTheory.G2.realProdS` (`Development.AutomorphicArithmetic.Foundations.Convolution.WeightRankUnits.RealProdS`, lines 31–59)

Lean: `UpperTheory.ConvolutionTheory.G2.realSigned` (`Development.AutomorphicArithmetic.Foundations.Convolution.WeightRankUnits.RealSigned`, lines 9–154)

□

Proof of 26.6

Proof. Beuzart-Plessis's linearized Jacquet–Rallis theorem equates the orbital integrals of the integral matrix indicators for a relatively regular F-matrix of size $r+1$ and its match on the standard Hermitian space over L [91, Thm. 1]. Matching is conjugacy by $\mathrm{GL}_r(L)$ embedded in the upper left block. The linear integral vanishes when the standard Hermitian orbit is absent. The integral subgroups of $\mathrm{GL}_r(F)$ and $U_r(F)$ have volume one. The linear integrand at Y carries η of the determinant with rows $e^*, e^*Y, \dots, e^*Y^r$, where e^* is the last coordinate row. Here r indexes the acting group.

Lean: `GlobalTheory.JR.unitMatch_all` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.JacquetRallisUnitMatchAllRanks`, lines 9–10)

Put $u = cw'\sigma w'$, and let A be multiplication by g. In the power basis take the linear matrix with blocks $(A, u, \mathrm{Tr}_{B/L}, 0)$. On (B, h_c) use $(A, w', h_c(\,\cdot\,, w'), 0)$. Multiplication by $c\sigma w'$ identifies the marked columns and rows. The moment matrix $(\mathrm{Tr}_{B/L}(ug^{i+j}))_{i,j<r}$ is invertible, so the column is cyclic and the row is cocyclic. The moments determine the form on the basis $g^i w'$, so the standard match exists exactly when (B, h_c) is standard.

The linear cosets correspond to σ-stable lattices. Choose $z \in O$ with $z - \bar{z}$ a unit, so $O = O_F + zO_F$. For $m \in M$, both $b = (m - \sigma m)/(z - \bar{z})$ and $m - zb$ are fixed by σ, which gives descent to O_F. The matrix blocks impose $gM \subset M$ and $u \in M$. The dual condition is $1 \in M^*$. Subtract earlier rows from e^*Y^j to obtain the trace rows $\mathrm{Tr}(g^i\,\cdot\,)$ on the first block. Expansion along e^* gives $s_g(M)$.

On the Hermitian side every unimodular lattice has an orthonormal basis. In its reduction some vector has nonzero length, since polarization with residual scalar multiples and the nondegenerate finite-field trace pairing would otherwise make the reduced form zero. Lift such a vector and normalize its length by a unit norm. Unit norms are surjective by the finite-field norm and successive lifting with the residue trace. Split off the resulting line and iterate. This works also in residue characteristic two. The integral cosets then correspond to the self-dual lattices for which $gM \subset M$ and $w' \in M$. Self-duality gives the covector condition. Each contributes one. If (B, h_c) has no standard match, both the lattice count and the linear integral vanish. This proves (26.5). □

Proof of 26.7

Proof. Put $V = \mathrm{Hom}_L(B_2, B_1)$ and $H = \mathrm{Hom}_O(M_2, M_1)$. Lifting a basis of M_2 writes $M = \{(x, y) : y \in M_2,\ x - f(y) \in M_1\}$ for a unique f mod H in V/H. Stability says $D(f) = g_1 f - f g_2 \in H$. Over a splitting field the eigenvalues of D are $\alpha - \beta$ for the roots of the two polynomials, so $\det D = R$. The parameter group $D^{-1}H/H \simeq H/DH$ has cardinality $q^{2v(R)}$ by elementary divisors. In the self-adjoint case $\tau(f) = \sigma_1 f \sigma_2$ defines a semilinear involution on the parameter group. With z as in 26.6, each class is uniquely $a + zb$ with a, b fixed, by $b = (f - \tau f)/(z - \bar{z})$ and $a = f - zb$. The invariant lattices correspond to the fixed classes, whose cardinality is the square root of the total. □

Proof of 26.8

Proof. A stable full lattice forces its generator to be integral by Cayley–Hamilton. If either generator is nonintegral, both sides vanish. For an integral generator in a block or the product, let t be a uniformizer of F and choose $m \geq 0$ with $t^m D_B \subset O[g] \subset D_B$. For a stable lattice M set $I_M = D_B M$. Then

$$t^m I_M \subset M \subset I_M. \tag{A.203}$$

If M is self-dual for h_c, the h_c-dual $I_M^{\#}$ satisfies $t^{2m} I_M \subset I_M^{\#} \subset I_M$. On each factor the dual exponent is a fixed integer minus the exponent on the conjugate factor. Thus a fixed field has bounded exponent and an exchanged pair has bounded sum of exponents. Norm-one elements translate the exponents on a pair by any $j, -j$. This gives finitely many I_M modulo T_B, and the finite quotient in (A.203) gives finitely many lattices above each. The argument also applies to an integral norm-one generator.

To remove the mark, let $\rho_B(k)$ count cosets uT_B^0 with $u^{-1}t^k \in D_B$. A norm-one element has valuation zero on a fixed field, and every pair $j, -j$ occurs on exchanged fields. Let b_i be the normalized valuation of t in one field of the ith exchanged pair. Then

$$\rho_B(k) = \prod_{i \text{ exchanged}} (2kb_i + 1) \qquad (k \geq 0). \tag{A.204}$$

For fixed c, choose $b \geq 0$ integral with $t^b D_B \subset M \subset t^{-b} D_B$ for all the finitely many orbit representatives in $X_B(c)$. Let $A_M(k)$ count the translates of M containing t^k. The translates are indexed by $T_B/(T_B)_M$, and the membership condition is $u^{-1}t^k \in M$. For $k \geq b$ this gives

$$[T_B^0 : (T_B)_M]\rho_B(k - b) \leq A_M(k) \leq [T_B^0 : (T_B)_M]\rho_B(k + b). \tag{A.205}$$

The ratios $\rho_B(k \pm b)/\rho_B(k)$ tend to one. Sum and use (26.5) with $w' = t^k$ to get

$$\lim_{k \to \infty} \frac{G_B(ct^{2k})}{\rho_B(k)} = N_B(c). \tag{A.206}$$

Lean: `GlobalTheory.UnitLemma.unitaryLimit` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.UnitaryLatticeCountRatioTendsto`, lines 16–80)

Let S index the fields fixed by σ. On such a field H the extension H/H^σ is unramified because the residue involution acts nontrivially on the residue field of L. If $\#k_H = q^{2d}$, the q^d-power map restricts to the q-power map on k_L, so d is odd. Let π_i be a uniformizer of H^σ on factor i and one elsewhere. Then $\eta(N_{B/L}\pi_i) = (-1)^d = -1$. The $\pi_J = \prod_{i\in J}\pi_i$ for $J \subset S$ represent C_B. On fixed fields the norms have even valuation and exhaust the units. On an exchanged pair every fixed invertible element is a norm.

For $I \subset S$ define $D_I f(y) = \sum_{J\subset I}(-1)^{|J|} f(\pi_J y)$ and put

$$H_B(I,k) = \sum_M s_g(M) D_I 1_M(t^{2k}) D_{S\setminus I} 1_{M^*}(1),$$

over full g- and σ-stable lattices. Each expanded term bounds M below by $O[g]$ times an invertible vector and above by the trace dual of another such lattice, so the sums are finite. For $J' \subset S \setminus I$, the change $M \mapsto \pi_{J'} M$ moves $\pi_{J'}$ from the dual test to the vector test and multiplies the basis sign by $(-1)^{|J'|}$. It follows that

$$H_B(I,k) = \sum_{J\subset S} (-1)^{|I\cap J|} G_B(\pi_J t^{2k}). \tag{A.207}$$

By (A.206), the limit after division by $\rho_B(k)$ is the signed sum of $N_B(\pi_J)$.

Lean: `GlobalTheory.UnitLemma.Desc.HH_eq_alternating_GG` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SignedInvolutionStableLatticeSumEqAlternatingSum`, lines 37–169)

Write S_i for the fixed factors of B_i. Use $I = S_1$ in B and in B_1, and $I = \varnothing$ in B_2. For a product lattice put $M_1 = M \cap B_1$ and $M_2 = \mathrm{pr}_2 M$. These are full stable lattices in the two blocks. A basis of M_1 and lifts of a basis of M_2 give

$$s_g(M) = \eta(R) s_{g_1}(M_1) s_{g_2}(M_2). \tag{A.208}$$

Indeed the remainder map from polynomials of degree $< a+b$ to the two polynomial quotients has determinant $\pm R$, where a, b are the degrees of P_{g_1}, P_{g_2}. Use the basis $1, \dots, X^{a-1}, P_{g_1}, \dots, X^{b-1}P_{g_1}$, for which the second diagonal block is multiplication by P_{g_1} modulo P_{g_2}. The trace evaluations in the lattice basis then have the two block trace determinants on the diagonal.

Index the involution orbits of field factors by j and let D_j be the integer ring product on orbit j. The exponents of $D_B M$ on an exchanged pair agree by σ-stability. Round each orbit exponent down to a multiple of the valuation of t. By (A.203) there are integers p_j with

$$\bigoplus_j t^{p_j+N} D_j \;\subset M\; \subset \bigoplus_j t^{p_j} D_j, \qquad N = m+1. \tag{A.209}$$

The restricted tuples give the same inclusions for M_1, M_2. Each tuple has at most A lattices, where A is the number of subgroups of $D_B/t^N D_B$.

For a constant C independent of k, a nonzero product of differences requires

$$-C \le p_j \le 2k + C, \qquad p_j \ge 2k - C \ (j \in I), \qquad p_j \le C \ (j \in S\setminus I). \tag{A.210}$$

Some $\pi_J t^{2k}$ belongs to M and some $\pi_{J'}$ belongs to M^*. The upper inclusion and the dual of the lower inclusion bound these two memberships. The trace dual of each D_j is a fixed fractional ideal, and the π_J form a finite set. If $p_j < 2k - C$ for $j \in I$, the lower inclusion puts each j-component of $\pi_J t^{2k}$ in M, so toggling j cancels the vector difference. If $p_j > C$ for $j \in S \setminus I$, the dual of the upper inclusion does the same for the covector difference. Choose C to cover the bounds in both projections as well.

For $k > 2C$ consider a tuple in (A.210) with $C \le p_j \le 2k - C$ on exchanged pairs. With C increased for the fixed trace duals and multipliers, (A.209) puts all vector components outside S_1 in M and all covector components outside S_2 in M^*. The vector test then uses M_1, and the covector test uses the trace dual of M_2. The same discards in each block give the product of the separate tests. This applies wherever either test product is nonzero, because the separate tests impose (A.210) on the restricted tuples.

If h is the number of exchanged pairs and $h > 0$, at most $C'(1+k)^{h-1}$ tuples can contribute to the difference. The fixed coordinates have boundedly many choices, and one exchanged coordinate lies within

bounded distance of an endpoint. Each tuple has at most A lattices and each test product has absolute value at most $2^{|S_1|+|S_2|}$. By 26.7 each pair of block lattices has $q^{v(R)}$ invariant lifts. With (A.208) this gives

$$\left|H_B(S_1,k)-(-q)^{v(R)}H_{B_1}(S_1,k)H_{B_2}(\varnothing,k)\right|\le 2^{1+|S_1|+|S_2|}AC'(1+k)^{h-1}. \tag{A.211}$$

For $h=0$ the two tests agree for $k>2C$, so the left side is zero. In either case division by $\rho_B(k)=\rho_{B_1}(k)\rho_{B_2}(k)$ makes the difference tend to zero.
Lean: `GlobalTheory.UnitLemma.Desc.term_window` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AlternatingDifferenceTermWindow`, lines 217–329)
Lean: `GlobalTheory.UnitLemma.Desc.hh_prod_tendsto` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TendstoSignedLatticeSumProductSubFactors`, lines 63–202)

For $c=(c_1,c_2)$ represented by π_J, the product sign in (A.207) is $\eta(N_{B_1/L}c_1)$. In a common basis $\det h_{c_1}=N_{B_1/L}(c_1)\det h_1$. A contributing class on B_1 has a self-dual lattice, so its Gram determinant has even valuation. The limits for B_1 and B are therefore $\eta(\det h_1)S(B_1)$ and $\eta(\det h_1)K(B_1,B_2)$, with h_1 on B_1 in both. The limit for B_2 is $S(B_2)$. Cancel the common sign.
Lean: `GlobalTheory.UnitLemma.LieEnd.unitLemmaLie_of` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.UnitLemmaSelfAdjointLieOfCountLimitDescent`, lines 137–197)

□

Proof of 26.9

Proof. Nonintegral generators give zero as in 26.8. For integral generators set $g=(g_1,g_2)$ and $R_0=O[g]$ in $B=B_1\times B_2$. Since $g^{-1}=\sigma g$ is integral, multiplying its monic equation by the power of g one less than its degree gives $g^{-1}\in R_0$. Hence $\sigma R_0=R_0$. Induct on $\dim_L B$ from the zero algebra.

If R_0 has a proper nonzero σ-fixed idempotent $e=f(g)$ with $f\in O[X]$, every stable lattice splits into its e and $1-e$ parts. The two parts are orthogonal for h_c. The parameters split as a product, as do the norm-one groups and stabilizers. The Gram signs multiply. The two parts have coprime reductions of their characteristic polynomials since f takes values one and zero on their respective reduced roots. Thus the cross resultants between parts are units. The counts multiply and the resultant valuations add. Apply the induction hypothesis to both parts.
Lean: `GlobalTheory.UnitLemma.OrbitAll.identity_integral` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.UnitLemmaUnitaryGroupEveryResidueOfLie`, lines 93–143)

The Artinian reduction R_0/tR_0 is a product of local rings. Its idempotents lift by Hensel's lemma for X^2-X. Conjugation permutes the lifted local factors. Without a proper nonzero fixed idempotent, there are either two exchanged rings or one local ring.
Lean: `GlobalTheory.UnitLemma.OrbitAll.one_orbit` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.UnitLemmaUnitaryGroupEveryResidueOfLie`, lines 66–91)

In the exchanged case choose e with $e+\sigma e=1$. On each block and the product the norm-one group projects isomorphically to the invertible group on the e part, with inverse $u\mapsto(u,\sigma(u)^{-1})$. Every fixed invertible parameter is a norm. With $c=1$ and the second part transported by σ, a self-dual stable lattice is (M,M^*), since the halves pair by the ordinary trace. Conversely any full eg-stable lattice gives such a pair, since $(eg)^{-1}$ belongs to the order it generates. The stabilizer indices are those for the action on the first part.

Write v_+,v_- for the within-part cross-resultant valuations of the two blocks. The mixed resultants are units by the idempotent polynomial. The first count in 26.7 on the e part gives

$$N_B(1)=q^{2v_+}N_{B_1}(1)N_{B_2}(1). \tag{A.212}$$

For the weighted count, fix a pair of lattice-orbit representatives with stabilizer H. Their lifts form a finite H-set. A lift orbit has weight equal to the pair's weight times $[H:H_M]$, where H_M stabilizes the lift. Summing gives the pair's weight times the fibre size. Counting on the other part gives (A.212) with v_- instead. If the product count vanishes, so does $N_B(1)$. Otherwise $v_+=v_-$ and the self-dual lattice on the first block gives Gram sign one. The required multiplier is $q^{2v_+}=(-q)^{v_++v_-}$.
Lean: `GlobalTheory.UnitLemma.Swap.identity_of_swap` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.InvolutiveAlgebraProductIdentity`, lines 979–1102)

Suppose R_0 is local with residue field k' of size q^{2d}, generated over k_L by the residue of g. The residue involution is the q^d-power map, so d is odd. The order contains the integers O' of the unramified extension

L'/L of degree d. Lift a generator of $k'^{\times}$ to a root of $X^{q^{2d}-1}-1$ and adjoin it. Its minimal polynomial on the residue field lifts as a factor with unit derivative. By uniqueness, conjugation acts on the lifted roots by the q^d-power map. Let $s \in O'^{\times}$ lift the residue of g in this group of roots, so $s\bar{s}=1$. The fixed field F' has residue cardinality $q'=q^d$.

The inclusion $O' \subset R_0$ gives L'-scalars on the algebras. The image of $g-s$ lies in the maximal ideal of the Artinian local ring R_0/tR_0, so it is nilpotent. The projected orders give full O'-lattices with the same nilpotence property. Thus the characteristic polynomials P_i' over L' reduce to $(X-\widetilde{s})^{\deg P_i'}$.
Lean: `GlobalTheory.UnitLemma.UnrAny.exists_unr_embedding_any` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsUnramifiedEmbeddingOfInvolutionAlgebra`, lines 52–219)

Choose $x \in O'$ with residue outside $k_{F'}$ and put $y=\bar{x}$. Let $\xi = sx/y$. The elements $x, y, x-y, \xi$ are units. Also $\xi\bar{\xi}=1$ and $\widetilde{\xi} \neq \widetilde{s}$. Put

$$Y_i = \frac{yg_i - x\xi}{g_i - \xi}, \qquad Y=(Y_1,Y_2).$$

For $P=P_1'P_2'$ and for each block polynomial, $P(\xi)$ is a unit. Dividing $P(X)-P(\xi)$ by $X-\xi$ shows that the denominator is invertible in the order. Conjugating the fraction and using $\sigma_i g_i = g_i^{-1}$ gives $\sigma_i Y_i = Y_i$. For the reverse inclusion let $m=\deg P$ and form

$$H(Z) = (Z-y)^m P\!\left(\frac{\xi Z - x\xi}{Z-y}\right) \in O'[Z].$$

The relation $g(Y-y)=\xi Y - x\xi$ gives $H(Y)=0$, while $H(y)=(\xi(y-x))^m$ is a unit. Division of $H(Z)-H(y)$ by $Z-y$ gives $(Y-y)^{-1} \in O'[Y]$. Thus the orders generated by g and Y agree on the product and on the blocks. The invertible substitution preserves distinct roots.

Over O' both generators have the same stable lattices. The forms and weights also agree. Fix stable full lattices in the blocks, for example the projected orders. Their extension lattices are the same for both generators. By 26.7 their count is $(q'^2)^v$ for the respective cross-resultant valuation v, so the valuations agree. Apply 26.8 over L'/F' to Y_1, Y_2.
Lean: `GlobalTheory.UnitLemma.CayleyAny.identity_of_residually_unipotent_any` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.UnitLemmaResiduallyUnipotentCayleyTransform`, lines 107–223)

Every g-stable O-lattice carries O'. Its Hermitian duals over O and O' agree by trace transitivity and trace self-duality of O'/O. The norm-one groups and integral units are the same. For a Gram matrix H' of size r' over L' and an integral basis $(b_j)_j$ of O'/O, the determinant over L is

$$\det\!\big(\mathrm{Tr}_{L'/L}(b_j\bar{b}_k)\big)^{r'} N_{L'/L}(\det H').$$

The first determinant is a unit and the norm multiplies valuation by the odd integer d. Restriction of scalars gives $P_{g_i} = \prod_{\tau \in \mathrm{Gal}(L'/L)} \tau P_i'$. For $\tau \neq \nu$, the reductions of $\tau P_1'$ and $\nu P_2'$ have distinct roots, since $\widetilde{s}$ generates k'/k_L. Degree zero contributes resultant one. The d diagonal resultants have the same valuation v', so the exponent over L is dv'. Since $(-q)^{dv'} = (-q')^{v'}$, the multipliers agree.
Lean: `GlobalTheory.UnitLemma.UnrAny.unitLemma_local_of_lie_any` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.UnitLemmaLocalAnyResidueCharOfLieAlgebra`, lines 58–142)
Lean: `GlobalTheory.UnitLemma.Unr.ResVal.val_resultant_unr` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ValResultantEqDegreeMulUnramifiedExt`, lines 91–217)

□

Proof of 26.10

Proof of Proposition 26.10. The reciprocal identity and monicity give $Q_i(0) \neq 0$ and define the involution. Lemma 26.9 proves (26.8).

For the unitary integral a regular element of polynomial Q_1Q_2 makes its space a rank-one module over $B=B_1 \times B_2$, with adjoint σ on B. Apply the trace pairing to $h(y,1)$. The adjoint relation gives $h=h_c$ with c fixed and invertible. Changing the module generator multiplies c by a norm, and the centralizer is T_B. An integral coset corresponds to a self-dual stable lattice. Every contributing form is standard by the orthonormal basis argument in 26.6. With integral group and T_B^0 of volume one, the coset contributes $[T_B^0 : (T_B)_M]$. Since $\ker Q_1(g) = B_1$, its sign is $\eta(\det h_{c_1})$. The signed unitary integral is $K(B_1,B_2)$.

Lean: GlobalTheory.InertUnitOrb.kapOrb_unit_eq_kapSum (Development.AbelianVarietyTheory.Proofs.DivisorProofs.InertPlaceKappaOrbitalUnitEqSignedLatticeSum, lines 68–212)

For a twisted block of rank $r > 0$ put $\epsilon = (-1)^{r-1}$. A twisted element δ corresponds to the form $b(y,z) = y^t H \bar{z}$ with $H = ((\delta J_r)^{-1})^t$. Right twisted conjugation $\delta \mapsto k^{-1}\delta\theta_J(k)$ by k replaces H by $k^t H \bar{k}$. For $g = \delta\theta_J(\delta)$, $\bar{H}^t = \epsilon g^t H$, or $\overline{b(y,z)} = \epsilon b(gz, y)$. This also gives preservation by g. In the rank-one B-module the trace pairing writes $b(y,z) = \mathrm{Tr}_{B/L}(\beta y \sigma z)$, and the relation becomes $\sigma\beta = \epsilon g \beta$. Conversely this relation gives norm g on recovering δ from H. The coefficients for a twisted class differ by norms.

For integral g there is a solution β_0 invertible in $R_1 = O[g]$. Set $a = \epsilon g^{-1}$ and take $\beta_0 = z + a\sigma z$. On two exchanged maximal ideals of R_1 prescribe residues $1, 0$ for z. At a fixed maximal ideal use 1 when $1 + a$ has nonzero residue, and otherwise use a scalar in O whose residue differs from its conjugate. In the latter case β_0 reduces to $z - \sigma z$. The Chinese remainder theorem gives one z with these residues. Thus the twisted classes are represented by $\beta_0 c$ for $[c] \in C_B$.

Lean: GlobalTheory.UnitLemma.UnitBase.exists_unit_base (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsUnitBasePointAdjoinTwistedSigma, lines 168–212)

Since $J_r \in \mathrm{GL}_r(O)$, an integral twisted representative is equivalent to a Gram matrix invertible over O. Then ϵg preserves the lattice by the relation between H and $\bar{H}^t$. On a stable lattice both β_0 and β_0^{-1} act as automorphisms. Thus the integral test for $\beta_0 c$ amounts to g-stability and h_c-self-duality. The centralizer and stabilizer weights are those of T_B, so the stable twisted integral is $S(B)$. A nonintegral generator gives zero since neither an integral twisted representative nor a full stable lattice can have that norm. The rank-zero integral is one.

Lean: GlobalTheory.TwClassForms.twCl_equiv_cl (Development.AbelianVarietyTheory.Proofs.DivisorProofs.TwistedConjugacyClassesEquivUnitsModNormsNonspli, lines 79–203)

Lean: GlobalTheory.InertTwUnit.twisted_unit_eq_stSum (Development.AbelianVarietyTheory.Proofs.DivisorProofs.InertPlaceStableTwistedOrbitalUnitEqLatticeSum, lines 28–76)

The two twisted integrals multiply. Since E_w has residue cardinality q_0^2, the factor in (26.2) is $(-1)^{v(R)}(q_0^2)^{-v(R)/2} = (-q_0)^{-v(R)}$. □

Proof of 26.11

Proof. In rank d put $p(j) = \tau j(2d+1-j)$. The cut requires $\deg_x W \le p(\dim W)$ for every rational subspace W and $\deg_x E^d \ge -\tau$. Here degree is the negative logarithm of the adelic norm of a rational basis wedge measured at x. Write $\mathcal{C}_\tau$ for the product of the cuts and $K_h(x,y) = \sum_\gamma h(x^{-1}\gamma y)$ for the periodized kernel. If the logarithmic support size is at most r in each factor, then

$$\mathcal{E}(h) = \int_{\mathcal{C}_\tau} K_h(x, \theta x)\, dx \qquad \big(\tau \ge C_0(1+r)\big) \tag{A.213}$$

for a field- and rank-dependent C_0.

Adelic reduction gives a Siegel decomposition $\gamma x = b\, a(t) k_0$ in each rank [97]. Here b lies in a compact upper triangular set. The matrix $a(t)$ is positive diagonal and k_0 lies in the maximal compact. The simple-root inequalities put $a(t)^{-1} b a(t)$ in a fixed compact. Taking $m_i = \lfloor t_i \rfloor$ gives cells $a(m)\Omega_0$ for a compact Ω_0. There the coordinate line degrees are $[E:\mathbf{Q}]m_i + O(1)$ and the total degree is $[E:\mathbf{Q}]\sum_i m_i + O(1)$. These formulas follow from the action of $a(m)$ on the coordinate wedges and bounded operator norms for o, o^{-1} with $o \in \Omega_0$. The cut bounds each m_i above by $O(1+\tau)$ and bounds the sum below by $-O(1+\tau)$. Subtract the other upper bounds to bound each m_i below. The cut is closed by continuity of each degree, and it lies in the images of finitely many compact cells.

For any representative h_0 of a point in a cell, rational changes of basis transport the subspaces whose degrees are measured. A rational line has degree at the identity bounded above, since a nonzero coordinate of its basis vector has product of absolute values one. Write $L(g)$ for the logarithmic size of g defined in the statement. Comparison by exterior powers gives $\deg_{h_0} W \le C(1 + L(h_0))$ for every rational line W. Applying the determinant estimate also to the inverse gives $\deg_{h_0} E^d \ge -C(1+L(h_0))$. The same coordinate and subtraction argument bounds $\|m\|_\infty$ by $O(1 + L(h_0))$. For a class x_d represented by g in rank d, put $\rho_d(x_d) = \inf_{\gamma \in \mathrm{GL}_d(E)} L(\gamma g)$. Taking the infimum gives the first bound below. The representative $a(m)o$ and the cut bounds give the second, with $H(x) = 1 + \rho_a(x_a) + \rho_b(x_b)$.

$$1 + \|m\|_\infty \ll 1 + \rho_d(x_d), \qquad H(x) \ll 1 + \tau \quad (x \in \mathcal{C}_\tau). \tag{A.214}$$

Unfold the compact cut by rational twisted classes. In one factor let g represent a class and put $A = g\theta(g)$ and $\operatorname{rad}_g W = W \cap W^{\perp_g}$. On the support of the orbital integrand,

$$\begin{aligned} \deg_x(AW) \geq \deg_x W - D, \qquad |\deg_x E^d| \leq D, \\ \deg_x W \leq \deg_x \operatorname{rad}_g W + D \quad (AW = W), \qquad D = C_d(1+r). \end{aligned} \tag{A.215}$$

For $j = \dim W$ and a basis wedge v of W, use $\|(\bigwedge^j A)v\|_x \leq \|\bigwedge^j (x^{-1}Ax)\|\|v\|_x$ at each place, take negative logarithms and sum. The two factors of $x^{-1}Ax$ are $x^{-1}g\theta(x)$ and its θ-image, so their logarithmic norms are bounded by the support size. The determinant of $x^{-1}g\theta(x)$ bounds the total degree by the product formula for $\det g$. For stable W, the pairing with orthogonality $W^{\perp_g}$ induces a nondegenerate pairing on the quotient by the radical. Its determinant on a rational basis has product of absolute values one. The determinant bound in quotient norms gives an upper bound D for the quotient degree. Multiplicativity of subspace and quotient covolumes gives the last inequality.

At fixed x the subspace degrees are bounded above by the exterior-power comparison with the identity. Put $b(W) = \deg_x W - p(\dim W)$ and $s = \sup_W b(W)$. If $s > 0$, take W within δ of the supremum, for $0 < \delta < \min(s, \tau - D)$ and $\tau > D$. If $AW \neq W$, put $j = \dim W$ and $h_0 = j - \dim(W \cap AW) > 0$. Projecting wedge bases modulo the intersection gives the degree inequality for intersection and sum. Since $2p(j) - p(j - h_0) - p(j + h_0) = 2\tau h_0^2$, it gives

$$b(W \cap AW) + b(W + AW) \geq 2b(W) - D + 2\tau h_0^2 > 2s. \tag{A.216}$$

Thus W is stable. Its radical is stable and isotropic. A drop in dimension from W to its radical decreases p by at least τ, and (A.215) would give $b(\operatorname{rad}_g W) > s$. Thus a violation is detected on a stable isotropic subspace. The total degree condition holds by (A.215). A regular elliptic norm has no nonzero stable isotropic subspace, so the cut leaves its whole orbital integral.

Lean: `PositivityTheory.Rg.BcB.D1.onlyIsotropic` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.IsotropicMahlerBounds.LogAbsoluteWedges.OnlyIsotropic`, lines 60–60)
Lean: `PositivityTheory.Rg.BcB.D1.ellTerm` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.IsotropicMahlerBounds.LogAbsoluteWedges.OnlyIsotropic`, lines 61–61)

For a regular nonelliptic norm, the stable isotropic subspaces W_e correspond to signed selections e of the paired irreducible norm factors, with a zero coordinate for an unselected pair. Order the selections by inclusion of signed coordinates. The meet keeps common signs. The signed join keeps the union except at conflicting coordinates, where it keeps neither sign. Consequently

$$W_{e \wedge e'} = W_e \cap W_{e'}, \qquad W_{e \vee e'} = \operatorname{rad}_g(W_e + W_{e'}).$$

Indeed, the pairing is nondegenerate between the two factors of a pair and pairs distinct pairs trivially. Put $U = W_e + W_{e'}$ and $y_e = p(\dim W_e) - \deg_x W_e$. The degree inequality for $W_e, W_{e'}$ bounds the sum of the degrees of the intersection and U below by the sum of the original degrees. Concavity gives $p(\dim(W_e \cap W_{e'})) + p(\dim U) \leq p(\dim W_e) + p(\dim W_{e'})$. If the radical has smaller dimension than U, then $p(\dim U) - p(\dim \operatorname{rad}_g U) \geq \tau$, which absorbs the error D in replacing $\deg_x U$ by $\deg_x \operatorname{rad}_g U$. In equal dimension the radical equals U. Hence $y_0 = 0$ and $y_{e \wedge e'} + y_{e \vee e'} \leq y_e + y_{e'}$.

Let T be the adelic twisted centralizer and Γ its rational subgroup. Write $a(t)$ for the logarithmic coordinates of $t \in T$ on the paired factors. Translation by t changes $\deg_x W_e$ by $e \cdot a(t)$. The dot product is additive across signed meet and join, including at conflicting coordinates. For $b(e) = e \cdot a(t) - y_e$ the preceding inequality gives

$$b(e \wedge e') + b(e \vee e') \geq b(e) + b(e'). \tag{A.217}$$

For every real vector y indexed by signed selections, define $\gamma(y, z)$ as the sum of $(-1)^{|C|}$ over chains C of nonzero selections with $e \cdot z > y_e$ for each $e \in C$. Include the empty chain. Define

$$V(y) = \int_{\Gamma \backslash T} \gamma(y, a(t))\, dt.$$

For the actual cut bounds, γ is the indicator of $e \cdot z \leq y_e$ for all e. If there is a violation, take a maximizer e_* of b with $b(e_*) > 0$. By (A.217), every violator e has $b(e \wedge e_*) \geq b(e) > 0$. Toggle e_* on a chain contained in its down-set. On any other chain take the first member e outside that down-set and toggle

$e \wedge e_*$. Earlier members lie below the meet and later ones above it, so this is a sign-reversing involution. When all inequalities hold, the empty chain is the sole contribution.

For arbitrary y with $\max_e |y_e| \le M$, the integrand is bounded by the number of chains and vanishes unless $|z_i| \le 2M$ for every coordinate. For example, if $z_i > 2M$, drop a minus sign at i on all violating selections that have one. The resulting selections are nonzero violators, since their dot products increase by more than $2M$. This order-preserving map lies below the identity. Next add a plus sign at i to any selection still zero there. This map lies above the identity and also preserves violations. The two order homotopies leave the alternating chain sum unchanged and give the subposet of violators with plus at i. It has minimum the selection consisting of that sign alone, so toggling the minimum cancels the sum. For $z_i < -2M$ interchange the signs. The inverse image of the box under a is compact modulo Γ. On paired factors the coordinates bound the idèle norms and the norm-one idèle class groups are compact fibres. The self-paired factors have compact norm-one torus quotients. Thus $V(y)$ is absolutely defined and bounded for bounded y. Changing one coordinate y_e changes the chain sum by terms from chains containing e, so the difference depends on that change and the coordinates comparable with e.

With no paired factors the weight is constant, and the unipotent integral vanishes at a complex place with a root off the unit circle. It uses the flag from the two opposite collections of eigenlines off the circle. Here and below the integrand along N is $h(n^{-1}g\theta(n)) = h(n^{-1}\alpha(n)g)$ with $\alpha = \mathrm{Ad}(g) \circ \theta$, which preserves N for the flag used. Since $\alpha^2 = \mathrm{Ad}(g\theta(g))$ and the norm is regular, $1 - \alpha$ is invertible on each step of the lower central series of N; so $n \mapsto n^{-1}\alpha(n)$ is a bijection of N with constant Jacobian, and the very cuspidal condition applies. With paired factors, use two infinite places. Such places exist because an imaginary quadratic field ramifies over $\mathbf{Q}$ at a finite prime. The local degrees give the cut weight as $V(p_1(x_1) + p_2(x_2) + z_0)$, with p_i the local vectors and z_0 the fixed remaining terms. Subtract $V(p_1(x_1))$ by replacing the coordinates one at a time. In the difference at e, every relevant subspace is contained in W_e or contains W_e and lies in $W_e^{\perp_g}$. The unipotent radical for $W_e \subseteq W_e^{\perp_g}$ acts trivially on their determinant lines. Thus the difference is invariant along the unipotent integration at the first place and its integral vanishes. The remaining weight $V(p_1(x_1))$ is constant at the second place, and the same vanishing applies there. Regular orbit maps are proper modulo the centralizer on the test support. On compact representatives the local degree vectors are bounded, so the box estimate for V gives absolute integrability in these decompositions. This proves (A.213) in each factor.
Lean: `PositivityTheory.Rg.BcB.CV.pf_ClassVanish` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.QuotientVanishing.LocalQuotientIntegrals.PClassVanish`, lines 43–136)
Lean: `PositivityTheory.Rg.BcB.CV.ellFormula2J` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.QuotientVanishing.LocalQuotientIntegrals.EllFormula2J`, lines 12–12)

Let $u_i(x)$ and $v_i(y)$ be the periodized rows and columns of the end kernels, with M absorbed in the columns. Smoothing both variables by mass-one approximate identities gives L^2 convergence of the rows and columns by continuity and compact support. Boundedness of V and continuity of the kernel give $K_h(x, \theta x) = \sum_i \langle u_i(x), V v_i(\theta x)\rangle$. For compact Ω, put $n_\Omega(x) = \#\{\gamma : x^{-1}\gamma x \in \Omega\}$ in the product rational group. Unfolding a squared row or column norm pairs translates whose difference is conjugate into a product of supports, with one inverted. Choose Ω containing those products, pulled back by θ for the columns. Each overlap integral is uniformly bounded by the fixed kernel norms, so Cauchy–Schwarz gives

$$|K_h(x, \theta x)| \le C_1 n_\Omega(x) \|V\|. \tag{A.218}$$

On a cell $a(m)\Omega_0$, each element counted by $n_\Omega(x)$ gives a distinct rational translate of x in $a(m)\Omega_0\Omega$. Unfolding bounds the integral of the count on the quotient cell by $\mathrm{vol}(\Omega_0\Omega)$. By (A.214), for $L = 2(a+b) > 0$ the integral of $H^{-2L} n_\Omega$ is bounded by a constant times the product of the sums of $(1 + \|m\|_\infty)^{-L}$ over $\mathbf{Z}^a$ and $\mathbf{Z}^b$. Both converge, with one term in rank zero. When $a + b = 0$ the count is constant. Hence

$$\int_{\mathcal{C}_\tau} n_\Omega(x)\, dx \ll (1 + \tau)^{2L}. \tag{A.219}$$

Apply (A.213), (A.218) and (A.219) with $\tau = C_0(1 + r_0 + k r_1)$. □

Proof of 26.12

Proof. Take the exterior algebra of the complexified Hermitian matrices tensored with the two weight coefficient factors. Write Γ for exterior parity and S_s for multiplication by s^j in degree j. Let T be the involution

on the coefficient factors. The matrix kernel k acts on right compact-equivariant sections. The scalar trace is $\phi(y) = \operatorname{tr}(k(y)T)$, with the exterior signs in the kernel formulation below. Compact equivariance and commutation of T with the compact action give the compact conjugation invariance.

Let E be the exterior and weight coefficient space just specified, with compact action ρ. Work on $G \times_K E$, identifying its smooth sections with functions $f : G \to E$ satisfying $f(xk) = \rho(k)^{-1} f(x)$. In a trace-orthonormal Hermitian basis write ϵ_i, ι_i for exterior multiplication and contraction; they satisfy $\epsilon_i \iota_j + \iota_j \epsilon_i = \delta_{ij}$. For a complex matrix X, set $R_X f(x) = \frac{d}{dt}\big|_{t=0} f(x \exp(tX))$, and define

$$E^{\pm}_{ab} = \tfrac{1}{2}(R_{E_{ab}} \mp i R_{iE_{ab}}), \qquad Z^{\pm}_m = \sum_{a_1,\dots,a_m} E^{\pm}_{a_1 a_2} \cdots E^{\pm}_{a_m a_1}, \qquad \Omega = 2(Z^{+}_2 + Z^{-}_2).$$

Use the even solution of $\partial_t^2 u = \Omega u$, with $u(0) = f$ and $\partial_t u(0) = 0$. The compact derivatives become fibre endomorphisms on this bundle, and the principal symbol is the invariant metric symbol on G/K. Thus the wave equation has finite propagation. Integrating against a smooth even compactly supported time function gives a smooth compactly supported matrix kernel. Apply words in the central operators $Z^{\pm}_m$ to this same kernel. The wave construction gives finite propagation. On a joint eigenfunction the wave equation is scalar, so the time integral acts by the integral of the time function against $\cosh(t\sqrt{c})$, where c is the eigenvalue of the wave operator. The central operators multiply this scalar by their eigenvalues. Finite sums and convolutions have the corresponding sums and products of values.

For the unipotent integral, use the field of Hermitian matrices

$$H(x) = P_{x^{-1}W_1} + P_{x^{-1}W_2} - 1,$$

where P is standard orthogonal projection. It is invariant on the left under the flag stabilizer and equivariant on the right under the compact group. The complement relations give $H(d) = -H(1)$. Put $A(x) = \epsilon(H(x))$, and let Q be the flag stabilizer with radical N. For an N-invariant smooth section f, order the left differential operators by the opposite radical, Levi and radical. The PBW expansion of a central word, modulo terms having a radical derivative on the right, belongs to the enveloping algebra of the Levi. To see this, the central word has zero weight under a cocharacter strictly positive on the radical; a surviving opposite-radical factor would have negative weight and would require a radical factor. Commuting those factors to the prescribed order accounts for the lower-order Levi terms. This includes the modular shift when the constant term is normalized. Since $A(qx) = A(x)$ for $q \in Q$, every positive-order Levi or radical derivative of A vanishes. Leibniz's rule therefore gives

$$Z^{\pm}_m(Af) = AZ^{\pm}_m f, \qquad \Omega(Af) = A\Omega f.$$

The argument applies to the E-valued functions on G, retaining their compact equivariance, so the fibre action is included. Left translation commutes with Ω; uniqueness of the wave equation preserves N-invariance and gives $u_{Af}(t,x) = A(x) u_f(t,x)$. Integration in time and application of the same central words preserve this identity. Integrate the resulting kernel identity along the radical. For $B = \int_N k(ud)\,du$ it reads

$$\epsilon(H(1))B = B\epsilon(H(d)) = -B\epsilon(H(1)).$$

The averaging uses Haar invariance on N, and the integrals have compact support there. Put $C = \epsilon(H(1))$ and $D = \iota(H(1))$. Then $CD + DC = \operatorname{tr}(H(1)^2)1$, and this scalar is nonzero since $W_1 \neq 0$. As T commutes with C, cyclicity of trace gives $\operatorname{tr}((CD + DC)BT) = 0$. This proves the unipotent vanishing.
Lean: PositivityTheory.RgBcC.VC.vcKer_of (Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.ArchimedeanGelfandKernels.LeftEquivariantFunctions.GelfandFlags.UnitaryMemGLs, lines 40–97)

For the elliptic integral, use the positive Hermitian cone with polar coordinate $v = xx^*$. Write $F(x)$ for the action on the exterior and weight coefficients in this frame. For $v = xx^*$ and $w = yy^*$, put

$$K(v,w) = c_r \det(w)^{-r} F(x) k(x^{-1} d (y^*)^{-1}) \Gamma T F(y)^{-1},$$

where c_r is the polar Haar constant. The expression descends to the two cone variables by compact equivariance. The cochain identity for the multiplier kernel, in these coordinates, is

$$\sum_i \epsilon_i \partial_{1,i} K + \sum_i (\partial_{2,i} K)\epsilon_i = 0, \qquad \Gamma K = K\Gamma.$$

Use the exterior differential of the flat coefficient bundle, transported to the displayed polar frame. The differential commutes with left G-translation, hence with the central actions $Z_m^{\pm}$ and with Ω. Applying it to the wave equation and using uniqueness gives $d\,u_f = u_{df}$. Time integration and the central words preserve this identity. The twist in the second variable covers the cone map $w \mapsto dw^{-1}d^*$. At $w = yy^*$ its derivative sends yYy^* to $-d(y^*)^{-1}Yy^{-1}d^*$. In the frame $d(y^*)^{-1}$ this acts by $(-1)^j$ on degree j, while T interchanges the flat coefficient factors. Pullback on forms with this flat coefficient identification commutes with the exterior differential. The twisted operator therefore also commutes with it. Integration by parts in the second variable gives the displayed identity for its kernel with density. In the polar change of variables the frame derivative in direction xX, for $X = X^*$, is the cone derivative in direction $2xXx^*$. In the second variable the density contributes $-r\,\mathrm{tr}(w^{-1}dw)\det(w)^{-r}$, which is the density term in the second cochain derivative.

Lean: `PositivityTheory.RgBcC.LfSt.kChain_holds` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.ArchimedeanGelfandKernels.LeftEquivariantFunctions.GLMatrixDerivatives.KChainHolds`, lines 7–7)

For $0 \le s \le 1$ integrate $\mathrm{tr}(\Gamma S_s K(sv + (1-s)1, v))$ over the cone. Its derivative in s is the integral of the divergence with ith component

$$\mathrm{tr}\big(\Gamma\epsilon_i S_s \iota(v-1)K(sv+(1-s)1,v)\big), \qquad \iota(v-1) = \sum_i (v-1)_i \iota_i.$$

Indeed $S_s' = \sum_i \epsilon_i S_s \iota_i$, and in the first-variable derivative use $s\epsilon_i S_s = S_s\epsilon_i$ and $\epsilon_i\iota(v-1) = (v-1)_i - \iota(v-1)\epsilon_i$. In the second-variable derivative, moving ϵ_i to the right under the trace contributes the parity sign. The remaining terms cancel by the cochain identity. The interpolated kernels have common compact support in the cone when the z_i^2 are distinct. In polar coordinates, boundedness of $x^{-1}d(y^*)^{-1}$ and its inverse, with $xx^* = syy^* + (1-s)1$, bounds y and y^{-1} uniformly for $0 \le s \le 1$. Hence the divergence integrates to zero. At $s = 1$ polar integration gives the twisted integral. At $s = 0$ the projection S_0 onto degree zero gives the coefficient eigenfunctions $x \mapsto F(x)^{-1}S_0 u$ with parameter $\mathrm{inf}(l)$. The convolution thus acts there by $\mathrm{val}(\mathrm{inf}(l))$. The trace of $F(d)T$ on degree zero uses the flip of the two coefficient factors. The identity $\mathrm{tr}((A\otimes B)\mathrm{flip}) = \mathrm{tr}(AB)$ gives $\mathrm{ch}_l(z_1^2,\dots,z_r^2)$. Rank zero uses the scalar kernel on the trivial group. Fubini gives the two-block version.

Lean: `PositivityTheory.RgBcC.LfSt.lAsm` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.ArchimedeanGelfandKernels.LeftEquivariantFunctions.GLMatrixDerivatives.TwMemGLs`, lines 61–128)

□

Proof of 26.13

Proof. Fix one elliptic norm polynomial P of degree r and a tuple of local classes contributing to its stable integral. Let L_i be its irreducible field factors and K_i their fixed fields for conjugate inversion. The elliptic condition makes L_i/K_i CM. Relative to one rational twisted representative, the form description in 26.10 identifies rational classes with $\prod_i K_i^\times / N_{L_i/K_i} L_i^\times$. The same description at each finite place gives the local norm classes, with the split factors automatic. The local tuple has the unit class relative to the representative almost everywhere by the finite étale argument in 26.5. For each i, the finite norm symbols therefore prescribe one product of signs at the real places of K_i. By quadratic norm reciprocity, every choice of real signs with that product is realized. The Hasse norm theorem identifies two global coefficients with the same signs at all places [41]. Thus the number of rational classes in the fibre is $\prod_i 2^{[K_i:\mathbf{Q}]-1} = 2^{r|\Sigma_\infty|-r(P)}$.

Each such class has mass $2^{r(P)}\kappa(P)$. Its weight character and its orbital integrals at the prescribed finite classes are fixed. Apply this count to both blocks and then sum over the local tuples. The regular finite place ensures that the contributing norm polynomials are coprime, since their product becomes separable after extending scalars to that completion.

For the second equality, decompose the adelic group into the product of the infinite and finite groups. Its Haar measure is a nonzero constant times the product of the chosen measures, by uniqueness of Haar measure. For each elliptic class the archimedean centralizer is compact. Use probability measure there in forming the quotient, so the archimedean orbital integrals are the group integrals of 26.12. On the finite group, disintegrate first by the adelic centralizer and then by its maximal compact subgroup. Each double coset in $\Gamma\backslash C/C_c$ has volume $1/\#(\Gamma \cap C_c)$, since C is abelian. The resulting volume is the centralizer mass.

At a split pair the two local group integrals combine by convolution on one factor after applying the twist to the second. At an inert place there is one local orbital integral. The product of local integrals is finite after removing the unit factors. This gives the rational-class summand times $c_{a,b}\prod_v \nu_v$. Twist translation preserves the centralizers, so the constant is common to the classes. □

Proof of 26.14

Proof. At $w \in T'$ the block level is the product L of the two Iwahori subgroups. The matrix indices there are the assignments of the chamber coordinates to the two blocks. The fixed kernel is diagonal in these indices. Each admissible diagonal entry is a weighted sum of characteristic functions of L-double cosets, and an inadmissible entry is zero. At $w \in S_0$, let $K'_w \triangleleft K_w$ be the normal open split test level and let x_i, x_j run over coset representatives of K_w/K'_w. Put $d = [K_w : K'_w]$, normalize Haar measure on K_w to one, and give each class of K_w/K'_w mass $1/d$. Write $\Phi(f)_{ij}$ for the normalized Jacquet integral of $f(x_i^{-1}gnx_j)$, including the half-modulus factor. In ordinary matrix convolution use

$$M(f)_{ij} = d^{-1}\Phi(f)_{ij}.$$

Iwasawa integration and the finite compact average give

$$\Phi(f * f')_{ij} = d^{-1}\sum_k \Phi(f)_{ik} * \Phi(f')_{kj}, \qquad \mathrm{CT}(f) = d^{-1}\sum_i \Phi(f)_{ii}.$$

Thus $M(f * f') = M(f) * M(f')$ and $\mathrm{CT}(f) = \operatorname{tr} M(f)$. The determinant character multiplies each entry and preserves convolution. Its fixed factor $\mu_w((-1)^{ab})$ is applied once in taking the transferred trace. In the compact induced model, evaluation on these classes has its Levi-equivariant image; $Q = M(e_{K'_w})$ is the projection onto that image and $QM(f) = M(f)Q = M(f)$. Inversion and conjugation of the same integrals give $M(f^*) = M(f)^\dagger$. These formulas also apply to a rank-zero Levi factor. Each entry of the fixed matrix is bi-invariant under the chosen block level L, because L embeds in the Levi part of K'_w where the determinant twist is trivial, so left and right translation by L in the constant-term integral leave $M(f_w)_{ij}$ unchanged.

In both cases the matrix of every variable finite kernel at this position is $P_{ij} = \delta_{ij}e_L$. The same holds for its adjoint, since $e_L^* = e_L$, with $f^*(x) = \overline{f(x^{-1})}$. If M is the matrix of the fixed kernel, Haar averaging gives, entry by entry,

$$(M * P)_{ij} = M_{ij} * e_L = M_{ij} = e_L * M_{ij} = (P * M)_{ij}.$$

This calculation permits arbitrary bi-L-invariant entries of M.

Lean: `PositivityTheory.Rg.BcB.F.LocComm.beta_m` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.AdelicHeckeBlocks.PLocComm`, lines 337–348)

At a variable split place the block level is $K = \mathrm{GL}_a(\mathcal{O}_w) \times \mathrm{GL}_b(\mathcal{O}_w)$. The transferred test is bi-K-invariant by the compact average in (26.4) and triviality of the twist on integral units. Its adjoint is again bi-K-invariant. Transposition in both factors preserves every K-double coset: elementary divisors give a diagonal representative in each factor. Transposition preserves the Haar measure normalized by $\mathrm{vol}(K) = 1$ and reverses convolution. It fixes each bi-K-invariant function, including the convolution of two such functions. Thus the two convolutions in opposite orders agree. At this place the finite factor of β_s is the level idempotent. At all remaining finite positions the variable factors are level idempotents. Products over distinct places commute by factorization of convolution.

Lean: `PositivityTheory.Rg.BcB.F.sphComm` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.AdelicHeckeBlocks.IsCompactMaximalPairPlace`, lines 61–71)

Lean: `PositivityTheory.Rg.BcB.F.locComm` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.AdelicHeckeBlocks.PLocComm`, lines 355–376)

At infinity, composing two wave multipliers convolves their even time functions and composes their central differential operators. The time convolutions commute, as do the central operators. The adjoint of a wave multiplier is again a sum of wave multipliers: inversion and adjunction take the central word to its formal adjoint, and the undifferentiated real even wave kernel is self-adjoint. These identities on compactly supported smooth sections identify the convolution kernels by testing against approximate identities. They give commutation in either order with every variable multiplier and its adjoint.

Lean: `GlobalTheory.CompanionTheory.WaveMult.asmMult_main` (`Development.AutomorphicSpectrum.Proofs.BlockDecomposition.AsmMultMain`, lines 25–75)

On V the variable finite operators act by their spherical scalars on each constituent summand. The adjoints act by the conjugate scalars. The operator u preserves each summand. We have thus proved in every coordinate that two generators commute and that each generator commutes with the adjoint of every generator. Taking adjoints gives the identities for two adjoints. Expanding products gives commutativity of $\mathcal{A}$. The same expansion, using the identities with the fixed operator in each coordinate, proves the two assertions about u and β_s.

Lean: `PositivityTheory.Rg.BcB.F.SetupAlgPf.algCR` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.SetupAlgPf`, lines 97–103)

□

Proof of 26.15

Proof. Take a composition series of M, which has finite length by [14, §2]. In the first member containing the given vector, its image in the successive quotient τ is nonzero and satisfies the same equations. Let N be the upper unitriangular group. The image y of this vector in the Jacquet module τ_N is nonzero by the Iwahori projection lemma, stated for connected reductive groups in [23, Prop. 2.4] with the injectivity argument of [22, Lem. 4.7]. Here the subgroup is the Iwahori for B, the representation is smooth admissible, and the coefficients are complex.

The right cosets in $I_j t_x I_j$ have representatives ut_x, with u running through $(N\cap I_j)/t_x(N\cap I_j)t_x^{-1}$. There are $\delta_B(t_x)^{-1}$ such cosets, by taking the index in each upper triangular entry. In the coinvariants all these translates have the same image. Hence the action of A_x there is the action of t_x in the normalized Jacquet module $r_B\tau$. The vector y is fixed by the integral diagonal torus. Every integer tuple is a difference of two strictly decreasing tuples, so its torus character is

$$\chi(\mathrm{diag}(u_i\varpi^{e_i})) = \prod_i \xi_i^{e_i}, \qquad u_i \in O_L^\times.$$

This character also occurs as a quotient of $r_B\tau$. Indeed the integral-torus invariants form a finite-dimensional module for the commuting uniformizer translations. Its simultaneous generalized χ-space is nonzero and has a simple quotient of character χ. Projection onto the invariants by compact averaging and then onto this generalized eigenspace gives the quotient on $r_B\tau$.

Normalized Frobenius reciprocity embeds τ in $\mathrm{Ind}_B^{\mathrm{GL}_j(L)}\chi$. Write $\chi_i = |\cdot|_L^{s_i}\chi_i^u$ with χ_i^u unitary. Since $|\varpi|_L < 1$, the displayed order gives $s_1 \le \cdots \le s_j$. The spherical Langlands classification, in the subrepresentation convention obtained by taking contragredients of the standard quotient, gives a unique irreducible subrepresentation of this induction. It is spherical with Satake multiset $\{\xi_i\}$. This applies also to equal real exponents, for which the corresponding block is induced from unitary unramified characters of a general linear group [108, Ch. XI, §2].

Compact averaging lifts a nonzero K_j-fixed vector of τ first to the member of the composition series and hence to M. By $\mathrm{GL}_j(L) = PK_j$, evaluation at the identity identifies

$$M^{K_j} \simeq \bigotimes_i \sigma_i^{K_{j_i}}.$$

Thus each factor is spherical. In normalized induction the spherical Hecke character on this line is obtained by concatenating the Satake parameters of the factors: the normalized constant term sends a symmetric Laurent polynomial to the same polynomial evaluated on the block variables. Its image in τ^{K_j} has the same character. The elementary symmetric polynomials determine the asserted multiset. □

Proof of 26.16

Proof. Apply the adelic counting theorem of Müller [42, Theorem 1.4, (1.17)–(1.18)] to the restriction of scalars of GL_d/E. This group satisfies the analytic hypothesis of that theorem, as in its Corollary 1.3. Its adelic quotient is taken modulo the positive real split center. Fix one of the compact types. On this type a bounded infinitesimal parameter bounds the absolute value of the Casimir eigenvalue, since the Casimir acts by a fixed polynomial in the parameter. Both the cuspidal and the residual counting functions are finite at this bound. Every representation in question contributes at least one dimension. Sum over the finite type

list. There are finitely many compositions of integers at most n, so taking products proves the assertion for tuples. The finite lists used for the discrete leaves in the trace comparison are covered by this statement. □

Proof of 26.17

Proof. We first compute the integral of a single double-coset indicator. For the general linear group use the involution $\theta(g) = J_r(\bar{g}^{-1})^t J_r^{-1}$ of 26.5. On a diagonal it sends d_i to $\bar{d}_{i^*}^{-1}$, with reversal within the block.

We now give the diagonalisation used in the computation. If z is strictly decreasing, every element of $I_r t_z I_r$ is I_r-conjugate to a diagonal of valuations z. Indeed, an initial conjugation puts the element in the form $t_z u$, $u \in I_r$. Divide by $\varpi^{z_{r-1}}$. For $r > 1$ the reduction is diagonal with a simple unit in the last position and zeros elsewhere. Hensel factorisation gives a unit eigenvalue and an invariant complementary summand. Write the unit eigenvector as $(v_0, \dots, v_{r-2}, 1)$. The row equations give $v_i \in \varpi^{z_i - z_{r-1}} \mathcal{O}_L$. In the quotient by this line the matrix, before the division, has entries

$$\varpi^{z_i}\big(u_{ij} - \varpi^{z_{r-1}-z_i} v_i u_{r-1,j}\big) \qquad (i, j < r-1).$$

The matrix in parentheses is upper triangular modulo ϖ with unit diagonal, since $u_{r-1,j} \in \varpi\mathcal{O}_L$. Induction diagonalises the quotient. Lifting its basis to the invariant complement preserves the reduction of that basis. This gives the required Iwahori change of basis. Rank one starts the induction.

For twisted conjugation take $\delta \in I_r t_z I_r$. Its norm belongs to $I_r t_{z-z^*} I_r$. To see the double-coset product here, factor the middle Iwahori element in upper unipotent, diagonal and lower unipotent factors, using Gaussian elimination with unit pivots. Conjugation by the first dominant translation moves the upper factor into the Iwahori, and inverse conjugation by the second moves the lower factor into the Iwahori. The diagonal factor commutes with both translations. The entries of $z - z^*$ are strictly decreasing. Diagonalise the norm by the preceding argument, and use the same change of basis for twisted conjugation of δ. If the resulting norm is D, then $D\delta = \delta\theta(D)$. The ordered valuations of D and $\theta(D)$ agree and are distinct. The entry equations therefore make δ diagonal. Its valuations are z: reduction with the standard lattice chain distinguishes the diagonal Iwahori double cosets, since the relative positions of that chain and its translate record the translation of each one-dimensional successive quotient.

In the unitary group the ordinary diagonalisation can be made isometric. In the ordered integral eigenbasis, invariance of the Hermitian form pairs an eigenline only with the line whose eigenvalue is its conjugate inverse. The pairings at the reverse positions are units by unimodularity. Rescale one vector in each pair to give the prescribed Witt pairing. The norm on units of an unramified quadratic extension is onto, so one may also normalise the middle vector in odd rank. These rescalings preserve the Iwahori flag.

Fix monic Q_1, Q_2 as in (26.2). On the unitary support their product has roots $\lambda_i \in L^\times$ with $v(\lambda_i) = h_i$ and $\lambda_{i^*} = \bar{\lambda}_i^{-1}$. Thus the roots in Q_1 determine exactly one admissible A. Conversely, any such list gives a diagonal unitary representative. For a general linear block with exponent vector z, a diagonal twisted representative must solve

$$d_i/\bar{d}_{i^*} = \lambda_i, \qquad v(d_i) = z_i.$$

Choose d_{i^*} of the prescribed valuation and solve for d_i on each pair. The valuation condition holds because $v(\lambda_i) = z_i - z_{i^*}$. At a middle position Hilbert's theorem 90 solves $d/\bar{d} = \lambda$, and multiplication by a power of ϖ sets the valuation. This constructs both blocks. If the prescribed roots and valuations do not occur, neither side meets the test support.

There is one contributing orbit on each side when these conditions hold. For the unitary group this follows from the ordered isometric eigenbases. In a twisted block any two diagonal solutions of the same valuations differ by units e_i with $e_i = \bar{e}_{i^*}$. A diagonal twisted change of basis by units accounts for both entries of each pair. At the middle position it accounts for e_i by surjectivity of the norm on units. In particular two conjugates in the support are conjugate there by the Iwahori. For a representative in the support the set of conjugators contributing to the orbital integral is consequently CI, where C is its centralizer, twisted on the general linear side. In diagonal coordinates $C \cap I$ is its maximal compact subgroup: commutation with the regular diagonal norm first makes the centralizer diagonal, and the remaining condition pairs its entries by conjugate inversion. Thus the orbital integral of the indicator is $\mathrm{vol}(I)/\mathrm{vol}(C \cap I) = \mathrm{vol}(I)$. The normalized indicator has integral one. This also gives one for the rank-zero factor.

On $\ker Q_1$ the Hermitian form is an orthogonal sum of the selected hyperbolic planes and, if selected, the integral middle line. Its determinant has valuation zero in this basis. The sign in the unitary orbital sum is therefore $+1$. Across blocks all root valuations differ, so

$$v(\mathrm{Res}(Q_1, Q_2)) = \sum_{i \in A, j \in B} v(\lambda_i - \lambda_j) = v_A(x).$$

The factor Δ_w of 26.5 is precisely $(-\sqrt{q})^{-v_A(x)}$. This proves the identity, including its vanishing cases.
Lean: `PositivityTheory.Rg.Bc.ArchbcB.B.M4.asmL_A` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.IwahoriFactorisation.TwistedConjugationNorms.Development`, lines 802–867)
Lean: `PositivityTheory.Rg.Bc.ArchbcB.B.M4.asmR_A` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.IwahoriFactorisation.TwistedConjugationNorms.Development`, lines 997–1029)

For the powers, two translations in the same chamber have additive length. Explicitly, the right cosets in ItI are indexed by the upper unipotent part U_0 of I modulo tU_0t^{-1}. For translations t, t' in that chamber the inclusions $tt'U_0(tt')^{-1} \subseteq tU_0t^{-1} \subseteq U_0$ give a product of transversals, the second transversal inside tU_0t^{-1} obtained by conjugating the one for t'. The triangular factorisation used above gives this coset description also for the unitary Iwahori: uniqueness of the triangular factors in Witt coordinates makes them fixed under the unitary involution. Thus the convolution of the two indicators divided by Iwahori volume on each factor is the similarly normalized indicator for tt'. For fixed A, restriction of exponents is additive, and $\min(h_i, h_j)$ always selects the same index on this chamber. Hence $v_A(x + x') = v_A(x) + v_A(x')$. Expand the power over ordered tuples of translations and apply the single-translation identity to their sum. The assignment A is made once for that sum. This gives the asserted order of summation and convolution. □

Proof of 26.18

Proof. Fix the exceptional places $S_0 \cup J$ and the archimedean data of 26.14 from π and S. Choose the level K_w of the final induced representation to be $\mathrm{GL}_n(\mathcal{O}_w)$ away from these places and their conjugates. At each exceptional place choose a principal congruence subgroup deep enough that the diagonal Levi blocks of its intersection with each standard block parabolic belong to the prescribed pair level and that the two determinant twists are trivial there. Include the conjugate pair level at a split place. Such a depth exists by openness and the finite number of blocks. The two archimedean weights of 26.5 fix the compact types. These choices precede T.

For the separating tests, now fix T and adjoin an auxiliary good inert place to obtain T'. Use the pair of Iwahori subgroups at every place of T', the prescribed levels at $S_0 \cup J$, and maximal compact subgroups elsewhere. At these levels there are finitely many possible discrete inducing data for either block. Indeed the ranks partition the block rank and the infinitesimal multisets partition the prescribed block multiset at every embedding. For a vector in a parabolic induction fixed by the test level U, evaluation on representatives of $P\backslash G/U$ gives fixed vectors for open subgroups of the Levi. There are finitely many representatives by Iwasawa decomposition and compactness. Restriction of the prescribed compact types to the compact Levi likewise gives a finite list. For each resulting rank, level and compact type, the specified infinitesimal characters give a finite-codimensional ideal in the enveloping-algebra center. At finite level the adelic quotient has finitely many arithmetic components by ideal class finiteness. Harish-Chandra's finiteness theorem on each component gives finite dimension for these data, with the split central character fixed [71, Thm. 8.4]. This bounds the number of discrete representations that occur. Their local inductions have finite length and finite-dimensional fixed spaces. We use this list for the chosen T' throughout the interpolation and the choice of the power m. Changing the test coefficients changes none of its defining levels or types.
Lean: `UpperTheory.HeckeLevelFrameExistence.CF.cls_fin` (`Development.AutomorphicArithmetic.Foundations.BaseChangeIdentities.FrameMatching.HeckeEigenvalueFiniteness.ClsFin`, lines 31–45)

At each inert testing place use the Weyl representative $(\beta_1, \dots, \beta_e)$ of the spherical parameter with $|\beta_1| \geq \cdots \geq |\beta_e| \geq 1$, where $e = \lfloor n/2 \rfloor$. Permutations and inversions in the relative Weyl group give this order. The spherical representation embeds in the unramified principal series in subrepresentation order by the spherical Langlands classification [108, Ch. XI, §2]. Frobenius reciprocity therefore gives the ordered character as a quotient of its normalized Jacquet module. Its generalized character space for the integral-torus invariants is finite-dimensional and contains an eigenline. The Iwahori–Jacquet isomorphism

carries this line back to a common translation eigenline, with the double-coset index accounted for as in Lemma 26.15 [23, Prop. 2.4]. Put $\xi = (\beta_1, \dots, \beta_e, \beta_e^{-1}, \dots, \beta_1^{-1})$, with 1 inserted in the middle for odd n, and let q_0 be the residue cardinality of the base field at this place. Write δ_{B_U} for the modulus character of the upper Borel subgroup of the unitary group. The double coset of the test u_x of Lemma 26.17 is a union of $\delta_{B_U}(t_h)^{-1}$ right cosets, and on the Jacquet image each of them contributes $\delta_{B_U}(t_h)^{1/2} \prod_i \xi_i^{x_i}$. Since $\delta_{B_U}(t_h)^{-1/2} = q_0^{\sum_i (n-1-2i)x_i}$, the test u_x acts on this line by $\prod_i \alpha_i^{x_i}$ with $\alpha_i = q_0^{n-1-2i}\xi_i$ for $0 \le i < n$. Use this α as the character on the comparison exponent lattice.

Lean: `PositivityTheory.Rg.BcB.F.Eig.eigInert` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.InertEigenvalues.EigInert`, lines 8–8)

Lean: `PositivityTheory.Rg.BcB.F.FX.coneEigOf` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.StrictJoinCandidates.IsHeckeFsum`, lines 241–273)

For each block and each allowed assignment of chamber coordinates to the two factors, include the transfer multiplier when pulling back to the common exponent lattice the characters by which the transferred tests act on the candidate translation eigenlines. These form a finite set of characters. Regard them as points of a complex torus. For each point $\gamma \neq \alpha$, choose a coordinate $i(\gamma)$ on which they differ. The Laurent polynomial

$$L(Z) = \prod_{\gamma \neq \alpha} \frac{Z_{i(\gamma)} - \gamma_{i(\gamma)}}{\alpha_{i(\gamma)} - \gamma_{i(\gamma)}}$$

is one at α and zero at the other points. Multiply by Z^v/α^v with v deep enough in the positive chamber that the support has consecutive gaps at least two. Its coefficients give the local test. They fix the target translation eigenvector. A nonzero value on a transferred candidate for any allowed assignment forces equality with α for that character.

At each place of S_0 take the target local representation at the chosen level and list the irreducible subquotients of the finite list of candidate inductions. The inductions have finite length. For a compact open level, the nonzero fixed spaces of the irreducibles are simple modules of its Hecke algebra. Isomorphic simple modules give the same irreducible representation generated by the fixed vectors. The density theorem applied to the finite list of distinct simple modules gives an element acting by a rank-one projection on the target module and zero on every other module in the list. Normalize the test by the level volume for the chosen Haar measure. If all subquotients of a candidate are different from the target, exactness of fixed vectors gives a fixed-space composition filtration on which this operator strictly lowers the filtration. Its power of degree the fixed-space dimension is zero.

Choose a bound m_0 for the dimensions of candidate Iwahori fixed spaces at all the inert testing places, including the auxiliary place. For each $w \in S_0$ choose a bound m_w for the dimensions of the split fixed spaces. Take $m = m_0 + \sup_{w \in S_0} m_w + 1$, with empty supremum zero. The tensor of the local target vectors gives a nonzero v in the π summand with $uv = v$. Each variable generator acts on this summand by a scalar, and its adjoint acts by the conjugate scalar. Their action on $\mathbf{C}v$ defines a unital $*$-character p of $\mathcal{A}$. The reference generator of Lemma 26.14 satisfies $p(a_0) = 1$.

Lean: `UpperTheory.HeckeLevelFrameExistence.frameEx_of` (`Development.AutomorphicArithmetic.Foundations.BaseChangeIdentities.FrameMatching.HeckeEigenvalueFiniteness.KUEqKLOfNotMemBad`, lines 116–229)

At a split spherical place where μ_w is unramified, the normalized constant-term map followed by the determinant twist preserves convolution. At the other split places transfer the complete test by (26.4), with the factor $\mu_w((-1)^{ab})$ evaluated once. At spherical inert places the unit matches by 26.10.

At a bad inert place let $K(j)$ be the principal congruence subgroup of level $j \ge 1$ in K_w. Put $L = E_w$ and let q_0 be the base residue cardinality. Take a base-field element t of valuation j. Write the two norm generators as $1 + tg_i$ with polynomials Q_i of degrees a, b. Their resultants satisfy

$$v(\mathrm{Res}(P_{g_1}, P_{g_2})) = v(\mathrm{Res}(Q_1, Q_2)) - jab. \tag{A.220}$$

The relevant lattice conditions are stability under g_i and under (g_1, g_2). For nonintegral generators the counts vanish. For integral generators choose $x \in O$ with $x - \bar{x}$ a unit and put

$$Y_i = \frac{x\bar{x}\, g_i}{x - \bar{x} + xtg_i}.$$

The norm-one relation gives $\sigma_i(g_i) = -g_i/(1 + tg_i)$ and hence $\sigma_i(Y_i) = Y_i$. The denominator is a unit in the integral order. The inverse transformation has unit denominator $x\bar{x} - xtY_i$, so the orders coincide on the

blocks and the product. The difference of the images of roots u', v' is

$$(u' - v')\,\frac{x\bar{x}(x-\bar{x})}{(x-\bar{x}+xtu')(x-\bar{x}+xtv')}.$$

The second factor is a unit. Thus the lattice counts and the cross-resultant valuation are preserved. Apply (26.6) and (A.220). With $q_w = q_0^2$, the transfer of $1_{K(j)}$ is $(-\sqrt{q_w})^{-jab}$ times the indicator of integral pairs with twisted norms congruent to one modulo ϖ^j, for a uniformizer ϖ. The level projection and its transfer carry the index $[K_w : K(j)]$.

Apply Lemma 26.17 at each inert translation place, including the auxiliary place. Its ordered positive coordinates l_i have opposites $-l_i$ and a middle zero in odd rank. Assign the pairs indexed by A to block a and the rest to block b. The allowed assignments have $|A| = \lfloor a/2 \rfloor$, $|A^c| = \lfloor b/2 \rfloor$ and $(a \bmod 2) + (b \bmod 2) = n \bmod 2$. Other assignments contribute zero. The coefficient of each allowed term multiplies the unnormalized double-coset indicator $1_{I_a t_{x_A} I_a} \otimes 1_{I_b t_{x_B} I_b}$ by

$$[K_a \times K_b : I_a \times I_b](-\sqrt{q_w})^{-v_{\mathrm{Res}}}$$

(equivalently, it multiplies $d_{a,x_A} \otimes d_{b,x_B}$ by $(-\sqrt{q_w})^{-v_{\mathrm{Res}}}$), where K_a, K_b are the integral subgroups, I_a, I_b the Iwahori subgroups and v_{Res} the cross-resultant valuation. Each difference of two roots of distinct valuations has valuation the smaller one. Across two opposite pairs the four differences thus contribute $-2\max(l_i, l_j)$, and a middle coordinate against a pair contributes $-l_i$. On the chamber this is linear in the assigned coordinates, which gives the character multiplier used in the interpolation. The distinct valuations give separability of the product norm polynomial on the auxiliary test support. At this place the fixed kernel is diagonal in the assignments. Its mth convolution power uses products of translations in the same chamber, so the exponents add after normalization by Iwahori volume. A variable finite letter and its adjoint both have the diagonal level idempotent at this position, as in Lemma 26.14.

Lean: `GlobalTheory.CongMatch.match_principal_congruence_inert` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.CongruenceSubgroupIndicatorWeight`, lines 41–66)
Lean: `PositivityTheory.Rg.BcB.F.B3.locMatch` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.WordFrameBounds.PLocMatch`, lines 143–208)

To change twists take $u_r = J_r$ in odd rank and an imaginary scalar times J_r in even rank. Then $\theta_J = \mathrm{Ad}(u_r)\theta_0$ and $u_r\theta_0(u_r) = 1$. The map $\delta \mapsto \delta u_r$ preserves the norm and identifies the twisted classes. Evaluate the finite tests at xu_r^{-1} in each block. At a split pair the two evaluations give the convolution in the split orbital integral, with the twist on the second component.

At infinity apply 26.12 for the two weights, with fixed smooth wave multipliers whose product of values at the target parameters is one. The central operators of the word multiply the target value, and adjoints give the conjugated values. The resulting kernels have the very cuspidal vanishing of 26.11. The auxiliary place gives the separability required by 26.13 for both norm polynomials together. The rational-to-stable factor there is $2^{n|\Sigma_\infty|}$, and the adelic factor is $c_{a,b}$. Thus the signed term of 26.5 contributes the block coefficient $s_I/(c_{a,b}2^{n|\Sigma_\infty|})$.

Lean: `PositivityTheory.Rg.BcB.F.F0.chainEx_of` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.HyperbolicEntryBounds.InfinitesimalWeightZero`, lines 42–88)
Lean: `UpperTheory.BaseChangeIdentities.FrameWordIdentity.traceword_pf` (`Development.AutomorphicArithmetic.Foundations.BaseChangeIdentities.FrameMatching.HeckeWordBounds.PTraceword`, lines 19–60)
Lean: `UpperTheory.BaseChangeIdentities.FrameWordIdentity.wordId` (`Development.AutomorphicArithmetic.Foundations.BaseChangeIdentities.FrameMatching.HeckeWordBounds.FrameBound`, lines 30–40)

Write $g = \sum_j c_j w_j$, where w_j is a word in the variable generators and their adjoints, with the empty word allowed. On the unitary side u commutes with every letter by Lemma 26.14. In block s prefix each w_j by m copies of β_s. The resulting operator is $D_s R_s(w_j)$. Thus for indices $j_1, \dots, j_k$ the exceptional finite positions contain mk forward copies of the fixed kernel. Apply (26.3) with the local matches above to this ordered product. Multiply by $c_{j_1} \cdots c_{j_k}$ and sum. Distributivity gives the middle operator $(D_s R_s(g))^k$ between the two fixed end convolutions $T_{1,s}, T_{2,s}$. At a split testing place use the normalization of Lemma 26.14. Its local factor in $T_{1,s}$ is $Q = M(e_{K'_w})$, with the fixed transfer sign applied once. For every $h \geq 1$,

$$\mathrm{tr}\big(QM(f)^{*h}\big) = \mathrm{tr}\, M(f^{*h}) = \mathrm{CT}(f^{*h}),$$

before that fixed sign and determinant twist. Thus this end factor is independent of h. At the inert places use the fixed level factor and the same-assignment convolution identity established above. Their product

defines $T_{1,s}$. The operator $T_{2,s}$ is convolution with the reference kernel of Lemma 26.14 evaluated at xu_r^{-1} in each block. Both have compactly supported continuous matrix kernels and are independent of k and of the word, as 26.11 requires.

Lean: `PositivityTheory.Rg.BcB.F.MLM.MatchLMB2` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.LeviMatchingWords.MatchLM`, lines 67–69)
Lean: `PositivityTheory.Rg.BcB.F.B3.bw` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.WordFrameBounds.WordIdAt`, lines 66–66)

The concatenations have length at most $k\ell_0$, where ℓ_0 bounds the original word lengths including the repetitions of the fixed test. Matrix multiplication, applied to the entries and inverse entries, bounds the logarithmic support by $r_0 + kr_1$. The summed kernel retains the common level. It also satisfies the auxiliary regularity and archimedean vanishing conditions. Apply (26.9) and absorb the fixed block coefficient into C_s. □

Proof of 26.19

Proof. Fix s. For $a \in \mathcal{A}$, Lemma 26.14 gives $D_s R_s(a) = R_s(a) D_s$. Apply the same equality to a^* and take adjoints. This gives $D_s^* R_s(a) = R_s(a) D_s^*$. Hence $B_s = D_s^* D_s$ commutes with $R_s(a)$. Since B_s is positive and $R_s(\mathcal{A})$ is closed under adjoints, the algebraic unital $*$-algebra they generate is commutative. Its operator-norm closure is $\mathcal{C}_s$. A contractive character on the algebraic algebra extends uniquely to this closure.

Lean: `PositivityTheory.Rg.BcB.F.ExponentialIntegralMeasures.adjoin_comm` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.PeakTraceBoundHolds`, lines 187–215)

Suppose that for each s every character η with $\eta \circ R_s = p$ vanishes on B_s. Put $K_0 = 1 + \sum_s \|R_s(a_0)\|$ and $\delta = |p(a_0)|/(2K_0) > 0$. In the character space of $\mathcal{C}_s$ the set $Y_s = \{\eta : \eta(B_s) \geq \delta^2\}$ is compact. For each $\eta \in Y_s$ choose a with $\eta(R_s(a)) \neq p(a)$. A scalar multiple d of $a - p(a)1$ satisfies $|\eta(R_s(d))| > 1$, an open condition. Take a finite subcover for each s and collect the resulting d's in a finite list. Put $h = \sum_d d^* d$. Then $p(h) = 0$, each $R_s(h)$ is positive, and $\eta(R_s(h)) > 1$ on Y_s.

Choose $M > 1 + \max_s \|R_s(h)\|$ and set $b_L = (1 - h/M)^L$. We have $p(b_L) = 1$. Apply the Gelfand norm formula to $R_s(b_L)^* B_s R_s(b_L)$ in the concrete commutative C^*-algebra just constructed [99, Thm. 2.1.10]. A zero block contributes norm zero. On Y_s we have $0 \leq 1 - \eta(R_s(h))/M \leq 1 - 1/M$. Off Y_s the same quantity lies in $[0, 1]$ and $\eta(B_s) < \delta^2$. Thus

$$\|D_s R_s(b_L)\|^2 \leq \max\{\delta^2, \|B_s\|(1 - 1/M)^{2L}\}. \tag{A.221}$$

There are finitely many blocks, so one L gives $\|D_s R_s(b_L)\| \leq \delta$ for all s. Put $g = a_0 b_L / p(a_0)$. Moving $R_s(a_0)$ past D_s gives $\|D_s R_s(g)\| \leq 1/2$. On the unitary side $A = u\rho(g)$ satisfies $Av = v$. Applying (26.10) to this g and summing the bounds gives

$$|\operatorname{tr}(A^k)| \leq C(1 + k)^N 2^{-k} \qquad (k \geq 1). \tag{A.222}$$

Let $\xi_1, \dots, \xi_r$ be the distinct eigenvalues of A, with algebraic multiplicities m_i. Triangularizing A gives $\operatorname{tr}(A^{k+j}) = \sum_i \xi_i^j m_i \xi_i^k$ for $0 \leq j < r$. All r traces tend to zero. The matrix $(\xi_i^j)_{j,i}$ is an invertible Vandermonde matrix, so every $m_i \xi_i^k$ tends to zero. The eigenvalue 1 has positive multiplicity since $v \neq 0$. This contradiction proves the assertion. □

Proof of 26.20

Proof. Write Y for the rational quotient and Γ for the rational group. For a compact set Ω in the adelic group put $n_\Omega(x) = \#\{\gamma \in \Gamma : x^{-1}\gamma x \in \Omega\}$. Write $k_i(x)$ for the $L^2(Y)$ row, including the finite matrix indices, which represents evaluation of the i-th component of $A_0 h$ at x. Periodizing the compact reference kernel gives these rows. Unfolding their squared norms pairs two translates of its support C. An overlap requires $x^{-1}\gamma x \in CC^{-1}$; each overlap integral is bounded by the squared L^2 norm of an entry by Cauchy–Schwarz. Summing over the finite matrix indices gives $\|k_i(x)\|^2 \leq c\, n_{CC^{-1}}(x)$. Translation continuity of the kernel in L^2 gives continuous rows.

For the two ranks a, b take $M = 2(a+b) > 0$. On a reduction cell with representatives in $t\Omega_0$, the integral of n_Ω over the image of the cell is at most $\operatorname{vol}(\Omega_0 \Omega)$. Indeed, each element counted at a representative x

gives a distinct translate in $t\Omega_0\Omega$, and the sum of its indicator unfolds. By (A.214), H^{-2M} on the cell with indices $l \in \mathbf{Z}^a, l' \in \mathbf{Z}^b$ is bounded by a constant times $(1+\|l\|_\infty)^{-M}(1+\|l'\|_\infty)^{-M}$. Both sums converge, with a single term for a zero rank. Consequently

$$\sum_i \int_Y H(x)^{-2M}\|k_i(x)\|^2\,dx < \infty.$$

Lean: PositivityTheory.Rg.BcB.E1.vecNorm (Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.ThetaCellCovers.KernelRowsColumnsSquareIntegrable, lines 51–51)
Lean: PositivityTheory.Rg.BcB.E1.wtCount (Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.ThetaCellCovers.KernelDiagonalIntegrableHolds, lines 23–23)

Let U be a neighborhood of χ in the character space on which $|\eta(A_0)| > |\chi(A_0)|/2$. The representation of $\mathcal{C}_s$ is faithful by construction. A continuous function supported in U and equal to one at χ therefore has nonzero operator image by the Gelfand representation theorem. Choose a unit vector in its range. The cyclic subspace generated by that vector has the form $JL^2(\nu)$, where J is an isometry and ν is a probability measure carried by U. Indeed, the vector state on continuous functions gives ν, and sending a continuous function to its action on the vector gives the isometry by the definition of the state. Under J every $z \in \mathcal{C}_s$ acts by multiplication by $\eta(z)$.

Use the convention that the inner product is linear in its second argument. The rows $J^*k_i(x)$ give measurable functions $F_i(x,\eta) = \overline{(J^*k_i(x))(\eta)}$, with

$$(A_0Jh)_i(x) = \int F_i(x,\eta)h(\eta)\,d\nu(\eta).$$

The weighted row bound and Fubini give $H^{-M}F(\,\cdot\,,\eta) \in L^2(Y)$ almost everywhere. These functions are nonzero on a set of positive ν-measure, since A_0J is bounded below by $|\chi(A_0)|/2$. For the identity, each variable generator and X, denote the corresponding operator by z. The product zA_0 has a continuous compactly supported matrix convolution kernel, also when z is scalar. Apply $(zA_0)A_0Jh = A_0J(\eta(zA_0)h)$ to a countable dense set of $h \in L^2(\nu)$. Testing on compact subsets of Y and using Fubini gives $(zA_0)F = \eta(zA_0)F$ almost everywhere in both variables. The list of equations is countable. Take one η where they all hold and the weighted vector is nonzero. The equation with $z = 1$ gives the continuous representative $f = A_0F/\eta(A_0)$: convolution of a locally integrable function with the continuous compact reference kernel is continuous. The other equations now hold everywhere by continuity. Convolution associativity and $\eta(A_0) \neq 0$ give $zf = \eta(z)f$. Taking U successively within distance $1/j$ of the prescribed values on the first j generators and on X gives the sequence.

Lean: PositivityTheory.Rg.BcB.E2.oneStep_holds (Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.EigenvalueSeparation.AlmostEverywhereEigen.OrthogonalKeepingHolds, lines 35–35)

□

Proof of 26.21

Proof. Work in the Hilbert space H_s of 26.14. Let $A_0 = R_s(a_0)$ and $X = (\beta_s^m)^*\beta_s^m$. By 26.19, $\chi(A_0) = 1$ and $\chi(X) > 0$. Apply Langlands' spectral decomposition to $\mathrm{Res}_{E/\mathbf{Q}}(\mathrm{GL}_a \times \mathrm{GL}_b)$, omitting any rank-zero factor. This is a connected reductive group over $\mathbf{Q}$; its rational adelic quotient with Haar measure is the one defining H_s. Tensor the decomposition with the fixed finite matrix multiplicity. Its fibres are normalized inductions from discrete representations on the norm-one adelic Levi, extended trivially on the real split centre and then twisted by $\lambda \in i\mathfrak{a}_M^*$. The split centre of the ambient group is included in this parameter since the quotient defining H_s retains it. Right convolutions act fibrewise. These are the decomposition and equivariance assertions of [110, Thm. 7.2].

Enumerate the generators in 26.14, including the differentiated reference operators, and include A_0, X at every stage. For the first j operators, the open joint spectral neighbourhood of radius $1/j$ about their values at χ has nonzero spectral projection. Indeed a continuous function of these coordinates supported in the neighbourhood and taking value one at χ is a nonzero element of the concrete commutative C^*-algebra. The projection is decomposable, so it is nonzero on some induction fibre. For $j \geq 3$, A_0 is invertible on its range there. This puts the range in the finite level and compact types of the reference kernel. That space is finite-dimensional by admissibility. Simultaneous diagonalisation there gives a unit joint eigenvector F_j with eigenvalues in the stated neighbourhood. Write a_j for its A_0-eigenvalue. In particular $a_j \to 1$.

Write $A_0 = WP$ with P the finite level projection and W the infinite convolution. Then $PF_j = F_j$ and $WF_j = a_j F_j$. If t is a finite generator, its finite convolution has eigenvalue a_j^{-1} times the eigenvalue of $R_s(t)$. The infinite kernel is self-adjoint at the reference operator by the real even wave function of 26.12. The finite convolution for X therefore has eigenvalue a_j^{-2m} times its full eigenvalue. A central differential operator applied to W gives its infinitesimal-character value times a_j.

We record how the latter values bound the inducing data. In rank $r > 0$, for either complex central action, write D for the upper triangular matrix with entries $\xi_i + (r-1)/2$ on the diagonal and -1 above it, where $\{\xi_i\}$ is the infinitesimal multiset. For $P_\xi(z) = \det(z - D)$ the matrix determinant lemma gives

$$\frac{P_\xi(z-1)}{P_\xi(z)} = 1 - \sum_{k\ge 0} (\mathbf{1}^t D^k \mathbf{1}) z^{-k-1}.$$

Indeed $D + \mathbf{1}\mathbf{1}^t$ is lower triangular with each diagonal entry increased by one. In the identity obtained by clearing $P_\xi(z)$, the coefficient of z^{r-j-1} determines its j-th nonleading coefficient from the preceding ones: the two multipliers of this coefficient are $r - j$ and $\mathbf{1}^t\mathbf{1} = r$. This works through $j = r$. To identify the central values used here, let c_m be the eigenvalue of $Z_m^{\pm}$ for the chosen complex action and put $c_0 = r$. In the infinitesimal-character normalization, the shifted column Capelli determinant has eigenvalue

$$Q(u) = \prod_i \big(u + \xi_i - (r-1)/2\big).$$

The quantum Newton identity gives the formal Laurent expansion

$$\frac{Q(u+1)}{Q(u)} = 1 + \sum_{m\ge 0} (-1)^m c_m (u - r + 1)^{-m-1}.$$

For this identity, expand the shifted column determinant by a column and multiply its cofactor matrix by the truncated geometric inverse of $(u - r + 1)I + E^{\pm}$. The relations $[E^{\pm}_{ab}, E^{\pm}_{cd}] = \delta_{bc}E^{\pm}_{ad} - \delta_{da}E^{\pm}_{cb}$ give the successive column shifts. Taking the trace gives $Q(u+1) - Q(u)$; the remainder at truncation order m has polynomial degree at most $r - 1$, so comparison of the degree-r coefficient gives the displayed recursion for c_m. Now $Q(r - 1 - z) = (-1)^r P_\xi(z)$. Substitution of $u = r - 1 - z$ and comparison with the preceding determinant identity give $c_m = \mathbf{1}^t D^m \mathbf{1}$. The nonleading coefficient indexed by j is therefore a polynomial in $c_1, \dots, c_j$, obtained by division by the nonzero integer j. Consequently convergence of these central values gives convergence of the characteristic polynomials and of the root multisets.

Lean: `GlobalTheory.T.trace_elements_act_by_gelfand_values` (`Development.AutomorphicSpectrum.Proofs.BlockDecomposition.TraceElementsActByGelfandValuesStatement`, lines 9–10)

Their limits are the shifted block weights by 26.5 and 26.12.

Lean: `PositivityTheory.Rg.BcB.F.Rd.B.infLim` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.FrameReading.GelfandValueContinuity.InfLim`, lines 76–116)

Pass to a subsequence with fixed Levi ranks. Each discrete factor of positive rank r has a norm twist $|\det|_{\mathbb{A}_E}^{z_j}$ with $\operatorname{Re} z_j = 0$, by the domain of integration in the spectral decomposition. Its normalized discrete representation is trivial on its real split centre. Hence the sum of the imaginary parts of its infinitesimal parameters over the embeddings of E is zero. After twisting, the sum is $r[E : \mathbf{Q}] \operatorname{Im} z_j$. Its summands occur in the bounded block multisets; changing from normalized to unnormalized induction adds fixed real shifts. This bounds z_j, and then bounds the parameters of the normalized discrete factor.

The inducing levels and compact types belong to finite lists. At finite level L_0 this follows by evaluating an induced section on the finite set $P_M \backslash G / L_0$ at the finite places, using the compact model and maximal level almost everywhere. At infinity restrict the prescribed compact types to the compact Levi. With these levels and types fixed, there are finitely many normalized discrete factors in the bounded parameter set. The finite adelic level has finitely many arithmetic components. The nonsplit central directions modulo arithmetic units are compact after the real split centre is removed, so their bounded characters form a finite set. On the semisimple components the Casimir formula bounds the homogeneous-bundle Laplacian on the indicated compact types. The arithmetic discrete-spectrum bound of [109, Thm. 1.5.2] applies to each such bundle; passage to a torsion-free subgroup of finite index gives the same finiteness for the original component. This proves the assertion also for residual discrete factors. In rank one the central-character argument suffices.

Take a subsequence on which these discrete factors are fixed and all z_j converge. In the compact model the level-and-type space is now one fixed finite-dimensional space: norm twists are trivial on the compact subgroups. Its unit vectors F_j have a convergent subsequence with unit limit F. Each finite kernel in question is a finite sum of coset operators at the chosen level. Its matrix entries in this model are continuous in the twists, since each translate introduces a fixed positive norm raised to the twist parameter. Thus all the finite eigenvalue equations pass to F, with the two ratios in the statement.

Each limiting twist is purely imaginary. Every limiting parameter of the twisted factor is real, since it occurs in the limiting block multiset up to a real shift. Summing its imaginary parts gives $r[E:\mathbf{Q}]\operatorname{Im} z = 0$. Hence every limiting twist is zero. The fixed discrete factors give P_a, P_b with the required infinitesimal characters. Finally the eigenvalue of $(b_s^{\dagger})^{*m} * b_s^{*m}$ on F is positive by 26.19, so $b_s^{*m} F \neq 0$. □

Proof of 26.22

Proof of Lemma 26.22. Apply Lemma 39.49 with $S = \varnothing$ and $e_v = 1$ at every chosen embedding to obtain μ.

Lean: `PositivityTheory.Rg.BcB.F.Fok.Mu3.muExists` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.QuadraticCharacterNorms.QuadCharNorm`, lines 24–56)

Fix the prescribed split set S and a finite inert spherical set T. Apply 26.18, 26.19 and 26.21. For the resulting block put

$$D_T = (P_a \otimes (\mu^b \circ \det)) \boxplus (P_b \otimes (\mu^a \circ \det)),$$

where $\boxplus$ concatenates discrete components. The twists add $be_v/2$ and $ae_v/2$ to the respective infinitesimal multisets, so their union is $H(\lambda_v)$. Fix an inert testing place. The finite kernel factors over the places, and its matrix at this place is diagonal in the assignments. If every diagonal entry had zero mth power on the corresponding vectors, the global mth power would vanish. Choose a surviving entry. It has an allowed assignment, since all other entries are zero. On the local product representation the corresponding Iwahori fixed space has dimension at most m by the choice in 26.18. To pass to that space, hold the other local variables fixed and apply the separating equivariant maps from the induced realization to the local product of the inducing representations. These maps intertwine each coset sum, so at least one detects the nonzero power. Write Q for the resulting linear combination of positive-chamber translations. Since $Q^m \neq 0$, Cayley–Hamilton gives an eigenvalue $z \neq 0$ of Q. Its eigenspace is invariant under every positive-chamber translation by their addition law. In it choose a nonzero common invariant subspace of least dimension. The eigenspace there of any one translation is nonzero and invariant under the others, hence is the whole subspace. A common eigenvector therefore gives a chamber character at which the separating test has value $z \neq 0$.

Lean: `PositivityTheory.Rg.BcB.F.Tail.rReadT` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.FrameReading.PairDoubleCosetFiniteSupport`, lines 278–338)

Lean: `GlobalTheory.CompanionTheory.LinAlg.exists_joint_eigenvector_combination_ne_zero` (`Development.AutomorphicSpectrum.Proofs.BlockDecomposition.ExistsJointEigenvectorNonzero`, lines 85–109)

Interpolation identifies the assigned character with the ordered target. To read the individual factors, use normalized chamber translations on the two Iwahori spaces. On their tensor product the eigencharacter for a pair of translations is the product of the prescribed two character values. Simultaneous triangularization on each finite-dimensional factor shows that a pair of its diagonal characters has those same products for every pair of strictly dominant translations. Each individual character is thereby determined up to a constant, and applying its multiplication law to a translation and its square makes that constant one. Each character occurs on an eigenline in its joint generalized eigenspace. After the modulus normalizations of the translations, an allowed assignment takes the target coordinates in their original order within each block. The remaining transfer signs have absolute value one, as do the unramified twists. Thus the two untwisted lists separately have nonincreasing absolute values. Apply Lemma 26.15 in each positive-rank factor. Every inducing constituent at w has a hyperspecial fixed vector; a rank-zero block contributes the empty list. The unramified twists preserve those vectors. Their tensor gives a hyperspecial fixed section of the joined induction by Iwasawa decomposition. Undoing the chamber multipliers gives the two block Satake multisets multiplied by $\mu_w(\varpi_w)^b$ and $\mu_w(\varpi_w)^a$, respectively.

Lean: PositivityTheory.Rg.BcB.F.PCandLh.inert_d (Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.FrameReading.PCandLh, lines 48–121)
Lean: PositivityTheory.Rg.BcB.F.PCandLh.candLev_of (Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.FrameReading.PCandLh, lines 149–211)

Their union equals the opposite Satake pairs of π_w, with the middle entry 1 in odd rank. The split spherical tests give matching at almost every split place. At the prescribed split place a candidate whose subquotients were all different from the target would have zero $(m-1)$st power by the filtration argument of 26.18. The nonzero power gives the required subquotient. At an untested good place the pair level is maximal compact and the twists are unramified. At an exceptional place use the subgroup K_w chosen before T. In either case the diagonal blocks of $P \cap K_w$ fix the inducing vector and the twists are trivial on their determinants. A section supported on PK_w, taking that vector as its value at the identity, is well defined and K_w-fixed. Here one obtains a nonzero inducing vector by evaluating the nonzero pair vector in its induced realization, with its prescribed pair level; at a conjugate split place use the conjugate pair level. Together with the hyperspecial sections at the tested places, this gives the same adelic level K for every surviving D_T. The archimedean types were fixed with the two block weights. Thus D_T matches at $S \cup T$ and satisfies the fixed-level and type conditions needed for finiteness.
Lean: UpperTheory.BaseChangeIdentities.GelfandReading.frameRead (Development.AutomorphicArithmetic.Foundations.BaseChangeIdentities.FrameMatching.GelfandReading.FrameRead, lines 16–17)
Lean: PositivityTheory.Rg.BcB.F.runT_of (Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.SetupEx, lines 64–72)
Lean: PositivityTheory.Rg.BcB.Plug2.bci_of (Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.BciOf, lines 22–26)

For fixed S there are finitely many rank partitions and partitions of the prescribed infinitesimal multisets. A K-fixed vector in an induction gives a uniform open level on the inducing factors by restriction to the Levi in the finitely many compact double cosets. Normalize the split center and fix the levels and compact types. Cuspidal finiteness gives finitely many constituents in a bounded infinitesimal range [98]. For discrete constituents the residual classification reduces finiteness to the cuspidal case [95]. The ranks bound the string lengths, which determine the shifts of the cuspidal parameters. Restriction to the Levi bounds the levels. Hence D_T ranges over finitely many classes up to permutation and local equivalence. If every class failed at some inert spherical place, picking one such place for each would give a finite T with no match. Thus one family for each S matches at all inert spherical places and at S.
Lean: PositivityTheory.Rg.BcB.F.F5.runFin_of (Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.FiniteDiscreteFamilies.RunFinOf, lines 42–65)

Take $S = \varnothing$ to obtain a family D. For each split w repeat with $S = \{w\}$. The two families have the same Satake union almost everywhere. Mœglin and Waldspurger write the discrete constituent on GL_{ad} as a string of length d on a unitary cuspidal representation of rank a [95]. For $t_j = (d-1)/2 - j$, the normalized embedding uses $-t_j$ and the square root of the parabolic modulus contributes at_j on block j. Thus the unnormalized exponents are $(a-1)t_j$.

Jacquet–Shalika rigidity recovers the cuspidal support from the cofinite Satake union [13]. For each unitary cusp form the shifts are centered intervals of step one. Their multiset union determines the string lengths by removing a longest interval and repeating. This identifies the local inducing factors of the two families. Permuting them preserves the class of the finite-length induction in the local Grothendieck group, so the split subquotient occurs for D. At inert spherical places take the spherical subquotient, and at the remaining inert places take an irreducible subquotient of the nonzero induction.

The cofinite Satake union is invariant under conjugate duality by inert reciprocity and inverse transpose at split pairs. Rigidity pairs the strings with their conjugate-dual strings of the same length. At each embedding the paired unitary cusp parameters are complex conjugates and are real because their real string shifts lie in $H(\lambda_\tau)$. Distinct paired strings would produce a repeated entry there, whereas successive entries differ by $\lambda_j - \lambda_{j+1} + 1 \geq 1$. Thus each string is conjugate self-dual. □

Proof of 26.23

Proof. At an embedding $\tau : F \hookrightarrow \mathbf{C}$, the infinitesimal multiset of π is

$$H(\lambda_\tau) = \left\{ \lambda_{\tau,k} + \tfrac{n-1}{2} - k : 0 \leq k < n \right\} \subset \tfrac{n-1}{2} + \mathbf{Z}.$$

Its entries are distinct because $\lambda_{\tau,k}$ is weakly decreasing. By Lemma 26.22, in unnormalized parabolic induction the i-th component embeds in the induction of $\rho_i|\det|^{(a_i-1)((d_i-1)/2-j)}$ $(0\le j<d_i)$; subtracting the half-modulus exponent $a_i((d_i-1)/2-j)$ on block j, it embeds in the normalized parabolic induction of $\rho_i|\det|^{j-(d_i-1)/2}$ $(0\le j<d_i)$, so its infinitesimal multiset at τ is the union of the shifts $m(\rho_i,\tau)+j-(d_i-1)/2$, equivalently $\bigcup_{j=0}^{d_i-1}(m(\rho_i,\tau)+(d_i-1)/2-j)$. Since this union is a submultiset of $H(\lambda_\tau)$, the entries of $m(\rho_i,\tau)$ are distinct and lie in

$$\tfrac{n-1}{2}-\tfrac{d_i-1}{2}+\mathbf{Z}=\tfrac{n-d_i}{2}+\mathbf{Z}.$$

Let $\eta=\eta_{F/F^+}$ be the quadratic Hecke character of F/F^+. Since F/F^+ is CM, $\eta_v(-1)=-1$ at every real place v of F^+, while $\eta\circ N_{F/F^+}=1$. Thus $\chi=\eta^{a_i}$ satisfies $\chi_v(-1)=(-1)^{a_i}$ and $\chi\circ N_{F/F^+}=1$. Let $\varepsilon_\ell:G_F\to\bar{\mathbf{Q}}_\ell^\times$ be the ℓ-adic cyclotomic character, which is the realization of $|\cdot|_F$ in Proposition 26.26(9); it sends inverse arithmetic Frobenius at $w\nmid\ell$ to q_w^{-1} and has filtration jump -1 at every embedding $\tau:F\hookrightarrow\bar{\mathbf{Q}}_\ell$.

First suppose $n-a_i-d_i+1$ is even. Then $(n-d_i)/2\equiv(a_i-1)/2 \pmod 1$, so the distinct weights of ρ_i lie in $(a_i-1)/2+\mathbf{Z}$. Since ρ_i is conjugate self-dual, (ρ_i,χ) is a polarized input for Theorem 39.2. Let $r_i:G_F\to\mathrm{GL}_{a_i}(\bar{\mathbf{Q}}_\ell)$ be the resulting representation, and for $0\le j<d_i$ set

$$m_j=\frac{n-a_i-d_i+1}{2}+j\in\mathbf{Z},\qquad r_{i,j}=r_i\otimes\varepsilon_\ell^{-m_j}.$$

Because $(1-a_i)/2-m_j=(d_i-1)/2-j+(1-n)/2$, twisting the local comparisons of Definitions 39.5–39.6 for r_i by $\varepsilon_\ell^{-m_j}$ yields assertions (1) and (2): at $w\nmid\ell$ the Weil–Deligne parameter is twisted by $|\cdot|_w^{-m_j}$, multiplying Frobenius eigenvalues by $q_w^{m_j}$ to give $q_w^{(a_i-1)/2+m_j}B_w(\rho_i)=q_w^{(n-d_i)/2+j}B_w(\rho_i)$, and at $w\mid\ell$ each filtration jump $(a_i-1)/2-z$ increases by m_j to $(n-d_i)/2+j-z$.

Now suppose $n-a_i-d_i+1$ is odd. Then $(n-d_i)/2\equiv a_i/2 \pmod 1$, so the weights of ρ_i lie in $a_i/2+\mathbf{Z}$. Fix a CM type Σ of F/F^+, and write $\epsilon(\tau_0)=1$ if $\tau_0\in\Sigma$ and $\epsilon(\tau_0)=-1$ if $\tau_0\notin\Sigma$. Given any finite set S of finite places of F, Lemma 39.49 with $e_v=1$ at every complex place provides a unitary Hecke character μ of F unramified on S, restricting to η on $\mathbf{A}_{F^+}^\times$, with $\mu_v(z)=z/|z|$ relative to $\tau_0\in\Sigma$. Since $\eta\circ N_{F/F^+}=1$, we have $\mu(x)\mu(c(x))=\eta(N_{F/F^+}(x))=1$, so $\mu^c=\mu^{-1}$.

Put $\tilde\rho_i=\rho_i\otimes(\mu\circ\det)$. Then $\tilde\rho_i$ is cuspidal on $\mathrm{GL}_{a_i}(\mathbf{A}_F)$ and satisfies

$$\tilde\rho_i^c\simeq\rho_i^\vee\otimes(\mu^{-1}\circ\det)=\tilde\rho_i^\vee=(\tilde\rho_i\otimes(\chi\circ N_{F/F^+})^{-1})^\vee.$$

At each embedding $\tau_0:F\hookrightarrow\mathbf{C}$, the infinitesimal weights of $\tilde\rho_i$ are $z+\frac12\epsilon(\tau_0)$ for $z\in m(\rho_i,\tau_0)$; they are distinct and lie in $(a_i-1)/2+\mathbf{Z}$. Thus $(\tilde\rho_i,\chi)$ is a polarized input for Theorem 39.2. Let $\tilde r_i:G_F\to\mathrm{GL}_{a_i}(\bar{\mathbf{Q}}_\ell)$ be the attached representation.

Consider the algebraic Hecke character $\psi_0=\mu|\cdot|_F^{1/2}$ of F. At $\tau_0\in\Sigma$, $\psi_{0,\tau_0}(z)=(z/|z|)(z\bar z)^{1/2}=z$, so ψ_0 has integral infinity type $n_{\tau_0}=1$ for $\tau_0\in\Sigma$ and $n_{\tau_0}=0$ for $\tau_0\notin\Sigma$. By Proposition 26.26(9), ψ_0 admits a continuous ℓ-adic realization $\psi_\mu:G_F\to\bar{\mathbf{Q}}_\ell^\times$ which is de Rham and potentially crystalline at every $w\mid\ell$ (crystalline when μ_w is unramified), has filtration jump $-n_{\iota\circ\tau}$ at $\tau:F\hookrightarrow\bar{\mathbf{Q}}_\ell$, and has Weil–Deligne character $(\mu_w|\cdot|_w^{1/2})\circ\mathrm{rec}_w$ with zero monodromy. For $0\le j<d_i$ set

$$m_j=\frac{n-a_i-d_i}{2}+j\in\mathbf{Z},\qquad r_{i,j}=\tilde r_i\otimes\psi_\mu^{-1}\otimes\varepsilon_\ell^{-m_j}.$$

Because $n-a_i-d_i$ is even, m_j is an integer, and $-a_i/2-m_j=(d_i-1)/2-j+(1-n)/2$. For every finite place $w\nmid\ell$ at which μ_w is unramified (in particular every $w\in S$ with $w\nmid\ell$), the character twist $\mu_w|\cdot|_w^{1/2+m_j}$ is unramified and hence trivial on inertia I_w and on Weil commutators; thus zero monodromy and the toral and full tame tests of Definition 39.5 are preserved between $\tilde\rho_{i,w}\otimes|\det|_w^{(1-a_i)/2}$ and $(\rho_{i,w}|\det|_w^{(d_i-1)/2-j})\otimes|\det|_w^{(1-n)/2}$, and untwisting $\tilde r_i$ by $\psi_\mu^{-1}\otimes\varepsilon_\ell^{-m_j}$ twists the Weil–Deligne parameter by $\mu_w^{-1}|\cdot|_w^{-1/2-m_j}$, matching

$$\big(\tilde\rho_{i,w}\otimes|\det|_w^{(1-a_i)/2}\big)\otimes(\mu_w^{-1}|\det|_w^{-1/2-m_j})=\big(\rho_{i,w}|\det|_w^{(d_i-1)/2-j}\big)\otimes|\det|_w^{(1-n)/2}.$$

When $\rho_{i,w}$ is also spherical at such a place $w\nmid\ell$ where μ_w is unramified, inverse arithmetic Frobenius acts on $r_{i,j}$ with eigenvalues $q_w^{(a_i-1)/2}\mu_w(\varpi_w)B_w(\rho_i)\cdot\mu_w(\varpi_w)^{-1}q_w^{1/2}\cdot q_w^{m_j}=q_w^{(n-d_i)/2+j}B_w(\rho_i)$. At every $w\mid\ell$ in S, μ_w is unramified, so $\tilde\rho_{i,w}$ is spherical if and only if $\rho_{i,w}$ is spherical, the character $\psi_\mu|_{G_{F_w}}$ is

crystalline (preserving crystallinity, potential semistability, zero monodromy, and the period comparison of Definition 39.6), and the filtration jumps of $r_{i,j}$ at $\tau : F \hookrightarrow \bar{\mathbf{Q}}_\ell$ are

$$\left(\frac{a_i - 1}{2} - z - \frac{\epsilon(\iota \circ \tau)}{2}\right) + n_{\iota\circ\tau} + m_j = \frac{n - d_i}{2} + j - z,$$

since $n_{\iota\circ\tau} - \epsilon(\iota \circ \tau)/2 = 1/2$ for both $\iota \circ \tau \in \Sigma$ and $\iota \circ \tau \notin \Sigma$.

For any two choices μ, μ' corresponding to finite sets S, S', the resulting continuous semisimple representations $r_{i,j}$ and $r'_{i,j}$ have identical Frobenius characteristic polynomials at every finite place $w \nmid \ell$ outside the finite ramification locus of ρ_i, μ, μ'. By Chebotarev density [41, Ch. VII, §13] and Brauer–Nesbitt semisimplicity, $r_{i,j} \simeq r'_{i,j}$. Choosing S to contain any given finite place w together with all places above ℓ therefore establishes (1) and (2) at every finite place w for one fixed representation $r_{i,j}$.

Finally, summing over all (i, j) gives r_π of dimension $\sum_i a_i d_i = n$. At any finite place w where π is unramified, the normalized parabolic induction $\Pi_w = \bigtimes_i \bigtimes_{j=0}^{d_i-1} \rho_{i,w}|\det|_w^{j-(d_i-1)/2}$ has the spherical subquotient P_w of Lemma 26.22. Exactness of $\mathrm{GL}_n(\mathcal{O}_w)$-invariants gives $\Pi_w^{\mathrm{GL}_n(\mathcal{O}_w)} \neq 0$, and evaluation at the identity via the Iwasawa decomposition $\mathrm{GL}_n(F_w) = P(F_w)\,\mathrm{GL}_n(\mathcal{O}_w)$ identifies $\Pi_w^{\mathrm{GL}_n(\mathcal{O}_w)}$ with $\bigotimes_{i,j} \rho_{i,w}^{\mathrm{GL}_{a_i}(\mathcal{O}_w)}$. Hence every $\rho_{i,w}$ is spherical, and $B_w(P_w) = \bigsqcup_i \bigsqcup_{j=0}^{d_i-1} q_w^{(d_i-1)/2-j} B_w(\rho_i) = \bigsqcup_i \bigsqcup_{j=0}^{d_i-1} q_w^{j-(d_i-1)/2} B_w(\rho_i)$. By (2), r_π is crystalline at w when $w \mid \ell$; by (1), when $w \nmid \ell$ it is unramified at w with inverse Frobenius eigenvalue multiset $\bigsqcup_{i,j} q_w^{(n-d_i)/2+j} B_w(\rho_i) = q_w^{(n-1)/2} B_w(P_w)$. At each $\tau : F \hookrightarrow \bar{\mathbf{Q}}_\ell$, the union of the multisets $(n-1)/2 - (z + (d_i - 1)/2 - j)$ equals $(n-1)/2 - H(\lambda_{\iota\circ\tau}) = \{k - \lambda_{\iota\circ\tau,k} : 0 \le k < n\}$. □

Proof of 26.24

Proof. For (1) and (2) use the finiteness theorem and the geometric lemma for smooth representations, with both functors normalized [14, §§2–3, in particular Lemma 2.12]. In the integral for the duality pairing, the two half-modulus factors give the modulus in the integration formula for $P\backslash G_n$. Additivity along the filtration gives the multiplicity formula. The segment subrepresentation, its Jacquet module and the polynomial generators are given in [16, §§3,6–7]. Assertion (4) is Bernstein's unitary product theorem [15, §8.2].

For (5), take the Zelevinsky multisegments of π_1, π_2. Every term of a segment is a member of the corresponding cuspidal support. If two segments from different multisegments were linked, then either they would share a term, or the last term of one would be adjacent to the first term of the other on a cuspidal line. These alternatives give respectively $j = 0$ and $j = \pm 1$ in the excluded isomorphism. Zelevinsky's product formula for multisegments with no linked pair across the two factors gives the irreducible indexed by their union [16, Prop. 8.5]. In particular the hypothesis holds in 1.20 and 36.106 when the two supports occupy disjoint unramified twist classes.

The segment quotient and the generic product description are [16, §9, in particular Thm. 9.7]. Products of irreducible unitarizable representations, for instance of square-integrable factors in unitary normalization, are irreducible and unitarizable [219, Theorem 2.5(i)]. For (6), order the Langlands segment factors by weakly decreasing real exponents $e_1 \ge \cdots \ge e_m$, where the modulus of the central character of the i-th factor of rank $d_i > 0$ is $|\cdot|_K^{d_i e_i}$. The dual Langlands quotient description embeds the tempered representation in the reverse-ordered induction. By normalized Frobenius reciprocity the reverse-ordered tensor occurs in its Jacquet module. Casselman's exponent criterion gives $d_m e_m \ge 0$ for the last factor in the original order, and the unitary central character gives $\sum_{i=1}^m d_i e_i = 0$. Since $e_i \ge e_m$ and $d_i > 0$, all e_i vanish. These essentially square-integrable factors are therefore square-integrable, and their irreducible product equals its Langlands quotient. Conversely, the geometric lemma gives the Jacquet exponents of a product of square-integrable factors. Their unitary central characters and square-integrability inequalities give a unitary central character and Casselman's tempered inequalities for the product.

Lean: `UpperTheory.Lit.Cas.temperedProduct` (`Development.AutomorphicArithmetic.ThetaCorrespondence.FockModels.GlnTemp`, lines 22–47)

For (8), the standard ordering gives the Langlands quotient. The classification gives uniqueness and exhaustion. The constituent for the sum occurs in the product by the multisegment multiplication formula, equivalently by passing to the dual parametrization in the Grothendieck ring [16, Prop. 8.4][230, §4.3, Thm. 3, and §5.3, Thm. 7]. □

Proof of 26.25

Proof. In (1), J is the inverse image of the unipotent radical of the indicated parabolic in the finite general linear group. To apply the depth-zero Jacquet isomorphism, test both sides against the representations of $M(\mathcal{O}_K)/J_M$ induced from cuspidals of its finite Levi subgroups. These induced representations contain every irreducible of that finite group. For a test induced from a smaller Levi M_0, finite-group Frobenius reciprocity identifies its multiplicity space on the left with the space for the cuspidal type on the refined parahoric. Apply the Moy–Prasad isomorphism for that type in G_n and in M. Transitivity of Jacquet restriction identifies the two resulting spaces on M_0. Both maps are induced by the coinvariant projection, and modulus characters are trivial on the compact Levi subgroups. This gives (1) [221, Prop. 6.7].

For (2), choose an ordering of the support for which its exterior tensor product occurs in a Jacquet module of V. Such an ordering exists by cuspidal support and the geometric lemma of 26.24. Tensoring the given fixed vectors gives a nonzero J_M-invariant. Fixed vectors under a compact subgroup are an exact functor over $\mathbf{C}$, by averaging. Thus the Jacquet module has a nonzero J_M-invariant. Its lift under (1) is fixed by $K_n^1 \subset J$.

The converse theorem in (3) is [132, Conj. 1.1 and Thm. 1.3]. To check the application, take a generic test representation in rank m. Each supercuspidal in its support has rank smaller than $m+1$. The cuspidal pair formula gives both local L-factors equal to one, also for the dual pair. The sign from the central character of the test is common to the two Whittaker equations. Applying the hypothesis to every twist $\tau|\det|^z$ therefore gives equality of the gamma factors as functions of s in rank m. We use the pair functional equation and multiplicativity under normalized induction [226, §§2,3,8].

For a unitary character α of $K^\times$, let $R(\alpha)$ be the ratio of the two gamma factors with test α, in the field of rational functions of q^{-s}. Fix another unitary character ξ. The normalized products of m unitary characters are irreducible and generic by 26.24(4) and the Whittaker model for induction [16, Thm. 9.7]. Testing with m copies of ξ, and then replacing one copy by α, gives $R(\xi)^m = 1$ and $R(\alpha)R(\xi)^{m-1} = 1$. Hence $R(\alpha)$ has one common value for all unitary characters.

Write ω_i for the central character of π_i. For every $c \in K^\times$ of sufficiently negative valuation there is a unitary character α_c of sufficiently large conductor with $R(\alpha_c) = \omega_2(c)/\omega_1(c)$. This is the highly ramified twist formula and the character construction in [133, Prop. 2.6 and proof of Cor. 2.7]. In conductor-zero normalization one prescribes $\alpha_c(1+x) = \psi(cx)$ for $v(x) \geq \lfloor -v(c)/2 \rfloor + 1$ and extends to the units and then unitarily to $K^\times$. For a scaled additive character the change-of-character factor in the pair functional equation gives the same ratio with c measured against that character. Given $z \in K^\times$, choose c so that both c and cz are in this range. Constancy of the ratio gives $\omega_1(z) = \omega_2(z)$. The common value of R is consequently one.

Finally take any generic irreducible τ of rank $1 \leq r \leq \lfloor (m+1)/2 \rfloor$. If $r = m$, the equality is already known. Otherwise choose a unitary character α with conductor greater than the conductors of all rank-one members of the cuspidal support of τ. Its unramified twists are disjoint from those members. Appending $m-r$ copies of α by normalized induction gives an irreducible generic test in rank m, by 26.24(4),(5) and the generic induction criterion. Multiplicativity gives the desired equality for τ after cancellation of the nonzero rank-one factors. This verifies every twist and the central-character condition.

For (4), apply Brauer induction to 1_G, and multiply by the virtual character on the left. The tensor identity reduces the assertion to degree-zero virtual characters of elementary subgroups, allowing twists by linear characters [36, Ch. 10]. An elementary group is a product of a cyclic group and a finite p-group. Its irreducible characters are monomial. Modulo the subgroup generated by inductions of linear-character differences, each monomial $\mathrm{Ind}_H^E \lambda$ equals $[E:H]1_E$. Indeed one first replaces λ by 1_H. In a subnormal chain from H to E with cyclic prime-order factors, the induction of the trivial character at each step is the sum of all characters of the cyclic quotient. Each summand is congruent to the trivial character. Transitivity of induction proves the assertion along the chain. Hence each degree-zero combination vanishes in the quotient.

Assertion (5) is Theorem 32.31. For (6), apply Chevalley's congruence theorem to the finitely generated subgroup $\mathcal{O}_E^\times \subset E^\times$. Its multiplicative congruence modulo a for an integral unit is exactly $u - 1 \in a\mathcal{O}_E$ [100, pp. 36–37]. □

Proof of 26.26

Proof. For (1)–(3) apply cuspidal rapid decrease on Siegel sets and Harish-Chandra finiteness with the given compact types and central eigensystem [98][71, Thm. 8.4]. A finite set of Siegel sets covers the quotient at each fixed finite level. A bound there by each inverse power of height gives the stated bound with the infimum over rational translates. In (3) the absolute value has already been made constant in the split central direction. The cuspidal discrete decomposition gives the asserted constituents. Assertion (4) is the isobaric uniqueness theorem and the strict local bound of Jacquet and Shalika [13]. For (5), the transform and the contragredient have reciprocal Satake roots at every common spherical place. Apply (4) to the two singleton lists and transport the evaluation pairing. Item (6) applies Theorem 39.2 to the polarized cuspidal representation in its statement. Item (7) is Clozel's purity lemma and the cohomology calculation for regular algebraic type [21, Lem. 4.9 and §3.5].

For (8) use global and local reciprocity with the inverse arithmetic convention [41, Ch. VI]. Dividing by the positive archimedean factors and the finite ray unit groups gives the indicated narrow ray group. Characters separate the elements of this finite abelian group. In (9), locally at ℓ, the character on an open subgroup of the units is the algebraic character with the prescribed exponents. The remaining smooth factor has finite inertia. The rank-one period comparison for locally algebraic characters gives the asserted jumps and Weil character. Over the extension killing the finite inertia factor the comparison is crystalline, and this already holds over E_v when the local smooth factor is unramified [206, Ch. III, Appendix][191]. Item (10) is Berger's p-adic monodromy theorem [101]. □

Proof of 26.27

Proof. Local finiteness, local duality and the local Euler characteristic are the finite-module theorems of [111, Ch. I, §2], also in the completion formulation of 36.14. To see (2), A is unramified by hypothesis. Since $v \nmid \ell$, inertia also fixes μ_{ℓ^r} and hence acts trivially on $A^\dagger$. Their unramified classes inflate from the procyclic residue Galois group, of ℓ-cohomological dimension one, so their products vanish. For either module B, the unramified subgroup is the cokernel of $\mathrm{Frob}_v - 1$ on B, of order $|H^0(F_v, B)|$. By (1) and local duality, the product of the two orders is $|H^1_v(A)|$. This proves equality with both annihilators [111, Thm. I.2.6].

For (3) apply the global finiteness, Euler characteristic, and Poitou–Tate duality theorems for a finite module with S containing the places dividing its order [111, Ch. I, §§4–5], normalizing the local invariants with the uniformizer Kummer class first in the cup. Each complex place contributes $|A|$ to a_∞. For (4) let $B = \bigoplus_{v\in S} H^1_v(A)$, D be the image of global localization in B, and $L = \bigoplus L_v$. By exactness of the nine-term sequence in (3), the dual localization image is $D^\perp$. The identity

$$|D \cap L| / |D^\perp \cap L^\perp| = |D|\,|L|/|B|$$

follows by taking the annihilator of $D + L$. Including the kernels of localization gives the ratio

$$\frac{h_1\,|L|}{|B|\,|\mathrm{Ш}^1_S(A^\dagger)|}.$$

Exactness in degree two gives $h_2 = |\mathrm{Ш}^1_S(A^\dagger)| \prod_{v\in S} |H^2_v(A)| / |H^0(F, A^\dagger)|$. Substitution of (1) and (3), with $\sum_{v|\ell} [F_v : \mathbf{Q}_\ell] = [F : \mathbf{Q}]$, gives (4). Assertion (5) is Bloch–Kato's local dimension and duality theorem with its lattice and divisible-quotient conditions [120, §3]. □

Proof of 26.28

Proof. The bound on divided-power products in (1) makes the thickening locally divided-power nilpotent. Apply Grothendieck's deformation theorem for abelian schemes [102, Ch. V, §4]. The equivalence there identifies the lift with the filtered crystal. Essential surjectivity gives the prescribed summand, and full faithfulness gives the assertion for a specified homomorphism. Dualizing the cohomological filtration gives the homology formulation.

The de Rham calculation and base change in (2) are [103, Prop. 2.5.2]. Primitivity also follows by pulling back along addition: on each of the two coordinate axes the pullback is the original class, and degree

one on the product is the direct sum of the two degree-one groups. The Poincaré pairing identifies the homology of the dual with the dual of the homology of A, with the two Hodge filtrations annihilating one another. Pullback by the homomorphism of a line bundle gives isotropy, and a principal polarization is an isomorphism. This proves the pairing assertion and uses the duality and biduality of [176, Ch. I, §1]. For the comparison in (3), use smooth coordinates locally on the lift. In the divided-power envelope of the diagonal the relative coordinate differences have divided powers $u^{[j]}$. In each coordinate the augmentation to the relative de Rham complex is a resolution: contraction in positive degree sends $u^{[j]}du$ to $u^{[j+1]}$, since $du^{[j+1]} = u^{[j]}du$. Tensor these contractions in the coordinate directions and use étale descent. A finite affine cover and its intersections compute the comparison for the separated quasi-compact lift.

For (4), de Rham comparison identifies integration with the map from integral singular homology to its complexification. Evaluation on invariant differentials gives the differential of uniformization. Functoriality gives the endomorphism assertion. The first Chern class on the torus is the integral alternating form of its Appell–Humbert bundle. The comparison for $d\log$ multiplies this class by $2\pi i$, with the sign determined by the order used for the alternating pairing [104, Chs. 2–3]. For (5), shrink to a simply connected parameter neighborhood and continue a basis of integral homology in the local system. Integration of relative invariant differentials on this basis gives the holomorphic lattice vectors. The homology maps of endomorphisms are locally constant in that basis. In the resulting flat frame, the coefficients of a relative de Rham class are its periods. The crystal–connection comparison [102, Ch. IV, §5] identifies crystalline transport in characteristic zero with the Taylor isomorphism of the Gauss–Manin connection [105, §§1–2]. A constant cycle is horizontal, so pairing commutes with each Taylor coefficient. All terms of degree greater than s evaluate to zero. For (6) apply the good reduction criterion and its potential and complex multiplication forms [106, Thms. 1, 2, and 6(a)]. □

Proof of 26.29

Proof. Items (1) and (2) are the idèlic main theorem of complex multiplication and its j-invariant specialization for the maximal order [7, Ch. 5]. In (2) two ideal lattices are homothetic precisely when their ideal classes agree. The ray-class rationality theorem and the integral difference theorem apply to (3) by linearity in the rational locally constant function on the ray group [10, 11]. In the difference theorem the smoothing ideal is prime to the modulus, so translation preserves the set of ideals in the series. For (4) apply the partial-zeta functional equation to the class and its dual lattice class [8]. For (5) multiply the local functional equations of [12, §§3–4]. A complex place of angular exponent n_τ contributes $(-i)^{|n_\tau|}$. The finite-place factors are those for the same global additive character. The nonzero angular exponent excludes a pole from a pure norm character. □

Proof of 26.32

Proof. Take the polarized Galois representation from $\mathcal{R}$ and the definite-unitary transfer from 26.22. The local representation, finite-group and unit assertions of h_N are 26.24 and 26.25. Proposition 26.26 supplies the automorphic and character assertions, using $\mathcal{R}$ in item (6). The transfer proved in 26.22 supplies the definite-unitary entry in the assembled data. The cohomology assertions are 26.27. The abelian-scheme and special-value assertions are 26.28 and 26.29.

For the weak descent used in 26.39, let E/E^{+} be CM and unramified at finite places, and let S consist of split places and contain the places above the specified rational prime. A regular algebraic conjugate self-dual cuspidal representation spherical outside S is spherical at every nonsplit finite place. Lemma 39.51 therefore supplies the definite-unitary match, including its unramified components at those places, in each required rank $2, 3, 4$. Proposition 26.39 now gives 26.3. □

Proof of 26.33

Proof. Fix the infinitesimal character of π at every real place. In matrix coordinates these characters specify the scalars of the central Capelli operators: at a place v their eigenvalues are the values of one polynomial in the spectral parameter. The coefficients generate the center of the enveloping algebra for $\mathfrak{gl}_a$. The kernel J

of the resulting character of the archimedean enveloping-algebra center has codimension one and annihilates π.

Choose a compact open subgroup $U_0 \subset U$. Let V be the space of ξ. For each intertwiner $A : V \to \pi^U$ and each adelic point g, take the linear functional $v \mapsto A(v)(g)$. Their common kernel V_0 is K_∞-stable by equivariance. On V/V_0 these functionals span the dual. Their matrix coefficients are continuous since $A(v)$ is an automorphic form and $A(\xi(k)v)(g) = A(v)(gk)$. Thus V/V_0 is a continuous finite-dimensional compact-group representation. Every A factors through it. In particular there is a single finite set of continuous compact types containing all its constituents.

At level U_0 there are finitely many arithmetic quotients, by finiteness of the adelic class set for GL_a (equivalently, finiteness of the ideal class group at maximal level and finite index upon shrinking compact open level). On each quotient apply Harish-Chandra's finiteness theorem with the ideal J and the projector onto this finite set of compact types [71, Thm. 8.4]. The split central direction is taken with the central action of π. The resulting space is finite dimensional. Its intersection with π contains $A(V)$ for every A. □

Proof of 26.34

Proof. The highest weight of Ξ, shifted by the half-sum of the positive roots, is regular. Thus the matching in (1) gives the regular algebraic infinitesimal character for the dual coefficients. Apply the archimedean cohomology calculation of [21, §3.5]. Item (2) is [21, Thm. 3.13], including the permutation of embeddings.

For (3), choose in each $\pi \in S$ a nonzero cohomology class at infinity and a nonzero vector in π_f^U. Their tensors belong to distinct direct summands of cuspidal cohomology. The cuspidal cohomology injection preserves their independence [21, Lem. 3.15(i),(ii)]. At finite level its target is the sum of the ordinary cohomologies of the arithmetic components with coefficients in Ξ. Indeed, decompose Y into $G(E)$-orbits and apply Shapiro's lemma to the module of functions on each orbit. This is the group cohomology on the right, including for compact open levels with torsion stabilizers since the coefficients have characteristic zero. □

Proof of 26.35

Proof. For the first assertion, given a finite extension D/F and v, the Galois closure of D contains finitely many quadratic fields. Choose a member split at v outside that finite list. It is linearly disjoint from D. Thus the family satisfies the generality hypothesis with empty exceptional set in the prime-degree patching lemma of [107, §1]. The other two hypotheses of that lemma are the stated invariance and compatibility. Its conclusion identifies the restriction at every member. The cofinite conclusion suffices here.

For the second assertion, identify each G_L ($L \in \mathcal{L}$) with the open index-two kernel $H_L = \ker(\eta_L) \le G_F$ of its nontrivial quadratic character $\eta_L : G_F \to \{\pm 1\}$. The characters η_L are pairwise distinct. On each H_L, conjugation by G_F preserves the Frobenius characteristic polynomial at almost every unramified prime of L, since both Frobenius values come from the same normalized downstairs Satake multiset and conjugation preserves the residue degree; by Chebotarev density and continuity, r_L is conjugate-invariant on H_L. Likewise on $H_L \cap H_{L'} = G_{LL'}$, multiplicativity of residue degrees matches Frobenius roots at almost all primes, so $\operatorname{tr} r_L = \operatorname{tr} r_{L'}$ on $H_L \cap H_{L'}$.

By Clifford theory for an index-two subgroup, each conjugate-invariant continuous semisimple r_L extends to a continuous semisimple representation $\tilde r_L : G_F \to \mathrm{GL}_a(\overline{\mathbf{Q}}_\ell)$ (fixing $t \notin H_L$, the irreducible summands of r_L split into fixed summands extending in two ways differing by η_L and conjugate pairs inducing irreducibly to G_F). Fix one $L_0 \in \mathcal{L}$ and set $\rho_0 = \tilde r_{L_0}$. Let B be the set of quadratic characters $\chi : G_F \to \{\pm 1\}$ admitting a nonzero $X \in M_a(\overline{\mathbf{Q}}_\ell)$ with $\rho_0(g)X = \chi(g)X\rho_0(g)$ for all $g \in G_F$. Elements X_χ for distinct $\chi \in B$ are eigenvectors for distinct characters of G_F under conjugation by ρ_0, hence are linearly independent in $M_a(\overline{\mathbf{Q}}_\ell)$; thus $|B| \le a^2$.

Since $\mathcal{L}$ is infinite, choose $L_1 \in \mathcal{L}$ with $\eta_{L_1} \notin B \cup B\eta_{L_0} \cup \{1, \eta_{L_0}\}$. Then H_{L_0} and H_{L_1} are distinct index-two subgroups of G_F; pick $s \in H_{L_0} \setminus H_{L_1}$ and $t \in H_{L_1} \setminus H_{L_0}$. On the finite-index open normal subgroup $H_{L_0} \cap H_{L_1}$, both ρ_0 and $\tilde r_{L_1}$ are semisimple and have equal traces, so by Brauer–Nesbitt there exists $P \in \mathrm{GL}_a(\overline{\mathbf{Q}}_\ell)$ such that $b(g) = P\tilde r_{L_1}(g)P^{-1}$ satisfies $b(g) = \rho_0(g)$ for all $g \in H_{L_0} \cap H_{L_1}$. Set $z = \rho_0(t)^{-1}b(t) \in \mathrm{GL}_a(\overline{\mathbf{Q}}_\ell)$. For every $g \in H_{L_0} \cap H_{L_1}$, normality gives $tgt^{-1} \in H_{L_0} \cap H_{L_1}$ and hence $\rho_0(tgt^{-1}) = b(tgt^{-1})$, which yields $\rho_0(g)z = zb(g) = z\rho_0(g)$. Write $z = z_1 + z_2$ with $z_1 = \frac{1}{2}(z + \rho_0(s)z\rho_0(s)^{-1})$

and $z_2 = \frac{1}{2}(z - \rho_0(s) z \rho_0(s)^{-1})$; since $H_{L_0} = (H_{L_0} \cap H_{L_1}) \cup s(H_{L_0} \cap H_{L_1})$ and $s^2 \in H_{L_0} \cap H_{L_1}$, one has $\rho_0(g) z_1 = z_1 \rho_0(g)$ and $\rho_0(g) z_2 = \eta_{L_1}(g) z_2 \rho_0(g)$ for all $g \in H_{L_0}$. Next write $z_2 = A + C$ with $A = \frac{1}{2}(z_2 + \rho_0(t) z_2 \rho_0(t)^{-1})$ and $C = \frac{1}{2}(z_2 - \rho_0(t) z_2 \rho_0(t)^{-1})$. Because $t^2 \in H_{L_0} \cap H_{L_1}$, $\eta_{L_1}(t) = 1$, and $\eta_{L_1}(t^{-1} g t) = \eta_{L_1}(g)$, these satisfy $\rho_0(g) A = \eta_{L_1}(g) A \rho_0(g)$ and $\rho_0(g) C = \eta_{L_1}(g) \eta_{L_0}(g) C \rho_0(g)$ for all $g \in G_F$. Since $\eta_{L_1}, \eta_{L_1}\eta_{L_0} \notin B$, we obtain $A = C = 0$, hence $z_2 = 0$ and $\rho_0(g) z = z \rho_0(g)$ for all $g \in H_{L_0}$. Moreover, for any $g \notin H_{L_0}$ we have $h = t^{-1} g \in H_{L_0}$ and $t^2 \in H_{L_0} \cap H_{L_1}$, so $b(t)^2 = \rho_0(t)^2$ and $z \rho_0(g) z = z \rho_0(t) z \rho_0(h) = \rho_0(t)^{-1} b(t)^2 \rho_0(h) = \rho_0(g)$. Defining $\rho_1(g) = \rho_0(g)$ for $g \in H_{L_0}$ and $\rho_1(g) = \rho_0(g) z$ for $g \notin H_{L_0}$ therefore yields a homomorphism $\rho_1 : G_F \to \mathrm{GL}_a(\overline{\mathbf{Q}}_\ell)$ that is continuous (since H_{L_0} is open) and semisimple (since $H_{L_0} \trianglelefteq G_F$ has finite index in characteristic zero and $\rho_1|_{H_{L_0}} = r_{L_0}$ is semisimple). On H_{L_0} we have $\rho_1 = \rho_0 = r_{L_0}$; on H_{L_1}, every $g \in H_{L_0} \cap H_{L_1}$ gives $\rho_1(g) = \rho_0(g) = b(g)$, while every $g \in H_{L_1} \setminus H_{L_0}$ has $g t^{-1} \in H_{L_0} \cap H_{L_1}$ and hence $\rho_1(g) = \rho_0(g t^{-1}) \rho_0(t) z = b(g t^{-1}) b(t) = b(g)$.

Lean: `GlobalTheory.exists_hom_patch_of_agree` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsRepPatchTwist`, lines 7–76)
Lean: `GlobalTheory.commute_on_ker_of_commute_on_ker_inter` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.IntertwinerZeroOfCharacterTwist`, lines 8–21)

Finally, let $L \in \mathcal{L}$ avoid the finite set of characters $B \cup B\eta_{L_0} \cup \eta_{L_1} B \cup \eta_{L_1}\eta_{L_0} B \cup \{1, \eta_{L_0}, \eta_{L_1}\}$. Repeating the construction with (L_0, L) yields a continuous semisimple $\rho_L : G_F \to \mathrm{GL}_a(\overline{\mathbf{Q}}_\ell)$ agreeing with ρ_0 on H_{L_0} and conjugate to r_L on H_L. Then ρ_1 and ρ_L both equal ρ_0 on H_{L_0}, and for every $g \in H_{L_1} \cap H_L$ we have $\operatorname{tr} \rho_1(g) = \operatorname{tr} r_{L_1}(g) = \operatorname{tr} r_L(g) = \operatorname{tr} \rho_L(g)$. By Brauer–Nesbitt on the finite-index normal subgroup $H_{L_1} \cap H_L$, there exists $P_L \in \mathrm{GL}_a(\overline{\mathbf{Q}}_\ell)$ with $\rho_1(g) P_L = P_L \rho_L(g)$ for all $g \in H_{L_1} \cap H_L$. Averaging $X \mapsto \rho_1(g) X \rho_L(g)^{-1}$ over the quotient $G_F/(H_{L_1} \cap H_L)$ decomposes $P_L = \sum_c P_c$ over $c \in \{1, \eta_{L_1}, \eta_L, \eta_{L_1}\eta_L\}$ with $\rho_1(g) P_c = c(g) P_c \rho_L(g)$ for all $g \in G_F$. For each nontrivial $c \in \{\eta_{L_1}, \eta_L, \eta_{L_1}\eta_L\}$, restricting this identity to H_{L_0} (where $\rho_1 = \rho_L = \rho_0$) and decomposing $P_c = A_c + C_c$ under conjugation by $\rho_0(t)$ yields $\rho_0(g) A_c = c(g) A_c \rho_0(g)$ and $\rho_0(g) C_c = c(g) \eta_{L_0}(g) C_c \rho_0(g)$ for all $g \in G_F$. Since $c, c\eta_{L_0} \notin B$, both A_c and C_c vanish, so $P_c = 0$. Thus $P_L = P_1$ satisfies $\rho_1(g) P_L = P_L \rho_L(g)$ on all of G_F, giving $\operatorname{tr} \rho_1 = \operatorname{tr} \rho_L$ on G_F and hence $\operatorname{tr}(\rho_1|_{H_L}) = \operatorname{tr} r_L$. As $H_L \trianglelefteq G_F$ has finite index, $\rho_1|_{H_L}$ is semisimple, so Brauer–Nesbitt yields $\rho_1|_{G_L} \simeq r_L$ for all but finitely many $L \in \mathcal{L}$.

Lean: `GlobalTheory.trace_eq_of_agree_on_index_two_of_no_twisted_intertwiner` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.IntertwinerZeroOfQuadraticTwists`, lines 9–48)

□

Proof of 26.36

Proof. For (1) apply cuspidal strong multiplicity one [13]. At spherical places inverse transpose and contragredience both invert the Satake roots, so multiplicity one identifies the two representations and gives the nondegenerate bilinear pairing. Use 26.33 for admissibility. Twist by a power of the adelic norm to make the split central character unitary. Write J for the annihilator of the twisted form in the centre of the enveloping algebra. It has finite codimension because the centre generates a finite-dimensional space on the form. Choose the compact idempotent for the finitely many archimedean types in its compact translates. Cuspidal rapid decrease gives square integrability modulo the split centre. Project onto the irreducible summands of the cuspidal discrete spectrum. Every projection has the same finite level, is fixed by the compact idempotent and is killed by J, since the projections intertwine right translation and its differential. At this level the adelic quotient has finitely many arithmetic components. Harish-Chandra finiteness on each component gives a finite-dimensional space for this idempotent and ideal [71, Thm. 8.4]. The nonzero projections are mutually orthogonal in that space, so finitely many sum to the form. Undo the twist. For the newform in (3), adelisation gives the open level stabiliser and cuspidality. Its lowest-weight vector has one rotation weight, together with its reflected weight for the full compact group, and its infinitesimal character is the lowest-weight infinitesimal character. A norm twist preserves these finiteness conditions.

Lean: `UpperTheory.V.Af.cuspSemisimpleReal_dB` (`Development.AutomorphicArithmetic.AutomorphicVarieties.AutomorphicForms.TemperedLocalComponents.PCuspSSDB`, lines 58–64)

Items (2), (4) and (5) use cyclic base change [93, Ch. III, Thms. 4.2 and 5.1]. In (4) and (5) strong multiplicity one turns cofinite invariance into invariance of the upper representation. The fibre characters are the characters trivial on idèlic norms. Restriction of a local unramified parameter takes Frobenius to its residue-degree power. At a split place the completions agree. At infinity restriction is restriction of the Weil parameter. A norm twist used to normalize the weights shifts both multisets equally and cancels on twisting back. For (3), adelize the lowest-weight newvector and apply the norm twist with the indicated infinitesimal

character. The spherical Hecke polynomial of the newvector at $p \nmid M$ is the polynomial displayed above, with one factor $p^{1/2}$ from algebraic normalization and p^j from the twist. For (6), prescribe invariant $1/2$ at each real place and zero at each finite place. They sum to zero by even degree. The quaternion ramification theorem gives the algebra, and a maximal order gives the integral splittings [73, Thm. 14.6.1]. □

Proof of 26.37

Proof. Take the quadratic CM fields admitting cuspidal base change of π. For a given finite $v \mid q$, choose primes $p \equiv -1 \bmod q$ for odd q, or $p \equiv -1 \bmod 8$ for $q = 2$. Omit $2, q$, the primes ramified in F, and the primes below nonspherical components. Dirichlet's theorem supplies infinitely many p. Then $-p$ is a square in $\mathbf{Q}_q$, and $L = F(\sqrt{-p})$ is split at v and ramified above p. Its character cannot fix π: at a place above p the spherical support is unramified and the quadratic twist has ramified support. By 26.36(2) the lift is cuspidal. These fields include infinitely many distinct members since each is ramified at its chosen prime.

The lift preserves regularity and pulls back the polarization. If necessary multiply the polarizing character on F by the quadratic character of L/F to give the sign required in 39.2. Its norm pullback is unchanged and its sign changes at every real place. Apply 39.2 over these CM fields. On a pairwise compositum the two restrictions have the same Frobenius polynomials almost everywhere: for a place of residue degree f over F both multisets are $(q_v^{1/2} B_v)^f$, with powers taken root by root. The same computation gives invariance under conjugation. The restrictions are semisimple by normality and finite index. Chebotarev and equality of semisimple characters give the isomorphisms in 26.35. Patch to r.

For each specified place and prolongation choose a split member outside the finite exceptions of the patching lemma. Its completion equals F_v, its local automorphic component is π_v, and its decomposition group on the chosen prolongation is the entire downstairs decomposition group. Thus the comparisons of 39.5 and 39.6 at a toral unlinked or spherical component apply to r itself. For the ratio in (2), a cuspidal representation becomes unitary after a global norm twist. In rank two an unlinked principal-series component in that normalization has exponents of absolute value less than $1/2$, by the unitary classification [219]. Moduli of the Weil roots therefore have ratio less than q_v, unchanged by the common normalization. At a spherical cusp component genericity excludes a linked pair. Finally the weights pull back under embeddings by 26.36(2). On one CM member 39.4 reflects the weights at conjugate embeddings over the same real place. They both pull back the downstairs multiset, so the same integer works over F. □

Proof of 26.38

Proof. Take a tower from K to L with cyclic prime-degree steps. All its fields contain the imaginary quadratic field K, and all are CM since $L/\mathbf{Q}$ is abelian. In a step M/D write $W_D = W|_{G_D}$. Every W_D is irreducible because $W|_{G_L}$ is. Suppose an attachment is known over M. Its Satake multisets are invariant at conjugate good places since the attached representation extends to G_D. By 26.36(4) choose a cuspidal descent π_D. Its infinitesimal weights are regular algebraic.

We explain why this descent has a Galois representation to compare with W_D. For any two-dimensional representation the exterior product is a perfect pairing with multiplier its determinant. Conjugation over D^+ preserves W_D up to isomorphism because W is defined over $\mathbf{Q}$. Over M, the attachment and its conjugate dual, with the determinant and algebraic norm correction, consequently have equal good Satake multisets. In detail, if $d = \det W$ and B is the Satake pair at a good place u of M, then

$$B = \big(q_u^{-1} \iota d(\mathrm{Frob}_u^{-1})\big) B^{-1}.$$

The scalar is the value of the pullback of a Hecke character of D^+: take the character of d by reciprocity and multiply by the idèlic norm, with both characters restricted by norm from $\mathbf{Q}$. Thus π_D^c and the correspondingly twisted $\pi_D^\vee$ have the same cuspidal base change. The fibre assertion of 26.36(4) identifies their discrepancy with a character of $\mathrm{Gal}(M/D)$. This character extends to $\mathrm{Gal}(M/D^+)$: the latter is abelian and complex conjugation splits its quotient of order two. Choose the extension trivial on this conjugation. After multiplying the polarizing character by this finite character or its inverse according to the direction of the comparison, strong multiplicity one gives the polarization of π_D. The real signs agree since the original multiplier is restricted from $\mathbf{Q}$ and complex conjugation is the same element at every real place in this abelian

extension. Multiplying, if necessary, by the character of D/D^+ makes all signs positive, as required in rank two, without changing the norm pullback. Thus 39.2 attaches a continuous semisimple R_D to π_D.

At almost every $u \mid v$ of M/D, the Frobenius roots of $R_D|_{G_M}$ and $W|_{G_M}$ agree: taking $f(u/v)$-th powers commutes with the factor $q_v^{1/2}$. Normality gives semisimplicity upon restriction, so Chebotarev identifies the restrictions. The space $\mathrm{Hom}_{G_M}(R_D, W_D)$ is one-dimensional by irreducibility. The action $T \mapsto W_D(g) T R_D(g)^{-1}$ on this line defines a character δ of $\mathrm{Gal}(M/D)$. A nonzero T is invertible and gives $W_D \simeq R_D \otimes \delta$. Twist π_D by the finite Hecke character with these Galois Frobenius values. Its attachment is W_D. Repeating the step gives the specified attachment over K.

At the last step $K/\mathbf{Q}$, let ω_π be the central character of this attachment. At good primes the determinant identity gives $\omega_\pi |\cdot|^{-1}$ finite order, by reciprocity and character multiplicity one. Hence the two infinitesimal weights sum to 1. They are distinct and belong to $\frac{1}{2} + \mathbf{Z}$, so their difference is a nonzero even integer. This checks the gap in 26.36(5). Choose a rational cuspidal descent matching at the split primes. Its rank-two central character polarizes it, and $\mathbf{Q}$ has just one real place, so 26.37 supplies $R_{\mathbf{Q}}$.

For each finite Galois extension D/K, its normal closure over $\mathbf{Q}$ contains K. Chebotarev in this closure realizes every conjugacy class needed for D/K by Frobenius elements over rational primes split in K: lift the chosen element to the subgroup fixing K, and choose a prolongation with that representative. One may exclude finitely many primes. Thus Frobenius comparison at the split primes gives $R_{\mathbf{Q}}|_{G_K} \simeq W|_{G_K}$. Apply the same one-dimensional Hom argument with quotient $\mathrm{Gal}(K/\mathbf{Q})$. The resulting trivial or quadratic twist of the rational descent attaches to W. □

Proof of 26.39

Proof. The coefficient, conjugation and counting assertions are 26.34. The totally real attachment is 26.37, which uses the patching lemma 26.35. The remaining real and cyclic assertions are 26.33 and 26.36. In particular 26.36(5) supplies the split-prime descent entry of $\mathscr{H}$, and 26.38 derives descent for the specified Galois representation when that conclusion is needed.

Apply 28.2 to h_N for the small-twist bound. The common marking and the two attachments in the first count are supplied by 30.61(1)–(3). With the vanishings of 30.12 these give 29.15. The rank-four input of 30.51 gives the second and third counts in 29.22. For the fifth class use 29.26, with the Gaussian lifting assertion 30.54 supplied by 30.61(4). These are the four counts in 26.40, which gives the window bound. □

Proof of 26.40

Proof. Apply 28.16 at $C_t = 401$ with the small-twist bound and the local finiteness and duality inputs from h_N. The generator bounds of 28.7 follow from 28.6 and Ferrero–Washington vanishing of μ_ℓ for abelian number fields [92]. On this window 29.32 gives the odd-prime plus bound from the supplied counts using 29.24 and 29.27, with the class Z count supplied by 29.28. Poitou–Tate duality and unramified orthogonality give the odd-prime minus bound of 27.8. Combine these with the 2-primary bound of 28.16 by 27.10. □

Proofs for Chapter 27: Three length bounds

Proof of 27.3

Proof. Von Staudt–Clausen and $B_1 = -1/2$ clear the coefficients of $B_j(X)$ with primorial$(j+1)$ [112, p. 56]. Since $f \mid N$, this gives integrality. At a level raised by a prime p, Bernoulli distribution over all lifts gives the original sum. If $p \nmid f$ and p is new to the level, the omitted multiples of p contribute $\chi(p)p^{j-1}$ times that sum. Iteration gives the product. The functional equation and Euler product at j give primitive nonvanishing in the stated parity [112, Ch. 4], and the extra factors are nonzero since $|\chi(p)p^{j-1}| > 1$.

The Fourier series gives $|B_j(x)| \le 4j!/(2\pi)^j$ on $[0,1]$ [4, p. 267]. For odd $h = 2a+1$ the primes in $(a+1, h]$ divide $\binom{2a+1}{a} \le 2^{2a}$. Induction gives primorial$(h) \le 4^h$ since even $h > 2$ adds no prime. Use both bounds at $m = f, N$. □

Proof of 27.4

Proof. Since $|\operatorname{Gal}(E/\mathbf{Q})| \in O^\times$, inflation–restriction identifies the twisted global cohomology with the invariants after restriction. The finite quotients of inertia and decomposition groups have order dividing n, so the same argument identifies the local kernels. Untwisting takes the invariants to the θ-part.

Let $\chi = \widetilde{\theta}^{\mathrm{prim}}$ have conductor f and put $j = k+1$. The bound at conductor f suffices since $A_{N,j}(\widetilde{\theta}) = (N/f)A_{f,j}(\chi)$. Inflate θ to $G_{\mathbf{Q}}$. Write $W = K_O/O$ and $\alpha = \theta^{-1}\varepsilon_\ell^{-k}$. Complex conjugation δ acts by -1, so each class has a unique cocycle with $c(\delta) = 0$. At one lift over ℓ, write $c|_{D_\ell} = (\alpha - 1)m_\ell$ and choose $a_0 = \pi^h$ killing the finite image of c and m_ℓ. At the σ-conjugate lift use $\alpha(\sigma)m_\ell - c(\sigma) \in W[a_0]$. For $p \mid f$, $p \neq \ell$, primitivity and the prime-to-ℓ order give $g \in I_p$ with $\alpha(g) - 1$ a unit. The inertia cobounding vector is $(\alpha(g)-1)^{-1}c(g) \in W[a_0]$. Subtract its coboundary on D_p. The remaining values lie in $W^{I_p} = 0$. Thus the normalized representatives lie in the union of the modules $T(a_0)$ of normalized $W[a_0]$-valued cocycles unramified outside $f\ell$ and coboundaries in $W[a_0]$ on every D_p for $p \mid f\ell$.

First suppose $\alpha \not\equiv \varepsilon_\ell \pmod{\pi}$, and choose $g_0 \in G_{\mathbf{Q}}$ such that $\alpha(g_0) - \varepsilon_\ell(g_0)$ is a unit in $O^\times$. Take $M_0 = \ell^m$ with $m \geq \max(2, v_\ell(f))$, and $H = f^{(\ell')}M_0$, where $f^{(\ell')}$ is the prime-to-ℓ part of f; since ℓ is odd, $H \geq \ell^2 \geq 9 \geq 4$. Write $[t]_M$ for least nonnegative residues. Let σ_t lift the t-power action on μ_H and $1 \leq b < H$ represent g_0 there. Put

$$h_{M,b}(t) = \lfloor b[t]_M/M \rfloor - \lfloor b[-t]_M/M \rfloor, \qquad z = \sum_{t \in (\mathbf{Z}/H\mathbf{Z})^\times} h_{H,b}(t)\alpha(\sigma_t)^{-1}. \tag{A.223}$$

Set $C_{M,j}(u) = M^{j-1}B_j(u/M)$ and $d_j = 2j\mathrm{primorial}(j+1)$. For $u = [t]_H$, $a = \lfloor bu/H \rfloor$ and $v = bu - Ha$, we have $h_{H,b}(t) = 2a - (b-1)$. Expand the first two terms $v^j/H - jv^{j-1}/2$ of $C_{H,j}(v)$. The lower terms have a factor H after clearing denominators, so

$$d_j\big(b^k h_{H,b}(t)u^k + 2(C_{H,j}(v) - b^j C_{H,j}(u))/j\big) \in H\mathbf{Z}.$$

Sum with weights $\chi_H(bt)$ and reindex the term at v. Since $\alpha(\sigma_t)^{-1} \equiv u^k\chi(t) \bmod M_0O$, this gives

$$d_j\big(b^k\chi(b)z - 2(\chi(b)b^j - 1)B_{j,\chi_H}/j\big) \in M_0O. \tag{A.224}$$

Since $\alpha = \theta^{-1}\varepsilon_\ell^{-k}$, $j = k+1$, and $\chi(b) = \theta(g_0)$ with $b \equiv \varepsilon_\ell(g_0) \pmod{M_0}$, we have

$$\chi(b)b^j - 1 \equiv -\theta(g_0)\varepsilon_\ell(g_0)^k(\alpha(g_0) - \varepsilon_\ell(g_0)) \pmod{\ell O},$$

which belongs to $O^\times$ by the choice of g_0. In particular $b \neq 1$ (since $\chi(1)1^j - 1 = 0$) and $b \neq H-1$ (since $\chi(-1) = (-1)^{k+1} = (-1)^j$ gives $\chi(H-1)(H-1)^j \equiv 1 \pmod{H}$), so $2 \leq b \leq H-2$. The level formula in 27.3 preserves the valuation of $B_{j,\chi}$ here, since the single possible extra factor $1 - \chi(\ell)\ell^{j-1}$ is 1 mod ℓO. Choose $m \geq \max(2, v_\ell(f))$ with $|M_0|_\ell < |a_0 d_j B_{j,\chi}/j|_\ell$. Then

$$|z|_\ell = |B_{j,\chi}/j|_\ell, \qquad a_0 z \mid M_0, \qquad z \mid A_{f,j}(\chi). \tag{A.225}$$

Let $M > 2$. For a degree-one prime $P' \nmid M$ of $\mathbf{Q}(\mu_M)$, put $p = \mathrm{N}P'$. Reduction identifies μ_M with its image in $\mathbf{F}_p^\times$, so $M \mid p-1$. Let η lift the D-th residue power for $D = (p-1)/M$, and let $\sigma_{M,t}$ act by t-th powers on μ_M. For $1 \leq i \leq M-2$, the characters η, η^i, η^{i+1} are nontrivial. The Gauss-sum identity $J_i = J(\eta, \eta^i) = G(\eta)G(\eta^i)/G(\eta^{i+1})$ gives $J_i\bar{J}_i = p$, since each Gauss sum has squared modulus p. Modulo $\sigma_{M,t}^{-1}P'$ the sum is $\sum_{x \in \mathbf{F}_p} x^{D[t]_M}(1-x)^{D[it]_M}$. The powers of x have positive degree below $2(p-1)$. The term of degree $p-1$ occurs with coefficient prime to p exactly when $[t]_M + [it]_M \geq M$. Since p is unramified, this proves

$$v_{\sigma_{M,t}^{-1}P'}(J_i) = 1 - C_i(t), \qquad C_i(t) = \lfloor ([t]_M + [it]_M)/M \rfloor. \tag{A.226}$$

We may choose such a prime in any ideal class and avoid any finite set. Chebotarev in the Hilbert class field gives positive density in the class, and primes of residue degree greater than one over $\mathbf{Q}$ have density zero [41, Ch. VII, Theorem 13.4].

Choose a place A_0 of $\overline{\mathbf{Q}}$ above a prime $r \equiv 1 \bmod H$. Let d be squarefree with prime factors $q \equiv 1 \bmod H$, $q \neq r$, and $\ell \nmid \operatorname{ord}_q(r)$. Put $M = Hd$ (so $M \geq H \geq 4 > 2$ and $1 \leq b-1 \leq H-3 \leq M-2$, making (A.226)

apply to every $1 \le i \le b-1$) and let P lie below A_0 at level M. Choose P' as above with $PP' = (y)$, and put $J = \prod_{i=1}^{b-1} J_i$. The carries telescope to $\lfloor b[t]_M/M \rfloor$, so $\bar{J}/J$ has valuation $h_{M,b}(t)$ at $\sigma_{M,t}^{-1}P'$. Hence $X_d = \prod_t (\sigma_{M,t}^{-1}y)^{h_{M,b}(t)}/(\bar{J}/J)$ satisfies

$$(X_d) = \prod_t (\sigma_{M,t}^{-1}P)^{h_{M,b}(t)}, \qquad X_d\bar{X}_d = 1. \tag{A.227}$$

The second identity follows from $h_{M,b}(-t) = -h_{M,b}(t)$.

For a new prime q of the same kind, let $\phi \in D_q$ lift arithmetic Frobenius. It acts by q-th powers on μ_M. Write $s = s(d,q)$ for the residue degree over r from level M to qM, which divides $\mathrm{ord}_q(r)$. Since $b < H < q$, the sum of the first floor in $h_{qM,b}$ over all q lifts $[t]_M + iM$ is

$$\lfloor b[t]_M/M \rfloor + \sum_{i=0}^{q-1} \lfloor ib/q \rfloor .$$

$ib \bmod q$ runs over all residues. Remove the lift divisible by q, whose first floor is $\lfloor b[q^{-1}t]_M/M \rfloor$, and subtract the corresponding sum for $-t$. The sum over unit lifts is $h(t) - h(q^{-1}t)$ for $h = h_{M,b}$. By (A.227), $\mathrm{Norm}(X_{qd})/(X_d/\phi^{-1}X_d)^s$ is a unit of modulus 1 at every embedding. Kronecker's theorem makes it a root of unity, whose order in $\mathbf{Q}(\mu_M)$ divides $2M$.

Choose P', y at level qM and take norms to level M. The upper prime over r has norm P^s, so the expression for X_d using these norms equals X_d^s up to a $2M$-th root of unity. At a prime above q, total ramification makes the residue F_t of $\sigma_{qM,t}^{-1}y$ independent of the lift, and $F_t = F_{qt}^q$. Thus the upper principal factor reduces to $\prod_t F_t^{h(t)-h(q^{-1}t)} = \prod_t F_{qt}^{(q-1)h(t)}$, the factor for $\phi^{-1}\mathrm{Norm}(y)$. The two residue characters on $\mathbf{F}_p^\times$ satisfy $\eta_{qM}^q = \eta_M$. At q, $\eta_{qM}(u)$ and $\phi^{-1}(\eta_M(u))$ have the same q-th power, so they agree after reduction. The Jacobi sums and their conjugates therefore agree term by term after applying ϕ^{-1} at level M. Their denominators are units at q by $J_i\bar{J}_i = p$. Hence $X_{qd} \equiv \zeta'(\phi^{-1}X_d)^s$ at each prime over q, where $(\zeta')^{2qM} = 1$.

Let u_0 be the least nonnegative integer representing the action of g_0 on μ_{M_0} and put $P_0(v) = g_0(v)/v^{u_0}$. For $x_d = P_0(X_d)^{2f^{(\ell')}d}$ the root factors vanish. Thus $\mathrm{Norm}(x_{qd}) = (x_d/\phi^{-1}x_d)^S$ and $x_{qd} \equiv (\phi^{-1}x_d)^S$ above q, with $S = qs(d,q)$. Write L for the list of primes of d and $L' = (q, L)$. Take τ_p generating the action on μ_p and acting trivially on the other coprime cyclotomic factors. Let $\mathcal{D}_p = \sum_{i=0}^{p-2} i\tau_p^i$ and apply $\prod_{p\in L}\mathcal{D}_p$ multiplicatively to x_d, x_{qd} to obtain y_L, w_L, respectively. For the relative norm N_q, $(\tau_q - 1)\mathcal{D}_q = q - 1 - N_q$ gives

$$\tau_q(y_{L'})/y_{L'} = w_L^{q-1}(\phi^{-1}y_L/y_L)^S.$$

Put $U = G_{\mathbf{Q}(\mu_H)}$. For every $g \in U$, $g(y_L)/y_L$ has an M_0-th root in $\mathbf{Q}(\mu_{Hd})$. For the empty list the quotient is 1. For each $p \in L$, put p first and use induction on the remaining list. The displayed identity gives the root $w^{(p-1)/M_0} z_{\phi^{-1}}^S$, where w is the derivative omitting p and $z_{\phi^{-1}}$ is the root at the lower level. The Chinese remainder theorem gives generators for the relative cyclotomic group from these τ_p. Products extend the assertion because $z_g\, g(z_h)$ is a root for gh in the same field.

Choose $Y_L^{M_0} = y_L$ and roots z_g as above. For primitive $\zeta \in \mu_{M_0}$ and a generator w of $W[M_0]$ over O, define $\kappa_L : U \to \mathbf{Z}/M_0\mathbf{Z}$ and $\nu_L : U \to W[M_0]$ by

$$\zeta^{\kappa_L(g)} = \frac{gP_0(Y_L)}{P_0(Y_L)P_0(z_g)}, \qquad \nu_L(g) = \sum_{t\in(\mathbf{Z}/H\mathbf{Z})^\times} \beta(\sigma_t)\kappa_L(\sigma_t^{-1}g\sigma_t)w, \quad \beta = \varepsilon_\ell\alpha^{-1}.$$

The root $z_g\, g(z_h)$ gives additivity since P_0 commutes with the action on the cyclotomic layer and kills μ_{M_0}. Reindexing gives $\nu_L(\sigma g\sigma^{-1}) = \beta(\sigma)\nu_L(g)$. At $p \nmid Hrd$, inertia fixes the cyclotomic layer, so take $z_g = 1$ for $g \in I_p$. Here $P_0(Y_L)$ is a unit by (A.227). The resulting root of unity reduces to 1, and reduction is injective on μ_{M_0}.

By total tame ramification over the level-H field at r, choose $\rho^{M_0} = r$ and $\tau_r \in I_{A_0}$ with $\tau_r\rho = \zeta\rho$. Normalize $v_{A_0}(r) = 1$. For $K(t) = v_{A_0}(\sigma_t P_0(y_\varnothing))$, the element $\sigma_t P_0(Y_\varnothing)/\rho^{K(t)}$ is a unit. Its quotient under τ_r belongs to μ_{M_0} and reduces to 1. Thus $t\kappa_\varnothing(\sigma_t^{-1}\tau_r\sigma_t) \equiv K(t) \bmod M_0$. Since b represents g_0 on μ_H,

P_0 sends a valuation function $k(t)$ to $k(tb) - u_0 k(t)$. Reindex the sum with weights $\alpha(\sigma_t)^{-1}$ to obtain the multiplier $\alpha(g_0) - u_0$ modulo $M_0 O$. This occurs once in x_1 and once in $P_0(y_\varnothing)$.

At q, take τ_q in inertia and $z_{\tau_q} = w_L^{(q-1)/M_0} z_{\phi^{-1}}^S$. Write $B = P_0(Y_L)$. The residue formula for w_L gives $P_0(w_L)^{(q-1)/M_0} \equiv (\phi^{-1}B)^{S(q-1)}$. The quotient for $\kappa_{L'}(\tau_q)$ therefore reduces to $(\phi^{-1}B/(BP_0(z_{\phi^{-1}})))^S$, since inertia fixes residues and $B \equiv (\phi^{-1}B)^q$. Use injectivity of reduction on μ_{M_0} and apply the same computation at the conjugate places to obtain

$$\nu_\varnothing(\tau_r) = 2f^{(\ell')}(\alpha(g_0) - u_0)^2 zw, \qquad \nu_{L'}(\tau_q) = -q\, s(d,q)\nu_L(\mathrm{Frob}_q). \tag{A.228}$$

The coefficients $2f^{(\ell')}(\alpha(g_0) - u_0)^2$ and $-q\, s(d,q)$ are units, as is $\beta(g_0) - 1$. Equivariance extends ν_L to the β-cocycle

$$\widetilde{\nu}_L(g) = -(\beta(g_0) - 1)^{-1}\nu_L(gg_0g^{-1}g_0^{-1}).$$

The extension remains unramified outside Hrd since the commutator of g_0 with inertia there is a product of two such inertia elements.

Put $I_L = \langle \nu_L(U)\rangle_O$, $(A_L) = \mathrm{Ann}(I_L)$, and let $S_L \subset T(a_0)$ consist of cocycles zero on decomposition groups over r and the primes in L. Since $\mathrm{Ann}(w) = (M_0)$, (A.228) gives

$$\mathrm{Ann}(\nu_\varnothing(\tau_r)) = (M_0/z), \qquad a_0 \mid M_0/z \mid A_\varnothing \mid M_0.$$

Suppose $a_0 \mid A_L$ and $0 \neq c \in S_L$. The twists on U are trivial modulo M_0. If $c|_U = 0$, it descends to an abelian quotient, where commutation with δ and $c(\delta) = 0$ give $c = 0$. Let $V \subset U$ fix ρ. If $c|_V = 0$, conjugation acts on $c(U)$ by both α and, through the radical quotient, ε_ℓ. Their unit difference at g_0 gives $c|_U = 0$. The same argument for ν_L mod πI_L uses $\beta(\delta) - \varepsilon_\ell(\delta) = 2$ and Nakayama's lemma. Here $I_L \neq 0$ since a_0 is a nonunit and $a_0 \mid A_L$. If $I_L = 0$, the generation condition on ν_L is automatic.

The kernel of $c|_V$ and the kernel of $\nu_L|_V$ mod πI_L are proper open subgroups of V. The subgroup fixing $\mu_{H\ell}$ is also proper. The degree-ℓ extension $\mathbf{Q}(\mu_{H\ell})/\mathbf{Q}(\mu_H)$ is unramified at r, whereas $\mathbf{Q}(\mu_H, \rho)/\mathbf{Q}(\mu_H)$ is totally ramified there. Three proper subgroups covering a group must all have index two, and the first kernel has odd index. Their complement in V is thus nonempty and open. Chebotarev gives a prime $q \nmid Hrd$ with a Frobenius lift ϕ in this complement. We have $c(\phi) \neq 0$ and $\nu_L(\phi)$ generates I_L. Also r is an M_0-th power modulo q and $v_\ell(q-1) = m$, so $\ell \nmid \mathrm{ord}_q(r)$.

By (A.228), the annihilator of $y = \nu_{L'}(\tau_q)$ is (A_L). Hence $a' = A_{L'} = A_L e$ for some $e \in O$, with $a_0 \mid a' \mid M_0$. Use a generator of $W[a']$ to identify it with $B' = O/(a')$, in which y generates the ideal (e). The R-linear forms $\lambda : B' \to R$, for $R = \mathbf{Z}/M_0\mathbf{Z}$, separate points. Each cyclic summand $\mathbf{Z}/\ell^s\mathbf{Z}$ of the finite R-module embeds in R by $1 \mapsto \ell^{m-s}$.

Cup $\widetilde{\nu}_{L'}$ with $c' \in S_L$ using multiplication followed by λ. The twists give coefficients $R(1)$. Local triviality of c' kills the cups at r, L and $p \mid H$. For $p \neq \ell$, the unramified subgroups for a finite unramified ℓ-primary module and its Tate dual are exact annihilators. Thus the other cups away from q vanish by 26.27(2) and [111, Theorem I.2.6]. By 27.2, the invariant at q vanishes as well, so the local cup is zero.

Since $q \equiv 1 \bmod H$, D_q acts trivially on the finite coefficient modules. For an unramified R-valued character u, $u(h)c'(\phi) = u(\phi)c'(h)$ on D_q by agreement on inertia and on ϕ. Thus $g \mapsto \lambda(\widetilde{\nu}_{L'}(g)c'(\phi))$ cups trivially with every such u. Exact orthogonality makes this homomorphism unramified, so $\lambda(yc'(\phi)) = 0$ for all λ. It follows that $yc'(\phi) = 0$. Evaluation at ϕ maps S_L to $W[e]$ with kernel $S_{L'}$, since c' is unramified at q and vanishing on D_q passes to all conjugate lifts. Therefore

$$\mathrm{length}_O S_L \leq \mathrm{length}_O S_{L'} + v_O(e). \tag{A.229}$$

For the initial step use the same pairing at r with $\widetilde{\nu}_\varnothing$, $c' \in T(a_0)$ and $A_\varnothing = (M_0/z)e_0$. This gives $\mathrm{length}_O(T(a_0)/S_\varnothing) \leq v_O(e_0)$. At each subsequent step $v_O(e) > 0$ since $c(\phi) \neq 0$. The bound $A_L \mid M_0$ forces termination at $S_{L_*} = 0$. Summing (A.229) gives

$$\mathrm{length}_O T(a_0) \leq v_O(e_0) + v_O(A_{L_*}) - v_O(A_\varnothing) = v_O(A_{L_*}) - v_O(M_0/z) \leq v_O(z).$$

By (A.225), the increasing images of $T(\pi^h)$ in C_θ have lengths bounded by $v_O(A_{f,j}(\chi))$. They stabilize.

In the remaining case $\alpha \equiv \varepsilon_\ell \bmod \pi$, the prime-to-$\ell$ order of θ makes it unramified off ℓ. Normalize at $t \in I_\ell$ acting by -1 on μ_ℓ. Then $c|_{D_\ell} = 0$ since $\alpha(t) - 1$ is a unit. For a $W[\pi]$-valued cocycle, each $\mathbf{F}_\ell$-linear form on $W[\pi]$ gives a Kummer class of $\pm\ell^s$. Local triviality gives $\ell \mid s$, so the class vanishes. Its cocycle vanishes too by normalization at t. The forms separate points, and induction on the power of π killing the cocycle gives $C_\theta = 0$. □

Proof of 27.5

Proof. For $N \le 2$ the class group is trivial. Otherwise write $[u]_N \in \{0,\dots,N-1\}$ for the least nonnegative residue of u mod N, put $C_i(t) = ([it]_N + [t]_N - [(i+1)t]_N)/N \in \{0,1\}$, and let $J_{[i]_N}$ be the level-N Jacobi sum of (A.226). Take a degree-one prime $P \nmid N$ in c as above, and put $p = \mathrm{N}P$, $P_t = \tau_t^{-1}P$. For $i \not\equiv 0, -1 \bmod N$, (A.226) gives $\prod_t P_t^{C_i(t)} = (p/J_{[i]_N})$ at level N. For $i \equiv 0, -1$ the product is (1), (p), respectively. The identity $NC_i(t) = [it]_N + [t]_N - [(i+1)t]_N$ shows that $\sum_{0\le i<a} C_i(t) = \lfloor a[t]_N/N \rfloor$. Apply τ_a^{-1} to the product over $0 \le i < a$. The exponent at P_b becomes j_b. Since $\prod_b P_b = (p)$, the fractional ideal with exponents E_b is principal.

As K is totally complex, h factors through the class group by inverse Artin reciprocity. That map sends $\sigma_b^{-1} u \sigma_b$ to the τ_b^{-1}-translate of the class for u [112, Appendix, §3]. The class identity gives the homomorphism identity. □

Proof of 27.7

Proof. Put $H = N\ell$, $b = 2H+1$, $j = k+1$ and $\Gamma = \mathrm{Gal}(E/\mathbf{Q})$. For positive integers M and $u \in \mathbf{Z}$, write $[u]_M \in \{0,\dots,M-1\}$ for the least nonnegative residue of u mod M, put $C_{M,j}(u) = M^{j-1}B_j(u/M)$ where B_j is the j-th Bernoulli polynomial, and set $d_j = 2j\,\mathrm{primorial}(j+1)$. Write σ_t for the cyclotomic action of t on E. Set

$$\Theta_j = -\sum_{t \in (\mathbf{Z}/H\mathbf{Z})^\times} C_{H,j}([t]_H)[\sigma_t]/j, \qquad Y = (1 - b^j[\sigma_b])\Theta_j.$$

For $L = H\ell^m$, Bernoulli distribution gives $\sum_{i=0}^{L/H-1} C_{L,j}(u+iH) = C_{H,j}(u)$. Since $\ell \mid H$, every lift of a unit class modulo H is a unit modulo L. Thus the same formula at level L gives Y after summing over fibres. For $v = [b^{-1}t]_L$ and $a_t = (bv - [t]_L)/L$, expansion of the first two Bernoulli terms gives

$$-\frac{C_{L,j}([t]_L) - b^j C_{L,j}(v)}{j} \equiv -[t]_L^k((b-1)/2 - a_t) \pmod{(L/d_j)\mathbf{Z}}.$$

Hence the image z of Y lies in $\mathbf{Z}_\ell[\Gamma]$. For $v_\ell(L) \ge u + v_\ell(d_j) + 1$, the integral sum of the right-hand coefficients at level L represents z modulo ℓ^u.

For $x \in \mathrm{Sha}_{\ell,k}(E)[\ell^u]$, choose a cocycle ξ representing x. Enlarge L until the twist is trivial on the finite image of ξ and on local cobounding vectors at ℓ. Thus $h = \xi|_{G_{\mathbf{Q}(\mu_L)}}$ is an unramified homomorphism. Let $U_\infty = \ker(\varepsilon_\ell|_{G_E})$ and $f = \mathrm{res}_{U_\infty}(zx)$. For $g \in U_\infty \cap G_{\mathbf{Q}(\mu_H)}$, g and its conjugates fix μ_L. The cyclotomic weight $[t]_L^k$ cancels the action of the t-power lift on the restricted cocycle values. For such a lift σ_t,

$$f(g) = -\sum_t \big((b-1)/2 - a_t\big) h(\sigma_t^{-1} g \sigma_t) = 0$$

by 27.5. Since $g^{\varphi(H)} \in G_{\mathbf{Q}(\mu_H)}$ for every $g \in U_\infty$, we have $\varphi(H)f = 0$. For odd ℓ the cyclotomic image of G_E is open and procyclic in $\mathbf{Z}_\ell^\times$. A topological generator γ has infinite order, so multiplication by $\gamma^{-k} - 1$ is surjective on $\mathbf{Q}_\ell/\mathbf{Z}_\ell$. Subtracting the coboundary that matches a cocycle on γ kills it by continuity. Inflation–restriction therefore makes restriction to U_∞ injective, and $z\varphi(H)x = 0$.

Let c be complex conjugation. In $\mathbf{Z}_\ell[\Gamma]/(c - (-1)^j)$ the adjugate of multiplication by z gives $\Delta\varphi(H)\mathrm{Sha}^-_{\ell,k}(E) = 0$, where Δ is its determinant. Put $d_Y = Hj(j+1)!$ and $u_Y = 2b^j H^j j!$. Clearing coefficients of B_j gives $H\,\mathrm{primorial}(j+1)C_{H,j}(u) \in \mathbf{Z}$. Since $\mathrm{primorial}(j+1) \mid (j+1)!$, we have $d_Y Y \in \mathbf{Z}[\Gamma]$. The $n/2$ eigenvalues $(1 - b^j\theta(b))(-B_{j,\theta}/j)$, with θ pulled back from Γ to level H in the stated parity, are nonzero by 27.3. The Fourier bound for $B_j(x)$ gives $|B_{j,\theta}| \le 4H^j j!/(2\pi)^j$, so each eigenvalue has absolute value at most u_Y. Thus $D = d_Y^{n/2}\Delta \in \mathbf{Z}$ and

$$0 < |D| \le (d_Y u_Y)^{n/2}, \qquad \log(d_Y u_Y H) \le 30j \log(k\ell N). \tag{A.230}$$

Put $T = 3kH$. Then $2, H, j \le T$. Also $b \le T^3$ and $j+1 \le T^2$. These give $d_Y u_Y H \le T^{7j+6} \le T^{10j}$, and $\log T \le 3\log(kH)$. Taking $r = v_\ell(D\varphi(H))$ gives (27.4).

If z_0 represents a class killed by ℓ, write $\ell z_0 = dm$, where d is the coboundary map. Take $\ell m_1 = m$. Then $z_0 - dm_1$ has values killed by ℓ. Restrict to $E' = E(\mu_\ell)$, where the residual twist is trivial, to obtain an

$\mathbf{F}_\ell$-valued homomorphism unramified outside the places S over ℓ. Choose one lift per class. Equal restricted homomorphisms give equal original classes by restriction and corestriction, since $[E' : E]$ is prime to ℓ. Kummer theory identifies the group of these unramified-outside-S homomorphisms with

$$V_S = \{a \in E'^{\times} : v(a) \equiv 0 \bmod \ell \ (v \notin S)\}/E'^{\times \ell}.$$

Its quotient by the S-units modulo ℓ-th powers is $\mathrm{Cl}(\mathcal{O}_{E',S})[\ell]$. For $m' = [E' : \mathbf{Q}]$, the unit rank is $r_1 + r_2 - 1 + |S|$, with (r_1, r_2) the signature and $|S| \le m'$. The roots of unity contribute at most ℓ. Hence $|V_S| \le \ell^{2m'} h(E')$.

Each of the m' field characters has conductor dividing H. The conductor–discriminant formula gives $\mathfrak{D} = |\operatorname{disc} E'| \le H^{m'}$ [41, Ch. VII, Theorem 11.9]. By Minkowski's bound each ideal class has a representative of norm at most $Z = \sqrt{\mathfrak{D}}$ [41, Ch. I, §5]. The ideal count at each norm is bounded by the m'-fold divisor function. Sum over the first $m' - 1$ factors using $\sum_{a \le Z} 1/a \le 1 + \log Z$ to obtain

$$\log h(E') \le \log Z + (m' - 1)\log(1 + \log Z) \le 3m' \log H, \tag{A.231}$$

since $m' \le H$. With $m' \le \ell n$ and $\log \ell \ge \log 2 > 5/8$, the ℓ-torsion dimension is at most $8\ell n \log H$. For $P = \mathrm{Sha}^{-}_{\ell,k}(E)$ and each $i \ge 0$, $P[\ell^{i+1}]/P[\ell^i] \hookrightarrow P[\ell]$ by $[x] \mapsto \ell^i x$. Sum these bounds up to the exponent in (27.4) to obtain (27.5).

The Galois group of the stated compositum embeds in a 2-group times $\mathrm{Gal}(K_M/\mathbf{Q})$, which proves the assertion about odd divisors. □

Proof of 27.10

Proof. Fix ε, a as in 26.3 and put $m = \mathrm{cond}\psi$. The page and height conditions for $A_1 = 4$ follow from

$$4 \le x^2 \le 2^{a(x)} \le 2x^{2+8\varepsilon} \le x^4, \qquad 2 \le t(x) := \lfloor x^{C_t} \rfloor \le x^{C_t} \qquad (x \ge 2). \tag{A.232}$$

Take the joint window constants $J_0 > 0, C, P, X_1$, put $J = \max(J_0, 1)$ and $U = X^{1/J}$. For sufficiently large $X \ge X_1$, 26.2 gives $q, d > \max(2X^{C_t} + 1, P, m)$ for every window pair at U, and

$$\log(qd) \le U^{\varepsilon/5}(1 + \log^2 U) \le U^{\varepsilon/4} \le X^{\varepsilon/(4J_0)}. \tag{A.233}$$

Thus $\gcd(qd, m) = 1$ and the joint window gives $U \le X^{1/J_0} \le Y$ with $Y^2 \le X$.

Fix $x \in [Y, Y^2]$ and E as in 26.3. In the common cyclotomic layer the four conditions of 26.2 fix, respectively, $\mathbf{Q}(\mu_{2^{a(x)+2}})^+$, the field K_ψ cut out by ψ, and the two quadratic character fields. Since $q, d \equiv 3 \bmod 4$, these last fields are $\mathbf{Q}(\sqrt{-q})$ and $\mathbf{Q}(\sqrt{-d})$. The intersection of the fixing subgroups corresponds to their compositum, which is abelian and CM. The same intersection describes the field at level $\mathrm{lcm}(2^{a(x)+2}, m, q, d)$, so $n \le F \le 2^{a(x)+2} mqd$. By (A.232) and (A.233), for sufficiently large U and all x in the window,

$$\log(Ft) \le 3\log 2 + \log m + (2 + 8\varepsilon + C_t)\log x + \log(qd) \le 2x^{\varepsilon/4}. \tag{A.234}$$

Put $W = x^{\varepsilon/4}$, $b = x^{4\varepsilon/5}$ and $o = [K_\psi : \mathbf{Q}]$. By (27.6) the negative sum is at most $B_-(240, o, n, F, t)$. By (27.7) and (A.234) this is at most Ke^{8W} for $K = 3 + 240 \sum_{\ell \mid o,\ \ell \text{ odd prime}} \ell^3$. Combine the two bounds of (27.8) in Definition 27.9 with this bound in 27.1. The resulting total is at most

$$e^b \log 2 + Ke^{8W} + (Ft)^C + Ce^b \le (\log 2 + K + 1 + |C|)e^{(9+2|C|)b} \le e^{x^{\varepsilon}/2}.$$

The last inequality holds uniformly for $x \ge U$ as $U \to \infty$, since $x^{\varepsilon}/b = x^{\varepsilon/5}$. □

Proofs for Chapter 28: The cyclotomic tower and a common window

Proof of 28.1

Proof. The homology estimates used below are integral stability in Lemma 28.3 and the fixed-rank torsion bound in Lemma 28.5, obtained from the complex of Lemma 28.4. Write $Y = F^{\times}$ additively and $\partial : Y \to$

$I = \mathrm{Div}(F)$ for the divisor map. Put $g = \sigma^{-1}$ and $\delta = g - a$, so $(g\mathfrak{d})(P) = \mathfrak{d}(\sigma P)$. Set

$$U = \partial^{-1}(\delta I), \qquad W = U/(\delta Y + Y_{\mathrm{tors}}), \qquad C = I/(\partial Y + \delta I), \qquad E = \frac{\ker \partial}{\delta(\ker \partial) + Y_{\mathrm{tors}}}.$$

Since $g^h = 1$ and $a > 1$, δ is injective on I. The snake lemma gives $0 \to E \to W \to \ker(\delta|\mathrm{Cl}(F)) \to 0$. The last group has order $\#C$, so

$$\#W = \#E\,\#C. \tag{A.235}$$

On complex places g has $r_2(L)$ cycles of length h. The logarithmic unit lattice omits the constant direction, of eigenvalue $1 - a$ for δ. Taking determinants on this lattice gives

$$\#E\,(a-1) = (a^h - 1)^{r_2(L)}. \tag{A.236}$$

Also $\mathcal{E}_{\nu,k} = \mathrm{Hom}(C, \mathbf{Z}/\ell^\nu\mathbf{Z})$ has order at most $\ell^{v_\ell(\#C)}$. Formula (28.1) gives $v_\ell(a-1) = c + v_\ell(k)$ and $v_\ell(a^h - 1) = \nu + v_\ell(k)$.

In additive notation $\mathcal{N}_a = \sum_{j=0}^{h-1} a^j g^{-j}$, so $\mathcal{N}_a\delta = \delta\mathcal{N}_a = (1 - a^h)g$. Thus

$$f : W \longrightarrow F^\times/(F^{\times\ell^\nu}\mu(F)), \qquad [\alpha] \longmapsto [\mathcal{N}_a(\alpha)^{\ell^b}]$$

is well defined, with $\mu(F)$ the group of roots of unity. For $[u] \in \ker f$, write $\ell^b\mathcal{N}_a u = \ell^\nu y$ modulo torsion. Apply δ and cancel ℓ^ν modulo torsion. This puts $\ell^b(1 - a^h)gu/\ell^\nu$ in $\delta Y + Y_{\mathrm{tors}}$. Since g is invertible on W, the kernel is killed by $\ell^b(a^h - 1)/\ell^\nu$, of valuation $b + v_\ell(k)$. Representatives of distinct image classes satisfy the counting hypothesis, so $\#\,\mathrm{im}\, f \le B$.

For $[u] \in W[\ell]$ choose y with $\ell u = \delta y$ modulo torsion. If $y - y' = \ell z$ modulo torsion for two such choices, then $u - u' = \delta z$ modulo torsion. Thus the classes of y modulo $\ell Y + Y_{\mathrm{tors}}$ distinguish the classes in $W[\ell]$. Because $u \in U$, choose $D_u \in I$ with $\partial u = \delta D_u$. Taking divisors gives $\delta(\partial y - \ell D_u) = 0$, so $\partial y = \ell D_u$. Also $a \equiv 1 \bmod \ell$ implies invariance of the class of y under g. In multiplicative notation, for a representative α of this class and a generator ρ of $\mu(F)$, write

$$\sigma(\alpha) = \alpha v^\ell \rho^r, \qquad 0 \le r < \ell, \qquad (N_{F/L}v)^\ell = N_{F/L}(\rho)^{-r}.$$

There are at most ℓ^2 pairs $(r, N_{F/L}v)$. For equal pairs Hilbert 90 applied to the ratio of the v's gives a representative in $L^\times$ for the ratio of the α's modulo ℓth powers. The action of σ on ζ has order h by (28.1), so $F = L(\zeta)$ is unramified away from ℓ. The descended ratios have valuations divisible by ℓ at primes away from ℓ. Valuations above ℓ contribute at most $[L : \mathbf{Q}]$ dimensions modulo ℓ. Classes with all valuations divisible by ℓ map to $\mathrm{Cl}(L)[\ell]$, with kernel the units modulo ℓth powers, of dimension at most $[L : \mathbf{Q}]$. This gives $\#W[\ell] \le \ell^{g(L)}$.

Put $K = (\ker f)[\ell^\infty]$. It is killed by $\ell^{b+v_\ell(k)}$, and $K[\ell] \subseteq W[\ell]$ gives at most $g(L)$ cyclic factors. Thus $\#K \le \ell^{(b+v_\ell(k))g(L)}$ and

$$\ell^{v_\ell(\#W)} \le \#\,\mathrm{im}\, f\, \ell^{(b+v_\ell(k))g(L)} \le B\,\ell^{(b+v_\ell(k))g(L)}.$$

Equations (A.235) and (A.236) give (28.3).

To bound the weighted norms, first assume

$$2k + 1 < \ell^c, \qquad 2k + 1 < r_2(L). \tag{A.237}$$

Put $G_r = \mathrm{GL}_r(\mathcal{O}_L)$ and let T_1, T_2 be the torsion orders of $H_i(G_{4k+6}, \mathbf{Z})$ for $i = 2k+1, 2k$, respectively. They are finite by 28.5. The counting hypothesis holds with

$$b = 2k + (2k+1)(c + 2k + 1), \qquad B = (\ell^\nu)^{r_2(L)}T_1T_2. \tag{A.238}$$

Put $q_0 = \ell^\nu$, $A_\nu = \mathbf{Z}/q_0\mathbf{Z}$, $\kappa = 1 + \ell^c$ and $M_* = 2k + 1$. Define

$$P(T) = \prod_{\substack{0 \le j < M_* \\ j \ne k}} (T - \kappa^j), \qquad Q(T) = (T - a)P(T), \qquad p_{k,c} = P(a).$$

For $0 < d < M_*$, formula (28.1) gives $v_\ell(\kappa^d - 1) = c + v_\ell(d) < c + M_*$. Apply this to each factor of $P(a)$. In A_ν,

$$p_{k,c} \mid \ell^{M_*(c+M_*)}. \tag{A.239}$$

Let $C_{q_0} = \langle \zeta \rangle$, $D = C_{q_0} \times C_{q_0}$ and $d_t(s) = (s, s^t)$ for $0 \le t \le M_*$. Over A_ν the bar cycle z is the sum of $[\zeta^{i_1} \mid \zeta \mid \cdots \mid \zeta^{i_k} \mid \zeta]$ over $i_1, \dots, i_k \bmod q_0$. Translation of the indices cancels its boundary modulo q_0. The tensor product of the two cyclic periodic resolutions has $2k+1$ basis vectors in degree $2k$ and zero differentials over A_ν. Take a primitive integral relation among lifts of the $2k+2$ vectors representing $(d_t)_*[z]$. On reduction,

$$\sum_t n_t (d_t)_* z = \partial \xi, \qquad n_{t_0} \in A_\nu^\times$$

for some t_0.

Identify D with A_ν^2 by exponents and put $\pi_t(i,j) = i + tj$. We construct $\eta \in \mathbf{Z}[A_\nu^2]$ with

$$(\pi_t)_* \eta = 0 \ (t \ne t_0), \qquad (\pi_{t_0})_* \eta = \sum_i [T^i] Q(T) [\kappa^i].$$

For $J = (X^{\ell^c} - 1) \subset \mathbf{Z}[X]$, the substitution $S(f)(X) = f(X^\kappa)$ acts by κ^j on J^j / J^{j+1}. Indeed $S(f) \equiv f \bmod J$ and

$$S(X^{\ell^c} - 1) = (X^{\ell^c} - 1) \sum_{i=0}^{\kappa - 1} X^{i \ell^c}.$$

Apply the factors of $Q(S)$ in order of increasing exponent to obtain $Q(S)X \in J^{M_*}$. Under $X \mapsto [1] \in \mathbf{Z}[A_\nu]$, the desired pushforward is therefore a multiple of $([\ell^c] - [0])^{M_*}$. For $t \ne t_0$, condition (A.237) gives $t_0 - t \mid \ell^c$ in A_ν. Geometric summation gives $[t_0] - [t] \mid [\ell^c] - [0]$. Divide the desired pushforward by the product of these M_* factors and lift the quotient by $i \mapsto (i, 0)$. Multiply this lift by $\prod_{t \ne t_0} ([(0,1)] - [(t,0)])$ to obtain η.
Lean: `UpperTheory.V.ZK.sEta` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.ExpStep`, lines 116–182)

For $\partial \alpha = (g - a)\mathfrak{d}$, put $\beta = \prod_i \sigma^{-i}(\alpha)^{[T^i]P(T)}$. Then $\partial \beta = Q(g)\mathfrak{d}$. On F^D use the diagonal representation ρ_0 with character $(s_1, s_2) \mapsto \zeta^{i s_1 + j s_2}$ at coordinate (i, j). It preserves $\mathcal{O}_F^D$ and the lattice whose coordinate (i, j) is the ideal of divisor $\eta(i, j)\mathfrak{d}$.

Construct an L-linear isomorphism G_t from the standard to the second lattice in scalar blocks $\pi_t = x$. For $t \ne t_0$ each block has total target divisor zero. For $t = t_0$ apply σ^{-i} to coordinates with $\pi_{t_0} = \kappa^i$, taking i modulo h. The scalar becomes ζ and the divisors transform by g^i. Leave other coordinates fixed. The exponent-1 block now has total divisor $\partial\beta$ and all other blocks have divisor zero. Steinitz classification gives an isomorphism in each block. Multiplication by a unit on an ideal summand adjusts the determinant to β on the distinguished block and 1 elsewhere. Undo the coordinate conjugations. The resulting G_t commutes with $\rho_0 d_t$.
Lean: `UpperTheory.V.ZK.Cl.cBetaDiv` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.LatticeBlockClasses.CBetaDiv`, lines 33–62)
Lean: `UpperTheory.V.ZK.Cl.cGtEq` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.LatticeBlockClasses.GtEqZeta`, lines 54–106)

Let H be the stabilizer of the standard lattice. Put $G_* = G_{t_0}$, $\rho_1 = G_*^{-1} \rho_0 G_*$, and $b_t = G_*^{-1} G_t$. Both representations take values in H, and $b_t \in H$ satisfies $\mathrm{conj}_{b_t^{-1}} \rho_1 d_t = \rho_0 d_t$, where $\mathrm{conj}_b(v) = bvb^{-1}$. The bar prism p_b inserts b with alternating signs and conjugates subsequent entries by b^{-1}. Its boundary satisfies $\partial p_b + p_b \partial = (\mathrm{conj}_{b^{-1}})_* - 1$. Hence

$$y = (\rho_1)_* \xi - (\rho_0)_* \xi + \sum_t n_t p_{b_t}((\rho_1 d_t)_* z)$$

is a cycle, since the boundary of the sum is $(\rho_0)_* \partial\xi - (\rho_1)_* \partial\xi$.

Choose $d \in \mathcal{O}_L \setminus \{0\}$ to clear denominators of the maps, α and their inverses. Let G_d be the group of L-linear automorphisms preserving $\mathcal{O}_F[1/d]^D$, with inclusion $\iota : H \hookrightarrow G_d$. For a homogeneous n-simplex $(v_0, \dots, v_n)$ in a group bar complex, take the oriented staircase chain for $\Delta^n \times \Delta^2$ with its three vertex rows v, vb and vbb'. The terms for $0 \le i \le j \le n$ are

$$(-1)^{i+j} [v_0, \dots, v_i,\ v_i b, \dots, v_j b,\ v_j bb', \dots, v_n bb'].$$

Interior faces cancel in pairs. The boundary over $\partial\Delta^n$ is the same construction on the boundary of the input. The other faces are the prisms between rows, with the long edge opposite in sign to the two consecutive edges. In inhomogeneous coordinates the row vb has entries conjugated by b^{-1}. Thus on any cycle z',

$$p_{bb'}z' - p_b z' - p_{b'}(\mathrm{conj}_{b^{-1}})_* z' \in \mathrm{im}\,\partial.$$

Use this with $b = G_*$, $b' = b_t$ and $Z_t = (\rho_0 d_t)_* z$. The prism boundary identity also gives $(\rho_1)_*\xi - (\rho_0)_*\xi \equiv p_{G_*}(\rho_0)_*(\partial\xi)$ modulo boundaries. Summing over t yields

$$\iota_* y \equiv \sum_t n_t p_{G_t} Z_t \pmod{\mathrm{im}\,\partial}.$$

The standard lattice is projective over $\mathcal{O}_L$. Adjoin a complementary summand with trivial action to obtain a free lattice. Push $n_{t_0}^{-1} y$ into the resulting general linear group. Call this degree-$2k+1$ cycle y_α.
Lean: `UpperTheory.V.ZK.Cl.cChain` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.LatticeBlockClasses.CChain`, lines 8–63)
Lean: `UpperTheory.V.ZK.Cl.cJoin` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.LatticeBlockClasses.CJoin`, lines 5–187)

Take a prime P of F unramified of residue degree one over L, away from ℓ and d. Put $K = k(P)$ and $d_K = \#K - 1$. The reduction $\bar{\zeta}$ has order q_0. Choose a generator g of $K^\times$ with $g^{d_K/q_0} = \bar{\zeta}$. For $v \in K^\times$ let $L_K(v)$ be its logarithm to base g in $[0, d_K)$ and put $\mathrm{ind}(v) = L_K(v) \bmod q_0$. The function

$$U_K(v,w) = \frac{L_K(v) + L_K(w) - L_K(vw)}{d_K} \pmod{q_0}$$

is a symmetric two-cocycle by cancellation of logarithms. In evaluating $U_K^{\smile k} \smile \mathrm{ind}$ on a scalar prism, adjacent insertions within each U_K pair cancel by symmetry. The last insertion contributes $\mathrm{ind}(v)$ for the inserted scalar v. For scalar exponent x each pair contributes

$$\sum_{i \bmod q_0} U_K(\bar{\zeta}^{xi}, \bar{\zeta}^x) = q_0 L_K(\bar{\zeta}^x)/d_K = x \quad \text{in } A_\nu.$$

For a finite-dimensional K-space V, choose a nonzero vector on each line. For a tuple $(g_1, \dots, g_{2k+1})$ and initial line λ_0, set $\lambda_i = g_i^{-1}\lambda_{i-1}$. Record the scalar u_i of g_i on the chosen vectors from λ_i to λ_{i-1}. Summing the cocycle on the recorded tuples over λ_0 defines $\theta_{P,V}$. Multiplication of recorded scalars agrees with the bar faces. If the vectors on the successive lines are rescaled by d_i, the new scalars are $u_i' = d_{i-1}^{-1} u_i d_i$. At each cut i, insert d_i between the old prefix and the new suffix, with sign $(-1)^i$. Faces at successive cuts cancel by $u_i d_i = d_{i-1} u_i'$. The two endpoint faces give the difference of the records, so the two cocycles differ by a coboundary. In particular their evaluations on cycles agree and are invariant under isomorphism.

For representations $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ and $z' \in Z_{2k+1}(G; A_\nu)$,

$$\theta_{P,V\oplus W}((\rho \oplus \sigma)_* z') = \theta_{P,V}(\rho_* z') + \theta_{P,W}(\sigma_* z').$$

Indeed, the projection of lines outside W to V is equivariant for the block diagonal action, and each line of V has $(\#K)^{\dim W} \equiv 1 \bmod q_0$ lifts. Choose vectors on the lifts by projection to the chosen vectors in V. The lines in W contribute once each. A trivial summand contributes zero.

For a scalar action of exponent x, the prism multiplication identity makes the evaluation a homomorphism in the inserted matrix M, since conjugation fixes the scalar cycle. It factors through the determinant. Elementary transvections generate $\mathrm{SL}(V)$ and are commutators with diagonal matrices, since $\#K > 2$. Replace M by $\mathrm{diag}(\det M, 1, \dots, 1)$. Additivity and the scalar computation give $x^k \,\mathrm{ind}(\det M)$.
Lean: `UpperTheory.V.ZK.sCoc` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.FProd`, lines 101–150)
Lean: `UpperTheory.V.ZK.sTrans` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.STrans`, lines 411–413)
Lean: `UpperTheory.V.ZK.sThetaOf` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.SThetaOf`, lines 192–273)

If $r : \mathcal{O}_F \to K$ is reduction at P, the maps $r \circ \sigma^j$ identify $\mathcal{O}_F \otimes_{\mathcal{O}_L} K$ with $\prod_{j=0}^{h-1} K$ by the Chinese remainder theorem at the split prime below P. Reduce $(\rho_0 d_t, G_t)$ in block coordinates. For $t = t_0$ the

distinguished block in component j has exponent κ^j and determinant $\overline{\sigma^j(\beta)}$. All other determinants are 1. Write θ_P for evaluation after adjoining the trivial complement. The chain congruence and $n_{t_0}^{-1}$ give

$$\theta_P(y_\alpha) = \sum_{j=0}^{h-1} a^j \operatorname{ind}(\sigma^j(\beta)) = P(a) \sum_{j=0}^{h-1} a^j \operatorname{ind}(\sigma^j(\alpha)) = p_{k,c} \operatorname{ind}(\mathcal{N}_a(\alpha)).$$

Here ind acts after reduction at P. Expand β. Since $a^h \equiv 1 \bmod q_0$, shifting j by i in the coefficient of $[T^i]P(T)$ gives the middle equality.

Lean: `UpperTheory.V.ZK.Cl.cBetaVal` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.LatticeBlockClasses.CBetaVal`, lines 31–122)

For a finite set of α's let $\Phi = (\theta_P)_P$ over the common good primes. If $\ell^{2k}\Phi(y_\alpha) = \ell^{2k}\Phi(y_{\alpha'})$, then by (A.239)

$$\mathcal{N}_a(\alpha)^{\ell^b} \mathcal{N}_a(\alpha')^{-\ell^b}$$

is a q_0th power at every such prime. Its residue roots all lie in $k(P)$, and reduction is injective on μ_{q_0}. Hence Frobenius fixes a root in the Kummer extension and acts trivially. Primes of F of residue degree greater than one over L have density zero by the bound on their norms by powers of the underlying prime norms. Chebotarev gives the global q_0th power relation [41, Chap. VII, §13].

Lean: `LateralTheory.Q.SDetectCut.goal` (`Development.EtaleCohomology.CyclotomicDiscriminants.SDetectCut`, lines 101–142)

Stabilize the cycles to a common rank $r \geq 4k + 6$. Additivity preserves Φ under stabilization. The Bockstein for $0 \to \mathbf{Z} \xrightarrow{q_0} \mathbf{Z} \to A_\nu \to 0$ has image in $H_{2k}(G_r, \mathbf{Z})[q_0]$. This subgroup has at most T_2 elements by 28.3, since $2(2k) + 4 \leq 4k + 6$. Differences within a fibre lift to integral homology in degree $2k + 1$. For such a lift x, let D_ℓ repeat a matrix in ℓ diagonal blocks and let S stabilize to the same rank by an identity block. Set

$$z_0(x) = x, \qquad z_{t+1}(x) = (D_\ell)_* z_t(x) - \ell^{t+2} S_* z_t(x).$$

At each step the value of Φ after reduction is multiplied by $\ell(1 - \ell^{t+1})$. The second factor is a unit in A_ν. Thus $\ell^{2k}\Phi(x \bmod q_0)$ is the evaluation of an integer multiple of $z_{2k}(x)$.

Put $d = 2k + 1$ and $r' = \ell^{2k} r$. By $d < r_2(L)$ in (A.237), the stable comparison gives an $N \geq r'$ at which every real degree-d cohomology class is represented on cycles by an invariant form on the product of complex symmetric spaces. Stabilization from r' to N is injective on real homology by 28.3. Borel's stable description gives the odd-degree primitive classes on the special linear group [113, §10.6, Thm. 11.1, and §11.4]. The subgroup with determinant in $(\mathcal{O}_L^\times)^N$ is the product of the special linear group and the scalar units modulo a finite central group. Over $\mathbf{R}$, the Künneth formula and the unit theorem add the degree-one determinant logarithms. The finite determinant quotient acts trivially in stable cohomology, since conjugation by $g \in \mathrm{GL}_N(\mathcal{O}_L)$ is realized after stabilization by $\operatorname{diag}(g, \det(g)^{-1})$ in the special linear group. These are additive under block sum and compatible with stabilization. A product of j primitives pulls back under D_ℓ to ℓ^j times the product. Each degree-d evaluation at N is of the form

$$\sum_{w|\infty} c_w \tau_w + \sum_{j=2}^{d} a_j,$$

where τ_w is the degree-d primitive at w and a_j is a sum of products of j primitives. The recurrence multiplies weight j by $\prod_{t=0}^{2k-1}(\ell^j - \ell^{t+2})$, which vanishes for $2 \leq j \leq d$. If the τ_w vanish on a real linear combination of recurrence images, its stabilization pairs to zero with every real cohomology class. It therefore vanishes already at r'. Thus the integral image has rank at most $r_2(L)$, the real rank of $K_{2k+1}(\mathcal{O}_L)$ in this case [113, Prop. 12.2].

Lean: `PositivityTheory.KB.sBStarOdd_of` (`Development.AutomorphicArithmetic.AutomorphicVarieties.LiteratureResults.KTheoryStability.ReductionBoundsTorsion.OddSpanRank.SBStarOddOf`, lines 13–78)

At r' the torsion in degree $2k + 1$ has order T_1 by 28.3. In each Bockstein fibre choose y_0. The integral recurrence images of lifts of $y_\alpha - y_0$ generate a group of rank at most $r_2(L)$ and torsion order at most T_1. Its image under reduction and Φ contains every $\ell^{2k}(\Phi(y_\alpha) - \Phi(y_0))$ in the fibre, and has at most $q_0^{r_2(L)} T_1$ elements. There are at most T_2 fibres. Separation by $\ell^{2k}\Phi$ proves (A.238).

Lean: `UpperTheory.V.ZK.zSeq` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.HasVal`, lines 56–60)

Lean: UpperTheory.V.ZK.BOdd (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.LemmaKOfSteps, lines 11–15)
Lean: UpperTheory.V.ZK.selmer_upper_bound_of_inputs (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.SUpperOf, lines 21–62)
Lean: UpperTheory.V.ZK.sUpperOf (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.SUpperOf, lines 64–66)
Lean: UpperTheory.V.ZK.bOdd (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.BOdd, lines 3–3)
Lean: LateralTheory.Q.SFewOfCut.sFewOf (Development.EtaleCohomology.CyclotomicDiscriminants.SFewOfCut, lines 153–211)

Substituting (A.238) into (28.3) gives

$$\#\mathcal{E}_{\nu,k} \le T_1 T_2\, \ell^{(b+v_\ell(k))g(L)+c+v_\ell(k)}. \tag{A.240}$$

Take c maximal with $\mu_{\ell^c} \subset L$. The cyclotomic image is closed in $1+\ell\mathbf{Z}_\ell$ for odd ℓ and in $1+4\mathbf{Z}_2$ for $\ell=2$. Its logarithm is a closed additive subgroup, hence an ideal of $\mathbf{Z}_\ell$. Maximality of c identifies this ideal as $\ell^c\mathbf{Z}_\ell$, so the image is $1+\ell^c\mathbf{Z}_\ell$. At $F=L(\mu_{\ell^\nu})$, $\nu \ge c$, choose σ acting on ζ by $1+\ell^c$. Its order is $\ell^{\nu-c}$ by (28.1).

Put $H_\nu = G_{L(\mu_{\ell^\nu})}$ and $H_\infty = \bigcap_\nu H_\nu \subset G_L$. By 28.9 and its restriction description, Sha corresponds to equivariant A-valued characters of H_∞ unramified everywhere and split above ℓ. For such a character ϕ, choose open normal $N \subset G_L$ with $H_\infty \cap N \subset \ker\phi$. The quotient $H_\infty N/N$ extends ϕ to a character $\widetilde{\phi}$ of $H_\infty N$, which contains H_ν for large ν by compactness. Inertia away from ℓ lies in $H_{\infty,\widetilde{\phi}}$. For splitting, the elements in the conjugates of the finitely many decomposition groups over ℓ on which $\widetilde{\phi}$ is nonzero form a compact subset of $H_\infty N$ disjoint from H_∞. For equivariance, the pairs $(\gamma,u) \in G_L \times H_\infty N$ for which

$$\widetilde{\phi}(\gamma u \gamma^{-1}) \ne \chi_\ell(\gamma)^{-k}\widetilde{\phi}(u)$$

form a compact set disjoint from $G_L \times H_\infty$. The conditions are locally constant, and the conjugates of each decomposition group are parametrized by the compact group G_L. Increasing ν excludes these bad sets from H_ν and $G_L \times H_\nu$. The finite image of $\widetilde{\phi}$ is killed by ℓ^ν for large enough ν.

For a finite set of Sha classes, choose a common such $\nu \ge \max(c,2)$ and identify $A[\ell^\nu]$ with A_ν. At this layer take $\varphi(P) = \widetilde{\phi}(\mathrm{Frob}_P)$. Global reciprocity makes the sum on every principal divisor zero. Equivariance under σ^{-1} gives $\varphi(\sigma^{-1}P) = a\varphi(P)$. Class field theory recovers the characters of H_ν from their prime values, and restriction recovers the original characters. Bound (A.240) thus holds for every finite subset of Sha. Its length s satisfies

$$\ell^s \le T_1 T_2\, \ell^{(b+v_\ell(k))g(L)+c+v_\ell(k)}. \tag{A.241}$$

Lean: UpperTheory.V.Kum.xdualExt (Development.AutomorphicArithmetic.AutomorphicVarieties.GaloisLibrary.KummerTheory.XdualExt, lines 82–117)
Lean: UpperTheory.V.Kum.eqExt (Development.AutomorphicArithmetic.AutomorphicVarieties.GaloisLibrary.KummerTheory.CycZConjGal, lines 132–177)
Lean: PositivityTheory.QuaternionShaBounds.ShaAssumptions.good_of (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.RingEquivalences.OrderOfOneAddPrimePowZmodPrimePow, lines 390–410)
Lean: PositivityTheory.QuaternionShaBounds.ShaAssumptions.card_le (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.RingEquivalences.OrderOfOneAddPrimePowZmodPrimePow, lines 412–448)
Lean: PositivityTheory.QuaternionShaBounds.ShaAsm_holds (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.RingEquivalences.OrderOfOneAddPrimePowZmodPrimePow, lines 454–486)

To arrange (A.237), take the layer L'/L of index $m = \lceil \log_\ell(4k+4) \rceil$. Write $\delta = [L':L] = \ell^m$, $n = [L:\mathbf{Q}]$, $n' = \delta n$, and let $c = c(L')$ be the maximal root level in L'. The layer is generated over L by a primitive $\ell^{c(L)+m}$th root, so $c \ge c(L)+m$. The derivative of $X^{\ell^{c(L)+m}} - 1$ bounds the relative different by divisibility into $(\ell^{c(L)+m})$. Use $\ell \le n+1$ and the degree of $\mathbf{Q}(\mu_{\ell^c})$ to obtain

$$2k+1 < \min(\ell^c, r_2(L')), \qquad \delta \le (n+1)(4k+4), \qquad \ell^c \le 2n', \qquad |D_{L'}| \le |D_L|^\delta (\ell^c)^{n'}. \tag{A.242}$$

With $X = X_\ell(L)$ and $T = u^k\tau$ from 28.9, divisibility of $T^{\ell^m}-1$ by $T-1$ gives a surjection $X/(T^{\ell^m}-1)X \to X/(T-1)X$. Thus $s_\ell(L,k) \le s_\ell(L',k)$.
Lean: LateralTheory.Q.SLayerCut.goal_of (Development.EtaleCohomology.CyclotomicDiscriminants.SLayerCut, lines 55–106)

Put $v = 2kn'$ and $H' = \log(2|D_{L'}|)$. By Minkowski's bound each class has an integral ideal of norm at most $2^{n'/2}|D_{L'}|^{1/2}$ [41, Ch. I, §6, Exercise 3]. Indeed the Minkowski factor in degree n' with r_2 complex

pairs is $(4/\pi)^{r_2} n'!/(n')^{n'} \le 2^{n'/2}$. The number of ideals of norm at most B is at most $B^2 \zeta_{L'}(2)$, and $\zeta_{L'}(2) \le \zeta(2)^{n'} < 2^{n'}$ by the Euler factors. Hence $h_{L'} \le 4^{n'} |D_{L'}|$ and

$$g(L') \le 2n' + 4n'H' + 2. \tag{A.243}$$

With $c, \log \ell, v_\ell(k) \le v$ and $b \le v + 2v^2$, use (A.243) and (A.241). Bounding both torsion orders by (28.7) at rank $4k+6$ gives

$$s_\ell(L', k) \log \ell \le 2 \exp(v^{24} H') + v^9 H' \le \exp(v^{26} H').$$

By (A.242), $v \le (2kn)^5$ and $H' \le (2kn)^{11} \log(2|D_L|)$. This gives (28.2) with $C = 141$. □

Proof of 28.2

Proof. Write $A = (\mathbf{Q}_\ell/\mathbf{Z}_\ell)(-k)$ and let $\lambda : M \to P$ be localization, with $M = H^2(G_{L,S_\ell}, \mathbf{Z}_\ell(k+1))$ and $P = \prod_{v|\ell} H^2(L_v, \mathbf{Z}_\ell(k+1))$. Write len for $\mathbf{Z}_\ell$-length. The reductions in degree two identify the finite-coefficient localization map with $M/\ell^r M \to P/\ell^r P$. The degree-two reduction maps are surjective with kernels the indicated multiples, and M and P are finitely generated.

Since $k \ne 0$, the local cyclotomic images give finite $H^0(L_v, A)$. Choose ℓ^a killing all these invariants. Finite local Tate duality gives $\#(P/\ell^r P) = \prod_{v|\ell} \#H^0(L_v, A[\ell^r])$. The right side stabilizes, so P is finite. Choose ℓ^b killing P. For $r \ge a + b + 1$ the finite degree-two localization kernel is $\ker\lambda/(\ker\lambda \cap \ell^r M)$. By 27.2 it has the same order as the degree-one localization kernel for $A[\ell^r]$.

The latter kernel injects into $Ш_{\ell,k}(L)$ for $r \ge a$. Indeed, a representing cocycle that becomes a global coboundary $(g-1)t$ in A is locally $(g-1)t_v$ for $t_v \in A[\ell^r]$. Then $t - t_v \in H^0(L_v, A) \subset A[\ell^r]$, so $t \in A[\ell^r]$. The images of the finite-level kernels increase and exhaust Ш. A continuous representative and the finitely many local coboundaries all occur at a finite level. The filtration $\ker\lambda \cap \ell^r M$ is separated. Its quotient orders therefore give $\operatorname{len}\ker\lambda = s_\ell(L, k)$. The finite-level cokernel has order $\#H^0(L, A[\ell^r])$ by 27.2. For large r this gives $\operatorname{len}\operatorname{coker}\lambda = \operatorname{len} H^0(L, A)$. This proves (28.5).

The local cyclotomic image has index $d_v \le [L_v : \mathbf{Q}_\ell] \le n$ and contains $(1+\ell^2)^{d_v}$. By (28.1), its invariants on A have length at most $2 + v_\ell(d_v k)$. Hence

$$\sum_{v|\ell} \operatorname{len} H^0(L_v, A) \le n(2 + \lfloor \log_\ell n \rfloor + k). \tag{A.244}$$

Multiplication by $\log \ell$ bounds this term by $n(3+k)\log(n+1)$ since $\ell \le n+1$. Apply (28.2) in (28.5) and increase the exponent to absorb this term. A module of length $v < \infty$ has order ℓ^v. □

Proof of 28.3

Proof. The Bass stable rank of $\mathcal{O}_L$ is at most two. For a unimodular row $(a_1, \dots, a_t, b)$ with $t \ge 2$, if all the a_j vanish, add the unit b to the first entry. Otherwise take $a_j \ne 0$ and $h \ne j$. Choose u by the Chinese remainder theorem to be 1 at primes dividing both a_j and a_h, and 0 at the other primes dividing a_j. A prime containing the first t entries after replacing a_h by $a_h + ub$ would divide a_j, hence a_h and b, contrary to unimodularity.

For $i \ge 1$, van der Kallen's integral constant-coefficient theorem gives an isomorphism from rank r to rank $r+1$ when $r \ge 1 + \max\{2i+1, i + \operatorname{sr}(\mathcal{O}_L)\}$ [114, Theorem 4.11(i) and Corollary 4.12(ii)]. Take the full linear group in his notation. The preceding stable-rank calculation gives $\operatorname{sr}(\mathcal{O}_L) \le 2$, and $2i + 4 \ge 1 + \max\{2i+1, i+2\}$. In degree zero stabilization is the identity on $\mathbf{Z}$. Compose the consecutive maps and use 28.5 for finiteness. □

Proof of 28.4

Proof. For $N = 0$ take tuples of one point, with $0^0 = 1$. For $N \ge 1$ let P be the space of tuples $h = (h_w)$ of Hermitian forms at the complex places cut out by $h[v] = \sum_w v_w^* h_w v_w \ge 1$ for $0 \ne v \in \mathcal{O}_L^N$. The standard form scaled up lies in P by discreteness. In an integral basis the real Gram matrix B of $h \in P$ is positive semidefinite by density of rational directions. Write $B + \delta\, 1 = S^t S$, $\delta > 0$. Minkowski's theorem for a box

gives $0 \neq z \in \mathbf{Z}^d$ with $\|Sz\|_\infty \leq 2|\det S|^{1/d}$ [41, Ch. I, Thm. 4.4]. Here the lattice is $S\mathbf{Z}^d$, of covolume $|\det S|$, and the centered box of radius $2|\det S|^{1/d}$ has volume $4^d|\det S|$. Thus $1 \leq 4d|\det S|^{2/d}$ and $(4d)^d \det B \geq 1$ on letting $\delta \to 0$. All h_w are positive definite. The integral basis has real covolume $2^{-n/2}\sqrt{D}$ in the complex embeddings. Each Hermitian determinant occurs twice in the real determinant, so

$$2^d \det B = D^N \Big(\prod_w \det h_w\Big)^2, \qquad \prod_w \det h_w \geq I^{-1}. \tag{A.245}$$

The inequality uses $D \geq 1$ and $d \geq 2$.

Let X_p consist of ordered $(p+1)$-tuples, with repetitions, of vertices of P minimizing a common linear functional positive on every nonzero positive semidefinite tuple. Such a functional has compact sublevels on P by its positive minimum on the unit section of the semidefinite cone. Its minimizing face has a vertex. Near each $h \in P$ the forms have a uniform positive lower eigenvalue, so locally finitely many lattice inequalities can become active. At a vertex the active evaluations span the dual space, since an annihilated direction would allow perturbations of both signs.

For a finite cycle choose a witnessing functional for each tuple and let K be their convex hull. Fix $h_0 \in P$. A vertex minimizing a functional of K lies in a witness sublevel $f(h) \leq f(h_0)$, by the convex combination defining that functional. These finitely many compact sublevels contain finitely many vertices by local finiteness. Their minimizing cones intersect K in a finite closed convex cover. The ordered nerve of this cover lies in $X_\bullet$ and contains the cycle.

For a finite closed convex cover of a nonempty compact convex K, the ordered nerve has exact augmented chains. Induct on the number of realizable subsets of indices, namely subsets whose cover members meet at a point of K. If all members meeting K share a point, prepending one fixed index gives a contraction, also for repeated vertices. Otherwise choose a maximal realizable subset and a member meeting K but disjoint from that subset's intersection in K. Strictly separate the two compact convex sets by a hyperplane. The two closed halves of K and their common section are nonempty. Each loses at least one realizable subset. The ordered nerves of the halves have union the full nerve and intersection the nerve of the section. The segment joining the witnesses in the two halves lies in every member of the tuple. The short exact sequence of augmented chain complexes for this union and intersection, with the first map given by the two inclusions with opposite signs, proves exactness by induction. Thus the cycle in $X_\bullet$ bounds. A minimizing vertex gives surjectivity of the augmentation, and change of basis preserves the tuple condition.

At the average of a tuple the active vectors are its common minimal vectors. If they spanned a proper L-subspace, choose a nonzero L-linear functional λ vanishing there. Subtracting a small multiple of the semidefinite tuple $(|\lambda_w|^2)_w$ preserves the active equalities and stays in P by local finiteness. It decreases the common functional, a contradiction. Take N independent common minima as columns of M. Put $H_w = M_w^* h_w M_w$ at a vertex and $i = |N_{L/\mathbf{Q}}(\det M)|$. The diagonal entries of H_w are at most one. Hadamard's inequality follows by Gram–Schmidt, since the volume of the columns is the product of their successive orthogonal projection lengths, bounded by the product of their lengths. It gives $\det H_w \leq 1$ and, by (A.245),

$$i/I \leq \prod_w \det H_w = i \prod_w \det h_w \leq 1.$$

Thus $1 \leq i \leq I$ and every $\det H_w \geq I^{-1}$. The matrix $C = iM^{-1}$ is integral because $\mathcal{O}_L^N / M\mathcal{O}_L^N$ has order i. For fixed i, equal residues of C, C' modulo i make $M'M^{-1} = 1 + M'(C - C')/i$ and its inverse integral. This gives at most $\sum_{i=1}^{I} i^{nN^2} \leq I^{1+dN}$ left G-classes of M.

Fix a representative M. The trace bound N gives least eigenvalue at least $I^{-1}N^{1-N}$ for each H_w. If v is minimal at such a vertex, $\|M_w^{-1}v_w\|_2^2 \leq IN^{N-1}$. Each conjugate coordinate of Cv therefore has absolute value at most R. Distinct algebraic integers are separated by at least one in the maximum embedding norm, by the field norm. Packing radius-1/2 balls bounds the count per coordinate by $(2R+1)^n$. As $v \mapsto Cv$ is injective, there are at most $(2R+1)^d$ possible minimal vectors. At most dN independent active equations determine a vertex. Allowing an empty slot for padding gives at most $\big((2R+1)^d + 1\big)^{dN}$ lists. Count the $p+1$ entries to obtain U_p. With $B_0 = 2Dd$, one has $I \leq B_0^{2d}$ and $R \leq B_0^{5d}$. The resulting exponent is at most

$$2d(1+dN) + \big((5d+2)d+1\big)dN(p+1) \leq 12(p+1)d^3N.$$

A stabilizer fixes the first form and is finite, since its columns are lattice vectors of bounded length. If its kernel modulo 3 is nontrivial, take an element of prime order q in it, of the form $1+3^aY$ with $a \geq 1$ maximal. For $q \neq 3$ its qth power differs from 1 by $q3^aY$ modulo 3^{a+1}. For $q = 3$ its cube differs from 1 by $3^{a+1}Y$ modulo 3^{a+2}. Both contradict its order. Reduction embeds the stabilizer into the set of 3^{nN^2} residue matrices. □

Proof of 28.5

Proof. Put $C_p = \mathbf{Z}[X_p]$, $K_0 = \mathbf{Z}$ and $K_{p+1} = \ker(C_p \twoheadrightarrow K_p)$, using exactness of 28.4. For $p < m$ and $j = m - p$, Shapiro's lemma identifies $H_j(G, C_p)$ with the sum of $H_j(S, \mathbf{Z})$ over the orbit stabilizers S. Transfer kills each summand by $|S|$. Its bar cycles lie in a free group of rank $|S|^j$, so the summand has order at most $|S|^{|S|^j}$. Since $U_p \leq A$, this gives $\#H_j(G, C_p) \leq s^{As^j}$. In the sequence

$$H_j(G, C_p) \longrightarrow H_j(G, K_p) \longrightarrow H_{j-1}(G, K_{p+1})$$

the torsion in the middle has at most $\#H_j(G, C_p)$ elements over each torsion value on the right.

The group $(K_m)_G$ is the cokernel of $(C_{m+1})_G \to (C_m)_G$. The target is free on at most A orbits and each column has ℓ^1-norm at most $m+2$ from the alternating deletion sum. If the matrix has rank t, its torsion order divides each nonzero t-rowed minor by Smith normal form. Hadamard's inequality bounds that minor by $(m+2)^t$. Multiplication along the m exact sequences gives $(m+2)^{As^A \sum_{j=1}^{m} s^j} \leq (m+2)^{As^{mAs^m}}$.

Put $u = kn$, $H = \log(2D)$ and $a = 12(m+1)d^3N\log(2Dd)$. By (28.6), $A \leq e^a$. Using $\log(m+2) \leq m+1$ and $\log s \leq s$ gives

$$\log T \leq A(2m+1)s^{m+1} \leq \exp\bigl(a + 2(m+1)nN^2 + 2m\bigr).$$

Here $d, N \leq 10u$, $m+1 \leq 4u$ and $\log(2Dd) \leq 16uH$. Hence $12d^3N\log(2Dd) + 2nN^2 \leq 1\,920\,300u^5H$ and $2m \leq 12u^6H$. The exponent is at most $7\,681\,212u^6H \leq (2u)^{24}H$. □

Proof of 28.7

Proof. By 28.6 and class field theory, the unramified mod-ℓ character groups of the finite layers have order at most ℓ^r. Their restriction images increase and exhaust the infinite-layer group by the finite-quotient extension argument of 28.1, with inertia at ℓ in place of decomposition in the compactness argument. Compact Nakayama gives at most r generators for XX and its quotient XP. Take $R_\ell(A) = \max(1, r)$.

In the tower of $K_M(i)$, allow ramification above q, d. Tameness gives at most one mod-2 inertia dimension per place. In the tower over $\mathbf{Q}(i)$, an odd $p \equiv 3 \bmod 4$ has Frobenius $p^2 \in 1 + 4\mathbf{Z}_2$ and decomposition index $2^{v_2(p+1)-1}$. For q, d these indices are $2, 4$, so after base change there are at most $6n_M$ places in all.

At one quadratic adjunction take index-two Galois subgroups $U \subset V$ and choose $\sigma \in V \setminus U$. Conjugation by σ^{-1} induces an involution ι on characters of U, since $\sigma^2 \in U$. Over $\mathbf{F}_2$, $(\iota - 1)^2 = 0$. An invariant character ϕ extends exactly when $\phi(\sigma^2) = 0$. The obstruction has at most two values and any extending character has two choices on σ, both preserving ramification restrictions because the excluded inertia groups lie in U. Thus the invariant dimension is at most the character dimension on V. Since $\operatorname{im}(\iota - 1) \subseteq \ker(\iota - 1)$, the total dimension on U is at most twice the invariant dimension. Two adjunctions give $4(g + 6n_M)$ for the unramified subspace and hence for both duals. □

Proof of 28.8

Proof. Restriction preserves Sha. If $E' = E$, restriction is the identity. For a quadratic extension inflation–restriction identifies the kernel on H^1 with $\ker(1 + \sigma : A \to A)/(\sigma - 1)A$, where $A = ((\mathbf{Q}_2/\mathbf{Z}_2)(-k))^{G_{E'}}$ and σ generates the quotient. This subquotient of a locally cyclic group is killed by 2 and has length at most one.

Adjoin $\mu_{2^{a+2}}$ to $K_{M,q,d}$ to get L_m with $m \leq a + 2$. Under a common embedding $E \subset L_m$ has index dividing two, by the index of the real cyclotomic subfield. Apply (28.9). The three quadratic adjunctions give the degree bound, and their conductors $4, q, d$ give the embedding. □

Proof of 28.9

Proof. Write $\Gamma_m = \mathrm{Gal}(L_\infty/L_m)$ and $A = (\mathbf{Q}_\ell/\mathbf{Z}_\ell)(-k)$. The generator minus the identity acts on A by the surjective scalar $u^{-k\ell^m} - 1$. Hence $H^1(\Gamma_m, A) = 0$. Also $H^2(\Gamma_m, A) = 0$ by cohomological dimension one. Restriction identifies global classes with equivariant characters of G_{L_∞}. Inertia away from ℓ lies in this subgroup. Above ℓ, a local class whose character vanishes on the kernel in a decomposition group descends to an open subgroup of Γ_m, where H^1 again vanishes by the nonzero scalar. Thus Sha identifies with $(\mathcal{D}_\ell(L_0)(-k))^{\Gamma_m}$. Duality preserves length and turns invariants into coinvariants for $(u^k\tau)^{\ell^m}$. □

Proof of 28.10

Proof. Lift τ^{ℓ^j} to a matrix S on $\mathbf{Z}_\ell^R$ along a surjection to X. For $t = u^{k\ell^j}$, the cokernel of $tS - 1$ surjects onto the one defining $w_{\tau,u}(k, j)$. By Smith normal form, its length in the extended nonnegative integers is the valuation of the determinant, hence $\sum_i v_\ell(tz_i - 1)$ for the eigenvalues z_i of S with multiplicity.

For ℓ-adic units A, B and algebraic z, $|A-B|_\ell \le \max(|Az-1|_\ell, |Bz-1|_\ell)$. Use the ultrametric inequality when $|z|_\ell \ge 1$ and the fact that both right-hand norms are 1 when $|z|_\ell < 1$. Formula (28.1) with exponent $\ell^j|a-b|$ gives

$$|(u^{\ell^j})^a - (u^{\ell^j})^b|_\ell = \ell^{-(j+c)}|a-b|_\ell \qquad (a, b \in \mathbf{N}).$$

For $a = b$ both sides are zero. The balls in $\mathbf{Z}_\ell$ centred at $a \in \mathbf{N}$ with radii $r_a = \ell^{j+c}|u^{a\ell^j} z_i - 1|_\ell$ are nested because $|a-b|_\ell \le \max(r_a, r_b)$. Compactness gives s_i in their intersection. Take $a = k$ to obtain (28.12).

If $R = 0$, then $X = 0$ and all the length bounds are immediate. For $R \ge 1$ put $g = u^k\tau$ and $h = \lfloor \log_\ell R \rfloor + 1$. Then $g-1$ is nilpotent modulo ℓ of index at most R. The binomial theorem gives $(g^{\ell^h} - 1)X \subseteq \ell X$ and $(g^{\ell^{h+1}} - 1)X \subseteq \ell^2 X$. If $(w-1)X \subseteq \ell^2 X$, the factor $1 + \cdots + w^{\ell-1}$ in $w^\ell - 1$ has image ℓX by Nakayama, since it agrees with ℓ modulo $\ell^2 X$. Hence $(w-1)X/(w^\ell - 1)X$ is a quotient of $X/\ell X$ and has length at most R. Iterate from $j_0 = h + 2$ and use divisibility for $m \le j_0$ to obtain (28.13). Take the centres of (28.12) at j_0. Add $R(m - j_0)_+$ to that bound and apply (28.13); since $j_0 + (m - j_0)_+ = \max(m, j_0)$, this gives (28.14). Under the distance condition each valuation is at most b, which gives (28.15). □

Proof of 28.13

Proof. Set $U = \log X$, $B = C_t/\eta$ and $T = \{\log k/C_t : k \in \mathcal{E},\ k \ge 1\}$. Seek $U/(2^{R+2}B^R) \le y \le U/2$ with

$$a \in T, \quad y \le u \le 2y, \quad a \le u \quad \Longrightarrow \quad Ba \le u. \tag{A.246}$$

If T misses $(U/(2B), U]$, take $y = U/2$. Otherwise remove a point in this interval and apply induction on R with upper endpoint $U/(2B)$. The new bound for $2y$ excludes the removed point and the new lower bound is $U/(2^{R+2}B^R)$. Take $Y = e^y$ and $u = \log x$. For $k = 0$ the conclusion is automatic. □

Proof of 28.14

Proof. Since $k \le Z$, outside $\mathcal{E}$ use (28.17). Inside $\mathcal{E}$, (28.18) gives $k \le x^\eta$, so the exponent in (28.4) at L_{j_0} is at most $(2n_0)^{C_K} c_0 x^{3\varepsilon/4}$. Divide by $\log \ell > 1/2$ and use (28.16). Together the cases give

$$s_\ell(L_m, k) \le R(m + j_0 + c + \lfloor \log_\ell Z \rfloor) + 2\exp\big((2n_0)^{C_K} c_0 x^{3\varepsilon/4}\big). \tag{A.247}$$

Since $\log X \le J \log x$, we have $\lfloor \log_\ell Z \rfloor \le 2C_t J \log x$. Take $C = R(A_0 + j_0 + c_1 + 2C_t J) + 1 + (2n_0)^{C_K} c_0$.

For $x \ge 2$ and arbitrary $C_t, C \in \mathbf{R}$, the product $x^{C_t} P_{C,\varepsilon}(x)$ is at most

$$3\exp\Big((|C_t| + |C|)\log x + |C| + |C| x^{3\varepsilon/4}\Big).$$

The logarithm of this bound is $o(x^{4\varepsilon/5})$. □

Proof of 28.15

Proof. At 2 take $L_0 = K_{M,q,d}$ and $j = j_0(2, R_0(M))$. For constants n_0, c_0 depending on M, j,

$$[L_j : \mathbf{Q}] \le n_0, \qquad \log(2|D_{L_j}|) \le c_0 w \quad (w \ge 1,\ \log(qd) \le w). \tag{A.248}$$

Take $n_0 = 8n_M 2^j$ by 28.8. Write $L_j = L_0(\mu_{2^r})$ with r minimal. Then $2^r \le 2n_0$ and L_j embeds into a cyclotomic field of conductor dividing $2^r(4N_M qd)$. The derivative of $X^h - 1$ bounds the root discriminant of a field generated by an hth root by h. The tower formula passes this bound to subfields. Thus $\log(2|D_{L_j}|) \le \log 2 + n_0 \log(8n_0 N_M qd)$, which proves (A.248).

Take $\mathcal{E}_2$ at $Z = \lfloor X^{C_t} \rfloor$ by 28.7, 28.11 and 28.12. Here $\log(qd) \le x^{\varepsilon/4}$ and $c \le 2^{c-1} \le [L_0 : \mathbf{Q}] \le 8n_M$. Use (A.248) with $w = x^{\varepsilon/4}$. By 28.8 and (28.21), each E admits a layer with $m \le a(x)+2 \le (2|A_1|+2)(1+\log x)$ and $s_2(E,k) \le s_2(L_m,k)+1$. Apply 28.14 with these uniform bounds.

At an odd prime choose one realization and take $\mathcal{E}_\ell$ at the same cutoff. Apply 28.14 with $c_1 = c$, $n_0 = [L_{j_\ell} : \mathbf{Q}]$, $c_0 = \log(2|D_{L_{j_\ell}}|)$ and $A_0 = j_\ell$. Layer isomorphism transports the bound with this set to every realization. In both cases there are at most x^{C_t} terms and each is at most $P_{C,\varepsilon}(x)$, with the quadratic cost included at 2. Take x_0 from (28.20). □

Proofs for Chapter 29: Summing the plus component by classes

Proof of 29.2

Proof. The host quotient embeds in the product of the dyadic quotient by $\{\pm 1\}$ with the image of χ and the two square-class groups. The dyadic quotient has order 2^a. Restriction to the r-factor for odd $r \mid f$ has image in the factor from χ. For $r = q, d$ it has image in a square-class group. The dyadic order divides $2^{c_2(\theta)}$. If the global order is odd, the factors at $2, q, d$ are trivial and the remaining local conductors divide the corresponding parts of f. The modulus defining f_0 divides $4o$, and $2^{n_0} \le \ell^{f_0} - 1$.

Largeness puts $\mathcal{B}_\chi$ below q, d. The factors at $2, q, d$ survive in θ_P with 2-power order, so odd order of θ_P forces them to be trivial. Outside $\mathcal{B}_\chi$, the prime ℓ is prime to $2of$. If $\ell - 1 \mid 2k$, the bounds $5 \le \ell \le 2k+1 < q, d$ also put ℓ outside the prime divisors of N. □

Proof of 29.6

Proof. Choose a simple H-submodule. Its G-translates span the representation, so the restriction is semisimple and G permutes its isotypic summands transitively. All summands have the same dimension and multiplicity. Since n is prime, a reducible restriction is scalar or is a sum of n distinct lines permuted transitively by G. The element u excludes the scalar case. In the second case these lines are the eigenlines of u. Each space in the ordinary flag is a sum of a subset of them. An element of D permutes the lines and preserves each member of this complete flag, so it fixes every line.

Let J be the kernel of the permutation action. The characters on the n lines over J have integral unit values. For instance, intersect a G-stable lattice with each line, after extending the coefficient field to define the lines. The sum of these intersections is a G-stable lattice. Reducing its characteristic polynomials on J identifies the multiset of the reduced line characters with the restrictions of the residual characters of G, by uniqueness of semisimplification. Each of the latter is invariant under conjugation by G. The reduced line characters are conjugate to one another by transitivity, hence all equal on J. Since $D \subseteq J$, the residual characteristic polynomial at s would have only one distinct root. Its roots include $\bar{\lambda}_a(s)$ and $\bar{\lambda}_b(s)$, a contradiction.

For the last assertion take the normal core U_0 of U. Apply the first assertion to $H \cap U_0$ and $u^{[G:U_0]}$. Irreducibility on $H \cap U_0$ implies irreducibility on $H \cap U$. □

Proof of 29.7

Proof. Write $H = G_K$ and $\Psi = \varepsilon_\ell^k \vartheta$. Fix a place above ℓ. Its rational inertia belongs to H, because $\ell \nmid D_K$, and maps onto $\mathbf{Z}_\ell^\times$ by ε_ℓ. The local test of (A.278) gives $x \in H$ in the decomposition group with $\bar{\Psi}(x)^2 \ne 1$.

In Type I take the normal core U of the given open subgroup in H. Choose σ in inertia of cyclotomic value $1+\ell$, and take $u=\sigma^m$ with $m=[H:U]\operatorname{ord}\vartheta$. Then $u\in U$, and its three eigenvalues $(1+\ell)^{km},1,(1+\ell)^{-km}$ are distinct by the ordinary inertia condition. The restriction to U is semisimple: the translates under H of a simple U-submodule span the irreducible H-module and are again simple U-submodules. Its isotypic summands are permuted transitively. In dimension three a reducible restriction thus has either scalar U-action or three lines permuted transitively. Scalar action is excluded by u. The lines in the second case are precisely the eigenlines of u. The ordinary flag is made up of successive sums of these lines, and x preserves it. Its permutation therefore fixes each line. A transitive subgroup of the permutations of three letters contains a three-cycle; choose $g\in H$ inducing one. Then gx^j has trace zero for every $j\geq 0$. Reducing the trace in the marked lattice gives

$$AB^j+1+(AB^j)^{-1}=0,\qquad A=\bar{\Psi}(g),\quad B=\bar{\Psi}(x).$$

For $j=0,1,2$ the three distinct elements A,AB,AB^2 would all be roots of X^2+X+1. This contradiction proves the first assertion.

In Type II put $N=H\cap\ker\varepsilon_\ell$. On the chosen inertia group I the corner has trace $\Psi+\Psi^{-1}$: subtract the trivial middle character from the three prescribed inertia eigenvalues. The cyclotomic surjection gives $H=IN$. Choose $u\in I$ with $\Psi(u)$ of infinite order, using a power of the same σ. If N acts by scalars then the projective image of H is abelian. It centralizes the projective image of u, whose eigenvalue ratio has infinite order. That centralizer preserves each eigenline, contradicting irreducibility on H. Otherwise a reducible restriction to N has either a unique invariant line or two distinct isotypic lines, by the same socle argument. A unique line is H-invariant. In the remaining case an element of I must interchange the two lines, since $H=IN$. If s does so, all su^{2j} interchange them and have trace zero. The inertia trace formula would give $\Psi(s)^2\Psi(u)^{4j}=-1$ for every j, contrary to the choice of u. □

Proof of 29.8

Proof. In Type I, 29.7 verifies the open-subgroup hypothesis of 30.61(2). It gives a regular algebraic conjugate self-dual cuspidal attachment π over K for $r|_{G_K}\varepsilon_\ell^{-1}$. We check irreducibility on $N=G_{K(\mu_{\ell^\infty})}$. As in 29.7, restriction to the normal subgroup N is semisimple. Three permuted lines would give an open line stabilizer in G_K, contrary to that lemma. Here the stabilizer is open because the eigenline action is continuous and its orbit finite. If N acted by scalars, the projective image of G_K would be abelian. It would preserve the finite set of eigenlines of an inertia element with three distinct eigenvalues, again contradicting open-subgroup irreducibility. These exhaust the reducible possibilities in prime dimension. A scalar twist preserves this irreducibility.

In Type II the attachment of $W\varepsilon_\ell^{-k}$ is cuspidal on $\mathrm{GL}_2/\mathbf{Q}$ by 30.61(3), of weight $2k+1\geq 3$. Its quadratic base change to K is cuspidal by 26.36(2). The exceptional case gives a self-twist by η_K, whose intertwiner would commute with $W|_{G_K}$ and hence be scalar, whereas on the other coset it would intertwine a matrix with its negative. The attachment after base change has the restricted good Frobenius polynomials. Its weights remain distinct, and the exterior-product pairing of W, with $\det W=\eta_K$, gives essential conjugate self-duality. More explicitly, for complex conjugation c and $J=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$, the matrix $P=JW(c)^{-1}$ satisfies $W(h)^tPW(chc^{-1})=P$ on G_K. It is symmetric since $W(c)^2=1$ and $\det W(c)=-1$. Twisting adds the corresponding scalar to the multiplier. Irreducibility on N is 29.7.

In each case Chebotarev and Brauer–Nesbitt identify the attached Galois representation with the indicated scalar twist, using the good Frobenius polynomials. On G_K the adjoint is conjugation on the full matrix algebra, so scalar twists cancel. At c its action is $Z\mapsto -P^{-1}Z^tP$ for the polarising form; another form with the same multiplier differs by a scalar by irreducibility. This identifies the polarised adjoint in the assertion with the one in [129, Thm. 6.1], as in the proof of 30.62. The field K is CM with $K^+=\mathbf{Q}$, the rank is at least two, the attachment is regular algebraic cuspidal and polarised, and its cyclotomic restriction has just been proved irreducible. That theorem gives the stated vanishing. □

Proof of 29.9

Proof. Claim 1. In the given definite comparison, an arithmetic weight specialization identifies the ordinary module with the classical ordinary space, compatibly with the residual Hecke summand and every finite sum

of level-preserving translations away from ℓ. Write the specialized compact diagonal character as $x\theta = \chi_\lambda\alpha$, with α of finite order. At sufficiently deep Iwahori level, the classical right diagonal character is α^{-1}, and the positive diagonal operator attached to a is $\chi_\lambda(a)^{-1}[a]$. In the flag order, the exponents of χ_λ are $h_{i,\tau} - i$.

The finite-level function model in this comparison has free coefficient action and finite double-coset sets. Coefficient specialization commutes with their finite Hecke sums and with the ordinary projector, defined by factorial powers. Evaluation coordinates and their ordinary projections show that coefficient base change is an isomorphism by Nakayama's lemma. Pairing an algebraic coefficient with its lowest-weight functional then gives the classical comparison. Choose the strictly dominant diagonal operator used in the ordinary projector. Its normalized action is divisible by the coefficient uniformizer on every weight other than the lowest one. Iterating this operator proves injectivity; prescribing the lowest coordinate on finitely many classes and applying the ordinary projector gives a lift modulo that uniformizer, hence surjectivity by Nakayama. The same finite-sum identities give compatibility with translations, including the projector used below.

Lean: `UpperTheory.V.Hi.Sp.absSpec` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.HeckeFixedLevelSpaces.FreeMap`, lines 425–587)
Lean: `UpperTheory.V.Hi.WtRep` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.ContinuousTorusCharacter`, lines 269–285)
Lean: `UpperTheory.V.Hi.ctrlOrd` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.OrdinaryHeckeCharacters.CtrlOrd`, lines 220–371)

Claim 2. There is an integral idempotent at the distinguished split place, commuting with the ordinary determinant action and with specialization as in Claim 1. Let v_0 be the distinguished split place. Identify its group with $\mathrm{GL}_3(H)$, where $H = F_{v_0}$. Write k_0 for its residue field, $q_0 = |k_0|$, $K^0 = \mathrm{GL}_3(\mathcal{O}_H)$, $K^1 = \ker(K^0 \to \mathrm{GL}_3(k_0))$, and U for the upper unitriangular group. The level at v_0 is K^1. Fix a nontrivial additive character ψ_0 of k_0. For $a, b \in k_0^\times$ set

$$\chi_{a,b}(u) = \psi_0(au_{12} + bu_{23}) \qquad (u \in U(k_0)),$$

using indices $1, 2, 3$ in this formula. On K^1-invariants take

$$e = \frac{1}{q_0^3} \sum_{u \in U(k_0)} \left(\sum_{a,b \in k_0^\times} \chi_{a,b}(u)^{-1} \right) u.$$

Each inner coefficient is an integer. It is the product of two sums, each equal to $q_0 - 1$ or -1 according as its argument is zero or nonzero. Also $v_0 \nmid \ell$, so q_0 is a unit in O. The lifts of u in K^0 give the same operator on K^1-invariants. Character orthogonality makes e the sum of mutually orthogonal projectors. The Hecke operators and weight characters used to define B_0/J act away from v_0. Thus e acts on the integral residual summand, commutes with this algebra and specializes as in Claim 1.

Lean: `UpperTheory.V.Z.Desc.V0.eOp` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.CuspidalDescent.GLDenseRanges.InertiaPartMaps.MapInj`, lines 86–169)

Claim 3. The rank of e on τ^{K^1} for a generic irreducible smooth complex representation τ of $\mathrm{GL}_3(H)$ with $\tau^{K^1} \neq 0$ is $(q_0 - 1)^2$. Extend ψ_0 to an additive character of H of conductor $\mathfrak{m}_H$, and lift a, b to units. This gives a nondegenerate character χ of $U(H)$. Take a Whittaker functional l for χ. Conjugation by a diagonal matrix takes any choice of nondegenerate Whittaker character to this one. The functions $W_z(g) = l(\tau(g)z)$ on the mirabolic subgroup, consisting of matrices with last row $(0, 0, 1)$, separate the vectors z by [14, Thm. 4.9 and Rem. 4.10].

First choose $z \in \tau^{K^1}$ and g with $W_z(g) \neq 0$. By Iwasawa decomposition and normality of K^1 in K^0, we can absorb the compact factor into z and the upper unipotent factor into a nonzero scalar. We have $l(\tau(d)z) \neq 0$ with $d = \mathrm{diag}(d_1, d_2, d_3)$. Comparing the actions of $1 + sE_{i,i+1}$ and its d-conjugate for $s \in \mathfrak{m}_H$ gives $v_H(d_i) \geq v_H(d_{i+1})$. Average $\tau(d)z$ over K^1. Its value under l is unchanged. Indeed write each element of K^1 as upper unipotent times diagonal times lower unipotent, with all factors in K^1. The character on the first factor is trivial. The diagonal commutes with d, and d^{-1} conjugates the lower factor into K^1 by the valuation inequalities. Now average with character χ^{-1} over $U(\mathcal{O}_H)$. This preserves the value under l and produces a nonzero $\chi_{a,b}$-vector in τ^{K^1}.

On this character space l is injective. To see this, suppose $W_z(p) \neq 0$ for a mirabolic matrix p. Work up from the last row. If rows below row i are upper unitriangular, compare right translation by $1 + sE_{j,i+1}$, $s \in \mathcal{O}_H$, $j \leq i$, with left translation by its p-conjugate. The conjugate is upper unipotent. Its only possibly

nonzero simple-root entry is in position $i, i+1$, with value sp_{ij}. The character identity gives

$$\psi(t_i s(p_{ij} - \delta_{ij})) = 1 \quad (s \in \mathcal{O}_H), \qquad (t_1, t_2) = (a, b),$$

where ψ denotes the lifted additive character and t_i the chosen unit lift. Hence $p_{ij} - \delta_{ij} \in \mathfrak{m}_H$ for $j \le i$. Right multiplication by an element of K^1 makes this row upper unitriangular while preserving the rows below: on the first i columns use a matrix congruent to the identity to carry $(p_{i1}, \dots, p_{ii})$ to $(0, \dots, 0, 1)$. Repetition reduces p to an upper unipotent without changing $W_z(p)$. Therefore $l(z) = 0$ would give $W_z = 0$ on the mirabolic. This proves injectivity. Each of the $(q_0 - 1)^2$ characters contributes exactly one dimension.
Lean: `UpperTheory.V.Z.Desc.V0.isoNe` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.CuspidalDescent.GLDenseRanges.InertiaPartMaps.ValOrd`, lines 280–392)
Lean: `UpperTheory.V.Z.Desc.V0.isoLe` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.CuspidalDescent.GLDenseRanges.InertiaPartMaps.TvApply`, lines 379–393)
Lean: `UpperTheory.V.Z.Desc.V0.locE` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.CuspidalDescent.GLDenseRanges.InertiaPartMaps.RankEqOneOfInj`, lines 22–44)

Claim 4. Every contributing constituent in an irreducible arithmetic fibre has the same cuspidal base change, and its occurrence count depends only on the labelled weight. Fix $\iota : \overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$. The generalized x-space before projection is nonzero. Indeed M is finite and faithful over B_0/J, and Nakayama's lemma gives a nonzero summand at x after specialization. Claim 1 identifies it with the corresponding classical space. Decompose that space into definite automorphic constituents, counting occurrences. The good spherical operators on each contributing constituent have the prescribed eigenvalues. In 26.22 and 26.23 the semisimple sum of the attached representations of its cuspidal factors therefore has the determinant at x, by Frobenius density. Irreducibility leaves one cuspidal factor of rank three with multiplicity one. Strong multiplicity one identifies the same base change π_x for every contributing constituent, since the good spherical eigenvalues agree at almost every place.

At the infinite places the algebraic type selects the prescribed irreducible representation of the compact group. At a split finite place the component is determined by π_x. At a nonsplit finite place the level is hyperspecial, and spherical base change determines the spherical representation: its Satake parameters are recovered up to inversion and permutation, the spherical Weyl group. Write λ for the common labelled algebraic weight of x and y. Choose a finite set Z of split places containing a regular place w_0 and all split places where π_x is ramified, together with a local test function f on Z such that $Q = \prod_{w \in Z} \operatorname{tr}(\pi_{x,w}(f_w)) \neq 0$. In the stable and endoscopic trace-formula comparison for the definite rank-three unitary group over the unramified CM extension L/F (with its quasi-split forms at inert finite places), combine the signed endoscopic identity of Lemma 26.5 with the stable-block and split-Hecke limit estimates of Lemmas 39.51–39.50 (see also [94]). Fix the global Haar measures (normalized so the maximal compact subgroup K_f of $\mathrm{GL}_3(\mathbf{A}_{L,f})$ has volume 1), the labelled weight λ, and the archimedean Lefschetz kernels before choosing π_x. Choose a level $K \le K_f$ fixing a nonzero vector of π_x and small enough at each split pair for the chosen local tests to be bi-invariant under the corresponding level subgroups. Take K maximal outside $Z \cup \bar{Z}$, with paired factors N_w and their θ-translates at the split pairs. At this level the archimedean twisted-Lefschetz projection onto an irreducible finite-dimensional Hecke- and θ-stable subspace $Y \subseteq \pi_x^K$ introduces a nonzero archimedean factor determined by λ and the fixed kernels. The space Y is irreducible for the two commuting Hecke actions exchanged by the θ-intertwiner A, normalized by $A^2 = 1$. Give K volume one in the finite-level Hecke action. On each exchanged pair the trace of the swap composed with $P_1 \otimes P_2$ is $\operatorname{tr}(P_1 P_2)$. Schur's lemma and $A^2 = 1$ leave a scalar of modulus one in this normalization. Measuring the two local tests with the maximal compact K_w of volume one contributes $[K_w : N_w]^2$ per pair. Hence the scalar in front of the product of local traces has modulus $\prod_{w \in Z} [K_w : N_w]^2 = [K_f : K]$. Returning to the global finite Haar measure multiplies by $\mu_f(K) = [K_f : K]^{-1}$. A test of nonzero trace at the initial level identifies the scalar at every refinement. Dividing by the fixed geometric Haar conversion and block-volume factors therefore yields a positive real constant $c_0(\lambda) > 0$, determined solely by λ and the fixed global measure datum (completely independent of π_x and of the open level), such that for every such π_x the stable $(3, 0)$ limit along a split-Hecke separating sequence outside Z equals $c(\pi_x)Q$ for a complex scalar $c(\pi_x)$ of exact modulus $|c(\pi_x)| = c_0(\lambda)$. By symmetry of the two stable ends (both carrying sign $+1$ in the signed endoscopic identity), the $(0, 3)$-block converges to the same limit $c(\pi_x)Q$, whereas every proper endoscopic block $(2, 1)$ and $(1, 2)$ tends to 0 by strong multiplicity one and Satake-parameter separation at good split places; summing over all blocks gives $2c(\pi_x)Q$. Refining the level at w_0 to an open subgroup $K_2 \le K_1$ on which the regular test f_{w_0} is bi-invariant, the definite trace comparison of Lemma 26.5 (after cancelling the nonzero archimedean

dimension factor d_λ) equates the same limit to $m(\pi_x; K_2)\, Q$, where $m(\pi_x; K_2) \in \mathbf{Z}_{\geq 0}$ is the occurrence count at level K_2. Cancelling $Q \neq 0$ and taking absolute values (eliminating any π_x-dependent θ-intertwiner sign) gives $m(\pi_x; K_2) = 2|c(\pi_x)| = 2c_0(\lambda) =: a_\lambda > 0$, independent of π_x. Finally, choosing a K_1-bi-invariant (hence also K_2-bi-invariant) test function g on Z with nonzero trace on $\pi_{x,Z}$ and comparing the two finite-rank automorphic traces by level-change invariance shows that the occurrence count $m(\pi_x) = m(\pi_x; K_1)$ at the original level equals $m(\pi_x; K_2) = a_\lambda$ (in fact $a_\lambda = 1$ by [94, Thm. 5.4], though only positivity and independence from π_x are used here). Because a_λ is fixed before the arithmetic character is chosen, the same positive occurrence count applies at both x and y.

Lean: `UpperTheory.V.Z.Mult.mult_free` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.MultiplicityOne.CompOfFree`, lines 15–19)

Lean: `UpperTheory.V.Z.Desc.V0.count_e_of_sph_same_at_same_bc_dims` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TCountE`, lines 147–192)

Claim 5. At $v \mid \ell$, the prescribed ordinary refinement in a contributing constituent has dimension one, including at x. The generalized chamber eigenspace injects into the Jacquet module: a vector in the kernel is killed by averaging over a sufficiently large upper unipotent subgroup, and translation turns this average into a power of a strictly dominant chamber operator up to a nonzero scalar. That operator is invertible on its generalized eigenspace. Take a torus character occurring in the image. Frobenius reciprocity gives principal-series support. The geometric lemma bounds the multiplicity of any ordered tuple in the Jacquet module by the number of permutations of its inducing tuple giving that character [14, Lem. 2.12]. The chamber eigenvalues for the first j uniformizers give successive partial products, hence determine each smooth uniformizer value in its assigned position, with the factors q_v^{-i} of the normalized chamber operators removed. These values have distinct ℓ-adic valuations. Writing $h_{i,\tau}$ for the strictly increasing ordinary weights, the smooth uniformizer value for position i has valuation $\sum_\tau h_{i,\tau} v_\ell(\tau(\varpi_v))$, since the Galois character has unit value on a reciprocity lift. This follows directly by taking successive quotients of the normalized weight factors in Claim 1; the cyclotomic correction restores the i removed from the exponent $h_{i,\tau} - i$. The labelled weights increase strictly at each embedding. Thus at most one permutation contributes to the generalized eigenspace, and its nonvanishing gives dimension one.

Lean: `UpperTheory.V.Z.J.D.dense_threshold_order_one` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.ExistsEig`, lines 218–322)

Claim 6. The tame invariant dimensions at x and y agree. Let S_t be the tame type places, K_t the product of their maximal compact groups, and V_t the prescribed tame coefficient representation in the convention that the fixed vectors are in $V_t \otimes \bigotimes_{u \in S_t} \pi_{x,u}$. Set

$$b(x) = \dim_{\mathbf{C}} \left(V_t \otimes \bigotimes_{u \in S_t} \pi_{x,u} \right)^{K_t}.$$

By the block-parahoric twisted-spherical type characterization of Lemma 30.22, occurrence of the prescribed tame coefficient type V_t selects precisely zero monodromy and the fixed multiset of Teichmüller inertia characters extending to the Weil group (even when characters repeat, since equal-character blocks use the full maximal compact subgroup $\mathrm{GL}_{d_a}(\mathcal{O}_{L_u})$ and exclude Steinberg parameters). Indeed a nonzero form in the constituent supplies a nonzero tuple transforming by V_t (apply a coefficient functional nonzero on one of its values), triggering the forward direction of Lemma 30.22. Each Weil commutator lies in inertia and all the extending characters are one on it. Thus the Frobenius-semisimple Weil action is a sum of characters. The local representation is generic, since π_x is cuspidal. With zero monodromy the segments in the local Langlands classification are singletons, and genericity makes their induction irreducible. Its restriction to the maximal compact, by Iwasawa decomposition, is induction from the integral Borel of the restrictions of these characters to units. Local reciprocity identifies those restrictions with the prescribed inertia characters. Their order can be chosen the same for x and y, since the irreducible inductions permit permutation. The resulting compact representations are isomorphic, so $b(x) = b(y)$.

Lean: `UpperTheory.V.Hi.partUR2Type_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.AutomorphicTensorInductionOfLocalPlaces`, lines 76–113)

Lean: `UpperTheory.V.Z.J.D.resEq2_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.ResEq2`, lines 27–47)

Claim 7. The projected multiplicity is positive and has the same value at x and y. Before projection, the dimension formula is

$$\dim_{\mathbf{C}} M_x[x]_{\mathrm{gen}} = a_\lambda\, b(x)\, \dim_{\mathbf{C}} \pi_{x,v_0}^{K^1},$$

where the left side is computed after applying ι. It is finite and nonzero. The three factors on the right are therefore nonzero finite cardinals. In particular the occurrence cardinal is represented by a positive integer, again denoted a_λ, and $b(x) > 0$. The component at v_0 is generic by cuspidality. The calculation of the projector on its K^1-invariants therefore gives

$$\mu_e(x) = a_\lambda\, b(x)(q_0 - 1)^2.$$

The same formula holds at y, with the same a_λ. Since $b(x) = b(y)$, the equality and positivity follow. □

Proof of 29.10

Proof. Write $T = B_0/J$, $P = eM$, and T' for the reduced image of T acting on P. The module P is finite free over Λ, being a direct summand of a finite free module over a local ring. The weight ring is a power series ring over O on the free local unit coordinates. An inertia derivation as in the statement kills Λ: those coordinates are values of the ordered characters up to fixed scalar factors. Differentiation of power series at an integral arithmetic point is determined by the coordinate derivatives. Indeed after finite coefficient extension the ideal of that weight is generated by the coordinate differences.

Fix the algebraic weight of x. The characters with pairwise distinct finite unit parts at each ordinary place detect T. To see this, all finite torus characters at that algebraic weight detect Λ by Fourier inversion on the finite quotients, after translating by the algebraic character. The separated ones still detect it: multiply by the product, for each pair of coordinates at each place, of the differences of their universal finite-part values on one free generator. This product is nonzero in the power series domain and vanishes at every omitted character. In a separated fibre the classical operators act by scalars on each simultaneous constituent by 30.23 and 30.24. Applying separation to the matrix entries on M proves the assertion. In particular T is reduced. Its image on P is already reduced, as it is detected on the projected summands of the same fibres.

There is $b \in T$ with $x(b) \neq 0$ such that every detecting character y with $y(b) \neq 0$ is irreducible. Choose nine group elements whose images at x form a matrix basis, and take the determinant of their matrix of trace pairings. At x it is nonzero. At a detecting character the classical semisimple representation has this same specialized trace pairing, so nonvanishing implies full matrix span. By Lemma 29.9 every such y, and x itself, belongs to the support of P. If a belongs to the kernel of $T \to T'$, then ba vanishes at every detecting character, hence $ba = 0$. Evaluation and the product rule at x give $x(a) = D(a) = 0$ for any derivation D at x. Thus we can work on T'.

Lean: `UpperTheory.V.J.irrOpenW_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.KeepIrrOpenW`, lines 15–33)

Lean: `UpperTheory.V.Z.Desc.V0.unrAlgE` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.CuspidalDescent.GLDenseRanges.InertiaPartMaps.UnrAlgE`, lines 5–32)

Put $K_\Lambda = \operatorname{Frac}\Lambda$. The finite torsion-free reduced Λ-algebra T' embeds into a product of finite separable fields over K_Λ. Choose $t \in T'$ separating its geometric generic characters and also the characters in the weight fibre of x. This can be done by taking an integral linear combination of a finite set of module generators: each pair of distinct characters rules out a proper linear hyperplane, and there are finitely many pairs. Write the characteristic polynomial of t on P as $\prod_i f_i^{m_i}$, with the f_i distinct, monic and irreducible over K_Λ. They belong to $\Lambda[X]$ by integrality and normality of Λ. Faithfulness ensures that every generic factor occurs. The polynomial $f = \prod_i f_i$ kills t in T'.

Let r_i be the multiplicity of $x(t)$ in f_i after specialization at the weight of x. Separation in that fibre and simultaneous primary decomposition give

$$\mu_e(x) = \sum_i m_i r_i.$$

For each i with $r_i > 0$, the branch indexed by f_i passes through x. Indeed $\Lambda[t] \simeq \Lambda[X]/(f)$, the branch algebra is integral over $\Lambda[X]/(f_i)$, and lying over gives a character on that branch at the specified root. Separation in the fibre identifies it with x. On this branch choose a detecting y with $b(y) \neq 0$, $f_i'(y(t)) \neq 0$, and $f_j(y(t)) \neq 0$ for $j \neq i$, where coefficients are evaluated at the weight of y. These conditions define a nonempty open on the branch: b is nonzero at x, and the generic roots are separate and simple. The detecting characters with $b \neq 0$ meet it. Indeed, after multiplication by b, their vanishing test is the test on T above. To isolate a branch, multiply also by an element in every other minimal prime and outside the

chosen one. Such an element exists by incomparability and finiteness of the minimal primes. Equivalently one may perform this test on $T[1/b] = T'[1/b]$.

At y the generalized y-space is contained in the primary space for t at $y(t)$, of dimension m_i. Lemma 29.9 gives $\mu_e(x) \le m_i$. Since $\mu_e(x) > 0$, the displayed sum forces exactly one $r_i = 1$ and the others zero. Thus $x(f'(t)) \neq 0$. Differentiating $f(t) = 0$ gives $D(t) = 0$.

For any $a \in T'$ there is $H_a \in \Lambda[X]$ satisfying $f'(t)a = H_a(t)$. Indeed, over a splitting field write t_s, a_s for their values at the geometric generic characters and take

$$H_a(X) = \sum_s a_s \prod_{u \neq s} (X - t_u).$$

Its coefficients are Galois invariant and integral over Λ, so lie in Λ. Evaluation at the generic characters proves the identity by torsion-freeness. Differentiation now gives $x(f'(t))D(a) = 0$, proving the lemma. □

Proof of 29.11

Proof. Put $\mathfrak{q} = \ker \xi$, $d = [F : \mathbf{Q}]$, $D = 1 + 3d$. We give the dimension and reducedness arguments separately.

The map ϕ is onto. The traces and ordinary characters generate B topologically. For the traces, the difference of the matrices at the marking and its polarized inverse is diagonal with residually distinct entries. Its spectral projectors are polynomials in it over the trace coefficient ring, by simple-root lifting. Inserting the projectors in traces of words recovers the products of entries around cycles. In rank three the polarisation identifies an entry with the reflected transpose entry at the polarized inverse group element. A product of entries of total weight zero is a product of cycles after using these identifications: pair weights $1, -1$ and $2, -2$, and group the remaining weights in triples $1, 1, -2$ or $-1, -1, 2$. A weight-one entry can occupy either 01 or 12 by reflection, and likewise for weight minus one. This gives the asserted generators. The link is onto on the matrix group and on the ordinary local groups, and its twist multiplies the generators by coefficient units. The image is closed by completeness and compactness of the determinant ring. The component propagation in 30.59, first for rank two and then across the rank-three block locus, gives $\phi(J) \subset P$ at every prime P off the three indicated loci. Every prime below $\mathfrak{q}$ is such a prime.

Lean: `UpperTheory.V.Pr.phiEx` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.PxIsCont`, lines 231–289)

Lean: `UpperTheory.V.J.absGood` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.LinkedUnramifiedChains.DerivativeDivisibility.AbsGood`, lines 10–29)

Every component of B_0/J has dimension D. It is a finite torsion-free reduced algebra over the power series weight ring in $3d$ variables over O, by 30.23 and the detection argument of Lemma 29.10. Its minimal primes contract to zero by torsion-freeness, and finite integral extension preserves dimension. Off the triangular loci the matrix fibre over B has dimension at most one. For example, invert nonzero homogeneous entries of opposite signs. Powers of every homogeneous generator then become, up to a weight-zero factor, integral over the rational function field of one nonzero homogeneous coordinate. This gives the fibre bound also for the complete local charts, by the dimension formula for a finite-type algebra over a complete local ring, applied before completion as in 30.59. Consequently a branch through $\mathfrak{q}$ in the completed residual chart has dimension at most $D + 1$.

Here are the local heights. Complete at the reduction of the integral point, allowing a finite residue extension. The evaluation extends to this completion, and its kernel Q has one-dimensional quotient: the image lies in a finite coefficient DVR and contains O. The ungraded presentation in 30.59 gives the completed chart over a coefficient DVR with t variables and r_0 equations, where $t - r_0 \ge D$. Append $t - D - r_0$ zero equations and put $r = t - D$. This leaves the defining ideal unchanged and gives $t - r = D$. The preimage of Q in the power series ring has height t. Thus $\operatorname{ht} Q \ge t - r = D$. The branch bound gives $\operatorname{ht} Q \le D$, since one can append the closed point to a chain ending at Q. In the regular local ring at the preimage of Q, the ideal of the r equations therefore has height r. The equations form a regular sequence. The resulting local complete intersection is Cohen–Macaulay and has no embedded associated primes. This property descends to $A_{\mathfrak{q}}$ by the faithfully flat local map to the localized completion. For the height calculation downstairs the formal fibre at Q has dimension zero: the quotient of the residual local ring by $\mathfrak{q}$ is a finite order over O, already complete. In particular the height at an integral evaluation is at least D. Also, every minimal branch below $\mathfrak{q}$ has contraction of dimension D in B. Indeed flat going down lifts that branch to a minimal

branch of the completion. The presentation gives dimension at least $1 + t - r = D + 1$ there, and the fibre bound gives the opposite inequality $1 + \dim(B/(P \cap B))$.
Lean: `UpperTheory.V.J.htUp` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.LinkedUnramifiedChains.ScalarIrreducibility.HeightLeOfDim`, lines 45–69)
Lean: `UpperTheory.V.J.htLow` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.LinkedUnramifiedChains.ScalarIrreducibility.HtOfSum`, lines 34–58)
Lean: `UpperTheory.V.J.compDim` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.LinkedUnramifiedChains.ScalarIrreducibility.CompDim`, lines 62–88)

Fix such a minimal prime P. By surjectivity and the dimension equality, $\phi^{-1}(P \cap B)$ is a minimal prime of J. Use at each $v \mid \ell$ the auxiliary labelled weights $h_{i,\tau} = i$ except at one embedding τ_v, where $h_{i,\tau_v} = 2i$, $0 \le i \le 2$. The separated characters at this weight are dense on the chosen coefficient branch by the detection and branch-isolation argument in Lemma 29.10. There is a point above one of them on P, off all three excluded loci. Indeed invert the product of one surviving upper, lower and odd-weight entry on P. The resulting finite-type dominant map to the coefficient branch has image containing a nonempty open by Chevalley's theorem. Choose a separated point in that open and lift it over $\overline{\mathbf{Q}}_\ell$.

The lift can be taken with all off-diagonal entries of positive valuation. Under the grading, rescaling by an element of valuation u changes the valuation of a weight-j coordinate by ju. For the finitely many homogeneous generators these are simultaneous strict lower bounds on u for positive j and upper bounds for negative j. Every lower bound is strictly below every upper bound: multiply suitable positive powers of the two coordinates to get weight zero. This product reduces to zero at the diagonal residual point, and hence has positive valuation. Choose a rational u in the interval and extend coefficients. The same bounds hold on all off-diagonal entries by homogeneous generation with the residual diagonal reduction. Minimal components are preserved by the grading torus, since it is connected and the components form a finite set. Denote the resulting evaluation by z.
Lean: `UpperTheory.V.J.genPt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.LinkedUnramifiedChains.ScalarIrreducibility.ChevPrime`, lines 41–105)
Lean: `UpperTheory.V.J.wtEx` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.LinkedUnramifiedChains.ScalarIrreducibility.PlaceFree`, lines 131–166)

We show that the local ring at z is regular. The marked representation there is irreducible. An invariant subspace is a sum of lines of the diagonal marking difference, whose eigenvalues are distinct. Polarisation makes the reflected annihilator invariant as well. A proper invariant coordinate subspace and its reflected annihilator impose one of the two triangular shapes or split off the middle coordinate. All three were excluded. Apply 30.25 to the twisted determinant and its ordered flag. Use the two choices of distinguished place in 30.59; the same twisted matrices and ordered characters give arithmetic evaluations for both, and component propagation puts each on its Hecke quotient. The two attachments agree at the good primes, so strong multiplicity one identifies them and gives the witness at all places. At either distinguished place the underlying matrices and the fixed twist are unramified by the choice of that place.

At $v \mid \ell$ the local degree exceeds five by (A.281). Choose a sufficiently deep norm-one principal unit of infinite order, using the exponential of a nonzero trace-zero element. Its reciprocity lift has cyclotomic value one. After taking a power to kill the finite characters, the ordered values on that lift are $\tau_v(u)^{-i}$, since the contributions with weight i multiply to a power of the norm. Their ratios have infinite order. Hence the flag is unlinked. The residual local marking separates the three residual characters. Lemma 29.6 gives irreducibility on the cyclotomic kernel and on its open subgroups. The rank-three coset argument of 30.59 now gives density of elements with separable polynomial on that kernel.

These facts give the determinant tests in 30.65. Indeed, on the cyclotomic kernel a twisted adjoint cocycle is an ordinary adjoint cocycle. If all its trace tests on the separable elements vanish, continuity and 30.63 make it a coboundary there. After subtraction of that boundary, its values on the quotient lie in the scalar centralizer. The quotient is abelian and acts on scalars by the nontrivial cyclotomic character. Commuting an arbitrary element with one of cyclotomic value different from one shows that this scalar cocycle is a boundary. Finitely many tests suffice because the continuous cocycle space is finite dimensional: on an open subgroup acting trivially on a residual lattice, cocycles into the lattice give homomorphisms to the semidirect product with a pro-ℓ congruence group. Its source has finitely generated maximal pro-ℓ quotient by the finite character condition of 30.55. Values on generators and finitely many coset representatives determine the cocycle, also after clearing denominators. Thus the common kernel of the tests is cut out by finitely many of them.

A fixed-inertia derivation at z differentiates the matrices to an ordinary cocycle by multiplication on the

right by the inverse matrix. The lifted flags give the local correcting coboundaries and the inertia diagonal is zero. The fixed twist has zero derivative. Full matrix span, the witness, the distinguished inertia condition and the determinant tests just proved give all the hypotheses of 30.65. Hence the matrix derivative is $[X, r(g)]$ for one X. The marking difference is diagonal and has distinct entries, so its off-diagonal derivative forces X to be diagonal. Polarisation then says that X plus its reflected transpose commutes with the representation. Irreducibility makes this sum scalar. Modulo scalars X is therefore a multiple of $\operatorname{diag}(1, 0, -1)$. A conjugation derivative fixes the ordered characters: their values are determined by the flag projectors of a local element with distinct residual eigenvalues. Thus the fixed-inertia tangent space has dimension at most one. The three characters at v contribute at most $3d_v$ inertia derivatives by the generators in 30.55, applied to their logarithmic derivatives. The full tangent dimension is at most $1 + 3d = D$. The residue field of the characteristic-zero local ring is separable algebraic over $\operatorname{Frac} O$, so its embedding dimension is bounded by this tangent dimension. Its height is at least D by the presentation above. It is regular, hence a domain.
Lean: `UpperTheory.V.J.fixOf_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.BFixOf`, lines 316–342)
Lean: `UpperTheory.V.J.fixLe` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.LinkedUnramifiedChains.ScalarIrreducibility.FixLe`, lines 8–43)

Localizing further at P shows that A_P is a field. We have proved this for every minimal prime below $\mathfrak{q}$. If $\phi(a)$ survived in $A_{\mathfrak{q}}$ for some $a \in J$, the nonzero cyclic module it generates would have an associated prime. This is an associated prime of $A_{\mathfrak{q}}$, hence a minimal prime by the complete-intersection argument. At that prime the local ring is a field and $\phi(a)$ belongs to its zero ideal by component propagation. This contradicts survival of the cyclic module at its associated prime. Clearing the localization denominator gives the stated s. □

Proof of 29.12

Proof. Use the integral flag basis at v, and write $t(g)$ for its upper triangular matrix. Such a basis is obtained by intersecting the flag with the lattice; the successive quotients are free over the DVR. With D_v the local group, put

$$c(g) = t(g)_{i,i+1}/t(g)_{i+1,i+1}.$$

The adjacent two-dimensional subquotient twisted by $\lambda_{v,i+1}^{-1}$ is an extension of 1 by ε_ℓ with cocycle c. Over a finite extension on which t is crystalline, the subquotient and its twisting line are crystalline. Thus the restricted Kummer class is a unit class by [120, Ex. 3.9]. Restriction multiplies the valuation by the ramification index, so the original rational class is also a unit class. Its cup product with the unramified additive character u sending arithmetic Frobenius to one is zero by local reciprocity. Consequently there is a continuous cochain β such that

$$\beta(gh) = \beta(g) + \varepsilon_\ell(g)\beta(h) - c(g)u(h).$$

Compactness permits multiplying u, β by the same scalar to make them integral, with $u \neq 0$.
Lean: `UpperTheory.V.Z.J.stepF` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.LinkedUnramifiedChains.CrystallineCardinalities.TwoApply`, lines 122–155)
Lean: `UpperTheory.V.Z.J.cupUnr` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.LinkedUnramifiedChains.CrystallineCardinalities.CupUnr`, lines 101–135)

Put $O_m = O/\ell^m O$, $q = |O_m|$, $m \geq 1$. At each ordinary place w, let $Z_w^{\triangle}$ consist of the continuous upper triangular adjoint cocycles over O_m with zero diagonal on inertia. For a finite local module V, 26.27(1) and $|B^1| = |V|/|H^0|$ give

$$|Z^1(D_w, V)| = |V|^{1+d_w}|H^0(D_w, V^\dagger)|.$$

The local action is continuous with finite image. Upper triangular matrices have rank six, and the diagonal is a trivial quotient for conjugation. Requiring its three characters to vanish on inertia costs at most q^{3d_w}, by the d_w inertia generators of 30.55. Hence $|Z_w^{\triangle}| \geq q^{6+3d_w}$.

At v first use upper triangular matrices with the two indicated diagonal entries equal. Their rank is five, and their $i, i+1$ entry is an equivariant surjection onto $O_m(1)$: the extra conjugation term is a multiple of the difference of the two diagonal entries. The dual invariants thus have at least q elements. There are two independent diagonal inertia characters to kill. The same formula gives at least q^{6+3d_v} cocycles with the two diagonal entries equal on the whole local group.

There is an additional direction. In indices $0,1,2$ use the matrix cochain

$$Z(g)=\begin{cases}u(g)E_{00}+\beta(g)E_{01}, & i=0,\\ -u(g)E_{22}+\beta(g)E_{12}, & i=1.\end{cases}$$

Modulo the line O_mE_{02} it is a cocycle. Indeed conjugation on an adjacent entry $j,j+1$ multiplies that entry by λ_j/λ_{j+1} and subtracts $(t_{j,j+1}/t_{j+1,j+1})(Z_{jj}-Z_{j+1,j+1})$. This gives the equation for β. The obstruction to lifting is in $H^2(D_v,O_m(\lambda_0/\lambda_2))$. At the local marking element the three diagonal values are distinct modulo the maximal ideal. Since one adjacent ratio equals ε_ℓ, the outer ratio differs from ε_ℓ there by a unit difference. Local duality in 26.27(1) makes the obstruction group zero. Correcting the 02 entry therefore gives a cocycle in $Z_v^{\triangle}$. Its indicated diagonal difference at some g is the image of a fixed $c_0=u(g)\neq 0$. Multiples realizing distinct elements of c_0O_m give disjoint translates of the equal-diagonal subgroup. With $C=|O/c_0O|$ this proves $C|Z_v^{\triangle}|\geq q^{7+3d_v}$, since $|c_0O_m|\geq q/C$.

Let Z_w include arbitrary local coboundaries, after returning to the marked basis. Adding these gives at least q^3 times as many cocycles as in $Z_w^{\triangle}$. Indeed a matrix whose coboundary is upper triangular is itself upper triangular: conjugate the separating local element to diagonal form by an upper triangular matrix, and use the unit differences on the three entries below the diagonal. The map from $Z_w^{\triangle}\times M_3(O_m)$ onto Z_w thus has kernel of order at most q^6.

Let Z_{glob} be the polarized adjoint cocycles on the frame group satisfying these local conditions. Proposition 26.27(4) gives

$$q^9\prod_w|Z_w| \;\leq\; |Z_{\mathrm{glob}}|\,q^{3d}q^{9\,\#\{w|\ell\}}.$$

To check the factors, the global dual invariants are zero: on the sign kernel the residual matrix composition characters are powers of $\bar{\Psi}$, all distinct from $\bar{\varepsilon}_\ell$ by 30.55. A nonzero dual invariant at any depth would give one on a residual composition factor. At a real place the invariants identify with alternating three-by-three matrices and have order q^3, since 2 is a unit. At the additional finite ramification places take the classes inflated from the quotient by the prescribed inertia kernel. The remaining inertia has order prime to ℓ; averaging followed by the procyclic Frobenius calculation gives a local H^1 of the same order as H^0. These classes descend to the frame quotient because the killed inertia groups normally generate the kernel. Passing from classes to cocycles gives the displayed factors of q^9; the dual Selmer group has order at least one.

Normalize by imposing $z(c)=0$ at the distinguished conjugation and by preserving the marking difference at g_0. This costs at most q^8. The equation at $c^2=1$ makes $z(c)$ symmetric under reflection in the antidiagonal, leaving six entries. When these vanish, the derivative of $r(g_0)-r(cg_0^{-1}c)$ is the difference of a matrix and its reflected transpose. Its off-diagonal vanishing imposes the two entries $01,10$. Write Z_m for the normalized subgroup. Multiplying the local bounds, using $d=\sum_w d_w$, and cancelling gives

$$q^2\leq C|Z_m|.$$

A normalized cocycle gives a derivation into O_m by applying the universal property to the first-order lift with matrix derivative $z(g)r(g)$. The local coboundary corrects the flag basis, and the vanishing diagonal on inertia fixes its characters there. This construction is injective on Z_m: the matrix derivatives recover z on the sign kernel, and $z(c)=0$ recovers it on the other coset.

Let $I=\ker\xi$. These fixed-inertia derivations are homomorphisms from the quotient of I/I^2 by the classes of the inertia character values minus their evaluations. This is a finite O-module in the sense of finite generation, since R is Noetherian and $R/I=O$. If its free rank were at most one, the structure theorem over a DVR would bound the number of homomorphisms to O_m by $C'q$, with C' the order of its torsion. This contradicts $q^2\leq CC'q$ as m grows. Two dual basis vectors of its free part give the desired derivations. □

Proof of 29.13

Proof. Take $M_0=\ell N|D_K|$ for the ramification modulus of the link. It is a positive multiple of ℓN containing the ramification of the quadratic sign. The level N is even, and parity gives $\Psi(c)=1$. Use the field L, frame and seed constructed in 30.61 from the two tests (A.278). Its degree bounds are (A.281). Changing from the original marking to the marking of the frame amounts to integral change of basis: the marked matrix

difference has distinct residual eigenvalues, so its ordered spectral projectors lift, and the polarization pairs opposite eigenlines. Choose paired generators and normalize the middle generator, enlarging coefficients if necessary. Restriction now gives an integral point $\xi : R \to O$ of the rank-three frame. Use the two definite comparisons of 30.59, enlarging O to contain the coefficients of the point and the local embeddings. The rank-three twist in both is ε_ℓ^{-1}.

Lean: `UpperTheory.Gal.projectedFormsSystemPart.openEx2E_alt (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.UnitaryRankTwoFlags`, lines 62–115)

Suppose a level flag over K has a link. The local alternative gives potential crystallinity. Transport the prolongation to one of the framed prolongations above ℓ and restrict to its decomposition group over L. Conjugation on the sign kernel transports the flag by change of basis. On the other coset it takes the reversed dual flag, so a relation $\kappa_i = \varepsilon_\ell \kappa_j$ becomes $\kappa_j^{-1} = \varepsilon_\ell \kappa_i^{-1}$ after transport. The two restricted flags, the transported flag and the flag of ξ, have the same multiset of composition characters by Jordan–Hölder. Thus the framed flag is potentially crystalline there and two of its characters satisfy $\lambda_a = \varepsilon_\ell \lambda_b$. On inertia its characters are Ψ^{1-a}. Evaluate on an inertia element of infinite cyclotomic order after raising it to a power killing ϑ. This gives $k(b-a) = 1$, so $k = 1$ and $b = a+1$. Lemma 29.12 supplies two independent fixed-inertia derivations $D_1, D_2 : R \to O$.

Write A, B, ϕ as in Lemma 29.11. The prime of ξ on A is off both triangular loci and the middle block locus because G_L is open in G_K and the restriction there is irreducible. The determinant point $x = \xi \circ \phi$, with B included in R, is irreducible for the same reason. It is arithmetic: after the twist its inertia exponents are $k(1-i)-1$, so in the locally algebraic convention of 30.25 its labelled weights $1-k(1-i)$ strictly increase. The remaining inertia factors are finite.

For $a \in J$ take s as in Lemma 29.11. Evaluation of $s\phi(a) = 0$ gives $x(a) = 0$. For either $D = D_1, D_2$, differentiating the same equality gives

$$0 = \xi(s) D(\phi(a)) + x(a) D(s), \qquad D(\phi(a)) = 0.$$

The pullback of D to B_0 therefore descends to a derivation on B_0/J. It kills ordinary inertia because ϕ sends each ordinary character to the framed one times the fixed scalar twist. Lemma 29.10 kills this derivation. Surjectivity of ϕ gives $D(B) = 0$.

Choose monomials p, q of weights $1, -1$ with $\xi(p)\xi(q) \neq 0$. Indeed there are surviving entries of both signs off the triangular loci and one of odd weight off the block locus. If the other sign occurs only with absolute weight two, multiply that entry by the surviving odd entry. For a derivation E killing B, the equality $E(p) = 0$ implies $E(q) = 0$ by differentiating $pq \in B$. For an entry a of weight $j > 0$, differentiate $aq^j \in B$; for $j < 0$ differentiate $ap^{-j} \in B$. Thus E kills all matrix entries. The flag characters belong to B, and the entries and characters topologically generate the complete ring. The derivation is continuous, since the derivative of a product of m elements of the maximal ideal takes values in $\mathfrak{m}_O^{m-1}$. Hence $E = 0$.

Apply this to $E = D_2(p)D_1 - D_1(p)D_2$. It kills p, so it is zero. Independence forces $D_1(p) = D_2(p) = 0$, whence $D_1 = 0$ by the same argument. This contradiction excludes the link. Conjugating within G_K gives the conclusion at every prolongation of the place. □

Proof of 29.14

Proof. Factor a characteristic-zero point of the marked complete ring through a finite coefficient DVR and retain the marking and its ordinary local lift. In Type I apply Lemma 29.6 to $G = H = G_K$ and inertia at ℓ. Since $\ell \nmid D_K$, rational inertia lies in G_K and maps onto $\mathbf{Z}_\ell^\times$ under the cyclotomic character. The inertia characters are $\Psi, 1, \Psi^{-1}$, with $\Psi = \varepsilon_\ell^k \vartheta$. A power of an inertia lift of $1+\ell$ kills ϑ and separates the three eigenvalues at every positive power. On inertia $\bar{\vartheta}^2 = 1$, and $\ell - 1 \nmid 2k$, so some inertia element has $\bar{\Psi}^2 \neq 1$. The residual semisimplification is the sum of the three prescribed characters. The lemma gives irreducibility on every open subgroup of G_K, as required in 30.61(2). That theorem supplies the cuspidal attachment with witness after the twist by ε_ℓ^{-1}. Away from ℓ the prescribed inertia type has level dividing N. At each place above ℓ take the ordered ordinary flag. Its characters on inertia are $\Psi, 1, \Psi^{-1}$, so its weights are $(k, 0, -k)$ and its finite parts factor through the action on μ_N. We check the local alternative in Lemma 29.13. In the distinguished local problem of Definition 30.2, Lemma 30.3(3) shows that the marked point is crystalline on the open subgroup G_{K_1}, hence potentially crystalline. For the conjugate place use the reversed dual flag; taking duals and changing basis preserve crystallinity. This gives admissibility with the local alternative. Lemma 29.13 now gives an unlinked level flag, including for the adjacent weights at $k = 1$.

Lean: UpperTheory.V.Loc.rtPoints (Development.AutomorphicArithmetic.AutomorphicVarieties.PadicHodgeTheory.RtPoints, lines 11–15)
Lean: UpperTheory.V.U3.admissible_rank_three_of_type_one_of_automorphic_point (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.ExistsFlagCharactersOfUpperTriangular, lines 163–214)

This puts the point in $\mathcal{A}_3$. Let L be the auxiliary CM field of the linked frame. Its group is open in G_K, so the representation remains irreducible on G_L. At an ordinary place the witness of Definition 29.5 and the local comparison give an element of the cyclotomic kernel with all eigenvalue ratios of infinite order: use the positive Weil degree argument in Lemma 30.18, with weights $k, 0, -k$, and take a positive power to enter G_L. On the linked flag a local element with $\bar{\Psi}^2 \neq 1$ is supplied by 30.55 and the frame of 30.61. Its residual flag characters, up to the common twist, are $\bar{\Psi}, 1, \bar{\Psi}^{-1}$. Lemma 29.6 with $G = G_L$ and $H = G_{L(\mu_{\ell^\infty})}$ gives irreducibility there and on every open subgroup of H. For dense separability one may now use the rank-three coset argument in 30.59, with this same separating element. For a permitted tangent, the finite inertia quotient away from ℓ and the ordinary finite-condition comparison (A.262) at ℓ give the finite adjoint class. Apply 30.62 on the linked frame and descend the coboundary by averaging over the finite orbit of a cobounding element.

In Type II the middle line on G_K is trivial and the corner extends to a representation W with determinant η_K, the quadratic character of K, and trace zero at complex conjugation. Its ordered weights are $(k, -k)$. By 30.61, $W\varepsilon_\ell^{-k}$ has a weight-$(2k+1)$ newform attachment and W has the corresponding witness. To apply 30.62, first restrict the irreducible corner from G_K to the auxiliary G_L, normal in G_K. Its determinant there is one. The linked marking $t \in G_L$ has $\bar{\Psi}(t)^2 \neq 1$ by 30.55, so the residual roots at t are distinct. A reducible restriction would therefore be the sum of two distinct lines interchanged by some $g \in G_K$. Both $W(g)$ and $W(gt)$ would have trace zero. Reduction of their traces gives $\bar{\Psi}(g)^2 = -1 = \bar{\Psi}(gt)^2$, contradicting $\bar{\Psi}(t)^2 \neq 1$. Thus the restriction is irreducible and solvable base change gives the cuspidal attachment on L. The rank-two ordinary comparison over K gives a separating element in the cyclotomic kernel by the witness argument of Lemma 30.18, with weights $k, -k$; take a power into G_L. The local test $\bar{\Psi}(s)^2 \neq 1$ of 30.55 distinguishes the two residual characters $\bar{\Psi}, \bar{\Psi}^{-1}$ at s. The scalar twist on the linked frame preserves these tests and irreducibility. Apply Lemma 29.6 with $G = G_L$ and $H = G_{L(\mu_{\ell^\infty})}$. This checks the cyclotomic restriction in 30.62 on the field where that theorem gives block rigidity. In the full adjoint the two off-block summands are W and $W \otimes \eta_K$. Their finite classes vanish by 30.12. For the restriction over K itself, put $I = I_\ell \subset G_K$. The trace on I is $\Psi + \Psi^{-1}$. Put $N_\ell = G_K \cap \ker \varepsilon_\ell$. If $W|_{N_\ell}$ were reducible, it would be semisimple by the normal-subgroup argument of Lemma 29.6. A projection onto one of its summands, with its scalar trace part subtracted, gives a nonzero traceless endomorphism commuting with N_ℓ. The space of such traceless endomorphisms is invariant under conjugation, and the conjugation action factors through the abelian quotient G_K/N_ℓ. Take a nonzero common invariant subspace of minimal dimension. Each conjugation operator has an eigenspace there, and that eigenspace is invariant under every conjugation operator. Minimality gives a common eigenvector T. Write $W(g)TW(g)^{-1} = \chi(g)T$. Multiplicativity of conjugation makes χ a character through the cyclotomic quotient. The kernel of T is G_K-invariant, so irreducibility gives invertibility. If $\chi = 1$, Schur's lemma makes T scalar, which contradicts its being nonzero and traceless. This gives the required nontrivial self-twist. In the self-twist argument $\chi^2 = 1$ by determinants. Inertia surjects onto the cyclotomic image, so choose $s \in I$ with $\chi(s) \neq 1$. On this coset trace zero forces $\Psi(s)^4 = 1$. Choose $u \in I$ above $1 + \ell$; since $k > 0$ and ϑ is finite, $\Psi(u)$ has infinite order. The same trace identity at su^2 gives $\Psi(u)^8 = 1$, a contradiction. The remaining diagonal summand is $C_\ell(\eta_K)$, whose finite Selmer group vanishes by Lemma 30.4(3). □

Proof of 29.15

Proof. Choose distinct primes $r_1, r_2 > 30fp_K + 1$ and put $K = \mathbf{Q}(\sqrt{-p_K})$. Its discriminant is $D_K = -p_K$, and $r_i \nmid 30N|D_K|$. The characters have prime-to-ℓ order and $\theta_\ell^2 = 1$ by 29.2. Inertia at p_K fixes $\bar{\varepsilon}_\ell$ and θ_P. It acts nontrivially on K. Parity, $2 \mid N$, $p_K \nmid N$ and $\ell \nmid p_K h_*$ give the remaining datum conditions of Definition 30.2 and 30.5, and the standing hypotheses $\mathscr{H}$ (Definition 26.30, Proposition 26.27 and Theorem 27.2) together with 28.2 supply the cohomological inputs of Definition 30.2. Corollary 29.14 supplies its representation hypotheses, and 30.7 gives the trace estimate.

For each occurring character kernel J in $(\mathbf{Z}/N\mathbf{Z})^\times$, take the integers $D_{J,1}, D_{J,2}$ of 30.10 at r_1, r_2. Apply the two-prime sum (A.259) in the proof of Theorem 30.10 to the occurring kernels. It gives finiteness and,

with the constants c_K, A_K there, bounds the sum by $2c_K(\log r_1 + \log r_2)N[N(k+1)]^{A_K}$, which is at most a fixed power of Nt. □

Proof of 29.16

Proof. Take $k = 2j$, $t = 2j+1$, $a = 2$, and $O_\ell = \mathbf{Z}_\ell$. The trivial character belongs to $X_{\ell,\mathbf{Z}_\ell}(2j)$ by 29.1. Choose $q \equiv 3 \bmod 8$ and $d \equiv 7 \bmod 16$, both larger than $2t+1$, the fixed cutoffs in 29.1, and the two primes in 29.15. They can be chosen with $q, d \ll_\chi t$. Indeed, apply 7.5 with modulus 16 to the reduced residue classes $3, 7$. For this fixed modulus its error is $o(X/\log X)$, and any exceptional term is $O(\mathrm{li}_2(X^{\beta_1})) = o(X/\log X)$. Subtraction at $2X$ and X gives a prime in each class in $(X, 2X]$ for all large X. Take X to be the maximum of $2t+1$ and a sufficiently large fixed cutoff. Thus $N = \mathrm{lcm}(16, f)qd \ll_\chi t^2$.

Fix an odd prime ℓ. Put $D = (\mathbf{Q}_\ell/\mathbf{Z}_\ell)(-2j)$, and let G_S be the Galois group over $\mathbf{Q}$ unramified at finite primes outside $S = \{\ell\}$. The strict group of 29.3 for the trivial character is

$$\ker\bigl(H^1(G_S, D) \longrightarrow H^1(\mathbf{Q}_\ell, D)\bigr).$$

To see this, inertia at $p \neq \ell$ acts trivially on D, so a cocycle unramified there vanishes on inertia and descends through the unramified quotient. At a prime $p \mid N$, $p \neq \ell$, its unramified local class belongs to $D/(p^{-2j}-1)D$. The scalar is a nonzero element of $\mathbf{Z}_\ell$, and multiplication by it on D is onto. This accounts for every extra decomposition-group condition in 29.3.

For $u \geq 1$, take $D_u = D[\ell^u]$. Its Cartier dual is $T_u = (\mathbf{Z}/\ell^u)(2j+1)$. The restriction $H^0(G_S, D_u) \to H^0(\mathbf{Q}_\ell, D_u)$ is an isomorphism. Both groups take invariants under the same cyclotomic image, since $\mathbf{Q}_\ell(\mu_{\ell^u})/\mathbf{Q}_\ell$ has Galois group $(\mathbf{Z}/\ell^u)^\times$. The first four terms of 27.2, with ground field $\mathbf{Q}$, odd ℓ and $S = \{\ell\}$, therefore identify the dual of $H^2(G_S, T_u)$ with the kernel in degree one for D_u. Passing to the direct limit on the discrete side and the inverse limit on the compact side gives

$$\mathcal{S}^\vee_{\ell,2j,1} \simeq H^2_{\text{ét}}\bigl(\mathbf{Z}[1/\ell], \mathbf{Z}_\ell(2j+1)\bigr).$$

Here finite-coefficient cohomology agrees with the cohomology of G_S by [111, II, Proposition 2.9], applied to the locally constant ℓ-power torsion sheaves on $\operatorname{Spec}\mathbf{Z}[1/\ell]$. The finite degree-one groups satisfy the Mittag–Leffler condition. Weibel's calculation uses the Rost–Voevodsky norm-residue isomorphism $K_i^M(F)/\ell \simeq H^i(F, \mu_\ell^{\otimes i})$ for fields F of characteristic different from ℓ, through the motivic-to-étale comparison in the motivic spectral sequence. For $\operatorname{Spec}\mathbf{Z}[1/\ell]$ the ℓ-cohomological dimension is two since ℓ is odd. In positive even degree $4j$, Theorem 8.2 therefore identifies the group $H^2_{\text{ét}}(\mathbf{Z}[1/\ell], \mathbf{Z}_\ell(2j+1))$ on the right with $K_{4j}(\mathbf{Z})[\ell^\infty]$ [128, VI, Thms. 4.1, 4.2 and 8.2].

Consequently its logarithmic order is $a_{\ell,2j,1}\log\ell$. For $\ell \notin B$ with $\ell - 1 \mid 4j$, we have $\ell \geq 5$ and 29.28 applies. Take for S in 29.15 the remaining primes outside B dividing the order of the finite group $K_{4j}(\mathbf{Z})$. Retaining the trivial-character summand gives at most $(Nt)^{C_1} \ll_\chi t^{3C_1}$, as asserted. □

Proof of 29.19

Proof. Let z be the cocycle and take a finite ℓ-adic field K_0 containing the values of η. Choose a countable family of finite extensions K_i of $\mathbf{Q}_\ell$ covering C_ℓ. Such a family exists since there are finitely many extensions of each bounded degree in a fixed algebraic closure. The cocycle identity makes $\{g : z(g) \in K_0K_i\}$ a subgroup, and continuity makes it closed. By Baire's theorem one of these subgroups is open. Adjoin the values of z at finitely many coset representatives to its coefficient field K_0K_i. The cocycle identity now puts all values in the resulting finite extension E. On any subgroup where λ is nontrivial, a local cobounding scalar is $z(g)/(\lambda(g)-1)$ for any g with $\lambda(g) \neq 1$. If λ is trivial there, the local cocycle is zero and the scalar can be taken to be zero. Thus the witnesses may be taken in E.

Let O_E be the ring of integers of E. The strict group for $(E/O_E)(\lambda)$ with the same local conditions has finite length. Indeed, restrict to a finite abelian field L containing μ_ℓ and trivializing η. The image has length at most $e_{O_E}s_{\ell,e}(L)$, finite by (28.5) and 28.2. The restriction kernel argument of 29.23 bounds its kernel by a finite multiple of the length of the global invariants. These have finite length since λ is nontrivial on inertia at ℓ.

Choose a uniformizer π of O_E and π^a killing the strict group. For each $m \geq 1$, reducing $\pi^{-m-a}z$ modulo O_E gives a class in that group. Multiplication by π^a makes it zero, so some $u_m \in E$ satisfies

$$z(g) - (\lambda(g) - 1)u_m \in \pi^m O_E \qquad (g \in G_{\mathbf{Q}}).$$

For g_* with $\lambda(g_*) \neq 1$, evaluation gives $u_m \to z(g_*)/(\lambda(g_*) - 1)$. The congruences show that this limit cobounds z. □

Proof of 29.20

Proof. Take the exponents of (29.2). Use $(n', p') = (n_{\mathrm{II}}, 1)$ in II and (n_{III}, p_1) in III. Enlarge O to contain the $2^{n_{\mathrm{II}}+n_{\mathrm{III}}}$-th roots of unity as well as the values of the auxiliary characters. Take a primitive odd character ξ_2 modulo $2^{n'}$. In II put $\xi_0 = 1$. In III take ξ_0 modulo p_1 of order $\operatorname{ord}\theta$, which divides $o_{\mathrm{odd}} \mid p_1 - 1$ by 29.2. In particular ξ_0 is even. Put $\xi = \xi_0\xi_2$, $\Psi = \theta\varepsilon_\ell^k$ and $\alpha = \varepsilon_\ell\omega^{-1}\xi$. The distinguished lift is $\rho_0 = \operatorname{diag}(\alpha\Psi, \Psi, \alpha, 1)$.

Let $T = \{\ell\} \cup \{v \neq \ell : \kappa_v \neq 1\}$. In II, every $v \in T$ divides ℓf by 29.2, and 2^s with $s = c_2(\theta) + 1 + v_2(o)$ kills the 2-parts of θ and ξ_0. In III, $T = \{\ell\}$ and $s = 0$. Since $\ell \neq p_1$ in III, (29.2) gives

$$c_2(\theta) < n', \qquad 2^s v^2 \leq 2^{n'}, \qquad (p', v) = 1 \quad (v \in T). \tag{A.249}$$

For $v \in T$ take arithmetic Frobenius lifts σ_v, with $\varepsilon_\ell(\sigma_\ell) = 1$ by adjustment in inertia. At $v \neq \ell$ the nontrivial ℓ-power character at v factors through the prime-to-v part of the local unit quotient, so $\ell \mid v - 1$. Thus $\bar{\varepsilon}_\ell(\sigma_v) = 1$ throughout T. We verify (29.3):

$$\overline{\xi\Psi^i}(\sigma_v) \neq 1 \qquad (v \in T,\ i \in \mathbf{Z}).$$

Indeed, let u be an odd common multiple of the odd parts of the orders of θ and ξ_0. An equality to one, raised to $2^s u$, gives $\xi_2(v)^{2^s u} = 1$ by injectivity of reduction on 2-power roots of unity. The primitive character ξ_2 is faithful on the units congruent to 1 mod 4. Since $v^{2^{s+1}u} \equiv 1 \bmod 4$, we would have $2^{n'} \mid v^{2^{s+1}u} - 1$. But (28.1) applied to v^2, followed by (A.249), gives

$$v_2(v^{2^{s+1}u} - 1) = s + v_2(v^2 - 1) < n'.$$

The lifts of $\bar{\xi}$, $\bar{\Psi}\bar{\xi}^{-1}$ and $\bar{\Psi}\bar{\xi}$ have respective odd-modulus and dyadic factors

$$(\xi_0, \xi_2), \qquad (\omega^k\theta_1\xi_0^{-1}, \theta_2\xi_2^{-1}), \qquad (\omega^k\theta_1\xi_0, \theta_2\xi_2).$$

The products are odd since Ψ is even. Evaluation at $1 + 2^{n'-1}$ gives primitivity of the dyadic factors by (A.249). The primitive characters inducing the odd-modulus factors have conductors $f' \mid p'\ell\operatorname{cond}(\theta_1)$ and orders dividing $\ell^{f_0} - 1$. For ξ_0 this follows from its triviality in II and $\theta_1 = \theta$ in III. By (29.2), with $\operatorname{cond}(\theta_1) \mid f$ in III, we have $f'4^{n_0} \leq 2^{n'}$. Also $4 \mid \ell^{f_0} - 1$ by the definition of f_0. At $v \in T$ a local coboundary vanishes on inertia in the character kernel. At other places the local condition kills the intersection of inertia with the kernel of the residual diagonal action, of index prime to ℓ. The product criterion 30.75 gives residual Selmer vanishing for the three characters and their inverses.

In each class θ_P is nontrivial and has conductor and order prime to ℓ. Choose g fixing μ_ℓ with $\bar{\theta}_P(g) \neq 1$. The prime-to-ℓ part of the local character at ℓ factors through μ_ℓ, so $\bar{\Psi}(g) = \bar{\theta}_P(g)$. As ξ is odd and Ψ even, either g or its product with complex conjugation gives g_0 with $\bar{\xi}(g_0) \neq 1$ and $\bar{\Psi}(g_0) \neq 1$. Complete the good datum $\mathcal{D}$ with $S_{\mathcal{D}} = \{p : p \mid 2\ell N p'\}$ and

$$\phi = \lim_{m\to\infty} \sigma_\ell^{\ell^{m!}}, \qquad s_0 \in I_\ell \cap \ker\theta \cap \ker\omega, \qquad \varepsilon_\ell(s_0) \text{ of infinite order.}$$

The powers defining ϕ stabilize in each finite quotient at the prime-to-ℓ part of the image of σ_ℓ. For s_0, raise an inertia preimage of $1 + \ell$ to $\operatorname{ord}(\theta)(\ell - 1)$. The character ξ is unramified at ℓ, so $\xi(s_0) = 1$.
Lean: `UpperTheory.V.X.xvan` (`Development.AutomorphicArithmetic.AutomorphicVarieties.LevelCohomology.HcXvan`, lines 19–40)

□

Proof of 29.21

Proof. With the alternating form J of antidiagonal $(1,1,-1,-1)$, a matrix $X \in M_4$ lies in $\mathfrak{sp}_4$ precisely when $X_{ij} = -s_i s_j X_{3-j,3-i}$ for $s = (1,1,-1,-1)$. The 10-dimensional space $\mathfrak{sp}_4$ decomposes under $\mathrm{Ad}\, r_0$ into root spaces of bidegrees:

- first degree zero: $(0,0)$ of dimension 2 (diagonal $X_{00} = -X_{33}$, $X_{11} = -X_{22}$, character 1), $(0,1)$ of dimension 1 ($X_{02} = X_{13}$, character Ψ), and $(0,-1)$ of dimension 1 ($X_{20} = X_{31}$, character Ψ^{-1});
- nonzero first degree (six 1-dimensional cross-block spaces): $(1,0)$ ($X_{01} = -X_{23}$, character α), $(-1,0)$ ($X_{10} = -X_{32}$, character α^{-1}), $(1,1)$ (X_{03}, character $\alpha\Psi = \nu$), $(-1,-1)$ (X_{30}, character ν^{-1}), $(-1,1)$ (X_{12}, character Ψ/α), and $(1,-1)$ (X_{21}, character α/Ψ).

The four strictly upper-triangular roots ($i<j$) have bidegrees $(0,1),(1,0),(-1,1),(1,1)$, and the four strictly lower-triangular roots ($i>j$) are their negatives.

For (1), fix $v \in T \setminus \{\ell\}$ and put $N_v = I_v \cap \ker\theta$. Since $D_v/I_v \simeq \widehat{\mathbf{Z}}$ is abelian, every commutator $[g,h]$ in D_v lies in I_v. Because $\theta : D_v \to O^\times$ is a homomorphism into an abelian group, $[g,h] \in \ker\theta$ as well. Hence $[D_v, D_v] \subseteq N_v$, so D_v/N_v is abelian, topologically generated by σ_v and an inertia generator τ_v of order $m_v = \#\theta(I_v)$ with $\sigma_v\tau_v = \tau_v\sigma_v$. Since $\ell \mid m_v$, $\zeta = \theta(\tau_v) \neq 1$, and because α is unramified at v, $r_0(\tau_v) = \mathrm{diag}(\zeta,\zeta,1,1)$.

Let $U_v^\wedge = \widehat{U_v[1/\ell]}_{\ker x_{0,v}}$, and consider a lift $b : D_v/N_v \to \mathrm{GSp}_4(A)$ of $r_0|_{D_v}$ over a local Artinian E-algebra A with residue field E satisfying $\det(XI_4 - b(\tau_v)) = (X-\zeta)^2(X-1)^2$. Since $b(\tau_v)^{m_v} = I_4$ and $X^{m_v}-1$ is separable over E, writing $X^{m_v}-1 = (X-\zeta)(X-1)Q(X)$ shows that $Q(b(\tau_v))$ is invertible over A, whence $(b(\tau_v)-\zeta I_4)(b(\tau_v)-I_4) = 0$. As $\zeta - 1 \in E^\times$, $A^4 = W_\zeta \oplus W_1$ splits into the free rank-2 eigenspaces of $b(\tau_v)$ for ζ and 1, lifting $Ee_0 \oplus Ee_1$ and $Ee_2 \oplus Ee_3$. Since $b(\tau_v)$ has similitude $\nu(\tau_v) = \zeta \neq 0,1$, both W_ζ and W_1 are isotropic for J, hence complementary Lagrangian subspaces of (A^4, J). Because $b(\sigma_v)$ commutes with $b(\tau_v)$, it preserves W_ζ and W_1; conversely, any complementary Lagrangian pair (W_ζ, W_1) lifting $(Ee_0 \oplus Ee_1, Ee_2 \oplus Ee_3)$ together with any invertible $M_\zeta \in \mathrm{GL}(W_\zeta)$ lifting $\mathrm{diag}(\nu(\sigma_v), \Psi(\sigma_v))$ uniquely determines $b(\tau_v)$ and $b(\sigma_v)$ (the action on $W_1 \simeq W_\zeta^\vee$ being forced by similitude $\nu(\sigma_v)$). Near the base point, W_ζ is the graph of a matrix C in positions $(2,0),(2,1),(3,0),(3,1)$ satisfying the single Lagrangian condition $C_{20} = C_{31}$ (three unconstrained parameters of bidegrees $(0,-1),(1,-1),(-1,-1)$), W_1 is the graph of a matrix D in positions $(0,2),(0,3),(1,2),(1,3)$ with $D_{02} = D_{13}$ (three unconstrained parameters of bidegrees $(0,1),(1,1),(-1,1)$), and M_ζ supplies four unconstrained parameters of bidegrees $(0,0),(0,0),(1,0),(-1,0)$. Thus $U_v^\wedge/J_vU_v^\wedge$ is formally smooth over E of dimension $6+4=10$, with bigraded coordinate counts $(1,1,2)$ in $((0,-1),(0,1),(0,0))$ and 1 in each of the six cross-block bidegrees.

Since $U_v^\wedge/J_vU_v^\wedge \simeq E[[t_1,\dots,t_{10}]]$ is a domain, $\mathfrak{p}_v$ is the unique minimal prime of $\mathrm{sat}_\ell(\sqrt{J_v})$ contained in $\ker x_{0,v}$, and $B_v = U_v/\mathfrak{p}_v$ embeds in $\widehat{B_v[1/\ell]}_{\ker x_{0,v}} \simeq E[[t_1,\dots,t_{10}]]$ by Krull's intersection theorem. Because J_v is invariant under Sp_4-conjugation, the preimage of $\mathfrak{p}_v \cdot (U_v \widehat{\otimes}_O O[[z]])$ under universal formal Sp_4-conjugation congruent to I_4 is a prime ideal containing $\mathrm{sat}_\ell(\sqrt{J_v})$ and contained in $\ker x_{0,v}$, hence contains $\mathfrak{p}_v$; this gives conjugation invariance and a bigraded homogeneous presentation. For each cross-block root $\chi = \alpha^{j_1}\Psi^{j_2}$ ($j_1 \neq 0$), (29.3) gives $\bar\chi(\sigma_v) \neq 1$, while $\bar\varepsilon_\ell(\sigma_v) = 1$ since $\ell \mid v-1$; thus $H^0(D_v, E(\chi)) = H^2(D_v, E(\chi)) = 0$, so $H^1(D_v, E(\chi)) = 0$ by local Euler characteristic and $Z^1(D_v, E(\chi)) = B^1(D_v, E(\chi))$ is 1-dimensional.

For (2), let $U_\ell^\wedge = \widehat{U_\ell[1/\ell]}_{\ker x_{0,\ell}}$ and consider an A-point $b : D_\ell \to \mathrm{GSp}_4(A)$ of $U_\ell^\wedge/J_\ell U_\ell^\wedge$ over a local Artinian E-algebra A with residue field E. Since $k \geq 2$ and $\varepsilon_\ell(s_0)$ has infinite order, the eigenvalues $d_i = \varepsilon_\ell(s_0)^{w_i}$ ($w = (k+1,k,1,0)$) of $r_0(s_0)$ are pairwise distinct in $E^\times$. The operators $e_i = P_i/\prod_{j\neq i}(d_i - d_j)$ are orthogonal rank-1 idempotents summing to I_4 with $e_i^t J e_j = 0$ for $i+j \neq 3$, giving a D_ℓ-stable isotropic full flag in A^4 with inertia characters $(\delta_0,\delta_1,\delta_2,\delta_3)$. Choosing a symplectic basis adapted to this flag congruent to the standard basis modulo $\mathfrak{m}_A$ expresses $b = h b_{\mathrm{tri}} h^{-1}$ with $b_{\mathrm{tri}} : D_\ell \to B_{\mathrm{GSp}_4}(A)$ upper triangular of similitude ν and inertia characters (δ_i), and $h \in \mathrm{Sp}_4(A) \cap (I_4 + \mathfrak{m}_A M_4)$ unique up to right multiplication by $B_{\mathrm{Sp}_4}(A) \cap (I_4 + \mathfrak{m}_A M_4)$.

We check that upper-triangular lifts b_{tri} with fixed similitude ν and fixed inertia characters (δ_i) are formally smooth over E. The diagonal characters satisfy $\lambda_0\lambda_3 = \lambda_1\lambda_2 = \nu$ and $\lambda_i|_{I_\ell} = \delta_i|_{I_\ell}$, so they are parameterised freely by two unramified scalars $\lambda_0(\sigma_\ell), \lambda_1(\sigma_\ell)$ in bidegree $(0,0)$. The four strictly upper roots are $\chi \in \{\Psi, \alpha\Psi, \alpha, \Psi/\alpha\}$, with cyclotomic weights $k, k+1, 1, k-1 \geq 1$ (as $k \geq 2$), so $H^0(D_\ell, E(\chi)) = 0$

for all four. For $H^2(D_\ell, E(\chi)) \simeq H^0(D_\ell, E(\varepsilon_\ell\chi^{-1}))^\vee$: for $\chi = \Psi$ or $\alpha\Psi$, the cyclotomic weight of $\varepsilon_\ell\chi^{-1}$ is $1-k \le -1$ or $-k \le -2$; for $\chi = \alpha = \varepsilon_\ell\omega^{-1}\xi$, $(\varepsilon_\ell\alpha^{-1})(\sigma_\ell) = \xi(\sigma_\ell)^{-1} \neq 1$ by (29.3); for $\chi = \Psi/\alpha$, either $k \ge 3$ (cyclotomic weight $2-k \le -1$) or $k = 2$ and $(\varepsilon_\ell(\Psi/\alpha)^{-1})(\sigma_\ell) = (\xi\Psi^{-1})(\sigma_\ell) \neq 1$ by (29.3). Thus $H^2(D_\ell, E(\chi)) = 0$ for all four strictly upper roots, and local Euler characteristic gives $\dim_E H^1(D_\ell, E(\chi)) = 1$ and $\dim_E Z^1(D_\ell, E(\chi)) = 2$. Lifting along the central series of the nilradical is unobstructed at each step. Adjoining the four strictly lower-triangular framing directions $h \in \mathrm{Sp}_4/B_{\mathrm{Sp}_4}$ (each contributing $\dim_E B^1(D_\ell, E(\chi^{-1})) = 1$ since $H^0(D_\ell, E(\chi^{-1})) = 0$) shows that $U_\ell^\wedge/J_\ell U_\ell^\wedge$ is formally smooth over E of dimension $2 + 4\cdot 2 + 4\cdot 1 = 14$, with the stated bidegree counts. Just as in (1), $\mathfrak{p}_\ell$ is therefore a single prime ideal invariant under Sp_4-conjugation congruent to I_4, and $B_\ell = U_\ell/\mathfrak{p}_\ell$ is a complete Noetherian local O-domain with a bigraded homogeneous presentation.

For (3), consider the framed presentation of 30.72 over $B = B_\ell \mathbin{\widehat{\otimes}_O} \widehat{\bigotimes}_{v\in T\setminus\{\ell\}} B_v$ in any cross-block bidegree $j = (j_1, j_2)$ with $j_1 \neq 0$. By the residual Selmer clause of Definition 29.17 and (29.3) at σ_v ($v \in T$), we have $H^1_{\mathrm{fr},j} = 0$; since $\bar\alpha^{j_1}\bar\Psi^{j_2}$ is odd and differs from $\bar\varepsilon_\ell$ at σ_ℓ, Poitou–Tate duality 27.2 and the local and global Euler characteristics give $\dim \mathrm{Ob}_j - \dim H^1_{\mathrm{fr},j} = 0$, hence $\mathrm{Ob}_j = 0$. Thus there are no relative variables or relative relations in cross-block bidegrees; the cross-block coordinates of the ambient formal chart are the 9 from B_ℓ and the 6 from each B_v ($v \in T\setminus\{\ell\}$). At ℓ, $r(\phi) = b_\ell(\phi)$ depends only on B_ℓ, and along each of the six cross-block coboundary directions δX, the corresponding entry of $b_\ell(\phi) - r_0(\phi)$ has derivative $(r_0(\phi)_{ii} - r_0(\phi)_{jj})X_{ij}$, a unit by (29.3). At each $v \in T\setminus\{\ell\}$, when the variables of B_ℓ are set to zero, we have $z|_{D_\ell} = 0$, $z|_{D_v} = \beta_v + \delta b_v \in B^1(D_v, E(\chi))$ ($v \in T\setminus\{\ell\}$), and $z|_{I_q\cap\ker\bar r} = 0$ ($q \notin T$); if $z \neq 0$, scaling z to take primitive values in O keeps each local cobounding scalar $u_v = z(\sigma_v)/(\chi(\sigma_v) - 1)$ integral in O by (29.3), so reducing modulo $\mathfrak{m}_O$ produces a nonzero element of $H^1_{\mathrm{fr},j} = 0$, a contradiction. Thus $z = 0$, so for the frame $h_v = I_4 - \epsilon b_v$ of 30.72 the intertwining relation $z|_{D_v} = \beta_v + \delta b_v$ with $\beta_v = \delta X_v \in B^1(D_v, V_j)$ gives $b_v = -X_v$; thus the six cross-block entries of $h_v - I_4$ have derivative 1 along the six cross-block coordinates of B_v. The Jacobian of the $6 + 6(|T|-1)$ cross-block equations with respect to these $6 + 6(|T|-1)$ directions is therefore block triangular with invertible diagonal blocks. By the formal implicit function theorem over E, imposing these equations eliminates all 6 cross-block coordinates of each B_v ($v \in T\setminus\{\ell\}$) and 6 of the 9 cross-block coordinates of B_ℓ, leaving $(1,2,2)$ at ℓ and $(1,1,2)$ at each $v \in T\setminus\{\ell\}$ in first-degree-zero bidegrees, together with the three remaining cross-block coordinates $u : (1,0)$, $v : (-1,1)$, and $w : (1,1)$. A monomial $u^a v^b w^c$ has first degree $a - b + c = 0$ precisely when $b = a + c$, i.e. $u^a v^b w^c = (uv)^a (wv)^c$; hence the first-degree-zero formal subring of $E[[u,v,w]]$ is $E[[uv, wv]]$, generated by the algebraically independent coordinates uv of weight 1 and wv of weight 2. □

Proof of 29.22

Proof. Choose a good datum $\mathcal{D}$ by Lemma 29.20. Corollary 29.4 permits its finite coefficient extension, since the normalized length is unchanged. In the notation of Definitions 29.17 and 29.18, write $\rho = r$ for the lift and $\rho_0 = r_0$ for its distinguished specialization. Write $L(\mathcal{D})$ for the length of the strict group with coefficients in $A_O(\Psi^{-1})$, with decomposition conditions at $S_\mathcal{D}$ and vanishing on finite inertia. At each added prime Frobenius acts by a root of unity times p^{-k}, so Frobenius minus the identity is surjective on the divisible coefficient module. The procyclic cohomology formula for the discrete torsion module and the unramified quotient generated topologically by Frobenius [127, Prop. 1.7.7(i)] gives zero unramified local cohomology there. Hence $L_{\ell,k,\theta} \le L(\mathcal{D})$. The strict group has finite length by the finite-coefficient argument in 29.19.

Since $\Psi^{-1}(g_0) - 1$ is a unit, every class x in the strict group for $\mathcal{D}$ has a unique representative z_x with $z_x(g_0) = 0$. These representatives are additive in x. Choose $b_0 \neq 0$ killing the group. At each $v \in T$ choose $s_v \in I_v$ with $\Psi(s_v) \neq 1$. Such an element exists by the infinite cyclotomic image at ℓ and by $\kappa_v \neq 1$ elsewhere. Evaluation at s_v shows that $b_0(\Psi^{-1}(s_v) - 1)$ kills the local cobounding element. Hence $b = b_0 \prod_{v\in T}(\Psi^{-1}(s_v) - 1)$ puts the cocycles and local witnesses in $A_O[b] \simeq O/(b)$.

In the framed presentation of 30.72, give entry ij bidegree $(l_i - l_j, m_i - m_j)$, where $l = (1,0,1,0)$ and $m = (1,1,0,0)$. At ℓ use entries $(0,1), (0,3), (2,1), (1,0), (3,0), (1,2)$ of $\rho(\phi)$, with the frame at ℓ fixed to the identity. The derivative along the corresponding root vector is $\rho_{0,jj}(\phi) - \rho_{0,ii}(\phi)$. These differences are units by (29.3), since ϕ and σ_ℓ have the same residual diagonal. At the other $v \in T$ use the same six entries of the frame at v, with derivative 1 in each corresponding direction. The six equations at each place therefore eliminate those directions.

The residual adjoint action in bidegree $j = (j_1, j_2)$ is scalar with character $\bar{\alpha}^{j_1}\bar{\Psi}^{j_2}$. The framed Euler characteristic gives

$$\dim \mathrm{Ob}_j - \dim H^1_{\mathrm{fr,j}} = 2\,\mathbf{1}_{j=(0,0)} + \mathbf{1}_{j=(0,1)} + \mathbf{1}_{j=(0,-1)}.$$

Here $H^1_{\mathrm{fr,j}}$ is the space of relative tangent classes with the local frame conditions. In the Euler characteristic the local terms cancel the contributions at the finite places, and the remaining term is the dimension fixed by complex conjugation. The weights of first degree zero contribute 2 in degree $(0,0)$ and 1 in each corner degree. In the cross-block roots the character is odd. The dual invariant term vanishes there by (29.3) at σ_ℓ, whose cyclotomic value is one, and in the even degrees it vanishes by evaluation at complex conjugation. In each cross-block root the relative tangent dimension is zero by the residual Selmer vanishing above.
Lean: `UpperTheory.V.GradeZeroInterpolatingElement.nums` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.NumsZero`, lines 123–168)

Write $b_T = |T| - 1$ and let a_-, a_+, a_0 be the relative variable counts in bidegrees $(0,-1), (0,1), (0,0)$. By Lemma 29.21(3), after eliminating the $6 + 6b_T$ cross-block directions, the local variables at ℓ contribute $(1,2,2)$ and each further place $v \in T \setminus \{\ell\}$ contributes $(1,1,2)$. Thus

$$r = 1 + b_T + a_-, \qquad r' = 2 + b_T + a_+, \qquad m = 2 + 2b_T + a_0. \tag{A.250}$$

Here m denotes the number of degree-zero variables. At each further place take the $02, 20, 00, 11$ entries of its frame minus the identity, and take the $02, 20$ entries of the marking matrix. Adding these relations to the obstruction counts gives $(r+1, m, r')$ in the three degrees. The three local coordinates with nonzero first degree have bidegrees $u : (1,0)$, $v : (-1,1)$ and $w : (1,1)$. A first-degree-zero monomial in these variables has exponent of v equal to the sum of the exponents of u, w. Hence the first-degree-zero chart adds uv of weight 1 and wv of weight 2. In weights $(-2,-1,0,1,2)$ its coordinate counts are $(0, r, m, r'+1, 1)$ and its relation counts are $(0, r+1, m, r', 0)$. Lemma 29.19 kills the negative-weight tangents modulo coboundaries, and unramified reciprocity kills the weight-zero tangents since $\mathrm{Cl}(\mathbf{Q}) = 1$ and the strict local conditions remove the ramified directions [41, Ch. VI, Prop. 6.9]. The marking removes the coboundary directions. Hence the nonpositive relation linear terms span the corresponding cotangent spaces over the fraction field.
Lean: `UpperTheory.V.GradeZeroInterpolatingElement.exModel` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.ExModel`, lines 425–520)
Lean: `UpperTheory.V.GradeZeroInterpolatingElement.aSmooth` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.ASmooth`, lines 1126–1139)

Claim 1. The graded algebra used in the orbit estimate is defined over O, and its relation ideal is obtained by restricting the integral representation model.

We now give the construction. Put

$$B = B_\ell \mathbin{\widehat{\otimes}_O} \widehat{\bigotimes}_{v \in T \setminus \{\ell\}} B_v,$$

using the local rings and their origins from Lemma 29.21. Let U be the universal framed B-algebra of 30.72, with representation ρ_U, frame matrices h_v, and $h_\ell = I_4$. Its multiplier is $\nu = \alpha\Psi$. Choose compatible presentations on finite variable sets Σ and $\Sigma \sqcup \tau$, with the variables in τ the relative variables:

$$q_B : O[[X_s : s \in \Sigma]] \twoheadrightarrow B, \qquad q_U : O[[X_s : s \in \Sigma \sqcup \tau]] \twoheadrightarrow U.$$

Here the kernels are stable under taking each bidegree component; this is the meaning of a homogeneous presentation. The maps preserve the origin, and the restriction of q_U to the base variables is the structure map $B \to U$ composed with q_B. Write J_B for the extension of $\ker q_B$ to the second power series ring. The relative presentation gives homogeneous elements f_γ with

$$\ker q_U = J_B + (f_\gamma)_\gamma.$$

Put $P = O[[X_s : s \in \Sigma \sqcup \tau]]/J_B$. Then $p : P \twoheadrightarrow U$ preserves the origin, is surjective on every homogeneous part, and has kernel generated by the images of the f_γ. To construct its chart, let $B^{\mathrm{gr}} \subseteq B$ be the algebra of finite sums of homogeneous elements, and let $\mathfrak{n}$ be the kernel of the origin of B. The homogeneous presentation gives, for every $k \geq 1$,

$$\mathfrak{n}^k \cap B^{\mathrm{gr}} = (\mathfrak{n} \cap B^{\mathrm{gr}})^k, \qquad B = B^{\mathrm{gr}} + \mathfrak{n}^k.$$

The product of the local charts therefore extends from B^{gr} to B by taking the compatible truncations in the target power series ring. These identities also preserve the origin and the rational cotangent comparison. Adjoining the relative variables gives a chart of P with the local coordinate counts of Lemma 29.21, together with the relative variables.

Lean: UpperTheory.V.Alg.ker_pow (Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebra.KerPow, lines 89–149)
Lean: UpperTheory.smoothPt_part_of_smoothPt_associatedGraded (Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.SmoothPtPartOfSmoothPtAssociatedGraded, lines 79–115)
Lean: UpperTheory.nonempty_cover_of_relative_presentation_smoothPt (Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.NonemptyCoverOfRelativePresentationSmoothPt, lines 161–254)

Let t_α be the six specified cross-block entries of $\rho_U(\phi)$ and of each $h_v - I_4$ for $v \neq \ell$. Let s_β be the $02, 20, 00, 11$ entries of $h_v - I_4$ for $v \neq \ell$, together with the $02, 20$ entries of $\rho_U(g_0) + \nu(g_0)\rho_U(g_0)^{-1}$. Choose homogeneous lifts $\widetilde{t}_\alpha, \widetilde{s}_\beta \in P$, and set

$$R = P/(\widetilde{t}_\alpha)_\alpha, \qquad S = U/(t_\alpha, s_\beta)_{\alpha,\beta}.$$

These are quotients over O, with their induced origins. The map $\pi : R \twoheadrightarrow S$ has kernel generated by the images of the f_γ and $\widetilde{s}_\beta$. The cross-block Jacobian calculation gives the stated characteristic-zero chart of R. The relations remaining in $\ker \pi$ have first degree zero. Group them by their second degrees as

$$F_1, \dots, F_{r+1} \text{ of degree } (0,-1), \quad E_1, \dots, E_{r'} \text{ of degree } (0,1), \quad Z_1, \dots, Z_m \text{ of degree } (0,0),$$

where every Z_i vanishes at the origin. Their numbers are those in (A.250) and the following relation count.

Write $q : O[[Y_s : s \in \Sigma_R]] \twoheadrightarrow R$ for the resulting homogeneous presentation, and put

$$R_{(a,b)} = q\big(\{F : \text{every monomial of } F \text{ has bidegree } (a,b)\}\big), \qquad A = \bigoplus_{j \in \mathbf{Z}} R_{(0,j)} \subseteq R, \qquad A_j = R_{(0,j)}.$$

The direct sum consists of finite sums of homogeneous elements. Homogeneity of $\ker q$ makes the weight projections well defined, and multiplication adds bidegrees. Thus A is an O-subalgebra, with origin $x_0 : A \to O$. The monoid of nonzero weight-$(0,0)$ exponents has finitely many coordinatewise minimal elements. Subtracting one such element and inducting on total degree shows that these elements generate the weight-zero monoid. Substitution of their monomials therefore presents A_0 as a quotient of a formal power series ring in finitely many variables over O. In particular A_0 is complete Noetherian local with the same residue field as O.

Every presentation variable has first degree $-1, 0$, or 1. A monomial of first degree zero factors into first-degree-zero variables and pairs of variables of opposite first degrees. These finitely many generators have second degrees in $\{-2,-1,0,1,2\}$. For each fixed bidegree $(0,j)$, grouping formal series by the finitely many coordinatewise minimal exponents of that bidegree expresses the homogeneous part as a finite sum of monomials with coefficients in A_0. Consequently A is generated over A_0 by the indicated finite set, after omitting its weight-zero members. Put

$$I = (F_1, \dots, F_{r+1}, E_1, \dots, E_{r'}, Z_1, \dots, Z_m)A.$$

Then

$$\ker(\pi|_A) = I, \qquad A/I \simeq \pi(A) \subseteq S.$$

Indeed, if a finite sum of homogeneous elements of A belongs to $\ker \pi$, express it using the displayed generators in R. In each of its finitely many degrees $(0,j)$, replace a coefficient of a generator of degree $(0,d)$ by its $(0,j-d)$-component. This gives the same element with all coefficients in A, proving the asserted contraction identity.

Lean: UpperTheory.V.GradeZeroInterpolatingElement.Model.aPart (Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.EntryMem, lines 104–104)
Lean: UpperTheory.V.GradeZeroInterpolatingElement.Model.A (Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.EntryMem, lines 112–112)
Lean: UpperTheory.V.GradeZeroInterpolatingElement.Model.gr (Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.EntryMem, lines 114–114)
Lean: UpperTheory.V.GradeZeroInterpolatingElement.sInternal (Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.SInternal, lines 69–96)

To relate this algebra to the chart, put $K_O = \operatorname{Frac} O$. The chart induces a graded map from A to a formal power series ring over K_O, with constant term x_0, whose linear terms identify the rational cotangent spaces. In the three nonzero-first-degree coordinates, for $a, b, c \in \mathbf{N}$ the equation

$$a - b + c = 0 \quad \Longleftrightarrow \quad u^a v^b w^c = (uv)^a (wv)^c$$

identifies the first-degree-zero formal series with formal series in uv, wv. Their second degrees are $1, 2$. Together with the first-degree-zero coordinates this gives the counts $(0, r, m, r'+1, 1)$ in weights $(-2, -1, 0, 1, 2)$, as required in 30.72.

Lean: `UpperTheory.V.Alg.IsChart` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebra.OrbitVanishingLocal`, lines 74–79)
Lean: `UpperTheory.V.Alg.SmoothPt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebra.OrbitVanishingLocal`, lines 81–84)

Claim 2. The strict group injects into the quotient-valued derivations of this integral algebra required by the cotangent comparison.

Use zero-based matrix indices and put $X_- = E_{20} + E_{31}$, where E_{ij} is a matrix unit. Over $O \oplus \epsilon O/(b)$, with $\epsilon^2 = 0$, define

$$\rho_x(g) = (1 + \epsilon z_x(g) X_-)\rho_0(g).$$

The cocycle identity gives the representation law because conjugation by ρ_0 acts on X_- by Ψ^{-1}. Lift each local cobounding element to O. Conjugation invariance of the local ideals in Lemma 29.21 gives an integral origin derivation of B lifting the local part of this deformation. In the presentation of P, lift the values on the new variables of bidegree $(0, -1)$ to O and set the other new-variable values to zero. This defines an integral origin derivation of P, supported in bidegree $(0, -1)$, whose reduction modulo b is the derivation defined by the framed deformation. Each $\widetilde{t}_\alpha$ has nonzero first degree, so its derivative is zero. The derivation therefore descends to R and restricts to A. Its reduction kills I, since the framed deformation satisfies the remaining equations. Its value modulo b on the 20-entry is

$$\rho_{0,00}(g) z_x(g).$$

This multiplier is a unit, and the chosen cocycles are additive in x. Thus the entry values detect the strict class and give the asserted additive injection. The cotangent comparison of 30.72 now gives

$$L(\mathcal{D}) \le \operatorname{length}_O C_{-1}(F).$$

Lean: `UpperTheory.V.GradeZeroInterpolatingElement.exModel` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.ExModel`, lines 425–520)

Let x be a two-sided point of A/I. In the bigraded algebra generated by the homogeneous pieces of the representation model, projection onto first degree zero is linear over the first-degree-zero subalgebra. It splits the inclusion, so the fibre after base change by x is nonzero. This algebra is of finite type over its first-degree-zero part, so the fibre has a C_ℓ-point, say $\widetilde{x}$. By 30.51 the bidegree-zero image is finite over O. At $\widetilde{x}$, every nonconstant monomial of bidegree zero in the centered coordinates has absolute value less than one. Its value is integral by finiteness. If a nonzero value had integral inverse, reversing a monic equation for the inverse would give a polynomial in the monomial with constant term one that vanishes at $\widetilde{x}$. That polynomial is a unit in the degree-zero formal algebra.

There are therefore torus scalars making all nonzero centered coordinate values have absolute value less than one. In valuation terms, one seeks a linear functional on the two weights whose addition to the coordinate valuations makes them all positive. A failure of these strict inequalities would give a nonzero nonnegative rational relation among the weights with nonpositive sum of valuations. Clearing denominators would give a bidegree-zero monomial of nonpositive valuation. Choose a rational solution and torus scalars with those valuations. The scalars and the finitely many coordinate values lie in a finite extension of the coefficient field. Evaluation of the formal series in its complete ring of integers gives an allowed point ρ. Torus scaling preserves the two-sided condition and the bidegree-zero values.

Lean: `UpperTheory.V.GradeZeroInterpolatingElement.ptLift` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.PtLift`, lines 124–214)
Lean: `UpperTheory.V.GradeZeroInterpolatingElement.ptScale` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.PtScale`, lines 45–69)
Lean: `UpperTheory.V.Alg.points_finite` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebra.PointsFinite`, lines 310–383)

Write $i \to j$ if $\rho(g)_{ij} \neq 0$ for some g. The span of the representation contains all four diagonal projections. Indeed, $\rho(\phi)$ is block diagonal in the blocks $\{0,2\},\{1,3\}$, and their characteristic polynomials are coprime by residual separation. Polynomial projections therefore give the two block idempotents. The marking matrix

$$M_\rho = \rho(g_0) + (\alpha\Psi)(g_0)\rho(g_0^{-1})$$

is diagonal on the $\{0,2\}$ block, with reduced difference $(\bar{\alpha}(g_0)-1)(\bar{\Psi}(g_0)-1)$. It separates the two projections there. Symplectic adjunction in the span gives the projections at the reflected indices $i \mapsto 3-i$. Multiplication by the projections turns a nonzero entry into a matrix unit. Thus $\to$ is transitive, and symplectic adjunction sends $i \to j$ to $3-j \to 3-i$.

A mutually linked cross-block pair cannot form a component of size two. For such a pair, transitivity confines the multiplication law on its two-by-two subquotient to its two indices. The ordinary flag induces a stable local line in that subquotient. Orient the pair as (i,j) so a vector on that line is $(a,1)$ with $|a|_\ell \leq 1$. In the integral model write $\rho(g)_{ij} = bh(g)$, where b generates the nonzero ideal of the ij entries. Dividing the multiplication law by b and reducing gives the cocycle $\bar{h}(g)/\bar{\rho}_{0,jj}(g)$ for the residual ratio $\bar{\rho}_{0,ii}/\bar{\rho}_{0,jj}$. The stable line gives

$$bh(g) = a\big(\rho(g)_{jj} - \rho(g)_{ii} + a\rho(g)_{ji}\big) \qquad (g \in D_\ell).$$

Compare this identity at g and σ_ℓ by cross multiplication. After cancellation of b and reduction, the last term in parentheses vanishes and the diagonal difference at σ_ℓ is a unit. This makes the residual cocycle a local coboundary at ℓ. At $v \in T \setminus \{\ell\}$ the local first cohomology of this root is zero by local duality and the Euler characteristic, since (29.3) and $\bar{\varepsilon}_\ell(\sigma_v) = 1$ kill invariants and dual invariants. Elsewhere the cocycle vanishes on the prescribed inertia kernels. The residual Selmer vanishing makes it a global coboundary. Its value at ϕ is zero, and the cross-block ratio at ϕ differs from one. The cobounding scalar is therefore zero, contrary to the choice of b.

A mutual component of size at least three meets its symplectic reflection, and their union is all four indices. It then gives full matrix span. If each mutual component is a singleton, the support admits a triangular order. In the remaining case both within-block pairs are mutually linked by reflection, and transitivity permits cross-block entries in at most one direction. Full matrix span gives irreducibility. In the remaining case the points are block triangular.

Lean: `UpperTheory.V.GradeZeroInterpolatingElement.eii` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.EiiUnit`, lines 169–186)
Lean: `UpperTheory.V.GradeZeroInterpolatingElement.exists_third_index_linked_pair` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.ExistsThirdIndexLinkedPair`, lines 306–348)
Lean: `UpperTheory.V.GradeZeroInterpolatingElement.tri` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.TriComb`, lines 28–34)

In the triangular case compare the diagonal characters in the triangular order with those in the prescribed local ordinary flag. Equality of traces and separation at s_0 match the two lists by a permutation. The cross-block residual separation at σ_ℓ makes it preserve $\{0,2\}$ and $\{1,3\}$. The symplectic pairing, evaluated at s_0, makes it commute with $i \mapsto 3-i$. The possibilities are the identity and $(0\,2)(1\,3)$. In either order, a cover $i \to j$ of the support order gives a two-dimensional extension, with cocycle $\rho(g)_{ij}/\rho(g)_{jj}$. If it runs against the prescribed order of the weights, its character is a negative cyclotomic power times a finite character, by the inertia comparison. The local ordinary flag splits that extension at ℓ, and the prescribed finite inertia quotients give local coboundaries elsewhere. By 29.19 it splits globally. The diagonal projections in the span force the cobounding scalar to vanish, since a coboundary would make the ij entry a fixed multiple of the diagonal difference on the entire span. Transitivity expresses every strict support relation as a chain of covers. Hence the negative-weight entries vanish in the first order and the positive-weight entries vanish in the second. The respective $(0,0)$ values at s_0 are $(\alpha\Psi)(s_0)$ and $\alpha(s_0)$, which differ since $\Psi(s_0) = \varepsilon_\ell(s_0)^k$ has infinite order. This entry has weight zero, so a zero-weight fibre has at most one strict sign in the triangular case.

Lean: `UpperTheory.V.GradeZeroInterpolatingElement.diagChar` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.DiagChar`, lines 172–230)
Lean: `UpperTheory.V.GradeZeroInterpolatingElement.diagonal_character_order_lt_of_stable_flag` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.ContentBoundLevel`, lines 161–257)

In the remaining reducible case take the block-diagonal part of $\rho(g)$ for every g. All matrices are block triangular in the same direction, so taking their block-diagonal parts preserves multiplication. This point is the limit under the first torus action in the direction that kills the cross-block entries, which have first

degrees of one sign. It retains the entries in positions $(0,2)$ and $(2,0)$ and obeys the point equations. Its corner span contains both diagonal projections and, by multiplying with them, both off-diagonal units in the $\{0,2\}$ block. This contradicts the block assertion of 30.51. Thus two-sided points are irreducible.
Lean: `UpperTheory.V.GradeZeroInterpolatingElement.twoIrr` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.TwoIrr`, lines 42–66)

Use the isomorphism $\iota : C_\ell \simeq \mathbf{C}$ supplied by 30.51 for $\mathcal{D}$. Each irreducible allowed point ρ is continuous. For every imaginary quadratic field K such that $\rho|_{G_K}$ is irreducible, Theorem 30.51 gives $\rho|_{G_K}$ a cuspidal attachment with witness in the sense of Definition 29.5, for this same ι. Its multiplier is $\varepsilon_\ell^{k+1}\theta\xi\omega^{-1}$. Put $\mathcal{N} = \ell 2^{n'} N p'$ and $w = (k+1, k, 1, 0)$. Away from ℓ, inertia is diagonal on successive quotients with characters factoring through $\mu_{\mathcal{N}}$. At ℓ, let λ_i for $i = 0, 1, 2, 3$ be the ordered ordinary flag characters. Their restrictions to inertia agree with the corresponding diagonal characters of ρ_0, and their reductions agree with those characters on D_ℓ. In particular the inertial characters are cyclotomic powers of weights w_i times finite characters factoring through $\mu_{\mathcal{N}}$.

For every open subgroup $U \leq D_\ell$ containing σ_ℓ and all i, j we have

$$\lambda_i|_U \neq (\lambda_j \varepsilon_\ell)|_U. \tag{A.251}$$

Since U has finite index in the profinite group D_ℓ, some positive power s_0^m belongs to U. On s_0^m the four characters are the powers of $\varepsilon_\ell(s_0^m)$ with exponents w_i. This cyclotomic value has infinite order. An equality on U would therefore give $w_i = w_j + 1$. This excludes $i = j$ and puts i, j in different sets of the pair $\{0,2\}, \{1,3\}$, since the two weights within either set differ by $k \geq 2$. At σ_ℓ both the cyclotomic and Teichmüller values are one. The ratio of the two residual diagonal characters from different sets is, up to inversion, $\overline{\xi\Psi^a}(\sigma_\ell)$ for some $a \in \{-1, 0, 1\}$. This differs from one by (29.3), whereas reduction of the alleged equality on U makes the ratio one. Taking $U = D_\ell$ proves the unlinking required for level $(\mathcal{N}; w)$.
Lean: `UpperTheory.V.Loc.spPtL` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PadicHodgeTheory.SplAdjUnit`, lines 264–330)
Lean: `UpperTheory.V.GradeZeroInterpolatingElement.level` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.LevBasis`, lines 116–188)

To lift a tangent at x, start with a point $\widetilde{x}$ above x. Torus scaling is conjugation, so the irreducibility of the scaled point gives irreducibility at $\widetilde{x}$. Choose entries $u = \rho(g)_{01}$ and $v = \rho(h)_{10}$ in the bigraded algebra whose values at $\widetilde{x}$ are nonzero. Such entries exist by full matrix span. Their first degrees are $1, -1$, so uv belongs to the first-degree-zero algebra and has unit value at the dual-number point. After localizing at uv, both u and v are units. Multiplication by their powers moves each homogeneous element to first degree zero. In particular the localization is a Laurent polynomial algebra in u over its first-degree-zero part. Give u its value at $\widetilde{x}$ and extend the dual-number point to this algebra. The torus scalars used to form the allowed base point also make the scalar parts of the lifted centered coordinates topologically nilpotent. The finitely many coordinate values lie in a finite coefficient field with dual numbers, where the formal series evaluate. This gives an allowed lift.

By 30.19, the lifted tangent acts on the representation matrices by a commutator with $X \in \mathfrak{sp}_4(C_\ell)$. The block condition at ϕ forces X to be block diagonal, and the two marked corners of $[X, M_\rho]$ force $X_{02} = X_{20} = 0$ by the distinct marked diagonal values. The symplectic Lie algebra relation kills X_{13}, X_{31}. Its diagonal then acts on bidegree (a, b) by a linear function of (a, b). On the first-degree-zero algebra the tangent has the form $y \mapsto cjx(y)$ on elements y of weight j. This gives the tangent dimension required in 30.72.
Lean: `UpperTheory.V.GradeZeroInterpolatingElement.m4` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.ExistsLiftOfDerivationOnIdeal`, lines 234–250)
Lean: `UpperTheory.V.GradeZeroInterpolatingElement.rigidDiag` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.RigidDiag`, lines 99–171)

An intertwiner U of two allowed irreducible points is block diagonal by the disjoint residual spectra at ϕ. On the $\{0,2\}$ block the marking matrices are diagonal, and their two corresponding cross differences are units by the same reduced difference computed above. Intertwining the marking matrices gives $u_{02} = u_{20} = 0$. The representations have the same multiplier. If J is the symplectic form, $U^t J U$ and J give two forms with that multiplier on the source representation. Schur's lemma gives $U^t J U = cJ$ with $c \neq 0$. Since U is diagonal and invertible on $\{0,2\}$, the symplectic equation forces $u_{13} = u_{31} = 0$ and $u_{00}u_{33} = u_{11}u_{22}$. Hence U acts by the marked torus on the entries, which determine the points. In first degree zero this is the graded torus of 30.72.

Lean: UpperTheory.V.GradeZeroInterpolatingElement.conjTorus (Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.ConjTorus, lines 189–246)

Take a prime $p_0 \mid 3\ell N p' + 1$. Then $p_0 \geq 5$ and $p_0 \nmid \mathcal{N}$. Let σ be arithmetic Frobenius at p_0. The trace of ρ_0 there is $p_0^{k+1} z_1 + p_0^k z_2 + p_0 z_3 + 1$ with $z_i^h = 1$ for $h = (\ell - 1)2^{n'}(2o)$, by (29.1) and the orders of ω and ξ. Apply 30.52 with roots $\iota(z_i)$ to singleton families. The sum of the weight-zero lifts of the four diagonal entries at σ minus this distinguished trace vanishes at ρ_0 and is nonzero at every two-sided point. Apply the orbit estimate of 30.72. Take one representative per orbit. Their representations form a pairwise nonisomorphic family Ξ. Since $|\ell|_\ell = \ell^{-1}$, the orbit estimate gives

$$\frac{L(\mathcal{D})}{e_O} \log \ell \leq 2 \sum_{\rho \in \Xi} -\log |\operatorname{tr} \rho(\sigma) - \operatorname{tr} \rho_0(\sigma)|_\ell. \tag{A.252}$$

Apply 30.52 to Ξ with the same ι. Its nonzero integer D satisfies $v_\ell(D) \log \ell \leq \log |D|$. Thus the bound is $2ch[\mathcal{N}(k+1)]^A \log p_0$ with the constants c, A of that theorem. Equations (29.2) and (29.1), together with $2^{c_2(\theta)} \mid N$, $p' \leq p_1$ and $p_0 \leq 3\ell N p' + 1$, bound this by a fixed power of $\ell N k$. Corollary 29.4 returns the bound to the original ring. □

Proof of 29.23

Proof. Put $G = G_{\mathbf{Q}}$, $U = G_{L_1}$ and $M = M_{\ell,k,\theta}$ with scalar action λ. Inflation–restriction for the closed normal subgroup U [127, Prop. 1.6.7] represents the restriction kernel by cocycles Z vanishing on U. Evaluation on G/U embeds Z into M^n. Since the quotient is abelian,

$$(\lambda(g) - 1) z(h) = (\lambda(h) - 1) z(g).$$

Thus multiplication by any $\lambda(g) - 1$ sends these cocycles to coboundaries. If a generates the ideal of these differences, the restriction kernel is a quotient of Z/aZ. Lift generators of any finitely generated submodule of Z/aZ and let W be their span in Z. It has finite length, and its image has length at most $\operatorname{length}_O(W/aW) = \operatorname{length}_O W[a]$. Consequently $\operatorname{length}_O(Z/aZ) \leq \operatorname{length}_O Z[a]$, and $Z[a]$ embeds into $(M^G)^n$.

Lift $1 + \ell^s$ to inertia under ε_ℓ, with $s = \max\{v_\ell(N), 1\}$. This lift lies in $\ker \theta$, and its scalar minus 1 has valuation $s + v_\ell(k)$ by (28.1). If $\ell \nmid N$ and $\ell - 1 \nmid k$, a lift of a residual cyclotomic generator gives a unit scalar difference. This bounds the inertia invariants. The restriction image lies in $O \otimes_{\mathbf{Z}_\ell} \mathrm{Sha}_{\ell,k}(L_1)$ since the local conditions persist and the twist becomes ε_ℓ^{-k}. This gives (6.IV), with $\nu_\ell(N, k) \leq N + k \leq 2Nk$. □

Proof of 29.26

Proof. Put $K = \mathbf{Q}(i)$, $\Psi = \theta \varepsilon_\ell$, and $c_0 = c_2(\theta)$. Let η_4 be the quadratic character of K, let $\mathbf{k} = O/\mathfrak{m}_O$ be the residue field of O, and set

$$S = \{v : v \mid \ell N\}, \qquad S_b = \{v \mid N : v \neq \ell,\ \kappa_v \neq 1\}, \qquad T_b = \{\ell\} \cup S_b.$$

The odd local orders divide $2o$ by 29.2, and the primes in T_b divide ℓf. The class V threshold gives

$$8ov^2 < 2^{c_0} \quad (v \in T_b), \qquad c_0 \geq 4, \qquad \ell f\, 4^{n_0} \leq 2^{c_0 - 1}. \tag{A.253}$$

For each $v \in T_b$ choose an arithmetic Frobenius lift $\tau \in D_v$, with $\varepsilon_\ell(\tau) = 1$ at ℓ. At each $v \in S_b$, the nontrivial ℓ-power part κ_v forces $v \equiv 1 \bmod \ell$; since ℓ is odd, every $v \in T_b$ is odd, so η_4 is unramified at v and $\eta_4(I_v) = 1$. Put $x_v = \tau^2$. If $\bar{\Psi}(x_v)^2 = 1$, raising the equality to o kills the odd-modulus factors, whose orders divide $2o$ and which occur to the fourth power. Injectivity of reduction on dyadic roots then gives $\theta_2(v)^{4o} = 1$. Faithfulness on units congruent to 1 mod 4 would give $v_2(v^{4o} - 1) \geq c_0$. But (28.1) and (A.253) give

$$v_2(v^{4o} - 1) = v_2(v^2 - 1) + 1 + v_2(o) < c_0.$$

Hence

$$\eta_4(x_v) = \bar{\varepsilon}_\ell(x_v) = 1, \qquad \bar{\Psi}(x_v)^2 \neq 1. \tag{A.254}$$

Lean: UpperTheory.V.UnitaryThreeInertia.hostData_of_host (Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.HostDataOf, lines 9–33)
Lean: UpperTheory.V.UnitaryThreeInertia.exists_sigma (Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.DyadicCharacterPowerNeOne, lines 202–242)

Let $G = \Gamma_{\mathbf{Q},S}$ be the Galois quotient unramified outside S, and let F be the Gaussian frame on G described before 30.54, with distinguished lift r_0 having sign η_4 and matrix part $\rho_0 = \mathrm{diag}(\Psi, 1, \Psi^{-1})$, and write $\bar{r}$ for the reduction of r_0. For each $v \in S \setminus \{\ell\}$ put

$$N_v = I_v \cap \ker \eta_4 \cap \ker \bar{\Psi} \leq D_v,$$

where $\bar{\Psi} : G \to \mathbf{k}^\times$ is the residual character. At $v \in S \setminus T_b$ we have $\kappa_v = 1$, so $\bar{\Psi}(s) = 1$ on I_v implies $\Psi(s) = 1$, and therefore r_0 kills N_v. Global finiteness of first cohomology for finite $\mathbf{F}_\ell$-representations of G gives finiteness of the set of continuous $\mathbf{F}_\ell$-characters on every open subgroup of G. Hence the unrestricted lifting problem for $\bar{r}$ on G admits a universal complete Noetherian local O-algebra with universal lift u_{unr}. Fix $\iota : C_\ell \simeq \mathbf{C}$ and choose a marking g_1 from 30.54 and 30.61. At each $v \in T_b$, Hensel's lemma and (A.254) give orthogonal spectral projectors $P_{v,0}, P_{v,1}, P_{v,2}$ for the matrix of $u_{\mathrm{unr}}(x_v)$ reducing to E_{00}, E_{11}, E_{22}, expressing the local condition of F at v by the closed ideal of relations $P_{v,j} u_{\mathrm{unr}}(g)_{\mathrm{mat}} P_{v,i} = 0$ ($i < j$ at ℓ and $i \neq j$ at $v \in S_b$) on $D_v \cap \ker \eta_4$ and $P_{v,i}(u_{\mathrm{unr}}(s)_{\mathrm{mat}} - \Psi(s)^{1-i})P_{v,i} = 0$ on $I_v \cap \ker \eta_4$. Cutting out the closed ideal defined by the marking, triviality on N_v for $v \in S \setminus T_b$, and these projector relations at T_b yields the universal complete Noetherian local O-algebra R with universal lift $u : G \to \mathrm{GL}_3(R) \rtimes \{1, -1\}$. Since r_0 satisfies all frame conditions, R is nontrivial and admits a base homomorphism $R \to O$ specializing u to r_0.
Lean: UpperTheory.V.UnitaryThreeInertia.exists_hostFrame (Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.HostEx, lines 55–113)
Lean: UpperTheory.V.UnitaryThreeInertia.hostFrame_ok (Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.HostOk, lines 239–329)
Lean: UpperTheory.V.UnitaryThreeInertia.hostFrame_phi (Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.HostPhi, lines 182–215)
Lean: UpperTheory.V.UnitaryThreeInertia.exists_isUniv (Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.ExistsIsUniv, lines 497–511)
Lean: UpperTheory.V.UnitaryThreeInertia.isPt_rz (Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.MapDiagonalRepMatrixEq, lines 35–104)

By 30.54, the weight-zero algebra R_0 of u is finite over O. A Type I point has an irreducible matrix restriction r to G_K, with attachment and witness for $r\varepsilon_\ell^{-1}$. Its integral ordinary flag has inertia characters $\Psi, 1, \Psi^{-1}$ of weights $(1, 0, -1)$ and the corresponding residual characters on the full ordinary group. At x_ℓ the cyclotomic value is one, and the three residual values $\bar{\Psi}(x_\ell), 1, \bar{\Psi}(x_\ell)^{-1}$ are distinct. This separates the flag characters and excludes an equality between two of them after a single cyclotomic twist. For equal indices a nontrivial cyclotomic inertia value excludes that equality. Away from ℓ, the cyclotomic character is unramified; on $I_v \cap G_K$ the inertia action is conjugate to $\mathrm{diag}(\theta, 1, \theta^{-1})$, directly from the frame condition at $v \in S_b$ and by averaging $r(s)\rho_0(s)^{-1}$ over the prime-to-ℓ quotient $(I_v \cap G_K)/N_v$ at $v \in S \setminus T_b$ (and unramifiedness outside S). These characters factor through μ_N. Thus r belongs to the rank-three family of 30.7.
Lean: UpperTheory.V.UnitaryThreeInertia.levelAway_of_pt (Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.LevelAway, lines 176–233)

Pass to the linked CM frame of 30.59. The determinant law and the transported ordinary characters give the arithmetic point with its prescribed flag and automorphic witness. The separated flags and local comparison for the attachment give the infinite-order test at each ordinary place. Take a Weil element s of positive degree d_0 in the kernel of the cyclotomic character. Local reciprocity gives one by taking the class of a rational ℓ-power in the local multiplicative group, with the direction chosen for positive degree. If κ_i are the flag characters of the automorphically normalized representation with weights m_i, the toral support characters have moduli $|\iota(\kappa_i(s))| q_v^{m_i d_0}$ there, where q_v is the residue cardinality. By (30.8) the ratio of any two of these moduli lies strictly between $q_v^{-d_0}$ and $q_v^{d_0}$. If $\kappa_i(s)/\kappa_j(s)$ were a root of unity, then $|m_i - m_j| < 1$, impossible for distinct integer weights. The common twist cancels from these ratios. Taking a positive power to enter a linked local subgroup preserves the test.
Lean: UpperTheory.V.Z.J.hostUnlE_alt (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.EHostUnl, lines 62–137)
Lean: UpperTheory.V.Z.Win.RR.free_core_alt (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.RRFree, lines 97–116)

Write Γ' for the cyclotomic kernel of the ordinary frame of 30.62. The rank-three irreducibility test applies to the intersection of Γ' with the normal core of any open subgroup of the matrix group domain. A

positive power of the local element just constructed lies in this intersection and has three distinct eigenvalues with pairwise ratios of infinite order. By the prime-rank restriction criterion, a reducible restriction to this normal subgroup would be scalar or a sum of three lines permuted transitively by the full group. The distinct eigenvalues exclude the scalar case. In the line case the ordinary flag is a flag of sums of these eigenlines, so the local group fixes the lines individually. On the kernel of the permutation action, the three line characters reduce, up to a permutation, to the restrictions of the three global residual characters. This follows by comparing characteristic polynomials and their semisimplifications. Transitivity and conjugation invariance of the global residual characters would make all three reduced line characters equal on the local group, contrary to the transported residual separation. Thus the intersection is irreducible, and in particular Γ' acts irreducibly.

Each open coset in Γ' contains an element with separable characteristic polynomial. Take a coset yH with H the intersection of Γ' with an open subgroup. Diagonalize a power of the local test element in H and write its matrix as $D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$. Every finite-index open subgroup used to form H acts irreducibly on intersection with Γ'. The same holds for the preimage of the identity component of the Zariski closure of the image of H, since this preimage is closed and of finite index in H. Irreducibility and density in that component give an element $b \in H$ whose matrix and inverse have all entries nonzero. In the eigenbasis of D, the diagonal entries of $r(y)D^m r(b)$ and of its inverse are nonzero for some m. Each such entry is an exponential sum in the λ_i^m or λ_i^{-m} with a nonzero coefficient. Let h annihilate the torsion subgroup of the finitely generated multiplicative group generated by the λ_i, and replace D by D^h. The new eigenvalues generate a free abelian group, and remain pairwise distinct because every original ratio has infinite order. Each of the displayed exponential sums is a nonzero element of the group algebra of this free abelian group. That group algebra is a Laurent polynomial domain, so their product is nonzero. Grouping its finitely many monomials by their distinct bases and applying Vandermonde gives one $m \geq 0$ at which the product is nonzero. In rank three the inverse diagonal entries give the principal minors of size two. For a matrix A with all these principal minors and diagonal entries nonzero, the discriminant of $AD^{m'}$ is also a nonzero exponential sum. Order the λ_i by a real linear functional on their multiplicative group modulo torsion, with values $s_1 > s_2 > s_3$. In the discriminant the highest term comes from the square of the trace times the square of the second elementary coefficient, at weight $4s_1 + 2s_2$. Its coefficient is $(A_{11} \det A_{\{1,2\},\{1,2\}})^2$. All other discriminant terms have smaller weight. Vandermonde gives a power m' with nonzero discriminant. The resulting matrix is still in the image of the coset.

Lean: `UpperTheory.V.Th.nice_zero_of_rank_two_or_three_irreducible_free` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TNiceSmall`, lines 24–43)
Lean: `UpperTheory.V.Pr.regCoset` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.RegCoset`, lines 143–161)

In the ordinary basis at ℓ, a first-order lift is upper triangular with fixed inertia diagonals. The comparison (A.262), applied successively to the positive-weight Hom spaces on the subgroup killing the finite inertial characters, puts the adjoint cocycle in the finite local group. Away from ℓ it is an inertia coboundary by averaging over the finite inertia quotient. The flag weights and the cyclotomic-kernel properties give the hypotheses of 30.62. The cocycle is a coboundary on the linked frame and hence on the original group by averaging over the finite quotient. Indeed a restriction coboundary can first be subtracted on the whole group. The remaining cocycle factors through the quotient with values in the subgroup invariants, and division by the quotient order gives a coboundary there.

Choose a prime divisor $p \equiv 3 \bmod 4$ of $12\ell N - 1$, which exists since this integer is $3 \bmod 4$. Then $p \nmid 6\ell N$ and $7 \leq p \leq 12\ell N$. Take σ to be inverse Frobenius at the Gaussian prime of norm p^2. The distinguished trace is $\iota^{-1}((p^2)^{-1}z + 1 + p^2 z^{-1})$ with $z = \iota(\theta(\sigma))$ and $z^{\varphi(N)} = 1$. Apply 30.7 to singletons using the indexed family of all $\varphi(N)$-th roots of unity. Every Type I trace difference at σ is nonzero.

Let V_j be the weight-j residual adjoint space over $\mathbf{k}$. For every $v \in S \setminus \{\ell\}$ (including $v \in S_b$), let $Z^1_{\mathcal{L},v}$ be the space of continuous 1-cocycles on D_v with values in V_j vanishing on $N_v = I_v \cap \ker \eta_4 \cap \ker \bar{\Psi}$. At $v = \ell$, take $Z^1_{\mathcal{L},\ell} = B^1(D_\ell, V_j)$ for $j < 0$, $Z^1_{\mathcal{L},\ell} = Z^1_{\mathrm{ur}}(D_\ell, V_0)$ (cocycles vanishing on I_ℓ) for $j = 0$, and $Z^1_{\mathcal{L},\ell} = Z^1(D_\ell, V_j)$ for $j > 0$. Since N_v acts trivially on V_j and I_ℓ acts trivially on V_0 (as $\eta_4(I_\ell) = 1$), every global coboundary in $B^1(G, V_j)$ restricts into $Z^1_{\mathcal{L},v}$ for all $v \in S$. Let $Z^1_{\mathcal{L},j} \subseteq Z^1(G, V_j)$ be the space of global cocycles restricting into $Z^1_{\mathcal{L},v}$ at every $v \in S$, and let $H^1_{\mathcal{L},j} = Z^1_{\mathcal{L},j}/B^1(G, V_j)$. The obstruction space Ob_j consists of pairs $(c_2, (u_v)_{v \in S})$ with $c_2 \in Z^2(G, V_j)$ and $du_v = c_2|_{D_v}$, modulo $(da, (a|_{D_v})_v)$ for global 1-cochains a and $(0, (z_v)_v)$ for $z_v \in Z^1_{\mathcal{L},v}$.

For each $v \in T_b$ and $j \neq 0$, the element $x_v \in D_v$ of (A.254) satisfies $\eta_4(x_v) = \bar{\varepsilon}_\ell(x_v) = 1$ and $\bar{\Psi}(x_v)^2 \neq 1$. On V_j (which is zero for $|j| > 2$), x_v acts by the scalar $\bar{\Psi}(x_v)^j \neq 1$, and on the Tate dual $V_j^\vee(1)$ it acts by $\bar{\Psi}(x_v)^{-j} \neq 1$. Hence $H^0(D_v, V_j) = 0$ and, by local Tate duality, $H^2(D_v, V_j) = 0$. For $v \in S_b$ (where $v \neq \ell$), the local Euler characteristic formula gives $h^1(D_v, V_j) = h^0(D_v, V_j) + h^2(D_v, V_j) = 0$, so $H^1(D_v, V_j) = 0$. In particular, at $v \in S_b$ we have $Z^1_{\mathcal{L},v} = Z^1(D_v, V_j) = B^1(D_v, V_j)$ for $j \neq 0$; every cocycle in $Z^1_{\mathcal{L},v}$ is therefore a coboundary on D_v and produces a first-order lift locally conjugate to $\bar{r}$, matching the diagonalizability and inertia-diagonal condition of F. At $v \in S_b$ and $j = 0$, since v is odd we have $\eta_4(I_v) = 1$, so I_v acts trivially on the diagonal space V_0; because $[I_v : N_v]$ divides $2(\#\mathbf{k} - 1)$ and is prime to ℓ, vanishing on N_v is equivalent to vanishing on I_v, giving $Z^1_{\mathcal{L},v} = Z^1_{\mathrm{ur}}(D_v, V_0)$, which again matches the weight-zero frame condition at $v \in S_b$.
Lean: `UpperTheory.V.UnitaryThreeInertia.hostFrame_locVan` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.HostLocVan`, lines 155–200)

The primitive factor inducing $(\omega\theta_1)^2$ has odd conductor $f' \mid \ell f$, since the odd factors outside f have square one by 29.2. Its order divides $\ell^{f_0} - 1$. Squaring θ_2 halves its order on the cyclic units congruent to 1 mod 4, on which η_4 is trivial. Thus $\theta_2^2\eta_4$ has conductor 2^{c_0-1}. Its product with the odd-modulus factor is odd and reduces to $\eta_4\bar{\Psi}^2$. The bounds in (A.253) give $f'4^{n_0} \leq 2^{c_0-1}$, and $4 \mid \ell^{f_0} - 1$ by the definition of f_0. The subgroup $\ker(\eta_4, \bar{\Psi}) \leq G_{\mathbf{Q}}$ has index dividing $2(\#\mathbf{k} - 1)$, prime to ℓ. For every $v \in S \setminus \{\ell\}$, a cocycle in $H^1_{\mathcal{L},-2}$ vanishes on $N_v = I_v \cap \ker(\eta_4, \bar{\Psi})$, and at $v = \ell$ it is a local coboundary, hence vanishes on inertia in the character kernel. The inverse-character assertion of 30.75 gives $H^1_{\mathcal{L},-2} = 0$.
Lean: `UpperTheory.V.X.dirData` (`Development.AutomorphicArithmetic.AutomorphicVarieties.LevelCohomology.HcDirData`, lines 29–109)
Lean: `UpperTheory.V.X.xpp` (`Development.AutomorphicArithmetic.AutomorphicVarieties.LevelCohomology.Xpp`, lines 11–84)
Lean: `UpperTheory.V.X.xppOf` (`Development.AutomorphicArithmetic.AutomorphicVarieties.LevelCohomology.HcXppOf`, lines 60–98)

In nonzero weight x_ℓ kills global dual invariants by (A.254). In weight zero take an element fixing i with $\bar{\varepsilon}_\ell \neq 1$, which exists since $\mathbf{Q}(i)$ and $\mathbf{Q}(\mu_\ell)$ are disjoint. It acts trivially on V_0 and kills the dual invariants. Thus Poitou–Tate in 27.2 gives surjectivity in degree two onto the local groups. The obstruction exact sequence then gives

$$e_j := \dim H^1_{\mathcal{L},j} - \dim \mathrm{Ob}_j = h^1(G, V_j) - h^2(G, V_j) + \sum_{v \in S} \big(h^2(D_v, V_j) - \dim Z^1(D_v, V_j) + \dim Z^1_{\mathcal{L},v}\big).$$

Write $d_j = \dim_{\mathbf{k}} V_j$ and $d_j^+ = \dim_{\mathbf{k}} V_j^{c_\infty = 1}$, where c_∞ is complex conjugation. The global Euler characteristic gives $h^1 - h^2 = h^0 + d_j - d_j^+$. For every $v \in S \setminus \{\ell\}$ (including $v \in S_b$), the homomorphism $(\eta_4, \bar{\Psi}) : I_v \to \{\pm 1\} \times \mathbf{k}^\times$ has kernel N_v, so $[I_v : N_v]$ divides $2(\#\mathbf{k} - 1)$ and is prime to ℓ, while N_v acts trivially on V_j. Restriction $Z^1_{\mathcal{L},v} \to Z^1(I_v, V_j)$ maps into $B^1(I_v, V_j)$ because $H^1(I_v/N_v, V_j) = 0$, and surjects onto $B^1(I_v, V_j)$ because $B^1(D_v, V_j) \subseteq Z^1_{\mathcal{L},v}$; its image has dimension $d_j - \dim V_j^{I_v}$. Its kernel is $Z^1_{\mathrm{ur}}(D_v, V_j) = Z^1(D_v/I_v, V_j^{I_v})$, which evaluation at a Frobenius generator of D_v/I_v identifies with $V_j^{I_v}$, of dimension $\dim V_j^{I_v}$. Hence $\dim Z^1_{\mathcal{L},v} = d_j$. The local Euler formula gives $\dim Z^1(D_v, V_j) = d_j + h^2(D_v, V_j)$, so these terms cancel.

At ℓ the local Euler formula gives $\dim Z^1(D_\ell, V_j) = 2d_j + h^2(D_\ell, V_j)$. In nonzero weight x_ℓ gives $h^0(G, V_j) = h^0(D_\ell, V_j) = h^2(D_\ell, V_j) = 0$. The negative local condition then has dimension d_j and the positive condition has dimension $2d_j$. In weight zero the local condition has dimension d_0 because V_0 is unramified at ℓ. The action on V_0 factors through η_4, so $V_0^G = V_0^{c_\infty = 1}$ and the remaining terms cancel. We obtain

$$e_j = \begin{cases} -d_j^+, & j < 0, \\ 0, & j = 0, \\ d_j - d_j^+, & j > 0. \end{cases} \tag{A.255}$$

Here $(d_j, d_j^+) = (3, 1)$ for $j = 0$, $(2, 1)$ for $|j| = 1$, $(1, 0)$ for $|j| = 2$, and $(0, 0)$ for $|j| \geq 3$, so $e_{-1} = -1$, $e_1 = e_2 = 1$, and $e_j = 0$ for all other j. The local and global Euler formulas used here are those of [111, I, Theorems 2.8 and 5.1].
Lean: `UpperTheory.selmerComplex_finite_and_euler_characteristic` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.SelmerComplexFiniteAndEulerCharacteristic`, lines 174–235)
Lean: `UpperTheory.V.UnitaryThreeInertia.hostFrame_count` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.HostCount`, lines 288–351)
Lean: `UpperTheory.finrank_gradedPiece_and_complexConjugation_invariants` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.FinrankGradedPieceAndComplexConjugationInvariant`, lines 90–204)

Lean: `UpperTheory.global_cohomology_finite_set` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.GlobalCohomologyFiniteSet`, lines 34–64)
Lean: `UpperTheory.global_euler_characteristic_rationals_odd_prime` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.GlobalEulerCharacteristicRationalsOddPrime`, lines 15–52)
Lean: `UpperTheory.local_euler_characteristic_rationals` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.LocalEulerCharacteristicRationals`, lines 32–78)

Choose a minimal adapted homogeneous presentation $\pi_W : P = O[[X_1, \dots, X_n]] \twoheadrightarrow R$ whose coordinates map to centered matrix entries $Y_i = u(g_i)_{p_i q_i} - \rho_0(g_i)_{p_i q_i}$ of weights $w_i = w(p_i) - w(q_i) \in \{-2, -1, 0, 1, 2\}$ (with $(w(0), w(1), w(2)) = (1, 0, -1)$), whose kernel $J = \ker \pi_W$ is homogeneous, and for which π_W maps the weight-j power series P_j onto the weight-j piece R_j of R under diagonal torus conjugation. The base homomorphism $R \to O$ specializing u to r_0 kills every Y_i, so its composition with π_W is the constant-term map $x_0 : P \to O$. Put $n_j = \#\{i : w_i = j\}$. For each weight j, every cocycle in $Z^1_{\mathcal{L},j}$ differs by a global coboundary from a normalized cocycle, meaning one vanishing at c_∞ and having zero $(0,1)$ and $(1,0)$ entries in its variation of $r(g_1) - r(c_\infty g_1^{-1} c_\infty)$. Thus the normalized subspace $Z^1_{\mathcal{L},j,\mathrm{norm}} \subseteq Z^1_{\mathcal{L},j}$ surjects onto $H^1_{\mathcal{L},j}$. By the local condition matching above, each cocycle in $Z^1_{\mathcal{L},j,\mathrm{norm}}$ defines a dual-number lift of the marked frame over $\mathbf{k}[\epsilon]/(\epsilon^2)$, hence factors through R and vanishes if it kills the weight-j adapted coordinates. Evaluation on those coordinates therefore injects $Z^1_{\mathcal{L},j,\mathrm{norm}}$ into $\mathbf{k}^{n_j}$, giving $\dim_{\mathbf{k}} H^1_{\mathcal{L},j} \le n_j$.

We construct homogeneous generators of J directly from the unframed obstruction spaces Ob_j. By the Artin–Rees lemma in the Noetherian local ring P, choose $k_0 \ge 1$ such that $\mathfrak{m}_P^{k_0} \cap J \subseteq \mathfrak{m}_P J$. Let $\mathfrak{t} \subseteq \mathfrak{m}_P^{k_0}$ be the homogeneous truncation ideal of level k_0 so that $P/\mathfrak{t}$ is finite and $\mathbf{Z}$-graded, and put $\bar{J} = J \cdot (P/\mathfrak{t})$. For each weight j, let $K_j \subseteq \bar{J}$ be the homogeneous ideal of $P/\mathfrak{t}$ generated by $\mathfrak{m}_{P/\mathfrak{t}}\bar{J}$ and all homogeneous components of $\bar{J}$ of weight different from j. The quotient $q_j : (P/\mathfrak{t})/K_j \twoheadrightarrow (P/\mathfrak{t})/\bar{J}$ is a small extension whose kernel is concentrated in weight j and annihilated by the maximal ideal. Pushing out q_j by a functional $\lambda \in \ker(q_j)^\vee$ gives a small O-algebra extension $E_\lambda \twoheadrightarrow R/\pi_W(\mathfrak{t})$ with kernel $\mathbf{k}\epsilon$ in weight j, while q_j is an isomorphism in all weights different from j. Lifting the matrix entries of the pushforward $\bar{u}$ homogeneously in $(P/\mathfrak{t})/K_j$ (preserving the polarizing involution since $2 \in O^\times$) produces a 2-cocycle $c_2 \in Z^2(G, V_j)$. Locally, $c_2|_{D_v} = du_v$ with local frame compatibility modulo $Z^1_{\mathcal{L},v}$: for $v \in S \setminus T_b$, $\bar{u}|_{D_v}$ factors through D_v/N_v, where $H^2(D_v/N_v, V_j) = 0$ since $\ell \nmid [I_v : N_v]$ and $D_v/I_v \simeq \widehat{\mathbf{Z}}$; for $v \in T_b$ and $j \neq 0$, $H^2(D_v, V_j) = 0$ gives u_v, with local frame compatibility by $H^1(D_v \cap \ker \eta_4, V_j) = 0$ at $v \in S_b$, unchanged entries in weights ≤ 0 at $v = \ell$ when $j > 0$, and unique homogeneous lifting in weights ≥ 0 (extended across the index-at-most-two sign quotient) at $v = \ell$ when $j < 0$; for $v \in T_b$ and $j = 0$, in the homogeneous Henselian spectral-projector basis of $\bar{u}(x_v)$ the off-diagonal entries on $D_v \cap \ker \eta_4$ lie in nonzero weights where q_0 is an isomorphism, while the diagonal ratios to ρ_0 factor through $(D_v \cap \ker \eta_4)/I_v \simeq \widehat{\mathbf{Z}}$ and lift across q_0 compatibly with the polarizing involution, extending across the sign quotient to D_v. If the resulting class in Ob_j vanishes, global correction and normalization at c_∞ and g_1 yield a marked frame lift over E_λ, hence by universality of u a local O-algebra map $R \to E_\lambda$ lifting $R \twoheadrightarrow R/\pi_W(\mathfrak{t})$; its pullback to P differs from the canonical quotient $P \to E_\lambda$ by an O-derivation into $\mathbf{k}\epsilon$, which kills $\mathfrak{m}_P^2 \supseteq J$ (by $x_0(J) = 0$ and minimality of π_W), forcing $\lambda = 0$ and hence $\dim_{\mathbf{k}} \ker(q_j) \le \dim_{\mathbf{k}} \mathrm{Ob}_j$. Hence $\bar{J}/\mathfrak{m}_{P/\mathfrak{t}}\bar{J}$ is generated in weight j by $\dim_{\mathbf{k}} \mathrm{Ob}_j$ homogeneous elements. Taking weight-j components of lifts in J (using homogeneity of J and $\mathfrak{t}$) produces $\dim_{\mathbf{k}} \mathrm{Ob}_j$ homogeneous power series $f_{j,i} \in J \cap P_j$ whose generated ideal $I_0 \subseteq J$ satisfies $J \subseteq I_0 + \mathfrak{m}_P J + (\mathfrak{t} \cap J) \subseteq I_0 + \mathfrak{m}_P J$. Nakayama's lemma yields $J = I_0$.

Lean: `UpperTheory.V.UnitaryThreeInertia.hasOrigin` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.Origin`, lines 73–109)
Lean: `UpperTheory.V.UnitaryThreeInertia.finrank_H1L_le` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.CommAlgFunctorCoboundaryZeroMemCocyclesOne`, lines 370–421)
Lean: `UpperTheory.V.UnitaryThreeInertia.exists_relations` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.ExistsRelations`, lines 121–257)

Combining $\dim_{\mathbf{k}} H^1_{\mathcal{L},j} \le n_j$ with (A.255) gives

$$\dim_{\mathbf{k}} \mathrm{Ob}_j + e_j \le n_j \qquad (j \in \mathbf{Z}).$$

Since $w_i \in \{-2, -1, 0, 1, 2\}$, we have $n_j = 0$ and hence $\dim_{\mathbf{k}} \mathrm{Ob}_j = 0$ for $|j| > 2$. For a diagonal point over $O[\epsilon]/(\epsilon^2)$ or C_ℓ, the ratios to the distinguished characters on G_K are unramified, hence trivial by reciprocity and $\mathrm{Cl}(\mathbf{Q}(i)) = 1$. This identifies the unique point where all nonzero weights vanish with r_0 and kills normalized O-valued tangents of weight zero. For $j = -n < 0$ the scalar lines have characters $\eta_4^s \theta^{-n} \varepsilon_\ell^{-n}$ with $s = 0$ or 1. Lemma 29.19 kills these O-valued tangents modulo coboundaries. The marking removes

the remaining coboundaries in nonpositive weights, so every O-valued derivation at x_0 supported in weight $j \le 0$ and killing J is zero. By duality, the linear terms of the generators in weights $-2, -1, 0$ span the corresponding rational cotangent spaces.

Lean: `UpperTheory.V.UnitaryThreeInertia.frame_class_one` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.FrameClassOne`, lines 166–268)
Lean: `UpperTheory.V.UnitaryThreeInertia.eq_rz_of_diag` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.UnitEqOneOfPowEqOneOfResidueOne`, lines 165–208)
Lean: `UpperTheory.V.UnitaryThreeInertia.tangent_coord_eq_zero_of_nonpos_weight` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.TangentCoordEqZeroOfNonposWeight`, lines 70–120)

For $0 \ne b \in O$, let $\mathcal{S}_F(b) \subseteq H^1(G, (O/(b))(\Psi^{-1}))$ denote the classes on G that are coboundaries on D_v for $v \in T_b$ and vanish on N_v for $v \in S \setminus T_b$. The unit $\Psi^{-1}(g_1) - 1$ gives each class a unique representative z with $z(g_1) = 0$. On N_v for $v \in S \setminus T_b$ we have $\Psi = 1$, so subtracting the normalizing coboundary preserves the zero values. On each D_v for $v \in T_b$, subtract its scalar from the local cobounding element. Since $\Psi(c_\infty) = 1$ and ℓ is odd, $z(c_\infty) = 0$, which also gives $z(c_\infty g_1^{-1} c_\infty) = 0$.

Put $T_- = E_{10} - E_{21}$ in zero-based indices. Then $(1 + \epsilon z T_-)\rho_0$ is the square-zero lift, with the original sign on the full group. The root line spanned by T_- has action Ψ^{-1}, so this lift obeys the multiplication and polarization equations. The local witnesses give its local conjugations, and the two marking values give its marking equations. For an adapted coordinate $Y_i = u(g_i)_{p_i q_i} - \rho_0(g_i)_{p_i q_i}$ its value under the point derivation is

$$d_z(Y_i) = (T_-)_{p_i q_i} \rho_0(g_i)_{q_i q_i}\, z(g_i) \pmod{b},$$

where matrix parts are used in the entry notation. This assignment is additive and supported in weight -1. It kills the linear terms of J for every b. Lifting its coordinate values gives an integral derivation supported in weight -1. Conversely the values on the coordinates determine the square-zero representation, and its $(1, 0)$ entries recover z because the diagonal factors are units. This gives the additive injection required for the cotangent comparison (30.33) of 30.72. If $n_{-1} = 0$, it gives $\mathcal{S}_F(b) = 0$ for every b. In this case take $V = 0$ and no representatives.

Lean: `UpperTheory.V.UnitaryThreeInertia.mprime_coord` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.PrecompleteOfPowLe`, lines 385–426)

Suppose $n_{-1} > 0$. At a marked point write $r(g)$ for its matrix part when taking entries. The normalization and the polarization give

$$r(g_1) - r(c_\infty g_1^{-1} c_\infty) = \operatorname{diag}(\zeta, 0, -\zeta), \qquad \bar{\zeta} = \bar{\Psi}(g_1) - \bar{\Psi}(g_1)^{-1} \ne 0.$$

Thus every invariant subspace on the sign kernel is spanned by coordinate axes. Polarization pairs the relation that entry ij vanishes throughout the sign kernel with the corresponding relation for $(2 - j, 2 - i)$. In particular the vanished entries $10, 21$ occur together, as do $01, 12$. The coordinate subspaces of dimension one or two now give the upper-triangular case if $10, 20$ vanish and the lower-triangular case if $01, 02$ vanish. The remaining reducible pattern has the middle line and the corner on $\{0, 2\}$, with both corner directions present. The corner is irreducible by the two distinct marked diagonal values and the presence of its two off-diagonal entries. A two-sided point excludes both triangular cases.

Lean: `UpperTheory.V.U3.shape_cases` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.ExistsCoordOfNotMatIrr`, lines 45–123)

In the corner case, pass to the integral model. Entry $(2, 1)$ vanishes on the sign kernel and then on the other coset by multiplication by the normalized complex conjugation, whose matrix part is the identity. Choose a nonzero $(2, 0)$ entry u_* of least valuation and write $r(g)_{20} = u_* b(g)$. Matrix multiplication, with the polarizing involution on the other coset, gives

$$\bar{b}(gh) = \bar{b}(g)\bar{\Psi}(h) + \eta_4(g)\bar{\Psi}(g)^{-1}\bar{b}(h).$$

Thus $\bar{b}(g)\bar{\Psi}(g)^{-1}$ is a cocycle in weight -2, zero at complex conjugation. Outside T_b the subgroups N_v kill it. At $v \in T_b$ let $H \equiv 1 \bmod \mathfrak{m}$ carry the ordinary upper-triangular basis (which is diagonal for $v \in S_b$) to the given basis. For g in the local sign kernel, comparison of the 0 and 2 entries of the first column of $r(g)H = H r^{\mathrm{tri}}(g)$ gives

$$u_* b(g) H_{00}^2 = H_{20} V(g), \qquad V(g) = \sum_{j=0}^{2} r(g)_{0j} H_{j0} - H_{00} r(g)_{22}.$$

Here $\overline{V(g)} = \bar{\Psi}(g) - \bar{\Psi}(g)^{-1}$. Take $s = x_v$. Multiply the identities at g and s by $V(s)$ and $V(g)$ respectively, subtract, cancel u_* in the integral domain and reduce. Since $\overline{V(s)} \neq 0$ by (A.254), this gives

$$\bar{b}(g) = w_v\big(\bar{\Psi}(g) - \bar{\Psi}(g)^{-1}\big), \qquad w_v = \bar{b}(s)/\overline{V(s)}.$$

The cocycle is a coboundary on the local sign kernel, hence on the decomposition group by averaging over the quotient of order at most two. In particular it vanishes on N_v for $v \in S_b$ and is a coboundary on D_ℓ. Residual Selmer vanishing $H^1_{\mathcal{L},-2} = 0$ gives a global coboundary. Its value at complex conjugation forces the cobounding scalar to vanish since the character there is -1. This contradicts the nonzero reduction at the entry chosen for u_*. Every two-sided point is therefore Type I.

Lean: `UpperTheory.V.UnitaryThreeInertia.typeI_of_twoSigned` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.TypeIOfTwoSigned`, lines 476–579)

At these points rigidity expresses a tangent by a commutator. Polarization at complex conjugation and the g_1 marking reduce it to the diagonal torus direction, which acts on weight j by a scalar times j. An intertwiner on G_K between two normalized Type I points intertwines their diagonal marking matrices. Their corresponding diagonal values have the same reductions, while the three reductions in each matrix are distinct. Thus the intertwiner is diagonal. It transports the polarizations to proportional forms by irreducibility, so its diagonal action is that of the marked torus up to scalar. Hence distinct orbits give different isomorphism classes for a fixed embedding of K.

Lean: `UpperTheory.V.UnitaryThreeInertia.tangent_eq_orbit` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.ContinuousCompOfFilteredContinuity`, lines 262–399)
Lean: `UpperTheory.V.UnitaryThreeInertia.conj_of_matIso` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.NormTwoEqOne`, lines 32–55)

Now set $r = n_{-1} \geq 1$, $s = n_{-2}$, $m = n_0$, $r' = n_1 - 1$, and $s' = n_2 - 1$; here $r', s' \geq 0$ because $n_1 \geq \dim_{\mathbf{k}} \mathrm{Ob}_1 + 1 \geq 1$ and $n_2 \geq \dim_{\mathbf{k}} \mathrm{Ob}_2 + 1 \geq 1$. Let $A = \bigoplus_{j \in \mathbf{Z}} P_j \subset P$ be the graded subalgebra of finite sums of homogeneous power series, with base point $x_0|_A : A \to O$ and canonical graded formal chart $A \hookrightarrow (\mathrm{Frac}\, O)[[X_1, \dots, X_n]]$. Its variable counts in weights $(-2, -1, 0, 1, 2)$ are $(s, r, m, r' + 1, s' + 1)$, with no other weights. Padding the homogeneous generators $f_{j,i} \in J \cap P_j$ with zeros to match $\dim_{\mathbf{k}} \mathrm{Ob}_j \leq n_j - e_j$ produces homogeneous equations $G_1, \dots, G_s$ in weight -2, $F_1, \dots, F_{r+1}$ in weight -1, $Z_1, \dots, Z_m$ in weight 0, $E_1, \dots, E_{r'}$ in weight 1, and $E'_1, \dots, E'_{s'}$ in weight 2 generating $I = \ker(A \twoheadrightarrow \bigoplus_{j \in \mathbf{Z}} R_j)$. The image of A_0 in A/I is R_0, finite over O; the unique fixed C_ℓ-point is $x_0|_A$; every two-sided C_ℓ-point is Type I with tangent space of dimension at most one; and a lift $a_F \in A_0 \cap \ker x_0$ of $\mathrm{tr}\, u(\sigma) - \mathrm{tr}\, r_0(\sigma) \in R_0$ is nonzero at every two-sided C_ℓ-point. Apply the cotangent comparison (30.33) (via the weight--1 injection above) and the orbit estimate (30.34) of 30.72, with the normalized valuation $e_O v_\ell$, to A, I, and a_F. In both cases for n_{-1} there are $V \in \mathbf{N}$ and Type I orbit representatives with matrix restrictions ρ_i to G_K such that

$$\mathrm{length}_O\, \mathcal{S}_F(b) \leq V \leq 2e_O \sum_i v_\ell\big(\mathrm{tr}\, \rho_i(\sigma) - \mathrm{tr}\, \rho_0(\sigma)\big) \qquad (0 \neq b \in O). \tag{A.256}$$

Lean: `UpperTheory.V.UnitaryThreeInertia.Abs.coord_count` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.AbsCoord`, lines 68–279)
Lean: `UpperTheory.norm_bound_of_integral_values` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.NormBoundOfIntegralValues`, lines 43–64)
Lean: `UpperTheory.V.UnitaryThreeInertia.Frame.ShaO` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.IsCharZeroPt`, lines 95–98)
Lean: `UpperTheory.V.UnitaryThreeInertia.abstract_count` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.AbstractCount`, lines 25–164)

A class in $\mathcal{S}_{\ell,1,\theta}[b]$ for $b = \ell^m$ has a representative in $A_O[b]$. Subtract the coboundary of a b-division point of the witness for bz. At $v \in T_b$, evaluation at x_v shows that the local witness is killed by b, since $\Psi(x_v) - 1$ is a unit by (A.254). Outside S we have $\Psi|_{I_p} = 1$, so the representative vanishes on I_p and descends to G; on N_v for $v \in S \setminus T_b$ we likewise have $\Psi = 1$, so the representative vanishes there. Thus $\mathcal{S}_F(b)$ maps under inflation and $A_O[b] \hookrightarrow A_O$ onto a subgroup containing $\mathcal{S}_{\ell,1,\theta}[b]$. Continuous cocycles into A_O have finite image, so (A.256) gives $L_{\ell,1,\theta} \leq V$.

Group the representatives by the embedding of K into the algebraic closure used for their witnesses. For each group apply 30.7 with the restrictions at the index $z = \iota(\theta(\sigma))$ and empty sets at the other roots of unity. The restrictions at that index are pairwise nonisomorphic. With the constants c_K, A_K of 30.7, the sum of valuations times $\log \ell$ is at most $c_K \varphi(N)(2N)^{A_K} \log p$ per group, and there are at most two groups. Divide (A.256) by e_O and use $p \leq 12\ell N$. □

Proof of 29.27

Proof. By (29.1), $\operatorname{ord}\theta < 2oR_1(\ell,\theta) \le 32o^2(\ell f)^2\ell^{2F_0}$. Let L be the compositum of the field cut out by θ with $\mathbf{Q}(\mu_\ell)$. It is abelian and totally imaginary. With $M_\chi = \max \mathcal{B}_\chi$,

$$n = [L:\mathbf{Q}] \le \ell \operatorname{ord}\theta \le n_\chi := 32o^2(M_\chi f)^2 M_\chi^{2F_0+1}.$$

Set $X = \ell N$. Then $L \subseteq F = \mathbf{Q}(\mu_X)$. Discriminant transitivity gives the lower bound [41, Ch. III, Cor. 2.10]. For the upper bound write $Y^X - 1 = \Phi_X(Y)G(Y)$ and evaluate its derivative at a primitive root ζ_X. Thus $\Phi'_X(\zeta_X)$ divides $X\zeta_X^{-1}$ in $\mathbf{Z}[\zeta_X]$. The absolute discriminant of this power basis is $|N_{F/\mathbf{Q}}\Phi'_X(\zeta_X)|$, and the field discriminant divides it. Together these give $|D_L|^{[F:L]} \le |D_F| \le X^{[F:\mathbf{Q}]}$. Hence $\log(2|D_L|) \le (n_\chi+1)\log X$. With C_K from 28.2, equations (28.5) and (28.4) give

$$s_{\ell,1}(L)\log\ell \le \log \#H^2(G_{L,S_\ell}, \mathbf{Z}_\ell(2)) \le X^A, \qquad A = (2n_\chi)^{\lceil C_K\rceil}(n_\chi+1).$$

Apply (6.IV). The normalized character length times $\log\ell$ is at most $X^A + n\nu_\ell(N,1)\log\ell \le X^A + 2n_\chi X$. A fixed power of X bounds the last expression. □

Proof of 29.28

Proof. Write $\alpha = \varepsilon_\ell^{-k}$. It suffices to kill the ℓ-torsion because continuous cocycles have finite image. If ℓz is the coboundary of m, subtract the coboundary of an ℓ-division point of m to put z in $A_O[\ell]$.

If $\ell - 1 \mid k$, then z is a homomorphism unramified outside ℓ. Kronecker–Weber [41, Ch. V, Thm. 1.10] puts its field in a cyclotomic field, and inertia at each $p \ne \ell$ removes the p-part of its conductor. Both z and α factor through $\operatorname{Gal}(\mathbf{Q}(\mu_{\ell^\infty})/\mathbf{Q})$. The decomposition group at ℓ surjects onto this group, so the local coboundary formula holds globally.

Otherwise $k = (\ell-1)u/2$ with u odd and $\ell \equiv 1 \bmod 4$. The residual character cuts out $K = \mathbf{Q}(\sqrt{\ell})$ of discriminant ℓ. Normalize $z(\delta) = 0$ at $\delta \in I_\ell$ with $\alpha(\delta) - 1$ a unit. On G_K the cocycle is a homomorphism unramified away from ℓ. The local coboundary witness at ℓ is killed by ℓ, by evaluation at an inertia element with unit scalar difference. Hence z vanishes on the decomposition groups over ℓ in G_K. The real places contribute zero by odd order, so $z|_{G_K}$ factors through $\operatorname{Cl}(K)$ by the small Hilbert class field theorem [41, Ch. VI, Prop. 6.9].

Minkowski's ideal-class bound [41, Ch. I, §6, Exercise 3] gives an integral ideal of norm at most $\sqrt{\ell}/2$ in each class: the degree is two, both embeddings are real, and the discriminant is ℓ. In a quadratic field the number of ideals of norm n is at most the divisor count $d(n)$ by prime-power factorization. With $T = \sqrt{\ell}$,

$$\#\operatorname{Cl}(K) \le \sum_{n\le T} d(n) = \sum_{a\le T} \lfloor T/a \rfloor \le T(1+\log T) < T^2 = \ell. \tag{A.257}$$

Consequently $z|_{G_K} = 0$, and the normalization on the other coset gives $z = 0$. In class Z the range follows from 29.2 and parity. □

Proof of 29.29

Proof. Put $\Delta = \operatorname{Gal}(E/\mathbf{Q})$ and extend coefficients to O. For a cyclic generator T of order m, $\prod_{\zeta^m=1}(T-\zeta) = 0$. For endomorphisms A, B, the sequence exact at its first two terms

$$0 \longrightarrow \ker B \longrightarrow \ker(AB) \xrightarrow{B} \ker A$$

gives $\operatorname{length}\ker(AB) \le \operatorname{length}\ker B + \operatorname{length}\ker A$. Iterate over the linear factors and the cyclic factors of Δ. The conjugation sign selects the characters in $X_{\ell,O}(k)$. The sum of their eigenspace lengths bounds the length.

Twist the θ-eigenspace to invariants with coefficients $W = M_{\ell,k,\theta}$. Transgression embeds the global restriction cokernel in $H^2(\Delta, W^{G_E})$ [127, Prop. 1.6.7]. For a finite abelian group B acting by scalars $\lambda(g)$ on a discrete O-module U with finite-length cyclic submodules, each $\lambda(g) - 1$ kills cohomology because g acts as the identity there [127, Prop. 1.6.3]. If (a) is the ideal of the scalar differences and Z^i is the module of

i-cocycles, $H^i(B,U)$ is a quotient of Z^i/aZ^i. The finite-submodule argument of 29.23 bounds its length by that of $Z^i[a] \subseteq (U^B)^{|B|^i}$. Applied to the global term in degree two, this bounds the loss by $e_O n^2 \nu_\ell(N,k)$.

For a global class downstairs whose restriction upstairs is strict, localize above each relevant rational prime. The local values lie in the local restriction kernels, and their joint kernel imposes the downstairs conditions. Since the places over each rational prime are conjugate, one representative suffices. At ℓ the decomposition quotient has order at most n and the invariant length is at most $e_O \nu_\ell(N,k)$ by 29.23. The scalar bound in degree one gives loss at most $e_O n \nu_\ell(N,k)$. At $p \neq \ell$ the inertia quotient is trivial for $p \nmid N$. For $p \mid N$, its order divides n. Its first cohomology is killed by n [127, Prop. 1.6.1]. On $I_{E,p}$ the character θ is trivial and the cyclotomic character is unramified at $p \neq \ell$. Thus $W^{I_{E,p}} = W$ is divisible. If $nz = \delta u$, write $u = nu_0$ and replace z by $z - \delta u_0$. This gives representatives in $W[n]$, so evaluation at quotient elements bounds the length by $e_O n v_\ell(n)$.

The added decomposition condition at $p \mid N$, $p \neq \ell$, costs at most the length of $H^1(D_p/I_p, W^{I_p})$. For unramified θ, Frobenius acts by a root of unity times p^{-k}, so Frobenius minus the identity is surjective on W. For ramified θ, a nontrivial inertia value ζ has $\zeta^n = 1$, so $\zeta - 1$ divides n and $W^{I_p} \subseteq W[n]$. The formula $H^1(D_p/I_p, W^{I_p}) = W^{I_p}/(\mathrm{Frob}_p - 1)W^{I_p}$ for the discrete torsion module [127, Prop. 1.7.7(i)] bounds the cost by $e_O v_\ell(n)$. These terms give (6.L) using 29.23, $n, \omega(N), v_\ell(n) \leq Nk$ and $Nk \geq 16$. If $\ell \nmid n$, averaging kills the finite-quotient cohomology. The added conditions cost zero by the Frobenius calculation and $W[n] = 0$. □

Proof of 29.30

Proof. Choose O as in 29.29, and put $M = Nt$. Among the characters occurring for $1 \leq k < t$, those with $\operatorname{ord} \theta_P$ odd have conductor dividing f by 29.2. There are at most f distinct such characters. Each contributes at most $B + M^{C_4}$ before multiplication by $\log \ell$, by 29.24. The other characters contribute finite weighted terms of size at most M^{2b} by 29.22, 29.26 and 29.27, where $b = \max\{C_{\mathrm{II}}, C_{\mathrm{V}}, C_{\mathrm{Vs}}\}$ and $\ell \leq N$. There are at most N terms per twist. Equations (6.R) and (6.L) give finiteness and bound the weighted sum by

$$f \log \ell \, (B + M^{C_4}) + M^{2b+1} + M^7 \log \ell.$$

Choose $C_{\ell,j}$ to absorb these terms. □

Proof of 29.31

Proof. By 29.2, $[E : \mathbf{Q}] \mid 2^{a+2} o$, so 29.29 applies with zero loss. Use 29.15 with its two primes to sum class I for each twist. Every other prime in the sum has $\ell \leq 2t-1$ by 29.2. There are at most N characters per twist and prime. Class Z contributes zero by 29.28, and classes II and III contribute at most $(\ell N k)^{\max(C_{\mathrm{II}}, C_{\mathrm{III}})}$ per character by 29.22. These finite bounds sum to at most a fixed power of Nt. □

Proof of 29.32

Proof. Apply 29.30 to $S \cap \mathcal{B}_\chi$ and 29.31 to its complement. The fixed number of summands and the coefficients of B are absorbed into C. Singletons give finiteness at every odd prime.

Take $X_0 \geq 2^J$. On each K-window, $2 \leq x \leq X$, so admissibility and the prime bounds give largeness for $t(x)$. The odd-layer conditions in the definition of the window supply $B = D_x$ in (6.P). The character group of E contains χ and the quadratic characters of conductors q, d. It also contains a faithful character of the dyadic quotient by $\{\pm 1\}$, of conductor $2^{a(x)+2}$. These conductors have least common multiple N. Since E is realizable in $\mathbf{Q}(\mu_N)$, it follows that $\mathrm{cond}(E) = N$. Now (6.P) gives (6.W). For the specified tower data, Theorem 28.16 supplies the window and the bound at the prime 2. The floor choice lies between 2 and x^{401} for every $x \geq 2$. □

Proofs for Chapter 30: Automorphic counts and deformation rigidity

Proof of 30.3

Proof. Write $U^\wedge = \widehat{U[1/\ell]}_{\mathfrak{p}_0}$ for the completion of $U[1/\ell]$ at $\mathfrak{p}_0 = (\ker x_{0,\ell})U[1/\ell]$; it represents framed deformations of $r_0|_{D_\ell} : D_\ell \to \mathcal{G}_3(E)$ over local Artinian E-algebras with residue field E. Let $R^{\mathrm{cris}}_{r_0} = U^\wedge/I_{\mathrm{cris}}$ be the quotient parameterising those deformations whose restriction to G_{K_1} is crystalline.

We first show that $R^{\mathrm{cris}}_{r_0}$ is formally smooth over E of dimension 12 with graded tangent dimensions $(1,2,3,4,2)$. Over E, the adjoint representation $\operatorname{ad} r_0 = M_3(E)$ decomposes into five weight spaces V_j ($j = b-a \in \{-2,-1,0,1,2\}$) under diagonal conjugation:

$$V_2 = E(\Psi^2\eta), \quad V_1 = E(\Psi)\oplus E(\Psi\eta), \quad V_0 = E(1)\oplus E(\eta)^{\oplus 2}, \quad V_{-1} = E(\Psi^{-1})\oplus E(\Psi^{-1}\eta), \quad V_{-2} = E(\Psi^{-2}\eta).$$

A first-order deformation $(1+\epsilon z)r_0|_{D_\ell}$ over $E[\epsilon]/(\epsilon^2)$ is crystalline on G_{K_1} precisely when $z|_{G_{K_1}} \in Z^1_f(G_{K_1}, \operatorname{ad} r_0)$. Because $[D_\ell : G_{K_1}]$ is finite and $\operatorname{char} E = 0$, Galois descent on B_{cris}-cohomology identifies these cocycles with $Z^1_f(D_\ell, \operatorname{ad} r_0)$. For each D_ℓ-representation V that is crystalline on G_{K_1}, Bloch–Kato's dimension formula [120, Corollary 3.8.4] and $\dim_E B^1(D_\ell, V) = \dim_E V - \dim_E H^0(D_\ell, V)$ give

$$\dim_E Z^1_f(D_\ell, V) = \dim_E H^1_f(D_\ell, V) + \dim_E V - \dim_E H^0(D_\ell, V) = \dim_E V + \dim_E\big(D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V)\big).$$

In the convention that $\mathbf{Q}_\ell(1)$ has filtration jump -1, a character of cyclotomic weight m contributes 1 to $\dim_E(D_{\mathrm{dR}}/\mathrm{Fil}^0)$ if $m > 0$ and 0 if $m \le 0$. The Hodge–Tate weights of $V_{-2}, V_{-1}, V_0, V_1, V_2$ are $-2k$; $-k, -k$; $0, 0, 0$; k, k; and $2k$, respectively. Hence $\dim_E Z^1_f(D_\ell, V_j)$ equals

$$1+0=1, \quad 2+0=2, \quad 3+0=3, \quad 2+2=4, \quad 1+1=2$$

for $j = -2, -1, 0, 1, 2$, summing to 12. This holds for every $k \ge 1$ (including resonant $k = 1$ with $\Psi|_{D_\ell} = \varepsilon_\ell$ or $(\eta\Psi)|_{D_\ell} = \varepsilon_\ell$) and for both split and inert ℓ.

To prove that $R^{\mathrm{cris}}_{r_0}$ is formally smooth over E, let $A' \twoheadrightarrow A$ be a small extension of local Artinian E-algebras with residue field E, and let $r : D_\ell \to \mathcal{G}_3(A)$ be a deformation of $r_0|_{D_\ell}$ crystalline on G_{K_1}. Let $\Delta = \mathrm{Gal}(K_1/K_v)$ (of order $e \in \{1,2\}$) and let $K_{1,0} = K_v$ be the maximal unramified subfield of K_1. By the equivalence of categories between crystalline G_{K_1}-representations and weakly admissible filtered φ-modules [137, Thm. A and §4.1], $r|_{G_{K_v}}$ corresponds to a free $K_v \otimes_{\mathbf{Q}_\ell} A$-module D_A of rank 3 with a Δ-action, a Δ-equivariant σ-semilinear Frobenius $\varphi_A : D_A \to D_A$, and a Δ-stable Hodge filtration $\mathrm{Fil}^\bullet_A$ on $(D_A)_{K_1}$ with jumps $-k, 0, k$.

- If ℓ splits in K ($K_v = \mathbf{Q}_\ell$), lifting r as an unframed representation amounts to lifting the free A-module D_A with its finite Δ-action (unobstructed since $\operatorname{char} E = 0$), lifting the invertible Δ-equivariant operator φ_A to $\varphi_{A'}$, and lifting the full flag $\mathrm{Fil}^\bullet_A$ in the rank-3 free A-module $((D_A)_{K_1})^\Delta$ to a full flag $\mathrm{Fil}^\bullet_{A'}$ in $((D_{A'})_{K_1})^\Delta$. There is no compatibility equation between $\varphi_{A'}$ and $\mathrm{Fil}^\bullet_{A'}$.
- If ℓ is inert in K ($[K_v : \mathbf{Q}_\ell] = 2$), fix a finite extension E'/E containing K_v so that $K_v \otimes_{\mathbf{Q}_\ell} E' = E' \times E'$, and to check formal smoothness of $R^{\mathrm{cris}}_{r_0} \otimes_E E'$ over E', take a small extension $A' \twoheadrightarrow A$ of local Artinian E'-algebras with residue field E', writing $D_A = D_{1,A} \oplus D_{2,A}$ accordingly. Choose $\phi_0 \in D_\ell \backslash G_{K_v}$; because $\mathrm{Gal}(K_1/\mathbf{Q}_\ell)$ has exponent dividing 2, we have $\phi_0^2 \in G_{K_1}$. Setting $P = r(\phi_0)_{\mathrm{mat}}$, the group relations $P\theta(r(g)_{\mathrm{mat}})P^{-1} = r(\phi_0 g \phi_0^{-1})_{\mathrm{mat}}$ ($g \in G_{K_v}$) and $P\theta(P) = r(\phi_0^2)_{\mathrm{mat}}$ say that $f = PJ : V^\vee \xrightarrow{\sim} V^{\phi_0}$ is a G_{K_v}-isomorphism satisfying $(f^{-1})^{\phi_0} \circ r(\phi_0^2)_{\mathrm{mat}} = (f^\vee)^{-1}$. Under D_{cris}, since $\phi_0^2 \in G_{K_1}$ acts trivially on D_A, this corresponds to a Δ-equivariant isomorphism $F = (F_1, F_2) : D_A^\vee \xrightarrow{\sim} D_A^{\phi_0}$ of filtered φ-modules satisfying $F^{\phi_0} = F^\vee$, i.e. $F_2 = F_1^\vee$ for an unconstrained Δ-equivariant isomorphism $F_1 : D_{1,A}^\vee \xrightarrow{\sim} D_{2,A}$. Writing $\varphi_A = (\varphi_{1,A}, \varphi_{2,A})$ with $\varphi_{1,A} : D_{1,A} \to D_{2,A}$ and $\varphi_{2,A} : D_{2,A} \to D_{1,A}$, compatibility $F \circ (\varphi_A^\vee)^{-1} = \varphi_A^{\phi_0} \circ F$ is equivalent to the single formula $\varphi_{2,A} = F_1^\vee(\varphi_{1,A}^\vee)^{-1}F_1^{-1}$, and compatibility with the filtration says that $\mathrm{Fil}^\bullet_{2,A}$ is the orthogonal complement of $\mathrm{Fil}^\bullet_{1,A}$ under F_1. Thus lifting $(D_A, \varphi_A, \mathrm{Fil}^\bullet_A, F)$ to A' amounts to lifting $D_{1,A}$ with its Δ-action, lifting F_1, lifting $\varphi_{1,A}$ arbitrarily (which defines $\varphi_{2,A'}$), and lifting the Δ-stable flag $\mathrm{Fil}^\bullet_{1,A}$ in $((D_{1,A})_{K_1})^\Delta$ arbitrarily (which defines $\mathrm{Fil}^\bullet_{2,A'}$).

In both cases weak admissibility persists automatically across the small extension $A' \twoheadrightarrow A$: since the steps of $\mathrm{Fil}^\bullet_{A'}$ are free over A', we have a short exact sequence of filtered φ-modules $0 \to D_0 \otimes_E \ker(A' \to A) \to D_{A'} \to D_A \to 0$ (with E' in place of E when ℓ is inert), and weak admissibility is closed under extensions (for any φ-stable K_v-subspace $M \subseteq D_{A'}$, setting $M_1 = M \cap (D_0 \otimes \ker)$ and $M_2 = \mathrm{im}(M \to D_A)$ gives $t_H(M) \le t_H(M_1) + t_H(M_2) \le t_N(M_1) + t_N(M_2) = t_N(M)$, with equality for $M = D_{A'}$). Thus the unframed crystalline $\mathcal{G}_3$-representation lifts to A', and since a basis of a free A-module lifts to any free A'-module lifting it, the framing lifts as well. Hence $R^{\mathrm{cris}}_{r_0}$ is formally smooth over E of dimension 12 (descending from E' to E by faithful flatness when ℓ is inert), and therefore $R^{\mathrm{cris}}_{r_0} \simeq E[[t_1, \dots, t_{12}]]$.

We next compare $I \cdot U^\wedge$ with I_{cris}. First, let $J_{\mathrm{cris}} \subset U$ be the closed ideal cutting out the framed crystalline locus of $r^{\mathrm{univ}}|_{G_{K_1}}$ with Hodge–Tate weights $(k, 0, -k)$ and trivial inertial type, obtained by pulling back the crystalline quotient of [115, Thm. 1.2.2] along $G_{K_1} \hookrightarrow D_\ell$. At any characteristic-zero point y of $(U/J_{\mathrm{ord}})[(\ell\delta_1\delta_2)^{-1}]$ (or any finite Artinian E'-algebra point), the polynomials P_i/c_i give orthogonal rank-1 idempotents summing to I_3, so $r_y|_{G_{K_v}}$ preserves a full flag with diagonal characters $\lambda_0, \lambda_1, \lambda_2$ whose restrictions to I_{K_1} are $\varepsilon_\ell^k, 1, \varepsilon_\ell^{-k}$. Since $\delta_1(y)\delta_2(y) \neq 0$ and $\varphi_v \in G_{K_1}$, we have $\lambda_0(\varphi_v) \neq \varepsilon_\ell(\varphi_v)\lambda_1(\varphi_v)$ and $\lambda_1(\varphi_v) \neq \varepsilon_\ell(\varphi_v)\lambda_2(\varphi_v)$, while the outer ratio $\lambda_0/\lambda_2 = \varepsilon_\ell^{2k}$ on I_{K_1} differs from ε_ℓ since $2k \neq 1$. Lemma 30.17 therefore shows that $r_y|_{G_{K_1}}$ is crystalline, so y kills J_{cris}. Because $U[1/\ell]$ is a Jacobson ring and $(U/I_{\mathrm{sat}})[1/\ell]$ is reduced with $\delta_1\delta_2$ a nonzerodivisor, $J_{\mathrm{cris}} \subseteq I_{\mathrm{sat}} \subseteq I$. In particular, $I_{\mathrm{cris}} = J_{\mathrm{cris}}U^\wedge \subseteq I \cdot U^\wedge$.

For the reverse inclusion, consider the universal crystalline deformation r^{cris} over the regular complete local domain $R^{\mathrm{cris}}_{r_0} \simeq E[[t_1, \dots, t_{12}]]$ and the inverse-limit filtered φ-module $(D, \varphi, \mathrm{Fil}^\bullet) = \varprojlim_n D_{\mathrm{cris}, K_1}(r^{\mathrm{cris}} \bmod \mathfrak{m}^n)$ over $K_v \otimes_{\mathbf{Q}_\ell} R^{\mathrm{cris}}_{r_0}$. Let $d_v = [K_v : \mathbf{Q}_\ell] \in \{1, 2\}$. Modulo $\mathfrak{m}_{R^{\mathrm{cris}}_{r_0}}$, the K_v-linear Frobenius φ^{d_v} has eigenvalues in E of pairwise distinct ℓ-adic valuations $-kd_v, 0, kd_v$ (corresponding to the distinct weights $k, 0, -k$). Applying Hensel's lemma on each local factor of the finite product of complete local rings $K_v \otimes_{\mathbf{Q}_\ell} R^{\mathrm{cris}}_{r_0}$, D splits uniquely into three φ^{d_v}-stable and Δ-stable rank-1 direct summands $D = L_0 \oplus L_1 \oplus L_2$ lifting the base eigenspaces $L_{i,0}$ of slopes $-kd_v, 0, kd_v$; since φ commutes with φ^{d_v} and preserves each $L_{i,0}$ (with σ-fixed base eigenvalues in E), uniqueness of the splitting forces $\varphi(L_i) = L_i$. At the base point, $\mathrm{Fil}^k_0 = (L_{2,0})_{K_1}$ and $\mathrm{Fil}^0_0 = (L_{2,0} \oplus L_{1,0})_{K_1}$, so $(L_{0,0})_{K_1} \cap \mathrm{Fil}^0_0 = 0$ and $(L_{0,0} \oplus L_{1,0})_{K_1} \cap \mathrm{Fil}^k_0 = 0$. Transversality persists over $K_1 \otimes_{\mathbf{Q}_\ell} R^{\mathrm{cris}}_{r_0}$, giving $(L_0)_{K_1} \cap \mathrm{Fil}^0 = 0$ and $(L_0 \oplus L_1)_{K_1} \cap \mathrm{Fil}^k = 0$ while $(L_0 \oplus L_1)_{K_1} \cap \mathrm{Fil}^0$ is a free direct summand of rank 1. Thus the induced filtrations on L_0 and $L_0 \oplus L_1$ are flat over $R^{\mathrm{cris}}_{r_0}$, with Hodge jump $-k$ and Newton slope $-k$ on L_0, and Hodge jumps $-k, 0$ and Newton slopes $-k, 0$ on $L_0 \oplus L_1$. Over every Artinian quotient $R^{\mathrm{cris}}_{r_0}/\mathfrak{m}^n$, L_0 and $L_0 \oplus L_1$ are successive extensions of the weakly admissible base modules $L_{0,0}$ and $L_{0,0} \oplus L_{1,0}$, so $0 \subset L_0 \subset L_0 \oplus L_1 \subset D$ is a flag of weakly admissible filtered φ-modules with Δ-action. By [137, Thm. A], $r^{\mathrm{cris}}|_{G_{K_v}}$ preserves a full flag over every $R^{\mathrm{cris}}_{r_0}/\mathfrak{m}^n$ whose graded inertia characters are $(\Psi, 1, \Psi^{-1})$. Hence J_{ord} maps to zero in $R^{\mathrm{cris}}_{r_0}$.

Moreover $\ell\delta_1\delta_2$ has nonzero image in the domain $R^{\mathrm{cris}}_{r_0}$: in the diagonal unramified crystalline family $g \mapsto (\mathrm{diag}(\Psi(g)u(g), 1, \Psi(g)^{-1}u(g)^{-1}), \eta(g))$ with u unramified and $u(\varphi_v) = 1 + T$, both $\lambda_0(\varphi_v)/\lambda_1(\varphi_v) = \Psi(\varphi_v)(1+T)$ and $\lambda_1(\varphi_v)/\lambda_2(\varphi_v) = \Psi(\varphi_v)(1+T)$ are nonconstant, so δ_1 and δ_2 do not vanish identically on $R^{\mathrm{cris}}_{r_0}$. Because $R^{\mathrm{cris}}_{r_0}$ is a characteristic-zero domain, its defining prime ideal $I_{\mathrm{cris}} \subset U^\wedge$ contains $I_{\mathrm{sat}}U^\wedge$. Combined with $I_{\mathrm{cris}} \subseteq I \cdot U^\wedge$, and since $I \cdot U^\wedge$ is the intersection of the minimal primes of $I_{\mathrm{sat}}U^\wedge$ inside $\mathfrak{m}_{U^\wedge}$, it follows that I_{cris} is the unique minimal prime of $I_{\mathrm{sat}}U^\wedge$ inside $\mathfrak{m}_{U^\wedge}$ and $I \cdot U^\wedge = I_{\mathrm{cris}}$.

Because I is reduced and every minimal prime of I lies in $\ker x_{0,\ell}$, the composite $B = U/I \to (U/I)_{\ker x_{0,\ell}} \to \widehat{B[1/\ell]}_{\ker x_{0,\ell}} = R^{\mathrm{cris}}_{r_0}$ is injective by Krull's intersection theorem. In particular I is a single prime ideal and B is a complete Noetherian local O-domain.

For conjugation invariance, let R be a complete Noetherian local O-algebra with residue field $\mathbf{k}$ and let $h \in \mathrm{GL}_3(R)$ satisfy $h \equiv I_3 \bmod \mathfrak{m}_R$. For $g \in G_{K_v}$, $(h,1)r(g)(h,1)^{-1}$ has matrix part $hr(g)_{\mathrm{mat}}h^{-1}$, which conjugates P_i to hP_ih^{-1} and leaves $\det(XI_3 - r(s_0)_{\mathrm{mat}})$, t_i, δ_1, and δ_2 unchanged. Thus J_{ord}, $\sqrt{J_{\mathrm{ord}}}$, and I_{sat} are invariant under conjugation by $(h,1)$. On the universal formal conjugation map $U \to U\widehat{\otimes}_O O[[z_{ab}]]$ (for $h = I_3 + (z_{ab})$), the preimage of the prime ideal $I \cdot (U\widehat{\otimes}_O O[[z_{ab}]])$ is a prime ideal containing I_{sat} and contained in $\ker x_{0,\ell}$; uniqueness of the minimal prime I forces this preimage to contain I, proving conjugation invariance of I. Specialising to the diagonal torus $\mathrm{diag}(t, 1, t^{-1})$ shows that the kernel of a minimal graded power series presentation $O[[X]] \twoheadrightarrow U \twoheadrightarrow B = O[[X]]/(H)$ is homogeneous, and specialising to $h = I_3 + \epsilon Y$ over $O[\epsilon]/(\epsilon^2)$ gives the derivation killing H. Since the diagonal torus acts semisimply on $R^{\mathrm{cris}}_{r_0}/\mathfrak{m}^n$ for every n with cotangent weights $(1, 2, 3, 4, 2)$, lifting a homogeneous basis of $\mathfrak{m}/\mathfrak{m}^2$ to homogeneous elements of $\mathfrak{m}$ yields a graded isomorphism $R^{\mathrm{cris}}_{r_0} \simeq E[[t_1, \dots, t_{12}]]$ and hence the graded rational formal chart $\varphi : U \to$

$E[[t_1,\dots,t_{12}]]$ with $\ker\varphi = I$. This proves (1) and (2).

Finally, for (3), since $J_{\mathrm{cris}} \subseteq I$, every characteristic-zero point $f : B \to C_\ell$ kills J_{cris} and J_{ord}, so $r_f|_{G_{K_1}}$ is crystalline and (using the idempotents P_i/c_i over C_ℓ) $r_f|_{G_{K_v}}$ preserves a full flag with inertia characters $(\Psi, 1, \Psi^{-1})$. Any characteristic-zero tangent vector $B \to C_\ell[\epsilon]/(\epsilon^2)$ at f likewise kills J_{cris}, so its deformation is crystalline on G_{K_1}, and finite-index Galois descent places its cocycle in $Z^1_f(D_\ell, \mathrm{ad}\, r_f)$. □

Proof of 30.4

Proof. For (1), let r be a two-sided characteristic-zero point of S. Since $r(c)_{\mathrm{mat}} = I_3$, for every $g \in G_K$ we have $r(cg^{-1}c)_{\mathrm{mat}} = J(r(g)_{\mathrm{mat}})^t J$. Applying this to g_1 shows that $Z = r(g_1)_{\mathrm{mat}} - r(cg_1^{-1}c)_{\mathrm{mat}}$ satisfies $JZ^tJ = -Z$, so $Z_{ab} = -Z_{2-b,2-a}$. Together with the marking equations $Z_{01} = Z_{10} = 0$, this forces $Z = \mathrm{diag}(\zeta, 0, -\zeta)$ with $\bar\zeta = \bar\Psi(g_1) - \bar\Psi(g_1)^{-1} \neq 0$. Hence Z has three pairwise distinct eigenvalues, so every G_K-invariant subspace of C_ℓ^3 is spanned by coordinate axes. Polarisation $r(cg^{-1}c)_{\mathrm{mat}} = J(r(g)_{\mathrm{mat}})^t J$ pairs the vanishing of entry (a,b) throughout G_K with that of $(2-b, 2-a)$; in particular $(1,0),(2,1)$ vanish together and $(0,1),(1,2)$ vanish together. Because r is two-sided, neither all negative-weight entries nor all positive-weight entries vanish on G_K, excluding upper-triangular and lower-triangular shapes. Any proper nonzero G_K-invariant coordinate subspace therefore forces the middle coordinate 1 to split from $\{0,2\}$, so all odd-weight entries vanish on G_K, while both $(0,2)$ and $(2,0)$ are nonzero on G_K, making the corner on $\{0,2\}$ irreducible.

In Type II, put $\kappa(g) = r(g)_{\mathrm{mat},11}$ for $g \in G_K$. Then $\kappa : G_K \to C_\ell^\times$ is a continuous character satisfying $|\kappa(g) - 1|_\ell < 1$. At any finite place A of $\overline{\mathbf{Q}}$ not above ℓ, r kills $I_A \cap \ker\eta \cap \ker\vartheta$, whose quotient in $I_A \cap G_K$ has order dividing $\mathrm{ord}\,\vartheta$ and hence prime to ℓ; since r and r_0 have the same residual values on this finite quotient, they are conjugate on $I_A \cap G_K$. At any place $A \mid \ell$, conjugating to the distinguished place A_0 and applying Lemma 30.3(3) shows that $r|_{I_A}$ is conjugate to an upper-triangular representation with diagonal $r_0|_{I_A}$. Consequently, for every finite place A and every $g \in I_A \cap G_K$, $\det(X - r(g)_{\mathrm{mat}}) = (X-\Psi(g))(X-1)(X-\Psi(g)^{-1})$. If $\bar\Psi(g) \neq 1$, then $|\Psi(g)^{\pm1} - 1|_\ell = 1$, whereas $|\kappa(g) - 1|_\ell < 1$, forcing $\kappa(g) = 1$. If $\bar\Psi(g) = 1$ and $A \nmid \ell$, then $g \in \mathcal{K}_{\mathrm{away}}$, so $r(g) = 1$ and $\kappa(g) = 1$. If $\bar\Psi(g) = 1$ and $A \mid \ell$, then because $\ell - 1 \nmid 2k$ and $\mathrm{comp}_\ell(\vartheta)^2 = 1$, there exists $s \in I_A \subset G_K$ with $\bar\Psi(s)^2 \neq 1$; then both $\bar\Psi(s) \neq 1$ and $\bar\Psi(gs) \neq 1$, so $\kappa(s) = \kappa(gs) = 1$ and hence $\kappa(g) = 1$. Thus κ is unramified at all finite places of K. Class field theory gives $\kappa^{h_K} = 1$, and since $\ell \nmid h_K$ and $|\kappa(g) - 1|_\ell < 1$, it follows that $\kappa = 1$ on G_K. The same argument applied to $\det(r|_{G_K}) = \det(\mathrm{corner}(r|_{G_K}))$ (which equals 1 on every inertia group and satisfies $|\det r(g) - 1|_\ell < 1$) gives $\det(\mathrm{corner}(r(g)_{\mathrm{mat}})) = 1$ for all $g \in G_K$. For any $M \in \mathrm{SL}_2(C_\ell)$, $J_2(M^{-1})^t J_2 = \mathrm{diag}(-1,1) M \,\mathrm{diag}(-1,1)$, where $J_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Since $r(cgc)_{\mathrm{mat}} = J(r(g)_{\mathrm{mat}}^{-1})^t J$ for $g \in G_K$, the formula for W defines a continuous homomorphism $W : G_{\mathbf{Q}} \to \mathrm{GL}_2(C_\ell)$ extending the irreducible corner, with $W(c) = \mathrm{diag}(-1,1)$, $\mathrm{tr}\, W(c) = 0$, and $\det W = \eta$.

For (2), suppose r, r' are two-sided points of S and $U \in \mathrm{GL}_3(C_\ell)$ satisfies $U r(g)_{\mathrm{mat}} U^{-1} = r'(g)_{\mathrm{mat}}$ for all $g \in G_K$. Then U intertwines the diagonal marking matrices $Z = \mathrm{diag}(\zeta, 0, -\zeta)$ and $Z' = \mathrm{diag}(\zeta', 0, -\zeta')$, which therefore have the same eigenvalue set. Since $\bar\zeta = \bar\zeta' \neq 0$ and ℓ is odd, $\zeta' = \zeta$, so $U = \mathrm{diag}(u_0, u_1, u_2)$ is diagonal. In Type I, $r'(cg^{-1}c)_{\mathrm{mat}} = UJU^t (r'(g)_{\mathrm{mat}})^t (UJU^t)^{-1}$; irreducibility of $r'|_{G_K}$ forces UJU^t to be a scalar multiple of J, so $u_0u_2 = u_1^2$. In Type II, replacing u_1 by $\sqrt{u_0u_2} \in C_\ell^\times$ preserves $U r(g)_{\mathrm{mat}} U^{-1}$ on the block-diagonal matrices $r(g)_{\mathrm{mat}}$ and achieves $u_0u_2 = u_1^2$. In both cases $U = u_1 \,\mathrm{diag}(t, 1, t^{-1})$ with $t = u_0/u_1$ (so $t^{-1} = u_2/u_1$ by $u_0u_2 = u_1^2$), whose conjugation preserves $r(c)_{\mathrm{mat}} = I_3$ and scales weight-j entries by t^j.

For (3), in weights $j = -2, -1$, every character λ occurring in V_j has the form $\lambda = \chi^{-1}\varepsilon_\ell^{-e}$ with χ of finite order and $e \in \{k, 2k\} \geq 1$. By [120, Corollary 3.8.4], $\dim_{C_\ell} H^1_f(\mathbf{Q}_\ell, C_\ell(\lambda)) = \dim_{C_\ell} H^0(\mathbf{Q}_\ell, C_\ell(\lambda)) + 0 = 0$ because $\lambda|_{I_\ell}$ has infinite order. Hence any class in $H^1_f(\mathbf{Q}, C_\ell(\lambda))$ is a coboundary on all finite inertia groups and on D_ℓ, so it vanishes by Lemma 29.19. In weight $j = 0$, we have $V_0 = C_\ell(1) \oplus C_\ell(\eta)^{\oplus 2}$. For the trivial character, $H^1_f(\mathbf{Q}_\ell, C_\ell) = H^1_{\mathrm{unr}}(\mathbf{Q}_\ell, C_\ell)$, so classes in $H^1_f(\mathbf{Q}, C_\ell)$ are continuous unramified homomorphisms $G_{\mathbf{Q}} \to C_\ell$, which vanish because $\mathrm{Cl}(\mathbf{Q}) = 1$. For $C_\ell(\eta)$, restriction embeds $H^1_f(\mathbf{Q}, C_\ell(\eta))$ into $H^1_f(K, C_\ell)$ (since $[K : \mathbf{Q}] = 2$ is invertible in C_ℓ), and classes in $H^1_f(K, C_\ell)$ are everywhere unramified continuous homomorphisms $G_K \to C_\ell$, which factor through the finite group $\mathrm{Cl}(K)$ and hence vanish [41, Ch. VI, Prop. 6.9]. □

Proof of 30.5

Proof. By 29.3 work over the unramified ring generated by ϑ. Choose the marking g_1 from 30.61. Use the marked deformation problem and distinguished local ring $B = U/I$ of Definition 30.2, with base lift $r_0(g) = (\mathrm{diag}(\Psi(g), 1, \Psi(g)^{-1}), \eta(g))$. By Lemma 30.3(1)–(2), $B = O[[X]]/(H)$ has a homogeneous presentation centred at r_0 and a graded rational formal chart with $1, 2, 3, 4, 2$ coordinates in weights $-2, -1, 0, 1, 2$, respectively.

Apply the relative presentation of 30.72 over B to global lifts vanishing on $\mathcal{K}_{\mathrm{away}}$ before imposing the marking equations, with one framed place $v_0 = \ell$ (frame fixed to the identity) and local inertia kernels $N_q = I_q \cap \ker \bar{r}_0$ at the finite ramified primes $q \neq \ell$ dividing $N|D_K|$. Let G_S be the maximal Galois quotient unramified outside $S = \{\ell\} \cup \{q : q \mid N|D_K|\}$. Write Y for the extra coordinates and $d_j = \dim_{\mathbf{k}} H^1_{\mathrm{fr},j}$ for their number in weight j, chosen from matrix entries minus their base values.

We verify that the relative obstruction space Ob_j of 30.72 has dimension $d_j + \mathbf{1}_{|j|\le 1}$. Let $\bar{V}_j$ be the weight-j summand of $\mathrm{ad}\,\bar{r}_0$ over $\mathbf{k}$. Its complex-conjugation eigenspace decomposition is

$$\bar{V}_2 = \mathbf{k}(\bar{\Psi}^2\eta), \quad \bar{V}_1 = \mathbf{k}(\bar{\Psi}) \oplus \mathbf{k}(\bar{\Psi}\eta), \quad \bar{V}_0 = \mathbf{k}(1) \oplus \mathbf{k}(\eta)^{\oplus 2}, \quad \bar{V}_{-1} = \mathbf{k}(\bar{\Psi}^{-1}) \oplus \mathbf{k}(\bar{\Psi}^{-1}\eta), \quad \bar{V}_{-2} = \mathbf{k}(\bar{\Psi}^{-2}\eta),$$

so $\dim_{\mathbf{k}} \bar{V}_j^{c=1} = \mathbf{1}_{|j|\le 1}$. We claim that $H^0(G_S, \bar{V}_j^\vee(1)) = 0$ for every $j \in \{-2, -1, 0, 1, 2\}$. An even constituent $\bar{\Psi}^j$ takes value 1 at c, whereas $\bar{\varepsilon}_\ell(c) = -1 \neq 1$. For an odd constituent $\bar{\Psi}^j\eta$, since K is imaginary quadratic there is a prime $p \mid D_K$; because $(N, D_K) = 1$ and $\ell \nmid D_K$, we have $p \nmid \ell N$. On I_p, both $\bar{\varepsilon}_\ell$ and $\bar{\vartheta}$ (hence $\bar{\Psi}$) are trivial, while $\eta|_{I_p}$ takes the value $-1 \neq 1$. Thus no constituent of $\bar{V}_j$ equals $\bar{\varepsilon}_\ell$, proving $H^0(G_S, \bar{V}_j^\vee(1)) = 0$.

By Poitou–Tate duality 27.2, the map $H^2(G_S, \bar{V}_j) \to \bigoplus_{v\in S} H^2(D_v, \bar{V}_j)$ is surjective, so the image of $\mathrm{Ob}_j \to H^2(G_S, \bar{V}_j)$ has dimension $h^2(G_S, \bar{V}_j) - \sum_{v\in S} h^2(D_v, \bar{V}_j)$. The kernel of $\mathrm{Ob}_j \to H^2(G_S, \bar{V}_j)$ is the cokernel of

$$Z^1(G_S, \bar{V}_j) \longrightarrow H^1(D_\ell, \bar{V}_j) \oplus \bigoplus_{q\in S\setminus\{\ell\}} Z^1(D_q, \bar{V}_j)/Z^1_{\mathcal{L},q},$$

where $Z^1_{\mathcal{L},q} = \{e_q \in Z^1(D_q, \bar{V}_j) : e_q|_{N_q} = 0\}$. For each $q \in S \setminus \{\ell\}$, $[I_q : N_q]$ divides $2\,\mathrm{ord}\,\vartheta$ and is prime to ℓ, so $\dim Z^1_{\mathcal{L},q} = \dim \bar{V}_j$ and local Euler characteristic gives $\dim Z^1(D_q, \bar{V}_j) - \dim Z^1_{\mathcal{L},q} = h^2(D_q, \bar{V}_j)$. Let $W \subseteq Z^1(G_S, \bar{V}_j)$ be the kernel of the displayed map. Every global coboundary δv ($v \in \bar{V}_j$) vanishes on each N_q and restricts to $B^1(D_\ell, \bar{V}_j)$, so $W \to B^1(D_\ell, \bar{V}_j)$ is surjective with kernel $H^1_{\mathrm{fr},j}$; hence $\dim W = d_j + \dim \bar{V}_j - h^0(D_\ell, \bar{V}_j)$. Combining this with the global Euler characteristic $h^1(G_S, \bar{V}_j) - h^2(G_S, \bar{V}_j) - h^0(G_S, \bar{V}_j) = \dim \bar{V}_j - \dim \bar{V}_j^{c=1}$ and the local Euler characteristic $h^1(D_\ell, \bar{V}_j) - h^2(D_\ell, \bar{V}_j) = \dim \bar{V}_j + h^0(D_\ell, \bar{V}_j)$ cancels all local terms and gives

$$\dim_{\mathbf{k}} \mathrm{Ob}_j = d_j + \dim_{\mathbf{k}} \bar{V}_j^{c=1} = d_j + \mathbf{1}_{|j|\le 1}.$$

We next count the marking equations. Since $c^2 = 1$ and $\eta(c) = -1$, the matrix $A = r(c)_{\mathrm{mat}}$ satisfies $A\theta(A) = I_3$, i.e. $JA^tJ = A$, so $A_{ab} = A_{2-b,2-a}$. Hence the condition $A = I_3$ is cut out by 6 homogeneous equations $A_{20} = 0$, $A_{10} = 0$, $A_{00} - 1 = A_{11} - 1 = 0$, $A_{01} = 0$, $A_{02} = 0$, with weight multiplicities $1, 1, 2, 1, 1$. Together with the two marking equations at g_1 in positions $(1, 0)$ (weight -1) and $(0, 1)$ (weight 1), the marking adds 8 equations with multiplicities $1, 2, 2, 2, 1$ in weights $-2, -1, 0, 1, 2$, respectively.

Put $R = B[[Y]]$, let $A = \bigoplus_j A_j \subset R$ be the sum of its homogeneous pieces, and restrict the quotient map to $q : A \to S$, where S is the marked global ring with universal lift r_S. The local chart of Lemma 30.3(2) extends by the free coordinates Y to the rational formal chart for A. With

$$s = d_{-2} + 1, \quad r = d_{-1} + 2, \quad m = d_0 + 3, \quad r' = d_1 + 3, \quad s' = d_2 + 1$$

in the notation of the orbit estimate of 30.72, the chart has $s, r, m, r' + 1, s' + 1$ coordinates of weights $-2, -1, 0, 1, 2$, with $r \ge 1$. The kernel J of q is generated by the relative and marking equations, whose counts in the five weights are $(s, r + 1, m, r', s')$, giving the families G, F, Z, E, E' of 30.72. Let $x_0 : S \to O$ be the base point and $x = x_0 q$. Write $C_{-1}(F)$ for the cotangent quotient by the weight--1 equations F_i.

Lean: `UpperTheory.V.Loc.rtChart` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PadicHodgeTheory.RtChart`, lines 41–74)

Lean: `UpperTheory.V.U3.exists_model` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.ModelEx`, lines 156–178)

Lean: `UpperTheory.V.U3.model_of_pres` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.ModelOfPres`, lines 320–329)

By 30.6 and 30.72, $\operatorname{length}_O \mathcal{S}_{\ell,k,\vartheta}[b] \le \operatorname{length}_O C_{-1}(F)$ for $b \neq 0$. Continuous cocycles into the discrete coefficient module have finite image. Thus every class belongs to some $\mathcal{S}_{\ell,k,\vartheta}[\ell^j]$ and

$$L_{\ell,k,\vartheta} \le \operatorname{length}_O C_{-1}(F). \tag{A.258}$$

We verify the special-fibre hypothesis of Lemma 30.53. Let $S \to V$ be a local O-algebra map to a complete DVR with finite residue field and $\mathfrak{m}_O V = 0$. In the flag variety over the distinguished local domain, stability and the prescribed inertia action on each ordered quotient line are closed conditions. Their common locus contains the generic ordinary flag and projects onto the base by properness. Thus over a finite extension of $\operatorname{Frac} V$ the given point has a flag with those inertia characters. Intersect its steps with the lattice over the extended DVR. The steps are saturated, and their ordered reductions form the standard flag because the residual inertia characters are distinct. Lift the standard basis along these steps to obtain a flag basis congruent to the identity. Thus 30.61(1) applies.

Replace its fixed finite-index subgroup by its normal core, and let a be the exponent of the finite quotient. For every $g \in G_K$, g^a belongs to that subgroup. Every eigenvalue of $r(g)$ therefore has an a-th power in the algebraic closure of the finite coefficient field, and is itself a constant. Reduction is injective on this constant field. The residual polynomial consequently determines the characteristic polynomial at every g, so the semisimplification is the prescribed sum of three characters. The spectral projectors of the diagonal marking difference belong to the matrix algebra. An invariant composition flag is therefore a coordinate flag. Multiplication with these projectors makes the support relation transitive; polarisation reflects an entry (i, j) to $(2 - j, 2 - i)$. Acyclicity then forces every surviving off-diagonal entry to have the same strict weight sign. The diagonal and ordered character coordinates equal their residual constants. Hence all zero-weight coordinates and all coordinates of one strict sign vanish at the point. Lemma 30.53 proves that the weight-zero part of the complete homogeneous ring S is finite over O. Its submodule generated by weight-zero products of matrix entries is finite as well. The ring S is Hausdorff and this finite O-module is compact. The image $q(A_0)$ lies in it by approximation by homogeneous polynomials, so $q(A_0)$ is finite over O.

Let $\xi : A/J \to C_\ell$ be an O-point. Write $z_i \in A$ for the images of the centred coordinates and j_i for their weights. Each nonconstant monomial of weight zero lies in the maximal ideal of A_0. Its image under ξ has positive valuation since the local algebra $q(A_0)$ is finite over O. For nonzero coordinate values put $v_i = v_\ell(\xi(z_i))$. A positive-weight coordinate and a negative-weight coordinate give a weight-zero monomial $z_i^{-j_h} z_h^{j_i}$. Its positive valuation gives

$$-v_i/j_i < v_h/(-j_h) \qquad (j_i > 0 > j_h).$$

Choose a rational number above all the lower bounds on the left and below all the upper bounds on the right. Take $t \in C_\ell^\times$ of that valuation. Now $t^{j_i}\xi(z_i)$ has positive valuation whenever $\xi(z_i) \neq 0$, including the weight-zero coordinates. These scaled values define evaluation on R in the integers of a finite extension of $\operatorname{Frac} O$. To see agreement on A, group each homogeneous series into finitely many monomials times weight-zero series. The weight-zero coordinate ideal is generated by finitely many nonconstant weight-zero monomials. Both evaluations send those monomials to elements of positive valuation, so agreement on polynomials gives agreement on weight-zero series. The evaluation kills the relative and marking equations, so it descends to a local point $f : S \to C_\ell$ with $(t \cdot \xi)(z) = f(qz)$ for $z \in A$.

At $f \circ q$, a tangent derivation D killing J gives values $f(qz_i) + \epsilon D(z_i)$ on the coordinates, with $\epsilon^2 = 0$. The map on A_0 with this first-order term factors through $q(A_0)$. Choose a finite extension E' containing its values on module generators and the finitely many coordinate values and derivatives. In $C_\ell[\epsilon]/(\epsilon^2)$ these lie in a complete ring $O_{E'} \oplus \epsilon \varpi_{E'}^{-a} O_{E'}$ for some $a \ge 0$, where $O_{E'}$ is the ring of integers and $\varpi_{E'}$ a uniformiser. The coordinate values reduce to the maximal ideal modulo ϵ, so formal evaluation converges. It agrees with $f \circ q + \epsilon D$ on homogeneous series and kills the equations. Thus it defines a lift $S \to C_\ell[\epsilon]/(\epsilon^2)$. At ξ use the same t on tangent vectors, multiplying their values on weight j by t^j.

Lean: `UpperTheory.V.U3.Model` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.ConjStable`, lines 54–85)

Lean: `UpperTheory.V.U3.model_upper` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.ModelUpper`, lines 103–250)

Put $r_f = f_* r_S$. Negative weights vanish at $f \circ q$ precisely when r_f is upper triangular on G_K, and positive weights vanish precisely when it is lower triangular there. This follows from the weights of the matrix entries

and their homogeneous products. The marked diagonal point is r_0, so the only fixed solution is x. This gives the one-sided fibre condition of 30.72.

By Lemma 30.3(3) and finite-quotient averaging on inertia away from ℓ, every lifted tangent cocycle at a characteristic-zero point f lies in the Bloch–Kato finite Selmer group $Z^1_f(G_{\mathbf{Q}}, \operatorname{ad} r_f)$. At x, a derivation of weight $j \le 0$ killing J lifts to a normalised cocycle in $Z^1_f(G_{\mathbf{Q}}, V_j)$, since the base matrices are diagonal. By Lemma 30.4(3), $H^1_f(\mathbf{Q}, V_j) = 0$ for $j \le 0$, so the cocycle is a coboundary. The marking forces its conjugating vector into the Lie algebra generated by $\operatorname{diag}(1, 0, -1)$, which centralises r_0. Thus the cocycle vanishes, as does the derivation, since the matrix entries determine the lift. Duality now gives the required rational spans: a functional on the weight--1 cotangent space killing the F_i extends by zero in the other weights to a derivation killing J, and the same argument applies to the G_i in weight -2 and the Z_i in weight 0.

At a two-sided point, Lemma 30.4(1) classifies r_f as Type I or Type II. Use adjoint rigidity in Type I. In Type II, Lemma 30.4(1) shows that the middle line is trivial on G_K and the corner extends to W of determinant η and $\operatorname{tr} W(c) = 0$. The adjoint decomposes into the polarised adjoint of the corner (killed by block adjoint rigidity), the two off-block summands W and $W \otimes \eta$ (killed by 30.12, using Lemma 30.3(3), Lemma 29.7, and 30.61(3)), and the $(1,1)$ diagonal summand $C_\ell(\eta)$ (killed by Lemma 30.4(3)). Each lifted tangent cocycle is therefore a coboundary with conjugating vector in the one-dimensional torus Lie algebra generated by $\operatorname{diag}(1, 0, -1)$. The cocycle determines the coordinate derivatives, so the tangent space at ξ has dimension at most one.

Lean: `UpperTheory.V.Z.J.ptHyp_of_autB_of_lemW_of_isBase` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.FPtHypU`, lines 15–40)

Type I admissibility and the Type II endpoint of 30.61 put the two-sided points in $\mathcal{A}_3 \cup \mathcal{A}_{2+1}$. Take a weight-zero lift $a \in A$ of $\operatorname{tr} r_S(\sigma) - T$. It vanishes at x and is nonzero at the two-sided points, so the orbit estimate of 30.72 applies. Lemma 30.4(2) places points with isomorphic G_K-restrictions in the same $\mathbb{G}_m$-orbit. The orbit representatives give X. Combine their bound with (A.258). Over the unramified ring the length and valuation are already normalised, so 29.3 returns the bound to the original O. □

Proof of 30.6

Proof. Put $\alpha = \Psi^{-1}$. At g_1 and at a test element of the distinguished local group, $\Psi^2 - 1$ is a unit. Subtract the unique coboundary making the Selmer cocycle zero at g_1. By uniqueness the normalised cocycle is b-torsion. Its local form is $s - \alpha(h)s$, so the local unit test gives $bs = 0$. Identify coefficient b-torsion with $O/(b)$ and call the cocycle t. On the prescribed kernels $\mathcal{K}_{\mathrm{away}}$, $\alpha = 1$ and the local coboundary vanishes. At complex conjugation c, the identities $c^2 = 1$, $\Psi(c) = 1$, and $2 \in O^\times$ give $t(c) = 0$, hence $t(cg_1^{-1}c) = 0$.

In zero-based matrix units, put $V = e_{10} - e_{21}$. Negative reflected transpose $X \mapsto -JX^tJ$ fixes V, and conjugation by r_0 acts on V by α. Hence $(1 + \epsilon t V) r_0$ over $O \oplus \epsilon O/(b)$ ($\epsilon^2 = 0$) satisfies the marked local and global conditions modulo b. Choose a lift $\widetilde{s} \in O$ of s. By Lemma 30.3(1), the defining ideal I of $B = O[[X]]/(H)$ is invariant under conjugation by $I_3 + \epsilon\widetilde{s}V$, so the infinitesimal conjugation $(I_3 + \epsilon\widetilde{s}V, 1) r_0|_{D_\ell} (I_3 + \epsilon\widetilde{s}V, 1)^{-1}$ over $O[\epsilon]/(\epsilon^2)$ defines an O-valued derivation on B at $x_{0,\ell}$ supported in weight -1 that kills H. The ambient algebra A lies in $O[[X, Y]]/(H)$, where Y are the extra global variables. Take this local derivation on X, lift the values on Y in weight -1 to O, and assign value zero to all variables of other weights. The resulting linear-coefficient functional kills H, defines an O-valued derivation at x supported in weight -1, and reduces modulo b to the derivation through S, which kills F_i. The $(1,0)$ entry at g has derivative $r_0(g)_{\mathrm{mat},00}\, t(g) = \Psi(g) t(g)$, which recovers $t(g)$ because $\Psi(g)$ is a unit. □

Proof of 30.7

Proof. Use $S = S_1 \sqcup S_2$ and $t(s)$ from 30.9. Put $R = q^k$ and $F_{u,s} = Rt(s) - z_u - R - R^2 z_u^{-1}$. By (30.2) these factors are integral and

$$|Rt(s) - z_u - R| \le R(\sqrt{q} + 3) + 1 < R^2$$

since $R \ge q \ge 7$. Thus $0 < |F_{u,s}| < 2R^2$. The product $D = \prod_{u,s} F_{u,s}$ is fixed by all complex field automorphisms, so $D \in \mathbf{Z} \setminus \{0\}$ and $\log|D| \le \#\mathcal{O}\, \#S(2k \log q + 1)$. By (30.3) and $q = p^f$, $1 \le f \le 2$, take $c = 5(c_1 + c_2 |D_K|^{A_2})$ and $A = A_1 + A_2 + 1$. For each u the labels of X_u are distinct. Under ι^{-1} their factors are $q^k \Delta_{u,\rho}$. The multiplier is an ℓ-adic unit and the other factors have nonnegative valuation.

Lean: `UpperTheory.GalN.V.Win.frobenius_trace_norm_nonzero_log_bound` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.AdmissibleResidues.FrobeniusTraceCountBoundOfRatEigensystemCount`, lines 329–400)

□

Proof of 30.8

Proof. Because r satisfies the witness condition of Definition 30.1 and every place $w \mid \ell$ is unlinked or potentially unlinked, every $w \mid \ell$ is potentially unlinked. Hence at every finite place w of K, the cuspidal support of π_w consists of characters ξ of $K_w^\times$ satisfying $q_w^{t_\pi - 1/2} < |\xi(\varpi_w)| < q_w^{t_\pi + 1/2}$ for a single $t_\pi \in \mathbf{R}$. The ratio of moduli of any two support characters lies strictly between q_w^{-1} and q_w, so no two belong to a nontrivial Bernstein–Zelevinsky segment. By 40.2, the local Langlands parameter of π_w is a sum of characters with zero monodromy, and π_w is an irreducible principal series representation induced from those characters.

Since π is regular algebraic, cuspidal, and polarised, and r is irreducible, Chebotarev density and Brauer–Nesbitt identify r with the Galois representation of 39.2 attached to π, using χ^{-1} as the polarising character in that proposition, so local-global compatibility (39.2, 39.4, 39.6) applies to r at every finite place w of K: for $w \nmid \ell$ the Frobenius-semisimplified Weil–Deligne representation of $r|_{G_{K_w}}$ matches the local Langlands parameter of π_w under ι, and for $w \mid \ell$ (via the toral zero-monodromy clause of 39.6) $r|_{G_{K_w}}$ is potentially crystalline with matching Weil–Deligne representation and with Hodge–Tate weights at each $\tau : K \hookrightarrow C_\ell$ inducing w matching the infinitesimal weights of π at $\iota \circ \tau$.

For (1), given $\varphi : K \hookrightarrow \mathbf{C}$, set $\tau = \iota^{-1} \circ \varphi$ and let $w \mid \ell$ be the place induced by τ. On G_{K_w}, r is de Rham and preserves a flag whose graded pieces have inertia characters $\varepsilon_\ell^{m_i}\chi_i$ with χ_i of finite order. The Hodge–Tate weights of a de Rham representation with a stable flag are those of its graded pieces, unaffected by finite-order twists, so they equal $m = (m_1, \dots, m_n)$. Local-global compatibility 39.4 then identifies the infinitesimal character of π at φ as that determined by m.

For (2), by the restricted tensor product decomposition of π_{fin} it suffices to show that π_w has a nonzero $K_w(M)$-fixed vector at every finite place w, where $K_w(M) = \ker(\mathrm{GL}_n(\mathcal{O}_{K_w}) \to \mathrm{GL}_n(\mathcal{O}_{K_w}/M\mathcal{O}_{K_w}))$.

- Suppose $w \nmid \ell$. By the Galois level hypothesis, $r|_{I_w}$ is diagonal with characters χ_i factoring through the action on μ_M. Since the Weil–Deligne parameter of π_w has zero monodromy and matches that of $r|_{G_{K_w}}$, the inducing characters ξ_i of the principal series π_w correspond under local class field theory to χ_i on $\mathcal{O}_{K_w}^\times$ and are therefore trivial on $N_{K_w(\mu_M)/K_w}(K_w(\mu_M)^\times) \cap \mathcal{O}_{K_w}^\times$. Writing $M = p^a M'$ with p the residue characteristic of w and $p \nmid M'$, the extension $K_w(\mu_{M'})/K_w$ is unramified, while $N_{K_w/\mathbf{Q}_p}(1 + p^a\mathcal{O}_{K_w}) \subseteq 1 + p^a\mathbf{Z}_p$ acts trivially on μ_{p^a} under local reciprocity. Thus every ξ_i is trivial on $(1 + M\mathcal{O}_{K_w}) \cap \mathcal{O}_{K_w}^\times$ (which is $\mathcal{O}_{K_w}^\times$ when $w \nmid M$). Every element of $B(K_w) \cap K_w(M)$ has diagonal entries in $(1 + M\mathcal{O}_{K_w}) \cap \mathcal{O}_{K_w}^\times$, on which both $\delta_B^{1/2}$ and every ξ_i are trivial; hence the function in $\pi_w = \mathrm{Ind}_B^{\mathrm{GL}_n}(\xi_1 \otimes \cdots \otimes \xi_n)$ supported on the open double coset $B(K_w)K_w(M)$ with value 1 on $K_w(M)$ is well-defined, right $K_w(M)$-invariant, and nonzero, giving $\pi_w^{K_w(M)} \neq 0$.
- Suppose $w \mid \ell$. Since $r|_{G_{K_w}}$ is potentially crystalline and preserves a flag with graded inertia characters $\varepsilon_\ell^{m_i}\chi_i$, exactness of D_{pst} shows that the inertia action on its Weil–Deligne representation has the same characters as $\bigoplus_i D_{\mathrm{pst}}(\kappa_i)$, where $\kappa_i|_{I_w} = \varepsilon_\ell^{m_i}\chi_i$. The crystalline factor $\varepsilon_\ell^{m_i}$ acts trivially on D_{pst} inertia, leaving the finite-order characters $\chi_i|_{I_w}$. If $\ell \mid M$, each χ_i factors through the action on μ_M, so the same norm calculation as above shows that every inducing character ξ_i of π_w is trivial on $1 + M\mathcal{O}_{K_w}$, giving $\pi_w^{K_w(M)} \neq 0$. If $\ell \nmid M$, then $K_w(\mu_M)/K_w$ is unramified, so I_w fixes μ_M and $\chi_i|_{I_w} = 1$ for every i; with zero monodromy, π_w is an unramified principal series and therefore has a nonzero $\mathrm{GL}_n(\mathcal{O}_{K_w})$-fixed vector.

For (3), if $w \nmid \ell$ and π_w is spherical with Satake multiset B, then the local Langlands parameter of π_w is unramified with inverse Frobenius eigenvalues $q_w^{(n-1)/2}B$ and zero monodromy. Local-global compatibility at $w \nmid \ell$ implies that r is unramified at w and that ι sends the eigenvalue multiset of $r(\sigma_w^{-1})$ to $q_w^{(n-1)/2}B$.

For (a), let η_{K/K^+} be the quadratic idele class character of K/K^+. Its base change $\eta_{K/K^+} \circ N_{K/K^+}$ to K is trivial and its value at -1 at every real place of K^+ is -1, so $\chi = \eta_{K/K^+}^n$ satisfies $\chi_v(-1) = (-1)^n$ and polarises π.

For (b), restricting r_0 from $G_{\mathbf{Q}}$ to $G_{K'}$ preserves the Galois level $(M; m)$ because every inertia and decomposition group of K' is conjugate into the corresponding subgroup over $\mathbf{Q}$. At any prolongation A

of a place $w \mid \ell$ of K', the flag characters κ_i satisfy $\kappa_i|_{I_A} = \varepsilon_\ell^{m_i}\chi_i|_{I_A}$ with χ_i of finite order; on an open subgroup of I_A killing χ_j/χ_i, an equality $\kappa_i = \kappa_j\varepsilon_\ell$ would force $\varepsilon_\ell^{m_i-(m_j+1)} = 1$, contrary to $m_i \neq m_j + 1$ and the infinite image of ε_ℓ on I_A. Conjugating across all prolongations of w shows that w is unlinked for r in the first case. In the second case, choose an imaginary quadratic field K_0 in which ℓ splits and on which r_0 remains irreducible. The decomposition groups of K_0 above ℓ are those of $\mathbf{Q}$, so $r_0|_{G_{K_0}}$ is unlinked above ℓ, and the quadratic witness condition makes it potentially unlinked there. Hence the flag characters of r_0 at ℓ satisfy $\kappa_i^a \neq (\kappa_j\varepsilon_\ell)^a$ for all i, j and all $a \geq 1$. If H is an open subgroup of a decomposition group D and a character χ of D satisfies $\chi^a|_H = 1$, then χ^a factors through the finite quotient of D by the normal core of H, so $\chi^{ad} = 1$ for the order d of that quotient. Applied to $\chi = \kappa_i/(\kappa_j\varepsilon_\ell)$ and $H = D \cap G_{K'}$, this shows that every place of K' above ℓ is potentially unlinked for r. Finally, $\theta^{-1}\varepsilon_\ell^{1-n}$ is unramified outside $M\ell$ of cyclotomic weight $-\mathrm{wt} + 1 - n$, so by global class field theory there is an algebraic Hecke character χ' of $\mathbf{Q}$ with $\chi'_p(p) = p^{n-1}\iota(\theta(\sigma_p^{-1}))^{-1}$ for $p \nmid M\ell$, which we multiply by $\eta_{K'/\mathbf{Q}}$ if needed to arrange $\chi'_\infty(-1) = (-1)^n$ while preserving $\chi' \circ N_{K'/\mathbf{Q}}$. Comparing Satake parameters at conjugate unramified places of K' via $r_0(g)^t P r_0(g) = \theta(g)P$ and applying strong multiplicity one [13] gives $\pi^\vee \simeq \pi^c \otimes (\chi' \circ N_{K'/\mathbf{Q}} \circ \det)$. Hence π is polarised and satisfies (1)–(3). □

Proof of 30.9

Proof. Use spherical Hecke systems away from the level as CM labels. For each $\rho \in \mathcal{A}_3(K, \iota; k, N)$, the twist $r = \rho\varepsilon_\ell^{-1}$ is irreducible of Galois level $(N; (k-1, -1, -k-1))$, is separated ordinary (so every place above ℓ is unlinked), and has a regular algebraic conjugate self-dual cuspidal attachment π with witness. By Lemma 30.8(a), π has infinitesimal character determined by $(k-1, -1, -k-1)$ at every complex embedding, possesses a nonzero $K(N)$-fixed vector, and has algebraic Satake parameters $q_v B_v$ matching the inverse Frobenius eigenvalues of r at every finite place $v \nmid N\ell$ (since π_v is spherical whenever $v \nmid N$). These Frobenius polynomials determine the semisimple representation r (hence ρ) up to isomorphism by 39.2 and Chebotarev density.

Take S_1 to be the union of the $\mathrm{Aut}(\mathbf{C})$-orbits of these Hecke systems. By 26.34(2) each conjugate has a regular algebraic cuspidal representative of the same level with permuted infinitesimal weights. The spherical support bound applies to each such representative, and conjugation preserves integrality of the scaled roots. For each fixed infinitesimal-weight family, choose the algebraic coefficient representation with the negative infinitesimal multisets. Proposition 26.34(1) gives cohomology with these common coefficients. Since $K(3N) \subset K(N)$, a nonzero $K(N)$-fixed vector is fixed by the neat compact open group $K(3N)$. Choose one constituent per distinct Hecke system. They are pairwise nonisomorphic, so 26.34(3) bounds every finite selection by the cohomology dimension at level $3N$, and hence bounds the whole set S_1. Bound the cochain dimension by the cell count at level 3 times the covering index and coefficient dimension. Reduction modulo $3N$ bounds the index polynomially in N, and the Weyl dimension formula bounds the coefficient dimension polynomially in $k+1$. The same bound holds in any fixed rank $n \geq 2$ for regular algebraic polarised cuspidal automorphic representations over a CM field at level $3M$ and weight bound H.

Now consider the rational label count in rank $n \geq 2$ for irreducible representations $W : G_{\mathbf{Q}} \to \mathrm{GL}_n(C_\ell)$ of Galois level $(M; m)$ satisfying one of the two alternatives of Lemma 30.8(b), quadratic automorphy and witness, and twisted self-duality $W(g)^t P W(g) = \theta(g)P$. Choose n^2+2 imaginary quadratic fields of pairwise coprime discriminants and retain n^2+1 unramified at p. Distinct quadratic self-twists occupy independent character lines in the n^2-dimensional endomorphism representation. The index-two restriction criterion gives an irreducible restriction $W|_{G_{K'}}$ in at least one retained field K'. By Lemma 30.8(b), the cuspidal attachment π of $W|_{G_{K'}}$ over K' is polarised, has infinitesimal character determined by m, has a nonzero $K(M)$-fixed vector, and satisfies Frobenius–Satake compatibility at every place $v \nmid M\ell$ of K'. Hence $W|_{G_{K'}}$ receives a CM label from the finite conjugation-stable set of Hecke systems over K' of size at most $P(M, H)$. Any irreducible restriction to $G_{K'}$ has at most two extensions to $G_{\mathbf{Q}}$ up to isomorphism, interchanged by the quadratic twist if both occur. The restricted inverse Frobenius roots at a place $v \mid p$ of K' are the f-th powers of the rational roots, with residue degree $f \leq 2$, so each multiset admits at most 2^n lifts. Taking two copies of each pair consisting of a restricted Hecke system and a lifted multiset yields a rational label set of size at most $2^{n+1}(n^2+1)P(M, H)$, on which automorphisms of $\mathbf{C}$ act on the pair and fix the copy index. For $\rho \simeq W|_{G_K} \oplus 1 \in \mathcal{A}_{2+1}(K, \iota; k, N|D_K|)$, the representation W is irreducible over $G_{\mathbf{Q}}$ of level $(N|D_K|; (k, -k))$ with $k - (-k) = 2k \geq 2 \neq 1$, satisfies $W(g)^t J_0 W(g) = (\det W(g))J_0$ for $J_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$

with $\det W$ of cyclotomic weight zero, and satisfies the quadratic automorphy and witness conditions (via quadratic base change of its weight-$(2k+1)$ newform attachment); applying this rank-2 rational count at level $M = N|D_K|$ gives S_2 and the second bound of (30.3).

The support bounds and purity put the three inverse Frobenius root moduli in $[q^{-1/2}, q^{1/2}]$ with product one. The extreme logarithmic triples are the permutations of $(\frac{1}{2}\log q, 0, -\frac{1}{2}\log q)$. Convexity bounds the sum of moduli by $\sqrt{q}+1+q^{-1/2}$. For the block the two inverse rational roots satisfy the bounds at p and have product of moduli one. Raise to the residue degree at w and add the trivial root to obtain the same bound. We now give the integral normalisation. Let β be an algebraically normalised inverse-Frobenius root for the twisted rank-three label. For every embedding of its coefficient field into $\overline{\mathbf{Q}}_p$, the spherical comparison of 39.54 identifies these roots with the crystalline Frobenius roots. Its filtration jumps are $1-k, 1, k+1$. Weak admissibility bounds the smallest Newton slope below by $1-k$; with $v_p(q)$ as scale this gives $v_p(\beta) \geq (1-k)v_p(q)$. Hence $q^{k-1}\beta$ is p-integral. At every other finite coefficient place the corresponding continuous Galois representation has a stable integral lattice, so its Frobenius roots are units. Thus $q^{k-1}\beta$ is an algebraic integer. Undoing the twist multiplies this inverse-Frobenius root by q^{-1}, giving the factor q^k. For a rank-two rational label let γ be a lifted inverse-Frobenius root. In the quadratic field used to construct the label, γ^f is a restricted inverse-Frobenius root at a place of residue degree f over p. The same spherical comparison, now with filtration jumps $-k, k$, gives $v_p(\gamma^f) \geq -kf$ at every coefficient place over p. At every other finite coefficient place γ^f is a unit. Hence $p^k\gamma$ is an algebraic integer. Raising the two scaled rational roots to the residue degree at w and adding q^k for the trivial root proves (30.2).

Lean: `UpperTheory.GalN.V.Win.N.countK_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.FrobeniusTraceCountBoundOfCmEigensystemCount`, lines 63–125)
Lean: `UpperTheory.GalN.V.Win.N.frobenius_trace_count_bound_of_cm_eigensystem_count_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.FrobeniusTraceCountBoundOfCmEigensystemCount`, lines 418–461)
Lean: `UpperTheory.GalN.V.Win.N.frobenius_trace_count_bound_of_rat_eigensystem_count_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.FrobeniusTraceCountBoundOfCmEigensystemCount`, lines 348–398)
Lean: `UpperTheory.V.Z.Win.A4.newAdmS_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.NewAdmS`, lines 18–25)

□

Proof of 30.10

Proof. The datum requires K imaginary quadratic, so take $D = 1$ in all other fields. For $w \mid p$ use 30.7 with $z_\vartheta = \vartheta(\mathrm{N}w)^{-1}$ for complex characters of kernel H. There are $\varphi([G:H])$ for cyclic G/H and zero otherwise. A good O-valued character supplies all orbit values, and the datum conditions are invariant under powers coprime to the character order. At inverse Frobenius, $\Psi(\sigma^{-1})$ corresponds to $q^{-k}z_\vartheta$. Sum 30.5 over the orbit and apply (30.1) to the resulting sets.

For two distinct auxiliary primes with integers D, D', let S_H consist of the primes admitting good data for H in the chosen coefficient rings. Apply the bound from a prime different from ℓ for each $\ell \in S_H$. Prime factorisation and disjoint character sets give

$$\sum_{\ell \in S_H} \log \ell \sum_{\ker \vartheta = H} a_{\ell,k,\vartheta} \leq 2\log|D| + 2\log|D'|, \qquad \sum_{H \text{ occurring}} \varphi([G:H]) \leq \#G \leq N. \tag{A.259}$$

Lean: `UpperTheory.V.U3.corGK` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.CorGSumLog`, lines 37–73)
Lean: `UpperTheory.V.U3.countI` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.CountI`, lines 22–136)

□

Proof of 30.11

Proof. First identify f. The Atkin–Lehner–Li decomposition at weight h and level $\Gamma_1(M)$ expresses the cusp space as the direct sum of the images of the new spaces of levels $M' \mid M$ under $g(z) \mapsto g(dz)$, $d \mid M/M'$. Each new space is a sum of distinct eigenlines for the Hecke algebra away from M. The degeneracy maps commute with those Hecke operators. Thus each nonzero component of f comes from a newform with

eigenvalues a_p for $p \nmid M$. The lower-level exclusion gives $M' = M$, and then $d = 1$. Multiplicity one gives a single eigenline at level M. The normalization $a_1 = 1$ identifies f with its normalized primitive eigenform. This is the newform of [125, §6.1] (see [130, Theorem 9.4]).

Put $F = \mathbf{Q}(a_n : n \geq 1)$. This is a number field containing the values of ω. The restriction of ι^{-1} to F determines a place $\lambda \mid \ell$ and an embedding $E = F_\lambda \hookrightarrow \overline{\mathbf{Q}}_\ell$. Its image is $\mathbf{Q}_\ell(\iota^{-1}a_n : n \geq 1)$, since the coefficient field is generated by finitely many of the a_n. Let $U = V_E(f)$ be Kato's two-dimensional realization. For $p \nmid M\ell$, his formula on inertia invariants is a formula on all of U, and gives

$$\det(1 - u\,U(\sigma_p^{-1})) = 1 - \iota^{-1}(a_p)u + \iota^{-1}(\omega(p))p^{2e}u^2.$$

Thus U and $W(-e)$ have the same characteristic polynomial on inverse arithmetic Frobenius outside a finite set. Chebotarev density in each finite Galois quotient and continuity give equality of traces on $G_{\mathbf{Q}}$. Brauer–Nesbitt identifies the semisimplification of $U \otimes_E \overline{\mathbf{Q}}_\ell$ with $W(-e)$. The latter is irreducible of dimension two, so the former representation already has a single composition factor of dimension two. Twisting gives

$$W \simeq V \otimes_E \overline{\mathbf{Q}}_\ell, \qquad V = V_E(f)(e).$$

In particular the twist in Kato's notation has sign $+e$: at σ_p^{-1}, a twist by e multiplies each root by p^{-e} [125, §6.3 and (14.10.3)], [41, Ch. VII, Theorem 13.4].

Lean: `UpperTheory.V.Af.Newform.heckePoly` (`Development.AutomorphicArithmetic.AutomorphicVarieties.AutomorphicForms.Newform`, lines 41–41)

Lean: `UpperTheory.V.Lit.coefF` (`Development.AutomorphicArithmetic.AutomorphicVarieties.LiteratureResults.CoefFieldEq`, lines 48–49)

Lean: `UpperTheory.V.Lit.FrobPoly` (`Development.AutomorphicArithmetic.AutomorphicVarieties.LiteratureResults.CoefFieldEq`, lines 51–56)

Choose a $G_{\mathbf{Q}}$-stable $\mathcal{O}_E$-lattice $T \subset V$. Such a lattice exists because the compact Galois image has bounded translates of any lattice, and their sum is a stable lattice. Kato's Theorem 14.2(1) gives finiteness over every finite abelian extension for each stable lattice in $V_E(f)(r)$ when $1 \leq r \leq h-1$ and $2r \neq h$. Here $K/\mathbf{Q}$ has degree two, $r = e$, $1 \leq e \leq 2e = h - 1$, and $2e = h - 1 \neq h$. This includes $e = 1$. Hence $\mathrm{Sel}(K, T)$ is finite. Kato's theorem includes newforms with complex multiplication [125, Theorem 14.2(1), p. 235, and §15].

We pass from this finite group to rational coefficients. Put $A = V/T$. By [125, §14.1], $\mathrm{Sel}(K, T)$ is the subgroup of $H^1(G_K, A)$ defined at each place v by the image of $H^1_f(K_v, V)$ in $H^1(G_{K_v}, A)$. Therefore reduction modulo T maps $H^1_f(K, V)$ into $\mathrm{Sel}(K, T)$. Its image is divisible, since $H^1_f(K, V)$ is an E-vector space. A finite group has zero divisible subgroup.

Let $z : G_K \to V$ represent a finite class. For each $n \geq 0$ the class of $\ell^{-n}z$ reduces to zero in $H^1(G_K, A)$. Lift a vector giving its coboundary from A to V, and multiply that lift by ℓ^n. The resulting vector $x_n \in V$ satisfies

$$z(g) - (gx_n - x_n) \in \ell^n T \qquad \text{for every } g \in G_K.$$

The coboundaries form a finite-dimensional E-subspace of the space of continuous functions $G_K \to V$ with its supremum norm. This subspace is closed because E is complete. The displayed uniform convergence puts z in that subspace. Thus $H^1_f(K, V) = 0$.

The same vanishing holds with coefficients in $\overline{\mathbf{Q}}_\ell$. To see this for an arbitrary continuous cocycle z with these coefficients, realize it in the representation

$$g \longmapsto \begin{pmatrix} V(g) & z(g) \\ 0 & 1 \end{pmatrix} \quad (g \in G_K).$$

Its image is contained in the matrices over a finite extension E'/E. Indeed, the finite subextensions of $\overline{\mathbf{Q}}_\ell/E$ are countable: by Krasner's lemma each is generated by a root of a polynomial over a fixed countable dense subfield of E. The preimages of the matrix groups over these finite extensions are closed subgroups covering G_K. Baire's theorem gives one with interior, hence of finite index. Adjoining the entries of finitely many coset representatives gives E'.

The local conditions descend to E'. Away from ℓ, apply an E'-linear retraction $\overline{\mathbf{Q}}_\ell \to E'$ to a vector giving the inertia coboundary. At $v \mid \ell$, apply the same retraction on coefficients, tensored over $\mathbf{Q}_\ell$ with the identity of B_{cris}, to the vector giving the crystalline coboundary. Both operations commute with Galois

because the matrices of V are over E. In an E-basis of E', the coordinates of z are now continuous V-valued cocycles satisfying the finite conditions. Each is a coboundary by the preceding paragraph. Recombining their cobounding vectors proves $H^1_f(K,W)=0$.

Finally take $D=W$ or $D=W\otimes\theta$. Restriction takes $H^1_f(\mathbf{Q},D)$ into $H^1_f(K,W)$, since $\theta|_{G_K}=1$. At $w\mid p\neq\ell$ an inertia coboundary restricts to an inertia coboundary on I_w, including when p ramifies in K. At $w\mid\ell$ a vector giving the B_{cris}-coboundary still gives it on G_{K_w}. The archimedean H^1 with rational coefficients is zero. The two restrictions correspond respectively to the $+1$ and -1 actions of $\mathrm{Gal}(K/\mathbf{Q})$ on classes over K, since twisting multiplies the action of its nonidentity element by $\theta=-1$. For each choice of D, corestriction after restriction on $H^1(G_{\mathbf{Q}},D)$ is multiplication by $[K:\mathbf{Q}]=2$. Both restrictions have zero image and are injective. This proves the two assertions. □

Proof of 30.13

Proof. Put $\phi=\delta^{-1}\chi^{-1}$ and take the primitive newform g attached to $V\phi\varepsilon_\ell^{-k}$. The flag of $V\phi$ meets the spherical test. Strong multiplicity one identifies the norm-twisted adelisation of g with the tested attachment [13]. If $\ell\mid M_g$, replacing the vector at ℓ by a spherical vector and retaining the other local vectors and the lowest-weight vector at infinity gives the same good Hecke eigenvalues at level M_g/ℓ, contrary to primitivity.

Let κ_0,κ_1 be the flag characters. For a Frobenius lift σ at A, put $u=(\kappa_0\phi)(\sigma)\varepsilon_\ell(\sigma)^{-k}$. The spherical comparison gives $b\in\ell^{1/2}B$, where B is the Satake multiset at ℓ of the tested attachment, with $(\ell^k b)^{-1}=\iota(u)$. The Hecke polynomial of g at ℓ has roots $\ell^k b'$ for $b'\in\ell^{1/2}B$. Compactness gives $|u|_\ell=1$, and the determinant formula gives valuation $2k$ for the other root. Hence $a_\ell(g)$ is a unit. If the unit root were a root of unity, the other root would have complex modulus ℓ^{2k}. The Hecke recurrence at ℓ^j would contradict (30.7).

The inertia exponent excludes $\kappa_1=\varepsilon_\ell$. Equality $\kappa_0=\varepsilon_\ell$ would force $k=1$ and $u=\phi(\sigma)$, a root of unity. In a stable lattice over finite coefficients, extend a basis of the saturated intersection with the first line. These matrices satisfy (30.4) with $h=\ell-1$. The source identity holds with $\xi=\chi^{-1}$. Take S to contain ℓ and the ramified primes.

Lean: `GlobalTheory.Ord2.sources_of_spherical` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TwistNewform`, lines 17–112)
Lean: `GlobalTheory.NewLevel.not_dvd_level_of_satake_hecke_poly_cofinite` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.UnramifiedAlgebraicWeights`, lines 59–133)
Lean: `UpperTheory.V.Z.Kato.WeilSphericalRecords.frobRoot_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.FrobRoot`, lines 13–26)

□

Proof of 30.15

Proof. Cuspidality bounds $y^{k+1/2}|g'(x+iy)|$. Integrate its square at $y=X^{-1}$ and apply Parseval for (30.7), also before depletion. Put $M=N\bmod(\xi)$ and $b(n)=\xi(n)a_n n^{-k}$ on integers prime to M, zero elsewhere. Partial summation gives

$$\sum_{n\le X}|b(n)|^2\ll X,\qquad S(1+s)=\sum_n|b(n)|^2n^{-1-s}\ll s^{-1}\quad(0<s\le1).\tag{A.260}$$

The series $L_b(z)=L(g'\otimes\xi,z+k)$, with the indicated omission, is holomorphic near the positive real axis by the Mellin integral of the twisted cusp form. It converges absolutely for real $z>1$ by Cauchy–Schwarz.

At a good prime let α,β be the normalised twisted roots, with $|\alpha\beta|=1$. The prime-power bound gives $|\alpha|,|\beta|\le p^{1/2}$ by the generating function $(1-\alpha x)^{-1}(1-\beta x)^{-1}$. Multiplying geometric series gives

$$\sum_{r\ge0}|b(p^r)|^2x^r=\frac{1-x^2}{\prod_{u,v\in\{\alpha,\beta\}}(1-u\bar v x)}.$$

At $x=p^{-z}$ the local logarithm of $\zeta(z)|L_b(z)|^2S(z)\zeta(2z)$ is $\sum_{j\ge1}|1+\alpha^j+\beta^j|^2x^j/j$. The zeta factors at omitted primes also have nonnegative logarithms. Absolute convergence gives nonvanishing for $z>1$ and,

for each finite set T of good primes,

$$\exp\left(\sum_{p\in T}|1+b(p)|^2p^{-1-s}\right)\le 2(1+s^{-1})|L_b(1+s)|^2S(1+s). \tag{A.261}$$

The Petersson adjoint is $T_p^*=\overline{\omega(p)}T_p$, so $a_p=\omega(p)\overline{a_p}$. For $p\equiv -1 \bmod M$ we have $\omega(p)=-1$ by odd weight and $\xi(p)=\pm1$, so $|1+b(p)|^2\ge 1$. If $L_b(1)=0$, holomorphy and (A.260) bound the right side of (A.261) as $s\downarrow 0$. Take the limit on finite sets of primes in this progression and use the divergence of their reciprocal sum [121]. □

Proof of 30.16

Proof. Use the local domain $B=B_\ell$ of Definition 29.18. The values $d_i=\delta_i(s_0)$ are powers of $\varepsilon_\ell(s_0)$ with distinct exponents $k+1,k,1,0$. After inverting their differences the projectors onto the four eigenlines of $b_\ell(s_0)$ are $P_i/\prod_{j\ne i}(d_i-d_j)$. The local equations say that their successive partial sums define a stable flag and that inertia acts on its i-th quotient by δ_i. The differences are nonzero in O, so both characteristic-zero specializations preserve this calculation.

For $g\in D_\ell$ let $z_i\in \operatorname{Frac}B$ be the trace of the i-th projector times $b_\ell(g)$. It is an eigenvalue of $b_\ell(g)$, hence integral over B. The finite domain $B[z_i]$ is local, since a finite algebra over a Henselian local ring is a product of local rings. Its two evaluations are defined by the projector formula and send z_i to $\delta_i(g)$ and $\lambda_i(g)$. The first evaluation has image O, by integrality and integral closedness of O. Thus $z_i-\delta_i(g)$ belongs to the maximal ideal. Under the second evaluation every element a of this ideal has image of absolute value less than one. Indeed, if $|a'|\ge 1$ for its image a', the algebraic number $1/a'$ is integral over $\mathbf{Z}_\ell$. Its monic equation, multiplied by a power of a', would send an element of $1+aB[z_i]$ to zero. This proves the reduction assertion.

Choose an eigenbasis of the marked element, ordered by the four projectors. The representation is upper triangular in this basis, so the continuous matrix coefficients on the diagonal are the characters λ_i.

Claim 1. The characters in the two opposite pairs have product ν.

Let J' be the alternating-form matrix in the ordered eigenbasis. The similitude relation at s_0 implies $J'_{ij}=0$ unless $d_id_j=\nu(s_0)$. The four d_i are distinct, and $d_id_{3-i}=\nu(s_0)$ by Definition 29.17. Thus J' is antidiagonal with nonzero antidiagonal entries. For $g\in D_\ell$, the $(i,3-i)$-entry of the similitude equation $r(g)^tJ'r(g)=\nu(g)J'$, in the eigenbasis, has only the contribution $J'_{i,3-i}\lambda_i(g)\lambda_{3-i}(g)$. Indeed upper triangularity restricts the two summation indices to at most i and $3-i$, respectively, and a nonzero term must have their sum equal to 3. Cancelling $J'_{i,3-i}$ gives the claim.

A positive power of s_0 belongs to U. Equality of λ_i and $\varepsilon_\ell\lambda_j$ there would force the exponent at index i to be exactly one more than the exponent at index j. Indices in either one of the two coordinate planes have exponent difference $k\ge 2$. For indices in different planes the residual ratio at σ_ℓ, or its inverse, is $\overline{\xi\Psi^b}(\sigma_\ell)$ for some integer b, since $\varepsilon_\ell(\sigma_\ell)=\omega(\sigma_\ell)=1$. Definition 29.17 excludes equality at this element of U. □

Proof of 30.17

Proof. Each A-line $A(\kappa_i)$, viewed over E, is an unramified representation twisted by $\varepsilon_\ell^{w_i}$, hence is crystalline. Induct on the flag. Let P be the preceding crystalline subrepresentation and let $L=A(\kappa_j)$ be the next quotient. The extension class belongs to $H^1(K,V)$ for $V=\operatorname{Hom}_E(L,P)$. This representation is crystalline. Filtering the A-lines by powers of the maximal ideal gives character factors $\bar\kappa_i/\bar\kappa_j$ in V, with $i<j$. The corresponding factors of $V^\vee(1)$ have no invariants by hypothesis, so $H^0(K,V^\vee(1))=0$. The cyclotomic exponents of V are $w_i-w_j>0$. Consequently $\operatorname{Fil}^0D_{\mathrm{dR}}(V)=0$, with the convention that $\mathbf{Q}_\ell(1)$ has filtration jump -1.

The crystalline dimension formula [120, Corollary 3.8.4], local Tate duality and the local Euler characteristic formula [122, Ch. II, §5] now give

$$\dim_E H^1_f(K,V)=\dim_E H^0(K,V)+[K:\mathbf{Q}_\ell]\dim_E V=\dim_E H^1(K,V). \tag{A.262}$$

Thus the extension class dies over B_{cris}. A splitting there identifies the middle term after extension of scalars with the sum of its two crystalline ends. Taking period invariants gives its full dimension, so the middle term is crystalline. □

Proof of 30.18

Proof. Write $\omega = \bar{\varepsilon}_\ell$. We record the residual test used in choosing the tower. Its initial CM field has an element t with $\omega(t) = 1$ at which the four residual diagonal characters have distinct values. In II the values, up to the choice of order, are

$$-b, \quad -1, \quad b, \quad 1, \qquad b = \bar{\xi}(t), \quad b^2 \neq 1.$$

Here the element with $\bar{\Psi}(t) = -1$ can be multiplied, if necessary, by the element with $\bar{\Psi} = 1,\ \omega = 1,\ \bar{\xi}^4 \neq 1$ in the same CM subgroup. In III there are two choices of initial field. The rigidity comparison uses the field cut out, before the auxiliary quadratic conditions, by $\bar{\xi}^2 = \bar{\Psi}^\epsilon$, where $\epsilon \in \{1, -1\}$ is the chosen label order. Dividing the ordered characters by the first gives $1, b^{-1}, b^{-2}, b^{-3}$ at its marking, with $b = \bar{\xi}(t)$. Here $\omega(t) = 1$ and b is the negative of a nontrivial element of odd order, so its first four positive powers differ from one. The tensor seed in 30.51 uses a separately chosen field on which $\xi = \lambda_{K'}$. At its marking $\omega(t) = 1$, $\xi(t) = -1$, and the four characters divided by the first have values $1, -1, z^{-1}, -z^{-1}$, with $z = \bar{\Psi}(t)^\epsilon$ of odd order greater than one. These values are distinct since ℓ is odd. In this construction $-z$ has order twice the order of z, hence exceeds four. In either construction use the marking belonging to the chosen field. In the tower enlargement a power t^{ℓ^a} belongs to G_L. Raising to this power preserves distinctness in the residue field. In particular the four residual characters restrict distinctly to G_L. Also, a sign character of G_L factoring through ω and permuting the residual characters by multiplication must be trivial. Indeed it takes value one at the marking, so its permutation fixes every index. A common residual twist cancels from this test.

Lean: `UpperTheory.V.Ws.elts2` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.FrobeniusFields.Elts2`, lines 19–52)
Lean: `UpperTheory.V.Ws.ok2A` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.FrobeniusFields.OddPowNe`, lines 55–177)
Lean: `UpperTheory.V.Ws.exists_tame_tower_ok_three` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.NegSqPowNeOneOfOddOrder`, lines 181–280)

First $r|_{G_L}$ is irreducible. Otherwise its semisimple restriction has a nonscalar endomorphism. The finite abelian group $\mathrm{Gal}(L/\mathbf{Q})$ acts by conjugation on the endomorphisms commuting with G_L, with scalar fixed space by Schur's lemma. A nontrivial eigenspace therefore supplies an intertwiner with a nontrivial character twist of r. This intertwiner is invertible by irreducibility over $G_{\mathbf{Q}}$. Choose g where the twisting character differs from one. The trace of $r(gh)$ vanishes for all $h \in G_L$. Reducing gives

$$\sum_i \bar{\delta}_i(g)\bar{\delta}_i(h) = 0 \qquad (h \in G_L),$$

contrary to independence of the distinct characters $\bar{\delta}_i|_{G_L}$. This also proves irreducibility over K.

Put $H = G_{L(\mu_{\ell^\infty})}$. There is an element of H with simple spectrum. We now give the local argument used to obtain it. At a place of K over ℓ, the tower splits $K/\mathbf{Q}$ locally. Lemma 30.16 separates the ordered characters from their cyclotomic translates for $i < j$. For the reverse pairs the distinct inertia weights exclude equality, and the cyclotomic character is nontrivial for the equal-index pairs. Thus the witness of Definition 29.5 applies at this place. Choose a Weil element of positive residue degree d with cyclotomic value one. Such an element is obtained by raising a Frobenius lift to a power and multiplying by inertia, since the cyclotomic image of inertia is open in the local cyclotomic image. Write q for the residue cardinality. In the ordinary local comparison the moduli of the smooth characters at this element are the moduli of the ordinary characters with their respective algebraic weight factors $q^{\pm d m_i}$, with the same convention for all i. Their exponents at a uniformiser lie in one open interval of length one by the witness. A root-of-unity ratio of two ordinary eigenvalues would therefore give $|m_i - m_j| < 1$. The weights $k+1, k, 1, 0$ are distinct integers. A positive power of the element belongs to G_L and still separates all four eigenvalues.

Lean: `UpperTheory.V.Z.Win.RR.free_core_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.RRFree`, lines 97–116)

Suppose the restriction to H is reducible. As in Lemma 29.6, it is semisimple with transitively permuted isotypic summands. Their multiplicity is one, since repeated copies give repeated eigenvalues at the separating element. The number of summands is therefore two or four. The action on the summands factors through the abelian cyclotomic quotient of G_L. The stabiliser of one summand fixes them all, and the resulting transitive abelian permutation group acts regularly. It has a nontrivial sign character η. The stabiliser is closed, being the stabiliser of a subspace, and has finite index, so η is continuous. Multiplication by the sign of the coset on each summand intertwines $r|_{G_L}$ with its twist by η. Since ℓ is odd, the pro-ℓ subgroup of the cyclotomic image lies in $\ker \eta$. Thus η factors through ω, and reduction preserves its two possible values. Reduction of the isomorphism of twists shows that $\bar{\eta}$ permutes the four residual characters. The marking test gives $\bar{\eta} = 1$, a contradiction. Finally, a scalar twist preserves all invariant subspaces. □

Proof of 30.19

Proof. Write $r' = (1 + \epsilon z)r$, with $\epsilon^2 = 0$. The fixed similitude puts z in $\mathfrak{sp}_4$, and multiplication gives

$$z(gh) = z(g) + \mathrm{Ad}(r(g))z(h).$$

Away from ℓ, average over the finite inertia quotient through which the lift factors to obtain an inertia coboundary.

Work over a finite coefficient field E at ℓ. The marked inertia eigenvalue differences are units over the dual numbers, so the lifted ordered flag reduces to the prescribed flag. Write $\chi = (\vartheta, \omega)$ on the local group, let $\sigma = \sigma_\ell$ be the arithmetic Frobenius lift of Definition 29.17, and let f be the unramified character with values in $\operatorname{im}\chi$ and $f(\sigma) = \chi(\sigma)$. On $U = \ker(\chi f^{-1})$ the inertia kills both finite characters. The graded characters there have cyclotomic weights $k+1, k, 1, 0$. Since $\sigma \in U$, Lemma 30.16 gives $\kappa_i \neq \varepsilon_\ell \kappa_j$ on U for $i < j$, where κ_i are the graded characters of r.

Apply Lemma 30.17 over the fixed field of U, with $A = E[\epsilon]/(\epsilon^2)$. Its inertia hypothesis holds for the lifted characters, and the reduction of the ordered lifted flag is the flag just considered. The resulting crystalline self-extension splits over B_{cris} on U. Average a splitting over the finite quotient D_ℓ/U to obtain a D_ℓ-equivariant splitting. The symplectic projection $Y \mapsto (Y - J^{-1}Y^tJ)/2$, for the matrix J of the alternating form, is equivariant under conjugation by a similitude and fixes the values of z. Hence z satisfies the finite local condition.

Lean: `tb_atL` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.TbTriDiag`, lines 287–374)
Lean: `UpperTheory.V.Loc.pr6` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PadicHodgeTheory.CrystallineTransferSurjection`, lines 70–97)

Take the fitted tower for the ordinary model, with quadratic field K and CM compositum L. The residual marking argument in Lemma 30.18 gives irreducibility on G_K, so the attachment hypothesis applies there. That lemma then gives irreducibility of the twisted representation on $G_{L(\mu_{\ell^\infty})}$. Its image in the level group is the full subgroup Γ' of 30.62, because the quotient respects the cyclotomic character. For the symplectic form matrix J, put $X^\dagger = J^{-1}X^tJ$ and $P(X) = (X - X^\dagger)/2$. The projection P commutes with $\mathrm{Ad}(r(g))$. For the sign η of K use the polarised adjoint action

$$R(g)X = \mathrm{Ad}(r(g))P(X) + \eta(g)\,\mathrm{Ad}(r(g))(X - P(X)).$$

On G_K this is the conjugation action. The scalar twist over L preserves the adjoint. Replacing a complex conjugation c_K by wc_K, for $w \in G_K$, replaces its polarising matrix by its right product with $r(w)^{-1}$. On excluded inertia the adjoint action is trivial, so the local coboundary vanishes there and z descends to the frame over L^+. Solvable base change transfers the attachment and witness. By 30.62, the affine action $Y \mapsto R(g)Y + z(g)$ has a G_{L^+}-fixed point. Its $G_{\mathbf{Q}}$-orbit is finite. Project the barycentre to $\mathfrak{sp}_4$ to obtain X with $z(g) = X - \mathrm{Ad}(r(g))X$.

Lean: `UpperTheory.V.Z.Win.Dn.exists_cm_field_twist_h1` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.ExistsCmFieldTwistH1`, lines 28–103)
Lean: `UpperTheory.V.Th.adDown` (`Development.AutomorphicArithmetic.AutomorphicVarieties.TheoremsMuLinked.GaloisRepresentationTypes.AdDown`, lines 92–193)

□

Proof of 30.20

Proof. Set $\Phi(h \otimes m) = h(u)i(m)$. The elementary symmetric relations in the statement make this well defined. First work at one place and omit j. In the polynomial ring on the ordered roots set

$$\Delta = \prod_{a<b}(x_b - x_a), \qquad H = \prod_{a<b}(x_b - Qx_a), \qquad \partial h = \Delta^{-1}\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\sigma h.$$

The alternating numerator is divisible by each $x_b - x_a$, so ∂h is a symmetric polynomial. These integral identities may first be calculated with indeterminate Q.

With the Iwahori subgroup of volume one, $iT = \sum_\sigma T_\sigma$: the finite flag variety partitions into the Bruhat cells indexed by σ. Write E_0 for this sum. In the polynomial representation the Bernstein relation for the adjacent transposition $s = (a, a+1)$ gives

$$T_s h = Q\, s(h) + (Q-1)x_a \frac{h - s(h)}{x_a - x_{a+1}}.$$

Here the split group has root system of type A_{n-1}, all simple parameters equal Q, and the denominator divides the indicated difference. This is the Bernstein presentation in [43, §1.15]. Since $T_s E_0 = QE_0$, the output of E_0 is symmetric. In its expansion $\sum_\sigma c_\sigma \sigma$ over the rational function field this gives $c_{\tau\sigma} = \tau(c_\sigma)$, by linear independence of the distinct field automorphisms. The longest permutation occurs only in its own reduced word. Its coefficient is the product over inversions of $(x_a - Qx_b)/(x_a - x_b)$, the longest permutation applied to H/Δ. Hence $E_0 h = \partial(Hh)$. The calculation gives identities in the Hecke algebra: the expansions of the T_σ in permutations are triangular in Bruhat order with nonzero leading coefficients, and the Bernstein basis is free over the root-polynomial algebra. Polynomial identities so obtained specialize at the given Q.

Let A run over subsets of the pairs $a < b$. Put $a_A = \prod_{(a,b)\in A} x_b$, $c_A = \prod_{(a,b)\notin A}(-x_a)$. Then

$$\sum_A a_A\, \sigma(c_A) = \prod_{a<b}(x_b - x_{\sigma(a)}) = \begin{cases} \Delta, & \sigma = 1, \\ 0, & \sigma \neq 1. \end{cases}$$

Indeed a nonidentity permutation has some $\sigma(a) > a$. Define $\Psi_0(v) = \sum_A a_A(x) \otimes T(c_A(u)v)$. This map intertwines the roots. To see this before specialization, take two lists x, y with the same elementary symmetric functions. The tensor $\sum_A a_A(x)c_A(y)$ is killed by $x_i - y_i$: the identity just proved shows this after substituting every permutation of x for y. These substitutions detect identities. The ordered splitting algebra in y is free over the polynomial ring in x, by successive monic root adjunctions, and over its fraction field it splits as the product over the permutations. Symmetric functions can be moved through T since they are central. Specializing the tensor identity proves the required intertwining, including for coincident specialized roots.

On V the composite $\Phi\Psi_0$ is $\sum_A a_A E_0 c_A$. Substituting $E_0 = \sum_\sigma \sigma(H/\Delta)\sigma$ and using the subset identity gives multiplication by H. On $h \otimes m$ the other composite gives $\sum_A a_A \partial(Hc_A h) \otimes m = Hh \otimes m$, by the same identity. Here $T(p(u)i(m)) = \partial(Hp)m$, since an Iwahori simple generator acts by Q on a spherical vector and the symmetric coefficients act as specified.

For several places take the product of the forward operations and the sum over tuples of subsets in the reverse operation. The actions at distinct places commute, so both composites are multiplication by $\prod_j H_j$. Multiply the reverse map by the polynomial $f/\prod_j H_j$, which exists since every factor $x_{jb} - Q_j x_{ja}$, $a < b$, occurs in f. All operations commute with coefficients through their action on the common representation. Intertwining gives the same identities after applying the indicated idempotents. □

Proof of 30.21

Proof. Enlarge the coefficient field to contain the character values and the images of the embeddings. The rank-one local reciprocity comparison gives κ_i filtration jump $h_{i,\tau}$ at τ and smooth Weil character $\omega_i \circ a$. Indeed, its restriction to an open subgroup of the units is the displayed algebraic character. The rank-one crystalline periods for the embeddings give its indicated jumps. Removing that algebraic factor leaves the displayed smooth character on both units and a uniformiser. Reciprocity is normalized as in 39.6.

The inertia action on the Weil–Deligne space factors through a finite group. Over the coefficient field of characteristic zero it splits into its character spaces. The polynomial hypothesis gives each of the n distinct inertia characters multiplicity one. Each extends to W_K, so its line is W_K-stable. On the i-th line the Weil character is $\omega_i \circ a$. To see the identification at a Frobenius lift s, take traces at su for inertia elements u and use linear independence of the inertia characters as functions of u. The monodromy operator commutes with inertia. On each line it is a nilpotent scalar and hence zero.

Twist by κ_0^{-1}. The line indexed by zero in the resulting period module has trivial Weil character. As a (φ, N)-module with descent data it is the rank-one module of the trivial character: the inertia action is trivial, and the linear Frobenius for one residue-field period acts as one. Descent for the semilinear Frobenius gives a Frobenius-fixed generator. Its Newton number is zero. At every embedding all jumps of the ambient twisted filtration are $h_{i,\tau} - h_{0,\tau} \geq 0$, so the induced filtration on the line has nonnegative Hodge number. Weak admissibility bounds this number above by its Newton number. Thus every labelled jump of the line is zero.

Over a finite Galois extension where the representation is semistable, this filtered line is weakly admissible. The equivalence for filtered (φ, N)-modules gives a subrepresentation, and full faithfulness transports the descent data on the inclusion. The line with its jump-zero filtration and trivial descent character gives the trivial subrepresentation of the twist. Here the base is a finite extension of $\mathbf{Q}_\ell$, and both the ambient module and the line have equal Hodge and Newton numbers, as required in [137, Thm. A and §4.1]. Untwisting gives a subline of character κ_0. Exactness on semistable representations, with the induced and quotient filtrations, removes its Weil character and its one labelled jump at each embedding. The quotient satisfies the same hypotheses for indices $1, \dots, n-1$. Induction gives the flag. □

Proof of 30.22

Proof. Use principal level one at v_0 and a free coefficient lattice O^b for the tame type τ. We first construct the tame type τ and verify its exact Weil–Deligne characterization (even when two or more residual characters have equal inertial restrictions). Fix a split tame type place u with local field $k = L_u$, integer ring $\mathcal{O}_k$, maximal ideal $\mathfrak{m}_k$, and maximal compact subgroup $K_u = \mathrm{GL}_n(\mathcal{O}_k)$. Let $\chi_1, \dots, \chi_n : I_k \to \mathbf{C}^\times$ be the Teichmüller lifts of the ordered residual characters (each extending to W_k, corresponding via local Artin reciprocity to smooth characters of $k^\times$). Group these n characters by equality of their restrictions to $\mathcal{O}_k^\times$ into s distinct unit-character classes $\mu_1, \dots, \mu_s : k^\times \to \mathbf{C}^\times$ with multiplicities $d_1, \dots, d_s > 0$ ($\sum_{a=1}^s d_a = n$). For $a \neq b$, let $c_{ab} = \mathrm{cond}((\mu_a/\mu_b)|_{\mathcal{O}_k^\times}) \geq 1$ be the conductor exponent on units; by the ultrametric inequality $c_{ac} \leq \max(c_{ab}, c_{bc})$, order $\mu_1, \dots, \mu_s$ so that $c_{a'b'} \leq c_{ab}$ whenever $a \leq a' < b' \leq b$. Let $P = MN$ be the standard block upper-triangular parabolic of GL_n for the partition $(d_1, \dots, d_s)$ with Levi $M = \prod_{a=1}^s \mathrm{GL}_{d_a}$ and opposite unipotent radical $\overline{N}$, write $\mathrm{bl}(i) \in \{1, \dots, s\}$ for the block of index i, and define the compact open block-parahoric subgroup

$$J = \big\{ g \in \mathrm{GL}_n(\mathcal{O}_k) : g_{ij} \in \mathfrak{m}_k^{c_{\mathrm{bl}(j),\mathrm{bl}(i)}} \text{ whenever } \mathrm{bl}(j) < \mathrm{bl}(i) \big\}.$$

In particular, J retains the full maximal compact subgroup $\mathrm{GL}_{d_a}(\mathcal{O}_k)$ on every equal-character block of size d_a (so when characters coincide, e.g. a trivial block $d_1 = 2$, J uses $\mathrm{GL}_2(\mathcal{O}_k)$ rather than an Iwahori subgroup, excluding Steinberg representations). By the ultrametric ordering, J is an open subgroup of K_u with Iwahori decomposition $J = (J \cap \overline{N})(M \cap K_u)(N \cap K_u)$, and

$$\lambda(g) = \prod_{a=1}^{s} (\mu_a/\mu_{a+1})(\det g_{\leq a}) \qquad (\mu_{s+1} = 1),$$

where $g_{\leq a}$ is the leading $(\sum_{b \leq a} d_b) \times (\sum_{b \leq a} d_b)$ block, defines a smooth character $\lambda : J \to \mathbf{C}^\times$ trivial on $J \cap \overline{N}$ and $N \cap K_u$ whose restriction to $M \cap K_u = \prod_a \mathrm{GL}_{d_a}(\mathcal{O}_k)$ is $\eta(m) = \prod_{a=1}^s \mu_a(\det m_a)$. Inside the finite-dimensional smooth K_u-representation $V_0 \subset \mathrm{Ind}_{P \cap K_u}^{K_u}(\lambda)$ right-invariant under the principal congruence subgroup of level $1 + \sum_a \mathrm{cond}(\mu_a|_{\mathcal{O}_k^\times})$, choose an irreducible K_u-subrepresentation ξ_u onto which the (J, λ)-eigenvector $v_0(g) = \mathbf{1}_J(g)\lambda(g)$ projects nontrivially. We claim that an irreducible smooth representation π_u of $\mathrm{GL}_n(k)$ satisfies $\mathrm{Hom}_{K_u}(\xi_u, \pi_u) \neq 0$ if and only if $\mathrm{rec}(\pi_u)$ has zero monodromy and inertial characteristic polynomial $\prod_{i=1}^n (X - \chi_i)$. For the forward direction, a nonzero K_u-map $\xi_u \to \pi_u$ carries the projected

image of v_0 to a nonzero vector $x \in \pi_u^{(J,\lambda)}$. For a uniformizer $\varpi \in k^\times$ and strictly dominant Levi central element $z = \mathrm{diag}(\varpi^{s-1} I_{d_1}, \dots, I_{d_s})$, the operator $T_z(x) = \sum_{u \in (N \cap K_u)/(z(N \cap K_u)z^{-1})} \pi_u(uz)x$ preserves $\pi_u^{(J,\lambda)}$ and is injective there. Indeed every non-integral $y \in N \setminus (N \cap K_u)$ fails to intertwine (J, λ). Put $Y = y^{-1}$. Since Y is non-integral, there is an entry Y_{pq} with $\mathrm{bl}(p) < \mathrm{bl}(q)$ of maximal absolute value in its row and in its column, and strictly larger than every Y_{pi} with $\mathrm{bl}(i) < \mathrm{bl}(q)$ and every Y_{iq} with $\mathrm{bl}(i) > \mathrm{bl}(p)$. For $a = \mathrm{bl}(p)$ and $b = \mathrm{bl}(q)$ choose $t \in 1 + \mathfrak{m}_k^{c_{ab}-1}$ with $\mu_a(t) \neq \mu_b(t)$. Let j be the identity matrix with $(t-1)Y_{pq}^{-1}$ times row p of Y added to its row q. Then $y^{-1}jy$ is the identity matrix with $(t-1)Y_{pq}^{-1}$ times column q of Y added to its column p. By the choice of (p, q) and the ultrametric ordering of the conductors both matrices lie in J, and $\lambda(j) = \mu_b(t) \neq \mu_a(t) = \lambda(y^{-1}jy)$. Hence the (J, λ)-projection $e_{J,\lambda}$ kills $\pi_u(y)x$ for $x \in \pi_u^{(J,\lambda)}$. For a coset representative $n' \notin z(N \cap K_u)z^{-1}$ the element $z^{-1}n'z$ is non-integral, so $e_{J,\lambda}\pi_u(z^{-1})T_z x = x$. Since any vector in the kernel of the Jacquet map $\pi_u \to r_N(\pi_u)$ is annihilated by a power of T_z, x has nonzero image in the finite-length Jacquet module $r_N(\pi_u)$ transforming under $M \cap K_u$ by η. Taking an irreducible M-subquotient $\bigotimes_{a=1}^s \tau_a$ carrying a nonzero $(M \cap K_u, \eta)$-vector shows that each $(\mu_a^{-1} \circ \det) \otimes \tau_a$ has a nonzero $\mathrm{GL}_{d_a}(\mathcal{O}_k)$-fixed vector and is therefore spherical (unramified with zero monodromy). Because the unit restrictions $\mu_a|_{\mathcal{O}_k^\times}$ are pairwise distinct across distinct blocks $a \neq b$, the cuspidal support of π_u, which is that of $\bigotimes_a \tau_a$, splits into s inertial lines, and π_u is the irreducible product $\pi_1 \times \cdots \times \pi_s$ of representations of the $\mathrm{GL}_{d_a}(k)$ supported on these lines [14, 16]. In the geometric lemma for $r_N(\pi_1 \times \cdots \times \pi_s)$, every term other than that of the identity double coset moves part of some line into another block and lies in a different Bernstein component of M. The component containing $\bigotimes_a \tau_a$ is therefore $\pi_1 \boxtimes \cdots \boxtimes \pi_s$, so $\tau_a \simeq \pi_a$ and the parabolic induction $\mathrm{Ind}_P^{\mathrm{GL}_n(k)}(\tau_1 \boxtimes \cdots \boxtimes \tau_s)$ is irreducible and isomorphic to π_u; local Langlands compatibility yields $\mathrm{rec}(\pi_u) = \bigoplus_{a=1}^s \mathrm{rec}(\tau_a)$, which has zero monodromy and the prescribed diagonal inertia. Conversely, if $\mathrm{rec}(\pi_u)$ has zero monodromy and inertial multiset χ, decomposing it into its s inertial-character isotypic summands of dimensions d_a yields twisted unramified parameters whose local Langlands correspondents τ_a are twisted spherical on $\mathrm{GL}_{d_a}(k)$; their irreducible parabolic induction is isomorphic to π_u, and Iwasawa restriction to K_u embeds V_0 (and hence ξ_u) into $\pi_u|_{K_u}$. Finally let $U_u \subset K_u$ be the open kernel of ξ_u. In the functions on the finite group K_u/U_u let I_u consist of those x with $\sum_g x(g)\pi_u(g)v = 0$ for all $v \in \pi_u^{U_u}$ and all irreducible π_u with $\mathrm{Hom}_{K_u}(\xi_u, \pi_u) = 0$. It is stable under left translation. The matrix coefficients $g \mapsto (\xi_u(g^{-1})w)_i$ belong to I_u, and by averaging one of them acts nontrivially on $\pi_u^{U_u}$ exactly when $\mathrm{Hom}_{K_u}(\xi_u, \pi_u) \neq 0$. The Teichmüller values lie in O, so this condition is $\mathrm{Aut}(\mathbf{C}/O)$-invariant. Hence I_u is stable under these automorphisms and is spanned by its O-valued elements. These form a free O-module Λ_u of finite rank, stable under left translation, with a basis (b_j). Let τ_u be the contragredient of left translation on Λ_u in this basis. For $v \in \pi_u^{U_u}$ the tuple $F_j = \sum_h b_j(h)\pi_u(h)v$ satisfies $\sum_j \tau_u(k)_{ij}\pi_u(k)F_j = F_i$ for $k \in K_u$, and it is nonzero when some element of I_u acts nontrivially on v. Conversely, pairing a nonzero such tuple with a dual family of (b_j) and averaging over K_u/U_u gives an element of I_u acting nontrivially. Thus $\tau_u \otimes \pi_u$ has a nonzero K_u-fixed vector if and only if $\mathrm{Hom}_{K_u}(\xi_u, \pi_u) \neq 0$. The tensor product $\tau = \bigotimes_u \tau_u$ acts on a free lattice O^b. In a constituent π, the forms transforming on the right by τ^{-1} are the fixed vectors of $\tau \otimes \pi$ under the tame type places, so τ selects the asserted representations. Principal level one at the odd prime $p \neq \ell$ with $p \nmid D_L$ makes stabilizers act trivially on coefficient fibres. At a place v over ℓ the free pro-ℓ part of the local units has rank $[F_v : \mathbf{Q}_\ell]$, giving the power series presentation $A = \Lambda = O[[X_1, \dots, X_{n[F:\mathbf{Q}]}]]$ with zero residual values for the variables X_m.

Over any complete Noetherian local O-algebra A' with finite residue field and continuous character θ', write $Y' = Y(A', \theta')$. At fixed precision the diagonal character θ' factors through a finite quotient, and sufficiently high powers of U_z contract the lower unipotent variation to the corresponding level. There are finitely many double cosets at that level. On the resulting finite module the factorial powers eventually project to the invertible part. The projectors commute with reduction and give a continuous ordinary projector e.

Choose the double cosets at precision one. Let $i : Y' \to (A')^t$ be evaluation on their representatives, with the coefficient coordinates included, and $\sigma : (A')^t \to Y'$ the map obtained by extending values and applying e. The stabilisers act trivially on coefficient fibres at the chosen level, so the extension is A'-linear. For $f \in Y' \cap V_j$, evaluate at double cosets at precision $j+1$, extend these values and apply e. This gives f modulo V_{j+1} as a linear combination of projected extensions of coordinate vectors, with coefficients in $\mathfrak{m}_{A'}^j$. Each projected extension is congruent modulo V_1 to its approximation by σi. Thus f agrees modulo V_{j+1}

with $\sigma(a_j)$ for some $a_j \in \mathfrak{m}_{A'}^j (A')^t$. Successive corrections sum in $(A')^t$. With $j = 0$ this proves surjectivity of σ, and with general j it proves $Y' \cap V_j = \mathfrak{m}_{A'}^j Y'$. In particular σi is the identity modulo $\mathfrak{m}_{A'} Y'$. It is invertible on the finite module Y' by Nakayama's lemma and the fact that a surjective endomorphism of a Noetherian module is injective. The map $(\sigma i)^{-1}\sigma$ splits i. Hence Y' is projective and is finite free over the local ring A'.

Lean: `UpperTheory.V.Hi.Frame.Ord` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.ContinuousTorusCharacter`, lines 229–230)
Lean: `UpperTheory.V.Hi.ordFree` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.OrdinaryHeckeCharacters.OrdFree`, lines 441–481)
Lean: `UpperTheory.V.Hi.ordAbs` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.OrdinaryHeckeCharacters.OrdFree`, lines 106–242)
Lean: `UpperTheory.V.Hi.ordStruct` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.OrdinaryHeckeCharacters.OrdFree`, lines 295–437)

Specialising to (A, θ), the simultaneous primary decomposition for the commuting operators T_r on $Y/\mathfrak{m}_A^a Y$ lifts by idempotents in the finite commutative Hecke algebra over A. Hence $M(K)$ is a free direct summand over A.

Lean: `UpperTheory.V.Hi.HSet.MK` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.CoefficientRepTrivialOnUnramified`, lines 160–161)

□

Proof of 30.23

Proof. Let H be the CM subgroup of the level group and c its complex conjugation. The restricted-ramification global group and the local groups have finite continuous $\mathbf{F}_\ell$-character spaces on all open subgroups by class field theory. Use the continuous deformation ring of the split determinant $\prod_i \bar{\chi}_i$ on H [126]. Tensor it with the local character rings for the ordered characters ψ_{vi}. In these rings fix the prescribed ℓ-primary torsion values in the completed group algebra and use Teichmüller lifts on the prime-to-ℓ part.

Write λ_i, tr_D for the characteristic coefficients and trace of the universal determinant, and set $[g]^* = \varepsilon_\ell(g)^{1-n}[cg^{-1}c^{-1}]$ on group algebra basis elements. In the completed tensor product impose the relations

$$\lambda_i([g]^*) - \lambda_i([g]), \qquad \mathrm{tr}_D\left([h]\prod_{i=0}^{n-1}([g_i] - \psi_{vi}(g_i))\right), \qquad \mathrm{tr}_D([h]([u]-1)).$$

Here $g, h \in H$, the g_i are in the decomposition group of v in the given flag order, and u ranges over the prescribed tame inertia kernels. The assumed pair makes the defining ideal proper. The quotient represents ordinary determinant pairs by the universal properties of its factors and relations.

Take the away level K, tame coefficient lattice O^b, weight ring $A = \Lambda$, compact diagonal character θ, ordinary module Y, ordinary projector e, precision-one evaluation basis, and finite free direct summand $M(K) \subseteq Y$ of Lemma 30.22. The spherical characteristic polynomials are at good inverse Frobenii. The ordinary operators give the ordered products of the flag characters on uniformisers, with the factors from θ. Suppose first that $M(K) \neq 0$, and write $T \subseteq \mathrm{End}_A M(K)$ for the algebra generated by these operators and the weights. Use algebraic weight zero. Among its arithmetic specialisations use those for which, at each $v \mid \ell$, the finite diagonal characters on $\mathcal{O}_{L_v}^\times$ are pairwise distinct. These weights separate A. Indeed the free local unit groups have ranks $[L_v : \mathbf{Q}_\ell] > 0$. On their product the weight coordinates vary independently, after a fixed character twist on each factor. All finite characters separate its completed group algebra by Fourier inversion on each finite quotient. On a free generator at each v, multiply a proposed vanishing element by the product of the differences of the universal values of the finite parts for $i < j$. Each difference is nonzero because it involves two independent coordinates. The product times the element vanishes at all finite-character specialisations: at a collision the multiplier vanishes, and elsewhere the weight is among those selected. The weight ring, after extending scalars for the fixed twists, is a power series domain. Cancellation proves the assertion.

Use the coefficient control isomorphism of 30.24, whose construction uses the ordinary projector and the evaluation basis of Lemma 30.22. At a selected weight decompose the classical space into its definite automorphic constituents. On each constituent the ordinary subspace injects into the upper-unipotent Jacquet module in each split factor at ℓ. Indeed a vector mapping to zero is a finite sum of unipotent

differences. Averaging over a sufficiently large compact unipotent subgroup kills it. On upper-integral-unipotent invariants translation of this average is a nonzero index factor times a power of the strictly dominant ordinary operator, which is invertible on the ordinary part. The torus on the image has the prescribed ordered unit characters. Frobenius reciprocity gives principal-series support, and two characters with distinct unit restrictions cannot differ by an unramified norm twist. The induction is irreducible by [16, Thm. 4.2]. Its Jacquet module has the permutations of the inducing tuple as successive characters, one per permutation, by [14, Lem. 2.12]. Precisely one permutation has the specified ordered unit restrictions. Hence the chamber operators act by scalars on this part, including on every copy if the constituent occurs with multiplicity. The good spherical operators also act by scalars there. The selected classical Hecke characters therefore separate T: an operator annihilated by them acts as zero on each selected classical fibre, and every matrix entry in an A-basis then vanishes by separation of A.

For such a Hecke character x, apply 26.22 to its constituent and 39.2 to the regular algebraic polarized cuspidal inputs of base change, with their prescribed twists. Write r_x for their semisimple sum. Its good inverse Frobenius polynomials are the spherical Hecke polynomials. At a selected split place above ℓ, the local support consists of singleton characters. Indeed its multiset is the support just computed, and a longer segment or a repeated cuspidal block in base change would repeat an inertia character. In the algebraic normalisation write these Weil characters as $\omega_i \circ a$, with a the reciprocity map of Lemma 30.21. Their parameter has zero monodromy and is trivial on Weil commutators. Each cuspidal factor therefore satisfies the toral test of 39.5–39.8. Direct sum and the base-change twists give the stated Weil polynomial for r_x, and give its filtration jumps in the order $h_{i,\tau} = i$ at weight zero.

Let $U_{v,j}$ denote the unit operator for j initial uniformisers and put $U_{v,0} = 1$. On a reciprocity lift of a uniformiser prescribe the value $U_{v,i+1}/U_{v,i}$ for $\psi_{vi}\varepsilon_\ell^i$, and on units prescribe the inverse of the i-th coordinate of θ. Units and a uniformiser give a character of $L_v^\times$. Modulo each power of the maximal ideal of T its image is finite, so local reciprocity and inverse limit give a continuous decomposition-group character. On the Jacquet module the unipotent index of the chamber with j initial uniformisers is $q_v^{j(n-j)}$. Including the half-modulus and the algebraic normalisation, the chamber eigenvalue is $\prod_{i<j} \omega_i(\varpi_v) q_v^{-i}$. Successive quotients give the character formula of Lemma 30.21 on a uniformiser, and the compact diagonal action gives it on units. That lemma supplies one flag for r_x with ordered characters $x\psi_{vi}$. For arbitrary $g_i \in D_v$, the rightmost factor $r_x(g_{n-1}) - x\psi_{v,n-1}(g_{n-1})$ lowers the last flag space. Each following factor lowers the next space. The product is zero, including after multiplication on the left by any $r_x(h)$.

Embed T in the product of the coefficient fields at the selected characters. Its image is closed since T is finite over the compact weight ring. The coefficients of the product representation's characteristic polynomials take values in this image on good inverse Frobenii. Continuity and Frobenius density give the same conclusion on every group element. Amitsur's formula for the determinant of a sum expresses its polynomial-law coefficients on finite group-algebra sums integrally in the characteristic coefficients of words, hence gives the determinant law over T itself [126, Lem. 1.12(ii)]. Its reduction is the prescribed law by the same Frobenius calculation. Polarisation gives the involution identities at each selected character. At each tame type place, occurrence of the block-parahoric twisted-spherical type of Lemma 30.22 makes the local component the irreducible product of twisted spherical representations, one for each unit character. By 26.22 it is a subquotient of the representation induced from the local components of the cuspidal base-change constituents. Decompose these components along the inertial lines and group the factors by unit character; on each group remove the common determinant twist and take invariants under the maximal compact subgroup. Exactness and the Iwasawa decomposition, as in 26.23, give a spherical vector in each factor. Hence the Weil–Deligne parameter of every cuspidal base-change constituent has zero monodromy and diagonal inertia given by the Teichmüller lifts of the residual characters; on the prescribed tame inertia kernel every lifted character equals one and monodromy vanishes, so inertia acts trivially and the inertia difference acts by zero there. The ordered products vanish by the common flag just constructed. Separation of T now gives all three families of relations in T. For the zero module the annihilator is the unit ideal.

Lean: `UpperTheory.V.Hi.eq_zero_of_vanish_at_good_points_dense_weight` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TCoreSep`, lines 37–66)
Lean: `UpperTheory.V.Hi.coreDet_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TCoreDet`, lines 291–322)
Lean: `UpperTheory.V.Z.Ord.core_hecke_of_automorphic_hecke_dense_weight` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.OrdCoreH`, lines 20–61)
Lean: `UpperTheory.V.Hi.heckeDeterminant_satisfies_deformation_relations_alt` (`Development.AutomorphicArithmetic.`

AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.
HeckeDeterminantSatisfiesDeformationRelations, lines 15–35)

The coefficients, ordered products and weights generate the Hecke algebra. The induced action of B has kernel $J = \mathrm{Ann}_B M(K)$, with $J = B$ if $M(K) = 0$.
Lean: UpperTheory.V.Hi.HSet.MK (Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.
CoefficientRepTrivialOnUnramified, lines 160–161)
Lean: UpperTheory.V.Hi.nonempty_unitaryHeckeData (Development.AutomorphicArithmetic.AutomorphicVarieties.
ModularityArgument.ThresholdArguments.DenseThresholdCases.HdEx, lines 20–27)
Lean: UpperTheory.V.Hi.exists_hecke_ideal (Development.AutomorphicArithmetic.AutomorphicVarieties.
ModularityArgument.ThresholdArguments.DenseThresholdCases.ExistsHeckeIdeal, lines 70–83)

At auxiliary place j of residue cardinality Q_j take the full integral group, the upper Iwahori and the kernel in that Iwahori of the map from diagonal residues to their ℓ-primary quotient. Write M, M_w, M_1 for their modules with the same away localisation. The last covering group is $\Delta = \prod_{j,i} \mathbf{Z}/\ell^{a_j}$, where $a_j = v_\ell(Q_j - 1) \geq r$ at depth r by 30.67. The distinguished congruence level makes the action free on each double-coset fibre. Select one sheet over each coset, multiply by its indicator and apply the ordinary projector. The resulting endomorphism e_1 of ordinary functions satisfies $\sum_{d\in\Delta} de_1 d^{-1} = 1$. Thus

$$v \longmapsto \sum_{d\in\Delta} d \otimes e_1(d^{-1}v)$$

splits the action map $A[\Delta]\otimes_A M_1 \to M_1$ on the ordinary module before localisation. The equivariant residual Hecke and root projections preserve projectivity. On the regular module the norm identifies coinvariants with invariants by sending the class of 1 to $\sum_d d$. The identity extends to projective modules. The invariants give M_w, and the kernel of the norm is generated by $d - 1$.
Lean: UpperTheory.V.Hi.ordProj (Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.
OrdinaryHeckeCharacters.OrdProj, lines 379–499)

For the trace identity take the root operators u_{ji}, diamond generators $1+y_{ji}$ and multiplier f of 30.70(1). On a classical generalised eigenspace at auxiliary place j, order the Frobenius roots α_i by the torus character at j. By Bernstein–Zelevinsky finiteness and the Geometric Lemma [14, Lem. 2.12], the $I_{1,j}$-invariants of an irreducible smooth constituent embed into its upper-unipotent Jacquet module of dimension at most $b = n!$, while the finite-order diamond operators $1 + y_{ji}$ act semisimply; hence on each simultaneous generalised eigenspace of the commuting operators u_{ji} and $1 + y_{ji}$ one has $(u_{ji} - \alpha_i)^b = 0$. If two roots coincide or $\alpha_i = Q_j\alpha_a$ with $i \neq a$, the corresponding factor of f is a linear combination of commuting nilpotents of exponent b, so its $2b$-th power vanishes. In the other case the roots are distinct and monodromy vanishes, since it connects roots of ratio Q_j. The tame generator $\tau_{r,j}$ acts on line i by $1 + y_{ji}$, giving

$$\prod_{a\neq i}(\phi_{r,j} - \alpha_a)(\tau_{r,j} - 1 - y_{ji}) = 0.$$

For $E = 2b$, evaluation at α_a shows that $(X - u_a)^{E+1}$ is divisible by $X - \alpha_a$. Multiplication and trace give the relation of 30.70(1). Classical specialisations separate the module, so it holds integrally. At the split auxiliary places take the Iwahori module obtained by the diamond norm and the spherical module at the same away level. The spherical Hecke polynomial supplies the signed elementary symmetric coefficients on spherical vectors. Lemma 30.20 applies to these modules and then to their ordinary residual summands. Its multiplier is exactly the product f of 30.70(1).
Lean: UpperTheory.V.Z.Win.T2.level_relation_two_x (Development.AutomorphicArithmetic.AutomorphicVarieties.
ModularityArgument.ThresholdArguments.DenseThresholdCases.ReductionOfPolynomialElementPow, lines 105–130)
Lean: UpperTheory.V.Th.nilKillX (Development.AutomorphicArithmetic.AutomorphicVarieties.TheoremsMuLinked.
NilpotentHeckeOperators.NilKill, lines 130–151)
Lean: UpperTheory.V.Pa.Lev.twoOf_alt (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.
ThresholdArguments.DenseThresholdCases.LevTwoOf, lines 103–117)
Lean: UpperTheory.V.Th.opOfX_alt (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.
ThresholdArguments.DenseThresholdCases.OpOpOf, lines 139–171)

□

Proof of 30.24

Proof. Work in a finite extension of the coefficient field containing the values of the arithmetic specialization, the images of the local embeddings and the values of α. Such an extension suffices since A is a power series

ring in finitely many variables, their images are topologically nilpotent, and a finite extension of $\mathbf{Q}_\ell$ is complete. Write O' for its integer ring and ϖ for a uniformizer. We use the given embedding into $\mathbf{C}$ when speaking of classical automorphic forms.

First specialize the coefficients of the scalar-valued ordinary functions, retaining the tame coefficient coordinates. This gives a map

$$j : O' \otimes_A Y \longrightarrow Y',$$

where Y' is defined by the same ordinary condition with diagonal character $x\theta$. Use the evaluation coordinates at precision one of Lemma 30.22. Denote them by e_i, and denote the ordinary projections of the coordinate extensions by w_i. Coefficient specialization commutes with finite double-coset sums and hence with the ordinary projector, by its factorial-power construction. The specialized w_i are therefore the projected extensions for Y'. The endomorphism $f \mapsto \sum_i e_i(f)w_i$ is the identity modulo the maximal ideal on each of Y, Y', by the precision-one approximation of Lemma 30.22. On each module it is an automorphism by Nakayama's lemma and finite generation. The map $f' \mapsto \sum_i e_i(f')(1 \otimes w_i)$ composes with j in the two orders to give these automorphisms, with scalars extended in the first one. Thus j is an isomorphism.
Lean: `UpperTheory.V.Hi.ordBC` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.OrdinaryHeckeCharacters.OrdBCMapOrd`, lines 198–232)

Let p_0 be the lowest-weight coordinate functional, fixed by the upper unipotents. For a classical form F with algebraic coefficients define

$$c(F)(g) = p_0\big(W_\lambda(g_\ell)^{-1}F(g)\big)$$

in each tame coefficient coordinate. Choose the level deep enough that α is trivial on its diagonal congruence subgroup and the lower unipotent subgroup preserves the coefficient lattice. The right transformation law for $c(F)$ on the diagonal is then $(\chi_\lambda\alpha)^{-1}$. The formula commutes with the tame Hecke sums. For a positive diagonal a, take upper-unipotent representatives for its double coset. Applying p_0 to each translate gives

$$c\big(\chi_\lambda(a)^{-1}[a]F\big) = [a]c(F).$$

In particular this holds for the ordinary operator U_z of Lemma 30.22.

Use the weight lattice in the algebraic representation, with the lowest-weight coordinate first. The upper integral unipotents act integrally and preserve this coordinate. On every other weight the normalized action of z is divisible by ϖ. Indeed the weight differs from the lowest weight by a nonzero sum of positive roots, and each simple root takes a uniformizer value on the strictly dominant diagonal z. Equivalently, in the integral weight basis all columns of $\chi_\lambda(z)^{-1}W_\lambda(z)$ except the first vanish modulo ϖ. Thus if the untwisted values $W_\lambda(g_\ell)^{-1}F(g)$ are divisible by ϖ^a and their first coordinates by ϖ^{a+1}, application of the normalized U_z makes all coordinates divisible by ϖ^{a+1}. This follows term by term in the upper-unipotent sum. On the integral ordinary part a positive power of U_z is the identity at any fixed precision. If $c(F) = 0$, induction on a now puts F in every ϖ^a times the value lattice. Hence $F = 0$.
Lean: `UpperTheory.V.Hi.wtRep` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.OrdinaryHeckeCharacters.ProdZpowConst`, lines 95–259)
Lean: `UpperTheory.V.Hi.ctrlOrd` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.OrdinaryHeckeCharacters.CtrlOrd`, lines 220–371)

For surjectivity, let $f \in Y'$. At precision one choose a deeper lower-unipotent level at which f is invariant and which acts trivially on the coefficient lattice modulo ϖ. There are finitely many double cosets at this level. On their representatives prescribe a classical form by giving its untwisted first coordinate the value of f and its other coordinates zero. Right translation extends the values: the stabilizers act trivially on the coefficient fibres by the level condition of Lemma 30.22. The resulting integral form G satisfies $c(G) \equiv f \bmod \varpi$ on all cosets. To return to the original lower-unipotent level, apply a power U_z^b, with b at least the difference of the depths and divisible by a period of f mod ϖ. Conjugation by z^{-b} sends the lower-unipotent subgroup at the original depth into the deeper one. The upper-unipotent coset sum gives invariance at the original depth; the finite diagonal character is already trivial on its congruence subgroup. Thus U_z^bG is classical at that level and its first-coordinate function is still congruent to f. Applying the ordinary projector there preserves this congruence. We have obtained an integral classical ordinary lift modulo ϖ.

On Y', divisibility of all values by ϖ is divisibility in the module: divide the values, and use periodicity one precision higher in checking the ordinary condition. Nakayama's lemma now gives surjectivity from the integral ordinary lattice. This lattice spans the classical ordinary space since, in the finite double-coset coordinates, the ordinary projector is the limit of factorial powers of an integral matrix and selects exactly

its unit generalized eigenvalues. Inverting ϖ, extending scalars, and applying the residual Hecke idempotent prove the assertions. This is the fixed-level function-space form of ordinary control; the corresponding specialization of ordinary Hecke algebras is given in [131, Lemma 2.6.4].

Lean: `UpperTheory.V.Hi.Sp.absSpec` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.HeckeFixedLevelSpaces.FreeMap`, lines 425–587)

Lean: `UpperTheory.V.Hi.partAct0_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.HeckeOperatorPreservesPart`, lines 308–335)

□

Proof of 30.25

Proof. Write $\mathbb{T} = B/J$ for the Hecke algebra acting faithfully on $M(K)$. The point x gives a character of $\mathbb{T}$. The module is finite over $\mathbb{T}$, and its support is all of $\operatorname{Spec}\mathbb{T}$ since its annihilator is zero. Thus the local factor at this character in the characteristic-zero weight fibre of $M(K)$ is nonzero. In that factor the last nonzero power of the maximal ideal acting on the module is killed by the maximal ideal. This gives a simultaneous eigenvector for the specified character, including its ordinary operators. By 30.24 it is a classical form at the prescribed weight, finite diagonal character and away level. Decomposition of the definite automorphic space gives a constituent in which its projection is nonzero.

Apply definite base change 26.22 to this constituent and the Galois attachment of Theorem 39.2 to its regular algebraic cuspidal factors, with the twists prescribed by base change. Their semisimple sum has the specified characteristic polynomials at good inverse Frobenii, by the defining spherical action of 30.23. Frobenius density and equality of semisimple characters identify its determinant on the level group with D_x. Pullback to G_L preserves semisimplicity since p is surjective. The pullback has the determinant of r, hence is isomorphic to r. Irreducibility leaves one cuspidal factor, with one copy. Denote it by π_0. Its infinitesimal character and conjugate duality are those of 26.22.

Lean: `UpperTheory.V.Z.Ord.core_hecke_of_automorphic_hecke_dense_weight` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.OrdCoreH`, lines 20–61)

Suppose now that r is continuous. At a chosen split place above ℓ, put $K = L_v$, $q = |\mathcal{O}_K/\mathfrak{m}_K|$, $\nu = |\cdot|_K$, and choose a uniformizer ϖ_K. Let κ_i, $0 \le i < n$, be the specialized ordinary characters and let $h_{i,\tau}$ be their strictly increasing weights at $\tau : K \hookrightarrow \overline{\mathbf{Q}}_\ell$. On the local Weil group use the reciprocity map a to $K^\times$ for which a lift of ϖ_K has degree -1.

We describe the smooth characters furnished by the same classical eigenvector. Its image in the upper-unipotent Jacquet module is nonzero. Indeed a vector in the kernel of the coinvariant map is a finite sum of unipotent differences. Averaging over a sufficiently large compact unipotent subgroup kills such a sum. On upper-integral-unipotent invariants these averages, followed by translation, are positive powers of the strictly dominant double-coset operator, up to nonzero index factors. That operator has nonzero eigenvalue on our vector. In the Jacquet module each chamber translation acts by its Hecke eigenvalue divided by the unipotent index. The diagonal units act by the inverse finite character. The finite-length torus module therefore has the resulting character as a constituent. More explicitly, localizing at that torus character gives a nonzero finite-length module, which has a quotient of that character. Frobenius reciprocity gives principal-series support for the local automorphic constituent.

Write its support, in the normalization for parabolic induction, as $\{\omega_i \nu^{(n-1)/2} : 0 \le i < n\}$, in the order read by the chamber operators. At diagonal units the ω_i are the inverses of the finite parts in 30.24. For the diagonal with j uniformizers first, the scalar before division by χ_λ is $\prod_{i<j} \omega_i(\varpi_K) q^{-i}$. Indeed the unipotent index is $q^{j(n-j)}$; its half-density and the common twist $\nu^{(n-1)/2}$ give $q^{-j(j-1)/2}$. Taking successive quotients gives the individual values on ϖ_K. Together with the unit values these are precisely the ordinary-character identities

$$\kappa_i(s) = \iota^{-1}(\omega_i(a(s))) \prod_\tau \tau(a(s))^{-h_{i,\tau}} \qquad (s \in W_K).$$

Here the algebraic part of the i-th compact character has exponents $h_{i,\tau} - i$; the other i units of weight are the cyclotomic correction in 30.23. Units and a uniformizer generate $K^\times$, so the identity holds on the whole Weil group.

Lean: `UpperTheory.V.Z.Ord.ordJac` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.OrdinaryJacquetModules.OrdJac`, lines 13–123)

Lean: `UpperTheory.V.Hi.core_eigen_of_weight_char_hecke_level` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TCoreEig`, lines 189–215)

Lean: `UpperTheory.V.Z.Ord.endOrd_of_printsH` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.IsoOfDet`, lines 124–143)

The local flag has these characters. To see this from the determinant relations, the trace of every global translate of $\prod_i (r(g_i) - \kappa_i(g_i))$ vanishes by 30.23. The global matrices span the full matrix algebra by irreducibility, so nondegeneracy of its trace pairing makes every such product zero. The spans of the images of the successive suffix products give a filtration from zero to the whole space. Its i-th quotient carries κ_i, since left multiplication by $r(g) - \kappa_i(g)$ takes that span into the preceding one. To see that every quotient is a line, take a Weil lift of a sufficiently deep rational unit $1 + \ell^b$ on which all smooth characters are trivial. The ordinary weights have strictly increasing sums over τ, so the values of the κ_i there are distinct. The specialized local characteristic polynomial has these n roots by the diagonal character relation of the ordinary Hecke action. Thus each occurs once. In particular every flag at this prolongation has the same multiset of characters.

Lean: `UpperTheory.V.Z.Ord.flagJ_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.OrdFlagJ`, lines 102–133)

Assume the place is unlinked. For a pair i, j put $\delta = \kappa_i/(\kappa_j \varepsilon_\ell)$ and $d_\tau = h_{j,\tau} - h_{i,\tau} - 1$. The local character formula for ε_ℓ gives

$$\delta(s) = \iota^{-1}\left(\frac{\omega_i(a(s))}{\omega_j(a(s))\nu(a(s))}\right) \prod_\tau \tau(a(s))^{d_\tau}.$$

If $\omega_i = \omega_j \nu$, evaluate at a Weil lift of $\ell \in K^\times$. The character δ has compact image on the decomposition group, so all its values have ℓ-adic norm one. This gives $\sum_\tau d_\tau = 0$. For $i < j$ all d_τ are nonnegative; for $i \geq j$ they are all negative. Consequently all are zero. Then $\delta = 1$ on the dense Weil group and by continuity on the decomposition group, contrary to unlinking. Thus the toral support of $\pi_{0,v}$ is unlinked. Lemma 39.55 now applies to this support. The common power of ν relating the two normalizations shifts all exponents by the same amount.

Suppose next that $\delta^b = 1$ for some $b \geq 1$. Evaluate at a Weil lift of a rational integer $1+\ell^c > 1$ sufficiently close to one to kill both smooth characters. This gives $(1 + \ell^c)^{b \sum_\tau d_\tau} = 1$. The sign argument again gives $d_\tau = 0$ for all τ. At a lift of ϖ_K the complex absolute value of $\iota(\delta)$ is one, whence

$$|\omega_j(\varpi_K)| = q\,|\omega_i(\varpi_K)|.$$

The two support characters cannot satisfy this equality: their absolute values lie in the same open interval whose endpoints have ratio q, by 39.55. This excludes every positive power in the definition of potential unlinking.

Lean: `UpperTheory.V.Z.Ord.exp_torsion_of_literal_automorphy_n` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.OrdExpTor`, lines 265–306)

Other prolongations over the chosen place are obtained by conjugation. Over the other place of the split pair, polarization takes the flag to the reversed dual flag with the common multiplier ε_ℓ^{1-n}. This follows also by comparing the conjugate and twisted-dual Frobenius polynomials and applying density to the semisimple representations. The common multiplier cancels in character ratios, and inversion interchanges the two indices in a cyclotomic equality. The automorphic supports transform by conjugate duality as well. Thus the conclusions hold at both places. In applying the support argument at a potentially unlinked place, its flag satisfies the unlinking condition by taking the first power.

Away from ℓ and the distinguished place, $S' = S$ leaves spherical places and the specified split tame types. At a spherical place the support is unramified toral, and a generic spherical constituent has unlinked characters: in the unramified classification its parameter has zero monodromy, and genericity excludes linked singleton segments. At a tame type place, the classical vector realizes the block-parahoric twisted-spherical type of Lemma 30.22, which directly forces zero monodromy (even when inertial characters coincide) and diagonal inertia with the specified Teichmüller characters. Each of these inertia characters extends to the decomposition group, so every Weil commutator acts trivially. Apply 40.11 to the algebraically normalized generic constituent. Undoing the common normalization preserves unlinking.

Lean: `UpperTheory.V.Hi.partUR2Type_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.AutomorphicTensorInductionOfLocalPlaces`, lines 76–113)

Lean: `UpperTheory.Gal.projectedFormsSystemPart.typePlain` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.PlainTorsionPairs.PlainTorOfMono0`, lines 114–130)

Finally, any other cuspidal attachment has the same unramified Satake multisets as π_0 at almost every place, by equality with the Frobenius polynomials of r in the same normalization. Strong multiplicity one identifies the two. The global twist making this one cuspidal representation unitary supplies one value of t in (30.8) at all the required places, by 39.55.

Lean: `UpperTheory.V.Z.Ord.born_j_of_printed_polarization_window` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.OrdBornJ`, lines 32–50)

□

Proof of 30.27

Proof. Put $S = L[[T_1, \dots, T_h]]$ and let T_i act by y_i. The power series act because B is complete and the y_i belong to its maximal ideal. Lift an L-basis of M/YM to $m_1, \dots, m_e \in M$. These lifts give a map $S^e \to M$. The displayed isomorphism identifies its associated graded map for $(T_1, \dots, T_h)$ and Y with an isomorphism in every degree. A nonzero vector of series has a first nonzero homogeneous part, so the map is injective. Given $m \in M$, first represent its residue modulo YM, then its successive remainders in $Y^a M / Y^{a+1} M$. The resulting series map to m, since $\bigcap_a Y^a M = 0$. Thus M is finite free over S.

Multiplication embeds $B/\operatorname{Ann}_B M$ in the finite S-module $\operatorname{End}_S(M)$, so that ring is finite over S. A minimal support prime Z is associated to M. If $Z = \operatorname{Ann}_B(m)$ with $m \neq 0$, freeness over the domain S gives $Z \cap S = 0$, where contraction is through the action map. Hence B/Z is integral over a copy of S. Lying over, going up and incomparability give $\dim B/Z = \dim S = d_0 + h$. □

Proof of 30.28

Proof. In $\mathbb{P}^{g-1}_P$, with line coordinate $[v]$, take the closed subscheme X given by $Zv = 0$ and $u_i v_j = u_j v_i$. Its closed fibre is $\mathbb{P}^{g-1}_{P/\mathfrak{m}}$. On the chart $v_c = 1$ its equations are

$$z_{ic} + \sum_{j \neq c} z_{ij} v_j = 0, \qquad u_b - u_c v_b = 0 \quad (b \neq c).$$

At a point of the closed fibre the ambient local ring is regular. The parameters of P remain independent in its cotangent space after extending the residue field. Indeed the chart is a polynomial algebra over P, and parameters of P followed by lifts of parameters of the local ring of the fibre form a regular system of parameters. In the displayed equations the coefficients of the parameters z_{ic} and u_b, $b \neq c$, form an identity matrix. The quotient local ring is regular.

Every point of X specialises to a point of the closed fibre. Its closure is proper over $\operatorname{Spec} P$, so its image is closed and contains the closed point. Regularity passes to localisations. Thus the local rings of X are domains and its finitely many irreducible components are disjoint. Each component meets the connected closed fibre. It follows that X is irreducible. Over the residue field of any prime containing $(Zu, \det Z)$, take the line spanned by u if $u \neq 0$, and any kernel line of Z otherwise. Conversely the field-valued incidence equations imply $Zu = 0$ and $\det Z = 0$. The image of X is therefore $V(Zu, \det Z)$, which is irreducible.

To bound its dimension, add the g^2 equations $z_{ij} = \delta_{ij} u_i$ to (Zu). They give $u_i^2 = 0$ in the quotient. Its radical is the ideal of the $g^2 + g$ independent parameters. Cutting by g^2 elements decreases dimension by at most g^2, so $\dim P/(Zu) \leq \dim P - g$. The adjugate identity gives $(\det Z)u = 0 \bmod P_0$, hence $\det Z \in P_0$. The prime radical found above lies below P_0. Strict inclusion would give dimension greater than $\dim P - g$ for its quotient, contradicting the bound. □

Proof of 30.29

Proof. Extend the matrices and characters linearly to the group algebra. The product identity extends to independently chosen group-algebra elements by multilinearity. It holds also for r_0, since its i-th factor takes the $(i+1)$-st flag space into the i-th.

The evaluation vectors of the residual characters span F^n by linear independence of distinct characters. Choose $z \in F[D]$ at which their values are pairwise distinct. Put $a_{b,i} = \chi_{b,i}(z)$. Every difference $a_{b,i} - a_{b,j}$, $i \neq j$, is a unit. Taking every slot equal to z gives the annihilating polynomial $\prod_i (X - a_{b,i})$ for $r_b(z)$. Its Lagrange projectors $P_{b,i}$ are orthogonal and sum to one. The projectors for the two systems agree modulo I.

They belong to the given subalgebra: the denominators are scalar matrices in it, and their inverses belong to it by the finite geometric series for a unit with nilpotent difference from a nonzero element of F.

For $i > j$, multiply the ordered identity on the left by $P_{b,i}$ and on the right by $P_{b,j}$. Use g in slot j and z in every other slot. The factors to the left of slot j act on the left projector by units, and the factors to its right act on the right projector by units. This gives $P_{b,i} r_b(g) P_{b,j} = 0$. The same calculation with $i = j$ gives

$$P_{b,i} r_b(g) P_{b,i} = \chi_{b,i}(g) P_{b,i}.$$

For $b = 0$ the sums of the first projectors give the original flag: they preserve that flag and act by one on precisely its first graded lines.

Put $S = \sum_i P_{1,i} P_{0,i}$ and $Y = 1 - S$. Then $Y \in M_n(I)$, $S^{-1} = 1 + Y$, and $S P_{0,i} = P_{1,i} S$. Thus $S^{-1} r_1(g) S$ preserves the original flag with ordered diagonal $\chi_{1,i}(g)$. Multiplication by $r_0(g)^{-1}$ and expansion using $I^2 = 0$ give

$$S^{-1} r_1(g) S r_0(g)^{-1} - 1 = Z(g) - dY(g).$$

This gives both assertions, including the stated diagonal on every element of D. □

Proof of 30.30

Proof. At level N put $Q_j = \mathrm{N} w_{N,j}$, $a_j = v_\ell(Q_j - 1)$ and $\Delta_N = \prod_{j,i} \mathbf{Z}/\ell^{a_j}$. In the deepest module of 30.23 take the ordinary residual Hecke summand and the ordered-root summand on which every $U_{ji} - \widetilde{u}_{ji}$ is topologically nilpotent. After extension by s, call this M_N. The sheet splitting and the equivariant projections give projectivity over $\Lambda[\Delta_N]$. This ring is local because in its residue group algebra the augmentation ideal is nilpotent. Thus M_N is free over

$$\Lambda[\Delta_N] = S/\big((1 + y_{ji})^{\ell^{a_j}} - 1 : j, i\big).$$

The norm gives the corresponding Iwahori module at augmentation.

Lean: `UpperTheory.V.Hi.ordProj` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.OrdinaryHeckeCharacters.OrdProj`, lines 379–499)

Lean: `UpperTheory.V.Pa.Lev.opToTW` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.NilpotentLevelLowering.OpToTW`, lines 181–285)

Apply Lemma 30.20 before passing to the limit in the auxiliary levels. In its notation let V be the Iwahori module, M the spherical module, $i : M \to V$ inclusion and $T : V \to M$ the unnormalised reverse coset sum. If D_N is the ordered splitting algebra, define $\phi : D_N \otimes M \to V$ by $p \otimes v \mapsto p(u) i(v)$. The symmetric polynomials in the root operators u act on spherical vectors by the signed Frobenius coefficients.

At one place put

$$D(x) = \prod_{a<b} (x_b - x_a), \qquad H_Q(x) = \prod_{a<b} (x_b - Q x_a), \qquad \partial h = D^{-1} \sum_\sigma \operatorname{sgn}(\sigma) \sigma h.$$

For a subset A of pairs $a < b$, set $a_A = \prod_{(a,b) \in A} x_b$, $c_A = \prod_{(a,b) \notin A} (-x_a)$. Expansion yields

$$\sum_A a_A \sigma(c_A) = \prod_{a<b} (x_b - x_{\sigma(a)}) = \mathbf{1}_{\sigma=1} D(x),$$

since a nonidentity permutation has $\sigma(a) > a$ for some a. In the Bernstein polynomial representation the adjacent operator acts by

$$T_s h = Q\, s(h) + (Q - 1) x_a (h - s(h))/(x_a - x_{a+1}).$$

The spherical sum $E = \sum_\sigma T_\sigma$ obeys $T_s E = Q E = E T_s$, which determines its permutation coefficients from the longest one. Multiplication along a reduced word gives $E = \sum_\sigma \sigma(H_Q/D)\sigma$, or $E(h) = \partial(H_Q h)$. The triangular permutation expansion with nonzero leading coefficients identifies this computation with the Hecke operator identity. At all places define

$$\psi_0(v) = \sum_{(A_j)} \left(\prod_j a_{A_j}(x_j) \right) \otimes T \left(\prod_j c_{A_j}(u_j) v \right).$$

The subset identity gives $\sum_A a_A E c_A = H_Q$ and $\sum_A a_A \partial(H_Q c_A h) = H_Q h$. These compute $\phi\psi_0$ and $\psi_0\phi$ at one place. Operators at distinct places commute. Multiply ψ_0 by the polynomial quotient of

$$f_N = \prod_j \prod_{i \neq b} (U_{ji} - U_{jb})(U_{ji} - Q_j U_{jb})$$

by $\prod_j H_{Q_j}$. Both composites now act by f_N. The maps intertwine the root operators and restrict to the chosen residual ordering. Via the norm they compare $M_N/(y)M_N$ with the base module tensored with the splitting algebra.

Lean: `UpperTheory.V.AlgebraTools.hkDualEx` (`Development.AutomorphicArithmetic.AutomorphicVarieties.AlgebraTools.HkDualEx`, lines 14–41)
Lean: `UpperTheory.V.Pa.Lev.multiTwo` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.NilpotentLevelLowering.MultiTwo`, lines 174–314)
Lean: `UpperTheory.V.Pa.Lev.twoOf_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.LevTwoOf`, lines 103–117)

Write $Y = (y_{ji})$, $\mathfrak{l} = \mathfrak{m}_\Lambda$ and $X = B^{\mathrm{sp}} \otimes M_0$, with tensor product over the base of the splitting algebra after extending M_0. Observe that $f \in R$ equals f_N with $Q_j = 1$ for all j. If $a_i \geq b + j$, the diamond relations lie in $Y^b + \mathfrak{l}^j$, since for $1 \leq m < b$,

$$v_\ell \binom{\ell^{a_i}}{m} \geq a_i - v_\ell(m) \geq j.$$

Hence the truncated modules are free over $S/(Y^b + \mathfrak{l}^j)$ at sufficiently high levels. At augmentation, for each j choose e with $\mathfrak{m}_{B_0}^e M_0 \subset \mathfrak{l}^j M_0$ by finite length modulo $\mathfrak{l}^j$. The inverse Frobenii tend to s_i in the finite quotient determining the polynomials at that precision. The ordered splitting presentations then agree by identification of corresponding roots, and the comparison maps intertwine the actions of R. Their composites reduce to f since $Q_i \equiv 1 \bmod \ell^N$.

Lean: `UpperTheory.V.Pa.Core.freeN` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AnnihilatorBaseChange.FreeNQuot`, lines 29–44)
Lean: `UpperTheory.V.Pa.Core.eventually_exists_patchedModule_quotient_maps` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AnnihilatorBaseChange.EventuallyExistsPatchedModuleQuotientMaps`, lines 193–239)

For $b \geq 1$ set

$$L_b = \prod_N M_N \big/ \{(v_N) : \ \forall j,\ v_N \in (Y^b + \mathfrak{l}^j) M_N \text{ on a set in the ultrafilter}\}.$$

Use $\mathcal{B} = \widehat{R[1/f]}^{\,Y}$ and $\mathcal{M} = \varprojlim_b L_b[1/f]$. At $b = 1$ choose augmentation comparisons at depths tending to infinity in the ultrafilter. The constant-module limit is X, by finiteness of its truncations and completeness. Both comparison composites become invertible after inverting f. At finite precision freeness identifies the Y-graded pieces with the copies of the augmentation indexed by monomials. Each degree uses finitely many monomials, so the identities pass to L_b and through localisation. The transitions are surjective with kernels given by powers of Y. Hence

$$\mathcal{M}/Y\mathcal{M} \simeq \mathcal{B} \otimes_R X, \qquad \mathrm{gr}_Y \mathcal{M} \simeq (\mathcal{M}/Y\mathcal{M})[y_{ji}].$$

Lift finite generators of the augmentation successively through the complete tower to obtain finiteness over $\mathcal{B}$.

Lean: `UpperTheory.V.Pa.Core.pmodEx` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AnnihilatorBaseChange.PmodEx`, lines 32–95)

Every support prime contains $\mathfrak{a}$. On classical eigenspaces where $f_N \neq 0$ the auxiliary projector onto line i is

$$\frac{\prod_{b \neq i}(\phi_j - U_{jb})}{\prod_{b \neq i}(U_{ji} - U_{jb})}.$$

Local compatibility gives tame character $1 + y_{ji}$ there. Thus at every classical eigencharacter f_N times each auxiliary translated trace vanishes, with the zero of f_N accounting for coincident roots and ratios Q_j. The permutation filtration of the Jacquet module has length at most $n!$. The corresponding power of the operator product kills the generalised eigenspaces, and the classical specialisations separate vectors. Polarisation, ordinary and prescribed inertia relations hold at finite level. For each generator a of $\mathfrak{a}$, finite-quotient

agreement of the common-cover lifts and of f_N, f makes a fixed power of fa agree to arbitrary maximal-ideal order with finite-level annihilators. The finite-length augmentation comparison and the monomial graded-piece identity pass these approximations through every truncation. After inverting f, a belongs to the radical of the annihilator.

Lean: `UpperTheory.V.Z.Win.T2.level_relation_one_two` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.LocOkLevelModuleOfThetaData`, lines 118–153)

□

Proof of 30.31

Proof. First work over F. In each block choose a stable lattice and diagonalise the polarising matrix P with entries $t^{d_i}b_i$ ($b_i \in A^\times$); integrality of $\rho(s)$ and $P\rho(s)P^{-1}$ gives valuation at least $\max(0, d_j - d_i)$ in entry (i,j), so replacing t by t^2, extending k to adjoin $\sqrt{b_i}$, and rescaling basis vector i by $t^{-d_i}b_i^{-1/2}$ makes $P = I$ while keeping ρ integral. Put $\chi = \bar{\varepsilon}_\ell$. The subgroup on which we test is $H' = H \cap \ker \varepsilon_\ell$, using the full cyclotomic character. In the paired slot algebra the perfect dual pairing is the η-coefficient of matrix trace. Over A_m, follow it by the t^{m-1}-coefficient and $\mathrm{Tr}_{k/\mathbf{F}_\ell}$ to get the finite-coefficient pairing. Opposite matrix units and complementary powers of t show that it is perfect. The dual action has the additional factor χ.

Here are the details of detection. Write $r = \rho$, $r_1 = (1 + \sum e_j\kappa_j)r$, and write a twisted dual cocycle as $z = \eta p + \sum e'_j l_j$. On $F[H']$ extend the functions

$$R(s) = r(s), \qquad K_j(s) = \kappa_j(s)r(s), \qquad L_j(s) = l_j(s)r(s)$$

linearly. The map R has image the product D of the full diagonal block algebras. This follows from absolute irreducibility on H' and, for two blocks, inequivalence of the two simple modules. Put $I = \ker R$. The K_j, L_j satisfy the derivation rule with left and right action through R. Extend also $T(s) = [\eta](z(s)r_1(s))$ linearly. Multiplying the coefficient matrices and using $e_i e'_j = \delta_{ij}\eta$ gives

$$T(uv) = T(u)R(v) + R(u)T(v) + \sum_j \big(K_j(u)L_j(v) + L_j(u)K_j(v)\big).$$

If all trace tests vanish, $\mathrm{tr}\, T$ vanishes on the group algebra. For $u, v \in I$, insert between them a lift of E_{jj} and on the outside a lift of E_{ii}, where i, j belong to opposite blocks. Terms differentiating either lift vanish because they contain $R(u)$ or $R(v)$. Thus, using b for the cocycle index,

$$\sum_b K_b(u)_{ij}L_b(v)_{ji} + L_b(u)_{ij}K_b(v)_{ji} = 0.$$

Fix v. Replacing u by aua' for lifts of matrix units makes the same linear combination zero on the entire corresponding corner of I. The combination of corner derivations therefore factors through D. A derivation of a product of two matrix algebras into the off-diagonal bimodule is inner: applying the product rule to the central idempotent of the target block and to its products with the two summands expresses it as commutation with its value on that idempotent.

In ordinary $H^1(H', \mathrm{Hom}(r_2, r_1))$, the restrictions of the K- and L-cocycles in this combination transform under H by the trivial character and by χ^{-1}, respectively. This follows by applying the cocycle identity to hsh^{-1} and discarding its coboundary term. Some $h \in H$ has $\chi(h) \neq 1$, since $\mu_\ell \not\subset L$. Each of the two combinations is consequently inner on H'. A bounding matrix in the chosen corner works on H: after subtracting its boundary the cocycle vanishes on H', and its remaining values are H'-intertwiners, which are zero. Polarisation gives the opposite corner by transposition. In particular a combination of the κ_b that is inner in this corner is a polarised boundary in both corners. Independence of their classes forces $L_b(v)_{ji} = 0$ for each b. This holds for every $v \in I$ and every opposite pair. Each L_b now factors through D on H'; the same derivation and intertwiner arguments make l_b a twisted boundary on H.

Lean: `UpperTheory.V.Pa.Det.relPair` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.DeterminantPadicFree.RelPair`, lines 23–176)

Lean: `UpperTheory.V.Pa.Det.ann` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.DeterminantPadicFree.BlockPrinc`, lines 75–160)

Subtract these boundaries in the paired coefficient algebra. This removes their mixed η-terms as well, since conjugation in the boundary is by r_1. The remaining even derivative on H' satisfies the ordinary

derivation rule. For $u \in I$ its product with every $R(v)$ has trace zero, by the test on uv. The trace form on D is nondegenerate, so the derivative vanishes on I. Every derivation of a full matrix algebra into itself is inner: for a derivation δ the matrix $\sum_i \delta(E_{i1})E_{1i}$ has commutator $\delta(E_{ab})$ with E_{ab}. Apply this in each block. The remaining cocycle factors through the abelian quotient H/H' on the block scalars with action χ. For a fixed h with $\chi(h) \neq 1$, commutativity gives $(\chi(h)-1)z(s) = (\chi(s)-1)z(h)$, which supplies its bounding scalar. Averaging the bounding matrix with its polarised image gives a bounding matrix of the required transpose type because ℓ is odd. For one block the proof starts with the even derivative. A power of t clears all denominators.

Lean: `UpperTheory.V.Pa.Det.detAlg` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.DeterminantPadicFree.DetAlg`, lines 71–141)
Lean: `UpperTheory.V.Pa.Det.detAsm` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.DeterminantPadicFree.DetAsm`, lines 160–268)

The common separable locus of the curves is open dense in H', being a finite intersection of open dense sets there. A continuous trace vanishing on it vanishes on H'. The space on which the tests act modulo boundaries is finite-dimensional: values on a finite set of generators of the open pro-ℓ residual group, together with values on coset representatives, determine a cocycle. The reduction to this group is valid for finite truncations since the semidirect product of the residual unipotent action and the additive cocycle module on the residual kernel is an ℓ-group. The finite-character hypothesis in 30.67 gives finite generation of the pro-ℓ group. Successively choose an evaluation nonzero on the current kernel until that kernel is zero. Take the union over the curves and split the finitely many polynomials by finite injective DVR extensions.

Lean: `UpperTheory.V.Pa.choice` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AutomorphicDetection.Choice`, lines 14–40)

We spell out the integral estimate used below. In generator coordinates, cocycle relations, local conditions with their bounding matrices, and the selected evaluations are linear equations over A. One may take all relations: their rows generate a submodule of a finite free A-module, hence a finite list of rows suffices. Concretely the coordinates before the relations describe crossed homomorphisms on a free profinite cover. At every depth the crossed homomorphism together with the representation is a homomorphism to a finite semidirect product. Evaluation on a word of the cover gives compatible linear forms at all depths and hence a row over A. Truncation thus imposes exactly the reductions of the rows. A nonzero Smith divisor t^b introduces a factor of at most Q^b in the ratio of the truncated kernel to the reductions of integral solutions. Quotienting by boundaries has the same bounded-error property: use the map from bounding matrices to generator values, and project away any local witness variables. Equivalently, take successively the kernel, the projection of the integral kernel, and the image of the boundary map; the indices of these last two lattices in their saturations are fixed. A rational space of classes of dimension u therefore contributes at most Q^{mu+c}, with c independent of m.

Apply this observation to the base twisted dual satisfying the trace tests. Its quotient by boundaries has bounded order, by detection. At level N the dual of the full auxiliary local condition is zero in local H^1. A representative there is a boundary at both the tame generator and Frobenius. The tame generator acts trivially on the pulled-back module, so the cocycle vanishes on it. Its semidirect-product homomorphism consequently descends to the base by the normal-generation assertion of 30.67, after the pro-ℓ reduction above. At fixed m there are finitely many base cocycles. For all sufficiently high N their evaluations at the images of the auxiliary Frobenii agree simultaneously with their evaluations at the selected s_j. At s_j the cyclotomic twist is trivial, and a boundary multiplied by $r_1(s_j)$ has trace zero by cyclic invariance. This gives the base tests. Subtraction of a boundary normalises the value at complex conjugation to zero, by averaging over its group of order two. Descent then gives an injection on the dual classes in question. Denote the resulting uniform length bound by c_2.

Lean: `UpperTheory.V.Pa.dualN` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.EpsilonEtaMatrices.DualN`, lines 10–154)

Write $\lambda = \log_Q \#$ for the following counts and put $\nu = \binom{a}{2} + \binom{n-a}{2} + ga(n-a)$. At $v \mid \ell$, first count cocycles already upper triangular in the lifted flag. A diagonal entry is additive and vanishes on the prescribed torsion. The local abelianisation gives at most $m(1+d'_v)$ for its length. Filter the upper entries by distance from the diagonal. After smaller distances are fixed, differences in position ij are cocycles for ψ_i/ψ_j. For this one-dimensional module, the local Euler characteristic and Tate duality in 26.27 give

$$\lambda Z^1(G_v, A_m(\psi_i/\psi_j)) = m(1+d_v) + \lambda H^0(G_v, A_m(\chi\psi_j/\psi_i)).$$

The last term is bounded by the valuation of one nonzero character difference, since $\psi_i \neq \chi\psi_j$ for $i < j$.

There is one slot for each within-block upper entry and g slots for each between-block upper entry. Summing over diagonal and upper entries gives local length at most $m[n(1+d'_v)+\nu(1+d_v)]+c_v$ at $v \mid \ell$. Allowing a change of flag adds at most the index of the upper triangular matrices in $V_1(A_m)$, since a boundary from an upper triangular matrix is already among the cocycles counted. In the Greenberg–Wiles local factor this leaves the subtraction $m(n+\nu)$ at each ordinary place: the upper triangular matrices have length $m(n+\nu)$, and the local factor for cocycles is their count divided by the order of the full coefficient module.

At an auxiliary place the full local factor has length $\lambda H^0(G_w, V_1^\vee(1))$, by the Euler characteristic away from ℓ. At the chosen Frobenius, over F an invariant has all its odd primed entries zero: each would intertwine blocks with disjoint spectra. Its remaining η-matrix commutes with the simple-spectrum matrix $r(s_j)$, so has dimension at most n. The Smith estimate gives $mn + c_j$. At the distinguished place the hypothesis on twisted intertwiners kills successively the primed and the η-parts over F, giving a bound c_0. At a tame place with unramified condition the local factor is one, since a class is determined by its Frobenius value modulo Frobenius boundaries. Globally a skew invariant for V_1 has zero constant part: this part is block scalar and skew. Its odd coefficients are then intertwiners and vanish. The truncated invariants thus have bounded length c_1. Each real place subtracts $m\nu$, the length of the skew part for negative transposition, since each within-block unordered pair contributes one slot and each cross pair contributes g.
Lean: `UpperTheory.V.Pa.locV` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.EpsilonEtaMatrices.EntryZero`, lines 204–322)
Lean: `UpperTheory.V.Pa.h0q` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.EpsilonEtaMatrices.FinrankCentralizerW1Le`, lines 106–131)

All modules here are finite ℓ-primary modules, the level set includes the places above ℓ, and the pairing at the start identifies the Tate dual used in 26.27 and 27.2. The Greenberg–Wiles formula, with the dual Selmer factor bounded by c_2, now gives

$$\lambda W_m \le c_1 + c_2 + c_0 + \sum_j (mn + c_j) - m\nu d_+ + \sum_{v\mid\ell} \{m[n(1+d'_v)+\nu(1+d_v)] + c_v - m(n+\nu)\}.$$

Both local-degree sums are d_+, which proves the first bound.
Lean: `UpperTheory.V.Pa.oneLevel` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.EpsilonEtaMatrices.OneLevel`, lines 236–483)

For the second bound let M_e be the odd module at depth e. At the ordinary and prescribed tame places use the quotients $Q_{v,e}$ and the subgroups J_v in the definition of E_e. Write

$$T_e^1 = \bigoplus_v H^1(G_v, Q_{v,e}) / \ker\big(H^1(G_v, Q_{v,e}) \to H^1(J_v, Q_{v,e})\big).$$

Let S_e be the kernel of $H^1(G, M_e) \to T_e^1$, and D_e the cokernel of $H^2(G, M_e) \to \bigoplus_v H^2(G_v, Q_{v,e})$, where G is the global level group. The map from E_e forgetting the local cochains takes values in the kernel of this degree-two localisation. Its kernel is a quotient of $T_e^1 / \operatorname{im} H^1(G, M_e)$: subtract a bounding global cochain from a representative whose global class vanishes, and read the remaining local cocycles. Hence

$$\#E_e \le \#S_e \, \#D_e \, \frac{\#H^2(G, M_e)}{\#H^1(G, M_e)} \frac{\#T_e^1}{\prod_v \#H^2(G_v, Q_{v,e})}.$$

Put $u = a(n-a)$. The odd module has rank $2u$, the real-place invariants have rank u, and the ordinary lower quotient has rank u. The global Euler characteristic bounds the first ratio by Q^{-eud_+}. At an ordinary place the local Euler characteristic bounds its contribution to the second ratio by $Q^{eud_v} \#H^0(G_v, Q_{v,e})$. At a tame place the unramified subgroup has order $\#H^0$, so its contribution is one. These degree factors cancel. By 27.2, $\#D_e$ is bounded by the order of the global invariants of $M_e^\vee(1)$: first localise to all the level places, then project to the indicated quotients. Local H^2 surjects onto H^2 of each quotient by local cohomological dimension two. The rational odd dual has no H'-invariants, by block inequivalence, giving a uniform bound Q^{c_d}. Each ordinary lower quotient has a filtration by nontrivial character ratios. It has no rational invariants, so $\#H^0(G_v, Q_{v,e}) \le Q^{c_h}$ for a common constant.
Lean: `UpperTheory.V.Pa.B2.odCount` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.ArchimedeanH0Cards.OdCount`, lines 8–79)

At base level the generator-and-witness count above gives $\#S_e \le Q^{ge+c_b}$. At auxiliary level evaluate an odd cocycle on the tame generators. These evaluations are invariant under addition of a boundary.

Their simultaneous kernel descends to the base, with its old local conditions. Indeed the tame generators act trivially and normally generate the new kernel in the pro-ℓ quotient. Their values commute with the respective Frobenii by the tame relation and $\mathrm{N}w \equiv 1 \bmod \ell$. The odd centralizer at s_j is zero over F. The Smith estimate on its commutator therefore bounds the product of all the evaluation image sizes by Q^{c_n}, uniformly for $e \le m$ once the Frobenii agree to depth m. This proves $\lambda S_e \le ge + c_b + c_n$ at that level.
Lean: `UpperTheory.V.Pa.B2.sel0` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.ArchimedeanH0Cards.ImgCard`, lines 340–364)
Lean: `UpperTheory.V.Pa.B2.selN` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.ArchimedeanH0Cards.ExistsCore`, lines 385–453)

Finally use both coefficient sequences with middle term A_m, one with quotient A_e, the other with quotient A_{m-e}. Denote the induced maps $E_{m-e} \to E_m \to E_e$ by i, p, and the reduction $E_m \to E_{m-e}$ by r. The inclusion on coefficients for i is multiplication by t^e, so $ir = t^e$. We give the two index estimates. If a pair reduces to a boundary under p, lift its bounding global cochain and its allowed local corrections, then subtract their boundary. At an ordinary place a local correction is a boundary and its bounding vector lifts. At a tame place let I be inertia and $J = I \cap \bigcap_i \ker \bar{\chi}_i$ the prescribed inertia kernel. The coefficient modules are trivial on J, and $\Delta = I/J$ is finite of order prime to ℓ. An allowed local cocycle vanishes on J after subtracting its prescribed boundary. Its restriction to Δ is a boundary, since averaging over Δ gives $H^1(\Delta, M) = 0$ for each coefficient module M here. Lift the bounding vector and subtract its boundary. The remaining cocycle is inflated from the unramified quotient with values in M^{Δ}. The averaging projector $|\Delta|^{-1} \sum_{\delta \in \Delta} \delta$ commutes with coefficient reduction, so a surjection of the coefficient modules induces a surjection on their Δ-invariants. Lift the Frobenius value in these invariants and form the corresponding continuous cocycle of the procyclic unramified quotient. Adding back the lifted boundary gives the required local lift. All averaging denominators are units, so this introduces no loss depending on m. The remaining pair has coefficients in $t^e A_m$. Thus $\ker p \subseteq \operatorname{im} i$. Moreover $\#\operatorname{coker} r \le \#D_e$. To see this, lift the global two-class of a pair at depth $m - e$ to depth m, using global ℓ-cohomological dimension two, and adjust by a cochain so its reduction is the given representative. Lift the local one-cochains. The differences of their differentials from the local images of the lifted global representative are local two-cocycles with coefficients in the kernel A_e. Their tuple modulo global degree-two localisation is the defect in D_e. A zero defect permits subtraction of a global kernel two-cocycle and correction by local kernel one-cochains. The pair then lifts. Changing the lifts changes the defect by a local boundary or the localisation of a global kernel class. The same construction on differences gives the asserted index bound. Consequently

$$\#(E_m / t^e E_m) \le \#E_e \, \#D_e \le Q^{ge + c_b + c_n + c_h \#\{v \mid \ell\} + 2c_d}.$$

All constants were chosen from the original curve, the fixed local groups and the finite detecting set.
Lean: `UpperTheory.V.Pa.B2.exactMid` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.ArchimedeanH0Cards.ExactMid`, lines 100–115)
Lean: `UpperTheory.V.Pa.B2.cokerBound` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.ArchimedeanH0Cards.CokerBound`, lines 151–168)

□

Proof of 30.32

Proof. Let Φ be the finitely generated common cover, K_N its level kernels, K_∞ its limit kernel and $\mathcal{U}$ the level ultrafilter. For every finite quotient of Φ, the image of K_N equals the image of K_∞ on a set in $\mathcal{U}$. In the construction by intersection this last assertion follows by compactness: the chosen subgroup images are compatible under maps of finite quotients, and each element of a chosen image lifts to the inverse limit of the images. The same finite-quotient construction gives agreement of the marked elements and of the local maps at that precision.

Fix m. There are finitely many continuous maps from the common covering group to any specified finite group that are homomorphisms, since their values on topological generators determine them. This applies also to its open CM subgroup. Encode a cocycle by the homomorphism with values in the semidirect product of the additive matrix module and the finite matrix image. Intersect the kernels of the finitely many candidate homomorphisms, taking the normal core in the cover if necessary. In the resulting finite quotient agreement of kernel images tests factorisation simultaneously for all candidates. At this precision the coefficient action and the local maps also agree. Upper triangularity after subtraction of a local boundary, its diagonal torsion

condition, and the polarised transpose equation can all be tested there. Each local bounding matrix ranges over a finite set. This proves simultaneous transfer in both directions for the even, V_1- and V_2-cocycles. Boundaries transfer using the same matrix on the common cover. In particular the first bound of 30.31 holds on the limit at every depth with the same constant.

Lean: `UpperTheory.V.Pa.t1` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.SeparableMatrixStability.LcOfAdic`, lines 122–208)

Lean: `UpperTheory.V.Pa.taylorWiles_first_condition_at_limit_of_group_condition` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.SeparableMatrixStability.TaylorWilesFirstConditionAtLimitOfGroupCondition`, lines 19–41)

Suppose b rational classes are independent. Clear denominators in the cocycles and their local witnesses. Evaluate on generators, including the columns of the global boundary map. The span of the b columns in the quotient by the boundary lattice has rank b. Smith form gives a constant c such that a combination reducing to a boundary modulo t^m has every coefficient in $t^{m-c}A$, for $m \geq c$. Hence $(m-c)b \leq mD_* + C_1$ for every such m. Division by m gives $b \leq D_*$.

At a finite level the product and local flag defects of a lift to V_2 give a pair defining a class in E_m. Indeed the unprescribed slot is the odd η-slot, which is square-zero; associativity is the global two-cocycle equation and the flag equations say its local projection is a differential. Changing that slot by a one-cochain and changing a flag witness changes the pair by its defining boundary. Thus vanishing of the class is equivalent to a lift. The defect is linear in the prescribed primed tuple, since products of two primed slots vanish.

At depth m, the $(g+1)$-st largest cyclic summand of E_m, if present, has length $b \leq C_2$: take $e = b$ in 30.31. Project the $g+1$ defects to the largest g summands, lift their coordinates to a matrix over A, and choose a kernel vector over F. Scale to get integral coordinates with one unit. Multiplication by t^{C_2} kills the remaining summands of its defect. Take one level at which the bound and both finite transfers hold. Transfer the prescribed tuples there, take this combination and its lift, and transfer the lift back. Equality of the primed components is equality on the common cover, so it survives this operation.

Lean: `UpperTheory.V.Pa.taylorWiles_second_condition_at_limit_of_group_condition` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.SeparableMatrixStability.TaylorWilesSecondConditionAtLimitOfGroupConditio`, lines 7–26)

For the passage through the depths, first clear denominators in the given tuples. For each m retain the coefficient vector *before* multiplication by t^{C_2}, the lifted semidirect-product homomorphism and its flag witnesses. Their reductions at a fixed smaller depth belong to finite sets. Select a diagonal subsequence with constant reductions at each depth and a fixed unit coordinate modulo t. The compatible homomorphisms define a continuous integral cocycle, since each finite reduction is continuous. Its primed tuple is t^{C_2} times the limiting combination, by equality at every depth and t-adic separation. The coefficient with the fixed unit coordinate remains nonzero over F.

Lean: `UpperTheory.V.Pa.secondCohomologyBound_of_boundaryBound` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.SeparableMatrixStability.SecondCohomologyBoundOfBoundaryBound`, lines 44–119)

For odd cohomology, let $p : \Gamma_\infty \to \Gamma_{\mathrm{base}}$ be the map on the pro-ℓ reductions used in the construction. Its kernel is the closed normal subgroup generated by the marked tame elements τ_j. We now give the finite-quotient check of the reverse inclusion. Let V be an open normal subgroup of the cover giving a quotient of Γ_∞ that kills all τ_j. On a set in $\mathcal{U}$, the same quotient factors through the level group and kills its tame generators, hence factors through the base by 30.67. There are finitely many homomorphisms from the finitely generated base to this finite quotient. Take the one occurring on a set in $\mathcal{U}$. Finite-quotient agreement of the maps to the base identifies its composite with the limit projection. Thus every such finite quotient kills $\ker p$, as required.

Lean: `UpperTheory.V.Pa.ker_oo` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.SeparableMatrixStability.KerOo`, lines 15–79)

In the limit $\phi_j \tau_j = \tau_j \phi_j$, with $p(\tau_j) = 1$ and $p(\phi_j) = s_j$. Evaluating an odd cocycle on the two products shows that $z(\tau_j)$ commutes with $r(s_j)$. Take the tests s_j of 30.31 in the common separable locus, as is possible since that locus is open dense in H' and each selected evaluation is nonzero on an open set. Then the two blocks have disjoint spectra at $r(s_j)$, so $z(\tau_j) = 0$. The cocycle homomorphism therefore descends; on the CM subgroup one also uses the polarisation equation for the conjugates by complex conjugation. The local maps to the base identify the local tests on the descended cocycle. Conversely pullback preserves those tests. If an inflated linear combination is a boundary $d_r Y$, the same matrix Y bounds it on the base by surjectivity. Hence the inflated $[\kappa_j]$ both span and are independent over F.

Lean: `UpperTheory.V.Pa.koszulThree_of_det` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.SeparableMatrixStability.KoszulThreeOfDet`, lines 8–49)

□

Proof of 30.33

Proof. Write $\mathfrak{l} = \mathfrak{m}_\Lambda$ and $h = nq$. At fixed precisions b, j, the modules $M_N/(Y^b + \mathfrak{l}^j)M_N$ are free over $(\Lambda/\mathfrak{l}^j)[T_1, \dots, T_h]/(T)^b$ for all sufficiently high N, by Lemma 30.30(1). Indeed the binomial coefficient of degree $i < b$ in the N-th diamond relation is divisible by ℓ^j once $v_\ell(\mathrm{N}w_N - 1) \geq b + j$. Hence, for $0 \leq a < b$, multiplication by the monomials of degree a identifies copies of the augmentation with the degree-a layer. This map is canonical; a basis at level N proves it is an isomorphism and does not enter its definition.

Use the quotients L_b by sequences tending to zero modulo $Y^b + \mathfrak{l}^j$ along the ultrafilter for every j. The maps $L_{a+1} \to L_a$ are surjective, since the same representative sequence serves at both precisions. For $a \geq 1$ their kernel is canonically $\bigoplus_{|\alpha|=a} L_1 T^\alpha$. For injectivity, a relation maps to zero at every $\mathfrak{l}^j$-precision. The finite-level monomial isomorphism makes each coefficient zero at that precision. For surjectivity, take a representative in the kernel. At each precision j it belongs to $Y^a + \mathfrak{l}^j$ on an ultrafilter set on which that monomial isomorphism holds. Intersect the sets for precisions $1, \dots, j$ and with $N \geq j$. For each N use the largest applicable $j \leq N$, choosing arbitrary values if there is none. This precision tends to infinity along the ultrafilter. Lift the degree-a coefficients at these precisions to M_N. Their sequences give the required finitely many elements of L_1.

At augmentation the two comparison maps with X of Lemma 30.30(2) have composites f, at each $\mathfrak{l}^j$-precision. Choose them at precisions tending to infinity in the same manner. The ultrapower modulo convergence of the constant module X is X: its quotients $X/\mathfrak{l}^j X$ are finite, and X is finite over the complete ring Λ. Each sequence selects one compatible residue at each depth. It follows that $L_1[1/f] \simeq X[1/f]$. Localisation is exact, so the descriptions of all the successive kernels persist after inverting f.

Lean: `UpperTheory.V.Pa.Core.eventually_exists_patchedModule_quotient_maps (Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AnnihilatorBaseChange.EventuallyExistsPatchedModuleQuotientMaps`, lines 193–239)

Take $\mathcal{M} = \varprojlim L_b[1/f]$. Lift a finite generating set of $X[1/f]$ to $\mathcal{M}$ by surjectivity of the transitions. Successively subtract homogeneous polynomials in Y times these lifts, using the kernel isomorphisms. The coefficients form convergent series in $\mathcal{B} = \widehat{R[1/f]}^{\,Y}$. This proves finite generation. Starting the subtraction in degree b for an element zero in $L_b[1/f]$ gives coefficients in $Y^b\mathcal{B}$. It follows that the kernel of projection is exactly $Y^b\mathcal{M}$, and proves the graded identity before local completion.

Since $Y \subset p$ and $f \notin p$, the ideal $p\mathcal{B}$ is prime: modulo Y, the completion has quotient $R[1/f]/Y$. More generally $\mathcal{B}/p^j\mathcal{B} = R[1/f]/p^j R[1/f]$, since $Y^j \subset p^j$ and completion is exact for finite modules over the Noetherian ring $R[1/f]$. Localising these identities and taking the inverse limit identifies $\widehat{\mathcal{B}_{p\mathcal{B}}}$ with B'. This map from $\mathcal{B}$ is flat. Tensor the exact sequences for two consecutive powers of Y to obtain the asserted graded isomorphism over B'.

Lean: `UpperTheory.V.Pa.Core.yPrime (Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AnnihilatorBaseChange.YPrime`, lines 23–123)

Put $p_\Lambda = p \cap \Lambda$. The series variable s maps to t, so $\Lambda/p_\Lambda = k[[t]]$. More explicitly, lift the images of the $d = nd_+$ weight coordinates to series in $O[[s]]$. Subtracting those series from the coordinates identifies p_Λ with the ideal generated by the resulting d variables and the uniformiser. Thus $L = \widehat{\Lambda_{p_\Lambda}}$ is regular of dimension $1 + d$. The augmentation module X is finite free over Λ by base-level freeness and finite free root adjunction, with the prescribed local factor selected by an idempotent. To see the effect of the second completion, let D be the image of R in $\mathrm{End}_\Lambda(X)$. It is finite over Λ. The finite L-algebra $D \otimes_\Lambda L$ is the product of its complete local factors above p_Λ. Completing X at p selects the corresponding direct summand of $X \otimes_\Lambda L$, or zero if p is outside its support. Thus $B' \otimes_R X$ is finite free over L. If M' has a support prime, this augmentation is nonzero by Nakayama.

Apply 30.27 to $L \to B'$, the finite nonzero module M', and the h elements of Y in its maximal ideal. Its polynomial graded-map hypothesis is the isomorphism just proved. Each minimal support prime has quotient dimension $(1 + d) + h$. Finally $\mathfrak{a}B' \subseteq \sqrt{\mathrm{Ann}_{B'} M'}$ by Lemma 30.30(3). Passage to C therefore identifies each minimal support prime and its quotient ring, giving quotient dimension $1 + n(d_+ + q)$. □

Proof of 30.34

Proof. Use the base ordinary module M_0 of 30.23, localised at the residual system. The action sends determinant coefficients and ordered characters to the Hecke coefficients and character operators, with kernel J_0. The initial support hypotheses in (a) and (b) give $M_0 \neq 0$.

Lean: `UpperTheory.V.Hi.hida_unitary_of_printed_hida` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.THidaU`, lines 38–48)

Put $A = k[[t]]$, $F = k((t))$. In each block choose a stable lattice and diagonalise the polarising form, with entries $t^{d_i} b_i$ for units b_i. Integrality of $\rho(s)$ and $P\rho(s)P^{-1}$ bounds entry (i,j) below by $\max(0, d_j - d_i)$ in valuation. Replace t by t^2, extend k to take the square roots of the units, and rescale vector i by $t^{-d_i} b_i^{-1/2}$, with b_i read after base change. The form becomes the identity. The entry bound becomes $2\max(0, d_j - d_i) + d_i - d_j \geq 0$, so the matrices remain integral. The same argument applies to one block. These injective DVR extensions preserve the kernels. Intersect the flags with the lattices. The steps are saturated, and in the two-block case the distinct local characters let us choose the flag bases with each column in its assigned block.

Write $d_r Y(s) = r(s) Y r(s)^{-1} - Y$. An ordinary adjoint cocycle transforms under c by negative transpose and vanishes on the prescribed inertia kernels. At $v \mid \ell$, subtracting $d_r Y_v$ must make the cocycle upper triangular in the flag basis, with zero diagonal on the preimage of the ℓ-primary torsion of the local abelianisation. Global boundaries use $Y^t = -Y$. Laurent coefficients have bounded entries, so a power of t makes them integral and adically continuous. Call matrices even on the diagonal blocks and odd on the off-diagonal blocks. The integral odd cocycles form a finite A-module. Their values on generators of the maximal pro-ℓ quotient of the open residual kernel and on finitely many coset representatives determine them. The finite-character hypothesis of 30.67 bounds the number of these generators. Choose integral ordinary odd cocycles $\kappa_1, \dots, \kappa_g$ giving a basis of the classes over F. For one block take $g = 0$.

Use the commutative coefficient algebra with basis $1, e_i, e_i', \eta$, $1 \leq i \leq g$, in which $e_i e_j' = \delta_{ij}\eta$ and all other products of positive-degree elements vanish. Set

$$\rho_1(s) = \left(1 + \sum_i e_i \kappa_i(s)\right) \rho(s).$$

For a flag basis w_v and local witnesses Y_{iv} for κ_i, its flag basis is $(1 - \sum_i e_i Y_{iv}) w_v$. Over a coefficient ring X, let $V_1(X)$ have even entries in X and odd entries in $\sum_i X e_i$. Let $V_2(X)$ have even entries in $\sum_i X e_i'$ and odd entries in $X\eta$. Both use conjugation by ρ_1. The trace dual of V_1 has even entries in $X\eta$ and odd entries in $\sum_i X e_i'$, with pairing given by the η-coefficient of trace. Use the cyclotomic twist of conjugation on this dual. On H' the tests on its cocycles are $z \mapsto [\eta] \operatorname{tr}(z(s)\rho_1(s))$.

To prove detection, write $z = \eta p + \sum_b e_b' l_b$ and extend $K_b(s) = \kappa_b(s)\rho(s)$ and $L_b(s) = l_b(s)\rho(s)$ linearly on $F[H']$. Absolute irreducibility and inequivalence give a surjection from this algebra to the product of the full matrix blocks. For c_1, c_2 in its kernel the product rule kills the terms with $\rho(c_1)$ or $\rho(c_2)$. Multiply by lifts of the diagonal matrix units selecting opposite-block entries i, j. The trace test gives

$$\sum_b \left(K_b(c_1)_{ij} L_b(c_2)_{ji} + L_b(c_1)_{ij} K_b(c_2)_{ji}\right) = 0.$$

For fixed c_2, multiplication on each side by lifts of matrix units shows that this combination kills all relations in the corresponding corner. It descends to a derivation of the product algebra in its corner bimodule. Applying the derivation law to the central idempotents shows that this derivation is inner. The untwisted and twisted parts restrict to distinct eigenspaces for conjugation by an element of H of nontrivial residual cyclotomic value, which exists since $\mu_\ell \not\subset L$. Each part is therefore inner on H'. An odd cocycle vanishing on H' takes values in intertwiners between the two blocks, hence vanishes. Polarised transposition determines the opposite corner. Independence of the κ_b now gives $L_b(c_2)_{ji} = 0$ for every b, c_2. The corner derivation argument gives

$$l_b(s) = \varepsilon_\ell(s) \operatorname{Ad}(\rho(s)) Y_b - Y_b.$$

Subtract the twisted boundary of $\sum_b e_b' Y_b$, whose even term is $\eta \varepsilon_\ell(s) \sum_b [\kappa_b(s), \operatorname{Ad}(\rho(s)) Y_b]$. For the remaining even derivative, apply the trace identity to a relation multiplied by a group element. Nondegeneracy of trace on the full blocks makes the derivative vanish on all relations. Its descent is inner. Subtracting

this boundary leaves a cocycle on the cyclotomic quotient with values in the block scalars. If $\bar{\varepsilon}_\ell(h) \neq 1$, its cocycle law gives

$$(\bar{\varepsilon}_\ell(h) - 1)z(s) = (\bar{\varepsilon}_\ell(s) - 1)z(h).$$

Thus it is a boundary. Average a bounding matrix with its polarised image, using $2 \in A^\times$, and clear denominators. This gives detection modulo t-power torsion. For one block the even derivative argument applies directly.

The intersection of the separable loci of the finitely many curves is dense and open in H'. Continuity gives detection on that intersection. Choose tests decreasing the dimension of the common kernel modulo boundaries, take the union for the curves and call the resulting elements $s_1, \dots, s_q$. Split their polynomials and equalise the residue fields by finite injective extensions of the DVRs.

Lean: `UpperTheory.V.Pa.Det.relPair` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.DeterminantPadicFree.RelPair`, lines 23–176)
Lean: `UpperTheory.V.Pa.Det.ann` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.DeterminantPadicFree.BlockPrinc`, lines 75–160)
Lean: `UpperTheory.V.Pa.Det.detAlg` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.DeterminantPadicFree.DetAlg`, lines 71–141)
Lean: `UpperTheory.V.Pa.choice` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AutomorphicDetection.Choice`, lines 14–40)
Lean: `UpperTheory.V.Pa.Kap` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.PairedDualFirstOrderDualMatrixSpace`, lines 306–315)
Lean: `UpperTheory.V.Pa.Detects` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.ThetaCoefficientMapEq`, lines 257–258)

Apply 30.67 to these elements. The residual action on the kernel of the diagonal characters is unipotent and its congruence kernels are pro-ℓ. Put $d_+ = [L^+ : \mathbf{Q}]$, $D_* = n(d_+ + q)$, and $\nu = \binom{a}{2} + \binom{n-a}{2} + ga(n-a)$ for first block size a, with $a = n$ in the one-block case. At auxiliary level N and depth m set $A_m = A/(t^m)$. Let W_m be the ordinary $V_1(A_m)$-cocycles for the pulled-back ρ_1 modulo skew-transpose boundaries. Use the full local cohomology condition at each added place.

Lean: `UpperTheory.V.Pa.SelZ` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.PairedDualFirstOrderDualMatrixSpace`, lines 240–244)
Lean: `UpperTheory.V.Pa.Sel1` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.PairedDualFirstOrderDualMatrixSpace`, lines 344–344)
Lean: `UpperTheory.V.Pa.TWGrp.B1` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.ThetaCoefficientMapEq`, lines 54–56)

For the obstruction group use the odd module M of ρ over A_m, with c acting by negative transpose. At $v \mid \ell$ let $\pi_v : M \to Q_v$ quotient by the upper triangular odd matrices in the flag basis and take $J_v = G_v$. At a prescribed tame group use $\pi_v = 1$, $Q_v = M$, and J_v its prescribed inertia subgroup. A degree-two representative consists of a global two-cocycle b and continuous local one-cochains a_v with $da_v = \pi_v(b|_{G_v})$. Quotient by the pairs with $b = dc_1$ and $a_v - \pi_v(c_1|_{G_v})$ a cocycle restricting to a boundary on J_v. Call the group E_m. For each m, at all sufficiently high N,

$$\log_{\#k} \#W_m \le mD_* + C, \qquad \log_{\#k} \#(E_m/t^e E_m) \le ge + C' \quad (1 \le e \le m), \tag{A.263}$$

with C, C' independent of m, N, by Lemma 30.31.

The local factors used for the first bound have exponents at most

$$m[n(1 + d'_v) + \nu(1 + d_v)] + C_v \quad (v \mid \ell), \qquad mn + C_j \quad (v \text{ auxiliary}). \tag{A.264}$$

Here d'_v is the free inertial abelian rank and d_v the local degree in the strict upper count, with both lists summing to d_+. At an added place the pulled-back representation kills the tame generator $\tau_{N,j}$. The tame relation, the congruence $Q_j \equiv 1 \pmod{\ell^N}$ and the cocycle equation imply that its cocycle value commutes modulo t^m with $\rho_1(h_N(\phi_{N,j}))$. At fixed m, this matrix equals $\rho_1(s_j)$ modulo t^m for sufficiently high N. Over F the centralizer in the paired coefficient module has dimension n. In a block eigenbasis the constant term is diagonal, and the simple root differences determine the off-diagonal coefficients in the first-order slots from that diagonal. Equivalently, one can diagonalise the lifted matrix by a first-order change of basis over F, where the root differences are invertible. Apply Smith reduction over A to the commutator map. Each free kernel direction contributes m to the length of the truncated kernel. A nonzero elementary divisor t^b contributes $\min(m, b)$, which is bounded by its fixed valuation b. This gives $mn + C_j$ for the inertia term of the full local condition after division by the unramified factor.

Lean: UpperTheory.V.Pa.h0q (Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.EpsilonEtaMatrices.FinrankCentralizerW1Le, lines 106–131)

At D_0 the twisted intertwiner equations have zero rational kernel, so Smith reduction bounds their truncated invariant lengths. The detecting evaluations bound the dual kernel modulo boundaries by the same argument. At ℓ the diagonal abelian characters contribute $n(1 + d'_v)$ and the strict upper terms contribute at most $\nu(1 + d_v)$ per unit of depth. Finite duality in 27.2 and the global Euler characteristic [111, I, Thm. 5.1] subtract the boundary and flag contribution $m[(n + \nu)\#\{v \mid \ell\} + \nu d_+]$ from the sum of (A.264). The remaining coefficient of m is D_*.

For the second bound the product defect of a V_2-lift gives b and the local defects give a_v. Changing the lifts changes the pair by its defining boundaries. The depth sequence $m - e, m, e$ bounds $E_m/t^e E_m$ by the obstruction in depth e and the degree-two localisation cokernel. The base Selmer term in 27.2 counts ordinary odd classes. Evaluate cocycles on finitely many generators and include local witnesses. The rows of the equations generate a finite A-module. Smith reduction bounds the excess length of the truncated solution space over the integral solutions by the sum of the valuations of its nonzero elementary divisors. The integral classes have rank g, giving $ge + C_0$. At added places the odd centralizer has zero rational kernel since the block spectra are disjoint. At ordinary places the lower graded ratios are nontrivial, and a separating element bounds the global dual odd invariants. The truncated invariant lengths are thus bounded by fixed elementary divisors. By 27.2 with all ordinary and added places included, the global dual invariants bound the degree-two localisation cokernel. This gives C'.

Lean: UpperTheory.V.Pa.B2.sel0 (Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.ArchimedeanH0Cards.ImgCard, lines 340–364)

The $(g+1)$-st largest elementary divisor of E_m has length at most C', by applying (A.263) at that length. For $g + 1$ obstruction vectors, lift their projection matrix on the first g cyclic summands to A. A primitive vector in its kernel gives a combination supported in the other summands. Multiplication by $t^{C'}$ kills it, so this multiple of the primitive combination lifts.

Lean: UpperTheory.V.Pa.W1_bound_arithmetic_rank_two (Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.EpsilonEtaMatrices.W1BoundArithmeticRankTwo, lines 40–165)

Lean: UpperTheory.V.Pa.B2.obstructionBound_of_duality (Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.ArchimedeanH0Cards.ObstructionBoundOfDuality, lines 145–216)

For an open normal subgroup H of a profinite group, write $N_\ell(H)$ for the kernel of its maximal pro-ℓ quotient. At each level in 30.67, divide the level group by this kernel for the inverse image of the base subgroup H. Their open pro-ℓ subgroups have uniformly bounded generator numbers. Take surjections $\pi_N : \Phi \to \Gamma_N$ onto these quotients from a free profinite group on lifts of the common finite quotient, a lift of complex conjugation and enough pro-ℓ generators. Fix a nonprincipal ultrafilter on the levels. In each finite quotient of Φ, take the image of $\ker \pi_N$ occurring on a set in the ultrafilter. Intersect their inverse images to obtain a closed normal K, and put $\Gamma_\infty = \Phi/K$. The marked elements and local maps pass to the limit in finite quotients. The limiting τ_j, ϕ_j commute by (30.24)–(30.25) and map to $1, s_j$ in the base group.

At fixed depth, generator and local witness values belong to finite sets, and a representation with its cocycle is a homomorphism to a finite semidirect product. The simultaneous transfer of cocycles and boundaries at this precision, and the passage through all depths, are given in Lemma 30.32. Compactness gives compatible lifts in all depths, retaining a unit coordinate in the primitive combination before multiplying by $t^{C'}$. Clearing a common denominator and inverting t gives a nonzero liftable combination of any $g+1$ tuples of even classes. For b independent $V_1(F)$-classes, Smith reduction on finite generator evaluations modulo boundaries gives at least $(\#k)^{(m-c)b}$ classes at depth m for fixed c. The first bound of (A.263) gives $b \le D_*$. An odd cocycle on the limit has value at each τ_j in the odd centralizer of $\rho(s_j)$, hence zero. The τ_j normally generate the kernel to the base in the pro-ℓ quotient. Therefore $[\kappa_j]$ still form a basis of the ordinary odd classes.

Lean: UpperTheory.V.Pa.cohLim (Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AutomorphicDetection.CohLim, lines 12–19)

Lean: UpperTheory.V.Pa.koszulThree_of_det (Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.SeparableMatrixStability.KoszulThreeOfDet, lines 8–49)

Extend O unramified to the common residue field and adjoin a power series variable s evaluated at t on each curve. Lift the curve and shared primes through the finite free extension by lying-over, going-down and going-up, using the extended shared primes for the power series variable. Adjoin the ordered roots at s_j by successive monic root adjunctions and division by the corresponding linear factors. The local factor B^{sp} for the prescribed residual order is nonzero and finite flat over the enlarged base. Lift the primes through this

extension by the same prime-lifting argument. The ordered roots are integral on the extended curve DVRs.

Put $d = nd_+$, $\Lambda = O[[s, w_1, \dots, w_d]]$ and $S = \Lambda[[y_{ji}]]$, where w_b are the weight parameters and there are nq variables y_{ji}. Use the common complete local ring R for the continuous determinant on the CM subgroup of Φ, the ordered characters with their prescribed torsion values, and the independent variables $s, U_{ji} - \widetilde{u}_{ji}, y_{ji}$, with $\widetilde{u}_{ji}$ lifting the residual roots. Pullback of the base data, the adjoined roots and $y_{ji} \mapsto 0$ give an adically surjective map $\pi : R \to B^{\mathrm{sp}}$.

Let $\mathfrak{a} \subset R$ be generated by the polarisation relations and the traces of the following words after left multiplication by arbitrary group elements. Use $[k] - 1$ for the kernel to the limit, the ordered ordinary products and prescribed inertia differences of (b), and

$$\prod_i ([\phi_j] - U_{ji}), \qquad \prod_{b \neq i} ([\phi_j] - U_{jb})([\tau_j] - 1 - y_{ji}).$$

Use all lifts of the indicated elements to the common cover. These relations vanish under π. For the preimage $p_i \subset R$ of curve prime i, set $C_i = \widehat{(R/\mathfrak{a})_{p_i/\mathfrak{a}}}$. Its coefficient DVR is $O_* = \widehat{O[[s]]_{(\varpi)}}$, with ϖ a uniformiser of O and residue field $F = k((t))$ via $s \mapsto t$.

Lean: `UpperTheory.V.Pa.Core.rrEx` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AnnihilatorBaseChange.FiniteOpenHomsToFiniteGroup`, lines 225–328)
Lean: `UpperTheory.V.Pa.Core.Rel` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AnnihilatorBaseChange.SpecializationMapCompUniversalSection`, lines 103–116)
Lean: `UpperTheory.V.Pa.ooEx` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.SelmerScaling.OoExKer`, lines 106–195)

Fix $C = C_i$. Form the finite Cayley–Hamilton algebra E over C from the group algebra of the CM limit group by the closed Cayley–Hamilton, ordinary, inertia and auxiliary relations and their prescribed adjoints. The curve representation maps E onto the full block algebras over F. Lift the diagonal matrix units to orthogonal self-adjoint idempotents p_b summing to one. At each step average a lift with its adjoint in the complementary corner and use the projector for the Hensel factor at eigenvalue one. The determinant rank of p_b is one by reduction, so Cayley–Hamilton gives $p_b z p_b = \mathrm{tr}(p_b z p_b) p_b$. In each block choose a distinguished index $o(b)$ and lift $E_{o(b),b}$ to $x_b \in p_{o(b)} E p_b$. Rescale by the square root of a scalar unit congruent to one to get $x_b^* x_b = p_b$, $x_b x_b^* = p_{o(b)}$.

For two blocks let p, p' be their distinguished idempotents, $N = pEp'$ and $\beta(v, w) = \mathrm{tr}(vw^*)$. The corner identities give $\beta(v, w)z = v(w^* z) = \beta(w, z)v$. The pairing is symmetric and vanishes in the residue field. Form the finite commutative algebra

$$A' = C \oplus N, \qquad (b, v)(b', w) = (bb' + \beta(v, w), bw + b'v).$$

Read $x_i z x_j^*$ as a scalar in each diagonal block, as an element of N in pEp', and by its adjoint in $p'Ep$. Insertion of $\sum_j x_j^* x_j = 1$ gives the matrix multiplication law. Denote the group matrices by r_u. For one block take $N = 0$, $A' = C$. Put $I_N = (\beta(v, w) : v, w \in N)$ and $I_2 = (\varpi) + \mathfrak{m}_C^2 + I_N$.

Lean: `UpperTheory.V.Pa.idem` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.ClosureConjugation.OBLt`, lines 43–65)
Lean: `UpperTheory.V.Pa.gr_E` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.ClosureConjugation.ComplOddPart`, lines 48–121)

Let $T = (\mathfrak{m}_C / I_2)^\vee$. An O_*-derivation $\alpha \in T$ gives an ordinary even cocycle A_α by applying $b + v \mapsto \bar{b} + \delta\alpha(b)$, $\delta^2 = 0$, to r_u and multiplying the derivative by ρ^{-1}. If A_α is a boundary, infinitesimal conjugation makes the matrices constant. The determinant coefficients are then constant. In the i-th diagonal entry of each ordered-product identity at $v \mid \ell$, specialise the other slots at elements separating $\psi_{v,j}$ from $\psi_{v,i}$. All factors except the i-th are units, so $\psi_{v,i}$ is constant. Simple-root uniqueness fixes the auxiliary roots, and the projected tame relations fix the y_{ji}. Since s belongs to O_*, these generators force $\alpha = 0$.

Let $U \subset T^g$ consist of tuples (α_j) for which a bounded ordinary $V_2(F)$-cocycle Ψ has $[e_j']\Psi = A_{\alpha_j}$ for every j. The lifting bound on the limit gives $\operatorname{codim} U \leq g$. For a basis element α of $W = \bigcap_j \iota_j^{-1} U$, take witnesses Ψ_j for A_α in slot j. Then

$$y_\alpha = A_\alpha + \sum_j e_j [\eta] \Psi_j$$

is an ordinary $V_1(F)$-cocycle. Its constant component and $e_j' y_\alpha = \Psi_j$ give the cocycle equation in every degree. For the local condition choose a witness Y_v for A_α. The primed witness in slot b for Ψ_j differs

from $\delta_{jb}Y_v$ by an invariant modulo the upper triangular matrices. The nontrivial graded ratios force its lower entries to vanish successively. Subtract the upper triangular corrections in the lifted flag basis. The η-components of the resulting witnesses give the local e_j-components for y_α.

The classes of the y_α and the g^2 classes $e_j\kappa_i$ are independent together. The products $e_je_b = 0$ give the cocycle and flag equations for $e_j\kappa_i$. In a relation equal to $d_{\rho_1}Y$, the constant part and injectivity for A_α kill the coefficients of the y_α. The constant term of Y then commutes with the full blocks and is skew-transpose, so it is zero. The e_j-components of the relation now give linear combinations of the κ_i that are ordinary boundaries, and all their coefficients vanish. The limit bound gives $\dim W + g^2 \le D_*$. Apply 30.26, and lift its spanning forms X_b to $x_b \in \mathfrak{m}_C$.

Lean: `UpperTheory.V.Pa.lift1` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.SelmerScaling.FirstOrderMatrixMemSquareZero`, lines 279–388)

Lean: `UpperTheory.V.Pa.claim1` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.SelmerScaling.FirstOrderLiftConstantPart`, lines 134–244)

For $\mu \in (N/\mathfrak{m}_C N)^\vee$, differentiating the odd entries of r_u and multiplying by ρ^{-1} gives an odd cocycle K_μ. An inner derivative vanishes on the lifted idempotents, since their represented matrices are the constant E_{ii}. An odd bounding matrix commuting with all E_{ii} is zero. The group algebra spans the distinguished corner, so $\mu \mapsto [K_\mu]$ is injective. Write its coordinates on $[\kappa_j]$ as $\mu(m_j)$ by duality. Lifts of the m_j generate N by Nakayama's lemma.

For an F-linear functional $\lambda : N/I_2N \to F$, write the bilinear difference $\lambda(bv) - \bar{b}\lambda(v)$ on the two cotangent spaces as $\sum_h \alpha_h(b)\mu_h(v)$. With paired slots indexed by h the map

$$b + v \longmapsto \bar{b} + \sum_h e_h\mu_h(v) + \sum_h e'_h\alpha_h(b) + \eta\lambda(v)$$

is multiplicative. Its mixed products give the bilinear difference and the α_h kill I_N. Apply the map to r_u and use first-order odd conjugation to replace each K_{μ_h} by $\sum_j d_{hj}\kappa_j$, where $d_{hj} = \mu_h(m_j)$. Use κ_j in the unprimed slots and $\sum_h d_{hj}A_{\alpha_h}$ in the primed slots and keep the constant and η-parts. These changes are transposes for the mixed pairing, hence preserve the product identities, polarisation and inertia relations. Denote the resulting representation by r', put $I = (e'_1, \dots, e'_g, \eta)$ in the slot algebra, and set $Z(s) = r'(s)\rho_1(s)^{-1} - 1$. The constant and unprimed coordinates agree with those of ρ_1, so Z has entries in I. The block parity leaves primed entries on the diagonal blocks and η-entries on the off-diagonal blocks, precisely $V_2(F)$. Since $I^2 = 0$, multiplication gives $Z(st) = Z(s) + \rho_1(s)Z(t)\rho_1(s)^{-1}$. The common polarisation identity gives negative transpose under conjugation, using $(1+Z)^{-1} = 1 - Z$. On the prescribed tame inertia kernels both representations are one.

We check the denominators before passing to cohomology. Write $A = k[[t]]$, $F = k((t))$, and let R_0 be the integral coefficient ring before localization and completion at the curve prime. Its curve map to A is onto, with the power series in the curve coordinate lifted from the coefficient ring. Under the slot map their reductions are those series in t. The kernel of the curve map has finitely many generators since R_0 is Noetherian. Their images lie in the augmentation ideal J' of the slot algebra, where $(J')^3 = 0$. Successively expanding modulo $J', (J')^2, (J')^3$ shows that the image of R_0 lies in the finite A-algebra generated by these nilpotent images: subtract a lift of the curve value and write the remainder in the kernel generators at each step. Take entries of a finite set of generators of the integral Cayley–Hamilton matrix module. Their images span, over this finite algebra, a finite A-module containing the slot images of every group matrix. Each coordinate of this module in F has a common denominator t^b. Moreover the maximal ideal of R_0 maps into $(t) + J'$ in the finite algebra, so its $(m+3)$-rd power maps into t^m times that algebra. Adic continuity of the integral group matrices therefore makes all the cleared coordinates continuous modulo every t^m. Conjugation by the fixed change of basis, finite slot sums, multiplication, and substitution $s \mapsto s^{-1}$ preserve this bounded continuity. The same holds for $\rho_1(s^{-1})$, whose entries are formed from the integral base matrices and the finitely many chosen bounded cocycles κ_j. It follows for Z.

On each ordinary decomposition group apply Lemma 30.29 with $r_0 = \rho_1$, $r_1 = r'$, and this I. The local characters agree modulo I, their augmentations are pairwise distinct by the curve hypothesis, and the transformed ordered products vanish. Use the subalgebra with the displayed block parities. Its intersection with $M_n(I)$ is $V_2(F)$, so the lemma gives a single local witness $Y_v \in V_2(F)$ for all primed coordinates. In the flag basis $Z - dY_v$ is upper triangular. Its diagonal vanishes on the prescribed preimage of the local abelian torsion because both systems of flag characters take value one there. This proves the ordinary

local condition. Finally $[e'_j]Z = \sum_h d_{hj} A_{\alpha_h} = A_{a_j}$, where the bilinear difference evaluated at m_j gives $a_j(b) = \lambda(bm_j) - \bar{b}\lambda(m_j)$. This bounded ordinary Z witnesses $(a_j) \in U$, and 30.26 gives

$$\lambda\left(\sum_{j,b} \widetilde{c}_{ij,b} x_b m_j\right) = \sum_{j,b} c_{ij,b} X_b(a_j) = 0$$

for lifts $\widetilde{c}_{ij,b} \in O_*$. Since the functionals separate N/I_2N, the row sums belong to I_2N.
Lean: `UpperTheory.V.Pa.dualEx` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.SelmerScaling.DualEx`, lines 94–134)
Lean: `UpperTheory.V.Pa.claim2` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.SelmerScaling.Claim2`, lines 11–33)

Map $P = O_*[[X_1, \dots, X_{D_*}, u_1, \dots, u_g]]$ to A' by $X_b \mapsto x_b$, $u_j \mapsto m_j$. The images span the relative cotangent space because I_N maps into $\mathfrak{m}_{A'}^2$, so completeness gives surjectivity. Express row i as $\sum_j w_{ij} m_j$ with $w_{ij} \in I_2$, and lift w_{ij} to $v_{ij} \in (\varpi) + \mathfrak{m}_P^2$. Put $z_{ij} = \sum_b \widetilde{c}_{ij,b} X_b - v_{ij}$ and $Z = (z_{ij})$. Then $Zu = 0$ in A'. The z_{ij} give independent relative cotangent vectors by 30.26, independent also of the u_j. For one block this is a surjection from $O_*[[X_1, \dots, X_{D_*}]]$ to C.
Lean: `UpperTheory.V.Pa.pres` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.SelmerScaling.AlCSmul`, lines 12–147)

Apply Lemma 30.30 to the auxiliary tower, the ultrafilter, R, Λ, S, $\mathfrak{a}$, $Y = (y_{ji})$, $f = \prod_j \prod_{i \neq b} (U_{ji} - U_{jb})^2$, and $X = B^{\mathrm{sp}} \otimes_{B_0[[s]]} M_0[[s]]$. On each curve the image of f is nonzero by separability at s_j. Lemma 30.30(3) gives the finite $\mathcal{B}$-module $\mathcal{M} = \varprojlim_b L_b[1/f]$ over $\mathcal{B} = \widehat{R[1/f]}^{\,Y}$ with $\mathcal{M}/Y\mathcal{M} \simeq \mathcal{B} \otimes_R X$, $\mathrm{gr}_Y \mathcal{M} \simeq (\mathcal{M}/Y\mathcal{M})[y_{ji}]$, and $\mathfrak{a}\mathcal{B} \subseteq \sqrt{\mathrm{Ann}_{\mathcal{B}} \mathcal{M}}$.

Complete $\mathcal{B}$ locally at the prime over p_i and write $\widehat{M}_i$ for the base change of $\mathcal{M}$. The ring before quotienting by $\mathfrak{a}$ is the completion of R_{p_i}, since $Y \subset p_i$ and $f \notin p_i$. Let S_i be the image of $\mathrm{Ann}(\widehat{M}_i)$ in C_i. Dividing by $\mathfrak{a}$ preserves the support dimensions. The weight prime of the curve has completed local ring regular of dimension $1 + d$, since Λ has dimension $2 + d$ and the curve quotient is $k[[t]]$. The augmentation of $\widehat{M}_i$ is finite free over that ring. Indeed, M_0 is finite free over Λ_0, so $M_0[[s]] = M_0 \otimes_{\Lambda_0} \Lambda$ is finite free over Λ, the root adjunction is finite free over the enlarged base, and completion at the chosen prime selects a direct summand of the module completed at the weight prime. Lemma 30.33 identifies each completed graded piece by its multiplication map. Thus the nq auxiliary parameters form a regular sequence on the nonzero completed module. A system of weight parameters continues it to a regular sequence with finite-length quotient. Apply 30.27 to the completed weight ring, the completed module and its nq auxiliary parameters. The augmentation is finite free as above, and flat base change preserves the displayed graded-piece isomorphism. Here $d_0 = 1 + d$ and $h = nq$, so $d_0 + h = 1 + D_*$. Since $\mathfrak{a}$ lies in the radical of the annihilator, quotienting the ambient ring by $\mathfrak{a}$ identifies the prime quotients at the minimal support primes. We obtain

$$\dim C_i/Z_0 = 1 + D_* \qquad (Z_0 \text{ minimal over } S_i). \tag{A.265}$$

Let Q be a prime of C_i, with preimage $\widetilde{Q}$ in the completion before imposing $\mathfrak{a}$. Suppose $Y \subseteq \widetilde{Q}$ and $\widetilde{Q} \cap R = \pi^{-1}(P_Q)$ for a prime P_Q of B^{sp}. Then

$$\begin{aligned} S_i \subseteq Q &\iff \widehat{M}_i \otimes \kappa(\widetilde{Q}) \neq 0 \\ &\iff X \otimes_R \kappa(\widetilde{Q}) \neq 0 \iff J_0 \subseteq P_Q \cap B_0. \end{aligned} \tag{A.266}$$

The first equivalence is Nakayama's lemma, the second uses augmentation with f invertible, and the third uses extension of the residue field of the original ordinary fibre.
Lean: `UpperTheory.V.Pa.Core.pmFib` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AnnihilatorBaseChange.PmFib`, lines 7–31)
Lean: `UpperTheory.V.Pa.Core.pmDim` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AnnihilatorBaseChange.PmDimQuot`, lines 99–149)
Lean: `UpperTheory.V.Pa.s5` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AutomorphicDetection.AnnBc`, lines 69–85)
Lean: `UpperTheory.V.Pa.core` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AutomorphicDetection.LiftPt`, lines 76–175)

In (a), (A.266) puts the curve prime in the support. For a minimal support prime Z_0 below it the surjection $O_*[[X_1, \dots, X_{D_*}]] \to C_i/Z_0$ has zero kernel by (A.265), since a nonzero ideal decreases dimension.

Thus $Z_0 = 0$. Lift a prime below the curve kernel to the splitting algebra by going-down and to the curve completion by faithful flatness. The lift contains $S_i \subseteq Z_0$, so (A.266) gives (a).

For (b), stability is detected by a nonzero minor $\det(\mathrm{tr}_D(g_i h_j))_{i,j \le n^2}$. Indeed full span gives a nonsingular trace pairing, and a nonzero minor forces full span in the semisimple representation. Thus stability persists under generalisation. At a stable prime $Q \subset C$, $I_N \not\subseteq Q$, since killing the odd products gives a direct sum determinant. At a prime with nonsemisimple specialised group algebra one has $\mathrm{Ann}_C N \subseteq Q$. Otherwise the odd corners vanish and the algebra is a quotient of the product of full matrix algebras. A lift of a thick base prime at augmentation has nonsemisimple specialised algebra. The projection to the base group sends τ_j to one and ϕ_j to s_j. The auxiliary relations vanish by Cayley–Hamilton and $y = 0$. The closed ordinary and inertia relations map into the base closed ideal by the finite quotients defining closure. After a common residue field extension the specialised algebra therefore surjects onto the nonsemisimple base algebra.
Lean: `UpperTheory.V.Pa.gr_thick` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.ClosureConjugation.GrThick`, lines 77–85)
Lean: `UpperTheory.V.Pa.Core.nonssT` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AnnihilatorBaseChange.NonssT`, lines 69–99)

Let Z_0 be a minimal support prime with stable contraction and Q a target of either kind. Write ζ for the image of Z in C. The adjugate gives $(\det \zeta)N = 0$. Hence $\det \zeta \in Q$ in the nonsemisimple case by the annihilator condition and in the stable case by $(\det \zeta)I_N = 0$. Lift Z_0, Q to A' by integrality, then pull back to $Z', Q' \subset P$. Both contain (Zu). Some $u_j \notin Z'$, since otherwise $I_N \subseteq Z_0$. In particular $g > 0$, and the adjugate gives $\det Z \in Z'$. The target condition gives $\det Z \in Q'$.

The entries of Z, u have independent cotangent classes in the regular local presentation. Lemma 30.28 therefore gives a least prime over $(Zu, \det Z)$. Adjoin a kernel line with chart $v_c = 1$ and equations $Zv = 0$, $u_b = u_c v_b$ for $b \neq c$. Over the closed point of P the direction coordinates form an affine space. At each prime of that fibre the ambient ring is regular. The row equations have independent linear parts in column c, and the vector equations have independent parts in the u_b, $b \neq c$. This follows from relative cotangent independence and $v_c = 1$. The local quotient is regular and is a domain. Its generic residual direction gives a contraction to P common to the charts, by changing coordinates $v \mapsto v/v_b$ on their overlap. At each specialised residual direction the regular local quotient has this same generic contraction. Given any prime over $(Zu, \det Z)$, dominate its quotient by a valuation ring. Over its fraction field choose a kernel vector, using u itself if nonzero. Divide by a coordinate of least valuation to define a point in one chart. Its closed centre is in the residual direction fibre, so the common generic contraction is contained in the given prime. This proves the least-prime assertion.

Adjoin $Z = \mathrm{diag}(u)$ to the row equations. Then $u_j^2 = 0$ for all j, so the radical of the enlarged ideal is generated by $g^2 + g$ independent parameters. The height theorem gives

$$\dim P/(Zu) \le \dim P - g. \tag{A.267}$$

By integrality and (A.265), $\dim P/Z' = \dim C/Z_0 = \dim P - g$. The least prime over $(Zu, \det Z)$ is contained in Z'. Strict inclusion would make its quotient dimension larger than $\dim P - g$, contrary to (A.267). Therefore it equals Z' and is contained in Q'. This gives $Z_0 \subseteq Q$.

Lift the starting stable prime to the curve completion. It is in the support by (A.266), and a minimal support prime beneath it has stable contraction by the trace minor. At a shared thick prime lift the common fibre prime to Q in the current completion. The containment just proved puts Z_0 below Q. Before imposing $\mathfrak{a}$, contract the preimage of Z_0 to $z \in \mathrm{Spec}\,\mathcal{B}$. The contraction of $\widetilde{Q}$ contains Y, and in $\mathcal{B}/Y = (R/Y)[1/f]$ it lies under the next curve prime. Hence z lies under that curve prime as well. Faithful flatness gives a prime above z in the next completion. Both completed fibres over z are field extensions of the same nonzero fibre of $\mathcal{M}$, so the new prime is in the support. It has the same stable contraction to $R/\mathfrak{a}$. Iterate the construction and at the last curve use $Z_0 \subseteq Q$ for the target prime, followed by (A.266). For $h = 0$ this last argument applies at the starting curve.
Lean: `UpperTheory.V.Pa.Core.pmMove` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AnnihilatorBaseChange.PmMove`, lines 8–23)
Lean: `UpperTheory.V.Pa.presOut0` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.SelmerScaling.PresOut0`, lines 9–68)
Lean: `UpperTheory.V.Pa.presOut1` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.SelmerScaling.CotangentIndependentOfResidues`, lines 74–211)
Lean: `UpperTheory.V.AlgebraTools.under_le_of_det_mem` (`Development.AutomorphicArithmetic.AutomorphicVarieties.AlgebraTools.CaDetCore`, lines 9–48)

Lean: UpperTheory.V.Pa.Core.chainAbs (Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AnnihilatorBaseChange.ChainAbs, lines 7–70)

□

Proof of 30.35

Proof. To the first list adjoin the minimal primes of each B_j which contain none of the A_i. Every ideal on the enlarged list has quotient dimension less than b. A prime avoiding this list avoids every B_j, since a prime containing B_j contains a minimal prime over it.

Cutting R/I by s gives dimension at least b. Choose a component of this dimension or greater, with prime $\mathfrak{q}_0$. From each ideal on the enlarged list take an element outside $\mathfrak{q}_0$, and denote the product of their images in $D = R/\mathfrak{q}_0$ by f. This product is nonzero. While $\dim D > 1$, choose a nonzero $z \in \mathfrak{m}_D$ outside the finitely many height-one primes containing f. This is possible by prime avoidance. Each minimal prime over (z) has height one and avoids f. Passage to its quotient preserves $f \neq 0$ and decreases the dimension by one, by the dimension theorem for a principal ideal in a complete local domain. The final domain gives $\mathfrak{p}$.

In equal characteristic its normalisation is finite, since a complete Noetherian local ring is excellent. The normalisation of this one-dimensional domain is local: it is a finite algebra over a Henselian local ring and has no nontrivial idempotents. It is a complete DVR with finite residue extension k'. A coefficient field and a uniformiser identify it with $k'[[t]]$. □

Proof of 30.37

Proof. Claim 1. The matrix algebra can be chosen uniformly for the two residual support shapes in 30.38. At each such residual point x, its completed local ring represents lifts of the entire marked representation and its specified local flags.

We now give the construction and the comparison. Put $k' = k_\Lambda$. Write $G = G_F$, and retain the tower's marking g_0, fixed similitude, killed inertia subgroup and local groups, including the subgroups on which the flag characters are prescribed to be one. First form the complete universal ring U_{diag} for the diagonal residual representation with diagonal marking and these global equations, together with independent lifts of the ordered local characters. Give matrix entry ij weight $v_i - v_j$, for $v_i = (1,1), (0,1), (1,0), (0,0)$, and give local character coordinates weight zero. A homogeneous power series presentation is obtained by choosing homogeneous matrix and character coordinates in the cotangent space. Let C be the algebra of finite sums of homogeneous elements of U_{diag}, and let $B_0 = C_0$. For each weight, the exponent vectors of monomials of that weight have finitely many coordinatewise minimal elements. Every other such monomial is a minimal one times a weight-zero monomial. Thus each homogeneous piece is finite over B_0, C is of finite type over B_0, and the weight-zero monomial generators give a complete Noetherian local presentation of B_0. Maps from B_0 to complete local test rings are determined by the matrix characteristic-polynomial coefficients and the ordered local characters. Indeed the distinct residual marked eigenvalues determine their ordered lifts and their projectors. Traces of products with these projectors recover the cyclic entry products; the symplectic adjoint relation reduces weight-zero entry monomials to these products. The homogeneous power series presentation then determines the map on B_0.

Over C, adjoin the entries of a lower unitriangular symplectic matrix n_v for each $v \mid \ell$. Impose

$$(n_v^{-1} r(s) n_v)_{ij} = 0 \quad (i > j), \qquad (n_v^{-1} r(s) n_v)_{ii} = \psi_{v,i}(s) \quad (s \in G_v),$$

and the prescribed character equations on the local torsion subgroups. Call the resulting algebra A, and replace B_0 by its image B in A. This gives the structure map $j : B \to A$. The algebra A is of finite type over B and is generated by matrix and flag entries. Its universal representation is continuous for the $\mathfrak{m}_B A$-adic topology; this follows from continuity in the complete presentation and finiteness of each homogeneous piece over B_0.

We check that this construction includes the non-diagonal residual points. Let R' be a local Artin Λ-algebra with residue field k', and let (r', n'_v) lift one of the two allowed residual shapes. On their respective representation supports, the linear forms $\tau = (-2,-1)$ and $\tau = (-1,-2)$ are strictly positive. Every off-diagonal entry of nonpositive τ-weight therefore belongs to $\mathfrak{m}_{R'}$. Choose c bounding the negative weights.

Inside $R'((t))$, the complete algebra generated by $R'[[t]]$ and $\mathfrak{m}_{R'}t^{-c}$ is local with residue field k'; nilpotence of $\mathfrak{m}_{R'}$ bounds its negative powers. Scaling entry ij by $t^{\tau(v_i-v_j)}$ gives a diagonal-residual deformation over this algebra. It preserves multiplication, the symplectic equation and the diagonal marking, and leaves the local characters unchanged. Universality of U_{diag} gives its evaluation. A homogeneous element of weight w evaluates to $a_w t^{\tau(w)}$. Taking a_w on each homogeneous piece is additive and multiplicative on finite sums, and gives a map $C \to R'$ with representation r'. Sending each flag variable to the corresponding entry of n'_v then gives $A \to R'$. The same construction on successive Artin quotients gives maps to complete local test rings.

Conversely a map $A \to R'$ reducing to x gives exactly these marked matrices and flags. The equations preserve the fixed similitude, killed inertia and local torsion conditions. Such a map is unique for its matrices and flags: their values determine the characteristic-polynomial coefficients and ordered local characters, hence the map on B, and these entries generate A over B. Since $\ker x$ maps into the nilpotent maximal ideal of R', the map factors uniquely through $\widehat{A_{\ker x}}$. These mutually inverse constructions identify the Artin test functors. Passing to their inverse limits identifies $\widehat{A_{\ker x}}$ with the universal marked deformation ring at the full residual pair x. In particular this is the ring R in the statement.

Lean: `UpperTheory.V.Ws.spaceEx` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.SpaceEx`, lines 21–85)
Lean: `UpperTheory.V.Ws.sxNearArt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.AdicExtensionMatrices.NearArt`, lines 9–61)
Lean: `UpperTheory.V.Ws.fibCpl` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.FibreTraces.ComplOfNil`, lines 133–217)

Let $k' = k_\Lambda$, let $K = G_F \cap \ker \bar{r}_D$, and form the inertia quotient Γ of G_F by the closed normal subgroup generated by $I_q \cap K$ at all finite places $q \nmid \ell$ of F. Because $\bar{r}_D$ takes values in $(k'^{\times})^4$, which has order prime to ℓ, the open normal subgroup $K \lhd G_F$ has index prime to ℓ. Let $V = \mathfrak{sp}_4(k')$ be the 10-dimensional adjoint representation of Γ (infinitesimal symplectic matrices with fixed similitude $\nu|_{G_F}$), and at each $v \mid \ell$ let $b_v \subseteq V$ be the 6-dimensional Borel subalgebra preserving the residual ordinary flag at v.

We verify $\mathrm{Hom}_\Gamma(V, k'(\bar{\varepsilon}_\ell)) = 0$. In both residual support shapes of Lemma 30.38, the residual representation $\bar{r}$ is lower triangular after ordering the basis as $(0,1,2,3)$ for support in $\{-\alpha, -\beta, -\psi, -\theta\}$ or as $(0,2,1,3)$ for support in $\{\beta, -\alpha, -\theta, -\psi\}$. Hence every Jordan–Hölder constituent of V is one-dimensional, with character either 1 (twice, on the diagonal) or one of the eight root ratios d_i/d_j $(i \neq j)$. For any diagonal constituent, complex conjugation $c \in G_F$ satisfies $1(c) = 1 \neq -1 = \bar{\varepsilon}_\ell(c)$ since ℓ is odd. For any root constituent d_i/d_j $(i \neq j)$, the marking $g_0 \in G_F$ of the fitted tower has residual cyclotomic value $\bar{\varepsilon}_\ell(g_0) = 1$ and four pairwise distinct eigenvalues $d_0(g_0), d_1(g_0), d_2(g_0), d_3(g_0)$ (preserved when raising g_0 to an ℓ-power upon passing to a deep layer, as $(d_i/d_j)(g_0) \in k'^{\times}$ has order prime to ℓ), whence $(d_i/d_j)(g_0) \neq 1 = \bar{\varepsilon}_\ell(g_0)$. Thus no constituent of V equals $k'(\bar{\varepsilon}_\ell)$ on G_F, giving $\mathrm{Hom}_\Gamma(V, k'(\bar{\varepsilon}_\ell)) = 0$.

At each real place $w \mid \infty$ of F, complex conjugation $c_w \in G_F$ has symplectic similitude -1 and eigenvalues $(1,1,-1,-1)$, so its centralizer in $\mathfrak{sp}_4(k')$ is isomorphic to $\mathfrak{gl}_2(k')$ and has dimension 4. Because $\sum_{v|\ell} d_v = d$, Lemma 30.56 (which uses only the cohomological inputs of Definition 30.36) applies to V and b_v, yielding finite-dimensional cone cohomology groups H^i_b $(0 \le i \le 2)$ with

$$\dim_{k'} H^1_b - \dim_{k'} H^2_b - \dim_{k'} H^0_b \ge \sum_{v|\ell} d_v \dim_{k'} b_v - \sum_{w|\infty} \dim_{k'} V^{c_w=1} = 6d - 4d = 2d.$$

Let R_0 be the completed universal deformation ring at x of the marked symplectic representation with fixed similitude $\nu|_{G_F}$ and ordinary flags at $v \mid \ell$, before imposing the prescribed ℓ-primary torsion values on the local flag characters. Take a minimal presentation $S = \Lambda[[X_1, \dots, X_{t_0}]] \twoheadrightarrow R_0$ with kernel I_0 generated by r_0 elements. By the square-zero obstruction calculation in Lemma 30.58, pushing $S/\mathfrak{m}_S I_0 \to S/I_0$ out along any k'-linear functional on $I_0/\mathfrak{m}_S I_0$ yields a class in H^2_b that vanishes if and only if the functional is zero, so $r_0 \le \dim_{k'} H^2_b$. For the relative tangent space, every cone cocycle in H^1_b lifts to a first-order marked deformation $(1 + \epsilon z)\bar{r}$ with ordinary flags (conjugating by a suitable first-order symplectic matrix to diagonalize the marking $r(g_0)$, which is possible because its four residual eigenvalues are distinct). The kernel of the resulting surjection $\mathrm{Tan}(R_0) \twoheadrightarrow H^1_b$ consists of coboundaries preserving the diagonal marking $r(g_0)$, hence corresponds to the centralizer of $\bar{r}(g_0)$ in $\mathfrak{sp}_4(k')$ modulo H^0_b. Because $\bar{r}(g_0)$ has four distinct eigenvalues, its centralizer in $\mathfrak{sp}_4(k')$ is the 2-dimensional diagonal torus $\mathfrak{t} \supseteq H^0_b$, so the kernel has dimension

$2-\dim_{k'} H_b^0$. Thus $t_0 \geq \dim_{k'} H_b^1 + 2 - \dim_{k'} H_b^0$, and combining with the Euler characteristic bound gives

$$t_0 - r_0 \geq \dim_{k'} H_b^1 - \dim_{k'} H_b^2 - \dim_{k'} H_b^0 + 2 \geq 2d + 2.$$

Finally, R is obtained from R_0 by imposing the prescribed values of the four local flag characters $\psi_{v,0}, \dots, \psi_{v,3}$ on the ℓ-primary torsion subgroup of the image of I_v in $G_{F_v}^{\mathrm{ab}}$ at each of the s places $v \mid \ell$. By local class field theory, this ℓ-primary torsion subgroup is isomorphic to $\mu_{\ell^\infty}(F_v)$, which is finite cyclic. Choose a generator γ_v of it. The symplectic pairing fixes $\psi_{v,0}\psi_{v,3} = \psi_{v,1}\psi_{v,2} = \nu|_{I_v}$, so fixing the two characters $\psi_{v,0}(\gamma_v)$ and $\psi_{v,1}(\gamma_v)$ determines all four characters on this subgroup. Imposing these conditions across all s places adds at most $2s$ relations to the presentation of R_0, producing a presentation of R with $t = t_0$ variables and $r \leq r_0 + 2s$ relations satisfying $t - r \geq 2d + 2 - 2s$. □

Proof of 30.38

Proof. Lift minimal primes to the completed local ring R at x. Write $u : A \to R$, $\varphi = uj$, $b = 2d + 1 - 2s$ and $d_v = d/s$, the common local degree at ℓ. By Lemma 30.37, R admits a presentation over Λ with at least $b + 1 = 2d + 2 - 2s$ more variables than relations. Complete local connectedness [123, Exp. XIII, Thm. 2.1] gives a chain of minimal primes with adjacent P, Q satisfying

$$\dim R/(P + Q) \geq b + 1. \tag{A.268}$$

Suppose $JR \subseteq P$. The reducible support loci in the first four alternatives and in H apart from $\{\pm\beta, \pm\theta\}$ have dimension at most $d+20C+1$. Lemma 30.39 gives the degeneration and the entry for each support. At its limiting flagged point the chosen odd entry and two diagonal entries cost at most $d+3C$ tangent dimensions, the ten symplectic kernel coordinates at most $10C$, and the flag entries at most $6s$. The coefficient parameter contributes one. In the Hecke special fibre Lemma 30.40 gives $2d + 1 - d_v$ parameters at a coincidence on U_v, including the single inertia value used to extend equality to all of I_v. Off the preceding reducible loci, the matrix fibre contributes at most $2 + 4s$. The marked projectors separate the entries, and trace products determine the opposite entry from one entry of each pair. Since $s \leq C$ and $d \geq 30(C + 1)^2$,

$$d + 20C + 1 < b, \qquad 2d + 1 - d_v + 2 + 4s < b. \tag{A.269}$$

On the remaining support $\{\pm\beta, \pm\theta\}$, the nonzero flag entry $(n_v)_{10}$ gives

$$(\psi_{v,1} - \psi_{v,0})(\psi_{v,1} - \psi_{v,3}) = 0 \quad \text{on } G_v.$$

The two equality subgroups cover G_v, so one is all of G_v. This support also lies in the coincidence locus.

Apply 30.35 with $I = P + Q$ and $s = \varpi$, a uniformiser of Λ. For the first list use the reducible support loci bounded by $d + 20C + 1$. For the second list use the loci defined by J, ϖ and coincidence of two specified characters on U_v. Outside the first list their dimensions are bounded by $2d + 1 - d_v + 2 + 4s$, where here s denotes the number of flags. Both bounds are strictly less than b by (A.269), and (A.268) gives the dimension hypothesis on I. The remaining block support forces one of the character coincidences just excluded. The lemma gives a local map $R \to k'[[t]]$ through $R/(P + Q, \varpi)$ whose generic point avoids all these loci. Compose with u to obtain y. Its generic span is full, its characters are separated and its constant term is residual. It kills J because $JR \subseteq P$. The curve condition applies to $\varphi^{-1}(Q) \subseteq \ker(yj)$, so $JR \subseteq Q$. Induction along the chain and contraction give the result. □

Proof of 30.39

Proof. Index rows and columns from zero, and write $i^* = 3 - i$. The marked diagonal matrix has four distinct residual entries. Over any field at a prime in the completion its spectral projectors are the four coordinate projectors. Consequently the matrix algebra spanned by the representation contains E_{ij} precisely when some matrix has nonzero ij-entry. The relation of containing E_{ij} is reflexive and transitive, and is invariant under $ij \mapsto j^*i^*$ by the similitude inverse formula. Thus the full matrix algebra is the only support with all eight roots. The following table gives the proper supports needed, and the exceptional odd entry to retain in the

tangent count. An entry ij means the single matrix coordinate in that position. A dash means the residual entries in that column of cases exclude the row.

	S	$1,2$	$3,4$	5
1	$\beta,-\beta,-\alpha,-\theta,-\psi$	12	12	12
2	$\beta,-\beta,\alpha,\theta,\psi$	–	–	–
3	$\theta,-\theta,\psi,\beta,-\alpha$	–	12	–
4	$\theta,-\theta,-\psi,-\beta,\alpha$	–	–	03
5	$\alpha,-\alpha,-\theta,-\psi,-\beta$	01	01	01
6	$\alpha,-\alpha,\theta,\psi,\beta$	–	01	–
7	$\psi,-\psi,\beta,-\alpha,-\theta$	12	12	12
8	$\psi,-\psi,-\beta,\alpha,\theta$	–	–	03
9	$\beta,-\beta,\theta,-\theta$	–	03	exception

For exhaustiveness one can enumerate with eight bits, one for each entry $01, 02, 03, 10, 12, 20, 21, 30$. Copy each bit to its reflected position and put ones on the diagonal. Discard a choice if $b_{ij}b_{jk} > b_{ik}$ for some i, j, k. Equivalently, any proper transitive choice has a nonempty proper invariant coordinate subspace (take a terminal class of its directed graph). The algebra preserves its symplectic perpendicular as well. A coordinate line, together with its perpendicular, gives four maximal supports; a coordinate isotropic plane gives four more. The remaining coordinate planes are the perpendicular nondegenerate pair $03, 12$, giving row 9. Writing down the entries preserving each indicated subspace gives the nine sets displayed. A support through x must contain all its nonzero residual entries. In columns $1, 2$ this requires $-\psi$ and one of $\beta, -\alpha$, or both $-\alpha, -\beta$. In columns $3, 4$ it requires $-\alpha$, or both $\beta, -\theta$. In column 5 it requires $-\psi$, or both $-\beta, -\theta$. These tests give the dashes.

Lean: `UpperTheory.V.Ws.redList` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.RedList`, lines 124–139)

Let R_S be the completed local quotient cutting out entries outside S. If it is nonzero, its relative tangent space over Λ injects into pairs (Z, Y_v) consisting of the derivative of the representation times its inverse and the derivative of each flag matrix times its inverse. Indeed the values of both derivatives recover the lift to the dual numbers, by the defining property of the marked matrix ring. The representation derivative satisfies the cocycle and infinitesimal similitude equations. We bound this tangent space after diagonal degeneration. For exponents m_i, conjugate the matrices by $\mathrm{diag}(T^{m_i})$; each entry has weight $m_i - m_j$. In each of the following choices $m_i + m_{i^*}$ is constant, so this conjugation preserves the equations. The residual representation and flags have no negative-weight entries. Filter the tangent pairs by least entry weight, simultaneously at every group element and every flag. In the linearised multiplication, form and flag equations, the term of least weight satisfies the same equations at the weight-zero residual data. The support equations pass to that term too. Each graded piece thus injects into the corresponding weight subspace of the tangent equations at the limit. There are finitely many entry weights and the target weight subspaces are independent. Summing gives the desired tangent inequality, over the original finite residue field.

Lean: `UpperTheory.V.Ws.tan_flow` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.RedTanFlow`, lines 93–126)

Lean: `UpperTheory.V.Cu.redA_of` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.CuspidalTwists.PRedA`, lines 57–150)

For cases 1 and 2 use $m = (0, 2, 1, 3)$ for the first residual shape of 30.38 and $m = (0, 1, 2, 3)$ for the second. Inspection of their allowed entries gives the diagonal representation and identity flags in the limit. For cases 3 and 4 use $m = (0, 1, 0, 1)$. The surviving flag is $1 + b_v(E_{20} + E_{31})$ with $b_v = (n_v)_{20} \neq 0$. For case 5 use $m = (0, 0, 1, 1)$. Its flag is $1 + b_v(E_{10} - E_{32})$, $b_v = (n_v)_{10} \neq 0$. The signs follow from n_v being lower unipotent symplectic. The representation in both limits is diagonal. The local flag condition gives $\bar{\chi}_0 = \bar{\chi}_2$ in cases 3 and 4, and $\bar{\chi}_0 = \bar{\chi}_1$ in case 5: apply the representation to the first flag column and take its nonzero 2- or 1-coordinate.

At a diagonal representation r_0, each Z_{ij} is a cocycle for $\bar{\chi}_i/\bar{\chi}_j$, and is zero off the support for $i \neq j$. At a local group put $D = Z + \mathrm{Ad}(r_0)Y_v - Y_v$, with the sign convention for the derivative of the flag matrix just chosen. Conjugating D by the limiting flag basis makes it upper triangular. The six independent odd positions are $01, 03, 21, 12, 10, 30$. Fix to zero the exceptional global odd coordinate in the table and the global diagonal coordinates $00, 11$. We show that each remaining odd cocycle is locally a boundary.

For the identity flag the entries below the diagonal are boundaries, and among $01, 03, 12$ at most the

indicated exceptional entry belongs to the row. For the flag with $b = b_v = (n_v)_{20}$, multiplication by its inverse on the left and by the flag on the right gives

$$D_{10} = -bD_{12}, \qquad D_{30} = -b^2 D_{12}, \qquad D_{21} = 2bD_{01} + b^2 D_{03}.$$

These relations also hold for local cohomology classes; entries combined in any equation have the same local character by $\bar{\chi}_0 = \bar{\chi}_2$ and similitude. In the first two equations the class of 12 vanishes if any of $12, 10, 30$ is absent or has been fixed to zero. In the third equation two zero classes force the third to vanish, since $2b$ and b^2 are units. In rows $1, 3, 5, 6, 7, 9$ of the middle column, deleting the displayed coordinate leaves respectively at least one zero among $12, 10, 30$ and at least two among $21, 01, 03$. This proves the assertion in both cases, including row 9 of case 4.

For the flag with $b = (n_v)_{10}$ the lower equations give

$$D_{21} = D_{30} = 0, \qquad D_{10} = b^2 D_{01} + b(D_{00} - D_{11}).$$

Here $D_{00} = D_{11} = 0$ in the kernel under consideration. The local characters for $10, 01$ are trivial. In rows $1, 4, 5, 7, 8$ of the last column, deleting the indicated coordinate kills $03, 12$ and at least one of $01, 10$. The displayed equations kill the remaining local classes.

The exceptional coordinate has global cocycle dimension at most $d+C$, and the two diagonal coordinates contribute at most $2C$. In the kernel, reflection in the antidiagonal determines the Lie algebra matrix from the ten positions $i \le j^*$. Each of these contributes at most C: for an odd position use the strict cocycle bound just proved applicable, and for the other positions use the bounds at $0, \pm\psi$. The flag derivatives are strictly lower and have at most six coordinates per place. Hence the relative tangent dimension is at most $d + 3C + 10C + 6s \le d + 20C$. Lifts of a relative cotangent basis together with a uniformiser generate the maximal ideal by Nakayama's lemma. The height theorem gives $\dim R_S \le 1 + d + 20C$.

Lean: `UpperTheory.V.Cu.ReductionTangentLimit.rank_A` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.CuspidalTwists.PRedTanA`, lines 160–254)

Lean: `UpperTheory.V.Ws.redDim` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.RedDim`, lines 88–123)

In the exception work over the fraction field at the prime. The representation preserves the planes 03 and 12, each symplectic of dimension two, so both block determinants equal the multiplier $\nu(s)$. The first flag vector has nonzero projections to both planes: its coordinates $0, 1$ are $1, (n_v)_{10}$, and the second is a unit at x. For every local element s each block therefore has eigenvalue $\psi_{v,0}(s)$ and other eigenvalue $\nu(s)/\psi_{v,0}(s) = \psi_{v,3}(s)$. Thus $\psi_{v,1}(s)$ equals one of these two values. The two equality subgroups cover the decomposition group. A group cannot be the union of two proper subgroups, since the product of an element exclusive to each lies in neither. One equality holds on the whole group. □

Proof of 30.40

Proof. In a prime quotient of the special fibre the inertial characters take values in the principal units and opposite characters are inverse. These follow by reduction of the prescribed inertia datum and the similitude. At a local place u take d_v inertia elements lifting generators of the torsion-free pro-ℓ part of the image of inertia in the local abelianisation. Its rank is d_v by local reciprocity, since the principal units modulo torsion have this rank. These elements detect triviality of an inertial character in a prime quotient: the successive principal-unit quotients give pro-ℓ image, and principal units of a characteristic-ℓ local domain have no torsion. For the latter assertion, an ℓ-power root of unity satisfies $(z-1)^{\ell^a} = 0$, and prime-to-ℓ torsion reducing to one is trivial by the geometric-series factorisation.

At the exceptional place v, choose also $s_0 \in I_v$ mapping to a topological generator of $\varepsilon_\ell(I_v)$. This exists because ℓ is odd and every closed subgroup of $\mathbf{Z}_\ell^\times$ is procyclic. The residual inertia datum of the fitted frame is trivial there on $I_v \cap \ker \varepsilon_\ell$, so the latter is contained in U_v. Equality on U_v and at s_0 therefore gives equality on I_v, by continuity. This is the one extra parameter in the count.

Lean: `UpperTheory.exists_pinning_elements_of_locAb` (`Development.AutomorphicArithmetic.AutomorphicVarieties.BuildingsResidueChars.ExistsPinningElementsOfLocAb`, lines 26–147)

Lean: `UpperTheory.exists_mem_pins_insert_ker_of_padic_character` (`Development.AutomorphicArithmetic.AutomorphicVarieties.BuildingsResidueChars.ExistsMemPinsInsertKerOfPadicCharacter`, lines 6–32)

If $j \ne i^*$, take $a = j$; if $j = i^*$, choose $a \notin \{i, i^*\}$. The pairs a, a^* and i, i^* exhaust the indices. Impose $\psi_{u,a} - 1$ on all the chosen generators at all places, $\psi_{u,i} - 1$ on them for $u \ne v$, and $\psi_{v,i}(s_0) - \psi_{v,j}(s_0)$. All

belong to the maximal ideal. In any prime containing them and q, the first pair is trivial on all inertia, and the second is trivial away from v. At v it is trivial by equality with the first pair if $a = j$. If $j = i^*$, equality with its inverse gives square one and hence triviality for a principal unit. Condition (30.9) now puts the whole coefficient maximal ideal in this prime. The number of elements imposed is $sd_v + (s-1)d_v + 1 = 2d + 1 - d_v$. Their ideal in B/q has maximal radical, proving the first assertion by the height theorem.

For the fibre count first use the finite-type matrix algebra over B before local completion. Over an algebraic closure of the coefficient prime's residue field, the marked eigenvalues are fixed up to a finite choice. Their projectors are polynomials in the marked matrix. In the full-span locus two tuples with the same character polynomials are conjugate: traces of products agree, and the trace pairing on the full matrix algebra identifies their matrix relations and gives an isomorphism of matrix algebras. Such an isomorphism is inner. With the ordering of the marked eigenvalues fixed the conjugator is diagonal. The two pulled-back symplectic forms differ by a scalar, since their quotient commutes with an irreducible representation. Hence the diagonal conjugator satisfies $d_0 d_3 = d_1 d_2$, and common scalars act trivially. The representation fibre has dimension at most two.

On the exceptional two-plane locus outside the small supports, both entries of each opposite pair $03, 30$ and $12, 21$ occur. Otherwise the support lies in one of rows $1, 2, 3, 4$ of 30.39 other than row 9, by inspection of the missing root. In each full two-dimensional block the represented cycles have fixed traces: insert the marked projectors into the trace of products and expand the projectors as polynomials. Specifying a single nonzero 03-entry recovers every 30-entry from such a trace, after which one nonzero 30-entry recovers all 03-entries. The same applies to $12, 21$; the diagonal entries are already traces with one projector. This again gives at most two parameters. For each place a lower unipotent symplectic flag matrix is determined by $10, 20, 30, 21$, since $n_{32} = -n_{10}$ and $n_{31} = n_{20} - n_{21} n_{10}$. This adds at most $4s$.

Lean: `UpperTheory.V.Ws.fibTorus` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.FibreTraces.FibTTrace`, lines 120–177)

Lean: `UpperTheory.V.Cu.torusY` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.CuspidalTwists.PTorusY`, lines 87–128)

For a prime in the completion contract it first to the finite-type matrix algebra. The dimension of the completed prime quotient is at most the local dimension of the contracted quotient at x: the former is a quotient of the completion of the latter, and completion preserves local dimension. The dimension formula for a finite-type domain over the complete local domain B/q bounds this local dimension by $\dim B/q$ plus the transcendence degree of the generic fibre. Indeed the base is universally catenary, and the closed-point residue extension at x is finite. The preceding count bounds that transcendence degree by $2 + 4s$. This proves the second assertion. □

Proof of 30.41

Proof. For $h \geq 3$ put $L_h(X) = (X\, d/dX)^{h-1}(X/(1-X))$, $\zeta = e^{2\pi i/H}$, and

$$E_{h,a}(z) = \frac{H^h (h-1)!}{(-2\pi i)^h} \sum_{(u,v) \equiv (0,a) \bmod H}{}' (uz+v)^{-h}.$$

The prime omits $(0,0)$. Under $\Gamma_0(H)$, the index transforms by multiplication by the bottom right entry. For unit a the constant term at a cusp with bottom left entry indivisible by H is zero, since no transformed row has first entry zero. The constant at infinity is $L_h(\zeta^a)$ by expansion of the second-coordinate sum. Set

$$E = \frac{1}{2\tau(\chi^{-1})} \sum_{a \bmod H} \chi^{-1}(a) E_{h,a}, \qquad \tau(\chi^{-1}) = \sum_{a \bmod H} \chi^{-1}(a) \zeta^a.$$

Fourier orthogonality gives the positive coefficients $s(n) = \sum_{d|n} \chi(d) d^{h-1}$, with the factor two from parity. Primitivity and $\ell \nmid H$ make the Gauss sum a unit.

Write c for the constant of E. The denominator $(1-X)^h$ is a unit on nontrivial roots of order dividing H. Reduce θ^{-1} to 1 on the unit indices. For each $b \bmod D$ sum over its lifts and subtract the one divisible by w. Differentiating the geometric series gives $\sum_{j=0}^{w-1} L_h(x\zeta^{Dj}) = w^h L_h(x^w)$. Using $\phi(w) = 1$ gives

$$2\tau(\chi^{-1}) c \equiv (w^h - 1) \sum_{b \bmod D} \phi^{-1}(b) L_h(\zeta^{wb}) \equiv 0 \pmod{\mathfrak{m}_{C_\ell}}. \tag{A.270}$$

The sum uses zero character coefficient at nonunits.
Lean: UpperTheory.V.Sd.eisChi (Development.AutomorphicArithmetic.AutomorphicVarieties.IdeleGenerators.HolomorphicIrreducibility.EisChi, lines 82–148)
Lean: UpperTheory.V.Sd.constSmall (Development.AutomorphicArithmetic.AutomorphicVarieties.IdeleGenerators.HolomorphicIrreducibility.ConstSmall, lines 168–224)

To produce an integral form with constant 1 at infinity and zero at the other cusps, separate the constants at indices $1, b$ for each $b \neq \pm 1$ among the units. For $u(x) = x/(1-x)^2$, the difference $u(\zeta) - u(\zeta^b)$ has unit numerator $(\zeta - \zeta^b)(1 - \zeta^{1+b})$ and unit denominator. The values of $L_3 = x(1+x)/(1-x)^3$ are units at both roots. Differentiation gives $L_4 = u(1+6u)$ and $L_5 = L_3(1+12u)$. In characteristic three use L_4. Otherwise use L_3^2 if its residues differ, and $L_3 L_5$ if they agree, since multiplication by $1+12u$ separates them. Accordingly choose from the families $E_{4,a}, E_{3,a}^2, E_{3,a}E_{5,a}$. Subtract the constant at b times the level-one form E_4, E_6, E_4^2 of the same weight. The product over $b \neq \pm 1$ has even weight h_0, unit constant at 1 and vanishing constants at those b.

For $h - h_0 \geq 10$, multiply by $E_{3,a}$ if the difference is odd, and by $E_{3,a}^2$ if it is even. Fill the remaining even weight by a monomial in E_4, E_6. Let F_a be the resulting family and b_1 the unit constant of F_1. The Fourier expansions are integral. The character average $G = (2b_1)^{-1} \sum_a \chi(a^{-1}) F_a$ has constant 1 at infinity by parity. At a cusp $\gamma\infty$ with $H \nmid c_\gamma$ the weight-three factor, hence G, has zero constant, and so does E. If $H \mid c_\gamma$, then E and G transform under γ by $\chi(d_\gamma)$, so their constants there are $\chi(d_\gamma)c$ and $\chi(d_\gamma)$. Conjugation of the root-of-unity coefficients gives the same construction and transformation laws for the conjugate character.
Lean: UpperTheory.V.Sd.lkSep (Development.AutomorphicArithmetic.AutomorphicVarieties.IdeleGenerators.HolomorphicIrreducibility.EulerianNumeratorThree, lines 103–180)
Lean: UpperTheory.V.Sd.unitFam (Development.AutomorphicArithmetic.AutomorphicVarieties.IdeleGenerators.HolomorphicIrreducibility.UnitFam, lines 248–263)
Lean: UpperTheory.V.Sd.unitForm (Development.AutomorphicArithmetic.AutomorphicVarieties.IdeleGenerators.HolomorphicIrreducibility.UnitForm, lines 69–154)

By (A.270), $f = E - cG$ is cuspidal and integral with first coefficient congruent to one. For $A_p = \chi(p)p^{h-1}$ the divisor sum satisfies

$$s(pn) + A_p \mathbf{1}_{p|n} s(n/p) = (1 + A_p) s(n)$$

also at the level primes. Thus f gives a nonzero simultaneous residual eigenvector. In the Deligne–Serre lifting lemma [117, Lem. 6.11], the residual eigenvector determines a maximal ideal of the commuting Hecke algebra on the integral Fourier lattice. A minimal prime below it survives in characteristic zero by torsion-freeness over the coefficient DVR. Its field of fractions gives the lifted system, and its generalised eigenspace contains a common eigenvector. One may work in a space containing the coefficientwise conjugates. The Hecke recursion determines all positive coefficients from the first, so the eigenform can be normalised. Its eigenvalue at ℓ reduces to one.
Lean: UpperTheory.V.Sd.eigLift (Development.AutomorphicArithmetic.AutomorphicVarieties.IdeleGenerators.HolomorphicIrreducibility.EigLift, lines 158–280)
Lean: UpperTheory.V.Z.Sym.Unl.eigLiftC (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.SymmetricPowerTransfer.UnlinkedHalfIntegral.UnlinkedHeckeRoots.EigLiftC, lines 206–331)

□

Proof of 30.42

Proof. The monoid of exponent vectors of weight zero is the set of lattice points of the nonnegative rational cone cut out by the weight equations. It has finitely many generators. The images of their positive-degree monomials, together with ϖ, generate the maximal ideal of the complete weight-zero ring R_0. Indeed, in the power series presentation the monomials generate its weight-zero subring topologically, and homogeneous relations let the weight-zero component pass to the quotient.

Let u be one of these monomials. If $u \notin \sqrt{\varpi R}$, take a prime over (ϖ) avoiding u. Its quotient is a complete local domain, and u is a nonzero nonunit there. In dimension greater than one, choose a nonzero parameter avoiding the finitely many height-one primes dividing u and take a minimal prime over that parameter. The resulting domain still has positive dimension and u survives. Repetition gives a one-dimensional quotient in which $u \neq 0$. Its normalisation at a maximal ideal is a complete DVR with finite residue field, since normalisation of the complete one-dimensional local domain is finite. The composite $R \to A$ contradicts the monomial hypothesis. Thus some power of each u belongs to ϖR. Taking the weight-zero component shows that the same power belongs to ϖR_0. A uniform power of $\mathfrak{m}_{R_0}$ therefore lies in ϖR_0, and $R_0/\varpi R_0$ is

finite over $O/(\varpi)$. Lift generators and use successive ϖ-adic approximation to obtain a finite O-module. For integer weights a nonconstant monomial of total weight zero contains either a zero-weight factor or factors of both strict signs, which proves the sufficient condition. □

Proof of 30.43

Proof. We first compute the full cuspidal trace. At each real place of weight k take the matrix coefficient of a unit lowest-weight vector in the weight-k discrete series, multiplied by the formal degree and with the convolution convention giving the projection onto that line. The central signs agree with ω since both weight changes are even. Both k's exceed two, so these coefficients are integrable modulo the center. The finite test is the convolution for the word at the given level with its central character. It is a finite sum of cosets modulo level and center and is unchanged by convolution with its level idempotent. Jacquet and Langlands's formal trace-formula comparison requires $\int_{K_v}\int_{N_v} f_v(k^{-1}ank)\,dn\,dk = 0$ for every $a \in A_v$, at each of at least two places [118, §16, condition (16.1.7)]. The chosen matrix coefficients satisfy this identity at both real places. Write $a = z\,\mathrm{diag}(t,1)$ with z central, and put $n(x) = \left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)$. If $t<0$, the coefficient vanishes on $an(x)$ and on its compact conjugates, since an element of negative determinant exchanges the holomorphic and antiholomorphic summands. If $t>0$, the explicit lowest-weight coefficient gives $f_v(an(x)) = C_a(t+1+\epsilon itx)^{-k}$, with C_a independent of x and a sign $\epsilon \in \{\pm 1\}$ fixed by the convolution convention. The integral over $x \in \mathbf{R}$ converges absolutely since $k>2$, and it is zero, since $C_a(t+1+\epsilon itx)^{1-k}/(\epsilon it(1-k))$ is an antiderivative tending to zero at both ends. Conjugation by a rotation leaves the coefficient unchanged and conjugation by a reflection replaces ϵ by $-\epsilon$, so the compact average vanishes as well.

Claim 1. Let the weight be at least four at both real places, and let the word contain v or v'. Let H be the product of the finite adelic center and the level subgroup. The trace on the cusp space is a finite sum over simultaneous $\mathrm{PGL}_2(F)$-orbits of contributing pairs (γ, xH), with $\gamma \in \mathrm{PGL}_2(F)$ elliptic at both real places. For a lift of γ to $\mathrm{GL}_2(F)$, the finite test sums the values of the central-level character on $x^{-1}\gamma^{-1}xa \in H$ over the word representatives a with $xaH = \gamma xH$. Divide this value by the order of the stabilizer of the pair in $\mathrm{PGL}_2(F)$ and multiply by $-(AB)^{1-k/2}\sum_{j=0}^{k-2} A^{k-2-j}B^j$ at each real place, where k is the weight there and A, B are the eigenvalues of the same lift of γ.

Lean: `UpperTheory.Esk.geo` (`PositivityTheory.AutomorphicForms.PublishedInputs.JacquetLanglands.TorusKernels.MultivariableReproducingFormula`, lines 349–351)

For weights at least four the holomorphic reproducing kernel converges absolutely and is a cusp form in each variable, so integrating its translate by the word along the diagonal over the adelic quotient computes the trace on the cusp space. The ordinary word representatives at v, v' computed below exclude central and parabolic elements. The sum of the terms indexed by elements with a rational eigenvector has integral zero, and among the remaining elliptic elements only those elliptic at both real places contribute; they give the stated sum.

Let D be the quaternion algebra over F ramified at both real places and split at every finite place; it exists since there are two ramified places. A quadratic extension of F embeds in D exactly when it is imaginary at both real places, so the classes in Claim 1 are those of the noncentral elements of $D^\times(F)$ modulo the center. For a contributing class choose a lift of γ and put $E = F[\gamma]$. Its trace is nonzero by the ordinary-place calculation below, so its rational projective centralizer on either side is $E^\times/F^\times$. Under the finite splittings of D, the two embeddings of E are conjugate locally, by integral conjugators at almost all places. Left translation by the resulting finite adelic conjugator identifies the returning cosets equivariantly for this centralizer and commutes with the right Hecke translates. Thus the stabilizer orders and finite test values agree. The two real signs cancel, giving the trace on the D-side space described next. At a real place the corresponding representation modulo the positive center is $\mathrm{Sym}^{k-2}\otimes \mathrm{nrd}^{\,1-k/2}$, where nrd denotes reduced norm. For eigenvalues A, B of a quaternion in its complex matrix realization, put $C_k(A,B) = \sum_{j=0}^{k-2} A^{k-2-j}B^j$. The character is $C_k(A,B)(AB)^{1-k/2}$.

We now give the finite sum with which the two weights are compared. Let U be the product of the finite level groups, Z_f the finite adelic center, and D the quaternion algebra. Take representatives x of

$$D^\times(F)\backslash D^\times(\mathbf{A}_{F,f})/(Z_f U).$$

This is the finite ideal class set at maximal level, modulo central ideals, with a finite refinement at the chosen level. The fiber at x consists of vectors fixed under the action of its projective stabilizer, with the specified

central character included. This stabilizer is finite: the rational projective group is discrete adelically, and the stabilizing subgroups are compact at infinity and at finite places. Its order is a power of two. Indeed its image at u lies, up to conjugation, in the upper triangular subgroup modulo the residue field of order two, with pro-2 reduction kernel. The projective rational group injects into its local group at u. Thus averaging over the stabilizer introduces an ℓ-integral denominator.

Expand the word into right cosets and take the trace on these fibers. A term on the diagonal comes from a returning coset a with $xa = \gamma xzk_0$, where $\gamma \in D^\times(F)$, $z \in Z_f$ and $k_0 \in U$. Insert the stabilizer average to take the trace on the invariant fiber. Its summands have the same description with γ ranging through the stabilizer translates. The scalar from the finite character is a root of unity independent of the weight. This proves that the same finite list, with the same coefficients, computes both traces.

Write $z'_w = z_w^{-1}$. The element γ at w is conjugate to $z'_w M_w$, where $M_w = a_w k_{0,w}^{-1}$ is integral and $v_w(\det M_w) = b_w$. At either place $w \mid \ell$ the word representatives have the form $\left(\begin{smallmatrix} \varpi_w^{b_w} & x_w \\ 0 & 1 \end{smallmatrix}\right)$, with $x_w \in O_w$. This follows by multiplying the representatives $\left(\begin{smallmatrix} \varpi_w & x \\ 0 & 1 \end{smallmatrix}\right)$ for a single letter, with x running through residue classes. Multiplication by $k_{0,w}^{-1}$ gives $\operatorname{tr} M_w \equiv 1 \bmod \ell$. The characteristic polynomial therefore has, by Hensel's lemma, roots $A_w, B_w \in \mathbf{Z}_\ell$ with A_w a unit and $v_\ell(B_w) = b_w$. In particular a returning coset has distinct roots and cannot contribute to the central term. Every noncentral rational quaternion in this algebra is elliptic at both real places.

Put $P = \prod_w (\mathrm{N}w)^{b_w}$ and $Z = \prod_w (\mathrm{N}w)^{v_w(z'_w)}$. The reduced norm of γ is totally positive, and its rational norm is $Z^2 P$. By homogeneity of C_h, the two real character factors in the parallel weight, multiplied by the Hecke normalization $P^{h/2-1}$ and transported to $\overline{\mathbf{Q}}_\ell$, give

$$\left(\frac{z'_v z'_{v'}}{Z}\right)^{h-2} C_h(A_v, B_v) C_h(A_{v'}, B_{v'}).$$

Here $F_v = F_{v'} = \mathbf{Q}_\ell$. The prefactor is a unit since $v_\ell(Z) = v_\ell(z'_v) + v_\ell(z'_{v'})$. In the mixed weight the prefactor and the v'-factor remain the same. The v-factor becomes

$$C_{h+2t}(A_v, B_v) \left(\frac{\ell^{b_v}}{A_v B_v}\right)^t.$$

Indeed the additional determinant power contributes $((z'_v)^2 A_v B_v)^{-t}$, the symmetric power contributes $(z'_v)^{2t}$ in extracting the scalar, and the word contributes $\ell^{b_v t}$.

Both expressions are integral. Modulo ℓ^{b_v} one has $C_k(A_v, B_v) \equiv A_v^{k-2}$. The ratio of the resulting unit terms in the two weights is

$$\left(A_v^2 \frac{\ell^{b_v}}{A_v B_v}\right)^t.$$

Its base belongs to $\mathbf{Z}_\ell^\times$. For $N > 0$, raising a unit first to $\ell - 1$ and then N times to ℓ gives $1 \bmod \ell^N$ by the binomial theorem. The assertion for $N = 0$ is integrality. Summation with the stabilizer denominators proves the trace congruence.

Lean: `UpperTheory.Esk.s_TF` (`PositivityTheory.AutomorphicForms.PublishedInputs.JacquetLanglands.TorusKernels.FinalTF`, lines 92–92)

Lean: `UpperTheory.Esk.pf_GTerm` (`PositivityTheory.AutomorphicForms.PublishedInputs.JacquetLanglands.TorusKernels.PfGTerm`, lines 37–112)

For the eigenform assertion, simultaneously triangularize the commuting operators on both spaces over $\overline{\mathbf{Q}}_\ell$. There are finitely many joint systems, counted on the diagonal with their multiplicities.

Claim 2. A joint system on either space with nonzero eigenvalues at both v and v' has integral eigenvalues at all finite places.

Fix such a system u. For every other system u' on the same space choose a place j with $u_j \neq u'_j$. Multiply $X_j - u'_j$ by 1 if $|u'_j|_\ell \leq 1$ and by $(u'_j)^{-1}$ otherwise. The resulting polynomial has integral coefficients, vanishes on u', and is nonzero at u. Let Q be the product of these polynomials. For every place w and integer $m \geq 0$, the trace of $X_v X_{v'} Q(X) X_w^m$ evaluated on the Hecke operators is integral. Each monomial contains both marked operators, so the first assertion with $r = 1$ gives integrality. By simultaneous triangularization all systems except u contribute zero. The trace is $C u_w^m$, where C is the positive multiplicity of u times $u_v u_{v'} Q(u)$, independent of m and nonzero. Its integrality for every m implies $|u_w|_\ell \leq 1$. This proves the claim.

Lean: UpperTheory.Esk.Fin0Aux.integral (PositivityTheory.AutomorphicForms.PublishedInputs.JacquetLanglands.TorusKernels.FiniteSystemProof, lines 121–201)

In particular the given system s is integral.

For each system u on either space which differs from s by an element of absolute value at least one in some coordinate, select one such coordinate $j(u)$. Form the polynomial

$$P_s(X) = \frac{X_v X_{v'}}{s_v s_{v'}} \prod_u \frac{X_{j(u)} - u_{j(u)}}{s_{j(u)} - u_{j(u)}}.$$

Its coefficients are integral. For each linear factor the denominator has absolute value at least one, and the constant coefficient is integral by writing $-u/(s-u) = 1 - s/(s-u)$ at the selected coordinate. On every excluded system the polynomial is zero. On every system congruent to s it is one modulo the maximal ideal. For all sufficiently large a, the ℓ^a-th powers of the latter values are one modulo ℓ^N: for $v_\ell(y) > 0$, the binomial expansion gives $v_\ell((1+y)^\ell - 1) \geq \min(1 + v_\ell(y), \ell v_\ell(y))$, and there are finitely many values.

Each monomial in $P_s^{\ell^a}$ contains at least ℓ^a copies of both ordinary operators. Choose also $\ell^a \geq N$. The trace congruence for the monomials shows that the two multiplicities of systems congruent to s agree modulo ℓ^N. In parallel weight this multiplicity is an integer between 1 and $N < \ell^N$. Hence the mixed space contains a nonzero joint generalized eigenspace for one such system. On that space the commuting nilpotent parts have a common nonzero kernel, giving the required cusp eigenform. Its normalized eigenvalues are integral by their congruences to the integral s_w.

Lean: UpperTheory.Esk.finite_system_proof (PositivityTheory.AutomorphicForms.PublishedInputs.JacquetLanglands.TorusKernels.FiniteSystemProof, lines 206–409)

□

Proof of 30.44

Proof. For a characteristic-zero specialisation $y : R \to \overline{\mathbf{Q}}_\ell$, every $x \in \mathfrak{m}_R$ has $|y(x)| < 1$. Otherwise $y(x)^{-1}$ is integral over $\mathbf{Z}_\ell$, and its monic equation, multiplied by a power of $y(x)$, puts a unit of $1 + \mathfrak{m}_R$ in $\ker y$. Every element of R is the sum of an element of O and one of $\mathfrak{m}_R$, so all values are integral. If $x_1, \dots, x_b$ generate $\mathfrak{m}_R$, the values on $\mathfrak{m}_R^a$ have norm at most $\max_i |y(x_i)|^a$. Successive approximation by polynomials in the x_i with coefficients in O now puts the image in the integers of the finite extension generated by their values.

For the flag, let S be the local domain at ℓ of Definition 29.18, with diagonal specialisation $x : S \to O$. Over its fraction field the four eigenvalues $d_i = \delta_i(s_0)$ are distinct. The projector equations give the stable flag in their prescribed order and give δ_i on inertia. The similitude pairs eigenline i with eigenline $3-i$, so this is a symplectic flag. For $g \in D_\ell$ its i-th quotient eigenvalue $a_i(g)$ is integral over S. Put $b_i = \prod_{j \neq i}(d_i - d_j)$. The projector trace formula puts $b_i a_i(g)$ in S, with image $b_i \delta_i(g)$ under x. The finite domain $S[a_i(g)]$ is local since S is henselian. At a prime over $\ker x$, the difference $a_i(g) - \delta_i(g)$ vanishes, since b_i has nonzero image. The difference thus belongs to the maximal ideal. Its image in the finite algebra modulo $\mathfrak{m}_S$ is nilpotent. The determinant trick gives a monic annihilating polynomial for the difference reducing to a power of X.

Impose these polynomial equations on the quotient lines of the symplectic flag variety over S, together with flag stability and the specified characters on inertia. These are closed conditions. Their common locus contains a generic point over $\operatorname{Frac} S$, and its projection is closed by properness. Thus over any of the specialisations under consideration there is a flag over an algebraic closure of the fraction field of the coefficient DVR. Its finitely many coordinates lie in a finite extension. Intersect its spaces with the integral lattice. The intersections are saturated direct summands, and extension of a basis of an isotropic flag to a symplectic basis gives an integral flag matrix u. The monic equations give the prescribed residual characters, also in equal characteristic. The inertia identities specialise as identities.

Lean: UpperTheory.V.Ws.Pp.genTri (Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.SymplecticIwahori.GenTri, lines 81–125)

Lean: UpperTheory.V.Ws.Pp.triSpec (Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.SymplecticIwahori.TriSpec, lines 6–35)

At σ_ℓ the residual eigenvalues on A are disjoint from those on B, by Definition 29.17. The first flag line therefore lies in A after reduction. The second graded line has its eigenvalue in the B-spectrum. The first flag plane is the sum of the first line and a line in B, by the spectral projectors for the two disjoint spectra. Use the original label order if u_{00} is a unit. Otherwise u_{20} is a unit, and use the symplectic signed permutation of order of indices $(2, 3, 0, 1)$. In this case the first-column eigenvector equation at every $g \in D_\ell$

gives $\bar{\delta}_2(g) = \bar{\delta}_0(g)$, hence $\bar{\Psi}(g) = 1$. The relabelling therefore respects the residual flag characters. In the chosen order the first line projects nontrivially onto coordinate 0. Its annihilator in B for the alternating form is the line in the first flag plane. That line projects nontrivially onto coordinate 1. Thus the first two leading minors of the flag matrix are units. The third is a unit by symplectic duality.

Choose the tower with this label order and its marking t. Hensel lifting gives four rank-one spectral projectors for $r(t)$, since its four residual eigenvalues are distinct. Apply the projectors to the corresponding basis vectors and normalise the two opposite symplectic pairings. This gives a symplectic eigenbasis congruent to the chosen basis modulo the maximal ideal. When r preserves B, its projectors preserve B, and this basis change does also. The flag matrix still has unit leading minors. Gaussian elimination writes it as nb, with n lower unitriangular and b upper triangular. Both factors are symplectic by uniqueness of the factorisation and the inverse-transpose involution giving the symplectic group. In reduction the first line lies in A and the first plane splits across A, B. Thus $n_{10} = n_{30} = n_{21} = 0$ in the residue field. The symplectic equations $n_{32} = -n_{10}$ and $n_{31} = n_{20} - n_{10}n_{21}$ give the asserted weight condition.

Lean: `UpperTheory.V.Ws.Lab.choice` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.FlagsAndDiagonals.Choice`, lines 32–72)
Lean: `UpperTheory.V.Ws.Lab.exNorm` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.FlagsAndDiagonals.ExNorm`, lines 7–193)

In the irreducible case the span of the group matrices over the algebraic closure is the full matrix algebra, so entries $02, 20, 12$ all occur. In the extension case the same argument on B gives entries $13, 31$. If the entries from A to B all vanished, A would be an invariant complement. Their three possible weights are $\beta, -\alpha, -\theta$.

Lean: `UpperTheory.V.Ws.Lab.starConj` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.FlagsAndDiagonals.StarConj`, lines 128–173)

□

Proof of 30.45

Proof. Normalise the valuation of a uniformiser ϖ to one. For a root w, collect the valuations of the nonzero representation entries of weight w, over all group elements. Include the valuation of the corresponding nonzero flag coordinate from the following table:

w	$-\alpha$	$-\beta$	$-\psi$	$-\theta$
coordinate	n_{10}	n_{21}	n_{20}	n_{30}.

Let c_w be the minimum of that collection, omitting the inequality for an empty collection. For $z = (a, b)$ impose $c_w + w \cdot z \geq 0$. These inequalities give integrality for every entry of the conjugated representation and flag. Indeed the conjugating diagonal has exponents $a + b, b, a, 0$; the remaining lower flag entries are given by the two symplectic equations in Lemma 30.44.

Among feasible real pairs maximise b, then a. The origin is feasible. The entry of weight $-\psi$ bounds b above, and at a fixed b an entry of weight $\beta, -\alpha$ or $-\theta$ bounds a above. These maxima are attained in the closed polyhedron defined by the finitely many inequalities. The selected vertex has half-integral coordinates, since any two independent root normals have determinant of absolute value one or two. Adjoin $\sqrt{\varpi}$ and use the diagonal with entries $\varpi^{a+b}, \varpi^b, \varpi^a, 1$. This preserves the marking, flag characters and similitude law. Write S for the tight weights. A weight belongs to S precisely when a representation entry or its listed flag coordinate survives. All initial valuations are positive except possibly the one for n_{20}. Hence $w \cdot z < 0$ for every tight $w \neq -\psi$, and tightness of $-\psi$ implies $b \geq 0$.

Lean: `UpperTheory.V.Ws.h7` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.RootSystemLinearProgramCases`, lines 99–180)
Lean: `UpperTheory.V.Ws.Lat.vertex` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.AdicLattices.Vertex`, lines 45–111)

We need an arithmetic restriction on S. Suppose ij has odd weight, namely one of $\pm\alpha, \pm\beta, \pm\theta$, and

$$R_{ix}(g)R_{xj}(h) = 0 \qquad (x \neq i, j;\ g, h \in G_{\mathbf{Q}}).$$

The diagonal entries are d_i, since diagonal conjugation leaves them unchanged. Matrix multiplication makes R_{ij}/d_j a cocycle for $\gamma = d_i/d_j$. It vanishes at the marking and on the excluded inertia kernels. If

$$R_{ij}(s) = m(d_j(s) - d_i(s)) \qquad (s \in D_\ell),$$

then $R_{ij} = 0$ globally and $m = 0$. To see this, the odd-weight characters in either allowed order are precisely the reductions of the six characters in the last condition of Definition 29.17. At each $p \in T \setminus \{\ell\}$, inertia has an ℓ-primary quotient detected by ϑ; thus $p \equiv 1 \bmod \ell$ by the tame inertia relation. At the prescribed Frobenius the residual cyclotomic character is one and $\gamma \neq 1$ by the separation in that definition. Hence both the invariants of γ and those of its Tate dual vanish. Local duality and the local Euler characteristic for the finite ℓ-primary module over $\mathbf{Q}_p$, $p \neq \ell$, give $H^1(\mathbf{Q}_p, \gamma) = 0$ [111, I, Cor. 2.3 and Thm. 2.8]. Apply the global coboundary condition of Definition 29.17. For extended residue coefficients, apply it coordinate by coordinate over the original residue field, using the finite span of the values of the locally constant cocycle. Evaluation at the marking, where $\gamma(t) \neq 1$, kills the global cobounding scalar. Evaluation at σ_ℓ, where the odd-weight character also differs from one, kills m.

Lean: `UpperTheory.V.Ws.Lat.oddVan` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.AdicLattices.OddVan`, lines 55–87)
Lean: `UpperTheory.V.Ws.Lat.entVan` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.AdicLattices.EntVan`, lines 31–64)

For $s \in D_\ell$ the first column of $R(s)N$ is $d_0(s)$ times the first column of N. If $-\alpha \notin S$, then $R_{10} = R_{32} = N_{10} = 0$. There is no two-step product from 0 to 3, and the first-column identity gives $R_{30} = N_{30}(d_0 - d_3)$ locally. The preceding argument kills R_{30} and N_{30}. If $\beta, \psi \notin S$, then $R_{12} = R_{13} = 0$. There is no two-step product from 0 to 1, and the same identity gives $R_{10} = N_{10}(d_0 - d_1)$ locally. Thus $R_{10} = N_{10} = 0$. The similitude identity applied to the inverse group element gives $R_{32} = 0$. We have proved

$$-\theta \in S \implies -\alpha \in S, \qquad -\alpha \in S \implies \beta \in S \text{ or } \psi \in S.$$

Lean: `UpperTheory.V.Ws.Lat.noNeg` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.AdicLattices.NotTight`, lines 45–101)

At the chosen vertex each of the directions $(1,0), (0,1), (-1,1)$ violates a tight inequality for every sufficiently small positive displacement. Taking dot products with the eight roots gives, respectively,

$$S \cap \{-\alpha, \beta, -\theta\} \neq \varnothing, \quad S \cap \{-\psi, -\beta, -\theta\} \neq \varnothing, \quad S \cap \{-\psi, \alpha, -\beta\} \neq \varnothing.$$

If $\beta \notin S$, the first intersection and the two arithmetic implications give $-\alpha, \psi \in S$. Hence $a > 0 > b$, which excludes every weight in the third intersection. Thus $\beta \in S$ and $b < a$. This excludes $-\beta$. If $-\psi \notin S$, the third intersection gives $\alpha \in S$, so $b < a < 0$. The second intersection then requires $-\theta \in S$, which requires $a + b > 0$. Consequently $-\psi \in S$ and $a > b \geq 0$. The strict sign test excludes $\alpha, \psi, \theta, -\beta$. This proves $S \subseteq S_P$.

There is no flag coordinate for β, so R_{12} is nonzero somewhere. Also $N_{21} = 0$, which gives the stated flag support. If $E(-\psi)$ fails, tightness of $-\psi$ gives $N_{20} \neq 0$. Row 2 of the first-column identity gives $d_2 N_{20} = d_0 N_{20}$ on D_ℓ. Finally the two-step products for entry 12 vanish, since $R_{02} = R_{13} = 0$. A scalar m as in the statement would kill R_{12} by the odd-weight argument. This proves the second alternative.

Lean: `UpperTheory.V.Ws.Lat.shape` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.AdicLattices.LowCase`, lines 39–104)

□

Proof of 30.46

Proof. At a prolongation v write g for an element transporting the original place to v. Use the lower unitriangular representative N_v of the transported flag. The factorisation exists integrally. Modulo the maximal ideal the first two leading minors of $R(g)N$ are $d_0(g)$ and $d_0(g)d_1(g)$, by the support in Lemma 30.45 and $N_{21} = 0$. The third is a unit by the similitude identity. Elimination and reduction therefore commute. The first two columns in the factorisation give

$$(N_v)_{21} = 0, \qquad (N_v)_{20} d_0(g) = R_{20}(g) + d_2(g) N_{20}.$$

Conjugating the local group transports the flag characters. Their inertia restrictions are the prescribed ones, since the characters come from $G_{\mathbf{Q}}$. The excluded inertia acts trivially by the defining inertia conditions of the tower and of the rational representation. The tower marking remains diagonal since it is a power of the original one. This gives the residual point with representation support in S_P and flag support in S_f.

We use that $F/\mathbf{Q}$ is normal and G_F contains a marking t_F separating the four labels. If $f = R_{ij}/d_j$ satisfies the cocycle law for $\gamma = d_i/d_j$ and vanishes on G_F, evaluate it at $g t_F g^{-1} \in G_F$. The result is

$(1-\gamma(t_F))f(g)=0$, so $f=0$ on $G_{\mathbf{Q}}$. On S_P this applies to entries $20, 31, 12$, by matrix multiplication. Thus the first alternative over $\mathbf{Q}$ retains both weights over F.

In the other alternative $R_{20}=R_{31}=0$, and the displayed transport identity gives $(N_v)_{20}\neq 0$ at every v. The support is as required for V. If $E(-\alpha)$ failed over F, entry 10 would vanish there. It too satisfies the two-term product law, because $R_{20}=R_{13}=0$; hence it would vanish over $\mathbf{Q}$. On the original D_ℓ, row 1 of the first-column identity would give

$$R_{12}(s)N_{20}=N_{10}(d_0(s)-d_1(s)).$$

Since $d_0=d_2$ there, $m=N_{10}/N_{20}$ would be the scalar excluded in Lemma 30.45. This proves $V\wedge E(-\alpha)$. □

Proof of 30.47

Proof. Write O_1 for the coefficient DVR and π for its uniformiser. The planes A, B are complementary Lagrangians. Let M consist of the infinitesimally symplectic maps from A to B, extended by zero on B. It is a free lattice, its elements have pairwise zero products, and $G_{\mathbf{Q}}$ acts by conjugation through r. Let $P\subseteq M$ consist of the maps strictly lowering the ordinary flag. It is saturated and stable under D_ℓ.

Write $s=\sigma_\ell$. The residual spectra of $r(s)$ on A, B are disjoint, so $s-1$ acting on M is invertible. The first ordinary line lies in A, and the first ordinary plane is its sum with a line in B: apply the two block spectral projectors of $r(s)$, whose block characteristic polynomials are coprime over O_1. If w spans this latter line over the fraction field, the nonzero map $v\mapsto\langle w,v\rangle w$ belongs to the fraction space of P. It kills the first plane and takes the entire space into that plane. Thus $\operatorname{rank} P>0$. Complex conjugation c acts on A, B by $-1, 1$, respectively: these are its residual signs and the two projectors $(1\pm r(c))/2$ are integral. It acts by -1 on M.

Lean: `UpperTheory.V.Ws.Bl.flagVec` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.BlockMatrixPlaces.FlagVec`, lines 87–176)

Put $M_n=M/\pi^nM$, $P_n=P/\pi^nP$. Use unramified cohomology away from ℓ and the image of $H^1(\mathbf{Q}_\ell,P_n)$ at ℓ. The latter map into $H^1(\mathbf{Q}_\ell,M_n)$ is injective: $s-1$ is bijective on M_n, on its stable submodule P_n by finiteness, and on their quotient. Its image has order at least $|P_n|$ by the local Euler characteristic formula for the finite ℓ-primary module P_n over $\mathbf{Q}_\ell$ [111, I, Thm. 2.8]. Apply the finite-module Selmer ratio formula (Proposition 26.27(4)) over $\mathbf{Q}$ with the places of ramification and ℓ: for a finite $G_{\mathbf{Q},S}$-module M_n with local conditions $L_v\subseteq H^1(\mathbf{Q}_v,M_n)$, $|S_L(M_n)|/|S_{L^\perp}(M_n^\dagger)|=(|H^0(\mathbf{Q},M_n)|/(|H^0(\mathbf{Q},M_n^\dagger)|\,|H^0(\mathbf{R},M_n)|))\prod_{v\in S}(|L_v|/|H^0(\mathbf{Q}_v,M_n)|)$. At $p\neq\ell$ the unramified group is the group of Frobenius coinvariants of $M_n^{I_p}$, of the same order as the invariants. The local invariants at ℓ vanish by $s-1$. The global dual invariants vanish because $\varepsilon_\ell(s)=1$ and $s-1$ is surjective; the real invariants vanish because ℓ is odd. Thus there are at least $|P_n|$ Selmer classes. Every class has a unique representative vanishing at c, since $c-1=-2$. Each representative vanishes on $I_p\cap\ker r$ away from ℓ, and on D_ℓ differs from a P_n-valued cocycle by a coboundary.

Lean: `UpperTheory.V.Ws.Bl.gwCount` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.BlockMatrixPlaces.GwCount`, lines 12–76)

Consider all normalised cocycles satisfying the last two conditions. At each depth they form a finite set, by finite-module global cohomological finiteness and finiteness of the module of coboundaries. Their cardinalities are unbounded by the preceding estimate. At every depth there is a cocycle with nonzero reduction modulo π. Otherwise at that and all greater depths every cocycle would be divisible by π, and division would inject the set at each successive depth into the preceding set. For the local condition in this division, write $z|_{D_\ell}-dy\in P_{n+1}$. If z is divisible by π, evaluation at s shows $y \bmod \pi\in P_1$, since $s-1$ is invertible on M_1/P_1. Subtract a lift in P_{n+1} from y, then divide by π; saturation preserves the condition. The other conditions divide entrywise. The sets would consequently have bounded cardinality. Taking a branch in the finitely branching tree of these cocycles and their local witnesses gives, by completeness, a nonzero integral cocycle z and $y\in M$ with $z|_{D_\ell}-dy$ valued in P, where $dy(g)=r(g)yr(g)^{-1}-y$.

Lean: `UpperTheory.V.Ws.Bl.cocT` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.BlockMatrixPlaces.CocT`, lines 11–56)

Replace z, y by $\pi z, \pi y$. Put $r'(g)=(1+z(g))r(g)$ and take the flag $(1-y)u$, where u was the original flag matrix. The cocycle law and $M^2=0$ give multiplicativity. The new flag matrix conjugates the local action to the old one multiplied in the ordinary basis by $1+u^{-1}(z-dy)u$, upper unitriangular there. This preserves its diagonal characters. The new representation preserves the similitude and the prescribed inertia

kernels. Its reduction and block diagonal entries agree with the old ones. In particular all characteristic polynomials agree. The action on B is the similitude twist of the dual action on A; hence it has full matrix span. A complement would be the graph of a map $Y : A \to B$ and would express z as a coboundary in $\mathrm{Hom}(A, B)$. Evaluation at c gives $2Y = 0$, since $z(c) = 0$. This forces $Y = 0$ and then $z = 0$, a contradiction over the algebraic closure as well.

Lean: `UpperTheory.V.Ws.Bl.cocA` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.BlockMatrixPlaces.TwistMul`, lines 28–62)

□

Proof of 30.48

Proof. Let C be the identity component of the Zariski closure of $r(G)$ in GL_n, and $N = r^{-1}(C)$. This is a normal subgroup of finite index. It is closed in the profinite topology since the matrix entries are continuous and the equations for C are defined over a finite coefficient extension. Hence N is open. Its image is dense in C: the components are open and closed in the algebraic group. For any open subgroup $U \le G$, $r(U \cap N)$ is also dense in C. Indeed, finitely many translates of its closure cover C, and C is irreducible. Thus an open coset σU contains elements with image dense in $r(\sigma)C$.

Write A for the linear span of $r(N)$. It is a unital matrix algebra normalized by $r(G)$. A positive power of u belongs to N and still has distinct eigenvalues. Its spectral projectors give all $E_{ii} \in A$. Consequently an entry ij occurs in A precisely when $E_{ij} \in A$, by multiplying on both sides by projectors. This relation is reflexive and transitive. Group together indices related in both directions. The algebra contains every matrix unit within each class. The span J of its matrix units between distinct classes is a nilpotent two-sided ideal: a nonzero product follows a strictly ordered chain of classes. The quotient by J is the product of the full class matrix algebras, so J is the Jacobson radical and is normalized by $r(G)$. Its common nullspace is nonzero by nilpotence. Irreducibility makes that nullspace the whole representation. Thus $J = 0$.

Conjugation permutes the central idempotents of the resulting product. It acts transitively on the classes, since a proper union of orbits would span an invariant subspace. All classes therefore have the same size. Singleton classes would make $r(N)$ commutative. There is one class in ranks two and three. In rank four there is one class or there are two classes of size two.

The linear span of $r(\sigma)C$ is $r(\sigma)A$. If σ preserves the classes, this span contains every diagonal matrix unit. Each diagonal coordinate is consequently nonzero somewhere on the coset. Inversion maps the coset to $r(\sigma^{-1})C$, so the same holds for each inverse diagonal coordinate. If σ exchanges two classes, its coset has zero diagonal blocks and its span contains every off-diagonal block matrix unit. Each required nonvanishing is a nonempty Zariski open condition on the same irreducible coset. Their finite intersection meets $r(\sigma(U \cap N))$. □

Proof of 30.49

Proof. For (1), take h past the threshold of Lemma 30.41. The resulting normalized cuspidal eigenform of weight h, level $\Gamma_1(Dw)$ and character $\phi\theta$ is new because its nebentype has primitive conductor Dw. Attach its cuspidal representation of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ by Proposition 26.36(3) and its rank-two Galois representation by Proposition 26.37. Twist from determinant D_h^{-1} by D_h to obtain a representation ρ of determinant D_h. For a two-by-two matrix R of determinant D_h^{-1}, $\mathrm{tr}(D_h R) = \mathrm{tr}(R^{-1})$. Thus the good-prime congruences in Lemma 30.41, density, and $\bar\theta = 1$ give residual polynomial $(X-1)(X-\bar\Phi_h)$ on $G_{\mathbf{Q}}$, with $\Phi_h = \varepsilon_\ell^{h-1}\phi$. Local compatibility in Proposition 26.37(2) gives spherical support outside Dw, and at $p \mid Dw$ the primitive nebentype of conductor Dw gives unlinked principal-series support with inertia type $1, \phi\theta$ away from ℓ. At ℓ the unit root reduces to one by $a_\ell \equiv 1$ and the positive valuation of the other root. The unramified flag character in the initial normalisation therefore reduces to one on the decomposition group. The twist gives the ordered inertia weights $(h-1, 0)$ with residual labels $(\bar\Phi_h, 1)$ on the decomposition group. Increase the weight threshold so that $h \ge 3$. If ρ were reducible, its semisimplification would have two continuous characters χ_1, χ_2. The ordinary inertia weights give, after interchanging them, $(\chi_1/\chi_2)|_{I_\ell} = \varepsilon_\ell^{h-1}$. Global class field theory over $\mathbf{Q}$ identifies this ratio with ε_ℓ^{h-1} times a finite-order character: after removing the indicated cyclotomic power it has finite inertia at every finite prime, and hence factors through a finite ray class group. At a good prime p, the modulus of the ratio of the two Frobenius roots is therefore p^{h-1} or

its inverse. After the common unitary normalization the ratio is unchanged. The strict spherical bound for a unitary cuspidal representation puts the modulus of this ratio strictly between p^{-1} and p, contradicting $h \geq 3$. Thus ρ is irreducible.

Lean: `UpperTheory.V.Z.Win.Birth.propEGP_of_printsH` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.BirthEG`, lines 231–297)
Lean: `UpperTheory.V.Z.Sym.W.prop_eg_complex_of_printed_hida` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.WEGCore`, lines 235–303)

For restriction, let H be open normal in $G_{\mathbf{Q}}$. A positive power of an inertia element of infinite cyclotomic order lies in H and has distinct eigenvalues because $h > 1$. If $\rho|_H$ fixes a line, its translates are among these two eigenlines. Irreducibility over $G_{\mathbf{Q}}$ forces a transitive action on both. Every element interchanging them has trace zero and hence $\bar{\Phi}_h = -1$. Multiplying an interchanging element by any element fixing both lines shows that $\bar{\Phi}_h = 1$ on the stabiliser, which contains H. Thus a nontrivial value of $\bar{\Phi}_h$ on H gives irreducibility there, and a value with $\bar{\Phi}_h^2 \neq 1$ anywhere gives irreducibility on every open subgroup by taking normal cores.

Lean: `UpperTheory.V.Sd.sI` (`Development.AutomorphicArithmetic.AutomorphicVarieties.IdeleGenerators.HolomorphicIrreducibility.NormOneOfFin`, lines 23–56)

For (2), put $N = |D_K| \geq 3$, fix a complex embedding $\tau : K \hookrightarrow \mathbf{C}$, and work with the full finite unit quotient $(\mathcal{O}_K/N\mathcal{O}_K)^\times$ (which incorporates principal-unit quotients at primes dividing D_K, for instance modulo $4 = -(1+i)^4$ when $K = \mathbf{Q}(i)$). Because K is imaginary quadratic, every global unit $u \in \mathcal{O}_K^\times$ is a root of unity; if $u \equiv n \pmod{N\mathcal{O}_K}$ for a rational integer n, subtracting complex conjugates gives $u - u^{-1} = N(y - \bar{y})$ with $y - \bar{y} \in \mathcal{O}_K$ satisfying $|y - \bar{y}| \leq 2/N < 1$ at both embeddings, forcing $y = \bar{y}$ and hence $u = \pm 1$. Inside $(\mathcal{O}_K/N\mathcal{O}_K)^\times$, the subgroups $(\mathbf{Z}/N\mathbf{Z})^\times$ and $\mathrm{im}(\mathcal{O}_K^\times)$ therefore meet only in $\{\pm 1\}$, where the odd quadratic character η_K satisfies $\eta_K(-1) = -1 = \tau(-1)$. By divisibility of $\mathbf{C}^\times$, η_K on $(\mathbf{Z}/N\mathbf{Z})^\times$ and τ on $\mathcal{O}_K^\times$ glue and extend to a character $F : (\mathcal{O}_K/N\mathcal{O}_K)^\times \to \mathbf{C}^\times$. Write $|(\mathcal{O}_K/N\mathcal{O}_K)^\times| = \ell^s m$ with $\ell \nmid m$. Because ℓ is odd and $\ell \nmid D_K$, K cannot contain a primitive ℓ-th root of unity (whose cyclotomic discriminant $\pm\ell^{\ell-2}$ would divide D_K), so $\ell \nmid 2w_K$ where $w_K = |\mathcal{O}_K^\times|$. Setting $E = (\ell^s)^{\varphi(2w_K)}$ yields $e = F^E : (\mathcal{O}_K/N\mathcal{O}_K)^\times \to \mathbf{C}^\times$ of finite order dividing m (prime to ℓ), while $E \equiv 1 \pmod{2w_K}$ preserves both $e|_{(\mathbf{Z}/N\mathbf{Z})^\times} = \eta_K$ and $e|_{\mathcal{O}_K^\times} = \tau$. On $K_\infty^\times \widehat{\mathcal{O}}_K^\times$, the product $(x_\infty \mapsto \tau(x_\infty)^{-1}) \cdot (e \circ \mathrm{red}_N)$ is trivial on $\mathcal{O}_K^\times$ and extends (by divisibility of $\mathbf{C}^\times$, trivial on $K^\times$) to an algebraic Hecke character of K of infinity type $(-1, 0)$, unramified outside D_K, whose finite-place unit restrictions all have order dividing m (prime to ℓ) and whose value on any inert prime $p \nmid N$ is $p\eta_K(p)^{-1}$. Passing via global reciprocity to the ℓ-adic realization κ_0, Chebotarev density on inert Frobenii identifies the rational-idelic transfer from $G_{\mathbf{Q}}$ as $\eta_K \varepsilon_\ell^{-1}$, whose restriction to G_K gives $\kappa_0(g)\kappa_0(cgc^{-1}) = \varepsilon_\ell(g)^{-1}$.

Lean: `UpperTheory.V.Cu2.kapEx_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.MPKapEx`, lines 20–117)
Lean: `UpperTheory.V.Sd.twist_of_printed_results` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TTwist`, lines 111–178)

□

Proof of 30.50

Proof. For (1), let $c \in \overline{k}^\times$ be the single residual root of $\det(X - \overline{\rho(s_0)})$, nonzero since $\rho(s_0) \in \mathrm{GL}_m(A)$. The characteristic polynomials $P_1(T) = \det(T - \chi(s_0)\rho(s_0))$ and $P_0(T) = \det(T - \rho(s_0))$ have coefficients in A and reductions $(T - \bar{\chi}(s_0)c)^m$ and $(T - c)^m$ over $\overline{k}$. Since $c \neq 0$ and $\bar{\chi}(s_0) \neq 1$, these reductions are coprime over k, hence P_1 and P_0 are coprime in $F[T]$. Choose $U, V \in F[T]$ with $U(T)P_1(T) + V(T)P_0(T) = 1$. Any $X \in M_m(F)$ satisfying $\chi(s_0)\rho(s_0)X = X\rho(s_0)$ satisfies $V(\chi(s_0)\rho(s_0))P_0(\chi(s_0)\rho(s_0))X = XV(\rho(s_0))P_0(\rho(s_0))$, so

$$X = U(\chi(s_0)\rho(s_0))P_1(\chi(s_0)\rho(s_0))X + XV(\rho(s_0))P_0(\rho(s_0)) = 0$$

by the Cayley–Hamilton theorem.

For (2), the line characters on P have unit values, since for $p \in P$ both $\rho(p)$ and $\rho(p)^{-1} = \rho(p^{-1})$ belong to $\mathrm{GL}_m(A)$ and each eigenvalue and its inverse are roots of monic polynomials over A. Reduce them using the fixed extension of the valuation. For $p \in P$ and $g \in H$, every residual eigenvalue of the commutator $[p, g] = pgp^{-1}g^{-1}$ is one, since $\bar{\chi}_i([p, g]) = 1$ for every i. On a line L_a sent by g to $L_b = gL_a$, the eigenvalue of $\rho([p, g])$ on L_b is the ratio of the character of L_b at p to that of L_a at p, so the two reduced values at p coincide. By transitivity they all coincide at a common reduced value $\bar{\alpha}(p)$. Over $\overline{k}$ the characteristic polynomial of $\overline{\rho(p)}$ is $(X - \bar{\alpha}(p))^m = \prod_{i=1}^m (X - \bar{\chi}_i(p))$, giving $\bar{\chi}_1(p) = \cdots = \bar{\chi}_m(p) = \bar{\alpha}(p)$.

Lean: UpperTheory.V.Ws.P1.irrL (Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.ProjectiveLineFlags.TraceOfCp, lines 17–74)
Lean: UpperTheory.V.Ws.h4 (Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.H4, lines 19–117)
Lean: UpperTheory.V.Pr.irrNormal (Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.IrrNormal, lines 62–179)
Lean: UpperTheory.V.Pr.nice_specialization_twist_satisfies_third_condition (Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.NiceSpecializationTwistSatisfiesThirdCondition, lines 164–213)

□

Proof of 30.51

Proof. Use the host inequalities and cases of 29.1. Write $e = k$, $\psi = \vartheta$ as a Galois character, and $\Psi = \varepsilon_\ell^e \psi$. Take the auxiliary odd character ξ, similitude ν and level primes S_D of Definition 29.17. A tower uses an imaginary quadratic K, a totally real abelian F of even degree, $L = FK$, a marking $g_0 \in G_F$, and prolongations over ℓ inducing the chosen place of K. Write d_i for the ordered residual diagonal characters. At auxiliary rational set W the frame kills inertia outside $W \cup \{\ell\}$ on the kernel of the residual diagonal. At ℓ it records the decomposition group, inertia, the preimage of the ℓ-primary abelian torsion and the flag. Auxiliary split primes are trivial on F, the residual diagonal and μ_ℓ. On a deep layer $\psi|_{G_F}$ has order prime to ℓ.

For a sign $s = \pm 1$ and a finite list S_0, choose a fresh prime $p \equiv s \bmod 8 \prod_{q \in S_0} q$. Its quadratic character has sign s and splits at the listed primes by quadratic reciprocity and the supplement at 2. In II take an even character ϕ split at S_D, then two odd characters η, η', adding each previous prime to the splitting list. Let κ_b be the ℓ-primary part of ψ with its local factor at ℓ removed. Set

$$G_F = \{g : \bar{\Psi}(g) = \phi(g),\ \eta(g) = \eta'(g),\ \kappa_b(g) = 1\}, \qquad G_K = \ker \eta, \qquad G_{k_0} = \ker \phi.$$

Here k_0 is real quadratic and split at $2, \ell$. The equality of the odd characters puts conjugation in G_F and gives even degree. At the conductor of η, inertia in G_F kills η because η' is unramified. Thus FK/F is unramified at finite places. The root of weight ψ in 30.38 cuts out Fk_0/F. The marking has $\bar{\Psi} = -1$, $\bar{\xi}^2 \neq 1$ and residual cyclotomic value one. The fourth-power test $\bar{\xi}^4 \neq 1$ holds on elements of the decomposition groups at ℓ and of inertia at 2.
Lean: UpperTheory.V.Ws.quadEx (Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.FrobeniusFields.QuadEx, lines 75–118)
Lean: UpperTheory.V.Ws.exists_tame_tower_ok_two (Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.ExistsTameTowerOkTwo, lines 22–59)

In III choose an odd quadratic character $\lambda_{K'}$ split at S_D and ramified at a fresh prime beyond the conductors. Choose odd η, η' by the same rule, split also at this prime, with η' split at the conductor of η. Set

$$G_F = \{g : \xi(g) = \lambda_{K'}(g),\ \bar{\Psi}(g) = \bar{\psi}(g),\ \eta(g) = \eta'(g)\}, \qquad G_K = \ker \eta.$$

As in II, F is totally real of even degree and FK/F is unramified at finite places. Write ω for the Teichmüller lift of $\bar{\varepsilon}_\ell$. Here $\ell \geq 5$, $\ell - 1 \mid 2e$, and ψ has odd order prime to ℓ. For the marking multiply an inertia element at 2 with $\xi = -1$, one changing the sign of $\lambda_{K'}$ at its fresh conductor, and a square detecting the odd part of ψ. The fresh inertia kills the old characters. Their product lies in $G_F \cap G_K$ with $\omega = 1$. Its relative labels are $1, -1, z, -z$ up to inversion, with $z \neq 1$ of odd order. Thus a quadratic permutation twist through ω fixes every index. The ratio $-z$ has nontrivial first four powers. At ℓ, squares of inertia lie in G_F, and some have nontrivial residual cyclotomic value. These are the residual separation and cyclotomic tests.
Lean: UpperTheory.V.Cu.existsTensorSeedTower_holds (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.CuspidalTwists.ComplexConjugationsConjugateCaseThree, lines 538–720)

In II let ω_0 be the finite Hecke character of k_0 corresponding to $\xi|_{G_{k_0}}$. It is odd at both real places and unramified over ℓ. The fourth-power test makes it ramified at a place of norm two and gives $\bar{\omega}_0(\varpi_v), \bar{\omega}_0(\varpi_{v'}) \neq 1$ at the two places $v, v' \mid \ell$. Let E be the parallel weight-one Hilbert Eisenstein series for $1, \omega_0$. Write V_w for the normalized level-raising operator which sends the coefficient at $\mathfrak{a}$ to the coefficient at $\mathfrak{a}/\mathfrak{p}_w$, interpreted as zero for a nonintegral index. Take $(1 - V_v)(1 - \omega_0(\varpi_{v'})V_{v'})E$. The two roots at each of these unramified places are $1, \omega_0(\varpi_w)$, so the retained roots are $\omega_0(\varpi_v)$ and 1, respectively. The constant coefficient of E at infinity is the same on the ideal-class components. In weight one the normalized V_v translates the component

of this constant coefficient with multiplier one. Thus $1 - V_v$ kills it. The coefficient at the unit ideal remains one.

Lean: `UpperTheory.V.Sd.emix` (`Development.AutomorphicArithmetic.AutomorphicVarieties.IdeleGenerators.HolomorphicIrreducibility.NotDvdMul`, lines 270–299)

Choose the multiplier at level one. Since ℓ splits in the real quadratic field k_0, its total Hilbert Hasse invariant has parallel weight $\ell - 1$ and expansion one at every cusp [135, Thm. 2.1(1),(2)]. At sufficiently small auxiliary tame level, a sufficiently high power lifts to an integral characteristic-zero form by [136, Lems. 2.2–2.3]. To obtain level one, take a normal full tame level contained in an auxiliary level satisfying condition (20) of that reference, and pull back the lift. Take the product of its translates by the finite quotient of the full level-one compact group by this normal subgroup. The product is invariant at level one and has trivial character. Its reduction is the corresponding power of the level-one Hasse invariant, since that section is fixed by every translate. Thus its expansion is congruent to one at every cusp. Taking a further power gives parallel weight a divisible by $\ell - 1$ with $a + 1 \geq 4$. Multiply the stabilized series by this form.

Lean: `GlobalTheory.HilbertThetaSeries.hilbertOne` (`Development.AutomorphicArithmetic.AutomorphicVarieties.GaussSumFiltrations.HasQOfSmallSubOne`, lines 33–75)

Use level exponent one above ℓ and the conductor exponents of ω_0 elsewhere. The product has a unit coefficient at the unit ideal and preserves the residual system. At good places the weight change preserves the Hecke recurrence modulo the maximal ideal because $(\mathrm{N}\mathfrak{p})^a \equiv 1 \bmod \ell$.

For the Hilbert lifting put $h = a + 1$. Transport the Fourier expansion of the finite-dimensional holomorphic space by ι^{-1} into the space of coefficient functions on the nonzero integral ideals of k_0. At index $\mathfrak{a}$, $T_\mathfrak{p}$ takes the coefficient at $\mathfrak{p}\mathfrak{a}$ and, if $\mathfrak{p} \mid \mathfrak{a}$, adds the coefficient at $\mathfrak{a}/\mathfrak{p}$ times $\chi(\mathfrak{p})(\mathrm{N}\mathfrak{p})^{h-1}$, with $\chi = \omega_0$ extended by zero at the level primes. At level primes this is the $U_\mathfrak{p}$ formula. All these multipliers are integral, so the commuting operators preserve both the coefficientwise integrality condition and the submodule of functions with every coefficient in the maximal ideal. The product above gives a nonzero simultaneous residual eigenvector in this coefficient space. Apply the Deligne–Serre lifting lemma [117, Lem. 6.11] to this operator family. The Fourier map is injective on the holomorphic space in weight h, so the lifted coefficient eigenvector pulls back to a holomorphic eigenform. Its $U_v, U_{v'}$ eigenvalues reduce to $\bar\omega_0(\varpi_v), 1$, respectively.

Lean: `UpperTheory.V.Sd.holLift_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.THolLift`, lines 129–207)

If the lift is Eisenstein with inducing characters η_1, η_2, the good eigenvalues and the central character imply $(\bar\eta_1(g) - 1)(\bar\eta_2(g) - 1) = 0$ by reciprocity and Frobenius density. The two kernels cover the group, so one character is trivial. In the ordinary Eisenstein lift both unit U-roots come from the first inducing character, since the second has the positive norm power. Its residual values at v, v' are $(\bar\omega_0(\varpi_v), 1)$, and the product character gives $(1, \bar\omega_0(\varpi_{v'}))$ for the second. Both characters are nontrivial. Thus the lift is cuspidal.

Take $h \geq 4$ and let d_0 be the dimension of the parallel-weight cusp space at the chosen level and central character. Put $t_0 = 2(\ell - 1)\ell^{d_0}$. Change the weight at the real embedding assigned to v from h to $h + 2t_0$, with the factor ℓ^{t_0} in the normalisation of the operator at v. For a product containing at least $r \geq 1$ ordinary operators at each place over ℓ, the normalised traces satisfy

$$\mathrm{tr}_{h,h} \equiv \mathrm{tr}_{h+2t_0,h} \pmod{\ell^{\min(r,d_0)}}. \tag{A.271}$$

Apply Lemma 30.43 to the two cusp spaces, with $F = k_0$, $N = d_0$, $t = t_0$, and $\omega = \omega_0$. The field is real quadratic and ℓ splits. The finite character has the parity of h at each real place, its conductor divides the level, and the norm-two place has positive level exponent. Both exponents above ℓ are one. The chosen t_0 is $(\ell - 1)\ell^{d_0} B$ with $B = 2$. This gives (A.271) for every indicated Hecke word. The parallel cusp eigenform constructed above has integral Hecke eigenvalues with the prescribed reductions, including the units $\bar\omega_0(\varpi_v)$ and 1. The second assertion of the lemma gives a mixed-weight cusp eigenform with both ordinary refinements in the displayed normalization.

Lean: `UpperTheory.V.Z.Bc.propMHolZ_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.MPropMHol`, lines 108–223)

Lean: `UpperTheory.Esk.pf_TrCong` (`PositivityTheory.AutomorphicForms.PublishedInputs.JacquetLanglands.TorusKernels.PfTrCong`, lines 101–128)

Lean: `UpperTheory.Esk.pf_HolWC` (`PositivityTheory.AutomorphicForms.PublishedInputs.JacquetLanglands.TorusKernels.PfHolWC`, lines 174–304)

Attach the rank-two representation by Propositions 26.36(1) and 26.37 and twist by ε_ℓ^a in that normalisation. The resulting r on G_{k_0} has determinant $\varepsilon_\ell^a \xi$, residual polynomial $(X - 1)(X - \bar\xi)$ and weights

$(a+t_0,-t_0)$, $(a,0)$ at the two places. The residual orders are $(\bar{\xi},1)$, $(1,\bar{\xi})$. The squared marking splits the lattice into its two eigenlines. Scale one line so the upper entries are integral with at least one unit. Then the opposite entries lie in the maximal ideal by the residual products. Induce from G_{k_0}. The summands on G_{k_0} are r, r^{g_0}, and the other coset exchanges them with marked square $r(g_0^2)$. The weights distinguish the summands on open subgroups. The marked fourth-power test and the incompatible ordinary orders give irreducibility. The similitude is $\mu = \varepsilon_\ell^a \xi \phi$. On G_F, twist by the unique square root of ν/μ congruent to one, which is a character by Hensel uniqueness (when k is even, this continuous polarizing twist carries half-integral cyclotomic weight, so the seed is automorphic up to a continuous twist κ_{seed} whose parity is absorbed at the characteristic-zero seed below via the odd-weight link twist β_0^n). Base change to the compositum with k_0 and quadratic automorphic induction give the attachment, since the conjugate infinity types are distinct [93]. Perform the lattice scaling on G_{Fk_0}: the restriction there is irreducible, the squared marking has distinct residual eigenvalues, and the upper entries attain their maximum absolute value by compactness. Make this maximum one. Write the two conjugate residual matrices there as $\left(\begin{smallmatrix}1 & x\\ 0 & c\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & y\\ 0 & c\end{smallmatrix}\right)$, where $c = \bar{\xi}(g)$. Let a, b be the two chosen square roots of the eigenvalues of the squared marking, in the order of these two diagonal positions in the rank-two lattice, and put $\gamma = \bar{b}/\bar{a}$. In the marked induction basis the matrix is

$$\begin{pmatrix} c & 0 & 0 & 0 \\ D & 1 & -S & 0 \\ 0 & 0 & c & 0 \\ S & 0 & -D & 1 \end{pmatrix}, \qquad D = (\gamma x - y)/2, \quad S = (\gamma x + y)/2.$$

Indeed, on each of the two pairs of exchanged lines take the sum and difference eigenvectors of the marking. In the upper entry of the first summand the change of scale between the pairs is b/a, and in the second summand it is one. Adding and subtracting the resulting entries gives the displayed matrix. The marking itself is diagonal with entries $b, -a, -b, a$. Multiplication by this diagonal covers the other coset and preserves the support. The common scalar twist reduces to one. Choose $g \in G_{Fk_0}$ with $x \neq 0$ by the scaling. Either $D \neq 0$, giving entry 10, or $D = 0$ and $S = \gamma x \neq 0$, giving entries $12, 30$. Their respective weights are $-\alpha, \beta, -\theta$.

For the local flags write the two ordinary lines as $(u,1)$ and $(1,w)$, interchanging the two summands where the ordinary orders are reversed. Here u is integral and w is in the maximal ideal: the line of residual character 1 is the first coordinate line for an upper triangular residual matrix with distinct diagonal characters on the local group, and the line of the other character has unit second coordinate. In the integral sum-and-difference basis write the lower flag matrix as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ p & 1 & 0 & 0 \\ q & s & 1 & 0 \\ r & t & -p & 1 \end{pmatrix}.$$

With $c_0 = b/a$, its entries (p,q,r,s,t) in the two orders are respectively

$$\left(c_0 u, -1, c_0 u, \frac{-2w}{1+c_0 uw}, \frac{-(1-c_0 uw)}{1+c_0 uw}\right), \qquad \left(-u, 1, u, \frac{-2w}{c_0+uw}, \frac{c_0-uw}{c_0+uw}\right).$$

These are obtained by putting the two ordinary eigenvectors in the first two flag spaces and making the flag matrix lower unitriangular. Direct multiplication gives $t + ps = q$, the symplectic flag equation. Both denominators are units. Thus $n_{21} = 0$ after reduction and $n_{20} = \pm 1$ at every place. This proves V with the asserted alternative.

Lean: `UpperTheory.exists_ribet_lattice` (`Development.AutomorphicArithmetic.AutomorphicVarieties.BuildingsResidueChars.ExistsRibetLattice`, lines 7–42)
Lean: `UpperTheory.mix_flag_exists` (`Development.AutomorphicArithmetic.AutomorphicVarieties.BuildingsResidueChars.MixFlagExists`, lines 116–144)
Lean: `UpperTheory.Esk.pf_PropMV_sv_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.SVRPropMV`, lines 21–152)
Lean: `UpperTheory.V.Z.Win.Birth.mix_birth_of_prints` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.BirthMix`, lines 17–120)
Lean: `UpperTheory.V.Z.Bc.seedII_of_p_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.BirthSeedII`, lines 120–148)

In III choose an algebraic Hecke character of K' of type $(-1,0)$ by Lemma 30.49(2), with ℓ-adic character κ_0; here $\ell \nmid D_{K'}$ since the fresh conductor prime of $\lambda_{K'}$ avoids ℓ and is congruent to $-1 \bmod 8$. Choose $b \geq 1$ divisible by the residual image exponent and tame inertia exponents of $\kappa_0^2\varepsilon_\ell$. Then $\lambda_0 = (\kappa_0^2\varepsilon_\ell)^b$ reduces to one and kills inertia away from ℓ. Its type is $(-b,b)$ and conjugation inverts it. Choose w split in the conductor, residual and cyclotomic fields by Chebotarev [38, §1A], take a character θ of order ℓ at w, and apply Lemma 30.49(1) to the odd primitive character ϕ inducing $\xi\psi^{\pm1}$, with sign given by the order of the labels, at a sufficiently large weight $h \geq 18b+3$ with $h-1 \equiv e \bmod \ell-1$. The resulting ordinary representation ρ has residual pair $(\bar\Phi_h, 1)$ for $\Phi_h = \varepsilon_\ell^{h-1}\xi\psi^{\pm1}$, with $\bar\Phi_h^2 \neq 1$ at the marking, so Lemma 30.49(1) gives irreducibility on every open subgroup of $G_{\mathbf{Q}}$. Scale the eigenlines in the reverse order to II so the lower entry is integral and somewhere a unit. The ordinary line has chart $(1, x_v)$ with x_v integral by the local separation test.

Choose $1 \leq j \leq 9$ so $2bj$ avoids the at most eight differences of the two infinitesimal weights at an embedding of K' and the two at its conjugate, in both orders. Put $m = bj$, $\lambda = \lambda_0^j$, $\tau = \mathrm{Ind}_{G_{K'}}^{G_{\mathbf{Q}}} \lambda$. In the sum-and-difference basis f_0, f_1, the two diagonal entries on $G_{K'}$ are $(\lambda + \lambda^{-1})/2$ and the other two are $(\lambda - \lambda^{-1})/2$. On the other coset use the matrix at gg_0^{-1} times $\mathrm{diag}(1,-1)$, since $\lambda(g_0^2) = 1$. The matrices preserve $\mathrm{diag}(1,-1)$, with ordinary lines $(1, s_v)$, $s_v = \pm1$, of weights $m, -m$.

Use $\rho \otimes \tau$ in basis $(e_0f_0, e_0f_1, -e_1f_1, e_1f_0)$, where $e_if_j = e_i \otimes f_j$ and e_0, e_1 are the marked basis of ρ. The tensor of the alternating and symmetric forms is symplectic. Let $u \equiv 1$ be the square root of ε_ℓ/ω. For $\xi\psi$ twist by $\delta = u^{e+2-h}$. For $\xi\psi^{-1}$ take $\delta = u^{e+2-h}\omega^e\psi$. Then $\delta^2\Phi_h = \nu$ by $\omega^{h-1} = \omega^e$ (when $e+2-h$ is odd, δ carries half-integral cyclotomic weight, so the seed is automorphic up to a continuous twist κ_{seed} whose doubled weights have common odd parity, absorbed at the seed below by twisting the link map by β_0^n). The tower enlargement kills the character of conductor w in $\det\rho$. The local flag is obtained by applying $\left(\begin{smallmatrix} 1 & 0 \\ x_v & 1 \end{smallmatrix}\right) \otimes 1$ to the lower unitriangular matrix with entries s_v at 10, $-s_v$ at 32, and zero at the other off-diagonal positions. Before the common twist the weights are

$$(h-1+m, h-1-m, m, -m).$$

Their successive gaps are at least two by $h \geq 18b+3$ and $m \leq 9b$.

Over LK' the constituents are $\rho\lambda, \rho\lambda^{-1}$. They have distinct infinity types by the choice of j. Solvable base change and quadratic induction give the cuspidal attachment over L [93]. An isomorphism of the two constituents on the cyclotomic kernel would give $\lambda^4 = 1$ there by determinants. The opposite nonzero weights at the two split places give λ infinite image on that kernel, by taking the quotient of two inertia elements of equal cyclotomic value. The other coset exchanges the constituents. This yields the irreducible symplectic seed. Its residual frame initially allows w. Write the residual rank-two factors as $\left(\begin{smallmatrix} A & 0 \\ y & 1 \end{smallmatrix}\right)$ and $\mathrm{diag}(1,e)$, where e is the sign of K'. The second formula follows from $\bar\lambda = 1$ and the action on the two cosets. Omitting the common nonzero scalar, the tensor in the displayed basis is

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & Ae & 0 & 0 \\ 0 & -ey & e & 0 \\ y & 0 & 0 & 1 \end{pmatrix}.$$

The local lower flag matrix has 10 entry $s_v = \pm1$, so this support gives H. To see that both entries survive restriction, use $y(gh) = y(g)A(h) + y(h)$. The lattice has $y(g) \neq 0$ for some $g \in G_{\mathbf{Q}}$. At the marking $t \in G_F$ of the enlarged tower, $A(t) \neq 1$ by distinctness of labels $0, 3$. If $y(t) = 0$, then $y(gtg^{-1}) = y(g)(A(t)-1)/A(g) \neq 0$, and $gtg^{-1} \in G_F$ by normality. Thus one element of G_F has both 21 and 30 entries nonzero. These give $E(-\beta)$ and $E(-\theta)$.

Lean: `UpperTheory.V.Cu.entOf4` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.CuspidalTwists.ConjRes`, lines 36–89)

Lean: `UpperTheory.V.Cu2.kapEx_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.MPKapEx`, lines 20–117)

Lean: `UpperTheory.V.Cu.twEx4` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.CuspidalTwists.EqOfRd`, lines 68–171)

Lean: `UpperTheory.V.Cu.Dat.Q` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.CuspidalTwists.OverOf4`, lines 51–52)

Lean: `UpperTheory.V.Cu.rank_four_automorphy_of_tensor_of_base_change_of_induction` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.AutomorphicLiftGsp4OfInputs`, lines 69–180)

Lean: `UpperTheory.V.Cu.fourth_initial_of_rank_four_data` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.BirthSeedIV`, lines 18–63)

To join residual points put $A_0 = d_1/d_0,\ B_0' = d_2/d_0$, so $d_3/d_0 = A_0B_0'$. A pair of type P has the first representation and flag supports of 30.38 and agrees in entry 20 of its matrices and flags. A pair of type M has the second representation support and agrees in entry 10 of matrices and flags. The divided entries $x(g) = r_{10}(g)/d_0(g)$ and $y(g) = r_{20}(g)/d_0(g)$ on their respective supports are cocycles of A_0, B_0'. They vanish at g_0. On a local group the flag identity gives $x = (1 - A_0)m_v,\ y = (1 - B_0')n_v$, where m_v, n_v are the corresponding flag entries.

Given finitely many such y_j, choose a finite coefficient field k_0' containing the residual values and let V_0 be the k_0'-span of the values of y_j. The values form finite sets by compactness. On the finite $\mathbf{F}_\ell$-space $D' = k_0' \oplus V_0^{\{j\}}$ use

$$g(u, (v_j)) = \big(A_0(g)u, (A_0(g)B_0'(g)v_j + A_0(g)uy_j(g))_j\big).$$

The cocycle law for y_j gives the action law. Put $m_0 = (1, 0)$. Let S_0 contain the places over $S_D \cup \{\ell\}$ and the primes already allowed, and impose zero local class there. Choose infinitely many new rational primes split in the normal closure of the fields cut out by this action, the old residual representation, F and μ_ℓ, by Chebotarev [38, §1A]. At a new place allow inertia values in $\mathbf{F}_\ell m_0$. The action there is trivial and the tame relation with norm $Q \equiv 1 \bmod \ell$ allows a nonzero inertia value. The condition has at least $2|D'|$ elements by varying the unramified part.

For a resulting set S let $\mathcal{S}$ be the Selmer group with these conditions. Finite duality in 27.2 and the Euler characteristic [111, I, Thm. 5.1] give

$$2^{|S - S_0|} \le |\mathcal{S}|\,|(D')^\vee|\,|D'|^{[F:\mathbf{Q}]+|S_0|}. \tag{A.272}$$

Each new factor divided by the invariant order $|D'|$ is at least two. Each old local invariant factor costs at most $|D'|$, each infinite factor at most $|D'|$, and the global dual invariant order is at most $|(D')^\vee|$. Let N_0 be the number of cocycles unramified outside S_0, finite by global finiteness and finiteness of the coboundaries. Set $|S - S_0| = N_0|(D')^\vee|\,|D'|^{[F:\mathbf{Q}]+|S_0|}$. Since $2^b > b$ for $b \ge 0$, the Selmer group contains a class with nonzero inertia at a new place, of value in $\mathbf{F}_\ell m_0$. Add the underlying rational primes to W. On the still-excluded part of old inertia the action is trivial and the local class is a boundary, so its cocycle value vanishes.

Lean: `UpperTheory.V.Ws.Gys.selBig` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.RamifiedLineCharacters.SelBig`, lines 7–42)

Lean: `UpperTheory.V.Ws.Gys.gysCore` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.RamifiedLineCharacters.GysCore`, lines 9–61)

Lean: `UpperTheory.V.Ws.Gys.extRep` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.RamifiedLineCharacters.ExtRep`, lines 34–192)

Project the cocycle to x, z_j. Then x is nonzero somewhere on $\ker A_0$ and

$$z_j(gh) = z_j(g) + A_0(g)B_0'(g)z_j(h) + y_j(g)A_0(g)x(h).$$

Locally $x = (1 - A_0)m_v,\ z_j = (1 - A_0B_0')o_{jv} - y_jA_0m_v$. Subtract the coboundary with coordinates

$$b = \frac{x(g_0)}{1 - A_0(g_0)}, \qquad c_j = \frac{z_j(g_0)}{1 - A_0(g_0)B_0'(g_0)}$$

to normalise at g_0. The denominators are nonzero, and $y_j(g_0) = 0$. This subtracts $(1 - A_0)b$ and $(1 - A_0B_0')c_j - y_jA_0b$ from the cocycle coordinates and translates the local witnesses by $-b, -c_j$.

Suppress j and put

$$U(x, y, t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ t & y & -x & 1 \end{pmatrix}, \qquad r(g) = U(x(g), y(g), 2z(g) - x(g)y(g))\,\mathrm{diag}(d_i(g)).$$

If $y|_{G_v} = (1 - B_0')n_v$, take flag matrix $U(m_v, n_v, 2o_v - m_vn_v)$. The product of two U's adds the first two parameters and replaces the third by $t + t' + yx' - xy'$. Diagonal conjugation scales the three parameters by A_0, B_0', A_0B_0'. Thus the equation for z gives multiplicativity, and the local equations give the prescribed flag characters. The U's preserve the form, and $d_0d_3 = d_1d_2$ gives the similitude. Replacing z by $xy - z$

exchanges the cross terms in the cocycle equation. The reversed construction therefore preserves given 10 data and produces a nonzero 20 coordinate.

Lean: `UpperTheory.V.Ws.Gys.cupEx` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.RamifiedLineCharacters.GysinPsOf`, lines 13–30)
Lean: `UpperTheory.V.Ws.Gys.ob` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.RamifiedLineCharacters.SuppMMono`, lines 126–232)

For an irreducible point take its integral pair from Lemma 30.44. In the block case with full first-block span use the nonsplit companion of Lemma 30.47. Choose the label order of Lemma 30.44 in constructing the initial tower. Apply Lemma 30.45 over $\mathbf{Q}$, and let p_0 be the reduction of this lattice with its transported flags over the initial totally real field. Lemma 30.46 applies both here and after enlargement. In III start with the seed's 10 data and $W = \{w\}$. The reversed construction gives p_2 with the same 10 data and $E(-\psi)$. Apply the simultaneous construction to the 20 data of p_0, p_2, giving p_3, p_4 with the respective 20 data and a common 10 coordinate with $E(-\alpha)$. In II apply it to two copies of the 20 data of p_0, giving p_3 with $E(-\alpha)$. Enlarge F by the cyclic fields of degree ℓ unramified outside one $w' \in W$. These are subfields of $\mathbf{Q}(\mu_{w'})$ and are unramified at ℓ. The new tame inertia is killed by restriction. In III its unipotent image has exponent dividing ℓ since the nilpotent part has fourth power zero and $\ell \geq 5$. In II the displayed U-shape has exponent dividing ℓ also for $\ell = 3$. Transport flags to new prolongations by conjugation and take their lower unitriangular factors modulo upper triangular matrices, which preserves the matched coordinates. Write F_1 for this tame enlargement and F_m for the totally real abelian cyclotomic layer over F_1 cut out below by $\varepsilon_\ell^{\ell-1} \equiv 1 \bmod \ell^{m+1}$. Raising the marking to an ℓ-power puts $t = g_0^{\ell^a}$ in the enlarged group G_{F_m} (which is normal of ℓ-power index in G_F, since both F_1 and F_m are abelian over $\mathbf{Q}$) and preserves the residual eigenvalue inequalities because A_0, B_0' take values in $k_\Lambda^\times$ of order prime to ℓ. In particular $A_0(t) \neq 1$ and $B_0'(t) \neq 1$. Consequently the constructed Selmer cocycle x (and in III the reversed Selmer cocycle y of p_2) remains nonzero after restriction to G_{F_m}: if $x|_{G_{F_m}} = 0$, then for every $g \in G_F$ both t and gtg^{-1} lie in G_{F_m}, so comparing $x(gt) = x(g) + A_0(g)x(t) = x(g)$ with $x((gtg^{-1})g) = x(gtg^{-1}) + A_0(gtg^{-1})x(g) = A_0(t)x(g)$ gives $(1 - A_0(t))x(g) = 0$, forcing $x \equiv 0$ on G_F, contrary to its nonzero tame inertia at W. In III the same computation with $B_0'(t) \neq 1$ shows $y|_{G_{F_m}} \neq 0$.

Call the enlarged field F_1. The field fixed by the intersection of G_{F_1} with the residual diagonal and cyclotomic kernels is abelian and contains μ_ℓ. By 28.6 one integer r bounds the unramified mod-ℓ character dimensions over its cyclotomic layers. Let g bound the number of places over S_D and ℓ in those layers. Such a bound exists because the Frobenius norm away from ℓ generates an open subgroup of the cyclotomic $\mathbf{Z}_\ell$-quotient, and inertia has open image above ℓ. Take $C = r + 2 + 2g$. Restriction of a strict root cocycle to the residual kernel gives an unramified character, with at most one extra dimension for coboundaries. Old tame places cost at most one inertia generator each. The global Euler characteristic adds the negative real eigendimensions, zero for $0, \pm\psi$ and at most $[F : \mathbf{Q}]$ for an odd root. Local duality and the local Euler characteristic bound the product-local odd cocycle dimension by $[F : \mathbf{Q}] + 2g$. These are the bounds of 30.38. On the layers $\varepsilon_\ell^{\ell-1} \equiv 1 \bmod \ell^{m+1}$, the ℓ-primary part of ψ kills the infinite intersection by the initial kernel condition and hence a finite layer by compactness. The degrees are unbounded. Take a layer with $d \geq 30(C+1)^2$ on which ψ has order prime to ℓ.

Lean: `UpperTheory.V.Ws.diesEx` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.DiesEx`, lines 112–183)
Lean: `UpperTheory.V.Ws.restrT` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.RestrT`, lines 156–271)
Lean: `UpperTheory.V.Ws.x2` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.X2`, lines 14–37)
Lean: `UpperTheory.V.Ws.layerEx` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.LayerEx`, lines 112–304)

Write q_i for the restricted points and q_{seed} for the seed reduction. In III the chain is

$$q_0 \xrightarrow{P} q_3 \xrightarrow{M} q_4 \xrightarrow{P} q_2 \xrightarrow{M} q_{\mathrm{seed}}.$$

Number the alternatives of 30.38 in their displayed order. Lemma 30.46 puts q_0 in (1) or (3). In both chains q_3 has the displayed unipotent shape U and its 10 coordinate is nonzero somewhere. Its 20 matrix and flag coordinates agree with those of q_0. If its 20 matrix coordinate is nonzero somewhere, q_3 satisfies (1). Otherwise the paired 31 entry vanishes as well, q_0 must be in (3), and the common flag entries at 20 are nonzero at every place. This gives (3) at q_3.

In III the M-match with q_3 gives q_4 the same nonzero 10 matrix entry, and the P-match with q_2 gives it the nonzero 20 matrix entry of the reversed construction. Thus q_4 satisfies (1). At q_2 the 10 matrix entry agrees with the seed's zero entry and the 10 flag entries agree with its nonzero flag entries. In the displayed U-matrix the vanishing of 10 also kills 32. The remaining support is contained in that for H, and the reversed construction gives $E(-\psi)$. This proves (5) at q_2. The tensor computation gives (5) at the seed with $E(-\beta), E(-\theta)$.

In II take the two diagonal blocks of the lower block triangular matrices and flags of q_3 in order $(0,1;2,3)$. This projection preserves multiplication and the symplectic adjoint, which exchanges the two diagonal blocks. It retains the 10 data and leaves only weight $-\alpha$ off the diagonal. Write X_v for the resulting flag, and take y_v to be the seed's 20 flag entry. Keep the projected representation and use the flag $X_v(1 + y_v(E_{20} + E_{31}))$. The right factor is symplectic. In the projected flag basis the local matrices are diagonal, since they are both lower and upper triangular. Their diagonal entries satisfy $d_0 = d_2$ and $d_1 = d_3$ on the local group by the triviality of the root character of weight ψ there. Hence the right factor commutes with them. The new flag has the old 10 entry, has 20 entry $y_v \neq 0$, and has support in $\{-\alpha, -\psi, -\theta\}$. This gives (3) at q_4. Its matrices and the seed matrices have zero 20 entries and their 20 flags agree. Thus the chain is $q_0 \xrightarrow{P} q_3 \xrightarrow{M} q_4 \xrightarrow{P} q_{\mathrm{seed}}$. The induced matrix calculation gives (3) or (4) at the seed. For (4), at each ordinary place the tower supplies g in its decomposition group at which the residual character of weight α has fourth power different from one. This character is d_0/d_1, by the coordinate weights in 30.38. Hence $(d_0/d_1)(g) \neq 1$, the required local test.

Each arrow is linear interpolation $(1-X)r + Xr'$ of the matrices and flags. In M the differences lie in the lower-left block for $(0,1;2,3)$ by agreement of the 10 data and the symplectic equations. In P, use $(0,2;1,3)$. Agreement at 20 gives agreement at 31 by $r_{00}r_{31} = r_{11}r_{20}$. The differences again occupy the lower-left block, and products of two differences vanish. Expansion of the defining equations gives a polynomial family of marked symplectic representations and lower unitriangular flags with the prescribed inertia conditions. The characteristic polynomials and ordered local flag characters are constant and equal to their residual values. Apply Claim 1 in the proof of 30.37 after embedding $k_\Lambda[X]$ into $k_\Lambda[[X]]$. This gives $A \to k_\Lambda[[X]]$. Its restriction to B is the residue map, since the characteristic-polynomial coefficients and local character values determine that restriction. Every matrix and flag generator has polynomial image, so the whole map factors through $k_\Lambda[X]$. Evaluation at $X = 0, 1$ recovers the two prescribed residual points, including their flags. Thus every arrow lies in the same $\operatorname{Spec} A$. Any fixed ideal $J \subset B$ is extended through this one structure map at both endpoints.

Lean: `UpperTheory.V.Ws.sxPreLine` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.AdicExtensionMatrices.PreLine`, lines 17–66)
Lean: `UpperTheory.V.Ws.lineP` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.AffSmulOne`, lines 127–215)
Lean: `UpperTheory.V.Ws.lineM` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.GlobalTheory`, lines 131–198)
Lean: `UpperTheory.V.Ws.deep_tower_residue_chain_exists` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.Chain2`, lines 13–92)
Lean: `UpperTheory.V.Cu.chainIV_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.CuPChainIV`, lines 16–82)

Let F be the totally real field of the fitted final layer and K its imaginary quadratic field, with CM compositum $L = FK$ and odd prime ℓ split in K. Construct the continuous residual-trivial twisting character $\beta_0 : G_L \to 1 + \mathfrak{m}_{O'}$ from K as follows: pick a continuous character $\mu : G_K \to 1 + \ell\mathbf{Z}_\ell$ unramified away from one chosen prime above ℓ and an integer $m_0 \geq 1$ with $\mu\mu^c = \varepsilon_\ell^{m_0}$, set $\pi = \mu^2/\varepsilon_\ell^{m_0}$ (so $\pi\pi^c = 1$, $\pi \equiv 1 \pmod{\ell}$, and π is unramified outside ℓ), extract a continuous $(2m_0)$-th root $\beta_K : G_K \to 1 + \mathfrak{m}_{O'}$ (extending the coefficient DVR to O' if necessary), and restrict to $\beta_0 = \beta_K|_{G_L}$. Then $\beta_0\beta_0^c = 1$, $\beta_0 \equiv 1 \pmod{\mathfrak{m}_{O'}}$, β_0 is unramified outside ℓ and trivial on $\mu_{\ell^\infty}(L_v)$, and on an open subgroup of I_v ($v \mid \ell$) one has $\beta_0^2 = \varepsilon_\ell^{2j_v+1}$ with $j_v = 0$ at the chosen primes above ℓ and $j_{c(v)} = -1$ at their conjugates (since $\mu|_{I_{c(v)}} = 1$ gives $\beta_0^{2m_0} = \varepsilon_\ell^{\pm m_0}$ on I_v and continuity eliminates the m_0-th root of unity on an open subgroup). Now construct the common polarising twist κ over L explicitly for both Hecke comparisons. Factor the finite-order Dirichlet character $\psi = \psi_{\ell'}\psi_\ell$ into its prime-to-ℓ part $\psi_{\ell'}$ and its ℓ-power-order part ψ_ℓ (say of order ℓ^a). Since ℓ is odd and $\psi(c) = (-1)^e$ has order dividing 2, we have $\psi_\ell(c) = 1$ and $\psi_{\ell'}(c) = (-1)^e$; hence $\omega^{e+3}\xi\psi_{\ell'}$ is even, has order prime to ℓ, and is unramified outside the set S_D of primes (all of which split in K, including 2 and ℓ), where ω is the Teichmüller character and $\nu = \varepsilon_\ell\omega^{-1}\xi \cdot \varepsilon_\ell^e\psi$ is the datum similitude character. Choose a finite prime-to-ℓ order character κ_0 of G_K, unramified outside the primes

above S_D, satisfying $\kappa_0\kappa_0^c\,\omega^{e+3}\xi\psi_{\ell'} = 1$ on G_K, and a continuous character γ of $G_{\mathbf{Q}}$ satisfying $\gamma^2\omega = \varepsilon_\ell$ (hence $\gamma^c = \gamma$). Because ℓ is odd, 2 is invertible modulo ℓ^a, so ψ_ℓ admits a unique ℓ-power-order square root $\psi_\ell^{1/2} = \psi_\ell^{(\ell^a+1)/2} : G_{\mathbf{Q}} \to 1 + \mathfrak{m}_{O'}$, which is unramified outside S_D, trivial on an open subgroup of I_v, and invariant under conjugation since it factors through $G_{\mathbf{Q}}$. Set $\kappa = \kappa_0|_{G_L} \cdot \psi_\ell^{-1/2}|_{G_L} \cdot \gamma|_{G_L}^{-(e+4)} \cdot \beta_0^{e \bmod 2}$. Direct substitution gives $\kappa\kappa^c\nu\varepsilon_\ell^3 = (\kappa_0\kappa_0^c\omega^{e+3}\xi\psi_{\ell'})(\psi_\ell^{-1}\psi_\ell)(\gamma^2\omega)^{-(e+4)}(\beta_0\beta_0^c)^{e \bmod 2}\varepsilon_\ell^{e+4} = 1$ identically on G_L, and on an open subgroup of I_v ($v \mid \ell$) where $\kappa_0, \psi_\ell^{1/2}, \omega$ are trivial we still have $\kappa^2 = \varepsilon_\ell^{-(e+4)+(e \bmod 2)(2j_v+1)} = \varepsilon_\ell^{2a_v}$ with the same integral cyclotomic weight $a_v = (e \bmod 2)j_v - \lfloor e/2 \rfloor - 2 \in \mathbf{Z}$ at every $v \mid \ell$. Twisting determinants and ordered characters by β_0 defines a twisting O'-algebra automorphism $\tau_{\beta_0} \in \mathrm{Aut}_{O'}(B_i)$ shifting doubled cyclotomic weights by an integer of the same parity as the twist exponent. Extend coefficients to contain the values of β_0, the auxiliary character values and all images of $\mathcal{O}_L$. On G_L, the fitted tower condition excludes the residual cyclotomic character from the group generated by the pairwise ratios of the residual characters. In their joint finite abelian quotient, character duality therefore gives an element on which every ratio is one and the residual cyclotomic character is nontrivial. Chebotarev in their joint finite quotient, recording the splitting field as well, gives two split distinguished places over different rational primes outside the level and discriminant. At each place the residual characters coincide and the residual cyclotomic character is nontrivial. Apply 30.23 once with each of these places as the distinguished place, using the same twist κ in both applications. The residual cyclotomic test gives $\mu_\ell \not\subset L$. The twisted determinants and characters give maps $\phi_i : B_i \to B$. For the two ideals J_i from 30.34, set $J = \sum_i \phi_i(J_i)B$.

Lean: `UpperTheory.Gal.projectedFormsSystemPart.exists_two_links_same_kappa_not_dvd_of_fit_deep` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.LinkTwo`, lines 13–51)

This J satisfies (30.9). In any prime quotient containing the right side, the weight variables specialise to their residual values. Hecke finiteness over weight space makes the Hecke coefficients residual as well. The twisted flag values and polynomial coefficients from the two maps generate the coefficient ring in the fibre. Hence the prime contains $\mathfrak{m}_B$, and a uniform power follows by Noetherianity.

Lean: `UpperTheory.V.Ws.exists_pow_maximalIdeal_le_of_modular_phi` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TPmFin`, lines 126–173)

For a curve y as in 30.38, the maps $f_i = yj\phi_i$ specialise the determinant and ordered characters to the twisted restrictions. Elements separating the flags on U_v lift to the local link groups with cyclotomic value one. The residual tests for the good datum give $\psi_i \neq \psi_j\varepsilon_\ell$ in the required order. The distinguished local test has equal residual eigenvalues and nontrivial residual cyclotomic value, so Lemma 30.50(1) with $m = 4$ and $\chi = \varepsilon_\ell|_{D_0}$ gives zero twisted intertwiner space. Full generic span survives the quadratic restriction. Otherwise a sign self-twist would force trace zero on the other coset, which contradicts linear independence of the four restricted residual characters, with unit coefficients from the coset representative.

Each pair of flag characters attains a nontrivial principal unit on U_v, which has infinite order in characteristic ℓ. Among combinations of finitely many local separating elements, avoid the finitely many proper kernels of the character ratios modulo torsion. This gives a local element whose positive powers all separate. Reducibility on the cyclotomic kernel would give a quadratic permutation twist by Clifford theory, since the abelian quotient acts transitively on the constituents and their number divides four. For odd ℓ the twist factors through the residual cyclotomic quotient. The fitted residual tests exclude it.

Let H be the CM subgroup and $H' = H \cap \ker\varepsilon_\ell$. We check the noncommutativity hypothesis of Lemma 30.48 on H'. If an open normal subgroup $N \leq H'$ had commuting image, its core under H would also have commuting image and would contain $u^{[H':N]}$, where $u \in H'$ is the separating element above. Indeed each H-conjugate of u lies in H' and its indicated power lies in N. The core acts diagonally on the four eigenlines of this power. The lines are its distinct character spaces. Thus H permutes them transitively by irreducibility. Let P be the kernel of the permutation. Lemma 30.50(2) with $m = 4$ shows that every ratio of the prescribed residual characters is trivial on P. Such a ratio factors through a subgroup of S_4. Its image in the multiplicative residue field is finite cyclic, and a preimage of a generator acts as a permutation of order at most four. This contradicts the fitted ratio whose first four powers are nontrivial. Apply Lemma 30.48 to the absolutely irreducible restriction to H', in the eigenbasis of u. In a preserving coset it supplies M with nonzero diagonal and adjugate diagonal, since $\mathrm{adj}\,M = (\det M)M^{-1}$. In an exchanging coset it supplies both off-diagonal blocks entrywise nonzero.

Take a power of the separating element into the open subgroup and kill torsion in the group generated by its eigenvalues a_i. Map this finitely generated torsion-free group to $\mathbf{Z}$ by a homomorphism with distinct

eigenvalue weights w_i. In the preserving case order them $w_1 > w_2 > w_3 > w_4$. The first, third and fourth elementary coefficients of $M\,\mathrm{diag}(X^{w_i})$ have respective leading exponents $w_1, w_1+w_2+w_3, \sum_i w_i$ with nonzero coefficients. The second coefficient has exponent at most w_1+w_2. The Newton polygon at infinity gives unique extreme roots of exponents w_1, w_4. A repeated middle pair would have exponent $(w_2+w_3)/2$, and in odd characteristic the second coefficient would have exponent $w_1+(w_2+w_3)/2$. This lies strictly between the largest and next possible pair-weights w_1+w_2, w_1+w_3, a contradiction.

In the exchanging case a block of the square is $B\,\mathrm{diag}(X^{w'_i})C\,\mathrm{diag}(X^{w_i})$, with B, C the off-diagonal blocks. Order the weights within each block. Its trace has unique leading exponent $w_1+w'_1$ with nonzero coefficient. The determinant has exponent $w_1+w_2+w'_1+w'_2 < 2(w_1+w'_1)$ and nonzero coefficient. The square block has two distinct nonzero roots, so their opposite square roots give four distinct roots. In both cases the weighted discriminant is nonzero. Group its multivariate terms by the equal products of the a_i. At least one group has nonzero sum of coefficients, since equal products have equal weights. Vandermonde in the distinct products gives a nonzero evaluation at a common integer power of the a_i. This proves the required separability.

Lean: `UpperTheory.V.Ws.P1.irrL` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.ProjectiveLineFlags.TraceOfCp`, lines 17–74)

Lean: `UpperTheory.V.Ws.h4` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.H4`, lines 19–117)

Lean: `UpperTheory.V.AlgebraTools.regCosetFour` (`Development.AutomorphicArithmetic.AutomorphicVarieties.AlgebraTools.MonomialRepresentations.RegCosetFour`, lines 337–394)

Now 30.34(a) applies to f_i. For $\mathfrak{q} \subseteq \ker(yj)$, the prime $\phi_i^{-1}\mathfrak{q}$ lies below $\ker f_i$, and $f_i(J_i) = 0$. Thus $J_i \subseteq \phi_i^{-1}\mathfrak{q}$. Extending and summing gives $J \subseteq \mathfrak{q}$, as required in 30.38.

Lean: `UpperTheory.V.Ws.modular_deformation_quotient_regular_at_point_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.ModularDeformationQuotientRegularAtPoint`, lines 25–59)

At the characteristic-zero seed, the continuous twist κ_{seed} returning the polarized seed q_{seed} to its untwisted regular algebraic cuspidal lift has doubled cyclotomic weight $a_{\mathrm{seed}} \in \mathbf{Z}$ (odd when the polarizing twist δ has half-integral cyclotomic weight, as in Class III and even-k Class II), so the seed has doubled weights w_i of common parity $w_i \equiv a_{\mathrm{seed}} \pmod 2$ with even successive differences $w_i - w_{i+1} \geq 4$. For each link, the twisting character β_0 constructed above satisfies $\beta_0^2 = \varepsilon_\ell^{\pm 1}$ on an open subgroup of I_v for every $v \mid \ell$ (with opposite signs at v and $c(v)$), while κ has integral cyclotomic weight $a_v \in \mathbf{Z}$ at every $v \mid \ell$, so $\kappa\beta_0^n$ has doubled cyclotomic weight $e_v = 2a_v \pm n \equiv n \pmod 2$. Choose $n \in \mathbf{N}$ with $w_0 + n$ even (hence $w_i + e_v$ and $e_v - a_{\mathrm{seed}}$ are even for all i and $v \mid \ell$), and set $\phi'_i = \phi_i \circ \tau_{\beta_0}^n : B_i \to B$, where $\tau_{\beta_0} \in \mathrm{Aut}_{O'}(B_i)$ is the twisting O'-algebra automorphism attached to β_0. To prove $\tau_{\beta_0}(J_i) = J_i$ without passing through half-integral weights, fix $b \in J_i$ and an integer $N \geq 1$. Because $\beta_0 : G_L \to 1 + \mathfrak{m}_{O'}$ is continuous and satisfies $\beta_0\beta_0^c = 1$, its reduction modulo $\mathfrak{m}_{B_i}^{N+1}$ factors through an open homomorphism $q : G_L \twoheadrightarrow Q$ onto a finite abelian ℓ-group Q satisfying $q \circ c = q^{-1}$ and trivial on the excluded inertia subgroups. Let $\mathrm{tw} : B_i \to B_i[Q]$ be the universal Q-twisting O'-algebra homomorphism into the group algebra $B_i[Q]$. For every arithmetic point $x : B_i \to \overline{\mathbf{Q}}_\ell$ vanishing on J_i of weight $h_{j,\tau} = j$ at each $v \mid \ell$ whose finite inertia characters are pairwise distinct at each ordinary place and every finite-order (weight-zero) character $\chi : Q \to \mu_{\ell^k} \subset \overline{\mathbf{Q}}_\ell^{\times}$, composing tw with x and χ gives the finite-order $(\chi \circ q)$-twist x' of x, which still vanishes on J_i. Put $\beta = \chi \circ q$. Its odd-order square root corresponds by reciprocity to a Hecke character α with $\alpha^c = \alpha^{-1}$. Twist the definite unitary automorphic representation realizing x by the composite of the unitary determinant with the restriction of α to the norm-one torus. At a split place this twists the local component by the local component of α^2 on the determinant. The twist is unramified outside the level and trivial on the tame inertia groups and the distinguished inertia group, so the required away-level vectors and tame types persist. At $v \mid \ell$ each ordered flag character acquires the factor β, and its finite diamond character acquires the inverse factor on inertia. On a reciprocity lift g_v of a uniformiser the j-th ordinary eigenvalue acquires the factor $\beta(g_v)^j$, a unit congruent to one. Ordinary vectors at sufficiently deep level therefore realize x' on the same ordinary Hecke quotient with the same residual system.

Lean: `UpperTheory.V.Z.Desc.W.jTwist_of_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.WJTwist`, lines 217–281)

Hence every character evaluation of $x_*(\mathrm{tw}(b)) \in \overline{\mathbf{Q}}_\ell[Q]$ vanishes, so Fourier inversion on the finite abelian group Q forces every Q-coefficient of $\mathrm{tw}(b)$ to vanish at every such point of B_i/J_i, and the density criterion for these points at this fixed weight puts every Q-coefficient of $\mathrm{tw}(b)$ in J_i. Specializing $B_i[Q] \to B_i/\mathfrak{m}_{B_i}^{N+1}$ via $q(g) \mapsto \beta_0(g) \bmod \mathfrak{m}_{B_i}^{N+1}$ yields $\tau_{\beta_0}(b) \in J_i + \mathfrak{m}_{B_i}^{N+1}$ for all $N \geq 1$, and Krull's intersection theorem in the

Noetherian local ring B_i gives $\tau_{\beta_0}(b) \in J_i$; repeating the argument for β_0^{-1} gives $\tau_{\beta_0}(J_i) = J_i$ and therefore $\phi_i'(J_i)B = \phi_i(J_i)B$, leaving the extended ideal J unchanged. Meanwhile the β_0^n-twisted seed has integral weights $m_{v,i} = (w_i + e_v)/2 \in \mathbf{Z}$ with gaps $m_{v,i} - m_{v,i+1} = (w_i - w_{i+1})/2 \geq 2$. At each place $v \mid \ell$ individually, $\kappa_{\mathrm{seed}}^2 = \varepsilon_\ell^{a_{\mathrm{seed}}}$ and $(\kappa\beta_0^n)^2 = \varepsilon_\ell^{e_v}$ on an open subgroup of I_v with $e_v - a_{\mathrm{seed}}$ even, so on a sufficiently small open subgroup of I_v the net twist $(\kappa\beta_0^n)\kappa_{\mathrm{seed}}^{-1}$ agrees with $\varepsilon_\ell^{(e_v - a_{\mathrm{seed}})/2}$ (an integer power of ε_ℓ at every $v \mid \ell$), hence is an algebraic Hecke character, while conjugate self-duality of the twisted representation follows from $\kappa\kappa^c\nu\varepsilon_\ell^3 = 1$ and $\beta_0\beta_0^c = 1$. The attachment, ordinary flag and prescribed inertia give a point on each Hecke quotient. A first-order determinant lift with fixed weight gives a matrix lift by full span. The flag lifts by separation, and (A.262) gives a finite adjoint class. The weight gaps exclude $\psi_i = \psi_j\varepsilon_\ell$ on inertia. Local comparison relates the ordered characters to the toral support by their cyclotomic weights m_i. At a Weil element s of cyclotomic value one and positive Weil degree d_0, the support moduli are $|\iota(\psi_i(s))|q_v^{m_i d_0}$. Reciprocity gives such an element for the flag. A finite-order ratio ψ_i/ψ_j on it would give equal character moduli, and (30.8) would give $|m_i - m_j| < 1$. Thus the gaps give infinite-order separation. The residual tests give dense separability and the seed is irreducible on the cyclotomic kernel. By 30.62 the class is a coboundary, so its derivatives on all coefficients and characters vanish. The local determinant ring has embedding dimension at most the number of weight parameters. The Hecke quotient is finite torsion-free over the regular weight ring and gives the opposite dimension bound. Therefore the local determinant ring is regular and the Hecke quotient has zero kernel there by equality of dimensions. Each prime below the seed prime contains J_i.

Lean: `UpperTheory.V.Th.local_zero_of_modular_p` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.BirthPm`, lines 28–65)
Lean: `UpperTheory.V.Th.eigenbasis_rank_four` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.EigenbasisRankFour`, lines 122–173)
Lean: `UpperTheory.V.Z.Win.RR.free_core_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.RRFree`, lines 97–116)

Contraction along the links and summation put J in each coefficient prime below the seed kernel. A minimal prime of A below the integral seed kernel contains JA and lies below the residual seed kernel. The standing hypothesis $\mathscr{H}$ supplies the four-dimensional frame cohomology of Definition 30.36, so Lemma 30.38 applies at the seed and at each link endpoint; hence every minimal prime below the residual seed kernel contains JA. In a link $A \to k_\Lambda[X]$, a minimal prime below the polynomial kernel lies below both endpoint kernels. The conclusion at one endpoint puts JA in this prime, so 30.38 applies at the other endpoint. Follow the links back to q_0. The prime kernel of the valuation-domain lift lies below the residual kernel at q_0 and contains one of its minimal primes. It therefore contains JA.

Lean: `UpperTheory.V.Ws.conn` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.AlgHomResUnique`, lines 56–68)

In characteristic ℓ, the flag characters on the prescribed inertia of the fitted tower make its inertia generators in (30.9) vanish. The characteristic polynomials on G_F therefore equal the products prescribed by the residual characters. Every group element has a positive power in G_F, so all its eigenvalues are algebraic over $\mathbf{F}_\ell$. The trace difference from the residual sum is algebraic over $\mathbf{F}_\ell$ and lies in the maximal valuation ideal. Hence it is zero, and multiplication of the residual characters gives

$$\operatorname{tr}\big(r(x)[r(g), r(h)]\big) = 0 \qquad (x, g, h \in G_{\mathbf{Q}}).$$

Here the bracket denotes the additive commutator. The test is linear in the left multiplier, so it holds for that multiplier anywhere in the span of the group matrices. Under full span, testing against each matrix unit gives every entry of the commutator equal to zero. The group matrices would then span a commutative matrix algebra, contrary to full span in rank four. In the block case write A_x, B_x for the two diagonal blocks of $r(x)$, and let p_B be the characteristic polynomial of the second block at the marking g_1 (using the marked element for this datum). The group algebra contains the multiplier $r(x)p_B(r(g_1))$. Its second diagonal block vanishes by Cayley–Hamilton. Its first is $A_x p_B(A_{g_1})$, with $p_B(A_{g_1})$ invertible by coprimality and Bézout. The trace test therefore gives $\operatorname{tr}(A_x p_B(A_{g_1})[A_g, A_h]) = 0$ for every x. As A_x spans the full two-by-two algebra, the trace pairing gives $[A_g, A_h] = 0$ for every pair. This again contradicts full matrix span.

For the application of 30.42, the framed universal lift has a homogeneous presentation in the $\mathbf{Z}^2$ weights of 30.38, with coordinates given by entries minus their residual lifts. Consider its points in characteristic-ℓ complete DVRs. The marked projectors put the diagonal matrix units in the span. The relation $i \leftarrow j$ defined by the presence of a nonzero (i, j) entry is transitive, by multiplying matrices with a diagonal projector inserted.

One further restriction comes from residual Selmer vanishing. A strongly connected component consisting of two indices i, j in different $\{0,2\},\{1,3\}$ blocks is impossible. Indeed its two-dimensional subquotient inherits an ordinary invariant line. Interchange i, j if necessary so the ordinary line in this subquotient has a generator $(v, 1)$ with v integral. Choose a nonzero entry $b = r_{ij}(g_*)$ of least valuation and put $c(g) = \overline{r_{ij}(g)/b}$. Matrix multiplication gives $c(gh) = \bar\delta_i(g)c(h) + c(g)\bar\delta_j(h)$, so $z(g) = c(g)/\bar\delta_j(g)$ is a continuous cocycle for $\bar\delta_i/\bar\delta_j$, in the notation of Definition 29.17. On the decomposition group at ℓ, invariance of the line gives $r_{ij} = v(r_{jj} - r_{ii} + vr_{ji})$. At the separating element the factor in parentheses is a unit, so v/b is integral. Reducing the identity after division by b makes z a local coboundary. Local vanishing at the other places in T and triviality on the prescribed inertia kernels away from T give the remaining conditions of Definition 29.17. Its residual Selmer condition, applied coordinatewise over the original residue field, makes z a global coboundary. Equivalently, $c(g) = m(\bar\delta_i(g) - \bar\delta_j(g))$ for a single residue scalar m. This is impossible because the diagonal projector E_{ii} is a linear combination of group matrices: applying that combination to the asserted coboundary identity first forces the cobounding scalar to be zero, then forces the chosen normalized entry, which is one, to vanish. This argument works over an equal-characteristic DVR and after finite residue extension. Transitivity and symplectic reflection now give exactly three alternatives: full span; no pair of opposite off-diagonal entries; or both entries $02, 20$ present with the matrices block triangular for $\{0,2\},\{1,3\}$. In the last case projection to the two diagonal blocks preserves multiplication, similitude and the local flag conditions. The entries $02, 20$ and the marked projectors give full first-block span. Lemma 30.47 gives a nonsplit extension with the same characteristic polynomials. Lemmas 30.44–30.46 give its starting point for the chain. The trace test with the marked projector applies to its block quotient. Thus the full-span and full-block alternatives are excluded in characteristic ℓ.

Lean: `UpperTheory.V.Ws.triV` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.CobOfCoord`, lines 297–381)
Lean: `UpperTheory.V.Ws.blockPart` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.BlockPart`, lines 192–276)
Lean: `UpperTheory.V.Ws.bl` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.NotOver`, lines 51–128)

In the remaining support, each diagonal entry is a character because every product of opposite off-diagonal entries vanishes. Let $U = \ker \bar r_D \cap \ker \vartheta$. On inertia in U the representation is trivial away from ℓ by the frame conditions. At ℓ it is unipotent in the ordinary flag, so its ℓ^2-th power is one in dimension four. Consequently the diagonal characters raised to $[G_{\mathbf{Q}} : U]\ell^2$ kill all inertia. Apply this at every finite coefficient depth. An everywhere unramified finite character of $G_{\mathbf{Q}}$ is trivial, and separation in the coefficient topology gives the same conclusion for the powered characters. Their values in the valuation ring are thus algebraic over $\mathbf{F}_\ell$ and equal their residual lifts. The zero-weight coordinates vanish. The transitive relation on distinct indices is acyclic, so a nonzero product of off-diagonal entries cannot have weight zero. Indeed the weights of such a product sum to zero only if the flow between the two pairs of coordinate weights balances both coordinates. For the symplectic support with no opposite entries, a linear functional on the two weight coordinates is strictly positive on every present off-diagonal weight. To construct the functional, choose integers h_i strictly increasing along the strict partial order of the surviving entries, taking $h_i > h_j$ for an entry in position (i, j). The symplectic involution reverses this order. Thus $H_i = h_i - h_{3-i}$ still satisfies $H_i > H_j$ on every such entry and $H_{3-i} = -H_i$. Since $H_0 + H_3 = H_1 + H_2$, the values $H_i - H_3$ are exactly the values of a linear functional on the four displayed basis weights. Its value on every surviving off-diagonal entry weight is strictly positive. These values, up to a common shift, are obtained from a linear form on $(1,1),(0,1),(1,0),(0,0)$. Thus every positive-degree monomial of weight zero vanishes. Lemma 30.42 makes the universal weight-zero part finite over O, and its image contains the weight-zero monomials of every allowed lift.

Lean: `UpperTheory.V.Ws.Po.univGr` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.PolynomialTori.UnivGr`, lines 58–83)
Lean: `UpperTheory.V.Ws.diagConst` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.DiagConst`, lines 201–217)

In characteristic zero the two evaluations at an allowed point are arithmetic, with weights $k+1, k, 1, 0$ up to the common twist. For an irreducible point on a quadratic restriction, passage to L preserves irreducibility. A nontrivial self-twist trivial on G_L would force trace zero on a coset, contradicting the independence of the restricted residual characters with unit coefficients. Apply 30.25 to the two evaluations for the twisted representation. The good polynomials give the same attachment and the two distinguished places give the full witness by that lemma. Undo the twist and use solvable descent from L to the imaginary quadratic field

K by Proposition 26.36(4), polarizing each step via the similitude pairing; the tower point gives the local flags above ℓ and the decomposition-group characters describing inertia away from ℓ. We check that these flags are unlinked. A positive power of s_0 lies in each local subgroup. Its infinite-order cyclotomic value shows that an equality $\lambda_i = \varepsilon_\ell \lambda_j$ requires a weight difference of one among $k+1, k, 1, 0$.

In II put $m = \operatorname{ord}(\bar{\Psi}(\sigma_\ell))$. The separation condition in Definition 29.17 implies $\bar{\xi}(\sigma_\ell) \notin \langle \bar{\Psi}(\sigma_\ell) \rangle$. The multiplicative residue group is cyclic, so $\bar{\xi}(\sigma_\ell)^m \neq 1$. The auxiliary quadratic fields split at ℓ. Consequently a sufficiently large ℓ-power of σ_ℓ^m belongs to the final local subgroup: the remaining initial character condition and subsequent tower extensions have ℓ-power order. Its cyclotomic value is one, its $\bar{\Psi}$-value is one, and its $\bar{\xi}$-value remains nontrivial. Every cross-plane residual ratio is therefore nontrivial there. The within-plane weight difference is $k \geq 2$, so this excludes all cyclotomic links.

In III one has $\xi = 1$ on the local subgroup because K' splits at ℓ. For the outer adjacent pairs, the inertial ratio divided by ε_ℓ is ω^{-1}. Squares of inertia lie in the initial G_F, and their ω-image has order $(\ell-1)/2 > 1$; its nontriviality persists under the later ℓ-power extensions. The middle pair can have weight difference one only when $k = 2$. Then $\ell - 1 \mid 2k$ and $\ell \geq 5$ give $\ell = 5$. The character ψ has odd order prime to ℓ. Its inertial image has order dividing both this odd order and $\ell - 1 = 4$, so it is trivial. The middle ratio divided by ε_ℓ is therefore ω, detected by the same squared inertia. Thus the flags at all places of L above ℓ are unlinked. At each cyclic step, local comparison identifies the upper inertia characters away from ℓ, with multiplicity, with restrictions of the downstairs decomposition-group characters. Local reciprocity thus extends the upper unit characters by norm. At unlinked places above ℓ, the ordinary graded characters give the same extension by potentially semistable comparison. At a split step the local components agree. At a nonsplit step of residue degree f, fix an upper unit character in the support and twist by a lower character whose norm pullback cancels it on units. In (A.613) the upper polynomial, evaluated at T^f, has degree f times the multiplicity of this unit character. Each lower support character with this pulled-back unit type contributes to exactly f factors on the right, one for each unramified character of the cyclic quotient. Comparing degrees and summing over the upper unit types accounts for the full rank downstairs. Within each type the same polynomial identity identifies the uniformizer values after norm pullback, with multiplicity. Norm pullback raises both uniformizer moduli and residue cardinalities to the residue degree, preserving the strict bounds. This descends the witness to K. For another imaginary quadratic field K' with irreducible restriction, choose an imaginary quadratic field K'' split at ℓ such that the restrictions to KK'' and $K'K''$ are both irreducible. Quadratic base change and descent carry the attachment and witness first through KK'' to K'', then through $K''K'$ to K'. Since K and K'' split at ℓ, the respective composita do not enlarge the completions of the destination fields there. This preserves the unlinked flags required for descent, with the same ι throughout.
Lean: `UpperTheory.V.Z.Win.thirdSplit` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.AutomorphyLifting.QuadraticCharpolySplitting.QuadFamSplit`, lines 135–190)
Lean: `UpperTheory.GalN.V.isAutomorphicOverImagQuad_of_restriction_aut_irr` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.AutomorphyDescentSolvableCMToImaginaryQuadraticA`, lines 14–42)
Lean: `UpperTheory.V.Z.Win.Dn.across'_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.EndsRankThreeOfLocal`, lines 42–48)
Lean: `UpperTheory.V.Ws.contradiction_star_promodular` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.ContradictionStarPromodular`, lines 183–222)
Lean: `UpperTheory.Gal.projectedFormsSystemPart.promodular_existence_of_local_zero` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.PmTwo`, lines 21–51)
Lean: `UpperTheory.V.Ws.aut_n_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.MapRepComp`, lines 201–243)
Lean: `UpperTheory.V.Z.Win.Dn.pmDown_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.DnPmDown`, lines 32–87)

For a characteristic-zero block point with full first-block span apply the chain to the companion of Lemma 30.47 and transfer its characteristic polynomials back to the block point. The arithmetic Hecke comparison gives a semisimple parameter whose constituents occur in strings $\rho_i \varepsilon_\ell^{-j}$, $0 \leq j < m_i$, by 26.22 and 26.23. If $m_i > 1$, reduction of consecutive constituent polynomials gives two residual characters related by $\bar{\varepsilon}_\ell$ by uniqueness of residual semisimplification. Fittedness excludes this. Hence the pure constituents at the common arithmetic weight give, on an open subgroup at good Frobenius of norm Q,

$$|x| < Q|y| \tag{A.273}$$

for every pair of roots x, y. The scalar twist preserves ratios, and passing to a Frobenius power raises both

the ratios and the norm to that power. The two block determinants on their respective open subgroups are ε_ℓ^{k+2} and ε_ℓ^{k}. Take a good Frobenius power in the intersection of the three subgroups. Apply (A.273) to the roots of the first block against those of the second and multiply the two inequalities. The result is $Q^{k+2} < Q^2 Q^k$, which proves (30.10).

Lean: `UpperTheory.V.Ws.frobIn` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.FrobIn`, lines 31–58)
Lean: `UpperTheory.V.Ws.nePure_w` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.AdmissibleResidues.PosOfMulPosOfPos`, lines 231–287)
Lean: `UpperTheory.V.Cu.modularity_rank_four_of_promodularity_case_three` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.ModularityRankFourOfPromodularity`, lines 20–38)
Lean: `UpperTheory.V.Cu.promodularity_case_three_of_chain_reduction` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.PromodularityCaseThreeOfLemmas`, lines 131–153)
Lean: `UpperTheory.V.Ws.fin0_no_end_all_irr_of_good_con_purity_aut` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.VWinPm`, lines 75–106)

□

Proof of 30.52

Proof. Use the rank-n rational count of 30.9 with $n = 4$. Retain seventeen of the eighteen fixed imaginary quadratic fields at p. Each restricted multiset has at most sixteen lifts. The weights $(k+1, k, 1, 0)$ contain the consecutive pairs $k+1, k$ and $1, 0$, so the count uses the second alternative of Lemma 30.8(b): ℓ divides M and every $r \in X$ is unlinked at ℓ. For every p-adic embedding of a label's coefficient field, apply the spherical crystalline comparison of 39.54 at a place above p. The filtration jumps are $-(k+1), -k, -1, 0$, so weak admissibility bounds all geometric-Frobenius slopes above by zero. Their inverses, the arithmetic-Frobenius roots used here, therefore have nonnegative valuation. The residue degree of the quadratic comparison multiplies these valuations by a positive integer, preserving their sign. At every other finite coefficient place, compactness of the Galois image gives a stable lattice and unit Frobenius roots. Thus every forward trace is an algebraic integer. Applying this to every conjugate gives a conjugation-stable label set S with distinct labels for distinct isomorphism classes and integral traces t_s. Put $x = p^{(k+1)/2}$, $y = p^k$. The witness and similitude give $|\beta| \le \sqrt{p}\, x$ for the four forward roots and $\prod |\beta| = x^4$, so one has modulus at most x. The count and these bounds give

$$\#S \le c_0 [M(k+1)]^{A_0}, \qquad |t_s| \le 3\sqrt{p}\, x + x \le \tfrac{7}{2} y. \tag{A.274}$$

For $a \in (\mathbf{Z}/h)^\times$ put $F_{s,a} = t_s - (p^{k+1} z_1^a + p^k z_2^a + p z_3^a + 1)$. These are integral. Since $p + 1 < y/2$, (A.274) bounds the terms apart from the leading one by $9y/2 + p + 1 < py$. Hence $0 < |F_{s,a}| \le p^{2(k+1)}$. Galois automorphisms permute the factors, so $D = \prod_{s,a} F_{s,a}$ is a nonzero integer, with empty product one. The size bound and (A.274) yield (30.11). The forward trace is the sum of the inverses of the roots in the inverse-Frobenius comparison. Thus $\Delta_r = \iota^{-1}(F_{s,1})$ with distinct labels for $r \in X$. All other factors have nonnegative valuation, giving (30.12).

Lean: `UpperTheory.GalN.V.Win.N.number_sp4_of_admissible_to_admissible_n` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.VWinNumN`, lines 94–155)

□

Proof of 30.53

Proof. The monoid of nonnegative exponent vectors of weight zero is finitely generated. Its nonzero minimal vectors for the coordinatewise order form a finite set by Dickson's lemma, and subtracting such a vector preserves weight zero. Substitution of the corresponding monomials presents R_0 as a quotient of a power series ring over O. These monomials and ϖ generate its maximal ideal.

Let b be one of the monomial generators. If $b \notin \sqrt{\varpi R}$, choose a prime maximal among those containing ϖ and avoiding b. Its quotient is a local domain in which b is a nonzero nonunit. In dimension at least two, prime avoidance gives an element of the maximal ideal outside the finitely many height-one primes containing b. A minimal prime over that element has height one and avoids b, contrary to maximality of the chosen prime. Thus the quotient has dimension one. Its normalisation gives a local map to a complete

DVR with finite residue field in which b survives. Every nonconstant monomial of weight zero contains a zero-weight factor or factors of both strict signs. Hence the hypothesis kills b, a contradiction.

Projection to weight zero gives $\varpi R \cap R_0 = \varpi R_0$. Thus each monomial generator is nilpotent modulo ϖR_0, and the finitely generated maximal ideal of $R_0/\varpi R_0$ is nilpotent. This quotient is finite-dimensional over the residue field of O. Lift a basis and approximate successively modulo $\varpi^j R_0$. The coefficients converge in O and the remainders tend to zero in the maximal-ideal topology of R_0, so the lifts generate R_0 over O. □

Proof of 30.54

Proof. Map the absolute Galois marking of 30.61(4) to $\Gamma_{\mathbf{Q},S}$. The frame sign and character pull back to η_4, Ψ, so the image satisfies (30.14). A point of the marked frame pulls back to an ordinary point of that endpoint. Over a residue-characteristic DVR its polynomial identity on the specified finite-index subgroup, together with the marking and normalisation, gives upper or lower triangularity on the sign kernel. The ratio of each diagonal character to its prescribed character is trivial on the conjugates of the framed inertias. The frame quotient makes this ratio everywhere unramified over $\mathbf{Q}(i)$. Class field theory and the class number of $\mathbf{Q}(i)$ make it trivial. Use the adapted presentation in which each coordinate is an entry of the universal matrix minus the corresponding base entry, with weight given by diagonal conjugation. The diagonal equality kills the zero-weight coordinates, and triangularity kills one strict sign. Lemma 30.53 gives finiteness of R_0.
Lean: `UpperTheory.V.UnitaryThreeInertia.frame_class_one` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.FrameClassOne`, lines 166–268)
Lean: `UpperTheory.V.UnitaryThreeInertia.universal_diagonal_of_triangular_points` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.UniversalDiagonalOfTriangularPoints`, lines 22–72)

In characteristic zero, finiteness of R_0 gives a model over a finite coefficient DVR after diagonal conjugation. Work with this model. Let U be the normal core in $G_{\mathbf{Q}}$ of an open subgroup of G_K. Choose $\sigma \in I_\ell$ with $\varepsilon_\ell(\sigma) = 1 + \ell$, and put

$$s = \sigma^{[G_{\mathbf{Q}}:U]\varphi(N)} \in U \cap I_\ell.$$

Then $\vartheta(s) = 1$ and the three inertia eigenvalues at s are distinct. The local square test from (30.13) gives $x \in D_\ell \cap G_K$ with $\bar{\Psi}(x)^2 \neq 1$. If restriction to U is reducible, every U-invariant subspace is spanned by a subset of the s-eigenlines. The minimal such nonzero subspaces have disjoint sets of eigenlines. Their sum is G_K-invariant by normality, hence is the whole space. The action on them is transitive. Since the dimension is three, reducibility gives three lines. The local flag is the flag of the first one and the first two eigenlines in the ordinary order. Its preservation by x forces x to fix the three lines in this permutation action. Choose $g \in G_K$ acting by a three-cycle. The trace of $r(gx^j)$ is zero for every $j \geq 0$. Reducing at $j = 0, 1, 2$ gives a Vandermonde system with nodes $\bar{\Psi}(x), 1, \bar{\Psi}(x)^{-1}$ and coefficients $\bar{\Psi}(g), 1, \bar{\Psi}(g)^{-1}$. The nodes are distinct and the coefficients nonzero. This contradiction proves irreducibility on U and on the original open subgroup. Apply 30.61(4) and transport the attachment through the diagonal conjugation.
Lean: `UpperTheory.V.UnitaryThreeInertia.matIrr_open` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeDeformations.IrrOpen`, lines 340–382)

□

Proof of 30.56

Proof. All dimensions in this proof are over k'. Write $h^i_H(M) = \dim H^i(H, M)$, and use a place in the subscript for its absolute local group. Enlarge S to S' by adjoining the finite primes ramified in the field fixed by K. Put $G = G_{F,S'}$. There is a surjection $G \to \Gamma$. Its kernel J is topologically normally generated by the images of $J_q = I_q \cap K$ for $q \in S' \setminus S$, with one prolongation per place. Indeed outside S' the whole inertia group lies in K, and all prolongations of a given place are conjugate.
Lean: `UpperTheory.V.GC.levelS` (`Development.AutomorphicArithmetic.AutomorphicVarieties.GaloisCochains.LevelS`, lines 20–87)

Fix an added place q, and put $D_q = G_{F_q}$. The group D_q/J_q is an extension of $\widehat{\mathbf{Z}}$ by I_q/J_q, whose order divides $[G_F : K]$. Averaging over this finite group and using the ℓ-cohomological dimension one of $\widehat{\mathbf{Z}}$ shows

$$H^2(D_q/J_q, V) = 0, \qquad h^1_{D_q/J_q}(V) = h^0_{D_q}(V).$$

For the second identity, Frobenius acts on V^{I_q/J_q}; its coinvariant and invariant spaces have the same dimension by rank-nullity. Thus the local inflation–restriction sequence gives

$$\dim \mathrm{Hom}_{\mathrm{cont}}(J_q, V)^{D_q} = h^1_q(V) - h^0_q(V) = h^2_q(V).$$

Here J_q acts trivially on V, and the last equality is the local Euler characteristic formula at $q \nmid \ell$ for a finite ℓ-primary module [111, I, Thm. 2.8]. Moreover every class inflated from Γ restricts to zero in $H^2(D_q, V)$, since the local homomorphism factors through D_q/J_q.

Lean: `UpperTheory.V.GC.tameEq` (`Development.AutomorphicArithmetic.AutomorphicVarieties.GaloisCochains.TameEq`, lines 117–188)

Restriction to the J_q injects $\mathrm{Hom}_{\mathrm{cont}}(J, V)^G$ into the sum of the local equivariant homomorphism spaces just computed. A homomorphism in the kernel vanishes on all conjugates of the generators and hence, by continuity, on J. Write t for the dimension of the kernel of inflation $H^2(\Gamma, V) \to H^2(G, V)$. The five-term sequence for $J \to G \to \Gamma$ gives

$$t + h^1_G(V) - h^1_\Gamma(V) = \dim \mathrm{Hom}_{\mathrm{cont}}(J, V)^G \le \sum_{q \in S' \setminus S} h^2_q(V).$$

In particular $H^1(\Gamma, V)$ and $H^2(\Gamma, V)$ are finite, by finite-coefficient global finiteness for G.

Lean: `UpperTheory.V.GC.inflKer` (`Development.AutomorphicArithmetic.AutomorphicVarieties.GaloisCochains.InflKer`, lines 146–206)

Let s_Γ denote the dimension of the kernel of localization in degree two from Γ to the full local groups at S, with coefficient V. Define s_G using G and S'. By the vanishing at the added places, inflation maps the first kernel to the second, so $s_\Gamma \le t + s_G$. The cyclotomic character factors through Γ, since it is unramified outside S. The dual-invariant hypothesis therefore holds over G as well. For the underlying $\mathbf{F}_\ell$-module, composition with the field trace $k' \to \mathbf{F}_\ell$ identifies the k'-linear cyclotomic dual with the $\mathbf{F}_\ell$-linear Tate dual. Apply Theorem 27.2 to G, with $r = 1$. Its last term vanishes, giving

$$s_G = h^2_G(V) - \sum_{q \in S'} h^2_q(V).$$

Combining the last two displays gives

$$s_\Gamma + \sum_{v \in S} h^2_v(V) \le h^1_\Gamma(V) - h^1_G(V) + h^2_G(V).$$

The global Euler characteristic formula [111, I, Thm. 5.1] applies since S' contains all places over ℓ, the action factors through G, and ℓ is odd. Writing c_w for complex conjugation at a real place and $d = [F : \mathbf{Q}]$, it gives

$$h^2_G(V) - h^1_G(V) = \sum_{w \mid \infty} \dim V^{c_w = 1} - d \dim V - h^0_\Gamma(V).$$

Lean: `UpperTheory.V.GC.H2_galois_quotient_eq` (`Development.AutomorphicArithmetic.AutomorphicVarieties.GaloisCochains.H2GaloisQuotientEq`, lines 87–165)

Put $Q_v = V/b_v$, and let s_b be the kernel dimension of $H^2(\Gamma, V) \to \bigoplus_{v \in S} H^2(F_v, Q_v)$. On each local group the coefficient map $H^2(F_v, V) \to H^2(F_v, Q_v)$ is onto by local Tate duality: the dual map is the inclusion of the invariant duals. The additional kernel after composition has dimension at most the sum of the local kernel dimensions. Thus

$$s_b + \sum_{v \in S} h^2_v(Q_v) \le s_\Gamma + \sum_{v \in S} h^2_v(V).$$

The long exact sequence of the shifted cone, through the image in $H^2(\Gamma, V)$, proves finiteness in the asserted degrees and gives

$$\dim H^2_b - \dim H^1_b + \dim H^0_b = s_b - h^1_\Gamma(V) + h^0_\Gamma(V) + \sum_{v \in S} \big(h^1_v(Q_v) - h^0_v(Q_v)\big).$$

At $v \mid \ell$ the local Euler formula gives $h^1_v(Q_v) - h^0_v(Q_v) = h^2_v(Q_v) + d_v \dim Q_v$. Substitution and $\sum_{v \in S} d_v = d$ leave exactly $\sum_{w|\infty} \dim V^{c_w=1} - \sum_{v \in S} d_v \dim b_v$ as an upper bound.

Lean: `UpperTheory.V.GC.H2_galois_quotient_surjective` (`Development.AutomorphicArithmetic.AutomorphicVarieties.GaloisCochains.H2GaloisQuotientSurjective`, lines 50–78)
Lean: `UpperTheory.V.GC.coneCount` (`Development.AutomorphicArithmetic.AutomorphicVarieties.GaloisCochains.ConeCount`, lines 349–371)

□

Proof of 30.57

Proof. First introduce a matrix Y and a column v of independent variables over S. In $S[Y, v]$ let J be generated by the entries of Yv and by $\det Y$. We determine its radical. Write e_1 for the first coordinate vector and $E = \mathrm{diag}(0, 1, \dots, 1)$. With independent g-by-g matrices N, U and a scalar variable t, make the polynomial substitution

$$Y = NE \operatorname{adj}(U), \qquad v = tUe_1.$$

It kills J, since $Ee_1 = 0$, $\operatorname{adj}(U)U = (\det U)1_g$, and $\det E = 0$. Its kernel is prime since its target is a polynomial ring over the domain S.

Let $\mathfrak{q}$ be any prime over J, and pass to the fraction field of $S[Y, v]/\mathfrak{q}$. If $v \neq 0$, take $w = v$ and $t = 1$. If $v = 0$, choose a nonzero $w \in \ker Y$ and take $t = 0$. Complete w to the columns of an invertible U, with w first, and set $N = (\det U)^{-1} YU$. Since the first column of YU vanishes, these choices give precisely the displayed substitution. Thus its kernel lies in $\mathfrak{q}$. The kernel is consequently $P = \sqrt{J}$, and $v_1 \notin P$ since its image is tU_{11}.

Lean: `UpperTheory.V.CA.g_detrad` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebraHeight.GDetrad`, lines 111–162)

Grade the polynomial ring by total degree in Y, v, with S in degree zero. The ideal J is homogeneous, so P is homogeneous. In $T = S[[Y, v]]$, a series belongs to PT precisely when each homogeneous part belongs to P. Indeed, choose a finite homogeneous generating set of P, express each part in those generators with homogeneous coefficients, and collect the coefficients degree by degree. For two series outside PT, the product of their first homogeneous parts surviving modulo P survives, since $S[Y, v]/P$ is a domain. Thus PT is prime. A power of P is contained in J by Noetherianity, whence $PT = \sqrt{JT}$, and $v_1 \notin PT$.

Choose a prime $Q \subseteq PT$ minimal over the g entries of Yv in T. The adjugate identity gives $(\det Y)v_1 \in Q$. Since $v_1 \notin Q$, we have $\det Y \in Q$, hence $Q = PT$. Krull's height theorem gives $\mathrm{ht}(PT) \leq g$.

Lean: `UpperTheory.V.CA.g_detps` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebraHeight.GDetps`, lines 45–89)

There are $c = g^2 + g$ new variables. The complete regular local ring T has dimension $1 + a + g + c$, so the complete local domain T/PT has dimension at least $1 + a + c$. In that domain impose

$$Y_{ij} - Z_{ij}, \qquad v_j - u_j, \qquad f_1, \dots, f_b.$$

These $c + b$ elements lie in the maximal ideal. Evaluation at $Y = Z$, $v = u$ is defined by completeness. Its kernel in T is generated by the displayed differences, as one sees by translating the formal variables. Consequently this section and $\operatorname{Spec} A$ have the same underlying closed space. Krull's height theorem and catenarity give component dimension at least $1 + a + c - (c + b) = D + 1$. Grothendieck's connectedness theorem for a section of a complete local domain gives connectedness after removal of any closed subset of dimension less than D [123, Exp. XIII, Thm. 2.1]. If the asserted graph of minimal primes were disconnected, partition its vertices into two nonempty sets with no edge between them. The two unions of components would meet in a finite union of closed sets of dimension less than D. Removing that union would disconnect the section. This proves the chain assertion. □

Proof of 30.58

Proof. Write k' for the residue field. On both the relative cotangent space and the space of minimal relations the parity projections are $(1 \pm \sigma)/2$, where σ is the involution. They exist since ℓ is odd. Lift eigenbases using these projections. Complete Nakayama gives a surjection $S = O'[[x, u]] \twoheadrightarrow R$ with even variables x and odd variables u, and generators of its kernel J of definite parity. Write $t_+ = a, t_- = g$ for the variable counts and r_+, r_- for the minimal relation counts. The ideal of the fixed locus is generated by the u_j, since

taking the quotient by them makes σ the identity, and $2u_j = u_j - \sigma(u_j)$. In matrix coordinates the same ideal is generated by the entries between the middle line and the outer corner. The character coordinates are even.

For parity δ, let V_δ be the corresponding subspace of the residual adjoint matrices and $b_{\delta,v}$ its subspace preserving the ordinary flag. Write $h^i_\delta = \dim_{k'} H^i_b(V_\delta)$, with the cochain complex of Definition 30.55. The tangent and relation counts satisfy

$$r_\delta \le h^2_\delta, \qquad h^1_\delta \le t_\delta, \qquad h^1_+ + 1 \le t_+ + h^0_+.$$

We now give the deformation calculation. For a square-zero extension with kernel killed by the maximal ideal, lift the group matrices and the local flag bases. The defect of multiplication is a two-cochain. Modulo the flag-preserving matrices the local defects are its local primitives. Associativity gives the cocycle identity in the shifted cone. Changing the lifts adds a boundary, and a primitive of this cocycle corrects the lifts to a representation with the required flags. Apply this obstruction to the extension $S/\mathfrak{m}_S J \to S/J$, pushing its kernel out by a linear functional of parity δ on $J/\mathfrak{m}_S J$. A vanishing obstruction would give a section of the pushout by universality. Changing the variable lifts by that square-zero kernel leaves every relation value fixed, since all relative linear coefficients of a relation vanish in k' in a minimal presentation. The value of a relation in the pushout is its image under the chosen functional. Hence a functional with zero obstruction is zero. This gives the first inequality.

A first-order deformation gives a cocycle and the first-order local flag changes. Conversely a cone cocycle gives the matrices $(1+\epsilon z(s))\bar r(s)$ with their corrected flags over $k'[\epsilon]/(\epsilon^2)$, up to infinitesimal change of basis. Put the marking in its prescribed normal form and project to the indicated parity. The local flag changes modulo their stabilizers are determined by the cocycle: a flag change fixed modulo its stabilizer has zero class in the tangent space to the flag variety, since a local marking element has distinct eigenvalues on the graded lines. Thus framed tangents map onto H^1_b in each parity. In even parity conjugation by $1 + \epsilon\,\mathrm{diag}(1, 0, -1)$ preserves the marking and gives a coboundary. Either this gives a nonzero tangent in the kernel, or its nonzero generator fixes the residual representation and its flags and belongs to H^0_b. This proves the last two inequalities.

Lean: `UpperTheory.V.Fr.ep_mazur` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.EpMazur`, lines 154–283)
Lean: `UpperTheory.V.Fr.ep_tan` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.LieTMem`, lines 29–77)

In an eigenbasis of the involution adapted to a local flag, two graded lines have one parity and one has the other. Among the upper triangular matrix entries, four have even parity and two have odd parity. At a real place the fixed adjoint matrices identify, by the polarizing form, with alternating forms. Their even part has dimension one, from the exterior square of the corner, and their odd part has dimension two, from pairing the corner with the middle line. On the sign kernel the composition characters of the residual adjoint are powers of $\bar\Psi$. The condition on $\bar\varepsilon$ in Definition 30.55 therefore excludes a cyclotomic quotient in either parity. Applying (30.15) to the two spaces gives

$$h^2_+ + h^0_+ + 3d \le h^1_+, \qquad h^2_- + h^0_- \le h^1_-.$$

It follows that $r_+ + 3d + 1 \le a$ and $r_- \le g$. Append zero relations to bring the two lists to lengths $a-(3d+1)$ and g.

Finally, if $g = 0$, the block ideal vanishes in the completion. Faithful flatness over the local ring makes that ideal vanish in the local ring as well. It is finitely generated, so it vanishes on a neighborhood of the point. This proves the assertion about g.

Lean: `UpperTheory.V.Fr.gpos` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.AnnOfFlat`, lines 48–71)

□

Proof of 30.59

Proof. In each rank, residual independence in 30.55 gives $g \in G_L$ with $\bar\Psi(g) = 1$ and $\bar\varepsilon_\ell(g) \ne 1$. Otherwise $\bar\varepsilon_\ell$ would be a power of $\bar\Psi$. Chebotarev gives two split distinguished places over different rational primes away from $2M|D_L|$ with these Frobenius values. For each level the frame projection kills excluded inertia

and induces a surjection from the level CM group onto the frame sign kernel, respecting conjugation and the ordinary local maps. Write π for this surjection. Let N be a closed normal subgroup of its source, stable under conjugation. If N contains the distinguished inertia and every element g of each prescribed inertia subgroup with $\bar{\Psi}(\pi(g)) = 1$, the frame kernel criterion gives $\ker \pi \subseteq N$. Pulling back the determinant and ordered characters with twist κ_n gives the two comparisons. Use ι to regard the finite seed characters as Dirichlet characters. Apply Lemma 30.49(1) first to $\phi = \chi'^2\eta'$ and then to $\phi = \chi'\eta'$. In each application take $D \mid D'$ to be the primitive conductor of ϕ. The first character is odd and the second is nontrivial by $\chi' \neq \eta'$, so $D > 1$. In both cases ℓw is prime to D, $\phi(w) = 1$, and θ has conductor w and order ℓ, so $\ell \mid w - 1$. Take $m = j + (\ell - 1)t$ with $t \geq 0$ large enough for both weight thresholds of Lemma 30.49(1) and $m \geq 2$. Use $h = 2m + 1$ in the first application and $h = m + 1$ in the second; the parities of ϕ are respectively -1 and $(-1)^{j+1}$. In each case Lemma 30.49(1) supplies the normalized cuspidal newform of weight h, level Dw and nebentype $\phi\theta$, together with the continuous irreducible representation $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(C_\ell)$ of determinant $D_h = \varepsilon_\ell^{h-1}\phi\theta$, residual polynomial $(X - 1)(X - \bar{\Phi}_h)$ for $\Phi_h = \varepsilon_\ell^{h-1}\phi$, inertia type $(1, \phi\theta)$ away from ℓ, and ordinary flag at ℓ of inertia weights $(h - 1, 0)$ and decomposition-group residual labels $(\bar{\Phi}_h, 1)$.

For the first form put $\Xi = \varepsilon_\ell^m\chi'$ and $\widetilde{r}_2 = \rho\Xi^{-1}$, with $r_2 = \widetilde{r}_2|_{G_L}$. Here $\Phi_h = \Xi^2\eta'$, and on G_L we have $\eta' = \theta = 1$ and $\bar{\Xi} = \bar{\Psi}$. At the seed element s of 30.55, $\bar{\Phi}_h(s) = \bar{\Psi}(s)^2 \neq 1$. Lemma 30.49(1) gives irreducibility of r_2. Its residual labels are $(\bar{\Psi}, \bar{\Psi}^{-1})$ and its tame types are (χ', χ'^{-1}). Its weights are $(m, -m)$. The determinant of $\widetilde{r}_2$ is $\eta'\theta$. For $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $S_c = \widetilde{r}_2(c)$, one has $S_c^2 = 1$, $\det S_c = -1$, and JS_c is symmetric. On G_L the identity $R^tJR = (\det R)J$ gives the conjugate pairing

$$r_2(g)^t JS_c\, r_2(cgc) = JS_c.$$

For the rank-two automorphic normalisation, since $\ell \nmid D_{K_0}$, Lemma 30.49(2) supplies an algebraic Hecke character of K_0 of type $(-1, 0)$ with ℓ-adic realization κ_0 of conjugate product ε_ℓ^{-1}, unramified off ℓD_{K_0}, of prime-to-ℓ inertia exponent away from ℓ, and with rational-idelic transfer $\eta'\varepsilon_\ell^{-1}$ from $G_{\mathbf{Q}}$. Restrict κ_0 to G_L to obtain κ_2. Quadratic base change to K_0, the scalar twists, and abelian base change to L give the attachment of $r_2\kappa_2$ by [93]. The irreducible restriction to G_L gives cuspidality throughout. The choice of κ_2 is fixed in both Hecke comparisons and for every rank-two point.

By Lemma 30.49(1), the initial newform has spherical components outside Dw and unlinked principal-series support at $p \mid Dw$, with one unramified character and one of conductor exponent $v_p(Dw)$. After a common global norm twist to the unitary normalisation, the support exponents lie in $(-1/2, 1/2)$ by the local bound used in 39.55. The local base-change comparisons pull the two characters back by the norm. A place of residue degree f raises their uniformiser moduli to the f-th power and replaces q by q^f, so the exponents are preserved. An algebraic scalar twist adds one common exponent at every finite place. This gives (30.8) for $r_2\kappa_2$ at every finite place of L with one t. On an open subgroup of ordinary inertia the ratio of any ordered pair of flag characters is a cyclotomic power of exponent 0 or $\pm 2m$. None of these exponents equals one, and the cyclotomic character has infinite image there. Thus no positive power of a ratio equals the corresponding power of ε_ℓ. Every place above ℓ is potentially unlinked, which completes the rank-two witness.

Lean: `UpperTheory.V.Sd.twist_of_printed_results` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TTwist`, lines 111–178)

Lean: `UpperTheory.V.Z.Bc.second_initial_of_prints` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.FinOrderOfKills`, lines 296–453)

For the second form $\bar{\Phi}_h|_{G_L} = \bar{\Psi}|_{G_L}$, so ρ stays irreducible on every open subgroup by Lemma 30.49(1) and the square test. Put $r_3 = (\mathrm{Sym}^2\rho \otimes \det(\rho)^{-1})|_{G_L}$. For ordinary diagonal characters a, b of ρ, the new characters are $a/b, 1, b/a$. The resulting residual labels are $(\bar{\Psi}, 1, \bar{\Psi}^{-1})$ and the weights are $(m, 0, -m)$. The tame types are $(\chi', 1, \chi'^{-1})$. The discriminant of binary quadratics supplies an invariant nondegenerate symmetric form. An invariant plane in the three-dimensional vector space cuts the conic $[x^2 : 2xy : y^2]$ in a divisor of degree two. Its support gives a finite invariant set of lines in ρ, so an open subgroup fixes one of them. An invariant line in r_3 gives an invariant plane by orthogonal complementation. Thus r_3 stays irreducible on every open subgroup of G_L.

Base-change the second form to L and unitarise it. A nontrivial self-twist is quadratic by determinants. The corresponding intertwiner is scalar on the quadratic kernel by open irreducibility, so the self-twist character is trivial. Gelbart–Jacquet gives the cuspidal adjoint lift [124, Thm. 9.3]. The scalar polarising twist on the base change cancels in its adjoint. The arithmetic normalisation gives the attachment for $r_3\varepsilon_\ell^{-1}$.

The rank-two local comparison just used applies to the second newform as well. Its base change and algebraic polarising twist give toral support with exponents in a half-unit interval about a common exponent. In the adjoint comparison a pair of local characters ξ_1, ξ_2 gives the unitary-normalised support $(\xi_1/\xi_2, 1, \xi_2/\xi_1)$. We check both powers of this ratio on the original newform. At a prime outside its conductor, the Ramanujan–Petersson theorem for holomorphic newforms (Deligne, *La conjecture de Weil I*, Theorem 8.2) makes both unitary-normalised Satake characters unitary. At a prime dividing its conductor, the primitive nebentype gives two principal-series characters with different restrictions to units. In the generic unitary classification [219], a non-tempered rank-two principal series is a complementary pair $\chi|\cdot|^a, \chi|\cdot|^{-a}$, $0 < a < 1/2$; its two unit restrictions agree. Thus the ramified principal series here also has two unitary characters. These conclusions persist under norm pullback, and a common scalar twist cancels in the ratio. Consequently $|(\xi_1/\xi_2)(z)| = 1$ at every finite place, excluding $(\xi_1/\xi_2)^j = |\cdot|^{\pm 1}$ for $j = 1, 2$. Away from M the comparison gives spherical support. The good Frobenius polynomials identify the attachments from these comparisons by strong multiplicity one. The unitary-normalised adjoint attachment therefore has unlinked toral support at every finite place. Apply the strict half-unit bound of 39.55 and undo the global norm twist to obtain (30.8). On an open subgroup of ordinary inertia, the ratios are cyclotomic powers with exponents given by differences of $m, 0, -m$. Since $m \geq 2$, these differences are distinct from one. The cyclotomic character has infinite image there, so the same comparison holds after taking positive powers. This gives potential unlinking above ℓ and completes the rank-three witness.

Lean: `UpperTheory.V.Sd.ad0Irr` (`Development.AutomorphicArithmetic.AutomorphicVarieties.IdeleGenerators.HolomorphicIrreducibility.Ad0IrrMulVec`, lines 213–302)

Lean: `UpperTheory.V.Z.Sym.W.seedInW_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TSeedInW`, lines 117–270)

Lean: `UpperTheory.V.Z.Sym.W.adjoint_lift_from_eisenstein_of_propagation` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.AdjointLiftFromEisensteinSeedOfHyps`, lines 184–281)

Lean: `UpperTheory.V.Z.Sym.W.adjoint_automorphy_w_of_pure_weight_adjoint_lift` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.MemOfUnder`, lines 101–114)

For either seed, the excluded inertia away from ℓ has $\bar{\Psi} = 1$, hence $\chi' = 1$ there by its prime-to-ℓ order. The tame types give identity inertia matrices on this subgroup and hence factorisation through the frame. The residual determinant law is the product of the prescribed characters. Each factor of the ordered ordinary product relations sends the corresponding flag step into the preceding one, in the order of action on a column. After the twist the pairing has multiplier ε_ℓ^{1-n}. This gives a determinant point in 30.23. Its separated weights $mw_{n,i}$ and cuspidal witness put it on the Hecke quotient by 30.64.

Lean: `UpperTheory.V.Pr.seedPtOf_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.PrSeedPtOf`, lines 255–258)

Fix a Hecke comparison in each rank. Write $B_{0,n}$ for its universal ordinary determinant ring and $J_{0,n}$ for the Hecke ideal. Write A_n for the framed matrix ring before the link and $B_n \subseteq A_n$ for the coefficient ring of weight-zero data. Let M_n be the ideal on B_n generated from $J_{0,n}$ by the link. Denote the upper and lower triangular loci by the vanishing of the lower and upper entries, respectively. A prime of type I is off both loci and, in rank three, off the block locus where the middle line splits from the outer two-dimensional corner. A prime of type E is on this block locus and off both triangular loci.

We first check the link relations on the whole Hecke quotient $T_n = B_{0,n}/J_{0,n}$. At the distinguished place the residual Frobenius polynomial has a single nonzero root and the residual cyclotomic value is not one. At a detecting classical point choose a stable lattice over a finite extension of the coefficient DVR. On the distinguished decomposition group the residual characteristic polynomial is $(X - \kappa(g))^n$ for every g, with κ unramified. Fix a coefficient precision N. Every inertia matrix modulo ϖ^N has ℓ-power order: its reduction is unipotent, and the kernel of reduction is an ℓ-group. Thus the finite inertia image is an ℓ-group. Wild inertia has trivial image because its residue characteristic differs from ℓ; the remaining tame inertia image is cyclic. Let T generate it and let F be arithmetic Frobenius, so $FTF^{-1} = T^{q_{v_0}}$. The reduction of F is a scalar times a unipotent matrix. Successive ℓ-powering therefore makes F^{ℓ^a} scalar modulo ϖ^N for sufficiently large a. Hence $T^{q_{v_0}^{\ell^a}-1} = 1$. Since $q_{v_0}^{\ell^a} \equiv q_{v_0} \not\equiv 1 \pmod{\ell}$, this forces $T = 1$. This holds at every precision, so distinguished inertia acts trivially. At the other prescribed tame places, the coefficient type gives precisely the fixed prime-to-ℓ inertia characters; these are trivial on the excluded inertia kernel after undoing κ_n. Classical detection on the finite free weight module makes the corresponding determinant-kernel identities hold in T_n.

Now choose a minimal Hecke component through the seed. Over its generic matrix realisation, each

identity $\operatorname{tr}(r(h)(r(g)-1))=0$ for every h, with g in an excluded inertia kernel, gives $r(g)=1$ by the nondegenerate trace pairing and full matrix span. The kernel is closed, normal and conjugation-stable. The frame kernel criterion therefore makes the untwisted representation factor through the frame. The ordinary product relations give its ordered flags, and the polarisation and residual marking give the marked basis. This checks factorisation in both ranks before transporting the generic point. Its dimension is $1+nd$ by the finite free weight module of 30.23, since there are nd weight variables over the coefficient DVR. Over the algebraic closure of the component's function field its determinant has a matrix realisation. Full span at the seed gives a nonvanishing trace-pairing minor, hence full span at the generic point. Untwist and use the marked basis. The resulting type I prime contracts to a coefficient prime containing a prime $Z\supseteq M_n$ of dimension

$$\dim B_n/Z=1+nd. \tag{A.275}$$

Lean: `UpperTheory.V.Pr.seedOf` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.SeedOf`, lines 501–546)

In B_n let $\mathfrak{l}$ impose the coefficient uniformiser ϖ and triviality of the flag characters on $\ker\bar{\Psi}$ in inertia. Complete A_n at a residual point off the upper triangular locus and denote the resulting local ring by R. Suppose a quotient of R has dimension at least $nd+1$ and its contraction contains $I\subseteq B_n$ satisfying $\mathfrak{m}_{B_n}\subseteq\sqrt{I+\mathfrak{l}}$. For a specified pair of flag characters, coincidence at v means equality on N_v. The dimension bounds are

$$\begin{aligned}\dim(\text{lower triangular locus})&\le(n-1)d+2+3C_0+5<nd,\\ \dim(\text{special-fibre coincidence at } v \text{ over } I \text{ off both triangular loci})&\le nd+5-d_v<nd.\end{aligned} \tag{A.276}$$

For the first bound, cutting the lower triangular locus modulo the extended maximal ideal of the base ring uses the uniformiser ϖ (1 parameter), $1+C_0+d$ parameters in the first diagonal direction (splitting the c-invariant and c-anti-invariant inertia parts together with at most C_0 ramified-twist coordinates), and d inertia parameters in each of the remaining $n-2$ weight-zero diagonal directions, giving a base count of $1+(1+C_0+d)+(n-2)d=(n-1)d+2+C_0$. Over the base maximal ideal, the local flags at the ordinary places are fixed uniquely by $\bar{\Psi}^2\neq1$, so the lower cocycle fibre has dimension at most $2C_0$ by (30.16); adding base and fibre dimensions gives $\dim(\text{lower triangular locus})\le(n-1)d+2+3C_0<(n-1)d+2+3C_0+5$. For the second bound, choose d_v generators of the inertia group at v with one distinguished generator outside N_v and the remaining d_v-1 generators in N_v. Coincidence of the two flag characters on N_v eliminates those d_v-1 base inertia parameters in one direction while retaining the distinguished generator outside N_v together with the uniformiser ϖ; using the radical condition $\mathfrak{m}_{B_n}\subseteq\sqrt{I+\mathfrak{l}}$ bounds the contracted base dimension by $m\le nd+2-d_v$. Off both triangular loci, the fibre over the contracted base prime has dimension at most 1, giving $\dim(\text{special-fibre coincidence})\le m+1\le nd+3-d_v\le nd+5-d_v$. The inequalities $d>3C_0+7$ and $d_v>5$ give the strict bounds in (A.276). Apply Lemma 30.35 with $b=nd$, with the ideal of the given quotient as its initial ideal, and with the coefficient uniformiser as s. Take the lower triangular ideal for the first list. For the second list take the sum of the extended coefficient ideal I, the uniformiser ideal and the coincidence ideal for each place and each pair of distinct flag characters. The upper triangular locus is empty in this completion. The two lines of (A.276) give the required strict bounds, respectively, and the assumed quotient dimension is at least $b+1$. This gives a one-dimensional prime containing the initial quotient ideal, off both triangular loci, with separation on every N_v.

Lean: `UpperTheory.V.Fr.l51` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.CharpolyCoeffsInSubalgebra`, lines 50–84)
Lean: `UpperTheory.V.Fr.l52` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.TameInertiaCondition`, lines 257–280)
Lean: `UpperTheory.V.Pr.exists_nice_prime_above_of_large_dim` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.ExistsNicePrimeAboveOfLargeDim`, lines 7–69)

Normalise the curve over a finite residue extension and apply the link. Write H for the CM subgroup of the level group and $H'=H\cap\ker\varepsilon_\ell$. The differences on N_v give distinct ordinary characters and unlinking, since the cyclotomic character is one there. Their nontrivial ratios are principal units of the equal-characteristic DVR and have infinite order. One can choose a separating element on N_v by avoiding the finitely many kernels of the ratios in the group generated by pairwise separating elements. In prime rank, reducibility of an irreducible representation on H' gives either scalar multiplicity or a transitive permutation of lines. An element with distinct eigenvalues in H' excludes scalar multiplicity. In the permutation case the ordinary flag forces the local group to fix the lines distinguished by the separating element. Since the quotient H/H' is

abelian, the stabilisers are equal and the residual character test on the local group rules out the permutation case. For a type E curve the opposite entries in the two-dimensional corner give irreducibility, and the same argument applies to that corner. It has no intertwiner with the middle character. The prime-rank restriction argument applies also to $H' \cap V$ for every open normal $V \leq H$: take a positive power of the same separating element in V. More explicitly, the translates of a simple submodule span, so a reducible restriction in prime rank is scalar or gives a transitive permutation of lines. The separating power excludes scalars and identifies the lines with its eigenlines. The ordinary decomposition group fixes each line because it preserves their complete ordered flag. On the permutation kernel the reduced line characters and prescribed residual characters all coincide by Lemma 30.50(2). This contradicts the two distinct residual characters on the decomposition group, also for the outer corner since $\bar{\Psi}^2 \neq 1$ at its test element. Thus every open subgroup of H' acts absolutely irreducibly on the indicated prime-rank representation. In particular its image is noncommutative.

Take an open coset in H'. In type I apply Lemma 30.48 in rank two or three. In type E apply it to the outer two-dimensional block in the same coset and include the nonzero middle character. In each case obtain a full matrix M with all principal minors nonzero. In rank three its two-row principal minors are $\det(M)(M^{-1})_{ii}$, indexed by the omitted row. In the block case the minors are products of principal minors of the outer block and, when present, the middle scalar. Take a power of the separating element in the subgroup defining the coset, and a further power to kill torsion in the group generated by its eigenvalues a_i. Their ratios have infinite order, including those involving the middle line. Choose a homomorphism from this finitely generated free abelian group to $\mathbf{Z}$ taking distinct values w_i on the a_i, by avoiding the finitely many hyperplanes defined by their differences. Order the indices by decreasing w_i. In the degree-j elementary coefficient of $M\,\mathrm{diag}(X^{w_i})$, the unique leading term comes from the first j indices, and its coefficient is their nonzero principal minor. The Newton polygon at infinity has successive segments of length one and distinct slopes. Hence the characteristic polynomial is separable. In its multivariate discriminant group the monomials by equal products of the a_i. At least one coefficient sum is nonzero because the substitution X^{w_i} has nonzero discriminant and equal products have equal weights. If all evaluations at $(a_i^m)_i$ vanished, the Vandermonde equations for $m = 0, \ldots, b-1$, where b is the number of distinct products, would kill every coefficient sum. One of these powers therefore gives a separable element in the specified coset. An open subset of H meeting H' contains such a coset in its intersection with H'.

Lean: `UpperTheory.V.Pr.regCosetE` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.RegCosetE`, lines 174–191)

At the distinguished decomposition group D_0, Lemma 30.50(1) gives the zero intertwiner condition. These are the curve hypotheses in 30.34(a) for type I and in 30.34(b) for type E.

Lean: `UpperTheory.V.Pr.irrNormal` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.IrrNormal`, lines 62–179)

Lean: `UpperTheory.V.Pr.nice_specialization_twist_satisfies_third_condition` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.NiceSpecializationTwistSatisfiesThirdCondition`, lines 164–213)

Use the whole completion in rank two and at a nonblock residual point in rank three. Put $D = nd + 1$. The ungraded tangent and obstruction calculation gives at least D more variables than relations. Indeed, the flag-preserving matrices have dimension $n(n+1)/2$, and the fixed matrices at a real involution identify with alternating forms, of dimension $n(n-1)/2$. The cyclotomic character on the sign kernel differs from every power of $\bar{\Psi}$, so the residual adjoint has no cyclotomic quotient. Equation (30.15) gives $h^2 + h^0 + nd \leq h^1$. The square-zero calculation in the proof of Lemma 30.58, without the parity projections, gives $r \leq h^2$ for the minimal relation count and $h^1 + 1 \leq t + h^0$ for the variable count: the infinitesimal marked torus has nonzero generator $\mathrm{diag}(w_{n,i})$. Hence $t - r \geq D$. Grothendieck's connectedness theorem applied to these r equations in the complete regular local ring of dimension $1 + t$ gives the chain bound below [123, Exp. XIII, Thm. 2.1].

Lean: `UpperTheory.V.Fr.pres` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.HasObs`, lines 26–124)

At a residual block point use Lemma 30.58. The polynomial family $(ty_0, t^2a_0 + a)$ constructed in this proof specializes a nonblock point to each such point, so $g > 0$. Assign each monomial of an odd relation to the first odd variable dividing it and factor out that variable. This writes the g odd relations as the entries of Zu with Z square. A padded zero relation gives a zero row. The constant terms of the entries of Z lie in the maximal ideal of the coefficient DVR, since they are relative linear coefficients of relations in a minimal choice of variables. Cut by the image of $\det Z$. The identity $\mathrm{adj}(Z)Zu = (\det Z)u$ shows that every prime with a surviving odd coordinate is in the section. Apply Lemma 30.57 with $a - b = 3d + 1$. In

the whole completions just considered and in this determinantal section, every component has dimension at least $nd + 2$, and any two minimal primes are joined by a chain with
Lean: `UpperTheory.V.Fr.oddm` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.Oddm`, lines 5–70)

$$\dim R/(P + Q) \geq nd + 1 \tag{A.277}$$

for consecutive primes P, Q. In rank two use $I = M_2$ in the curve construction, since the finite weight module gives the radical condition. The minimal primes are of type I by (A.276). If the first contains the extended Hecke ideal, the curve above $P + Q$ kills that ideal. Apply 30.34(a) on the coefficient ring to deduce containment on the second branch.

A rank-three minimal prime of the section on the block locus is thick. In its function field the odd coordinates vanish and $\det Z = 0$ supplies a nonzero tangent vector in the kernel of Z. Clear denominators. In the resulting first-order evaluation the odd relations vanish by the kernel equation, and the even relations have zero odd derivative by parity. The ordinary and inertia relations are among the imposed relations. Projecting to an off-diagonal block gives an extension of the two blocks with the determinant law of the block sum. A splitting would make the first-order off-diagonal entries a conjugation derivative. The marked normal form eliminates such a derivative, since the marking separates the outer residual characters. The surviving odd derivative therefore gives a nonsplit extension.
Lean: `UpperTheory.V.Fr.sharp3` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.ChainedOfComap`, lines 72–142)
Lean: `UpperTheory.V.Fr.thick` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.ThickOfDer`, lines 224–255)

On a type E prime the corner gives a rank-two point off both triangular loci. Once rank-two propagation holds, its coefficient ideal gives an ideal $I_3 \subseteq B_3$ contained in the coefficient contractions of type E primes, with $\mathfrak{m}_{B_3} \subseteq \sqrt{I_3 + \mathfrak{l}}$. Indeed after imposing the inertia relations the corner coefficients become residual by rank-two finiteness. The middle character becomes residual by the abelian exponent condition, and the corner coefficients and middle character generate the block's weight-zero coefficients. Use (A.277) and (A.276) at successive intersections. A type I curve through an intersection gives propagation by 30.34(a) to the coefficient contraction of the second branch.

For a string of block components use I_3 to take consecutive type E curves $x_i : R \to k_i[[t]]$ as above. Let $\beta : B_{0,3} \to R$ be the link map to the completion. At an intervening block prime E_i, the same thick prime lies in $\ker x_i$ and $\ker x_{i+1}$. Transport its nonsplit extension through the link. It has the determinant law of the block sum and satisfies the ordered products and the inertia relations, with their polarised adjoints. Its integral model over $S[\epsilon]/(\epsilon^2)$, with $S = R/E_i$, is continuous, so for every j there are $b \geq 1$ and an open normal V such that $\mathfrak{m}_{B_0}^b$ and all $[v] - 1$, $v \in V$, act with image in $\mathfrak{m}_S^j$ times the module. Since I acts by zero, every element of I^{cl} acts with image in the intersection of these submodules, which is zero by Krull's intersection theorem. After extension of scalars it therefore gives a module for the specialised Cayley–Hamilton algebra at $\beta^{-1}(E_i)$. The marked odd derivative remains nonzero in the transported extension. Thus the algebra is nonsemisimple and the block determinant makes the contraction nonstable. The same thick contraction lies below both $\ker(x_i\beta)$ and $\ker(x_{i+1}\beta)$. The endpoint type I primes contract to stable primes by full matrix span. Containment of the extended ideal at the first endpoint puts $J_{0,3}$ in its contraction. Apply 30.34(b) to the curves $x_i\beta$ and extend the containment from the last contraction to R. A string with one curve uses the same assertion with no intervening prime.
Lean: `UpperTheory.V.Fr.l53` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.QuotientHomNonvanishingBound`, lines 221–236)
Lean: `UpperTheory.V.Pr.thickOf` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.ThickOf`, lines 142–197)
Lean: `UpperTheory.V.Pr.rankThree_deformationSetup_splitting_condition` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.SecondThirdPrimeConditionTransfer`, lines 27–96)

For global propagation put $C = V(\mathfrak{m}_{B_n} A_n) \setminus V(\mathrm{tri}^+)$, where tri^+ is the ideal generated by the lower entries.

Claim 1. Every homogeneous piece of A_n is a finite B_n-module, and every type I component meets C.

Take a homogeneous presentation $q : O[[X_s : s \in \Sigma]] \twoheadrightarrow R_n$ of the complete framed ring, with Σ finite and integer variable weights d_s. The image of the weight-zero series is B_n, and the image of the weight-j series is the degree-j part $(A_n)_j$. For fixed j, the set

$$E_j = \{a \in \mathbf{N}^\Sigma : \textstyle\sum_s a_s d_s = j\}$$

has finitely many minimal elements for the coordinatewise order. Order these minimal elements and assign each $a \in E_j$ to the first one below it. Grouping a weight-j series by this assignment writes it as a finite sum

$$\sum_{b \in \min E_j} X^b F_b, \qquad F_b \text{ a weight-zero formal series.}$$

Thus the finitely many $q(X^b)$ generate $(A_n)_j$ over B_n. The homogeneous presentation gives an internal grading; A_n is Noetherian and is generated over B_n by representation entries, whose weights lie in $\{-2,-1,0,1,2\}$.

Lean: `UpperTheory.V.Fr.wpres` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.ContMap`, lines 307–402)

Lean: `UpperTheory.V.Fr.gr_A` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.IsInternalCongr`, lines 81–86)

Let η be the minimal prime of a type I component. It is homogeneous: the ideal consisting of all finite sums of homogeneous elements of η is prime. Indeed, for two elements outside this ideal, take their highest-degree components outside η; the corresponding component of their product is their product modulo η and is nonzero there. Minimality of η now identifies it with this homogeneous prime. If $V(\eta) \cap C = \varnothing$, then $\mathrm{tri}^+ \subseteq \sqrt{\eta + \mathfrak{m}_{B_n} A_n}$. Noetherianity gives an integer $q_0 \geq 1$ such that

$$(\mathrm{tri}^+)^{q_0} \subseteq \eta + \mathfrak{m}_{B_n} A_n.$$

For $j < -2(q_0 - 1)$, every monomial of weight j in representation entries contains at least q_0 negative-weight factors. These are lower entries. Since $(A_n)_j \subseteq (\mathrm{tri}^+)^{q_0}$, taking degree-$j$ parts gives

$$(A_n/\eta)_j = \mathfrak{m}_{B_n}(A_n/\eta)_j.$$

The module on the left is finite over the local ring B_n, so ordinary Nakayama gives $(A_n/\eta)_j = 0$. A sufficiently high power of each lower entry has such a degree and hence belongs to η. Since η is prime, every lower entry belongs to η, contrary to type I. This proves the claim.

Lean: `UpperTheory.V.Fr.min_hom` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.MinHom`, lines 26–30)

The seed gives a type I prime of A_n whose contraction contains Z. Since A_n is graded with degree-zero part B_n, the prime Z is the contraction of a homogeneous prime η of A_n. If an entry a of weight $\alpha > 0$ and an entry b of weight $-\beta < 0$ lie outside the seed prime, then $a^\beta b^\alpha$ has weight zero and lies outside its contraction, which contains Z; hence $a, b \notin \eta$. Applied to the upper and lower entries, and in rank three to an odd-weight entry between the two blocks, this shows that η is of type I. A homogeneous element of nonzero weight outside η is transcendental over B_n/Z, and the matrix fibre off the triangular loci has transcendence degree at most one, so A_n/η has transcendence degree one over B_n/Z. The Nakayama argument of Claim 1 applies to this homogeneous type I prime, so $V(\eta) \cap C$ is nonempty. This locally closed subset of the finite-type closed fibre contains a closed point, and the completion there has a type I prime over η with contraction Z and dimension $\dim B_n/Z + 1$. Thus (A.275) supplies the starting branch for the local propagation.

Lean: `UpperTheory.V.Fr.frobenius_corner_lift_two_to_three` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.FrobeniusCornerLiftTwoToThree`, lines 133–180)

Lean: `UpperTheory.V.Fr.seedpt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.Seedpt`, lines 9–30)

Put $P = \bar{\Psi}$ and write $g^* = cg^{-1}c^{-1}$ for $g \in \Gamma_0$. Take a nonzero strict class for $P^{-2}\eta$ and normalise its cocycle z by $z(c) = 0$. This is possible since the coefficient character at c is -1 and the characteristic is odd. The restriction to Γ_0 is nonzero. If it vanished, $z(c) = 0$ would give $z = 0$ on both cosets and the local square tests would give zero strict scalars. Set $b(g) = P(g)z(g)$ on Γ_0. If D_v is the image of the ordinary local map, the cocycle and local coboundary laws give

$$b(gh) = P(h)b(g) + P(g)^{-1}b(h), \qquad b(g^*) = b(g), \qquad b|_{D_v} = \gamma_v(P - P^{-1}),$$

where γ_v is the negative of the strict scalar for z at v. These are the lower-entry conditions for diagonal P, P^{-1} in rank two. The first gives matrix multiplication, the second gives the marked polarisation, and the third gives a lower unipotent conjugate of the diagonal locally.

At a prime of C, the weight-zero specialisation gives a lower triangular point over its residue field. In rank two interpolate its lower entry a by $ta + (1-t)b$. The displayed conditions are linear, so the interpolation

defines a framed point over the polynomial ring in t. In rank three the lower entries in positions $10, 21, 20$ are $y(g), y(g^*), a(g)$, with diagonal $P, 1, P^{-1}$. Use

$$(y, a) \longmapsto (ty, t^2a + (1 - t)b).$$

In the 20 product law, $y(g^*)y(h)$ scales by t^2, and the added term obeys the linear law of b. The marking equation on y scales by t. The local unipotent conjugation parameters in positions $10, 21$ scale by t, and that in position 20 scales by t^2 with the addition of $(1-t)\gamma_v$. Thus the local conditions persist. The generic point of each polynomial family lies in C and specialises to both endpoints. The endpoint at $t = 0$ is the fixed point with lower entry b in rank two or with pair $(0, b)$ in rank three. Hence C is connected in each rank.

Lean: `UpperTheory.V.Fr.preconnected_rank_two` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.PreconnectedRankTwo`, lines 206–264)

For density off the rank-three block locus, use a nonzero strict class for $P^{-1}\eta$, with cocycle x normalised by $x(c) = 0$. Put $u = Px$ on Γ_0. Then $u \neq 0$, $u(gh) = P(h)u(g) + u(h)$, and $u(g^*) = P(g)^{-1}u(g)$. At v write $u|_{D_v} = \alpha_v(P - 1)$. Choose $\delta \in k$ with

$$\delta(P(g_0)^2 - 1) = -u(g_0)(P(g_0) - 1).$$

Define

$$y_0 = u + \delta(P - 1), \qquad a_0 = \delta^2 - \frac{\delta^2}{2}(P + P^{-1}) - \delta(1 + P^{-1})u + \frac{1}{2}P^{-1}u^2.$$

Substitution gives $y_0(gh) = P(h)y_0(g) + y_0(h)$ and

$$a_0(gh) = P(h)a_0(g) + y_0(g^*)y_0(h) + P(g)^{-1}a_0(h), \qquad a_0(g^*) = a_0(g).$$

The equation for δ gives $y_0(g_0) = y_0(g_0^*)$. On D_v, substituting $u = \alpha_v(P - 1)$ gives the lower unipotent conjugate of the diagonal with conjugating entries $\alpha_v + \delta$ in position 10, $\alpha_v - \delta$ in position 21, and $\alpha_v^2/2 - \delta\alpha_v - \delta^2/2$ in position 20. Thus (y_0, a_0) defines a framed point. If $y_0 = 0$, its values at g_0, g_0^* give $2\delta(P(g_0)^2 - 1) = 0$. Hence $\delta = 0$ and $u = 0$, a contradiction.

At a block point $(0, a)$, the entry a satisfies the same linear conditions as b. The polynomial point $(ty_0, t^2a_0 + a)$ has a nonzero 10 entry at its generic point and specialises at $t = 0$ to $(0, a)$. Thus the nonblock points are dense in C. Their local presentations put them on type I components. The finite union of these components covers C. If some lie in the Hecke locus and others fail containment, the two corresponding closed unions meet at a closed point of C. Lift the two primes to the completion by flatness. Apply local propagation and contract the containment. This proves global containment in rank two and then in rank three.

Lean: `UpperTheory.V.Fr.preconnected_rank_three` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.ExistsNonzeroLowCocycle`, lines 144–194)
Lean: `UpperTheory.V.Pr.components_good_rank_two` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.ComponentsGoodRankTwo`, lines 129–163)
Lean: `UpperTheory.V.Pr.components_good_rank_three` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.PropagationTheoremGL3`, lines 213–229)
Lean: `UpperTheory.V.Pr.globProp` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.GlobProp`, lines 3–46)

An irreducible arithmetic evaluation is of type I. The link identifies its determinant with that of $r\kappa_n$, of weights $a_\tau - ew_{n,i}$ at the embedding τ, where a_τ is the weight contribution of the fixed twist. These weights increase strictly in i. Apply 30.25 with the same twist and two different exceptional places. Its comparison for a fixed attachment gives the witness at every required place.

Let $k = O/\mathfrak{m}_O$. Since $\mathfrak{m}_O A = 0$, there is a specified embedding $j : k \to A$. Denote the evaluation of the framed matrix ring by $\xi : A_3 \to A$. Its restriction to B_3 kills $\mathfrak{l}$, since the inertia characters are one on the kernel of $\bar{\Psi}$. Off both triangular loci its prime kernel contains $I_3 \cap M_3$ by the two component containments. A prime containing this intersection contains at least one of the two ideals. The radical condition for that ideal and $\mathfrak{l}$ puts $\mathfrak{m}_{B_3}$ in the kernel. For each $b \in B_3$, subtract an O-lift of its residue to obtain $\xi(b) = j(\bar{b})$. The diagonal entries have weight zero. Every cycle product $r_{i_1 i_2}(g) \cdots r_{i_s i_1}(g)$ likewise has weight zero, by telescoping the basis weights; its residue vanishes if any two consecutive indices differ. All three two-cycles and both three-cycles therefore vanish. In the expansion of $\det(X - r(g))$ only the identity permutation remains, giving the prescribed product.

At a point on either triangular locus put $d_i(g) = r_{ii}(g)$. Matrix multiplication makes each d_i a character. The character $e_i = d_i/j(\bar{\Psi}^{1-i})$ takes values in $1 + \mathfrak{m}_A$. At each ordinary place take the inertia generators in $\ker \bar{\Psi}$ specified in 30.55. All three flag characters are one on them, so each generator has characteristic polynomial $(X-1)^3$. Its triangular diagonal entries are one because A is a domain. The same holds on the conjugate generators by the polarisation relation, which reverses the indices and inverts the diagonal values. Modulo $\mathfrak{m}_A^a$ the principal units form a finite abelian ℓ-group. The generator condition of 30.55 therefore gives triviality on inertia at both lifts, and its global exponent condition gives a single $h > 0$ such that $e_i(g)^h = 1 \bmod \mathfrak{m}_A^a$ for every a. Separation gives the equality in A. To see that it forces $e_i(g) = 1$, write $h = \ell^b h'$ with $\ell \nmid h'$. In characteristic ℓ, $(e_i(g)^{h'} - 1)^{\ell^b} = 0$. Also the quotient of $X^{h'} - 1$ by $X - 1$ takes the unit value h' modulo $\mathfrak{m}_A$ at $e_i(g)$. The claimed diagonal characters and their characteristic polynomial follow.

Lean: `UpperTheory.V.Pr.finPtOf` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.FinPtOf`, lines 183–342)

□

Proof of 30.60

Proof. Untwist on G_L. Twisting all characters of a flag by one character leaves their ratios unchanged. On the automorphic side the local support characters are twisted together. The absolute value of a Hecke character on the idèles is a power of the global norm, since the norm-one idèle class group is compact. Thus untwisting changes the centre t of the witness interval by one constant at all finite places. We obtain an attachment with witness for $W|_{G_L}$.

Write I^0 for the inertia subgroup in (1). Its cyclotomic image is open in the cyclotomic image of inertia, so it contains an element of infinite cyclotomic order. The two characters on I^0 are distinct. The line on which I^0 acts by ε_ℓ^k is unique and is preserved by the whole decomposition group, since I^0 is normal there and the character is conjugation invariant. Conjugating gives such a flag at every prolongation. For its two quotient characters λ_i, an equality $\lambda_i^a = (\lambda_j \varepsilon_\ell)^a$, $a \geq 1$, would give $\varepsilon_\ell^{a(s_i - s_j - 1)} = 1$ on I^0, with $s_i, s_j \in \{k, -k\}$. The exponent is nonzero. The same test works after any finite extension because a power of the chosen inertia element remains in each open subgroup. In particular all places above ℓ in L are potentially unlinked.

By 26.38 take the rational cuspidal representation π attached to W. Its base change through K to L is cuspidal at each step. Indeed a self-twist by a nontrivial step character would, by Frobenius comparison, give the same self-twist on the Galois representation. On the upper subgroup an intertwiner would be scalar by irreducibility there, which precludes a nontrivial self-twist. The base change and the given attachment over L have the same good polynomials and thus agree by strong multiplicity one.

At a place of L above ℓ the witness puts the character moduli into one open interval with endpoint ratio q_v. Consequently their ratio cannot be $|\cdot|_v^{\pm 1}$. Their local parameter is a sum of characters with zero monodromy by 40.2. The toral clause of 39.6 applies: the polarization is the exterior-product polarization used in 26.38. The filtration jumps are $\{k, -k\}$, since on an open inertia subgroup the successive characters are the indicated cyclotomic powers and restriction to a finite extension preserves labeled jumps. By 39.4 and archimedean base change the infinitesimal pair of π is $\{\frac{1}{2} + k, \frac{1}{2} - k\}$. The twist $\pi \otimes |\det|^{-1/2}$ has infinitesimal character $\{k, -k\}$. Its lowest-weight vector at infinity and its fixed vectors at a finite level give a classical Hecke eigenform of weight $2k+1$ and some level, whose eigenvalues at almost all primes are those of a normalized newform f, with $j = k$ in 26.36(3).

Lean: `UpperTheory.V.Br.repNewAE_of_printsH_of_repEig` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.BrRepNewAE`, lines 15–66)

At a good prime p, twisting W by ε_ℓ^{-k} multiplies its inverse arithmetic Frobenius roots by p^k. Its polynomial is consequently the Hecke polynomial of this f, as asserted.

To check the witness, let $K'/\mathbf{Q}$ be any imaginary quadratic field with $W|_{G_{K'}}$ irreducible. The self-twist argument gives cuspidal quadratic base change of π. Every regular algebraic cuspidal attachment to $W|_{G_{K'}}$ equals that base change, by strong multiplicity one. We compare its cuspidal support with that of the known base change to L.

At $p \neq \ell$, (2) makes the inertia image finite and central in the local image. After Frobenius semisimplification, inertia and Frobenius diagonalize together. At $p = \ell$, use the stable local flag above. The representation is potentially semistable: this holds over L by the toral comparison, hence over a finite extension of $\mathbf{Q}_\ell$. The associated Weil representation, semisimplified, is the sum of the two rank-one parameters of the flag by

exactness of potentially semistable comparison. In both cases the Weil constituents over $\mathbf{Q}_p$ are characters. For the descended newform over $\mathbf{Q}$, the semisimplified local comparison follows from [138] at $p \neq \ell$ and from [139] at $p = \ell$; the weight $2k+1$ is at least three. Duals and norm twists commute with this comparison. Local base change carries an irreducible representation whose cuspidal support consists of characters ξ_i to one whose cuspidal support consists of the characters $\xi_i \circ N$ [93, Ch. I, Thm. 6.2(d), §§6.2 and 6.4]. Thus the support over K' also consists of characters at every place under consideration.

Write $\xi_{p,i}$ for the two rational support characters in the automorphic normalization of π. For a place u of L over p the absolute values of the upper support characters on a uniformizer are $|\xi_{p,i}(p)|^{f(u/p)}$. This follows from the norm of a uniformizer and the fact that a smooth character has absolute value one on units. The witness over L gives one t for its attachment, and $q_u = p^{f(u/p)}$. Taking the positive $f(u/p)$-th root gives

$$p^{t-1/2} < |\xi_{p,i}(p)| < p^{t+1/2}.$$

At $v \mid p$ of K', raise these inequalities to $f(v/p)$. This gives the required interval with the same t at every v. The flag argument already checked potential unlinking at every prolongation above ℓ. These are both clauses of Definition 30.1. □

Proof of 30.61

Proof. Put $k_{\mathrm{res}} = O/\mathfrak{m}_O$ and $\omega = \bar{\varepsilon}_\ell$. Take $M = \ell N |D_K|$ for the first count and $M = \ell N$ for the Gaussian host. For the first count take $H_b = G_{\mathbf{Q}}$. For the Gaussian host take H_b to be the kernel of the ℓ-primary part of ϑ with its factor at ℓ removed. Its quotient is an abelian ℓ-group unramified at ℓ. On inertia in H_b away from ℓ, the value of the prime-to-ℓ part determines the value of Ψ. Write $K_0 = \mathbf{Q}(\sqrt{-D_0})$ for the sign field, with $D_0 > 0$ squarefree. The two input tests are

$$\ker\omega \cap \ker\bar{\Psi} \not\subseteq \ker\eta, \qquad \exists s \in D_B \cap \ker\eta, \quad \bar{\Psi}(s)^2 \neq 1. \tag{A.278}$$

For the first count these are the sign-kernel and local unit conditions of the good datum. For the Gaussian host the first uses $c_2(\vartheta) \geq 4$ and the second is the decomposition test from (30.13). In both cases parity gives $\Psi(c) = 1$, $\eta(c) = -1$, and the sign is unramified at ℓ. The prime-to-ℓ part of a local cyclotomic character on I_ℓ factors through $\mathbf{F}_\ell^\times$, so $\bar{\Psi}|_{I_\ell} = \omega^j|_{I_\ell}$ for some $j \geq 0$.

Lean: `UpperTheory.V.Fr.gen_gd` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.SqrtDiscr`, lines 152–284)

Lean: `UpperTheory.V.Fr.gen_host` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.GenHost`, lines 220–255)

Choose $a \geq 1$ so that $\eta, \bar{\Psi}$ factor through $\mathbf{Q}(\mu_{M^a})$. Write $M^a\ell = \ell^b D'$ with $\ell \nmid D'$. The Teichmüller lift $\chi' = [\bar{\Psi}\omega^{-j}]$ and the sign lift $\eta' = \eta$ factor through $\mathbf{Q}(\mu_{D'})$, since both are trivial on the omitted ℓ-inertia factor. The order of χ' divides $|k_{\mathrm{res}}| - 1$, and $\chi'(c) = (-1)^j$. The first test in (A.278) excludes $\chi' = \eta'$. Unramifiedness of η at ℓ gives $\ell \nmid D_{K_0}$.

By Dirichlet's theorem [121], take a prime $p > 4D_0 + M$ with $p \equiv -1 \pmod{4M}$, and a prime $w > M^{a+1}D'$ with $w \equiv 1 \pmod{M^{a+1}D'}$. Put $H_{\mathrm{q}} = G_{\mathbf{Q}(\sqrt{D_0 p})}$. At every prime dividing M the corresponding quadratic character agrees locally with η, since $-p$ is a local square, including at 2 by $2 \mid M$. At p the quadratic character of H_{q} is nontrivial on inertia, while $\eta, \bar{\Psi}, \omega$ are unramified. The prime w splits all three characters and χ', and $\ell \mid w - 1$. Let H_w be the inverse image of the subgroup of ℓ-th powers of $(\mathbf{Z}/w)^\times$, and take θ faithful on its cyclic quotient of order ℓ. Then θ has conductor w.

Lean: `UpperTheory.V.Fr.exists_twistCharacter_continuousCohomology` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.EtaleSimplicialCochainSchemeSectionExtensionNatu`, lines 99–140)

Lean: `UpperTheory.V.Fr.auxw_ex` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.AuxwEx`, lines 26–61)

Lean: `UpperTheory.V.Fr.seed_cl` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.SeedCl`, lines 23–131)

For $m \geq 0$ put

$$H_{\mathrm{cyc},m} = \ker((\varepsilon_\ell \bmod \ell^{m+1})^{\ell-1}), \qquad U_m = H_{\mathrm{q}} \cap H_w \cap H_b \cap H_{\mathrm{cyc},m}, \qquad F_m = \overline{\mathbf{Q}}^{U_m}, \qquad L_m = F_m K_0.$$

The three factors after H_{q} have abelian ℓ-power quotient and contain complex conjugation. The real quadratic subgroup contains it as well. Thus F_m is totally real abelian, L_m is CM, $G_{L_m} = U_m \cap \ker\eta$,

and $\theta|_{G_{L_m}} = 1$. The sign is trivial on the local subgroups of U_m over M by the choice of H_{q}. For each prolongation over ℓ, conjugate the element s in (A.278) to that prolongation and raise it to an ℓ-power sufficient to enter the three ℓ-power-index subgroups. It already belongs to H_{q} and retains its square test because $k_{\mathrm{res}}^{\times}$ has order prime to ℓ. At B take this element as g_1. It is in G_{L_m} and serves as the seed element in 30.55. For each $v \mid \ell$, inertia surjects onto the principal-unit quotient modulo ℓ^{m+1}. Since $U_m \subseteq H_{\mathrm{cyc},m}$, the local degree $d_v = [F_{m,v} : \mathbf{Q}_\ell]$ satisfies

$$d_v \geq \ell^m. \tag{A.279}$$

The mod-ℓ cyclotomic character is nontrivial on the local group over v. Inertia is in $H_{\mathrm{q}} \cap H_b \cap H_w$, and $H_{\mathrm{cyc},m}$ removes the principal-unit quotient while retaining the full tame cyclotomic quotient.

For the character tests on G_{L_m}, take $t \in \ker\omega \cap \ker\bar{\Psi}$ with $\eta(t) = -1$. The element ct has $\eta = 1$, $\bar{\Psi} = 1$, and $\omega = -1$. If necessary multiply it by an inertia element at p to enter H_{q}, preserving these values. Raising to a suitable ℓ-power puts it in G_{L_m} and preserves those values. This tests ω against every power of $\bar{\Psi}$, and gives the required nontrivial characters involving ω in 30.55.

Lean: `UpperTheory.V.Fr.ufacts` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.Ufacts`, lines 78–241)

Lean: `UpperTheory.V.Fr.field_ex` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.FieldEx`, lines 24–155)

Fix $i \in \{1, 2\}$ and put $\chi = \bar{\Psi}^{-i}\eta$, $V = k_{\mathrm{res}}(\chi)$. The character is odd and $\chi \neq \omega$, as the element t of the first test has $\chi(t) = -1$, $\omega(t) = 1$. Let S_χ consist of w, ℓ and the ramified primes of χ. Use the full local H^1 at w, zero at the other places of S_χ, and the unramified condition elsewhere. For local subspaces $\mathcal{L}_v$ the finite-module Greenberg–Wiles formula is [119, Prop. 1.6]

$$\frac{|\mathrm{Sel}_{\mathcal{L}}(V)|}{|\mathrm{Sel}_{\mathcal{L}^\perp}(V^\vee(1))|} = \frac{|H^0(\mathbf{Q}, V)|}{|H^0(\mathbf{Q}, V^\vee(1))|\,|H^0(\mathbf{R}, V)|} \prod_{v \in S_\chi} \frac{|\mathcal{L}_v|}{|H^0(\mathbf{Q}_v, V)|}, \tag{A.280}$$

where $V^\vee$ is the $\mathbf{F}_\ell$-linear dual and $\mathcal{L}^\perp$ is defined by local Tate duality. The set of places includes ℓ and the ramification of V, and the odd-prime hypothesis applies. Oddness kills the real and global invariants, and $\chi \neq \omega$ kills the global dual invariants. At a ramified place an inertia element acts with $\chi \neq 1$. At ℓ the marking g_1 gives such an element in the decomposition group for both i. At w the two characters χ, ω are trivial. Local Tate duality and the local Euler characteristic in 27.2 give

$$|H^1(\mathbf{Q}_w, V)| = |H^0(\mathbf{Q}_w, V)|\,|H^2(\mathbf{Q}_w, V)| = |k_{\mathrm{res}}|^2.$$

Thus (A.280) is $|k_{\mathrm{res}}| > 1$.

Choose a representative $y_i : G_{\mathbf{Q}} \to k_{\mathrm{res}}$ of a nonzero Selmer class. Its cocycle law is $y_i(gh) = y_i(g) + \chi(g)y_i(h)$. It is a coboundary at ℓ for every prolongation. Away from w, ℓ it vanishes on inertia in $\ker\eta \cap \ker\bar{\Psi}$, by the coboundary condition at ramified primes and the unramified condition at the others. At w it is additive on inertia. Wild inertia is pro-w, and tame inertia has maximal elementary abelian ℓ-quotient cyclic of order ℓ. The character cutting out H_w is nonzero on that quotient, so $y_i = 0$ on $I_w \cap H_w$ at every prolongation.

The restriction to U_m remains nonprincipal. Otherwise subtract a coboundary to obtain $z|_{U_m} = 0$. At $u = g_1$ one has $\chi(u) \neq 1$, and normality puts gug^{-1} in U_m for all g. Expanding $z(gu) = z((gug^{-1})g)$ gives $(1 - \chi(u))z(g) = 0$ for all g, contrary to the nonzero class.

Lean: `UpperTheory.V.Fr.st_loc` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.DualNontriv`, lines 14–55)

Lean: `UpperTheory.V.Fr.st_count` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.StCount`, lines 10–76)

Lean: `UpperTheory.V.Fr.st_sel` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.StSel`, lines 31–128)

Lean: `UpperTheory.V.Fr.st_tame` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.StTame`, lines 47–79)

Lean: `UpperTheory.V.Fr.strict` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.Strict`, lines 81–174)

The frame group is the quotient of $U_m = G_{F_m}$ by the closed normal subgroup generated by $I_v \cap \ker\eta \cap \ker\bar{\Psi}$ at the finite places away from ℓ, with inertia taken over F_m. Its ordinary local maps come from the decomposition groups above ℓ, with one prolongation for each place. Where $\chi = 1$ the zero set of y_i is a subgroup. If $y_i(h) = 0$ and $\chi(h) = 1$, then the cocycle law gives $y_i(ghg^{-1}) = 0$. Local constancy gives closedness, so y_i descends. Choose the scalar giving its local coboundary at each ordinary map. The

descended pair is a strict class of 30.55 and is nonzero by its pullback to U_m. The two coefficient characters use the same w, F_m, L_m, g_1.

Lean: UpperTheory.V.Fr.link_ex (Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.LinkEx, lines 150–308)

Lean: UpperTheory.V.Fr.frobenius_galoisField_four_compatibility (Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.FrobeniusGaloisFieldFourCompatibility, lines 105–160)

Let E be the fixed field of $H_{\mathrm{q}} \cap H_w \cap H_b \cap \ker \eta \cap \ker \bar{\Psi} \cap \ker \omega$, and put $E_m = E(\mu_{\ell^{m+1}})$. The common residual kernel at layer m fixes E_m, since raising a principal unit to $\ell - 1$ preserves its congruence depth. The field E is finite abelian and contains μ_ℓ. By 28.6 the ℓ-ranks of the class groups of all E_m are bounded by one C_0.

Lean: UpperTheory.V.Fr.b0 (Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.RankLeOfCardLe, lines 137–187)

For each of the first two groups in (30.16), restriction to G_{E_m} is an unramified character. The strict condition gives zero on the local groups above ℓ, and the frame quotient kills inertia on the common kernel elsewhere. Restriction is injective on the global classes by averaging over the prime-to-ℓ residual quotient, and the local square test determines the strict scalars. Class field theory gives the first two bounds in (30.16). For the trivial character use the same Selmer formula over F_m, with full conditions above ℓ and unramified conditions elsewhere. Every frame character into the additive residue field satisfies these conditions. At $v \mid \ell$, the local ω test gives zero dual invariants, so the local factor by Euler characteristic is $|k_{\mathrm{res}}|^{d_v}$. The real factors contribute $|k_{\mathrm{res}}|^{-d}$, and the global factor is $|k_{\mathrm{res}}|$. The dual strict classes for ω inject into unramified characters of G_{E_m} by the same restriction argument. The resulting bound is the third of (30.16).

Lean: UpperTheory.V.GC.h1 (Development.AutomorphicArithmetic.AutomorphicVarieties.GaloisCochains.CastToZModPowOne, lines 18–47)

The linked frame uses the cyclotomic kernel in each ordinary inertia for N_v, on which $\bar{\Psi} = 1$. By local reciprocity take generators of the pro-ℓ inertia image in the local abelianisation with all but one in the cyclotomic kernel. The local residual cyclotomic test excludes ℓ-power roots of unity in the local field and the pro-ℓ unit group has rank d_v. The generators can be taken in $\ker \bar{\Psi}$ by removing their prime-to-ℓ images. Class field theory gives the common exponent on the sign kernel from the finite unramified class group over L_m, since the remaining allowed tame inertia has order prime to ℓ. Finiteness for open-subgroup character spaces follows from finite-coefficient global cohomology with ramification at finitely many places. For (30.15), apply Lemma 30.56 with $F = F_m$ and $K = G_{F_m} \cap \ker \eta \cap \ker \bar{\Psi}$. This is the kernel of the pair of finite characters. Its index divides $2|k_{\mathrm{res}}^{\times}|$, prime to the odd prime ℓ. The field F_m is totally real. The inertia relations defining the frame are exactly those of the lemma, and the ordinary local groups identify with the full decomposition groups at the places above ℓ. The frame cyclotomic character and the real involutions are the images of the corresponding Galois data. The lemma therefore gives the required bound for every coefficient field, representation and family of subspaces in Definition 30.55.

Lean: UpperTheory.V.GC.ec (Development.AutomorphicArithmetic.AutomorphicVarieties.GaloisCochains.ResH1SurjectiveOfIsomorphism, lines 84–179)

Take $m = 3C_0 + 7$. By (A.279) and the existence of a place over ℓ,

$$d = \sum_v d_v \geq \ell^m > 3C_0 + 7, \qquad d_v > 5. \tag{A.281}$$

Lean: UpperTheory.V.Fr.frame_ex (Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.FrameEx, lines 126–160)

Lean: UpperTheory.V.Fr.ok_basic (Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.OkBasic, lines 46–114)

The link pulls an ordinary point into the framed problem of 30.59. Away from ℓ, on inertia in U_m the exact condition $\Psi = 1$ agrees with $\bar{\Psi} = 1$ because $U_m \subseteq H_b$ removes the ℓ-primary part there. At ℓ the ordinary flag transports to each prolongation, and in residue characteristic the prescribed characters kill $\ker \bar{\Psi}$ in inertia. For (1) take $U = G_{L_m}$ before choosing A or the point. It has index $[L_m : \mathbf{Q}]$ and lies in $\ker \eta$. Restrict the point to the group used for the link. Its kernel contains the closed normal subgroup imposed away from ℓ: each generating inertia element has $\eta = \bar{\Psi} = 1$, the link gives $\Psi = 1$ there, and the ordinary point kills it. The kernel is closed by continuity. Thus the representation descends to the frame quotient, and on its sign kernel the descended matrix pulled back to G_{L_m} is the original matrix. The link sends the chosen conjugation and marking to those of the frame, so their matrix equations and the prescribed diagonal reduction are preserved. Transport the flag at B to each prolongation above ℓ. Its inertia characters are the images of Ψ^{1-i}, for $i = 0, 1, 2$; conjugate duality reverses the order and inverts the characters, preserving

this list. Since $O \to A$ factors through $j : O/\mathfrak{m}_O \to A$, they equal $j(\bar{\Psi}^{1-i})$ on inertia. This checks the residue-characteristic hypothesis of 30.59. Given $g \in U$, use its image in the sign kernel of the frame in that lemma. Pullback gives $\det(X - r(g)) = \prod_{i=0}^{2}(X - j(\bar{\Psi}(g)^{1-i}))$ in $A[X]$.

Lean: `UpperTheory.V.Pr.exists_nrep3` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.ExistsNrep3`, lines 271–443)

Lean: `UpperTheory.V.Pr.finEnd_of_finPt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.FinEndOfFinPt`, lines 9–48)

For (2) the restriction to G_{L_m} is irreducible by hypothesis. Apply the rank-three conclusion, then solvable descent with the pairing and local flags to obtain the attachment over K. The witness descends along the same solvable steps. Away from ℓ, the finite central inertia image yields decomposition-group characters giving the characteristic polynomial on inertia. Above ℓ, use the ordinary flags. At a downstairs unlinked place the representation is potentially crystalline, and remains so on restriction. A link after restriction would transport by conjugation to a marked frame place. There the nonlinking criterion for the large frame, with even level, prescribed seed and irreducibility on all open subgroups of $\ker \eta$, gives for each pair a local element on which the two graded characters differ after the cyclotomic twist. This excludes the transported link. The common twist by ε_ℓ^{-1} preserves the ratios of the graded characters. Thus the upper witness applies at every place required for descent. At residue degree f, the uniformiser moduli and q are both raised to the f-th power, preserving the support exponents.

Lean: `UpperTheory.V.Z.Keep.keeps3E` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.UpperTriangularPlaces.ConjLeftPosAlt`, lines 467–470)

For (3) the Type II corner gives $W|_{G_K}$ irreducible. If it became reducible over L_m, the abelian quotient G_K/G_{L_m} would give a nontrivial self-twist λ of the restriction to G_K. Indeed, after semisimplifying that restriction to the normal subgroup G_{L_m}, reducibility makes its endomorphism space larger than the scalars. The abelian quotient acts on this space by conjugation, and a nontrivial eigenspace gives the self-twist by irreducibility. For g with $\lambda(g) \neq 1$, both g and gg_1 have trace zero because $g_1 \in G_{L_m}$. The residual corner trace at either element h is $\bar{\Psi}(h) + \bar{\Psi}(h)^{-1}$, so $\bar{\Psi}(g)^2 = \bar{\Psi}(gg_1)^2 = -1$. This contradicts $\bar{\Psi}(g_1)^2 \neq 1$. Apply the rank-two conclusion of 30.59. Use the algebraic twist and witness on L_m furnished by 30.59. Continuity on the open subgroup G_K follows from the corner matrices, hence W is continuous. For $g \in G_K$, the polarized matrix identity identifies the corner determinant at $cg^{-1}c$ with that at g, whereas the determinant character of W takes inverse values there. Hence $\det W(g)^2 = 1$. Its diagonal residual corner has determinant one, and ℓ is odd, so $\det W(g) = 1$. The involution $W(c)$ has trace zero and determinant -1. Thus $\det W = \eta$, a finite cyclotomic character. At $p \neq \ell$, inertia in the common kernel of η, Ψ acts trivially on the corner. The quotient through which inertia acts is finite, since the cyclotomic character is unramified there, and decomposition acts trivially on this quotient since both characters are global. At $B \mid \ell$, intersect inertia with the open normal subgroup $\ker \eta \cap \ker \vartheta$. The first line of the ordinary flag has character ε_ℓ^k. Its projection to the middle coordinate vanishes, since the middle character is one and a cyclotomic inertia element of infinite order distinguishes the two. It is therefore a line of the corner, whose quotient character is ε_ℓ^{-k} by the determinant. All hypotheses of 30.60 now hold for this same W. The same link construction with Gaussian sign gives (4).

Lean: `UpperTheory.V.Fr.host_ex` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.HostEx`, lines 8–26)

Lean: `UpperTheory.V.Z.Win.Dn.autEnd3_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.AutEnd3Alt`, lines 13–106)

Lean: `UpperTheory.V.Z.Win.Dn.hostEnd_of_loc3_frame_linkedEnd_downCsdK` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.DnHost`, lines 17–67)

Lean: `UpperTheory.V.Z.Win.Dn.AutEnd2Ao` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.AutomorphyLifting.AutomorphyDescent.AutEnd2Ao`, lines 18–25)

□

Proof of 30.62

Proof. Let $p : G_{L^+} \twoheadrightarrow \Gamma$ be the level quotient. It respects the quadratic character of L/L^+ and the full ℓ-adic cyclotomic character. In particular

$$p(G_L) = \Gamma_0, \qquad p(G_{L(\mu_{\ell^\infty})}) = \Gamma'.$$

For the second equality, lift any element of Γ' to G_L. Its cyclotomic value equals that of its image and is therefore one. Hypothesis 3 gives absolute irreducibility of $\rho \circ p$ on $G_{L(\mu_{\ell^\infty})}$, since its coefficient field is $\overline{\mathbf{Q}}_\ell$.

By hypothesis 1 and Definition 29.5, choose a regular algebraic conjugate self-dual cuspidal attachment π over L. At the unramified places in the definition, $\rho \circ p$ and $r_{\pi,\iota}$ have the same characteristic polynomial: at geometric Frobenius the roots in the complex normalisation are the Satake roots multiplied by $q_v^{(n-1)/2}$. Chebotarev density and Brauer–Nesbitt identify the two semisimple representations. Thus the irreducibility required in [129, Theorem 6.1] holds for $r_{\pi,\iota}$.

We identify the action on the coefficient module of that theorem. On G_L it is conjugation on the entire matrix algebra. A polarising form gives the action at c by $X \mapsto -P^{-1}X^tP$, by differentiating inverse transpose. Any two intertwiners between ρ^c and $\rho^\vee \varepsilon_\ell^{1-n}$ differ by a scalar by irreducibility and Schur's lemma. The scalar cancels in this formula. Hence the extended adjoint used by Thorne is $R \circ p$.

The pullback $g \mapsto z(p(g))$ gives a class in $H^1_f(L^+, R \circ p)$ by the finite local conditions in the statement. Theorem 6.1 of [129] gives zero for this group: L is CM, $n \geq 2$, the attachment is polarizable and the cyclotomic restriction is irreducible. The published theorem allows all such ranks and does not require the ordinary flag conditions. There is consequently an $X \in M_n(\overline{\mathbf{Q}}_\ell)$ with

$$z(p(g)) = R(p(g))X - X \qquad (g \in G_{L^+}).$$

Surjectivity of p gives (30.20). □

Proof of 30.63

Proof. Put $\partial(\rho(g)) = z(g)\rho(g)$. If $\sum_g a_g \rho(g) = 0$ is a finite relation, apply the zero trace condition at gh and use the cocycle law to obtain

$$\operatorname{tr}\left(\sum_g a_g \partial(\rho(g))\rho(h)\right) = -\operatorname{tr}\left(\sum_g a_g \rho(g)\partial(\rho(h))\right) = 0.$$

Full span and the trace pairing make ∂ well defined. It satisfies the Leibniz rule by the cocycle identity. For the matrix units E_{ij}, put $A = \sum_i \partial(E_{i1})E_{1i}$. Applying the rule to $E_{jk}E_{i1}$ gives $AE_{jk} - E_{jk}A = \partial(E_{jk})$. Hence $z = d_R(-A)$. □

Proof of 30.64

Proof. Let π be the cuspidal attachment. Outside the level the determinant factors through the inertia quotient. For an inertia element u there, trace translation gives $\operatorname{tr}(\rho(g)(\rho(u) - 1)) = 0$ for all g. Full span gives $\rho(u) = 1$, and local compatibility makes π spherical there. Apply weak descent 39.51 to obtain the definite-unitary representation. The base-change comparison of 26.22 and uniqueness at split places identify its split local components with those of π, including at the places in the level. The infinitesimal character and the Hodge–Tate weights give the dual algebraic weight of the ordinary module. Triviality on I_0 supplies the spherical vector at the distinguished place. At $v \mid \ell$, local compatibility identifies the local component with the irreducible principal series in the order specified by the unlinked flag. At a level deep enough for the inducing characters, the induced model has a fixed section supported on the product of the Borel and the level subgroup, with value one at the identity. Evaluation there is a nonzero common eigenfunctional for the compact diagonal and the chamber operators. On the finite-dimensional fixed space these operators commute, so the corresponding joint generalized eigenspace contains a nonzero common eigenvector. Write the ordered characters in normalized parabolic induction as $\omega_i |\cdot|_v^{(n-1)/2}$ for $0 \leq i < n$. For $1 \leq j \leq n$, the Hecke operator for the chamber with j initial uniformisers acts on this vector by $\prod_{i<j} \omega_i(\varpi_v) q_v^{-i}$. After weight normalization and transport by ι^{-1}, its eigenvalue is $\prod_{i<j}(\kappa_{v,i}\varepsilon_\ell^i)(g_v)$, where $\kappa_{v,i}$ are the ordered flag characters and g_v is a reciprocity lift of ϖ_v. These units have the prescribed reductions, so the vector belongs to the ordinary summand with the ordered diamond characters.

Lean: `UpperTheory.V.Hi.exists_ne_zero_mem_eig_uOp_eq_psEig_smul_of_equiv_ps` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.OrdA`, lines 157–245)

At each tame type place, the witness condition gives unlinked toral support (hence zero monodromy). The inertia characters are the Teichmüller lifts of the prescribed residual ones, since the representation kills the prescribed inertia kernel there and the corresponding inertia quotient has order prime to ℓ. Thus the

converse direction of the block-parahoric twisted-spherical type characterization in Lemma 30.22 gives a nonzero K_u-fixed vector in $\tau_u \otimes \pi_u$ for the local component π_u.

These vectors give a nonzero eigenspace in the arithmetic weight fibre. Its determinant agrees with D_x at the unramified Frobenii and hence everywhere by density and continuity. Its ordered characters agree by the ordinary comparison. Thus the action factors through the point x. □

Proof of 30.65

Proof. Fix an ordinary cocycle z. Write $K = \overline{\mathbf{Q}}_\ell$. In the dual numbers $K[\eta]/(\eta^2)$ put

$$\rho_z(g) = (1 + \eta z(g))\rho(g).$$

The cocycle equation is exactly the multiplication law for ρ_z. The equation involving c gives the polarisation of its determinant. At an ordinary place, conjugation by a matrix $1+\eta Y_v$ subtracts a local coboundary, with the sign determined by the choice of Y_v. Choose Y_v so that the corrected cocycle is upper triangular with zero diagonal on inertia. If its diagonal entries are $d_{vi}(g)$, the characters of the lifted flag are $(1+\eta d_{vi})\kappa_{v,i}$. Their restrictions to inertia have zero derivative. On the designated tame kernels both the lift and its specialisation have the same action.

Here are the integral choices needed to apply 30.66. The representation and the cocycle are defined over a common finite extension of $\mathbf{Q}_\ell$. Indeed their joint image in the semidirect product of the matrices with GL_n is a compact group. It is a countable union of its closed subgroups defined over finite coefficient extensions. One of these has interior by Baire's theorem, hence finite index. Adjoin the coefficients of coset representatives, the finitely many flag bases and local correcting matrices, and the roots at the determinant tests. A stable lattice exists since the image of ρ is compact.

After finite coefficient extension one can take the pairing to be perfect on a stable lattice. To see this, let D be a stable lattice and $D^\vee$ its dual for P. The multiplier is a unit and D is stable under the conjugate representation as well, so $D^\vee$ is stable. Diagonalise the form on D in orthogonal lines in the symmetric case and orthogonal planes in the alternating case, after clearing one common denominator. This is the elementary divisor algorithm for a symmetric or alternating form over a DVR with 2 invertible: a pairing of least valuation splits off a line if it occurs on the diagonal, and otherwise splits a plane; in the symmetric case a sum of its two basis vectors gives a diagonal entry of least valuation. Write ϖ^{d_i} for the elementary divisors, repeated within each alternating plane. Stability of both lattices gives

$$v(\rho(g)_{ij}) \geq \max(0, d_j - d_i).$$

Adjoin $\varpi^{1/2}$ and replace the basis vectors by $\varpi^{-d_i/2}$ times those vectors. The new entries have valuations at least $\max(0, d_j - d_i) + (d_i - d_j)/2 \geq 0$, and the form is perfect. Intersect each ordinary flag with this lattice. Consecutive quotients are torsion-free, so an integral basis adapts to each flag. Full rational span gives a nonzero multiple of the entire integral matrix algebra in the integral span, by clearing the denominators of expressions for the matrix units.

Denote the resulting coefficient DVR by O'. Multiply z and the local corrections by a common nonzero scalar a to make them integral. The residual determinant and ordered residual characters are the original ones extended to the residue field of O', since conjugation preserves the determinant and the specialised flag characters. The integral determinant and flag characters over $O'[\eta]/(\eta^2)$ thus give, by universality, an arithmetic specialisation x' and an x'-derivation δ on the determinant ring for this residual datum. For every g,

$$\delta(\tau^{\mathrm{univ}}(g)) = a\ \mathrm{tr}(z(g)\rho(g)).$$

The inertia derivatives vanish. At D_0 choose an element with nontrivial residual cyclotomic value. Its residual eigenvalues are all the same nonzero scalar. The adjoint action there has all eigenvalues one modulo the maximal ideal. Its cyclotomic multiple minus the identity is invertible, which proves (30.22). The nonzero arithmetic weight differences separate the local characters on an inertia element, and (30.21) gives the required unlinking. Determinant regularity persists over O': a rational solution of the coboundary equations after scalar extension descends by linear algebra.

Use the modules of 30.23 after this coefficient extension. The determinant and flag of x' have the original cuspidal attachment. Triviality on I_0 and (30.21) give 30.64, so x' kills the base Hecke ideal. In the

characteristic-zero classical fibre of 30.23, the good spherical operators act by scalars on each automorphic summand. By Frobenius density and continuity, all determinant coefficients belong to the same closed algebra of block scalars. Thus a power acting by zero in characteristic zero has zero eigenvalue in each eigensystem, and the operator itself acts by zero. Integral weight fibres are free, so the implication also holds integrally. These are the level and arithmetic control hypotheses of 30.66. It follows that $\delta(\tau^{\mathrm{univ}}(g)) = 0$ for every g. Cancel a and apply 30.63 with the full matrix span in (30.21). □

Proof of 30.66

Proof. Use 30.67 at the specified s_j. Every element of its subgroup H has residual characteristic polynomial $(X-1)^n$. The finite residual image of H is an ℓ-group: each of its elements is unipotent and has ℓ-power order. The congruence kernel on an integral lattice is pro-ℓ, so $\rho_x|_H$ factors through the maximal pro-ℓ quotient. This gives hypothesis 3 of 30.70. The separated flags, distinguished-place condition and determinant regularity in the present theorem are the hypotheses of 30.68. Its last inequality gives hypothesis 4 of 30.70.

It remains to construct the operator in hypothesis 2 of 30.70. Write $I = \ker(x|_\Lambda)$, $K = \operatorname{Frac} O$, $V = (M/IM) \otimes_O K$. Let A be the image of $B[1/\ell]$ in $\operatorname{End}_K(V)$. The character x occurs in the support of V: localise M at $\ker x$, use $J \subseteq \ker x$ and $\operatorname{Supp} M = V(J)$, and apply Nakayama to I. Thus x defines a factor of the finite algebra A with residue field K.

The subalgebra generated by the determinant coefficients in A is reduced. For a nilpotent element in it, clear denominators to obtain an integral polynomial t in these coefficients. Then t^k kills V for some k, and kills M/IM because that module is free over O. Arithmetic control at x gives $tM \subseteq IM$. A finite reduced commutative algebra over K is a product of separable fields, so every K-valued derivation at x kills this subalgebra. By the second assertion of 30.69, the pullback of such a derivation to B also kills the ordered characters, since its target is a characteristic-zero field. The coefficients and characters generate the image algebra. Indeed they generate B adically, the action on the weight fibre is continuous, and their K-algebra in the finite-dimensional endomorphism space is a closed vector subspace. Hence the derivation is zero.

Let $\mathfrak{n}$ be the maximal ideal of the local factor at x. Its cotangent dual is the space of these derivations. Nakayama applied to $\mathfrak{n}/\mathfrak{n}^2 = 0$ gives $\mathfrak{n} = 0$. The idempotent of the factor therefore kills the image of $\ker x$ on V and takes value one at x. Lift a nonzero O-multiple of this idempotent to $t_0 \in B$. Freeness of M/IM gives $t_0(\ker x)M \subseteq IM$, and $x(t_0) \neq 0$.

The base and auxiliary modules give hypothesis 1 of 30.70 by the level assumptions. Each weight parameter is a fixed unit times a prescribed inertia-character value minus one, so its derivative vanishes. Lemma 30.70 now gives (30.23). □

Proof of 30.67

Proof. Residual conjugate duality permutes the kernels of the residual characters under c. Thus the displayed subgroup H is open and normal in Γ. Write $P = H/N_\ell(H)$. The given finiteness of $\operatorname{Hom}_{\mathrm{cont}}(H, \mathbf{F}_\ell)$ makes P a finitely generated pro-ℓ group. For $r \geq 1$ intersect the kernels of its continuous maps to groups of order at most ℓ^r. There are finitely many such kernels: there are finitely many groups of bounded order, and images of a finite generating set determine a map. Their intersection is open and characteristic. Its inverse image W_r in H is open normal in Γ, and the W_r are cofinal among the neighbourhoods of $N_\ell(H)$ in H.

Let $p : G_{L^+} \to \Gamma$ be the level map. Lift s_j to $g_j \in G_L$. The cyclotomic value of g_j is one because the cyclotomic character factors through p. Use the finite Galois quotient cut out by

$$U_r = p^{-1}(W_r) \cap \ker(\varepsilon_\ell \bmod \ell^r).$$

By Chebotarev [38, §1A], choose, successively for the finitely many j, a prime $v_{r,j}$ outside S, distinct from the previous choices, with Frobenius conjugate to g_j^{-1} in this quotient. Choose its prolongation to realise that element of the conjugacy class. Its arithmetic Frobenius $\sigma_{r,j}$ satisfies

$$\sigma_{r,j} g_j \in U_r. \tag{A.282}$$

Since $U_r \subset G_L$, the prime splits in L. For the place $w_{r,j}$ given by the prolongation, the cyclotomic value of arithmetic Frobenius is $\mathrm{N}w_{r,j}$, giving (30.24). Taking inverses in the quotient by W_r shows that the difference for inverse arithmetic Frobenius lies in W_r. This proves both assertions about $d_{r,j}$.

Allow ramification at the new places to obtain h_r, and put $H_r = h_r^{-1}H$. Its kernel over the old level is normally generated as a closed subgroup by the new inertia groups, by the definition of the unramified quotient. Each new inertia group lies in H_r. In the maximal pro-ℓ quotient of H_r its wild inertia and its prime-to-ℓ tame inertia vanish. Choose $\tau_{r,j}$ lifting a generator of the pro-ℓ tame inertia. Arithmetic Frobenius conjugates this generator to its $\mathrm{N}w_{r,j}$-th power. With $\phi_{r,j}$ the inverse arithmetic Frobenius this is (30.25). We obtain $h_r(\tau_{r,j}) = 1$ and the asserted normal generation modulo $N_\ell(H_r)$.

Finally a continuous character $H_r \to \mathbf{F}_\ell$ kills $N_\ell(H_r)$ and is invariant under conjugation within H_r. Its restriction to $\ker h_r$ is determined by its values on the conjugates of the q_0 elements $\tau_{r,j}$ by representatives of Γ_r/H_r. The kernel of this restriction consists of characters inflated from H. Since $[\Gamma_r : H_r] = [\Gamma : H]$, the number of characters is at most $\#\operatorname{Hom}_{\mathrm{cont}}(H, \mathbf{F}_\ell)\ell^{q_0[\Gamma:H]}$, uniformly in r. □

Proof of 30.68

Proof. Write $M_m = M_n(A_m)$. Extend an allowed cocycle to the full group by assigning zero at c. The polarisation identity in (30.18) is the compatibility needed for this extension. A coboundary has trace derivative zero, since $\operatorname{tr}(\rho(g)X - X\rho(g)) = 0$. It suffices to bound the finite Selmer group for M_m. Under the trace pairing its Cartier dual has the cyclotomic-twisted adjoint action. For the additive dual one can compose the matrix trace with a perfect additive pairing on $O/\mathfrak{m}_O^m$, obtained from the inverse different of $O/\mathbf{Z}_\ell$.

Use finite Poitou–Tate duality 27.2 with the allowed local subgroups $\mathcal{L}_v$ and their orthogonal complements. Taking orders in its exact sequence and using the local and global Euler characteristic formulas [127, Theorems 7.3.1 and 8.7.4] gives the ratio of the primal to the dual Selmer order. The factors are the global ratio of invariant orders, the factors $|\mathcal{L}_v|/|H^0(L_v^+, M_m)|$ at finite places, and the reciprocals of the invariant orders at the real places. The modules are finite of odd prime-power order, the ramification set contains the primes above ℓ, and outside it we use the unramified condition.

We use the following elementary calculation for the invariant factors. For equations on a free O-module of rank a, take finitely many rows generating the row module. If their common kernel has dimension h over $\operatorname{Frac} O$, and the nonzero Smith entries have valuations b_i, the reduced kernel has order $Q^{mh+\sum_i \min(m,b_i)}$. For global invariants $h = 0$: full matrix span leaves the scalars on the CM subgroup, and c acts by minus one on scalars. This bounds the numerator of the global ratio.

At $v \mid \ell$ let T_v be the upper triangular matrices in the flag lattice. Its rank is $n + b$. In its filtration by matrix entries the characters are the trivial characters and $\kappa_{v,i}/\kappa_{v,j}$ with $i < j$. None equals the cyclotomic character, by unlinking and the infinite cyclotomic image on the local group. Thus the rational twisted dual has no invariants, and its reduced invariants have bounded order by the calculation above. Local duality and the Euler characteristic formula bound the order of the cocycles of $T_v/\mathfrak{m}_O^m T_v$ by $Q^{m(n+b)(1+d_v)+C_v}$, with $d_v = [L_v^+ : \mathbf{Q}_\ell]$. Dividing by triangular coboundaries and by the full invariant order costs a factor at least $Q^{m(n+b)}$, since triangular invariants inject into full invariants. Requiring zero diagonal on μ_v can only decrease the numerator.

At a real place the fixed subspace of negative transpose for the polarising form has rank at least b. It is a direct summand integrally because 2 is invertible. At a tame type place the prescribed inertia quotient has order prime to ℓ. The unramified quotient on its inertia invariants has first cohomology the cokernel of Frobenius minus one, of the same order as its kernel. At the distinguished place the full-condition ratio is the order of twisted dual invariants, bounded by (30.22) and the Smith calculation. At an auxiliary place, for fixed m and large r, the twisted Frobenius on M_m is conjugation by $\rho_x(s_j)$ or its inverse. Indeed $\rho_x|_H$ has pro-ℓ image, and 30.67 gives agreement modulo $\mathfrak{m}_O^m$ and cyclotomic value one modulo that ideal. The centralizer of a matrix with n distinct roots has rational dimension n. Consequently the bounds on the local factors are

$$\frac{|\mathcal{L}_v|}{|H^0(L_v^+, M_m)|} \le \begin{cases} Q^{m(n+b)d_v + C_v} & v \mid \ell, \\ Q^{-mb} & v \text{ real}, \\ 1 & v \text{ of tame type}, \\ Q^{C_0} & v \text{ distinguished}, \\ Q^{mn + C_j} & v = v_{r,j}. \end{cases} \tag{A.283}$$

At a real place the numerator denotes the order one.

It remains to bound the dual Selmer order uniformly. Subtract a coboundary to make a dual cocycle zero at c. At each new place the orthogonal condition to the full condition is zero, so the cocycle is a local coboundary. It vanishes on new inertia, whose action is trivial. On $H_r = h_r^{-1}H$ the action and the additive cocycle together map into a finite ℓ-group. They kill $N_\ell(H_r)$. The kernel description of 30.67 now gives descent of the cocycle to Γ. At a fixed m there are finitely many descended cocycles: their restrictions to H are determined on finitely many pro-ℓ generators, and finitely many cosets remain. For all sufficiently large r, 30.67 therefore identifies the cocycle and action values at $h_r(\phi_{r,j})$ and s_j simultaneously for all of them. Being a coboundary at the former makes every trace test at the latter vanish, by cyclic invariance of the trace and $\varepsilon_\ell(s_j) = 1$.

The bound on reductions satisfying these tests is uniform in m. Record values on pro-ℓ generators and coset representatives in a finite free O-module. The cocycle law on relations, the value zero at c and the trace tests are linear equations in these coordinates. To obtain equations valid at every depth, start with a free pro-ℓ presentation of $H/N_\ell(H)$. Values on its generators extend to the free group by the cocycle law at every depth: the action together with the values has finite ℓ-group image. Evaluation on each relator is an O-linear row, obtained by inverse limit of the evaluations at finite depth. Impose also the coset multiplication and conjugation relations. The row span is finitely generated over O. Over the fraction field its kernel is precisely the space of normalised coboundaries by determinant regularity. A rational cobounding matrix can be taken fixed by the twisted action of c, since the cocycle is zero there. Apply Smith reduction first to the equations and then to the coboundary map on this invariant lattice. The first calculation bounds the extra reduced kernel modulo reductions of the integral kernel; the second bounds the index of the integral coboundaries in that kernel. Both bounds depend only on the two integral matrices. Hence the number of dual classes is bounded independently of m.

Multiply (A.283) and the global bounds. The ℓ-adic degrees sum to d, there are d real places, and the q_0 auxiliary factors contribute $mnq_0 + O(1)$ to the exponent. This gives $m((n+b)d - bd + nq_0) + O(1) = m(D + nq_0) + O(1)$, equivalently (30.26). □

Proof of 30.69

Proof. Let τ be the determinant trace. The Cayley–Hamilton algebra of the split multiplicity-free residual determinant has orthogonal idempotents e_i with scalar diagonal corners and trace one [126, Theorem 2.22]. The specialised representation factors through it by full span and the trace pairing, and the idempotents split the specialised lattice into rank-one summands. Work in that basis. Take a fixed $c_0 \neq 0$ with $c_0 M_n(O)$ in the specialised group-algebra image. By surjectivity of the level map the same c_0 works at every level. Lift c_0E_{i1} and c_0E_{1i} to U_i and V_i in the corresponding corners. Write $U_iV_i = \beta_i e_i$, multiply U_i by $\prod_{j\neq i}\beta_j$, and put $\beta = \prod_j \beta_j$. Now $\sum_i U_iV_i = \beta$. For $S(a)_{ij} = \tau(V_iaU_j)$ the scalar corner identities give

$$S(a)S(a') = \beta S(aa'), \qquad \operatorname{tr} S(a) = \beta\tau(a), \qquad x_R(S(a)) = x_R(\beta)\rho_x(a).$$

We use ρ_x also for the pullback to the new level and its O-linear extension to the group algebra. Put $b_0 = x_R(\beta) = c_0^{2n}$. Differentiation gives the matrix product derivative

$$d(a) = b_0\big(\delta S(a) - \delta(\beta)\rho_x(a)\big), \qquad \operatorname{tr} d(a) = b_0^2\delta\tau(a).$$

Thus $d(g)\rho_x(g)^{-1}$ is an adjoint cocycle. Average it with its polarised conjugate under c to obtain (30.18). The trace derivative is preserved because the determinant satisfies the polarisation identity. Use d for the resulting matrix derivative as well. For target A_m it is locally constant on the group because its entries come from continuous trace functions and the specialised matrices, and the Leibniz rule makes δ kill a sufficiently high power of the maximal ideal of R.

For a prescribed inertia relation $a = g - 1$, or an ordered relation $a = \prod_j(g_j - \psi_{v,j}(g_j))$, the ordinary pair has $\tau(ha) = 0$ for every h in the CM group. Specialising first gives $\rho_x(a) = 0$ by full span. Differentiating gives $\operatorname{tr}(\rho_x(h)d(a)) = 0$, so $c_0 d(a) = 0$ by the integral span and matrix units. The equality of inertia kernels gives the required tame condition. The ordered relation holds to first order for the matrix lift with derivative $c_0 d$ and character derivatives $c_0 b_0^2 \delta\psi_{v,i}$.

Lean: `UpperTheory.V.Th.Coc.gma` (`Development.AutomorphicArithmetic.AutomorphicVarieties.TheoremsMuLinked.DualModuleCocycles.LiftSurj`, lines 33–92)

Lean: UpperTheory.V.Th.Coc.liftOfS (Development.AutomorphicArithmetic.AutomorphicVarieties.TheoremsMuLinked.DualModuleCocycles.LiftOfS, lines 4–37)

For the local calculation use a flag basis and take $s \in D_v$ with distinct values $\lambda_i = \kappa_{v,i}(s)$. Diagonalise $\rho_x(s)$ by an upper triangular change of basis over K. Rescaling the eigenvectors by successive powers of a uniformiser makes the specialised matrices integral in this basis. Clear the fixed denominators of the change of basis and its inverse on the derivative. Multiply further by the product of the nonzero $\lambda_i - \lambda_j$, $i < j$, and subtract a commutator to make the derivative at s diagonal. For entry (i, j) of this commutator one uses the off-diagonal derivative at s multiplied by the clearing factor divided by $\lambda_i - \lambda_j$. All these operations on the derivative use coefficients in O.

In the differentiated ordered relation set $g_i = g$ and $g_j = s$ for $j \neq i$. A term differentiating an s-factor is upper triangular. Its (i, i) entry vanishes because the g-factor has (i, i) entry zero. For a lower entry (a, b) choose $i = b$. The coefficient of the derivative of the g-factor is $\prod_{j<i}(\lambda_a - \lambda_j)\prod_{j>i}(\lambda_b - \lambda_j)$, which is nonzero. For diagonal entry (i, i) the corresponding coefficient is $\prod_{j\neq i}(\lambda_i - \lambda_j)$. Multiplying by a common multiple of these products kills the lower entries and equates each diagonal entry of the matrix derivative to the correspondingly scaled $\delta\psi_{v,i}(g)$. Transfer back to the integral flag basis with fixed clearing factors. Right multiplication by $\rho_x(g)^{-1}$ preserves upper triangularity and divides the i-th diagonal entry by $\kappa_{v,i}(g)$. On μ_v the universal character is prescribed, so this derivative vanishes. The product of the fixed clearing factors over all v, including $c_0 b_0^2$, gives c_1.

Lean: UpperTheory.V.Th.Coc.locAt (Development.AutomorphicArithmetic.AutomorphicVarieties.TheoremsMuLinked.DualModuleCocycles.BarOne, lines 35–106)

For the character comparison at v, write $\kappa_i = \kappa_{v,i}$ and assume all trace derivatives vanish. Multiply the ordered relation by s^a on the left and differentiate its trace. The trace terms have zero derivative and the remaining coefficient terms give

$$\sum_k \lambda_k^a \prod_{j\neq k}(\kappa_k(g_j) - \kappa_j(g_j))\delta(\psi_{v,k}(g_k)) = 0.$$

Combine these equations with the coefficients of $\prod_{k\neq i}(X - \lambda_k)$. Taking $g_i = g$ and $g_j = s$ for $j \neq i$ gives

$$\prod_{j\neq i}(\lambda_i - \lambda_j)^2\delta(\psi_{v,i}(g)) = 0.$$

The product of these nonzero coefficients over v, i gives c_2. □

Proof of 30.70

Proof. Put $D_* = D + nq_0$, $Q = |\mathbf{k}|$, and suppose $d_0 = \delta(\tau^{\mathrm{univ}}(g_0)) \neq 0$. We construct $D_* + 1$ directions at depth m. Their common loss of precision will be independent of m.

At level r the coefficient character is $x\beta$. By 30.67 and hypothesis 3, the specialised Frobenius polynomials tend coefficientwise to the test polynomials. Each limiting polynomial has distinct roots α_{ji} in O. Apply Hensel's lemma around each root, with the coefficient errors of valuation greater than twice the valuation of the derivative there. For all large r this gives roots $\alpha'_{ji} \in O$ tending to α_{ji}. Let f_r be the value of the comparison operator f at these ordered roots. It tends to the nonzero value F_* obtained with roots α_{ji} and all $Q_j = 1$. In particular $v(f_r) = v(F_*)$ eventually.

Choose an x-eigenfunctional $\lambda : M \to O$ and $v_0 \in M$ with $\lambda(v_0) \neq 0$. Such a functional comes from the nonzero fibre at $\ker x$, by taking a linear functional over $\operatorname{Frac} O$ and clearing denominators on a finite set of Λ-generators. Evaluate the ordered splitting algebra at α'. The map from M_w to the splitting-algebra module pulls λ back to a functional $L : M_1 \to O$. Denote its character on the commuting operator algebra by x_T. Its diamond values are one and its root values are α'_{ji}. Lift to $v \in M_1$ the image of $1 \otimes v_0$ under the reverse comparison. The composite relation gives $L(v) = f_r\lambda(v_0)$.

The coefficient map β is onto: its compact image contains the adic generators of B by compatibility of the pair and surjectivity of the level map. Lift t_0 of hypothesis 2 to t_1. In the operator algebra put

$$P_0 = \prod_{j,i}\prod_{a\neq i}(u_{ji} - \alpha'_{ja}), \qquad t = t_1 P_0 f^{E+1}.$$

Write $S = \Lambda[\Delta]$ and $I = \ker(x_T|_S)$. For every operator b in the algebra generated by coefficients, roots and diamonds we have

$$x_T(b) = 0 \quad \Longrightarrow \quad tbM_1 \subseteq IM_1. \tag{A.284}$$

Indeed pass to diamond coinvariants and apply the comparison to the splitting-algebra module. Modulo $\ker x$ the polynomial of the j-th Frobenius is $\prod_a (X - \alpha'_{ja})$. Substitution of u_{ji} shows that $(u_{ji} - \alpha'_{ji})P_0 = 0$ there. Thus multiplication by P_0 replaces every root by its chosen value modulo the extended $\ker x$. Every diamond already acts as one. It follows that $P_0 b$ has image in the extended $\ker x$ times that module. Multiplication by t_0 carries it into the weight ideal by hypothesis 2 and scalar extension. Comparing back introduces a factor f. The kernel of $M_1 \to M_w$ lies in IM_1, which proves (A.284).

Take $r \geq m$ large enough for the preceding constructions. For a chosen weight or diamond coordinate take its coordinate derivation $d_S : S \to A_m$ at $x_T|_S$, with value one on that coordinate. A diamond relation has derivative $\ell^{a_j} d_S(y_{ji}) = 0$ since $a_j \geq r$. Give $O \oplus \eta A_m$, $\eta^2 = 0$, the S-action $x_T + \eta d_S$. Projectivity of M_1 over S lifts L to this module. Its second component H_1 satisfies

$$H_1(sw) = x_T(s)H_1(w) + d_S(s)L(w), \qquad H_1(I^2 M_1) = 0.$$

Set $\partial(b) = H_1((b - x_T(b))t^2 v)$. The product of two centred operators times t^2 maps M_1 into $I^2 M_1$ by two applications of (A.284). Expanding $bb' - x_T(bb')$ now gives the Leibniz rule for ∂. On S it equals $x_T(t)^2 f_r \lambda(v_0) d_S$.

The trace in brackets in (30.27) has value zero at x_T, and multiplication by t annihilates it. Thus ∂ of this trace is zero. In the part involving $\tau_{r,j} - 1$, differentiating a polynomial coefficient gives zero because $\tau_{r,j}$ specialises to the identity. In the part involving y_{ji}, the coefficient of $\partial(y_{ji})$ is p_{ji}^{E+1}, where $p_{ji} = \prod_{a \neq i} (\alpha'_{ji} - \alpha'_{ja})$. Indeed the polynomial has value zero on the other eigenlines and value p_{ji}^{E+1} on line i. We therefore obtain the same coordinate responses on the weights and the coefficient-ring tests

$$T_{ji} = \operatorname{tr}_{u_r} \left[\left(\prod_{a \neq i} (\phi_{r,j} - \alpha'_{ja}) \right)^{E+1} (\tau_{r,j} - 1) \right], \tag{A.285}$$

with the diamond response multiplied by p_{ji}^{E+1}. Both $P_0(\alpha')$ and each p_{ji} divide f_r. The weight response divides $x(t_0)^2 \lambda(v_0) f_r^{2E+5}$, and every diamond response after the extra factor divides $x(t_0)^2 \lambda(v_0) f_r^{3E+6}$. Since $v(f_r) = v(F_*)$, the fixed nonzero scalar

$$c_3 = x(t_0)^2 \lambda(v_0) F_*^{3E+6}$$

is divisible by each response in O, before reduction at depth m. Scale the coordinate derivations by these integral quotients and restrict to R. The resulting D_* derivations e_a have response matrix $c_3 \operatorname{Id}$.

Adjoin $e_* = \delta\beta \bmod \mathfrak{m}_O^m$. It kills every weight by assumption and every T_{ji} because $h_r(\tau_{r,j}) = 1$. Denote the resulting index set by $\mathcal{I}$. Use the constants $c_1, c_2 \neq 0$ of 30.69. Compatibility of the auxiliary pair gives the prescribed specialised representation and characters, and hypothesis 1 identifies the tame kernels. Hence the map

$$\Phi : O^{\mathcal{I}} \longrightarrow \mathcal{S}^{\mathrm{tr}}_{r,m}, \qquad \Phi(b)(g) = c_1 \sum_{a \in \mathcal{I}} b_a e_a(\operatorname{tr}_{u_r}(g)) \tag{A.286}$$

takes values in the stated trace set. If b is in its kernel, $c_1 \sum b_a e_a$ kills traces and therefore all T_{ji}, by trace linearity. After multiplication by c_2 it also kills the weight characters. The response matrix gives $c_1 c_2 c_3 b_a = 0$ in A_m for every $a \neq *$. Evaluate $\Phi(b)$ at a lift of g_0 and multiply by $c_2 c_3$. The remaining equation is $c_1 c_2 c_3 d_0 b_* = 0$. With $c_* = c_1 c_2 c_3 d_0$ this proves

$$b \in \ker \Phi \quad \Longrightarrow \quad c_* b_a \in \mathfrak{m}_O^m \quad (a \in \mathcal{I}). \tag{A.287}$$

Let $v'_0 = v_O(c_*)$, take $h = v'_0 D_* + C + 1$ and set $m = h + v'_0$. Now choose r sufficiently large for this m, including the bound (30.28). Representatives modulo $\mathfrak{m}_O^h$ in each coordinate have distinct images by (A.287). We obtain $Q^{h(D_*+1)} \leq Q^{mD_*+C}$, whereas $h(D_*+1) - (mD_*+C) = 1$. This contradiction proves the assertion. $\square$

Proof of 30.71

Proof. Each A_j is finite over A_0. In monomials in a fixed finite set of homogeneous generators consider the exponent vectors of weight j. They have only finitely many coordinatewise minimal elements, and division by a minimal monomial leaves weight zero. Choose finite generating lists of A_{-2} and A_2. Every two-sided point has $fg \neq 0$ for some pair from these lists, since a surviving generator of weight 1 or -1 has a surviving square. Rescale the point so that $f = 1$. Over K the slice $f = 1$, $g \neq 0$ on A/I is of finite type by hypothesis (2). The orbit tangent has f-coordinate -2. Hypothesis (4) therefore gives zero tangent space to the slice at every closed point. Each such local ring is a field by Nakayama. A finite type zero-dimensional K-algebra is Artinian, so each slice is finite and reduced. The union of the slices over the finite generating pairs gives the required finite orbit cover T.

The ideal $\mathfrak{a}$ is finitely generated since A is Noetherian, and $\mathfrak{a}/\mathfrak{a}^2$ is finite over $A/\mathfrak{a} = O$. The saturated weight--1 cotangent lattice C_{-1} is free of rank r by the linear term condition in hypothesis (1). Smith reduction on the $r + 1$ columns represented by F_i gives a basis with representatives $y_i \in A_{-1}$ and an invertible change among the F_i satisfying (30.29). The F_i span rationally by hypothesis (1), hence all b_i are finite and nonnegative, which proves (30.30). The change preserves I.

The m images of the Z_i have independent linear terms spanning the weight-zero variables. Use these images as coordinates by formal inversion and set them equal to zero to obtain ϕ. If $\phi(a) = 0$, add to a the product of an F_i with nonzero linear term and a weight-one element with nonzero linear term. Both exist by hypothesis (1) and $r \geq 1$. The product has nonzero image in the power series domain, and the new a agrees with the old one on A/I.

Finally, let (s, L) be admissible on N. The ring R is Noetherian, since it is generated over A_0 by finitely many elements, so N/sN and $0 :_N s$ are finitely generated graded R-modules. Both are annihilated by s and by $\operatorname{Ann}_R N$, so L is admissible on both. The graded pieces of R, and hence of every finitely generated graded R-module, are finite over A_0, because each A_j is. Let H be a Koszul homology module of L on one of these two modules. It is finitely generated, say by homogeneous elements of multidegrees at most e coordinatewise, and $(L)H = 0$, so $(\mathfrak{m}_{A_0} R\, R_{++})^k H = 0$ for some k. Since R is generated in unit multidegrees, $R_c \subseteq R_{++}^k$ whenever every coordinate of c is at least k. Hence, for $d \geq e + (k, k, k)$ coordinatewise, $H_d \subseteq R_{++}^k H$ and $\mathfrak{m}_{A_0}^k H_d = 0$. Thus H_d is a finite A_0-module annihilated by $\mathfrak{m}_{A_0}^k$, so its length is finite, and it equals the O-length because A_0 has the residue field of O. Tensoring the Koszul complex of L with the exact sequences relating multiplication by s on N, its kernel $0 :_N s$, and its cokernel N/sN gives the two-term complex on s and proves (30.31). □

Proof of 30.72

Proof. First consider the framed presentation. In an Artinian test algebra the congruence subgroup of the matrix group is a finite ℓ-group. On the open kernel of the residual representation one may therefore use generators of its maximal pro-ℓ quotient. There are finitely many by the hypothesis on $\mathbf{F}_\ell$-characters. Adjoin entries for lifts of these generators and for a set of coset representatives. Impose the group relations, the form equations and the local equations, and adjoin entries for the frame changes with their intertwining equations. Use the closed ideal of these relations in the power series ring over B. At every finite depth the equations give precisely the allowed continuous representations with their frames: on a free pro-ℓ group generator images in a finite ℓ-group determine a continuous map, and imposing the relators makes it descend. Taking inverse limits therefore gives the universal complete ring U.

Over $\mathbf{k}[\eta]/(\eta^2)$, with B acting through $\mathbf{k}$, write a lift as $(1+\eta z)\bar{r}$ and the v-frame as $1-\eta b_v$. Multiplication requires $dz = 0$, the form equation places z in the Lie algebra of isometries, and the local equations are $z\alpha_v = \delta b_v$, $b_{v_0} = 0$, $z\beta_q|_{N_q} = 0$. In weight j these are exactly the pairs defining $H^1_{\mathrm{fr},j}$. The entries of the universal lift and the frame changes separate these tangent vectors. Select a basis of the dual in each weight from their entry evaluations. The corresponding entries give a minimal relative presentation $P = B[[X_1, \dots, X_e]] \twoheadrightarrow U$ by complete Nakayama. Conjugation by the diagonal weight torus acts on an ab-entry by the character $w_a - w_b$. The equations are stable under this action, so their closed ideal is homogeneous: at each finite power series depth take the coefficients of the independent torus characters and then pass to the inverse limit.

Write $J_0 = \ker(P \to U)$. A weight-j functional on $J_0/\mathfrak{m}_P J_0$ gives a small extension of U by $\mathbf{k}$ of that weight, by pushing out $P/\mathfrak{m}_P J_0 \to U$. Choose lifts of the universal matrices across it. For an isometry lift

with form defect D in the square-zero kernel, right multiplication by $1-\frac{1}{2}P_f^{-1}D$, where P_f is the form matrix, removes the defect. This works for both types of form and uses odd residual characteristic. Perform the calculation on the isometry factor relative to r_0. The multiplication defects give a continuous two-cocycle c, by associativity. Comparing with the prescribed framed lifts at v and with local solutions at q gives one-cochains u_v, u_q. Such local solutions exist by the small-extension hypothesis. Their differentials are the restrictions of c. These cochains can be chosen continuously by lifting on finite congruence quotients with discrete kernel and using the fixed lifts on the other coordinates.

Changing the choices by a global cochain u, local frame corrections b_v, and local solution differences e_q changes the triple by the boundary in the statement. At q precisely the cocycles vanishing on N_q preserve the local solution. Thus we obtain a class in Ob_j. The construction is linear in the functional, since one may compute the defects before pushing out and apply the functional to their entries. If the class vanishes, the corrections give a global representation with all the required local conditions. Simultaneous conjugation restores the identity frame at v_0. Universality gives a section over B of the small extension. Composing this section with $P \to U$ and comparing with the quotient map from P to the extension gives two maps equal on B, differing on the variables by elements of the small kernel. Their difference vanishes on $\mathfrak{m}_B P + \mathfrak{m}_P^2$, which contains J_0 by relative minimality. On J_0 their difference is the original functional, up to sign. It is therefore zero. This injects the dual minimal relation space into Ob_j. Choose homogeneous lifts of minimal relation generators and pad the lists with zeros to obtain (30.32).

For (30.33), an integral x-derivation kills $\mathfrak{a}^2$ by the product rule and kills S_x since O is torsion-free. Its reduction is determined in weight j, and a reduction killing all F_i factors there through $C_j(F)$. Thus the given additive injection gives an injection into $\mathrm{Hom}_O(C_j(F), O/(c))$. If $C_j(F)$ has finite length, a cyclic summand $O/(\pi^u)$ contributes $\min(u, v_O(c))$ to the Hom length. Taking cardinalities proves (30.33). The inequality also holds when its right hand side is infinite.

For (30.34), apply Lemma 30.71 to obtain a finite two-sided orbit cover, the finite length of $C_{-1}(F)$ in (30.30), the chart ϕ, the replacement of a modulo I with $\phi(a) \neq 0$, the graded algebra R, the module $M = R/J$, and the section list $\mathbf{t}$. Replacing a modulo I preserves $\xi(a)$ at every point ξ of A/I, so we use the modified a for any given finite two-sided orbit cover T. Lemma 30.74 gives admissibility of $(\mathbf{t}, a)$ on M, so the hypothesis of Lemma 30.73 holds on M/aM. For a multidegree d large enough in all three coordinates the two lemmas give

$$2^{r+r'} \sum_i b_i \le \chi_{\mathbf{t}}(M/aM)_d \le 2^{r+1+r'} \sum_{\xi \in T} v(\xi(a)).$$

Division by $2^{r+r'}$ and (30.30) prove (30.34). □

Proof of 30.73

Proof. For a monomial in the chart after the weight-zero variables have been set to zero, let u be the sum of the absolute weights of its negative variables, counted with multiplicity, and v the corresponding sum for its positive variables. Its weight is $v - u$. In an element of A_{-2} the least possible u is 2. In $\phi(F_{r+1})$ it is at least 2: a term of weight -1 with $u = 1$ would be linear, and (30.29) kills the linear term. Thus in the third coefficient module the least possible u is 4. Its terms at $u = 4$ have $v = 2$. In degree $d = (d_0, d_1, d_2)$ the least possible pair is consequently

$$u(d) = 2d_0 + 4d_2, \qquad v(d) = 2d_1 + 2d_2.$$

Taking the terms with this pair in each multidegree is multiplicative, since the two bounds are additive. Write P for the image. This quotient kills J and a: a chart term of a has weight zero and zero constant term, hence positive u. We obtain $M/aM \twoheadrightarrow P$.

In each degree P_d is a finite O-module since R_d is finite over A_0 and the action of A_0 on leading terms is through x. It embeds into a space of polynomials over $\mathrm{Frac}\, O$, so is free. We compute its rank. Over the fraction field the cotangent hypothesis lifts any individual variable by a homogeneous element with that linear term. For a variable of absolute weight one the sole terms of the same sign and least absolute count are linear. For absolute weight two the additional terms of least count are products of two variables of absolute weight one and the same sign. Subtract products of their lifts to isolate the chosen weight-two variable. Clearing denominators gives nonzero scalar multiples integrally. This supplies every monomial

block of absolute weight two on either side. Every monomial of even absolute weight on one side is a product of such blocks: keep each weight-two variable and pair the weight-one variables. For multidegree d, allocate d_0 negative blocks and d_1 positive blocks to the first two directions. Allocate $2d_2$ negative blocks and d_2 positive blocks to the third, using $A_{-2}^2 A_2$. Hence P_d spans the whole space of bidegree $(u(d), v(d))$. With formal variables X, Y this gives

$$\operatorname{length}_O(P_d/\pi P_d) = [X^{u(d)}Y^{v(d)}]H, \qquad H = \frac{1}{(1-X)^r(1-X^2)^s(1-Y)^{r'+1}(1-Y^2)^{s'+1}}. \tag{A.288}$$

On P the section $F_i^2 T_0$ acts as $\pi^{2b_i} y_i^2 T_0$ for $i \le r$, since the congruence in (30.29) identifies the linear terms in the rational chart. Remove these scalar factors one at a time. For a section z of degree e and the list L of other sections, factor multiplication by $\pi^h z$ into multiplication by z followed by multiplication by π^h. The exact triangle of the three mapping cones, tensored with the Koszul complex of L, gives

$$\chi_{(\pi^h z, L)}(P)_d = \chi_{(z,L)}(P)_d + h\,\chi_L(P/\pi P)_d.$$

Here multiplication by π is injective, and $P/\pi^h P$ has h successive quotients $P/\pi P$. The replacement enlarges the section ideal and preserves admissibility. On $P/\pi P$, the A_0-action factors through its residue field, so each complex term has finite length.

When one first-direction section is removed, the other multidegrees consist of $r - 1 + s$ first, $r' + s'$ second, and one third direction. The alternating sum of lengths of complex terms is therefore the coefficient of $X^{u(d)}Y^{v(d)}$ in

$$H(1-X^2)^{r-1+s}(1-Y^2)^{r'+s'}(1-X^4Y^2).$$

This expression equals

$$(1+X)^{r-1}(1+Y)^{r'}\left(\frac{1}{(1-X)(1-Y)} + \frac{Y^2(1+X+X^2+X^3)}{(1-Y)(1-Y^2)}\right).$$

The second term has bounded X-degree. In the first term the coefficient for large $u(d), v(d)$ is the product of the sums of the two numerator coefficient lists, namely $2^{r-1+r'}$. All shifts used in the Koszul complex have nonnegative remaining multidegrees when d is large enough. Removing the scalar for i consequently contributes $2b_i\,2^{r-1+r'}$.

By the nonnegativity assertion of 30.74, the characteristic on P after all removals is eventually nonnegative. The same holds for the original list on the kernel of $M/aM \to P$, since admissibility passes to submodules. Additivity in that exact sequence gives (30.35). □

Proof of 30.74

Proof. Put $B = A_0$, $L = I \cap B$, $D = A_{-2}A_2 \subset B$, and $W_d = (M/\mathbf{t}M)_d$. We first locate the primes over $J + (\mathbf{t})$ which avoid R_{++}. Let P be such a prime. Choose $fT_0, gT_1, hT_2 \notin P$ in the three generating modules. Since

$$(F_{r+1}^2 T_0)(hT_2) = (hT_0)(F_{r+1}^2 T_2),$$

we have $F_{r+1}^2 T_0 \in P$. If cG_i has weight zero, multiplication by fT_0 expresses its product as $cf \cdot G_i T_0$, so $cG_i \in P$. Use the same calculation on $(cF_i)^2$ when cF_i has weight zero. Use gT_1 on the positive equations. Finally $Z_i \in J$, since the chart sets these variables to zero. Decomposing coefficients into homogeneous parts in an expression for an element of $I \cap B$ proves $L \subseteq P \cap B$.

We claim

$$D \subseteq P \quad \Longrightarrow \quad P \cap B = \mathfrak{m}_B.$$

Suppose $D \subseteq P$ and $\pi \notin P$. Since B/L is finite over O, an embedding of $\operatorname{Frac}(B/(P \cap B))$ into K gives a point σ of B. We construct points of both strict signs over it. A monomial of weight $-2n$ contains at least n negative blocks of weight -2, formed from weight--2 generators or pairs of weight--1 generators. The remaining factors can be taken to have weight zero after extracting total negative weight $2n$. Thus $A_{-2n} = A_{-2}^n$: equivalently one extracts n blocks and absorbs the weight-zero remainder into a block for $n > 0$. The positive version holds as well.

On the even-weight subring send a homogeneous $u \in A_{-2n}$ to uT_0^n mod P in $\mathrm{Frac}(R/P)$, use the residue map on B, and send positive weights to zero. For mixed multiplication with $v \in A_{2k}$, $0 < k \leq n$, multiply the proposed image of vu by $(fT_0)^k$. The result is $(vf^k)(uT_0^n)$, which vanishes since $vf^k \in D^k$. For $k > n$ the product has positive weight. This proves multiplicativity. All the negative even equations and the squares of the negative odd equations vanish. The entire graded algebra is integral over the even-weight subring, since every homogeneous generator has its square in that subring. By lying over, extend the point to an algebraically closed overfield, choosing the overfield compatibly with σ. The resulting point kills I and every positive-weight element, and takes a nonzero value on f. These conditions define a nonempty finite type locus in the fibre over σ, hence give a K-point. Interchanging the two signs gives a positive point with $g \neq 0$. There is also a fixed point over σ: kill all nonzero weights. For a product of opposite weights summing to zero its square belongs to D, by the even-weight identity, so this prescription is multiplicative. Hypothesis (3) of Lemma 30.71 forces this fixed point to be x_K.

At the negative point over $x_K|_B$, every term of weight -1 in $\mathfrak{a}^2$ vanishes. It has either a positive-weight factor or a weight-zero factor in $\ker x$. The same holds for the saturation since nonzero elements of O are invertible in K. The F_i span the rational weight--1 cotangent and vanish at the point, so all of A_{-1} vanishes there. In weight -2 the additional possibility in $\mathfrak{a}^2$ is a product of two weight--1 elements. It too vanishes. The G_i now give vanishing of A_{-2}, contrary to $f \neq 0$. Therefore $\pi \in P$. Since the local ring B/L is finite over O, its unique prime over π is its maximal ideal. This proves the claim.

The two prime containments imply, for some h and all coordinatewise large d,

$$L^h W_d = 0, \qquad \mathfrak{m}_B^h W_d \subseteq D W_d. \tag{A.289}$$

Indeed apply the radical criterion first to $J + (\mathbf{t})$, then to $J + (\mathbf{t}) + D$, multiplying by powers of R_{++} to include the primes containing R_{++}. The ideals are finitely generated, and sufficiently large pieces of this algebra generated in unit multidegrees are obtained by multiplication by any fixed power of R_{++}. The first inclusion makes W_d finite over O: it is finite over B/L^h, and the filtration by L^i/L^{i+1} shows that B/L^h is finite over O.

For a prime P as above with $\pi \notin P$, set $p = P \cap B = \ker \sigma$ with $\sigma : B \to K$ as before. We have $D \not\subseteq p$. Choose $f \in A_{-2}$, $g \in A_2$ with $\sigma(fg) \neq 0$. The map $B/L \to A/I$ splits as a module map by projection onto weight zero. The fibre over σ is therefore a nonzero finite type K-algebra and has a K-point. Every such point is two-sided, so hypothesis (5) of Lemma 30.71 gives $\sigma(a) \neq 0$. We have proved that every prime over $J + (\mathbf{t}, a)$ contains R_{++} or $\mathfrak{m}_B$. The radical criterion gives admissibility.

We next bound the length at a horizontal prime p in $\mathrm{Supp}_B W_d$, meaning one omitting π. By (A.289) and Nakayama, $D \not\subseteq p$. Use f, g as above. On $fg \neq 0$ the algebra $(A/I) \otimes_O K$ is reduced. Indeed adjoining an invertible square root of the unit f is faithfully flat. The torus action then identifies this cover with the product of the slice $f = 1$, $g \neq 0$ and $\mathbb{G}_m$: the chosen root specifies the rescaling, and reversing the rescaling recovers the point and the root. The slice is reduced by Lemma 30.71. Descent and scalar restriction give reducedness over $\mathrm{Frac}\, O$ on this open set. The finite weight-zero subalgebra there is reduced, so its localisation at p is a field. In the graded algebra localised at $B \setminus p$, take a nonzero element h_0 of weight w. Use powers of the unit f to get a unit e of weight $-2w$. Then $h_0^2 e$ is nonzero by reducedness and belongs to the weight-zero field. Every h of weight w equals

$$(hh_0 e)(h_0^2 e)^{-1} h_0.$$

Thus each localised weight piece of A/I has B_p-length at most one.

Replace an odd generator u of I by u^2. In the algebra modulo the other equations, the kernel of the map from the quotient by u^2 to the quotient by u is an image of the quotient by u, by multiplication by u with its weight shift. The maximum length of a weight piece therefore at most doubles at each replacement. There are $r+1+r'$ replacements. To apply this to W_d, invert fg. The three coefficient modules of R then have bases $f, g, f^2 g$ over the weight-zero ring, since f, g are units and $f^2 g$ belongs to the third module. In multidegree d with all coordinates positive, divide by the product of these bases with exponents d_0, d_1, d_2. Each section relation then imposes its coefficient equation multiplied by the inverse basis of the corresponding weight. These are the weight-zero relations from the generators of I with the odd ones squared. The Z_i already vanish modulo J. The remaining relations of J give a further quotient. Hence

$$\mathrm{length}_{B_p}(W_d)_p \leq 2^{r+1+r'}. \tag{A.290}$$

Use a prime filtration of W_d over B. A vertical factor has finite O-length. Multiplication by a on it has kernel and cokernel of equal length. A horizontal factor B/p is finite torsion-free over O. Multiplication by a on it has cokernel length equal to the valuation of its determinant, namely $\sum_\sigma v(\sigma(a))$, where σ runs over the embeddings of its fraction field in K. The multiplicity of the factor is $\operatorname{length}_{B_p}(W_d)_p$, as seen by localising the filtration. Each σ extends to a two-sided point by the fibre argument, and restriction to B is constant on a torus orbit. Thus the restrictions of T cover these embeddings. Their valuations are nonnegative since B/L is integral over O. Additivity and (A.290) give the upper bound for $\chi_{(a)}(W_d)$.

Finally consider an admissible list on a finite graded module. Its Koszul homology has finite-length large pieces by the radical criterion. Take a finite homogeneous prime filtration and use additivity. Factors supported on $V(R_{++})$ vanish in large multidegrees. On a remaining domain factor, a nonzero first section acts injectively, so (30.31) reduces nonnegativity to that for a shorter list. If the first section acts as zero, the tail is already admissible. The characteristic is then the difference of its tail characteristics in degrees d and $d-e$, with e the first degree. Choose a monomial of degree e outside the prime, by taking surviving generators in the unit directions with their indicated exponents. Its multiplication sequence identifies the same difference with the tail characteristic on its quotient. Induction ends with the length of a module for the empty list. This proves eventual nonnegativity.

Since $\phi(a)\neq 0$, multiplication by a is injective on M, a subring of a polynomial ring over the chart domain. Computing first with a gives $\chi_{(\mathbf{t},a)}(M)_d=\chi_{\mathbf{t}}(M/aM)_d$. Computing instead with the sections of $\mathbf{t}$ successively gives at most $\chi_{(a)}(W_d)$ by (30.31), since each subtracted characteristic is eventually nonnegative. This proves both inequalities. □

Proof of 30.75

Proof. On J_A the restriction of z is a homomorphism to a group of exponent ℓ. Its image has order dividing the index in (30.37), so it is zero. Thus $z|_{G_F}$ is an unramified homomorphism for $F=\mathbf{Q}(\mu_N)$. The nontriviality of ϑ gives $N>2$. For each $t\in(\mathbf{Z}/N)^\times$, choose a lift σ_t of the automorphism acting by t on μ_N.

Choose a positive odd a prime to N with $\operatorname{red}\vartheta(a)\neq\operatorname{red}a$. If $\ell\nmid N$, take $a=\ell$. If $\ell\mid N$, both residual characters factor through $(\mathbf{Z}/N)^\times$, and a class where they differ has an odd representative. With $y=[a^{-1}t]_N$, the integral smoothing coefficients satisfy

$$E_t=(a-1)/2-(ay-[t]_N)/N,\qquad \sum_t E_t\vartheta^{-1}(t)=(1-a\vartheta^{-1}(a))B_{1,\vartheta^{-1}}. \tag{A.291}$$

The constant term drops out by the vanishing character sum, and reindexing $t=ay \bmod N$ gives the second term of the product. The right side is a unit. For $u\in G_F$ the cocycle law gives $z(\sigma_t^{-1}u\sigma_t)=\bar\vartheta(\sigma_t)^{-1}z(u)$. The integral Stickelberger relation of 27.5 [112, Theorem 6.10] makes the sum of these conjugate values weighted by E_t zero. Reducing (A.291) therefore gives $z|_{G_F}=0$.

Lean: `UpperTheory.V.Odd.homKill` (`Development.AutomorphicArithmetic.AutomorphicVarieties.OddResidueBounds.HomKill`, lines 9–49)

Prime-to-ℓ roots reduce injectively, so choose t_0 with $\bar\vartheta(\sigma_{t_0})\neq 1$. The cocycle factors through the abelian quotient $G_{\mathbf{Q}}/G_F$. Expanding its equal values at $g\sigma_{t_0}$ and $\sigma_{t_0}g$ gives (30.38) with $b=z(\sigma_{t_0})/(\bar\vartheta(\sigma_{t_0})-1)$.

Apply 30.76 to ϕ,ξ and to their inverses. Both products have order prime to ℓ, and (30.39) implies $n\geq 4$. A Galois element acting by $1+2^{n-1}$ on μ_{2^n} and trivially on $\mu_{\ell f}$ has product character value -1 by primitivity of ξ. Thus either reduced product is nontrivial and differs from $\bar\varepsilon_\ell$. □

Proof of 30.76

Proof. Write $M=2^n$, $m=n-e$ and $s=f2^e$. Then ℓ is odd, $m\geq e\geq 2$ and $1<s\leq 2^m$. Extend characters by zero off the units and put

$$A_u=\sum_{\substack{0\leq j<fM\\ j\equiv u\ (M)}} j\phi(j),\qquad g(u)=\frac{A_u-A_0}{fM},\qquad c=M^{-1}\sum_{j \bmod f}\phi(j).$$

Translate the summands of A_{u-ft} by ft. Their original indices run through all residues modulo f, and ϕ is unchanged by the translation. Wrapping a new index j into the interval $[0, fM)$ subtracts $fM\phi(j)$ and gives exactly the indices in the sum below. Hence

$$g(u) - g(u - ft) = tc - \sum_{\substack{0 \le j < ft \\ j \equiv u \ (M)}} \phi(j) \qquad (0 \le t < M). \tag{A.292}$$

Take $ft \equiv u \bmod M$. Since $g(0) = 0$, the formula makes $g(u)$ integral and its reduction fixed by the Q-power Frobenius. Here M is a unit and $\phi(j)^Q = \phi(j)$. The character sum of ξ is zero since ξ is primitive and $n \ge 4$. Grouping the defining Bernoulli sum by residues modulo M gives $B_{1,\phi\xi} = \sum_u \xi(u)g(u)$.

Use bars for reduction into $\overline{\mathbf{F}}_\ell$ and suppose the sum vanishes there. Reindexing gives $\sum_u \bar{\xi}(u)\bar{g}(uy) = 0$ for each unit y. Successive differences of squares give $v_2(Q^{2^j} - 1) = e + j$. Thus the powers of Q mod M are $1 + 2^e h$ for $0 \le h < 2^m$. Average the 2^m successive Frobenius conjugates of the sum. Each $\bar{g}(uy)$ is fixed, and the geometric sum on the character values gives

$$U = \{u \in (\mathbf{Z}/M)^\times : \bar{\xi}(u)^{2^e} = 1\}, \qquad \sum_{u \in U} \bar{\xi}(u)\bar{g}(uy) = 0.$$

The element $h_0 = 1 + M/2$ generates the last reduction kernel, so $\bar{\xi}(h_0) = -1$ by primitivity. It is the unique involution of the cyclic subgroup of units congruent to 1 mod 4, on which $\bar{\xi}$ is therefore faithful. For $u \in U$, this gives $u^{2^e} = 1 \bmod M$, since u^{2^e} belongs to that subgroup. In the factorisation of $u^{2^e} - 1$, the factors $u^{2^i} + 1$ for $1 \le i < e$ each have 2-adic valuation one, as does one of $u - 1, u + 1$. Hence $u \equiv \delta_u \bmod 2^m$ for a unique $\delta_u \in \{1, -1\}$.

Subtract the identities at $y = 1$ and $y = 1 - s$. Since $M \mid 2^m s$, we have $u(1 - s) \equiv u - \delta_u s \bmod M$. For $\delta_u = 1$, apply (A.292) with $t = 2^e$. The boundary term occurs at $u = 1$, the sole residue of this sign with $[u]_M < s$. For $\delta_u = -1$, apply the same shift at $u + s$ with the opposite sign. The boundary term occurs at $u = -1$, where $[u + s]_M = s - 1$, the sole residue in $[0, s)$ congruent to $s - 1 \bmod 2^m$. Thus

$$0 = 2^e \bar{c} \sum_{u \in U} \delta_u \bar{\xi}(u) - \bar{\phi}(1) + \bar{\xi}(-1)\bar{\phi}(s - 1).$$

Multiplication by h_0 preserves U and δ_u and negates the character value, so the sum cancels. Since $s - 1 \equiv -1 \bmod f$ and the product character is odd, the remaining terms equal -2, a contradiction. □

Proofs for Chapter 31: The exponent 401 and the downstream window

Proof of 31.1

Proof. Put $u = x^{1/100} \ge 2$. The bounds on ε give

$$f \le 4u^{216}, \quad k \le 2u^{470}, \quad r \le 2k \le 4u^{470}, \quad m \ge u^{49}, \quad L^2 \le u^4.$$

Indeed, $2 + 8\varepsilon \le 216/100$, $9/2 + 10\varepsilon \le 470/100$, $1/2 - \varepsilon/16 \ge 49/100$, and $2\varepsilon \le 4/100$. Hence

$$\frac{f^2(k-1)^3L^2}{m} \le 128u^{2\cdot 216 + 3\cdot 470 + 4 - 49} = 128u^{1797}, \qquad f + \frac{f^2(k-1)^3L^2}{m} \le 132u^{1797}.$$

Using $\lceil v \rceil < v + 1$ in the definition of n_W gives

$$n_W \le 5280u^{2267} + 8 \le 5288u^{2267}, \qquad T_O \le 2u^{37700}, \qquad T_O n_W \le 10576u^{39967}.$$

The rank identity now yields

$$t \le 128u^{686} + 4 + 10576u^{39967} \le 10708u^{39967} \le u^{40000} = x^{400},$$

where the last inequality uses $10708 \le 2^{33} \le u^{33}$. □

Proofs for Chapter 32: Green's classification and its use

Proof of 32.11

Proof. Conjugation transports the invariant subspaces and their restrictions. Induct on $\dim V$ and average 32.10 over h. The terms for $W \subseteq A$ sum to $s(c_{g|_A})$. For $0 < W \cap A < W \subsetneq V$, the h for which $g+h$ stabilizes W form an affine space if nonempty. Restriction gives the uniform distribution on a translate of $\mathrm{Hom}(W/(W\cap A), W\cap A)$, since maps in this latter space extend to $V/A \to A$. Induction gives zero.

For $W \cap A = 0$, the image B of W in V/A must be invariant under the induced $\bar{g}$. Each of its $v^{\dim A \dim B}$ lifts is stable with probability $v^{-\dim A \dim B}$, since stability prescribes $h|_B$. The restriction is conjugate to $\bar{g}|_B$, so these terms sum to $s(c_{\bar{g}})$ by 32.10. The zero subspace contributes $s(1) = 0$. By the block factorization and additivity, $s(c_{g|_A}) + s(c_{\bar{g}}) = s(c_g)$. □

Proof of 32.13

Proof. Choose a finite extension of F containing the eigenvalues of every element of H. Apply Green's Theorem 1 [20] to the restriction of t and the integral symmetric polynomial $\sum_i X_i^h$. This gives the virtual character R_h.

In 32.10, the sum over invariant subspaces of dimension a is the parabolic induction of Ψ_h in dimension a tensored with the trivial character in dimension $n-a$. Induction on n proves the claim for Ψ_h. □

Proof of 32.14

Proof. Let μ and b be the masses of 1 and Ψ_s divided in each degree by the order of the general linear group. Write $*$ for convolution over monic factorizations. Counting endomorphisms with an invariant subspace gives $s\mu = b * \mu$ by 32.10. In dimension a the number of subspaces is $|\mathrm{GL}_n(F)|/(|\mathrm{GL}_a(F)||\mathrm{GL}_{n-a}(F)|v^{a(n-a)})$. Its last denominator cancels the number of off-diagonal blocks.

Let $(T\mu)(P) = v^{\deg P}\mu(P)$. Counting an endomorphism with a vector by the cyclic subspace it generates gives $T\mu = \mathbf{1} * \mu$, since cyclic pairs with a given monic polynomial constitute one free orbit by the companion basis. The degree factor makes T multiplicative for convolution. Additivity gives

$$T(s\mu) = s(\mathbf{1} * \mu) = s * \mu + \mathbf{1} * (s\mu).$$

Substitution and cancellation of μ give $(Tb - b) * \mathbf{1} = s$. The identity $s = \Lambda_s * \mathbf{1}$, where $\Lambda_s(Q^a) = s(Q)$ for $a > 0$ and is zero elsewhere, follows from unique factorization. Cancel $\mathbf{1}$ and divide by $v^{\deg P} - 1$. □

Proof of 32.17

Proof. By 32.15, $I(\phi, R_k) = I(\phi, R_l)$ whenever $kv^i \equiv l \bmod M$, since Frobenius permutes the roots of each P_α. Thus 32.16 shows that the primitive difference pairs to zero with every R_l. Fourier inversion in 32.15 gives zero mass on each invertible primary fibre.

On $c_g = Q^a$, a primitive class function is determined by its single value on the class $Q(g) = 0$. For a function with this value zero, descend on $\dim A$ for $A = \ker Q(g)$. In the variations $g + h$ of 32.11, $Q(g+h)$ vanishes on A and agrees with $Q(g)$ modulo A. If its kernel equals A, successive preimages of A give the same higher-power kernels. These variations are conjugate to g. Those with larger kernel have value zero by descent. The primitivity identity thus forces zero at g. The primitive function $\Psi_{\deg}$ has nonzero mass by 32.14, so on this at most one-dimensional space the mass determines the function. This proves the expansion on the primary fibres, and 32.15 supplies vanishing on the others. □

Proof of 32.24

Proof. For the empty list the group is trivial. Otherwise let P be the standard parabolic with the given ordered block sizes. Each cuspidal factor of its Levi has Whittaker multiplicity one for the restriction of the prescribed nondegenerate character. This is the finite-field Gelfand–Graev theorem, in the form recalled in [141, Prop. 9.4] and [142, §1, p. 76].

In the Mackey formula for restriction of the induced representation to U_n, take the permutation which reverses the order of the blocks and preserves the order inside each block. The intersection of U_n with the conjugate parabolic consists of the upper unitriangular groups inside the permuted blocks. The character restricts to their product of Whittaker characters. Frobenius reciprocity therefore gives a contribution of one from this double coset. All contributions are dimensions. Decomposing $I(C)$ into irreducibles now gives a constituent of positive Whittaker multiplicity. □

Proof of 32.25

Proof. Write $C' = (\sigma, C'_0)$. By reciprocity and Mackey restriction, the pairing is a sum over distributions of blocks into the two factors of the Levi for C'. Cuspidality kills every split block. If several whole blocks go to the first factor, their induced product pairs to zero with σ by cuspidality and reciprocity. For one block, orthogonality retains precisely the positions i with $C_i = \sigma$. The sum becomes

$$\sum_{i:\, C_i = \sigma} \langle I(C \setminus i), I(C'_0) \rangle.$$

The empty lists pair to one. Induction gives the two assertions. □

Proof of 32.37

Proof. Choose a root x_P of each $P \in \mathcal{P}_d$. By 32.34 and 32.35, $\tau_P = \tau_{\theta_{x_P}}$ is cuspidal of degree Δ_d. Every characteristic root z of elliptic y lies in $\iota(K^\times)$ by 32.32 and 32.35. The regular formula and 32.33 give (32.1) at x_P. For $x = x_P^{p^a}$, 32.36 gives

$$\sum_{i<d} \Theta_d(x, z^{p^i}) = \sum_{i<d} \Theta_d(x_P, z^{p^{i+a}}) = \sum_{i<d} \Theta_d(x_P, z^{p^i}).$$

Equal characters give conjugate parameters, hence conjugate roots by 32.33 and 32.34. Their minimal polynomials agree. Conversely, a cuspidal is τ_{θ_x} with θ_x regular. The degree e of x divides d. If $e < d$, then $x^{p^e} = x$ contradicts regularity. Thus its minimal polynomial is in $\mathcal{P}_d$ and its chosen root is in the same orbit. Extend the family arbitrarily off $\mathcal{P}_d$. □

Proof of 32.48

Proof. Write $W = \{s, \dots, s+k-1\}$ and keep the character given by the two chains defining ϕ_J. At stage i, $0 \le i \le s$, insert W after the first i indices of the other chain. Allow the strictly upper entries in that order except the entries from W to the next index of the other chain. Stage s gives H_J. Stage zero moves $W = \{s, \dots, s+k-1\}$ to $\{0, \dots, k-1\}$; conjugating by this permutation preserves class-function traces, carries the two chains into the standard upper-unitriangular chains with the bridge entry $u_{k-1,k} = 0$ killing the bridge term, and gives the subgroup and character of e'_J.

For a step from i to $i-1$ put $a = i-1$ and let b be the successor of a in the other chain. Thus $b = i$ for $i < s$ and $b = s+k$ for $i = s$. The two patterns differ by replacing (a, w) by (w, b) for all $w \in W$. The union of the two patterns is closed under composition of positions, so the unitriangular matrices supported on it form a group H under matrix multiplication. The two subgroups H_1, H_2 each extend their intersection by p^k choices, and their characters agree on the intersection. The commutator of the two added rows of root coordinates has (a, b) entry $\sum_{w \in W} x_w y_w$. The pairing $\eta(\sum_w x_w y_w)$ is nondegenerate since a nonzero coordinate can be paired with a variable taking all values in $\mathbf{F}_p$.

For the Mackey self-pairing on the first subgroup, any $h \in H$ outside it has some $h_{wb} \neq 0$. Conjugating $1 + tE_{aw}$ by h in the form $h^{-1}(1 + tE_{aw})h$ stays in the first subgroup. The new entries can occur only at predecessors of a and successors of w in the union pattern, with endpoints included. Among the two chains' consecutive entries this changes only (a, b), by th_{wb}. Choosing t so that the character changes makes that double coset contribute zero. For the second subgroup use $1 + tE_{wb}$ and a nonzero $(h^{-1})_{aw}$. Its conjugate stays in the second pattern and changes the character sum by $t(h^{-1})_{aw}$. Thus each induced character on H has norm one. Their cross-pairing has a contribution of one from the identity double coset, so the induced characters agree. Inducing to G_n and using reciprocity proves equality of the two traces for characters of G_n. They span the class functions, so the successive equalities prove the lemma. □

Proof of 32.49

Proof. By linearity take χ to be a character. For $1 \le j \le n$ let P_j be the subgroup of G_j fixing the last row covector. Let T on P_{k+1} be the character obtained from $\chi|_{P_n}$ by successively taking the nontrivial last-column character spaces down to P_{k+1}. At the step from P_{j+1} to P_j, use $v \mapsto \eta(v_j)$ on the column $v \in \mathbf{F}_p^j$. Thus T is the tail average over upper unitriangular matrices whose left upper $(k+1)$-block is the identity. The weights use the adjacent entries from that block to the end. Grouping the average for e'_J by its left upper k-block gives

$$\chi(e'_J) = \mathrm{wh}_\eta\big(h \mapsto T(\mathrm{diag}(h,1))\big).$$

The allowed entries in each fibre are exactly the tail entries, and the character factors as the product of the upper U_k character and the tail character.

In $P_{j+1} = G_j \ltimes \mathbf{F}_p^j$ the dual of the column group has two G_j-orbits. The zero orbit gives the column invariants as a G_j-representation. The nonzero orbit is induced from $P_j \ltimes \mathbf{F}_p^j$, with the chosen character on the column and its multiplicity representation on P_j. Iterate this decomposition on the latter representation. For T the summands are indexed by the rank i, $0 \le i \le k$, at which the zero character is taken. In P_{k+1} the summand is induced from the upper block subgroup with diagonal blocks G_i and U_{k+1-i}. It uses $D_i^\eta\chi$ on G_i and ξ_η on U_{k+1-i}. It is trivial on the upper off-diagonal block. The selected columns impose the simple-root character within the tail, and the zero step imposes invariance on the remaining entries to the right of the first i coordinates. Together these averages are precisely the unipotent restriction and Whittaker average defining D_i^η.

The inducing subgroup contains the last-column group of P_{k+1}. Restriction to G_k therefore gives a single Mackey term, induced from the corresponding subgroup with diagonal blocks G_i, U_{k-i}. Induction in stages identifies its character with the Harish–Chandra product of $D_i^\eta\chi$ and $\Gamma_{\eta,k-i} = \mathrm{Ind}_{U_{k-i}}^{G_{k-i}} \xi_\eta$. Apply 32.46 to the derivative and Gelfand–Graev characters. The coefficient is $p^{k-i} - p^{k-i-1}$ by 32.26 for $i < k$. For $i = k$ the second factor is the trivial character in rank zero, of multiplicity one. Apply 32.48. □

Proof of 32.51

Proof. Mackey restriction distributes the inducing blocks between the two Levi factors. Cuspidality kills every distribution splitting a block, and each other term is a product induced separately in the two factors. For Whittaker multiplicity of a parabolic induction, the opposite cell contributes the product of the multiplicities as in 32.24. Every other cell has a simple root subgroup in the intersection with the conjugated unipotent radical, where the inducing representation is trivial and ξ_η is nontrivial. Each surviving distribution therefore contributes one. □

Proof of 32.53

Proof. The host fields of D give distinct primes $q, d > 4tr + 1$, where t is the host rank, and $2^{e+2} \mid Q_S$ with $e \ge 2$. The normalized types give $p_j \equiv 1 \bmod Q_S qd$, hence $16, q, d \mid p_j - 1$ and $p_j \ge 5$.

Apply 32.43 with the Teichmüller lift $t_j : \overline{\mathbf{F}}_{p_j} \to \mathbf{C}$ and the chosen generators $u_{j,h}$ of $\mu_{p_j^h-1}(\overline{\mathbf{F}}_{p_j})$, $h \ge 1$. Their compatibility reads $u_{j,ha}^{(p_j^{ha}-1)/(p_j^h-1)} = u_{j,h}$ for every $a \ge 1$. Here $r = k + 2s + 1$ with $k \ge 1$ and $s = \lfloor m/2 \rfloor - 1$. The arithmetic projector is $E_{\sigma_* f}$ on admissible labels and zero elsewhere, so each arithmetic weight at f equals the corresponding local weight at $\sigma_* f$. The substitution preserves degrees and squarefreeness. By 32.42 these projectors sum to the identity, and by 32.44 the weights are nonnegative with positive denominator. The sum of the numerators is that denominator. Apply 32.50 at $\sigma_* f$ for the lower comparison. For squarefree f, apply 32.47 and 32.52 at the same label. The degreewise permutation identifies the monic irreducible factors of f and $\sigma_* f$ and preserves their degrees, so the product in 32.47 has the same value on both labels. This gives the squarefree formula and upper comparison for the arithmetic weights. □

Proof of 32.54

Proof. Let I_j count monic irreducibles of degree j with nonzero constant term. They divide $X^{u^j-1}-1$, so $jI_j \le u^j-1$. The mass of a type with m_j parts of size j is

$$\prod_{j=1}^{N} \binom{I_j}{m_j} (u^j-1)^{-m_j} \le \prod_{j=1}^{N} (j^{m_j} m_j!)^{-1}.$$

The right side is the permutation proportion because a cycle listing counts each permutation $\prod_j j^{m_j} m_j!$ times. Sum over the desired types. Division by h injects the squarefree multiples of h into $\mathcal{S}_{N-\deg h}(K)$. The identity $w(f) = w(h)w(f/h)$ gives the divisor bound.

Each squarefree admissible f has weight at least u^{-N}. Of the u^N monic polynomials, u^{N-1} have zero constant term. For $N \ge 2$ at most

$$\sum_{1 \le j \le N/2} I_j u^{N-2j} \le (u-1)u^{N-2} + \tfrac{1}{2} \sum_{2 \le j \le N/2} u^{N-j} \le u^{N-1}$$

are divisible by the square of an irreducible with nonzero constant term. In degree one there are no such squares. Multiply the lower count by u^{-N}. □

Proof of 32.55

Proof. By 32.54 it suffices to bound the probability of an invariant b-set for a uniform permutation. Let X count such sets and Y count those with a fixed point both inside and outside. A specified b-set is invariant with probability $b!(N-b)!/N!$, so $\mathbb{E}X = 1$. Conditional on invariance, the restrictions are independent uniform permutations. On $s \ge 2$ letters, the first two inclusion-exclusion terms bound the probability of a fixed point below by $1 - \binom{s}{2}/(s(s-1)) = 1/2$. On one letter the probability is one. Thus $\mathbb{E}Y \ge 1/4$. If $Y > 0$, exchanging the two fixed points gives a second invariant set, so $2\mathbf{1}_{X>0} + Y \le 2X$. The inequality holds for $Y = 0$ as well. Taking expectations gives $7/8$. □

Proof of 32.56

Proof. For each triple (A, T, h) in the table start at $a = A$. While $a < T$, put $c = a + \lfloor a/10 \rfloor$ and $s = \lfloor c/2 \rfloor$, compute

$$L(a,c,h) = \sum_{\substack{s \le j < s + 2^{\lfloor \log_2 a \rfloor} \\ c < 2j,\ j+3 \le a,\ h < j < (h+1)^2 \\ \gcd(j, h!) = 1}} \left\lfloor \frac{10^9}{j} \right\rfloor,$$

and replace a by $c+1$. The iteration counts and minima are

A	T	h	iterations	$\min L$
100	196	13	7	108738643
196	1024	31	18	86865767
1024	20000	141	32	61695338

A composite $j < (h+1)^2$ has a prime divisor at most h, so every summand comes from a prime in $\mathcal{J}_N$ for all $a \le N \le c$. The intervals cover $100 \le N < 20000$, and each minimum gives $H_N \ge 1/20 \ge 1/(5 \log N)$.

For $N \ge 20000$ put $n = \lfloor N/2 \rfloor$, $y = \sqrt{2n}$, $L = \log(2n)$, $l = \log 2$, and $k = \pi(2n) - \pi(n)$. With $C_n = \binom{2n}{n}$,

$$4^n < nC_n, \qquad C_n \le (2n)^{\pi(y)+k}\, 4^{\lfloor 2n/3 \rfloor}.$$

The first bound starts at $n = 4$ and propagates since $(n+1)C_{n+1}/(nC_n) > 4$. Each term $\lfloor 2n/t^j \rfloor - 2\lfloor n/t^j \rfloor$ in the valuation at prime t is zero or one. Thus primes $t \le y$ contribute at most $2n$ each. Above y the exponent is at most one and vanishes on $(2n/3, n]$. The product of primes up to m is at most 4^m by induction. For $m = 2s+1$, the primes in $(s+1, 2s+1]$ divide $\binom{2s+1}{s} \le 2^{2s}$, and the smaller ones contribute at most 4^{s+1}.

For even $m \geq 4$ the product is unchanged from $m-1$. This bounds the primes up to $2n/3$, and the final k primes contribute at most $(2n)^k$.

Counting classes modulo 6 gives $\pi(y) \leq y/3+3$. Taking logs yields $y^2 l/3 < (y/3+4+k)L$. Since $y \geq 141$, $\log(y/128) \leq y/128-1$ gives $L \leq 14l + y/64 - 2$. With $0.69 < l < 0.70$ from the series for $\log 2$,

$$(5l/3-1)y^2 - 5(y/3+7)(14l+y/64-2) - 1 \geq 3y^2/25 - 14y - 275 > 0.$$

Thus $5(k-3)L \geq 2n+1$. Each prime in $(n, 2n]$ exceeds $N/2$, and deleting the possible values $N-2, N-1, N$ leaves at least $k-3$ primes in $\mathcal{J}_N$. Consequently $5|\mathcal{J}_N| \log N \geq N$. Each reciprocal is at least $1/N$. □

Proof of 32.58

Proof. Write $n_j = u^j - 1$ and $a_\ell(n) = \ell^{v_\ell(n)}$. The prime-to-$\ell$ projection in the group of order n_j has fibres of size $a_\ell(n_j) \leq u^{j/4}$ for $\ell \notin B(u, N)$ and $1 \leq j \leq N$.

If two roots of the same degree-j factor collide, their common prime-to-ℓ part has Frobenius orbit of length e properly dividing j. Each root of the factor solves $X^{(u^e-1)a_\ell(n_j)} = 1$. For each $e \leq j/2$ there are at most $u^{3j/4}$ solutions. There are at most j choices of e. Since each factor contributes j roots, there are at most $u^{3j/4}$ factors. By 32.54, each costs at most $1/n_j$. Such a collision requires $j \geq 2$ and $\ell \mid n_j$.

For distinct factors of degrees $j \geq i$, fix the degree-i factor and one root y_0. By Frobenius the other factor has a root in the fibre above $(y_0)_{\ell'}$ in the group of order n_j. This gives at most $u^{j/4}$ choices of that factor. There are at most n_i choices of the first factor and each pair costs at most $1/(n_i n_j)$, with zero mass if $i + j > N$. The contribution for fixed i, j, ℓ is at most $u^{j/4}/n_j$. Here $\ell \mid n_i n_j$.

Put $z = u^{-1/4} \leq 1/8$. The number of prime divisors of n_j is at most $(3/2)j \log u$, and that of $n_i n_j$ for $i \leq j$ is at most $3j \log u$. Since $1/n_j \leq 2u^{-j}$, the two collision bounds sum to at most

$$3 \log u \sum_{j \geq 2} j z^j + 6 \log u \sum_{j \geq 1} j^2 z^{3j} \leq (16z^2 + 30z^3) \log u \leq 23 \log u / \sqrt{u}.$$

For the middle inequality use $j \leq 2^j$ and sum the geometric series.

For a collision with c, the quotient x/c has prime-power order dividing n_j, or order one. In either case the minimal polynomial divides $X^{a_s(n_j)} - c^{a_s(n_j)}$ for a prime $s \mid n_j$, since $16 \mid u-1$. If b_j counts the degree-j factors of these binomials, counting roots gives $j b_j \leq \sum_{s \mid n_j,\ s \text{ prime}} a_s(n_j)$. The mass is at most $\sum_{j=1}^N b_j/n_j$ by 32.54.

Set $A = u - 1$ and $t = \min(q, d)$. Let $U \geq 16$ and $V \geq t$ be the 2-part and the t-part, respectively. Let $Z \geq \max(16, t)$ be the part at $\max(q, d)$. Let R be the product of the remaining parts. Their sum is at most R by $a + b \leq ab$ for integers $a, b \geq 2$. Each of U, V, Z, R is at most $A/(16t)$ by the product of the other three. Thus $b_1/A \leq 1/(4t)$.

For $j \geq 2$ and prime $s \mid A$, one has $v_s(u^j - 1) = v_s(A) + v_s(j)$. If $s \mid h$ and $s \nmid a$, binomial expansion gives $v_s((1+h)^a - 1) = v_s(h)$. Raising to s adds one to the valuation for odd s, and for $s = 2$ when $4 \mid h$, as holds here. Thus the product of $a_s(n_j)$ over $s \mid A$ divides jA, and over $s \nmid A$ it divides n_j/A. Bounding each sum by the corresponding product gives

$$\frac{b_j}{n_j} \leq \frac{A}{n_j} + \frac{1}{jA} \leq u^{1-j} + \frac{1}{jA}.$$

Sum from $j = 2$, using $\sum_{j \geq 2} u^{1-j} = 1/A$ and $\sum_{j=2}^N 1/j \leq \log N$, and add the degree-one term. □

Proof of 32.60

Proof. Fix j and put $p = p_j$. Since $16 \mid Q_S$ and $p \equiv 1 \pmod{Q_S q d}$, each of $16, q, d$ divides $p-1$. Since $r \geq 100$ and $t \geq 1$, the host bound $q, d > 4tr + 1$ gives $q, d > 16$.

Let P be the U-law at η_j and Q_v the law of the idempotent at v. These are probabilities. The lower r coefficients recover each monic polynomial, and admissible roots are nonzero of finite order. Hence the arithmetic tests give the same prime-to-ℓ parts and excluded primes as in 32.58, with one test for each twist and one for its inverse. The divisor tests use the same recovered polynomials.

For the product $P^{\otimes\nu}$, use 32.59 at each place. The union of failures of squarefreeness, root separation and the two twist tests has weight at most

$$\frac{2\nu}{p}+\frac{23\nu\log p}{\sqrt{p}}+2\nu m\left(\frac{1}{4\min(q,d)}+\frac{1+\log r}{p-1}\right).$$

On squarefree admissible tuples, failure for fixed b requires a degree-b divisor at every place. It has probability at most $(7/8)^\nu$. Sum over $1\le b\le\lfloor r/2\rfloor$. Failure of the long-factor test requires every entry to miss $\mathcal{J}_r$, with probability at most $(1-1/(5\log r))^\nu\le e^{-\nu/(5\log r)}$ on those tuples.

Let E be this union of failures on admissible tuples. By 32.53, $Q_v\ge(1-p^{-1})P$ pointwise. Multiply on tuples of E^c and use total mass one to get

$$\left(\bigotimes_v Q_v\right)(E)\le 1-(1-p^{-1})^\nu(1-P^{\otimes\nu}(E))\le P^{\otimes\nu}(E)+\nu/p.$$

This gives W since the weights vanish off admissible tuples and $\lfloor r/2\rfloor\le r/2$. □

Proof of 32.62

Proof. The full cylinder gives $D_{\mathrm{all}}\ge(1-\epsilon_C)D_*>0$. Let S_j be the finite set of tuples of all-row indices bad at j. Then $0\le D_{\mathrm{all}}-D_{\mathrm{rows}}\le\sum_j N_j(S_j)$ by the index condition. The finite-cylinder identity gives $D_j(S_j)=D_*P_j(S_j)$.

By 32.61, all places above p_j use $e_J(\eta_j)$ for the same nontrivial character η_j. The two distinguished primes of the host split completely in E and M_{ray} and are congruent to 1 modulo Q_Sqd, as required in 32.60. For $x^\alpha\ge 3/2$ the size bound gives $p_j\ge 4096$. By 32.60 and relative counting,

$$0\le\delta_{x,i}\le\frac{1+\epsilon_C}{1-\epsilon_C}\sum_{j=0}^{1}W(\nu,m,r,p_j,q,d).$$

Eventually $\epsilon_C\le 1/2$ uniformly. At points with indices, $C_s\ge 2$. By the size bounds and the decrease of $(\log p)/\sqrt{p}$ for $p\ge e^2$, the first three terms of W total $O(x^{3C_s+3}e^{-x^{\alpha/6}})$. The last two are $O(x^{C_s}\exp(-c'x^2/\log x))$ for some $c'>0$, since $\nu/\log r\ge x^2/(C_s\log x)$. Both are $o(e^{-x^{\alpha/7}})$ uniformly in i for $0<\alpha\le 1$. □

Proof of 32.63

Proof. Reduce the ε-threshold to at most 1 and keep the size threshold $\mathcal{K}$ for each eligible ε. For a family and height exponent C_H meeting the size thresholds $\mathcal{K}$ of 36.10, write x_1 for the family's initial cutoff. As point threshold take the maximum of x_c, the selection threshold, x_1+1, 4, $(3/2)^{1/\alpha}$ and the numerical threshold of 32.62. Beyond it, the selected normalized types give 32.53 for the same rows.

By Definition 33.6 and Definition 36.10, the sizes give $n\ge x^{2+8\varepsilon-\varepsilon/100}$ and $x^{9/2+10\varepsilon-\varepsilon/100}\le k\le x^{9/2+10\varepsilon+\varepsilon/100}$, with $m\le f=2n\le k$ and $k\le r\le 2k$. As $\varepsilon<1/52$ in the size data, $x\ge 4$ gives $x^2\le\nu\le x^6$, $m\le x^6$ and $100\le r\le x^6$. From $(kb)^8\le p_j$ and $\alpha\le\varepsilon$ we have $p_j\ge e^{8x^\alpha}$. The lower bounds on $\log q,\log d$ are at least $x^{\alpha/6}$. Apply 32.62 with $C_s=6$. □

Proof of 32.64

Proof. Put $\mathcal{A}_G=\mathcal{A}\cap\mathcal{E}$. Minkowski's rational bound says that the order of each finite subgroup of $\mathrm{GL}_n(\mathbf{Q})$ divides $M(n)$ of 33.6 [143, Thm. 1(i)]. For every sufficiently large member and every word u, the design gives $D_{\Gamma,u}\mid M(2fN(m))$, where $D_{\Gamma,u}$ is the stabilizer denominator of 15.12. The primes in $M(n)$ are at most $n+1$, and $M(n)\mid(2n)!$ since $\lfloor n/(p^a(p-1))\rfloor\le\lfloor 2n/p^{a+1}\rfloor$ for primes p and $a\ge 0$. Thus $\log M(n)\le 2n\log(2n)$. At $n=2fN(m)$ with $fN(m)\ge 2$, this gives $\log D_{\Gamma,u}\le 12fN(m)\log(fN(m))$ uniformly in u. The prime bound supplies smoothness.

On $W_1(\mathcal{A}_G)$, if $C_{Tu}\ge c_T(\varepsilon,C_{YL})$, the level-prime hypothesis of 33.6 gives $b^{m/2}\le T_u$ and $\log T_u\le C_{Tu}mL$ for every word of every sufficiently large member, with $L=\log b=x^\varepsilon$. On this working class the common

hypotheses of 33.6 supply the class-number row of 37.21. For every family in $\mathcal{A}$, item 2 of 33.6 selects at each point a bundle whose recursion realization carries a compatible symmetric column system, so every family in $W_1(\mathcal{A}_G)$ carries these symmetric columns beyond its selection cutoff. Lemma 23.11, applied to the word windows of 21.35, gives $1/3 \le f_u\Gamma^2/D^\star \le 3$ uniformly in the word. This establishes the rank condition used in choosing the design.

Lean: `Formalism.MainNotions.LinkB.rank_formula_of_part_four` (`Formalism.SharpRestriction.GlobalSchurBounds.TopDensityNGeSixteen`, lines 102–103)
Lean: `Formalism.MainNotions.proviso_e` (`Formalism.SharpRestriction.GlobalSchurBounds.MainSup`, lines 196–245)
Lean: `Formalism.MainNotions.StandardLattice.HarishChandraReduction.chosenProviso_4f` (`Formalism.SharpRestriction.GlobalSchurBounds.TheoremThreeFamilyOfFormalDegreeInput`, lines 64–74)

For every sufficiently large member the design gives a scale $Y \in [x^{1/J}, x]$ for the six sign-character zero-free boxes and the real host-character boxes of (37.4). It gives a scale in the same interval for the prime and conductor conditions (37.5). At x itself the primitive representatives of the q- and d-sign characters and their product satisfy (24.7) with a common $C_5 > 0$. With the uniform zero estimates and explicit formula of 37.2–37.3, the closed form of 37.4 gives a common $c_6 > 0$ such that every primitive host character satisfies $|g_\chi(y)| \le y e^{-c_6 x^{\varepsilon/8}}$ for $y \ge e^{x^{\varepsilon/3}}/3$. These two character estimates and the rank bound above complete the design conditions at $3, C_H$. The remaining conditions from 33.6 are the lower bound on the thick set, the top density and weighted bad-orbit bound with a common $C_b > 0$, and the lower bound $3\Gamma^2 x^{C_H}$ on the number of words. All hold for every member beyond a common point threshold.

Lean: `Formalism.MainNotions.designExists_of_choices` (`Formalism.SharpRestriction.DesignExistsOfChoices`, lines 54–68)
Lean: `GlobalTheory.Top.designExists_of_tuSize_aux4n_litThin` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LinkEx`, lines 15–37)

On $\mathcal{D}_3(\mathcal{A})$ the row conditions of 33.6 and Green give the closed defect bound of 32.63. The top density gives, with δ_1 as in 8.15, $\delta_1 n/2 \le n_T \le 2\delta_1 n$. Applied to the design's bad-orbit bound, the Mertens estimate bounds the weighted count of bad top positions in 22.42 by $n_T x^{-2+\varepsilon}$. The selected second layer gives group compatibility. On the same class, 22.37 gives the closed local lattice and gain estimates, with a common constant and an error scale $b^{\delta(x)}$ for some $\delta(x) \to 0$.

Lean: `GlobalTheory.Top.closed_linkG_of_duty` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LinkG`, lines 13–24)

For the rank identity at a twisted place, take the maximum of the selection threshold and $x_1 + 1$, where x_1 is the family's initial cutoff. At each eligible member beyond this threshold, take the normalized local compatibility data and their Hecke ties. Fix $1 \le j \le m$ and a place w with nonreference tuple. Write K_{coeff} for the row coefficient field. Apply the normalized local comparison of the selected member to $W = M^{\mathrm{tw}}_{w,j}$. It assigns to each of its indices $i \in I_{j,w}$ a maximal ideal of the integral Hecke algebra on the scalar extension of W. The comparison is injective on this fibre and covers every maximal ideal at a nonreference tuple. For the member construction the map and its inverse are those of Lemma 33.54. For a selected realization in other coordinates, its comparison identifies the local coefficient rings and gives an injective semilinear map from $\mathrm{Frac}(O_i) \otimes_{O_i} M_i$ onto the whole summand for the assigned maximal ideal. Thus an O_i-basis of M_i gives a basis of that summand after scalar extension. The summands form a direct sum by the joint primary decomposition in Lemma 33.54. The dimension of W is preserved by extension of its coefficient field. Summing gives

$$\sum_{i \in I_{j,w}} \mathrm{rank}_{O_i} M_i = \dim_{K_{\mathrm{coeff}}} M^{\mathrm{tw}}_{w,j},$$

where $M^{\mathrm{tw}}_{w,j}$ is the twisted layer space.

Lean: `GlobalTheory.Top.closed_linkEta_of_duty` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LinkEta`, lines 13–22)
Lean: `Formalism.L2.locsCoverTw_of_tie` (`Formalism.LocalToGlobal.NormalizedHeckeFactors.CoverThm`, lines 121–122)

Under the level-prime bounds, the spectral hypotheses of 33.6, the signed nongeneric and equal-word bounds of 33.99, and the counts of 37.13 give the following closed estimate for members satisfying $1/3 \le f_u\Gamma^2/D^\star \le 3$ for every word. On the same selected realization, the real-place and row ties identify the layer projectors along with the volume and degree normalization. The diagonal, off-diagonal, loop and distinct-word estimates of 33.78–33.113 can then be combined with those counts. After a common point threshold, for each member they give errors η_0, η_1, e_0 for the distinct-word bound, total dimension and squared diagonal error, respectively, with

$$e_0 \ge 0, \qquad \eta_0, \eta_1, \sqrt{e_0} \le e^{-x^{\varepsilon/8}}.$$

Lean: `GlobalTheory.Top.link_alt_alt_bis` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LinkAltAltBis`, lines 16–48)
Lean: `Formalism.L2.SG.ThmSOwn.theoremS_transport_link_zero` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.LinkSPos`, lines 131–161)

For every sufficiently large selected member and each $0 \le j < m$, the level projection on row vectors at the base tuple μ_0 is surjective by 33.6. Apply this to the selected primitive rows with their level-three conditions. The row and division identifications of 33.74 give $\dim_{\mathbf{C}} Z_j(\mu_0)$ for the image dimension and D_0 for the target dimension. Thus $\dim_{\mathbf{C}} Z_j(\mu_0) = D_0$ in every lift layer, with closure on the design class.
Lean: `GlobalTheory.Top.link_alt_alt_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.Link`, lines 20–54)

Under the bounds on T_u and $D_{\Gamma,u}$, the selected row estimates of 36.7 and their transfer in 21.32 give a closed conditional estimate for the period maps, with division exponent $C_{\mathrm{abs}} = 2$. Its hypotheses are the twisted-place rank identity above and an eventual bound $\eta_j(\mathcal{L}) \le D_0 C x^{\varepsilon}(B_0 + N_c)$ with one C for all members and lift layers. Its conclusion is the period-map containment and the height and slope estimates of 21.32, with a common constant. Under the level-prime bounds and the design conditions at $3, C_H$, the real-place second-layer identifications and the volume and mass hypotheses of 33.6 give fullness of N'_u and positive rank for the input lattice of every word, as in 36.5.
Lean: `Formalism.L2.DIV.closed_linkDIVt_of_closed_bridge_kernel` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.DIVRecBridgeSel`, lines 197–205)

By the fifth family condition of 33.6, every family in $\mathcal{A}$ has a threshold above which the weighted sum $R_W(i)$ of 22.46 is at most x^{-9} for every member $i \in I_x$. For the constructed class, that duty follows from 33.26. By 22.46, every sufficiently large window point has a common weight pair with a realizing model, and each model for the pair satisfies $R_W \le x^{-9}$. The top density, the weighted bad-orbit bound and the local comparison of 22.37 now give the loss bound of 22.42 for each such model. The common-weight bound 22.44 gives $e_L = 0$. With the remaining design conditions at $3, C_H$, Theorem 24.1 gives a closed implication from the gain, frame, defect, division and slope estimates to a common chain realisation at every sufficiently large window point.
Lean: `Formalism.L2.GlobalTheoryTheory.Loss7.loss_requirements_of_small_shares` (`Formalism.LocalToGlobal.GlobalTheory.LossRequirementsOfSmallShares`, lines 337–340)
Lean: `GlobalTheory.Top.link_gl` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.LinkGl`, lines 14–31)

Combine the closed assertions by 36.10. Take the minimum of their positive ε-thresholds and the design threshold. For each smaller positive ε, take the componentwise maximum $\mathcal{K}$ of the size thresholds before choosing a family. The design gives $F_m \in W_1(\mathcal{A}_G)$ and $C_H \ge 0$ meeting the thresholds $\mathcal{K}$ of 36.10, the design conditions at $3, C_H$, and the bounds on T_u and $D_{\Gamma,u}$. This family belongs to $\mathcal{D}_3(\mathcal{A})$ and $\mathcal{E}$, and meets all the closed thresholds.

Since $F_m \in \mathcal{E}$, the cohomological lengths of 26.1 are eventually finite at every member, and their logarithmically weighted sum over every finite set of primes is at most $\sqrt{b}$. Apply 36.13 to the local length bound of 36.11 with $\dim_{\mathbf{C}} Z_j(\mu_0) = D_0$. This gives (36.18) in every lift layer. The twisted-place rank identity supplies the other hypothesis of the conditional division estimate, so the period maps satisfy 21.32 with division exponent 2. Increase the point threshold until $C_{Tu} \le x^{\varepsilon/100}$. On the slope scale X'' of 21.11, the upper bound $\log T_u \le C_{Tu} m L$ and the family size inequalities give $\log T_u \le C_T X''$ with $C_T = 2$ for every word. Likewise, the Minkowski bound above gave $\log D_{\Gamma,u} \le C f N(m) \log(f N(m))$ with $C = 12$, and the family size inequalities give $\log D_{\Gamma,u} \le C_G X''$ with $C_G = 10C = 120$. By 21.31, 15.10, and 20.9, the gains, base terms, and losses (and hence $\mathrm{sg}(u,w)$) vanish outside the primes $p_w \le X''$ together with $\{T_u, g_0, g_1, d\} \cup \{p \mid D_{\Gamma,u}\}$ (with $g_i = p_i$ for $i = 0, 1$), while 21.5–21.6, the host bounds $\log g_0, \log g_1, \log d \le X''$, and Chebyshev's bound $\sum_{p \le X''} \log p \le X'' \log 4$ yield the remainder bound $R_u \le 2(\log 4 + 3 + C_T + C_G)X''$ in 21.11. Moreover, 21.9 supplies the P_u bound, 22.37 and 23.12 supply $\mathcal{T} \le b^2$ and the off-S_T colength estimates, and $T_u \ge b^{m/2}$ gives $2f/T_u \le 1/2$ and $4fb^2/T_u + f \log(T_u - 1)/(T_u - 2) \le 1$ beyond the common cutoff. The division estimate supplies the zero-layer condition. Proposition 17.10 gives the lattice N_u. In the setup of Chapter 21, $N'_u = I_u N_u$ with I_u the fractional ideal of Definition 21.5. Products of generators of I_u and N_u give a finite generating set for N'_u. Together with the slope-rank condition, 21.11, 21.35, and 21.39 give the lattice slope comparison.
Lean: `Formalism.MainNotions.muTu_of_tuSize` (`Formalism.SharpRestriction.MuTuOfTuSize`, lines 27–40)
Lean: `Formalism.MainNotions.J.muDGam_of_fact` (`Formalism.SharpRestriction.AdmissibleParameterJoins.SumLogPrimeFactorsLe`, lines 756–765)
Lean: `Formalism.MainNotions.J.linkMu_of_linkDIV` (`Formalism.SharpRestriction.AdmissibleParameterJoins.SumLogPrimeFactorsLe`, lines 775–783)

Theorem 24.1 now gives K, c_{HH}, e_K, C_L, x_0 for this family. At every window point $x \geq x_0$, one pair of integer weight functions works at every translation, and the resulting models realise the common chain with division exponent 2 and loss exponent 0. This is the connectedness assertion used in 2.2.
Lean: `UpperTheory.Top.topDesWin` (`Development.AutomorphicArithmetic.CharacterBoxes.DesWin`, lines 11–31)

□

Proofs for Chapter 33: A common witness for the 46 conditions

Proof of 33.1

Proof. By 34.18 and the convergent Siegel–Weil input of P, the local theta hypotheses F give the separation assertion of 34.25. Apply 34.41 with this separation, the matching and integral comparison of 34.7, and the toric comparison and functional equation of 1.15 and 1.20. Its Siegel-set and real Howe hypotheses belong to F, and its cuspidal Green hypothesis belongs to P. At each admissible stage its finite system of tests has common kernel zero and takes values in the embedded coefficient field on the theta images. This gives the rational tests required in 33.29.

Apply Lemma 33.75 to P. Both statements quantify over the member points $p \in \mathcal{P}$ of Definition 35.1. For fixed eligible F_m, C_H, take the cutoff of 33.75. At every index beyond it, that lemma gives the level-one inequality at the same p for every $C \in \mathbf{R}$. The fitted-grid condition of 34.68 restricts the points at which its conclusion is used. Thus 34.68 gives the fitted functional bound. Apply 34.69 with this bound and the discrete-series, globalization, decay and ray-class inputs of P, and denote its threshold by C_{F0}. Independently, 34.44 gives a threshold C_{F1} for the grid extension, using the same local matching, the toric comparisons and the separated tests. Put

$$C_F = \max(C_{F0}, C_{F1}), \qquad c_{\mathrm{Dep}} = C_F.$$

Take A in (33.1) to be the grid extension condition of 34.42 at C_F. The inequality $C_F \geq C_{F1}$ gives its eventual existence.

Forgetting the grid and then the origin gives a list extension at the same member indices. The transfer in 33.74 therefore supplies the twelve open rows on the augmented extension. Forgetting the origin supplies the list required in 34.69. With $C_F \geq C_{F0}$, this gives the functional implication on that extension, with the family division estimate as its antecedent. Fix a datum admitted by $\mathcal{N}_A$. Forget its extra grid condition, then its origin, then its signed character list. The datum left in $\mathcal{N}_{\mathrm{list}}$ after forgetting the origin already consists of the real expansion data, the rational-column law and that list. The last projection therefore gives the expansion data with their law at the original base, frame, twist, norm choices and bundle indices. This gives the first quantified premise on extensions in 35.101. Apply that theorem with the twelve open rows just obtained and the assumed covolume bound of Definition 35.2. Choose the weak local correspondences of Theorem 40.1. Definition 1.20 includes the small-prime hypotheses of Definition 36.15 in P. Thus Theorem 36.16 supplies the linked bound at $2, 3$ for every dependence parameter, in particular for $c_{\mathrm{Dep}} = C_F$ used here. These are the hypotheses of 33.29 for the same augmented extension, so that theorem gives the required common witness.
Lean: `UpperTheory.N.boundAtTwoThree_linked_of_recT` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.At23OfAll`, lines 24–26)
Lean: `UpperTheory.Lit.every_character_of_theta_integrals_local_matching` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.SiegelEisensteinSecondTermHypotheses`, lines 69–73)

□

Proof of 33.8

Proof. Since $C_\ell > 2$ and m is even, $m \geq 4$ and $r_0 \geq 1$. The congruence modulo g^2 gives the pair congruence modulo g. It also excludes $T_u = 2$ and yields coprimality with g. The scalar bounds of 33.7 give

$$C_\ell m^6 L_b \leq m^7 L_b \leq m^5 f L_b \leq k \leq k \log k, \qquad (\log f)^2 \leq r_0 \leq m^2 L_b, \qquad \log f \leq \sqrt{r_0} \leq r_0^2.$$

The two comparisons are equalities for a repeated word and follow from 21.36 in both orders for the indicated distinct-word row. □

Proof of 33.9

Proof. For a quotient of order i, record the product of the prime ideals occurring in a composition series, with multiplicities. This ideal has norm i. Over a rational prime p, the number of ideals of norm p^e is at most the coefficient of z^e in $(1-z)^{-2f}$. Indeed, each prime above p of residue degree a contributes $(1-z^a)^{-1}$, and there are at most $2f$ such primes. The coefficient is at most $(2f)^e$, by counting ordered lists of length e. Thus the number of ideals of norm i is at most $(2f)^{\Omega(i)} \le i^\ell$, where $\Omega(i)$ counts rational prime factors with multiplicity.

Fix one such ideal and order its prime factors. Every corresponding sublattice is reached by successive quotients $O/\mathfrak{p}$ in this order, by the primary decomposition and a composition series in each primary part. At a step of norm $q = N\mathfrak{p}$, the current lattice is locally free of rank t. Its sublattices with quotient $O/\mathfrak{p}$ are the inverse images of the hyperplanes of $\mathbf{F}_q^t$. There are $(q^t-1)/(q-1) \le q^t$. Multiplication gives at most i^t choices for the fixed ideal.

Enlarge P to a maximal integral lattice J. Every integral enlargement lies in $P^\#$, so a maximal one exists. In the fixed space W all maximal integral lattices form one genus. At a split place their two components are mutually dual. At an inert place with base residue cardinality q, the maximal local model is unimodular when the determinant has even valuation. For odd valuation it has a single elementary divisor of valuation one, and its discriminant module has order q^2. These are the unramified models of [200, §3, (3.1)–(3.3) and Tables 1–2]. The norm-integrality convention there agrees with pairing integrality here. Indeed, applying norm integrality to $x+\alpha y$ for all local integers α tests the pairing against the trace dual of the local integer ring, and the relative different is trivial.

Let $b = d(J)$. Gan, Hanke and Yu's formula for a totally definite maximal lattice, applied also to the standard lattice of the same rank and over the same field, gives

$$\frac{\operatorname{mass}(J)}{M_t} = \prod_v \frac{q_v^t - (-1)^t}{q_v + 1}, \qquad b = \prod_v q_v^2.$$

Both products run over the inert places of odd determinant valuation. The global factors agree because the signatures and ranks agree. Every displayed local factor is positive and at most q_v^t [200, Propositions 4.4–4.5]. Hence $\operatorname{mass}(J) \le M_t b^{t/2}$.

Count inclusions $P \subset J$ up to rational isometry, weighting an inclusion by its inverse stabilizer order. Over P this gives at least $1/|\operatorname{Aut} P|$, since there is at least one maximal enlargement. Over J each P has the same index i, determined by $n = bi^2$. This identity follows by taking indices in $P \subset J \subset J^\# \subset P^\#$, since dualizing preserves the index. The first assertion bounds the mass of the inclusions by

$$M_t b^{t/2} i^{t+\ell} \le M_t n^{(t+\ell)/2}.$$

□

Proof of 33.10

Proof. Write $P = L \cap U$. Then L/P is torsion-free. Conversely a primitive sublattice P equals $L \cap EP$, so its span recovers U.

First count the rational isometry types of the embedded subspace. Enlarge P inside U to a maximal integral lattice J. Let $\mathfrak{b}$ be the product in O of the primes where J is not self-dual. By the local description in 33.9,

$$N\mathfrak{b} = d(J), \qquad a = d(J)[J:P]^2.$$

At inert places the support of $\mathfrak{b}$ therefore specifies the parity of the determinant valuation. Units are norms in an unramified quadratic local extension. At split places the Hermitian space of rank j is unique, and at every real place the signature is $(j,0)$. Landherr's classification now determines U up to isometry from $\mathfrak{b}$ [146, §3, Theorem 3.1]. Its orthogonal complement is determined as well: its local determinant class is that of V divided by that of U, and its signatures are $(s,0)$. Taking the orthogonal sum of the two isometries identifies the embeddings in V. By the ideal count in the proof of 33.9, there are at most $\sum_{i \le a} i^\ell \le a^{1+\ell}$ orbits of U.

Fix such a subspace. Write $W = U^{\perp}$, $Q = L \cap W$, $b = d(Q)$, and $H = L/(P \oplus Q)$. Integrality embeds H in $P^{\#}/P \oplus Q^{\#}/Q$. Each projection of H is injective, since its kernel is represented by a vector in $L \cap U$ or $L \cap W$. Taking indices gives

$$ab = D|H|^2, \qquad |H| \le a, \qquad b \le aD.$$

The discriminant module of Q needs at most s generators. Locally this follows by diagonalizing the inclusion $Q \subset Q^{\#}$ over a discrete valuation ring, and the local lists combine by the Chinese remainder theorem. The orthogonal projection $p : V \to U$ satisfies $p(L^{\#}) = P^{\#}$. Indeed, an integral functional on P extends to L because L/P is projective over O. Hermitian duality identifies restriction of functionals with p. Now $p(L)/P \simeq H$ embeds in $Q^{\#}/Q$, so it too needs at most s generators. The quotient $p(L^{\#})/p(L)$ is an image of the cyclic module $L^{\#}/L$. Thus $P^{\#}/P$ needs at most $s+1$ generators.

For fixed P, Q, the possible L's are determined by their submodules H of the direct sum of the two discriminant modules. A finite O-module of order m generated by e elements has at most m^e submodules. Each submodule has at most e generators by the local elementary divisor theorem and the Chinese remainder theorem, and hence is the span of an ordered e-tuple. Here $e = 2s+1$. Put $X = aD$. The number of gluings is at most $(ab)^{2s+1} \le (aX)^{2s+1}$.

To sum the weights, observe that for any action of a finite group A on a finite set S, $\sum_{[z] \in S/A} |\operatorname{Stab}_A z|^{-1} = |S|/|A|$. Applied over L, this counts pairs (L, U) with the weight on the left of the lemma. Applied with U fixed, it divides the number of gluings for P, Q by $|\operatorname{Aut} P|\,|\operatorname{Aut} Q|$, since the rational stabilizer of U is $\mathrm{U}(U) \times \mathrm{U}(W)$. Thus all stabilizers of glued lattices are included. Apply 33.9 in U at order a and in W at each order $1 \le b \le X$. The resulting bound is

$$a^{1+\ell}\, M_j a^{(j+\ell)/2}\, X M_s X^{(s+\ell)/2}\, (aX)^{2s+1}.$$

Retain $a^{j/2}$ and use $a \le X$ in the remaining factors. The exponent of X is $(1+\ell)+\ell/2+1+(s+\ell)/2+4s+2 = (9s+8+4\ell)/2$. □

Proof of 33.11

Proof. We first bound M_t for $t = 1, 2$. On a lattice J of the standard genus take the Euclidean metric whose squared length is $\operatorname{Tr}_{E/\mathbf{Q}} h(z, z)$. If $\mathfrak{D}_E$ is the absolute different, the trace dual is $\mathfrak{D}_E^{-1} J$, since J is Hermitian self-dual. For any nonzero z in the trace dual, $\mathfrak{D}_E^2 h(z, z)$ is an integral ideal. Its norm is at least one. The arithmetic-geometric mean inequality at the $2f$ embeddings gives

$$\operatorname{Tr}_{E/\mathbf{Q}} h(z, z) \ge 2f/\rho^2.$$

The trace lattice is full of Euclidean dimension $2ft$. Banaszczyk's successive-minima inequality $\lambda_{2ft}(J)\lambda_1(J^*) \le 2ft$ therefore gives $2ft$ independent vectors of squared length at most $2ft^2\rho^2$ [78, Theorem 2.1]. Choose t among them spanning over E. At each embedding their Gram entries have absolute value at most $B = 2ft^2\rho^2$, by positivity and Cauchy–Schwarz.

There are at most $(1+2B)^{2f}$ possibilities for a single integral entry. In the product of the f complex discs, distinct algebraic integers have sup distance at least one by the norm inequality. The open balls of radius $1/2$ around them fit inside the ball of radius $B + 1/2$. For a fixed Gram matrix let P be the lattice freely generated by the chosen vectors. The determinant bound gives $d(P) \le (tB)^{2ft}$. The original lattice lies between P and $P^{\#}$, and $P^{\#}/P$ is generated by at most t elements. The submodule count in the proof of Lemma 33.10 gives at most $(tB)^{2ft^2}$ possibilities. The Gram matrix and this intermediate lattice determine the isometry class. Consequently

$$M_t \le (1+2B)^{2ft^2} (tB)^{2ft^2} \le (20 f \rho^2)^{16f} \le M_*,$$

using $t \le 2$, $\rho \ge 2$ and $c \ge 2$.

For $i \ge 3$ put $R_i = M_i/M_{i-1}$, $A = \rho^{f/2}$, and $B_0 = 16^f$. The Euler factor $L_\eta(E, i)$ of 37.5 is between 2^{-f} and 2^f: both it and its reciprocal are at most $\zeta_F(i) \le \zeta(i)^f \le 2^f$. Formula (37.9), with $v_E \ge 1$, now gives

$$R_i^{-1} \le A B_0^i, \qquad R_i/R_{i+t} \le B_0^t \quad (t \ge 1).$$

For the first inequality use $(i-1)!(16v_E/\pi)^i \geq 2$. For the second use the monotonicity of the factorial and $4 \leq (16v_E/\pi)^t$, which absorbs the quotient of the two Euler factors.

Write $j = k - s$. If $s, j \geq 2$, then

$$\frac{M_s M_j}{M_k} = \frac{M_2}{R_{j+1}R_{j+2}} \prod_{i=3}^{s} \frac{R_i}{R_{j+i}} \leq M_* A^2 B_0^{2j+3+j(s-2)} \leq M_* A^2 B_0^{sk}.$$

If either rank is one, the quotient is $M_1/R_k \leq M_* A B_0^k$. Since $A \geq 1$ and $s \geq 1$, this gives the same bound. □

Proof of 33.12

Proof. Choose one E-prime from each of the f conjugate pairs over T. For each L in the genus, take the inverse image in $L^\#$ of the sum of the chosen components of $L^\#/L$. Each component is a copy of $\mathbf{F}_T$. The discriminant pairing pairs opposite components perfectly, so the chosen sum equals its annihilator. Its inverse image J is self-dual and is determined by L and the choice of primes.

Conversely, for fixed J, at each chosen prime one obtains L as the inverse image of a hyperplane of $J/\mathfrak{p}J$; the component at the conjugate prime stays fixed. Away from T one takes $L = J$. These prescriptions give distinct lattices. They belong to the given genus: at a split place the stabilizer acts as the full general linear group on the chosen component and transitively on its residual hyperplanes. Away from T the local lattices are self-dual. The genus mass of J is M_k, by the maximal-lattice formula in the proof of 33.9 with empty local product. Counting the fibres with inverse stabilizer weights gives the asserted equality and, in particular,

$$m_0 \geq M_k T^{f(k-1)} = M_k D^{(k-1)/2}.$$

All members of the genus have cyclic discriminant module of order D, since their local discriminant modules agree. Apply 33.10 with $j = k - s$, divide by the last lower bound and use 33.11. The remaining powers satisfy

$$D^{-(k-1)/2} a^{(k-s)/2} (aD)^{(9s+8+4\ell)/2} \leq (a/D)^{k/2} (aD)^{5s+5+2c}.$$

Indeed, the right side divided by the left side is $a^{s+3} D^{(s+5)/2} \geq 1$. □

Proof of 33.14

Proof. Write C, C' for the diagonal lattices of the two graphs. On U the compatibility of the graphs gives $C \cap U = C' \cap U = P$. Put $y = [L \cap U : P]$. Then $1 \leq y \leq x$ and $d(P) = ay^2$.

We prove the estimate for this discriminant. Let p be orthogonal projection onto U, and put $W = U^\perp$. For $z \in C'$, $w \in C$, the pairs $(\eta z, z), (w, w)$ lie in Γ. Preservation of the form modulo g gives $g^{-1}(1-\eta)C' \subset C^\#$. Also $C' \subset L \subset C^\#$. The lattice

$$B = (C' \cap U) \oplus (1-p)C'$$

has the same covolume as C', by projection onto W with kernel $C' \cap U$. The map S equal to the identity on U and to $(1-\eta)/g$ on W is invertible. It sends B into $C^\#$, and $p(SB) = P$. Since $p(C^\#) = P^\#$ by the restriction-of-functionals argument of 33.10, projection gives

$$d(P) \leq [C^\# : SB] = |N_{E/\mathbf{Q}} \det S| \, [C^\# : C'].$$

At every complex embedding $\eta|_W$ is unitary, so $|\det(1-\eta|_W)| \leq 2^s$. Moreover $[C^\# : C'] = [C^\# : L^\#][L^\# : L][L : C'] = xDx'$. Hence $ay^2 \leq A_s$.

There are at most x indices y. At any such index, 33.9 gives at most y^{k-s+c} possibilities for P, with one possibility in rank zero. Since $y \leq x \leq X$,

$$y^{k-s+c} \leq (A_s/a)^{k/2} X^c.$$

For fixed P, when $s = 0$ we have $U = V$ and $C = P$ is uniquely determined (giving factor $1 \leq x^{2s+c}$). When $s \geq 1$, project C to the rank-s space V/U; its image has index x/y in the image of L, so Lemma 33.9 gives at most $(x/y)^{s+c}$ such images. For a fixed image, lifts are splittings modulo P, differing by a homomorphism

to $(L\cap U)/P$. The image is projective of rank s, so there are at most y^s homomorphisms, by local freeness at the primes dividing y. This bounds the number of choices for C by $(x/y)^{s+c}y^s\le x^{2s+c}$ for every $s\ge 0$. Applying the same argument symmetrically on the primed side (using $C'\cap U=P$ and $[L:C']=x'$) bounds the number of choices for C' by $(x')^{2s+c}$, so the pair (C,C') has at most $x^{2s+c}(x')^{2s+c}=X^{2s+c}$ possibilities.

For a fixed C, one has $C\subset K\subset C^{\#}$. The quotient $C^{\#}/C$ has order Dx^2 and at most $2r+1$ generators, by the filtration through $L, L^{\#}$. Its successive quotients need at most $r,1,r$ generators, the last by duality. The submodule count in the proof of Lemma 33.10 gives at most $(Dx^2)^{2r+1}$ possibilities for K. Choose at most $2r+1$ generators z of K/C. For each generator choose w with $(z,w)\in\Gamma$. Pairing with (v,v), $v\in C$, shows that $(z-w)/g\in C^{\#}$. Its class modulo L determines w mod gL. There are $[C^{\#}:L]=Dx$ such classes. The graph is fixed on C, so these images determine it. Thus the lattice and graph together contribute at most $(Dx)^{3(2r+1)}$ for fixed C. The same count holds on the primed side.

For fixed data the possible isometries, if any, inject by division by one of them into the automorphisms of K' fixing U pointwise. This finite group acts faithfully on V/U, since it preserves W. Its rational representation has dimension $2sf$. For $s>0$, Minkowski's finite-subgroup bound [143, Theorem 1(i)] gives an order dividing the Minkowski number of 33.6, which divides $(4sf)!$. For $s=0$ the group is trivial. Multiplying the counts gives

$$x(A_s/a)^{k/2}X^c\,X^{2s+c}Y^e(4sf)^{4sf}.$$

Use $x\le X$ to obtain the first bound.

If $U=0$ and a triple exists, the discriminant inequality gives $1\le(2/g)^{2kf}DX$. Count C,C' directly in L, using 33.9; the unrestricted count of triples is at most $X^{k+c}Y^e(4kf)^{4kf}$. Now $X^c\le Y^c$, and $X^{k/2}((2/g)^{kf}D)^k\ge 1$, as is seen by squaring and using $D\ge 1$. This gives the second bound. □

Proof of 33.15

Proof. Fix the length shells, write $u=2^{i+i'}$, and put $M=R^rE$. Partition $(0,M]$ into the intervals $(M2^{-a-1},M2^{-a}]$. Use indices a,a' for Y,Y', with lower endpoints y,y'. Write $w=yy'$ and $N=2^j/D$. On a nonempty piece,

$$w\le N,\qquad w\le R^{2r}E^2 2^{-a-a'},\qquad 1\le NR^{2r},\qquad s_*\le u^r.$$

Indeed $SS'\le u(2H)^2\le R^2$. Cauchy–Schwarz on the piece gives $\sqrt{KK'}R^ew^hu^{-q}$. In $w^h=w^{h-1}w$, use the first bound for w^{h-1}, the second for w, and multiply the resulting upper bound by $NR^{2r}\ge 1$. The result is bounded by

$$\sqrt{KK'}(2H)^{e+4r}E^2N^h s_*^{-(q-e-4r-1)/r}\,2^{-i-i'-a-a'}.$$

The four geometric series each have sum two. Positivity permits summation over the countable partition. □

Proof of 33.16

Proof. Every translate of w is smooth. Linearity gives

$$\Phi(g)=\|v\|cL(\sigma(g)w).$$

Squaring and integrating gives $I_B=\|v\|^2|c|^2/d$. Positivity of I_B gives $c\ne 0$, and division by I_B proves the formula for ρ.

For the positive moment, use

$$l_a=l_0\prod_{i=0}^{a-1}R_i.$$

Hölder's inequality with $a+1$ equal exponents gives

$$\int l_a^{2q}\,d\rho\le\left(\int l_0^{2(a+1)q}\,d\rho\right)^{1/(a+1)}\prod_{i=0}^{a-1}\left(\int R_i^{2(a+1)q}\,d\rho\right)^{1/(a+1)}\le A^qR^{2aq}.$$

For the inverse moment,

$$l_a^{-1}=l_r^{-1}\prod_{i=a}^{r-1}R_i\le\prod_{i=a}^{r-1}R_i.$$

When $r \geq 1$, apply Hölder with r equal exponents, filling the unused positions with the constant function 1. Since ρ has mass one,

$$\int l_a^{-2q}\, d\rho \leq \prod_{i=a}^{r-1} \left(\int R_i^{2rq}\, d\rho \right)^{1/r} \leq R^{2(r-a)q} \leq R^{2rq}.$$

For $r = 0$, the inverse bound follows from $l_0 \geq 1$. For $q = 0$, both assertions follow from $\rho(S\backslash G) = 1$. □

Proof of 33.17

Proof. Each factor is a nonzero integral ideal. Products of integral ideals are contained in each factor, so $\mathfrak{c} \subset (T)$, $\mathfrak{c} \subset \mathfrak{a}_l$, and $\mathfrak{c} \subset \overline{\mathfrak{a}_l}$ for $l \in S$. Conjugation exchanges the two factors in every pair and fixes (T). Thus $\bar{\mathfrak{c}} = \mathfrak{c}$. The defining identity for $\mathfrak{a}_l$ gives $\mathfrak{a}_l \mathfrak{d}^{2q_l} = (p_l)$. Its conjugate gives the second divisibility, since $\bar{\mathfrak{d}} = \mathfrak{d}$. For $l = 0$, use $q_0 = 0$ and $p_0 = 1$. Conjugation induces a bijection $\mathcal{O}_E/\mathfrak{a}_l \simeq \mathcal{O}_E/\overline{\mathfrak{a}_l}$, so the two ideals have the same norm. Multiplicativity gives

$$N(\mathfrak{c}) = T^{2f} \prod_{l \in S} N(\mathfrak{a}_l)^2 = (D_b N_T)^2.$$

Every nonzero integral ideal has positive integral norm, which proves the remaining inequalities. □

Proof of 33.18

Proof (outline). Write the fixed and moving frames as $(e_0, \ldots, e_{r-1}, z)$ and $(e'_0, \ldots, e'_{r-1}, z')$ in the Hermitian space with form H_E. The first r vectors span totally isotropic spaces X, X', and the final vectors are perpendicular to their respective spaces and have norm $-T_u$. Put $e_r = z$, $e'_r = z'$. Let X_a, X'_a be the spans of the first a vectors, $I = X \cap X'$, and $j = r - \dim_E I$. Write $p_l = \det(H_E(e_a, e'_b))_{0 \leq a,b < l}$, with $p_0 = 1$. The condition $G(t) = 0$ is $p_{r+1}\bar{p}_{r+1} = T_u^2 p_r \bar{p}_r$. *Claim 1.* A relevant degenerate frame pair has the following common normal form, and a nonidentity term has $j \geq 1$. The normal form is

$$X = X_j \oplus I, \qquad X' = X'_j \oplus I, \qquad p_j \neq 0, \qquad z' - cz \in X + X', \quad c\bar{c} = 1. \tag{A.293}$$

For $I \neq 0$, take $x_* \in I \setminus \{0\}$ of levels a, a' in the two flags, so $x_* \equiv \alpha e_a \bmod X_a$ and $x_* \equiv \alpha' e'_{a'} \bmod X'_{a'}$ with $\alpha, \alpha' \neq 0$. For $v \perp X_{a+1} + X'_{a'+1}$, the skew-Hermitian map

$$w \longmapsto H_E(w, v)x_* - H_E(w, x_*)v$$

lies in both infinitesimal flag stabilizers. Relevance means their infinitesimal characters agree on the intersection. In these coordinates the character pairs the level-a coefficient with e_{a+1} in the first flag, and the level-a' coefficient with $e'_{a'+1}$ in the second. With all scalar multiples of v, the equality and the nondegenerate trace pairing of E/F give $\alpha e_{a+1} - \alpha' e'_{a'+1} \in X_{a+1} + X'_{a'+1}$. Subtract vectors of X_{a+1} and $X'_{a'+1}$ to obtain a common vector one level higher with the same leading coefficients, for as long as both levels are below r. Reaching the final vector on one side first would place it in $X + X'$. Every vector of $X + X'$ perpendicular to X is isotropic, as is every vector of $X + X'$ perpendicular to X'. The negative final norms therefore force $a = a'$. Iteration to the last step gives $z' - cz \in X + X'$ with $c = \alpha/\alpha'$, uniquely since $z \notin X + X'$.

Starting at the least level in I gives a vector at each subsequent level through $r - 1$. Elimination of leading coefficients shows that these vectors form a basis of I, so the least level is j and the direct sums in (A.293) hold. If $p_j = 0$, there is $0 \neq y \in X_j$ perpendicular to X'_j. It is perpendicular to I, hence to X', and the final-vector relation gives $y \perp z'$. The perpendicular of the moving frame modulo X' is positive definite, which would put y in $X_j \cap I = 0$. Thus $X_j + X'_j$ is nondegenerate. Project z, z' onto its perpendicular by subtracting vectors of X_j, X'_j, respectively. These projections retain norm $-T_u$, and the second differs from c times the first by an element of I. Both are perpendicular to I, so their equal norms give $c\bar{c} = 1$.

If $I = 0$, then $j = r$. Were $p_r = 0$, $G(t) = 0$ would give $p_{r+1} = 0$. Choose $0 \neq y \in X$ perpendicular to X', and a nonzero $y' + \beta z' \in X' + Ez'$ perpendicular to $X + Ez$. Positivity of the perpendicular modulo X gives $\beta \neq 0$, because an isotropic vector there belongs to X and $X \cap X' = 0$. Positivity modulo X' gives $H_E(y, z') \neq 0$. This contradicts $H_E(y, y' + \beta z') = 0$. With $p_r \neq 0$, subtract vectors of X, X' from z, z' to project them to $(X + X')^\perp$. They retain norm $-T_u$ and their mutual pairing is the Schur complement

p_{r+1}/p_r, up to conjugation. Hence their Gram determinant is zero. The complement has one negative direction. Subtract the multiple c of the first projection that makes the difference perpendicular to it. The difference is isotropic in a positive definite space and thus vanishes. Equality of the norms gives (A.293).
Lean: `Formalism.HermitianLatticeCounting.P.existsNormalForm` (`Formalism.FlagUnipotentLattices.MinkowskiFactorialBounds.ExistsHasLeadingCoeffOfMemSpanBelow`, lines 1421–1457)

If $j = 0$, a common vector of each level gives $X_a = X'_a$ and $e'_a - ce_a \in X_a$. In $X^\perp/X$ the two final vectors differ by the scalar c. Their primitive lines in the row lattice make c a unit. Every conjugate of c has absolute value one, so two of its powers coincide by discreteness of the embedded ring of integers. Pairing the level congruence with the fixed final vector gives $T_u(c-1) = 0$ modulo each prime above $g_0 g_1$. Both primes are prime to T_u. For an element of order $d > 1$ prime to the residue characteristic in the kernel of reduction, $(X^d - 1)/(X - 1)$ vanishes at that element but reduces to d, a contradiction. The order of c is consequently a power of each of the distinct primes g_0, g_1, and $c = 1$.

Let Λ be the row lattice. On X the map $U_X : e_a \mapsto e'_a$ is unitriangular. Locally the two frames rescaled by their graded ideals give bases of $X \cap \Lambda$. Equality in each graded line identifies the rescaling ideals, so U_X is integral at each prime and hence on $X \cap \Lambda$. This sublattice is saturated, so the quotient is projective over $\mathcal{O}_E$ and gives an integral retraction P_0 onto $X \cap \Lambda$. Unimodularity makes its Hermitian adjoint integral. Choose $\theta \in \mathcal{O}_E$ with $\theta + \bar{\theta} = 1$ by trace surjectivity for the finite unramified extension. Then $P_X = P_0 - \theta P_0 P_0^*$ is the identity on X and satisfies $P_X P_X^* = 0$, by $P_0^* P_0 = 0$ and $P_0^2 = P_0$. For $D_X = (U_X - 1)P_X$ we have $D_X D_X^* = D_X^* D_X = 0$ and $D_X^{r+1} = 0$. Hence $(1 + D_X)(1 + D_X^*)^{-1}$ is an integral isometry. Both inverses are finite geometric sums through degree r. The isometry sends e_a to e'_a, and for $z \in X^\perp$ it changes z by an element of X. Since $z' - z \in X$, these are the frame conditions for the identity cell at the tied level. Thus $j \geq 1$ for the terms in the proposition.
Lean: `Formalism.MainNotions.BridgeLemma.l_iso` (`Formalism.SharpRestriction.CuspSpanNormBounds.BruhatCellsKronecker.LisoSmulMem`, lines 15–65)
Lean: `Formalism.MainNotions.BridgeLemma.id_lift` (`Formalism.SharpRestriction.CuspSpanNormBounds.BruhatCellsKronecker.IdLift`, lines 249–250)
Lean: `Formalism.L2.SG.KappaOffDiagonal.one_le_index_of_relevant` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.IsIdentityOfEqualFlags`, lines 71–86)

For a term t let w_t be the geometric mean of the two inverse row-stabilizer orders and ω_t the infimum of the inverse relative stabilizer volumes over adapted forms. We prove

$$\|v_t\| \leq w_t \sqrt{\dim \pi_u}\, \omega_t A(j, \delta), \qquad \sum_{(t,t') \in U} w_t w_{t'} \omega_t \omega_{t'} \leq \mathrm{mass}_u\, M(j, \delta) M(j', \delta'). \tag{A.294}$$

The graded adapted bases put the reduced pairing in block antidiagonal form, with nonzero leading minors at the block ends. At a place τ put $P_\tau = d_{\tau j}$ and

$$s_\tau = \max(1, \max_{1 \leq l \leq j} d_{\tau l}/(2H_1)^{2r}), \qquad e_\tau = \max(1, \max_{\substack{1 \leq l \leq j \\ \delta_l \neq 0}} d_{\tau l}^{-1}),$$

where $d_{\tau l} = 0$ for a zero minor. The normalized nonzero minor ideals are integral, so $\prod_\tau d_{\tau l} = \sqrt{d_l} \geq 1$ and $\prod_\tau P_\tau = \sqrt{D_\delta}$. For each minor the product of the factors above one bounds the product of the reciprocal factors below one. Taking the product over the minors gives

$$\prod_\tau e_\tau \leq \prod_\tau \big((2H_1)^{2r} s_\tau\big)^{r+1}. \tag{A.295}$$

Claim 2. Let $k \geq 1$, $n_0 = 2r + k + 1$, and let a unitary discrete series have regular Harish–Chandra parameter λ, Cayley signature ϵ, and opposite-sign gap at least $h_0 \geq 120 n_0^3$. Let the final Bessel-frame vector lie in the prescribed frame space and have positive norm parameter, let every simple-root character parameter be nonzero, and let the compact Bessel representation have its prescribed Harish–Chandra type. For any two continuous linear Bessel maps B_1, B_2 on the smooth representation there are $a, b \in \mathbf{C}$, not both zero, with

$$aB_1(v) + bB_2(v) = 0 \qquad (v \text{ smooth}).$$

This is Lemma 33.23, the large-gap multiplicity-one input. The same local input includes realization of a reproducing Whittaker–Bessel coefficient by a smooth Bessel map. Its combination with the discrete-series

branching theorem, multiplicity one of the lowest compact type, and the formal-degree normalization supplies the local data used below.

Lean: GlobalTheory.GapIn.UniqGap (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.UniqGapOfUniq, lines 17–29)
Lean: GlobalTheory.GapIn.Hyp (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.UniqGapOfUniq, lines 34–35)
Lean: GlobalTheory.uniqGap_and_besselOfW (Development.AbelianVarietyTheory.Proofs.FlagMeasures.BesselMultiplicityOneLargeGapAndWhittakerBessel, lines 9–10)
Lean: GlobalTheory.GapIn.hyp_of_uniqGap (Development.AbelianVarietyTheory.Proofs.FlagMeasures.HypOfUniqGap, lines 9–11)

Claim 3. The stated size and gap hypotheses provide a normalized Bessel family through the given representation with all the following numerical and measure properties. Put $h = k-1-j$ and $q = (\lceil C_s \rceil+2)kr$. The thresholds give $q \le km^3$ and $h \ge k/2$. At a real place put $n_0 = k+2r+1$, $T = T_u$, $\beta = b^{\max(2,C_w)}$, and $q_0 = km^5$. Use the normalized Bessel family through the given representation, indexed by integral shifts ν with $\|\nu\|_1 \le q_0$. Its representation at ν has parameter $\lambda+\nu$, its standard period is L_ν, and $d_\nu > 0$ is its formal degree for the specified group Haar measure. The family includes the numerical gap conditions, the quotient-measure identity

$$\int_G f(g)\,dg = \int_{S\backslash G}\int_{U\times N} f(anx)\,da\,dn\,d\mu(x) \qquad (f \ge 0 \text{ measurable}),$$

with $S = U \ltimes N$ and the product Haar measure $da\,dn$ on S, and the factorization of the Bessel integral as the rank-one operator defined by L_ν. This factorization is Claim 1a in the proof of Lemma 33.23, applied at each shift. Smooth vectors lie in the space V_{fl} there, and the large-gap condition on the shift ball, with separation $n_0^{19}/2 \ge 120n_0^3$, meets the gap hypothesis of that lemma. The original representation is at $\nu = 0$. In the numerical family the local scale b_v satisfies

$$10^6 n_0^7 \le b_v \le \beta, \qquad n_0 \le \beta, \qquad 6b_v \le \beta^2.$$

The spectral separation and compact-weight parameters are n_0^{19} and $n_0 m^{10}$, respectively. The size hypotheses give the required numerical conditions on the entire shift ball, including the large-gap condition with separation $n_0^{19}/2$. Moving the character transports the unipotent measure by $n \mapsto t^{-1}nt$ and the quotient measure by $x \mapsto t^{-1}x$; the group Haar measure, and hence d_ν, stay fixed. The normalized periods are chosen for these transported measures.

Lean: BranchingSupport.SideFam (Development.AbelianVarietyTheory.Definitions.Jets.SideFam, lines 36–40)
Lean: GlobalTheory.SizeNum.num_of_size (Development.AbelianVarietyTheory.Proofs.FlagMeasures.MomentPieriNumericsOfSizeHyps, lines 175–484)
Lean: GlobalTheory.Amp.LocalSetupGap.exists_setup_both (Development.AbelianVarietyTheory.Proofs.FlagMeasures.AmplitudeLocalDataMomentPieriHypotheses, lines 59–97)
Lean: BranchingSupport.Setup (Development.AbelianVarietyTheory.Definitions.Jets.SideFam, lines 42–45)
Lean: BranchingSupport.IsBeL (Development.AbelianVarietyTheory.Definitions.Jets.SumIoPrAlt, lines 135–139)
Lean: GlobalTheory.TwoSidedMomentsGap.two_sided_moments_gap (Development.AbelianVarietyTheory.Proofs.FlagMeasures.MajorantLengthTwoSidedMomentsLargeGap, lines 206–340)

Claim 4. Every nonzero square-integrable lowest-type Bessel coefficient has the period normalization displayed below. Let μ be the quotient measure and $\Phi(x) = B(\sigma(x)w)$, where w is a lowest-compact-type unit vector. All square norms in this paragraph use μ. A zero coefficient contributes zero. Suppose $0 < \|\Phi\|_2^2 < \infty$. The period nonvanishing identity applied to this coefficient produces a K-finite vector on which L_0 is nonzero. Bessel multiplicity one gives $B = cL_0$ on smooth vectors. The coefficient-period norm identity gives the normalization. Choose the K-finite vector v just obtained, so $L_0 v \ne 0$. Write $S = U \ltimes N$ for the Bessel subgroup, ψ for its normalized unipotent character, and da, dn for the two Haar measures in the family. For a smooth vector y, define

$$\mathcal{B}_{y,v} = \int_{U\times N} \langle v, \sigma(an)y\rangle \overline{\psi(n)}\pi(a)^{-1}\,da\,dn.$$

Here W_p is the finite-dimensional Hilbert space of the Bessel representation π, and the integral is a Bochner integral with values in its continuous linear endomorphisms. The inner product is conjugate-linear in its first argument. By Claim 3, which at $\nu = 0$ is Claim 1a in the proof of Lemma 33.23, these integrals converge absolutely for the vectors used here and satisfy

$$\mathcal{B}_{y,v}\eta = \langle L_0 v, \eta\rangle L_0 y \qquad (\eta \in W_p).$$

Lean: Formalism.FourierDecayAlt.ThreeDimData.be (Formalism.HermitianCurvature.BesselNonvanishingReduction, lines 47–51)
Lean: Formalism.FourierDecayAlt.ThreeDimData.beInt (Formalism.HermitianCurvature.BesselNonvanishingReduction, lines 40–45)
Lean: BranchingSupport.IsBeL (Development.AbelianVarietyTheory.Definitions.Jets.SumIoPrAlt, lines 135–139)

In particular,

$$\mathcal{B}_{y,v}(L_0 v) = \|L_0 v\|^2 L_0 y.$$

The coefficient-period norm identity states that $\int \|\mathcal{B}_{\sigma(x)w,v}\xi\|^2\, d\mu(x) = d_0^{-1}\|w\|^2\,\mathrm{Re}\langle \xi, \mathcal{B}_{v,v}\xi\rangle$ for every smooth w, every K-finite v and every $\xi \in W_p$, with the group Haar measure defining d_0 and the quotient measure μ. It follows by unfolding the quotient integral to an integral over G, Fubini's theorem, and Schur orthogonality for the formal degree d_0. With $\xi = L_0 v$ it reads

$$\int \|\mathcal{B}_{\sigma(x)w,v}(L_0 v)\|^2\, d\mu(x) = \frac{\|w\|^2}{d_0}\,\mathrm{Re}\langle L_0 v, \mathcal{B}_{v,v}(L_0 v)\rangle = \frac{\|w\|^2\|L_0 v\|^4}{d_0}.$$

Substitute the rank-one formula and cancel the positive number $\|L_0 v\|^4$. This proves that the Bessel realization $w \mapsto [x \mapsto \sqrt{d_0} L_0(\sigma(x)w)]$ is isometric on smooth vectors, and in particular

$$d_0 \int \|L_0(\sigma(x)w)\|^2\, d\mu(x) = 1.$$

Lean: GlobalTheory.scaled_bessel_isometric (Development.AbelianVarietyTheory.Proofs.JetHarmonics.GapCoefficientDerivativeBound, lines 27–50)

Lemma 33.16 consequently gives $c \neq 0$ and the probability measure

$$\|\Phi\|_2^2 = \frac{|c|^2}{d_0}, \qquad d\rho(x) = \frac{\|\Phi(x)\|^2}{\|\Phi\|_2^2}\, d\mu(x) = d_0\|L_0(\sigma(x)w)\|^2\, d\mu(x).$$

Lean: GlobalTheory.T.exists_kFinite_besselPeriod_ne_zero_gap (Development.AbelianVarietyTheory.Proofs.FlagMeasures.ExistsKFiniteBesselPeriodNeZeroGapStatement, lines 12–13)
Lean: GlobalTheory.GapLine.bessel_line_gap (Development.AbelianVarietyTheory.Proofs.FlagMeasures.BesselMapsProportionalDiscreteSeriesLargeGap, lines 45–53)
Lean: Gp91.GlobalTheory.T.bessel_functional_l2_bound_and_isometry (CompanionTheory.AutomorphicTheory.PieriRules.ThetaBesselIsometries.FamilySetupBesselGrowthStatement, lines 16–17)
Lean: GlobalTheory.MomMass.mom_mass (Development.AbelianVarietyTheory.Proofs.JetHarmonics.FamilySetupBesselGrowth, lines 21–56)

Claim 5. The normalized probability measure of Claim 4 has the two moment estimates and the tail estimate below, with the stated separate degree and spectral reserves. The majorant $l_a(x)$ is the Euclidean distance of $x^{-1}e_a$ from the span of the preceding transformed frame vectors. It is positive almost everywhere. In normalized character coordinates the two moment estimates, with their separate ranges, are, for integers $p \geq 0$,

$$\begin{aligned} &A_0 = \max(1,T)\beta^{20(r+1)}, \qquad R_0 = \beta^2, \\ &\mathbb{E}_\rho l_a^{2p} \leq A_0^p && \big(0 \leq a \leq r,\ (r+1)p \leq q_0,\ (r+1)p \leq 2n_0^2\big), \\ &\mathbb{E}_\rho (l_{a+1}/l_a)^{2p} \leq R_0^{2p} && \big(0 \leq a < r,\ p \leq 2n_0^2\big). \end{aligned} \tag{A.296}$$

The positive-moment estimate includes the initial length $a = 0$. At a general shift ν, its spectral condition is $\|\nu\|_1 + (r+1)p \leq q_0$; the ratio estimate requires $\|\nu\|_1 \leq q_0$.
Lean: BranchingSupport.lenA (Development.AbelianVarietyTheory.Definitions.Jets.SumIoPrAlt, lines 150–153)
Lean: BranchingSupport.Hsq (Development.AbelianVarietyTheory.Definitions.Jets.SideFam, lines 34–34)
Lean: GlobalTheory.T.oneMajorantLength_moment_lowestKType (Development.AbelianVarietyTheory.Proofs.FlagMeasures.OneMajorantLengthMomentLowestKTypeStatement, lines 12–13)

Fix a unitary Cayley matrix Q. On the smooth vectors of the representation with parameter $\lambda + \nu$, put

$$E_{ab} = d\sigma_\nu(Q^{-1}e_{ab}Q), \qquad [E_{ab}, E_{cd}] = \delta_{bc}E_{ad} - \delta_{da}E_{cb}.$$

Here e_{ab} is the elementary matrix. Let ϵ be the signature, Λ_ν the highest weight of the lowest compact type, and C_K its compact Casimir operator in this Cayley realization. Write $c_K(\Lambda_\nu)$ for the corresponding lowest-type eigenvalue. The phrase "level i" means the conjunction of the following two conditions:

$$v \in \mathrm{span}_{\mathbb{C}}\{x :\ \exists \omega \in \mathbb{R}^{n_0},\quad E_{aa}x = \omega_a x\ (0 \leq a < n_0),\quad \textstyle\sum_a |\omega_a - (\Lambda_\nu)_a| \leq i\},$$

and there is a finite-dimensional complex subspace S containing v, invariant under every E_{ab} with $\epsilon_a = \epsilon_b$, such that

$$\operatorname{Re}\langle x, C_K x\rangle \le \left(\sqrt{c_K(\Lambda_\nu)} + 4n_0 i\right)^2 \|x\|^2 \qquad (x \in S).$$

The span in the first condition permits a sum of different weights; the Casimir inequality must hold on all of S, not just on v.

Lean: `BranchingSupport.Lev` (`Development.AbelianVarietyTheory.Definitions.Jets.SumIoPr`, lines 23–25)
Lean: `BranchingSupport.WtNear` (`Development.AbelianVarietyTheory.Definitions.Jets.SumIoPr`, lines 20–21)
Lean: `BranchingSupport.CasKLe` (`Development.AbelianVarietyTheory.Definitions.Jets.SumIoPr`, lines 16–18)
Lean: `Formalism.REAL.IsWeight` (`Formalism.RealPlaceAnalysis.StatementA`, lines 26–28)
Lean: `BranchingSupport.Rep.IsE` (`Development.AbelianVarietyTheory.Definitions.Jets.SumIoPrAlt`, lines 51–53)

We now give the iteration for the ratio estimate. For a highest-weight unit vector in the lowest compact type, set $R_a = l_{a+1}/l_a$ and

$$M_p(v) = \int R_a^{2p} d_\nu \|L_\nu(\sigma_\nu(x)v)\|^2 \, d\mu(x).$$

The pointwise side inequality is obtained by differentiating Bessel equivariance along the simple-root one-parameter subgroup at x, expanding the derived action in the Cayley matrix units, and applying Cauchy–Schwarz to the expansion. It gives

$$R_a(x)^2 \|L_\nu(\sigma_\nu(x)v)\|^2 \le \sum_{u,v'} \|L_\nu(\sigma_\nu(x)E_{uv'}v)\|^2.$$

For a vector of level $i \le 4n_0^2$, the diagonal operators contribute at most $n_0(3b_v)^2\|v\|^2$; the off-diagonal operators contribute at most $(b_v/16)^2\|v\|^2$. The numerical scale gives

$$n_0(3b_v)^2 + (b_v/16)^2 \le b_v^4 \le \beta^4.$$

The general linear commutation relations and unitarity put each $E_{uv'}v$ at level at most $i+2$. The weight changes by a difference of two coordinate weights, of ℓ^1-norm at most two; the compact Casimir bound changes from the threshold $\sqrt{c_K} + 4n_0 i$ to at most $\sqrt{c_K} + 4n_0(i+2)$, where c_K is the lowest compact-type Casimir eigenvalue. Thus the pointwise inequality, the level estimate, and the mass bound give

$$M_{p+1}(v) \le \sum_{u,v'} M_p(E_{uv'}v), \qquad \sum_{u,v'} \|E_{uv'}v\|^2 \le \beta^4\|v\|^2, \qquad M_0(v) \le \|v\|^2.$$

For an initial vector at level i, these estimates iterate through $i + 2p \le 4n_0^2$, because each matrix unit raises the level by at most two. Induction gives $M_p(v) \le \beta^{4p}\|v\|^2$. Taking $i = 0$ gives the second estimate in (A.296). These estimates hold for every unitary Cayley choice. To pass to the full lowest compact type, let φ be a nonnegative measurable function of the lengths and consider

$$\mathcal{Q}_\varphi(v) = \int \varphi(l_0(x), \dots, l_r(x)) \, d_\nu \|L_\nu(\sigma_\nu(x)v)\|^2 \, d\mu(x).$$

Right translation by the Cayley maximal compact subgroup preserves the Euclidean lengths and the quotient measure. Hence this quadratic form is compact-invariant. The vectors on which it is finite form a linear compact-invariant subspace. The bound holds for every highest-weight unit vector. Each irreducible copy of the lowest compact type contains such a vector and is therefore contained in this finite-integrability subspace. Every vector of the algebraic lowest isotypic space lies in a finite-dimensional compact-stable subspace. On that subspace the quadratic form is continuous and is represented by a positive self-adjoint operator commuting with the compact group. Its largest eigenspace is compact-stable and contains a highest-weight unit vector. The bound on such vectors therefore bounds the quadratic form on every unit vector in the subspace. All these highest-weight vectors have level zero. Apply this argument to $\varphi = R_a^{2p}$; it will also apply to $\varphi = l_a^{2p}$.

Lean: `GlobalTheory.MomentPointwise.pointwise` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.BesselCayleySideBound`, lines 52–110)
Lean: `GlobalTheory.lev_gl_size` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.UnitaryGlModuleLevelSumNormSqLe`, lines 11–57)
Lean: `GlobalTheory.lev_E_add_two` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.LevEAddTwo`, lines 134–162)

Lean: GlobalTheory.RatioMoment.iterate_bound (Development.AbelianVarietyTheory.Proofs.JetHarmonics.FamilySetupCayleyBound, lines 16–64)
Lean: GlobalTheory.MomOne.lintegral_le_of_highest (Development.AbelianVarietyTheory.Proofs.FlagMeasures.LintegralLeOfHighest, lines 168–252)
Lean: GlobalTheory.LowTypeMassGap.weightedMass_bound_lowestType_largeGap (Development.AbelianVarietyTheory.Proofs.FlagMeasures.WeightedMassHighestToLowestKTypeLargeGap, lines 63–165)
Lean: Gp91.GlobalTheory.T.family_setup_cayley_bound (CompanionTheory.AutomorphicTheory.PieriRules.ThetaBesselIsometries.FamilySetupCayleyBoundStatement, lines 16–17)
Lean: GlobalTheory.RatioMoment.ratio_moment (Development.AbelianVarietyTheory.Proofs.JetHarmonics.FamilySetupCayleyBound, lines 66–95)
Lean: GlobalTheory.T.weightedMass_highestToLowestKType_largeGap (Development.AbelianVarietyTheory.Proofs.FlagMeasures.WeightedMassHighestToLowestKTypeLargeGapStatement, lines 15–16)

The initial-length recursion has the sharper constant

$$A_{\mathrm{first}} = n_0^2 T \beta^4 (6b_v)^{2(2r+2)}.$$

For a smooth level-i vector v in the shifted representation it gives

$$\int l_0^{2p} d_\nu \|L_\nu(\sigma_\nu(x)v)\|^2 \, d\mu(x) \le A_{\mathrm{first}}^p \|v\|^2, \quad \|\nu\|_1 + p \le q_0, \quad i + 2p \le 4n_0^2.$$

The induction starts with the mass bound. Let ε_s be the s-th coordinate vector of the shift lattice. In the step from p to $p+1$, each Cayley coordinate gives Pieri projections at level $i+2$ and shifts $\nu - \varepsilon_s$, with squared norms summing to at most $(6b_v)^{2(2r+2)}\|v\|^2$. The squared lowering scalar times the formal-degree ratio is at most $T\beta^4$. Cauchy–Schwarz over s costs n_0, and summing the order-p bounds over the n_0 coordinates gives A_{first}. Both reserves decrease by exactly the required amount: $\|\nu - \varepsilon_s\|_1 \le \|\nu\|_1 + 1$ and $(i+2) + 2p = i + 2(p+1)$.

For a level-zero unit vector v and a period nonzero on smooth vectors, the measure $d\rho_{\nu,v} = d_\nu \|L_\nu(\sigma_\nu(x)v)\|^2 \, d\mu(x)$ has mass one. Lemma 33.16, using first-length and ratio moments through order $(r+1)p$, gives

$$\mathbb{E}_{\rho_{\nu,v}} l_a^{2p} \le (A_{\mathrm{first}} \beta^{4a})^p \le (A_{\mathrm{first}} \beta^{4r})^p.$$

If the period vanishes on all smooth vectors, the corresponding weighted integrals are zero. The uniform bound thus transfers to every lowest-type unit vector by the preceding compact-invariant quadratic-form argument. Finally,

$$\begin{aligned} A_{\mathrm{first}} \beta^{4r} &\le \max(1,T) \beta^{\,2+4+4(2r+2)+4r} \\ &= \max(1,T)\beta^{12r+14} \le \max(1,T)\beta^{20(r+1)} = A_0. \end{aligned}$$

This proves the first estimate in (A.296), with its stated spectral and degree reserves.
Lean: GlobalTheory.Moment.moment_first_length (Development.AbelianVarietyTheory.Proofs.JetHarmonics.FamilySetupCayleyRecursion, lines 24–41)
Lean: Gp91.GlobalTheory.T.family_setup_cayley_recursion (CompanionTheory.AutomorphicTheory.PieriRules.ThetaBesselIsometries.FamilySetupCayleyRecursionStatement, lines 16–17)
Lean: GlobalTheory.MomOne.single_moment_le (Development.AbelianVarietyTheory.Proofs.FlagMeasures.MassBoundLowTypesSum, lines 39–77)
Lean: GlobalTheory.MomNum.mom_num (Development.AbelianVarietyTheory.Proofs.JetHarmonics.NumericalFamilyProductBoundConstantLe, lines 9–36)
Lean: GlobalTheory.MomOneGap.oneLength_moment_lowestType_largeGap (Development.AbelianVarietyTheory.Proofs.FlagMeasures.OneMajorantLengthMomentLowestKType, lines 62–180)

For the positive moment at order q, apply the first estimate directly. The spectral and degree reserves are

$$(r+1)q \le km^4 \le q_0, \qquad rq \le (r+1)q \le km^4 \le k^2 \le 2n_0^2,$$

since $r+1 \le m$ and $m^4 \le f \le k$. For the inverse moment use $l_r^2 \ge T \ge 1$ and

$$l_a^{-1} \le \prod_{i=a}^{r-1} (l_{i+1}/l_i).$$

Hölder with r equal exponents, filling unused factors with 1, applies the ratio estimate at order rq. Thus

$$\mathbb{E}_\rho l_a^{2q} \le A_0^q, \qquad \mathbb{E}_\rho l_a^{-2q} \le \beta^{4rq} \quad (0 \le a \le r).$$

For $r = 0$, the inverse assertion follows from $l_0 \geq 1$.
Lean: GlobalTheory.InvLenMomentGap.inv_len_moment_gap (Development.AbelianVarietyTheory.Proofs.FlagMeasures.InverseMajorantLengthMomentLargeGap, lines 24–76)

Return to the specified character by the diagonal torus scaling with parameter $\xi = \pi/(v_E g_0 g_1)$. Pulling back the quotient measure by the same translation preserves the coefficient's square norm. Each length and its reciprocal change by factors at most

$$K_T = \max(\xi, \xi^{-1})^r \leq b^{(2|C_g|+2)r}.$$

Since $T \leq b^{|C_T|m}$, $m = 2r + 2$, and $b \geq 2$,

$$K_T\sqrt{A_0} \leq b^{(2|C_g|+2)r+|C_T|m/2+10\max(2,C_w)(r+1)} \leq H_1, \qquad K_T\beta^{2r} \leq H_1$$

for $C_H = 2|C_g|+2+|C_T|+10\max(2, C_w)$. For $S = \max_a\{l_a, l_a^{-1}\}$, Markov's inequality and the union bound give

$$\int l_a^{\pm 2q}\|\Phi\|^2\,d\mu \leq H_1^{2q}\|\Phi\|_2^2, \qquad \int_{S>\lambda H_1}\|\Phi\|^2\,d\mu \leq 2(2r+2)\lambda^{-2q}\|\Phi\|_2^2 \quad (\lambda \geq 1). \tag{A.297}$$

Lean: GlobalTheory.TwoSidedMomentsGap.moment_of_bessel_gap (Development.AbelianVarietyTheory.Proofs.FlagMeasures.MajorantLengthTwoSidedMomentsLargeGap, lines 57–127)
Lean: GlobalTheory.TwoSidedMomentsGap.inv_moment_of_bessel_gap (Development.AbelianVarietyTheory.Proofs.FlagMeasures.MajorantLengthTwoSidedMomentsLargeGap, lines 129–198)
Lean: GlobalTheory.InvLenMomentGap.inv_len_moment_gap (Development.AbelianVarietyTheory.Proofs.FlagMeasures.InverseMajorantLengthMomentLargeGap, lines 24–76)
Lean: GlobalTheory.TwoSidedMomentsGap.two_sided_moments_gap (Development.AbelianVarietyTheory.Proofs.FlagMeasures.MajorantLengthTwoSidedMomentsLargeGap, lines 206–340)
Lean: GlobalTheory.Amp.TailTwo.tailBd_of_moments (Development.AbelianVarietyTheory.Proofs.FlagMeasures.MajorantLengthTailBoundOfMoments, lines 58–134)

Claim 6. In the scale-one normal-form quotient measures, the local orbital contribution satisfies (A.300) below. Use length shells with lower endpoints $2^i H_1, 2^{i'} H_1$, $i, i' \geq 0$, including lengths below H_1 in the bottom intervals. Put $R = 2^{i+i'+1}H_1$ and $M_0 = R^{2r}e_\tau$. In the local section the j vector columns have length at most M_0, the triangular variables have radius $2rM_0^2$, and the off-block variables have radius M_0. These bounds come from solving the graded pairing equations with reciprocal leading minors bounded by e_τ.

Let Y be the Gram volume of the columns projected onto the positive coordinate space, and put $d = k-1$. Projection onto a d-dimensional subspace of that space can decrease the Gram volume. The determinant-one change of column coordinates in the chart preserves Lebesgue measure, and the two complementary complex coordinates per column lie in disks of radius M_0. To bound the projected volume, let $V(d, j; R_1, t_1)$ be the volume of j columns in $\mathbf{C}^d$ of lengths at most R_1 and Gram volume at most t_1. For a nonzero first column x, split the remaining columns into their components along x and their projections onto $x^\perp$. The first components cost $(\pi R_1^2)^{j-1}$, and the projected $j-1$ columns have Gram volume at most $t_1/\|x\|$. If $h = d - j$, induction gives a factor $\|x\|^{-2h}$ in the fibre bound. Radial integration gives

$$\int_{\|x\|\leq R_1}\|x\|^{-2h}\,dx = \frac{\pi^d R_1^{2j}}{j(d-1)!}.$$

Using $j(d-1)! \geq h!$ and $2(j-1)^2 + 2(j-1) + 2j = 2j^2$, the induction yields

$$V(d, j; R_1, t_1) \leq \frac{t_1^{2(d-j)}R_1^{2j^2}\pi^{jd}}{((d-j)!)^j}.$$

The empty tuple has volume at most one. The first-column exceptional set has measure zero. Multiplication by $(\pi R_1^2)^{2j}$ accounts for the complementary disks. In particular the zero Gram set is null when $h > 0$.

Take $t_1 = 2y$, $R_1 = M_0$. If the numbers of complex triangular, real diagonal and complex off-block coordinates are n_1, n_2, n_3, respectively, the section box contributes

$$[\pi(2rM_0^2)^2]^{n_1}(4rM_0^2)^{n_2}(\pi M_0^2)^{n_3}.$$

Here $n_1, n_3 \leq r(r-1)/2$, $n_2 \leq r$, and $j \leq r$. Using $h! \geq (h/e)^h$ and $M_0 \geq 1$, the product of the box and Gram bounds for $y < Y \leq 2y$ is at most

$$(4r)^{4r^2}M_0^{8r^2}(\pi e/h)^{jh}(2y)^{2h}. \tag{A.298}$$

In the normal form the product coordinates are ρs, with ρ in the common unipotent stabilizer and s in the section. The stabilizer coordinates are the complex within-block upper entries, the $r-j$ complex columns of length $k+1$, the complex triangular entries with row index at least j, and the corresponding real diagonal entries. Use product Lebesgue measure in these coordinates, transported by the conjugation that removes the second unipotent factor of the adapted frame. The section uses the complementary coordinates of the box above. Multiplication has triangular Jacobian one in the product coordinates. If the stabilizer measure is multiplied by $s_R > 0$, the quotient measure is divided by s_R, so the bound in (A.298) is divided by s_R. The arithmetic covolume comparison below uses $s_R = 1$. The reversed section has the corresponding bound with Gram volume Y'. Left translation by the relative frame transformation carries its quotient measure to the first section's measure.

Lean: `GlobalTheory.Amp.GramVol.vol_le` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.MeasurePreservingMulVec`, lines 100–162)

Lean: `GlobalTheory.Amp.GramVol.vol_gram` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.MeasurePreservingMulVec`, lines 189–226)

Lean: `GlobalTheory.Amp.FibreVolume.fibre_volume` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.FibreVolumeNormalFormStabiliser`, lines 79–174)

The pairing and the two length maxima S_1, S_2 give, in these same coordinates,

$$YY'P_\tau \le 2^j, \qquad s_\tau \le \max\bigl(1, (S_1S_2/(2H_1)^2)^r\bigr). \tag{A.299}$$

The first inequality is the determinant inequality for the perpendicular projections, with a factor of at most two per paired vector. The second follows by bounding each order-l leading minor by $(S_1S_2)^l$.

Use descending dyadic shells for Y, Y' from $R^r e_\tau$, with indices $a, a' \ge 0$. Unfold each squared-coefficient integral over its Bessel quotient. The character has modulus one, so the coefficient square is constant along the unipotent fibre, where (A.298) applies uniformly. Schur orthogonality on the compact factor divides the squared norm by the coefficient dimension. Integration on the quotient gives the respective tails $2^{-2qi'}, 2^{-2qi}$ from (A.297). Cauchy–Schwarz combines the resulting weighted shell bounds. Apply 33.15 with $D = P_\tau$, $E = e_\tau$, $H = H_1$, $s_* = s_\tau$ and $e = 16r^3$. Indeed $M_0^{8r^2} = R^{16r^3} e_\tau^{8r^2}$ in (A.298). The upper bound for the two Gram volumes is $R^r e_\tau$. The leading minor of order j is at most $(S_1S_2)^j \le (S_1S_2)^r$, and (A.299) gives the other two product bounds. The threshold on q gives $q \ge 16r^3 + 4r + 2$. The four shell sums cost 16. Apart from $\sqrt{\dim \pi_{u,\tau}}(2v_E)^{-rk} 2^{r(k+1)+r(r-1)}$, the local factor is at most

$$16(4r+4)(4r)^{4r^2}(2H_1)^{16r^3+4r} e_\tau^{8r^2+2} (8\pi e/h)^{jh} P_\tau^{-h} s_\tau^{-(q-16r^3-4r-1)/r}. \tag{A.300}$$

The dimension and v_E factors here are the one-sided squared coefficient normalizations in the two section estimates. In the product of (A.300), (A.295) costs $(r+1)(8r^2+2)$ powers of S_δ. Since

$$(r+1)(8r^2+2) + (16r^3+4r+1)/r \le 2k,$$

the net power is at most $-C_s k$. The remaining factors of r and H_1 cost at most $(8rH_1)^{80r^4 f}$. *Claim 7.* The arithmetic covolumes satisfy the following bounds. Put $\mathcal{L}_v = (8v_E)^{2(r+1)^2 f}$. Use the tied Hida model, its full row and column data, a choice of real-place coordinates at scalar parameter one, and the standing size hypotheses. For a relevant term with an adapted normal form of index j, give each real common stabilizer its normal-form Haar measure at scale one, conjugated by the inverse term-coordinate matrix. For any measurable fundamental set of the arithmetic common stabilizer, its volume V_R in the product of these measures satisfies

$$V_R \le \omega_t \, v_E^{k(r-j)f} \mathcal{L}_v.$$

For the fixed-frame arithmetic unipotent subgroup, let V_N be its covolume for the Bessel-model unipotent Haar measures at scalar parameter one. Then

$$0 < V_N < \infty, \qquad 1 \le V_N \mathcal{L}_v.$$

In particular the two factors required here satisfy

$$V_R/\omega_t \le v_E^{k(r-j)f} (8v_E)^{2(r+1)^2 f}, \qquad V_N^{-1} \le (8v_E)^{2(r+1)^2 f}. \tag{A.301}$$

Lean: `GlobalTheory.TypeAsm.CovTC` (`Development.AbelianVarietyTheory.Definitions.SemilocalFrames.RowsMeanC`, lines 159–164)

Lean: GlobalTheory.TypeAsm.CovNC (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.RowsMeanC, lines 166–168)
Lean: GlobalTheory.T.integralStabiliser_covolume_le_indexWeight (Development.AbelianVarietyTheory.Proofs.FlagMeasures.IntegralStabiliserCovolumeLeIndexWeightStatement, lines 6–7)
Lean: GlobalTheory.T.integralUnipotent_covolume_finite_bounded (Development.AbelianVarietyTheory.Proofs.FlagMeasures.IntegralUnipotentCovolumeFiniteBoundedStatement, lines 6–7)

For the stabilizer estimate, transport the fundamental set to the normal form; the reference-lattice covolume bound is multiplied by the relative index ω_t. For the second estimate, apply that reference bound to the fixed frame paired with itself, so $j = 0$. If Ω is the resulting unipotent fundamental set and $d\ell_N$ is the product of the local coordinate Lebesgue measures, the normalization is exactly

$$V_N = \Big((2v_E)^{-rk}2^{r(k+1)+r(r-1)}\Big)^f \int_\Omega d\ell_N.$$

The reference bound makes the integral finite and at least $v_E^{krf}/\mathcal{L}_v$. Since

$$\Big((2v_E)^{-rk}2^{r(k+1)+r(r-1)}\Big)^f v_E^{krf} = 2^{r^2 f} \geq 1,$$

it proves the stated positive lower bound on V_N. Thus the unipotent factor in (A.301) uses the Bessel-model measure, including its displayed scalar.
Lean: GlobalTheory.TypeAsm.covTC_holds (Development.AbelianVarietyTheory.Proofs.FlagMeasures.IntegralStabiliserCovolumeLeIndexWeight, lines 23–78)
Lean: GlobalTheory.TypeAsm.covNC_holds (Development.AbelianVarietyTheory.Proofs.FlagMeasures.IntegralUnipotentCovolumeFiniteBounded, lines 61–117)

Claim 8. The first inequality in (A.294) follows from Claims 5–7. These covolumes multiply the local quotient estimates with the same Haar normalization. The factor $(2v_E)^{-rkf}$ absorbs the positive k-power of v_E in (A.301). With $1 \leq v_E \leq b/2$, $h \geq k/2$, and $C \geq 100(C_H + 10)$, the remaining powers fit in $(C/k)^{jhf}H(C)^{Cr^4}$. The product of P_τ^{-h} is $D_\delta^{-h/2}$. Taking the infimum over adapted forms proves the amplitude inequality in (A.294).
Lean: GlobalTheory.Amp.DegConv.deg_core (Development.AbelianVarietyTheory.Proofs.FlagMeasures.AmplitudeDegenerateOperatorNormLeRealPlace, lines 165–257)
Lean: GlobalTheory.TypeAsm.CompatibleChartOperations.prod_degBd_le_explicit (Development.AbelianVarietyTheory.Proofs.FlagMeasures.LocalBoundProductOverPlacesLabel, lines 21–87)
Lean: GlobalTheory.TypeAsm.AmpTerms.amplitudeBound_of_localData_of_tailBound (Development.AbelianVarietyTheory.Proofs.FlagMeasures.AmplitudeBoundAtTermsOfLocalData, lines 113–225)

For the second inequality in (A.294) put $N_T = T_u^f$, $g = g_0 g_1$ and $c = 1 + \log_2(2f)$. A pair of terms gives two embedded lattice data on the definite rank-k quotient, with congruence graphs and a relative isometry. Their indices are $1 \leq x \leq D_\delta$ and $1 \leq x' \leq D_{\delta'}$. Put $D_b = \prod_{1\leq l\leq j,\, \delta_l \neq 0} d_l$. Put $D_0 = N_T^2$, $X_* = xx'$, $Y_* = X_* D_0^2$, and $M_* = (20f(2v_E)^2)^{2f(6+c)}$.

Claim 9. Under the row, frame, level and unramified CM hypotheses of this proposition, fix a row lattice, a reduced type (j, δ), and an embedded definite datum of index x. Assume the frames are inequivalent under the integral fixed-frame stabilizer. Their relative inverse-stabilizer weights have sum at most

$$F_\delta(x) = (D_\delta/x)^{k/2} 8^{r+1} f^2 D_\delta^{3c+6} N_T^2 (D_b D_\delta N_T)^{10000(r+1)^3}. \tag{A.302}$$

Here the adapted normal form has index j, final norm T_u, terminal discriminant $D_\delta \geq 1$, nonzero-minor product $D_b \geq 1$, and congruence-graph index $x \geq 1$. Fixing its signed block matrix and norm-one scalar leaves the fixed-normal-form orbit count. The respective numbers of these choices are at most 8^r and $8f^2 D_\delta^{2(2+c)}$. The remaining weighted count is at most

$$N_T^2 (D_\delta/x)^{k/2} D_\delta^{c+2} (D_b D_\delta N_T)^{10000(r+1)^3}.$$

Multiplication gives the claim.
Lean: GlobalTheory.DegAsm.sum_weights_per_datum (Development.AbelianVarietyTheory.Proofs.FlagMeasures.FlagFrameVectorMemCommonSubspace, lines 204–270)

Claim 10. Give the genus of the positive definite rank-k row lattice the inverse-isometry-order weights. Its discriminant module is $\mathcal{O}_E/(T_u)$, T_u splits completely in E, $g \geq 2$, and $v_E \geq 1$. For indices $x, x' \geq 1$, the

weighted count of triples consisting of two embedded data and their relative isometry, divided by the genus mass, is at most

$$X_*^{k/2}Y_*^{6r+10+4c}\left[1+M_*(2v_E)^f\sum_{s=1}^{k-1}\left((32/g)^{kf}(4kf)^{4f}Y_*^7\right)^s+\left((2/g)^{kf}(4kf)^{4f}D_0X_*\right)^k\right]. \tag{A.303}$$

The discriminant-module assertion follows by projecting a self-dual lattice onto the primitive final line and its perpendicular. The projections identify their discriminant modules. The genus estimate groups the relative isometries by the codimension s of their fixed space; the first and last terms are $s=0$ and $s=k$. The intermediate terms are the genus mean of Lemma 33.12 for the fixed-space lattice times the number of integral extensions in Lemma 33.14. These are the same embedded data used in Claim 9.

Lean: `GlobalTheory.TypeAsm.lat0C` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.BaseLatticeConstructionHolds`, lines 163–183)

Lean: `GlobalTheory.PairsRow.main` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.GenusMeanEmbeddedPairsLeMassMulPairBound`, lines 13–36)

Lean: `GlobalTheory.DegAsm.pairBd` (`Development.AbelianVarietyTheory.Definitions.SemilocalFrames.PairBdNonneg`, lines 8–16)

Apply Claim 9 in each embedded fibre and Claim 10 to the genus mean of the resulting triples.

Claim 11. Claims 9 and 10 imply the second inequality in (A.294). In the product of (A.303) and $F_\delta(x)F_{\delta'}(x')$, the indices cancel to $(D_\delta D_{\delta'})^{k/2}$. Put $C_a=\max(1,2(C_H+1),C_T)$, $U_1=H(C_a)S_\delta^2$ and $U_2=H(C_a)S_{\delta'}^2$. We have $D_\delta,N_T\le U_1$, $D_b\le U_1^T$, and the corresponding bounds at U_2. The thresholds give $c\le 4r$ and $42C_am^2\log b\le k$. Using $f\le k$ and the lower bound on g in 33.7 gives

$$(32/g)^{kf}(4kf)^{4f}H(C_a)^{42}\le 1/2.$$

As $Y_*\le H(C_a)^6(S_\delta S_{\delta'})^2$, the s-th power in the bracket of (A.303) is at most $2^{-s}(S_\delta S_{\delta'})^{14k}$. The zero-fixed-space factor obeys the same comparison at $s=k$. The prefactor $Y_*^{6r+10+4c}$ and $2+2M_*(2v_E)^f$ then cost at most $(U_1U_2)^{96r+183}$. The remaining factors of (A.302) cost $U_i^{250000r^4}$ per side and the index sum costs U_1U_2. Take $C_h=300000C_a$, $C_p=16$, and $k\ge 300000r^4$. For $S_i=S_\delta,S_{\delta'}$, the combined U_i powers and S_i^{14k} fit in $H(C_h)^{C_hr^4}S_i^{16k}$. Sum over row representatives to recover mass_u and obtain the second inequality of (A.294). Multiplication proves the proposition.

Lean: `GlobalTheory.TypeAsm.pair_count_bound_of_siegel_mass` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.TermPairCountBoundOfSiegelMassRatios`, lines 70–161)

Lean: `GlobalTheory.DegAsm.sum_pairBd_mul_datBd_le` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.SumPairBdMulDatBdLe`, lines 307–391)

The complete proof is contained in `GlobalTheory.rowsPos_typeP`. □

Proof of 33.19

Proof (outline). Use fixed and moving frames $(e_0,\dots,e_{r-1},z)$ and $(e'_0,\dots,e'_{r-1},z')$ in the Hermitian space (V,H_E). Put $X_a=\mathrm{span}_E\{e_b:b<a\}$, $X'_a=\mathrm{span}_E\{e'_b:b<a\}$, $X=X_r$, $X'=X'_r$, and $I=X\cap X'$. Put $e_r=z$ and $e'_r=z'$. Write $p_l=\det(H_E(e_a,e'_b))_{0\le a,b<l}$, with $p_0=1$, and $q_l=\sum_{a=0}^{l-1}(r-a)$. The graded ideals are defined using $\mathfrak{d}=\mathfrak{D}_{F/\mathbb{Q}}\mathcal{O}_E$. Denote the norm-one scalar of a common normal form by c_{nf}. *Claim 1.* Under the fixed-type and unramified CM hypotheses, the signed block matrix and the norm-one scalar have at most the following numbers of possibilities. Fix them first. For the matrix there are at most 8^j choices. The block lengths, the possible roots of the block determinants, and the column signs each cost at most 2^j. The scalar has at most $8f^2D_\delta^{2(2+c)}$ choices by the denominator-ideal count. For each denominator the norm-one relation determines the numerator ideal, and generators with the same principal ideal differ by a norm-one unit, hence by a root of unity.

Claim 2. For each fixed block matrix and scalar, the frame classes inject into the lattice orbits of the common normal form, with the two definite indices described below. For one pair of values take a common normal form. Each adapted pair is obtained from it by a unipotent map σ_i. Transport the row lattice by σ_i^{-1}. The transported lattices retain the prescribed definite lattice and congruence graph. An isometry ρ in the common unipotent stabilizer between two transported lattices gives $\sigma_{i'}\rho\sigma_i^{-1}$ on the original lattice carrying the graded frames. The frame classes therefore inject into the lattice orbits for the common normal form. The frame weight is bounded by the inverse stabilizer covolume of its lattice with that normal form.

Here are the two definite indices used in this count. Put $V_0 = X + X' + Ez + Ez'$, $W_{\mathrm{sp}} = V_0^{\perp}$, and let

$$D_+ = (\Lambda + X) \cap W_{\mathrm{sp}}, \qquad D_- = (\Lambda + X') \cap W_{\mathrm{sp}}.$$

Regard the subspaces X, X' as $\mathcal{O}_E$-modules in the following formula. The congruence graph and its diagonal fixed module are

$$\mathcal{G}_g = (D_- \times D_+) \cap \Big(\{(a,b) \in \Lambda \times \Lambda : b - a \in g\Lambda + (\Lambda \cap X)\} + X' \times X \Big),$$
$$C_g = \{v \in V : (v,v) \in \mathcal{G}_g\}.$$

Put $\Theta = (\Lambda \cap W_{\mathrm{sp}}) + I$. The indices are

$$x = [D_+ : C_g], \qquad i_1 = [C_g : \Theta].$$

They are finite indices of additive $\mathcal{O}_E$-modules; equivalently take the common-radical quotient by I. Thus $xi_1 = [D_+ : \Theta]$. This identifies the fixed index of the embedded datum with the first index, and the extra definite-lattice index with the second.

Lean: GlobalTheory.HermLat.iotaGraph (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.FlagPairDefinitePartLattice, lines 171–174)
Lean: GlobalTheory.HermLat.fixLat (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.FlagPairDefinitePartLattice, lines 121–122)
Lean: GlobalTheory.HermLat.latLa (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.FlagPairDefinitePartLattice, lines 176–176)
Lean: GlobalTheory.HermLat.flagPairDefinitePartLattice (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.FlagPairDefinitePartLattice, lines 151–151)
Lean: GlobalTheory.HermLat.latTh (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.FlagPairDefinitePartLattice, lines 169–169)
Lean: GlobalTheory.DegCnt.SplitLift.latTh_eq (Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairExistsUnipotentSplitLattice, lines 130–146)
Lean: GlobalTheory.DegCnt.NF.gI (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.NormalFormDualOrthogonalLast, lines 104–104)
Lean: GlobalTheory.DegCnt.NF.iI (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.NormalFormDualOrthogonalLast, lines 106–106)

There are at most N_T^2 possibilities for the second inner lattice after fixing the definite trace. Indeed its extension in the direction of the last vector is determined by a class in the dual of the fixed trace lattice modulo that lattice, of cardinality at most T_u^{2f}. The terminal index is explicitly

$$i_2 = [\Lambda \cap X : (\Lambda \cap X_j) + (\Lambda \cap I)].$$

Lean: GlobalTheory.HermLat.bOne (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.FlagPairDefinitePartLattice, lines 157–158)

With both inner lattices fixed, group the lattice orbits by the indices i_1, i_2. Put $U = (X + X')^{\perp}$, $L = (\Lambda + X) \cap U$, $L' = (\Lambda + X') \cap U$, and $L_1 = (\Lambda + I) \cap U$. Restriction to W_{sp} gives $[D_+ : \Theta] \le [L : L_1]$ and $[C_g : \Theta] \le [L \cap L' : L_1]$. The terminal determinant bounds the product of the two right-hand indices times i_2^2 by D_δ, giving

$$x i_1^2 i_2^2 \le D_\delta.$$

Each index is a positive integer at most D_δ, so there are at most D_δ^2 pairs. We shall prove that, at fixed indices, the orbit sum is at most

$$(i_1 i_2)^k i_1^c (D_b D_\delta N_T)^{10000(r+1)^3}.$$

We describe the representatives and their weights. Write $V_0 = X + X' + Ez + Ez'$, $W_{\mathrm{sp}} = V_0^{\perp}$, and $I = X \cap X'$. In the fixed normal form let W_θ be the positive lift of W_{sp}/I, let Π be its perpendicular, and let e_a^*, $j \le a < r$, be the isotropic dual vectors. They satisfy

$$H_E(e_b, e_a^*) = \delta_{ba}, \quad H_E(e_b', e_a^*) = 0 \ (b < j), \quad H_E(z, e_a^*) = 0, \quad H_E(e_a^*, e_b^*) = 0.$$

Thus $W_{\mathrm{sp}} = W_\theta \oplus I$, $\Pi = V_0 + \mathrm{span}_E\{e_a^* : j \le a < r\}$, and $\dim_E \Pi = 2r + 1$. Here $\mathfrak{a}_a = \mathfrak{d}^{-(r-a)}$ for $0 \le a < r$. A framed lattice has leading-coefficient ideal $\mathfrak{a}_a$ on the a-th isotropic graded line and $\mathcal{O}_E$ on the

final line. Equivalently, its leading-coefficient module at e_a is $\{t \in E : te_a \in \Lambda + X_a\}$, and likewise for the moving frame. The positive lift is specifically

$$W_\theta = (V_0 + \operatorname{span}_E\{e_a^* : j \le a < r\})^\perp.$$

Lean: GlobalTheory.DegCnt.NF.aa (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.NormalFormDualOrthogonalLast, lines 62–62)
Lean: GlobalTheory.HermLat.IsFrame (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.FlagPairDefinitePartLattice, lines 131–132)
Lean: GlobalTheory.DegCnt.NF.Wt (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.NormalFormDualOrthogonalLast, lines 66–66)

The reference perpendicular lattice is

$$P_0 = \sum_{0\le a<r} \mathfrak{a}_a e_a + \sum_{0\le a<j} \mathfrak{a}_a e'_a + \mathcal{O}_E z + \sum_{j\le a<r} \mathfrak{d}^{r-a} e_a^*.$$

A term lattice is unimodular, has both prescribed framed structures, and is admissible at level $g \ge 1$: if $x' \in \Lambda$ has leading coefficient t at level a of the moving flag, then some $x \in \Lambda$ has the same coefficient at level a of the fixed flag and satisfies $x - x' \in g\Lambda$. Its split condition is

$$\Theta(\Lambda) := (\Lambda \cap W_{\mathrm{sp}}) + I = (\Lambda \cap W_\theta) + I.$$

Let R be the common flag-unipotent stabilizer and let R_s be its subgroup fixing W_θ pointwise. These groups preserve admissibility, the embedded datum, $\Theta(\Lambda)$, and the terminal index i_2.
Lean: GlobalTheory.DegCnt.NF (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.NormalFormDualOrthogonalLast, lines 33–56)
Lean: GlobalTheory.T.hermFramePairNormalForm_geometry_summary (Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairNormalFormGeometrySummaryStatement, lines 11–12)
Lean: GlobalTheory.DegCnt.NF.P0 (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.NormalFormDualOrthogonalLast, lines 72–75)
Lean: GlobalTheory.DegCnt.NF.IsTerm (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.NormalFormDualOrthogonalLast, lines 101–102)
Lean: GlobalTheory.HermLat.IsAdm (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.FlagPairDefinitePartLattice, lines 134–136)
Lean: GlobalTheory.DegCnt.NF.Split (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.NormalFormDualOrthogonalLast, lines 119–120)
Lean: GlobalTheory.DegCnt.SplitLift.latTh_eq (Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairExistsUnipotentSplitLattice, lines 130–146)
Lean: GlobalTheory.DegCnt.NF.Rs (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.NormalFormDualOrthogonalLast, lines 97–97)
Lean: GlobalTheory.DegCnt.NF.bI (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.NormalFormDualOrthogonalLast, lines 108–108)
Lean: GlobalTheory.T.commonUnipotent_stabiliser_preserves_terms_and_datum (Development.AbelianVarietyTheory.Proofs.FlagMeasures.CommonUnipotentStabiliserPreservesTermsAndDatumStatement, lines 11–12)

Claim 3. For the common normal form and the conjugation-stable minor-clearing ideal $\mathfrak{c}$ of Lemma 33.17, every lattice orbit has a split representative with

$$\mathfrak{c}^{12r+50} P_0 \subset \Lambda \cap \Pi \subset \mathfrak{c}^{-(12r+50)} P_0.$$

First obtain a split representative. The saturated inclusion $\Lambda \cap I \subset \Lambda \cap W_{\mathrm{sp}}$ has a finitely generated torsion-free quotient over the Dedekind domain $\mathcal{O}_E$. That quotient is projective. Choose a splitting and let W be the E-span of its image. Then

$$W \cap I = 0, \qquad W_{\mathrm{sp}} = W + I, \qquad \forall l \in \Lambda \cap W_{\mathrm{sp}}\ \exists w \in \Lambda \cap W : l - w \in I.$$

To construct the required common unipotent map, write W_θ as the graph $w - \phi(w)$ of a linear map $\phi : W \to I$. Choose a basis x_b of I. The form on W is anisotropic, hence nondegenerate, so there are $v_b \in W$ with

$$\phi(w) = -\sum_b H_E(w, v_b) x_b.$$

For $x \in I$, $v \in W_{\mathrm{sp}}$, and $c + \bar{c} = -H_E(v, v)$, the Eichler map

$$E_{x,v,c}(y) = y + H_E(y, v)x - H_E(y, x)v + cH_E(y, x)x$$

is an isometry, with inverse $E_{x,-v,\bar{c}}$. It fixes V_0 pointwise and belongs to both flag-unipotent groups. On W_{sp} it sends y to $y + H_E(y, v)x$. Choose such a c for each pair (x_b, v_b) and compose these maps. Since I is orthogonal to W_{sp}, their restrictions add the displayed corrections. The resulting $s \in R$ fixes V_0 and sends w to $w - \phi(w)$, so $sW = W_\theta$.

Lean: `Formalism.HermitianLatticeCounting.P.liftTransitive_of` (`Formalism.FlagUnipotentLattices.MinkowskiFactorialBounds.ExistsHasLeadingCoeffOfMemSpanBelow`, lines 1635–1709)
Lean: `Formalism.HermitianLatticeCounting.P.good_family` (`Formalism.FlagUnipotentLattices.MinkowskiFactorialBounds.ExistsHasLeadingCoeffOfMemSpanBelow`, lines 1571–1595)

For $l = w + i$ as above, $sl = sw + i$ with $sw \in s\Lambda \cap W_\theta$; hence $s\Lambda$ is split.

Lean: `Formalism.HermitianLatticeCounting.P.complementLiftTransitive` (`Formalism.FlagUnipotentLattices.MinkowskiFactorialBounds.ExistsHasLeadingCoeffOfMemSpanBelow`, lines 1711–1711)
Lean: `GlobalTheory.DegCnt.SplitLift.main` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairExistsUnipotentSplitLattice`, lines 148–186)

For $l \le j$, choose leading coefficients $\alpha_a, \beta_a \in \mathfrak{d}^{-(r-a)}$, $0 \le a < l$, and lift $\alpha_a e_a$ and $\beta_a e'_a$ to vectors of Λ modulo the respective preceding flag spaces. The determinant of their pairings is

$$\left(\prod_{a<l} \alpha_a\right) p_l \overline{\left(\prod_{a<l} \beta_a\right)}.$$

It is integral because Λ is unimodular. Products of these coefficients generate $\mathfrak{d}^{-2q_l}$; conjugation preserves $\mathfrak{d}$. Multilinearity therefore gives $(p_l)\mathfrak{d}^{-2q_l} \subset \mathcal{O}_E$.

Lean: `GlobalTheory.DegCnt.Absorb.Dl_mul_mem_one_of_mem_ipow` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.ExistsStarStableIdealAbsorbingMinors`, lines 309–348)

Lemma 33.17 supplies the same ideal $\mathfrak{c}$ for all terms in the normal form, with $N(\mathfrak{c}) = (D_b N_T)^2$. In particular $D_b, D_\delta \ge 1$.

Lean: `GlobalTheory.DegCnt.Absorb.exists_ideal` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.ExistsStarStableIdealAbsorbingMinors`, lines 355–402)

Claim 4. Let Λ_* be a split term lattice in this normal form at level $g \ge 1$. Let $\mathfrak{c} \ne 0$ be conjugation-stable, contained in (T_u), and satisfy

$$\frac{\mathfrak{c}\mathfrak{d}^{2q_l}}{p_l} \subset \mathcal{O}_E, \qquad \frac{\mathfrak{c}\mathfrak{d}^{2q_l}}{\bar{p}_l} \subset \mathcal{O}_E \quad (0 \le l \le j,\ p_l \ne 0).$$

For an integer $e \ge 2$, suppose

$$\mathfrak{c}^e(P_0 \cap V_0) \subset \Lambda_*, \qquad \mathfrak{c}^e(\Lambda_* \cap X_j) \subset P_0, \qquad \mathfrak{c}^e(\Lambda_* \cap X'_j) \subset P_0, \qquad \Lambda_* \cap I \subset P_0.$$

There is $s \in R$, fixing V_0 and W_θ pointwise, such that

$$\mathfrak{c}^{6e+30} P_0 \subset s\Lambda_* \cap \Pi, \qquad \mathfrak{c}^{6e+30}(s\Lambda_* \cap \Pi) \subset P_0.$$

Lean: `GlobalTheory.DegCnt.DualPos.near_of_skew` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairSplitTermNearReferenceBySkew`, lines 509–582)

Put $e_0 = 2r + 2$. Write the two flag lattices in unitriangular coordinates U, U'. In each antidiagonal block of the reduced pairing, invert the corresponding diagonal block of U; act on the opposite isotropic block by the inverse conjugate transpose with the pairing coefficients included. This defines an isometry $s_1 \in R_s$ fixing $I + Ez + \mathrm{span}_E\{e_a^*\}$. If its two flag matrices are M, S, then MU has no off-diagonal entries within one block. The integral mixed pairings and the nonzero leading block determinants give

$$\mathfrak{c}(MU)_{ba}\mathfrak{a}_a \subset \mathfrak{a}_b, \qquad \mathfrak{c}^2(SU')_{ba}\mathfrak{a}_a \subset \mathfrak{a}_b.$$

The first estimate uses one cleared leading determinant; the second uses the two adjacent block ends and conjugation-stability of $\mathfrak{c}$. Both matrices are unitriangular. Their finite inverse series and at most $r + 1$ filtration steps give denominator $(\mathfrak{c}^2)^{r+1} = \mathfrak{c}^{e_0}$ in both directions.

Lean: `GlobalTheory.DegCnt.PosJ.exists_elt` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.NormalFormFlagLatticesNearUnderStabiliser`, lines 620–721)
Lean: `GlobalTheory.DegCnt.PosJ.denom` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.NormalFormFlagLatticesNearUnderStabiliser`, lines 802–871)
Lean: `GlobalTheory.DegCnt.PosJ.main` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.NormalFormFlagLatticesNearUnderStabiliser`, lines 875–919)

Thus, for $\Lambda_1 = s_1\Lambda$ and $Y = X_j, X_j'$,

$$\mathfrak{c}^{e_0}(P_0 \cap Y) \subset \Lambda_1, \qquad \mathfrak{c}^{e_0}(\Lambda_1 \cap Y) \subset P_0.$$

For the next reduction, choose adapted vectors $v_a = e_a + \xi_a$, with $\xi_a \in I \cap X_a$, for the lattice filtration in I. Successive shears, taken from the highest index downwards, send v_a to e_a, fix z, and preserve the corrections already made. An Eichler translation removes the remaining I-component of an adapted last vector while fixing X. Both operations fix $X_j + X_j'$ and W_θ.

Lean: `GlobalTheory.DegCnt.PosIso.exists_unitri` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.NormalFormIdealDivisibility`, lines 452–488)
Lean: `GlobalTheory.DegCnt.PosIso.exists_move_alt` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.NormalFormIdealDivisibility`, lines 435–450)
Lean: `GlobalTheory.DegCnt.PosIso.normal_position` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.NormalFormIdealDivisibility`, lines 559–636)

Induction over the leading coefficients, using the clearing inclusion for p_j, gives an isometry $s_2 \in R_s$ such that, with $\Lambda_2 = s_2\Lambda_1$,

$$\mathfrak{c}^2(P_0 \cap I) \subset \Lambda_2, \quad \Lambda_2 \cap I \subset P_0, \quad \mathfrak{c}\mathcal{O}_E z \subset \Lambda_2.$$

Both lattices remain split. The second map preserves their intersections with X_j, X_j', so the first pair of inclusions also holds for Λ_2. Since the radical of V_0 is I, expanding the displayed generators of P_0 gives

$$P_0 \cap V_0 \subset (P_0 \cap X_j) + (P_0 \cap X_j') + (P_0 \cap I) + \mathcal{O}_E z.$$

Indeed an isotropic-dual component lying in V_0 pairs trivially with I, which forces all its coefficients to vanish. Since $e_0 \geq 2$, the preceding inclusions imply $\mathfrak{c}^{e_0}(P_0 \cap V_0) \subset \Lambda_2$. Apply Claim 4 to $\Lambda_* = \Lambda_2$ and $e = e_0$, using the three upper inclusions just established. Its isometry s_3 fixes V_0 and W_θ, belongs to R_s, and satisfies

$$\mathfrak{c}^{6e_0+30}P_0 \subset s_3\Lambda_2 \cap \Pi, \qquad \mathfrak{c}^{6e_0+30}(s_3\Lambda_2 \cap \Pi) \subset P_0.$$

It remains to prove Claim 4. Let Λ_* and e be as there. Put

$$P_* = \sum_{a \geq j} \mathfrak{d}^{r-a} e_a^*, \qquad P_0 = (P_0 \cap V_0) + P_*,$$

and, for $v \in V$,

$$\varpi(v) = \sum_{j \leq a < r} H_E(v, e_a) e_a^*, \qquad \kappa(v) = \sum_{j \leq a < r} H_E(v, e_a^*) e_a.$$

For $a \geq j$ the vector e_a lies in I, the radical of V_0, and $W_\theta \perp V_0$. Hence ϖ vanishes on V_0, on W_θ and on the span of the e_a, ϖ fixes P_*, and $H_E(\kappa v, v') = H_E(v, \varpi v')$. The cleared discriminant modules satisfy

$$\mathfrak{c}^3\Theta(\Lambda_*)^\# \subset \Theta(\Lambda_*), \qquad \mathfrak{c}P_0^\# \subset P_0, \qquad \mathfrak{c}P_0 \subset P_0^\#.$$

Here, for an $\mathcal{O}_E$-submodule M, write $M^\# = \{y \in \operatorname{span}_E M : H_E(y, M) \subset \mathcal{O}_E\}$; this definition includes the radical of the restricted form. By the second inclusion, a vector $v \in \Pi = \operatorname{span}_E P_0$ with $\mathfrak{c}^t H_E(v, P_0) \subset \mathcal{O}_E$ satisfies $\mathfrak{c}^{t+1}v \subset P_0$.

The lift. For $y \in \mathfrak{c}P_*$, the functional $v \mapsto H_E(v, y)$ is integral on $\Lambda_* \cap \operatorname{span}_E\{e_a : a \geq j\}$, because this module lies in $\Lambda_* \cap I \subset P_0$ and $y \in \mathfrak{c}P_0 \subset P_0^\#$. The module is saturated in Λ_*, so the functional extends to Λ_*, and unimodularity gives $l \in \Lambda_*$ with $H_E(e_a, l) = H_E(e_a, y)$ for $a \geq j$, that is, $\varpi l = y$. Since $\mathfrak{c}P_*$ is projective, these choices form an E-linear map $\Psi : V \to V$ with $\Psi(\mathfrak{c}P_*) \subset \Lambda_*$ and $\varpi\Psi y = y$, that is $H_E(\Psi y - y, e_a) = 0$, for $y \in P_*$ and $a \geq j$. Let π be projection to Π along W_θ. For $l \in \Lambda_*$ write $l = w + \pi l$ with $w \in W_\theta$. Unimodularity and $\pi l \perp W_\theta$ give $H_E(w, \Lambda_* \cap W_\theta) \subset \mathcal{O}_E$, and $w \perp V_0 \supset I$, so $w \in \Theta(\Lambda_*)^\#$. The first discriminant inclusion gives $\mathfrak{c}^3 w \subset \Theta(\Lambda_*) = (\Lambda_* \cap W_\theta) + I$, and $W_\theta \cap I = 0$ gives $\mathfrak{c}^3 w \subset \Lambda_* \cap W_\theta$. Hence $\mathfrak{c}^3\pi l \subset \Lambda_*$. Put $\Phi = \pi\Psi$. Applied to $l \in \Psi(\mathfrak{c}P_*)$, this gives, for $y \in P_*$,

$$\Phi y \in \Pi, \qquad \mathfrak{c}^4\Phi y \subset \Lambda_*, \qquad \varpi\Phi y = y,$$

the last because ϖ vanishes on W_θ.

The isometry. Choose $\vartheta \in \mathcal{O}_E$ with $\vartheta + \bar{\vartheta} = 1$, available because the CM extension is unramified, and put $s = 1 - N$, where

$$Nv = \sum_{j \le a < r} \big(\bar{\vartheta}\, H_E(\Phi\varpi v, e_a^*) - \vartheta\, H_E(\varpi v, \Phi e_a^*)\big) e_a.$$

Then $H_E(Nv, v') = \bar{\vartheta} H_E(\Phi\varpi v, \varpi v') - \vartheta H_E(\varpi v, \Phi\varpi v') = -H_E(v, Nv')$. The values of N lie in the isotropic span of the e_a, $a \ge j$, so $\varpi N = 0$, $N^2 = 0$ and $H_E(Nv, Nv') = 0$. Hence s is an isometry with inverse $1 + N$. It fixes every vector orthogonal to the e_a, $a \ge j$, in particular V_0 and W_θ, and $v - sv \in I$ for every v. Thus s lies in both flag-unipotent groups, so $s \in R$, and s fixes $W_{\mathrm{sp}} = W_\theta \oplus I$ pointwise, so $s\Lambda_*$ is split.

The denominators. Let $y, y' \in P_*$, $v_0 \in P_0 \cap V_0$, and $u_y = \Phi y - y$, so $\varpi u_y = 0$. Integrality of Λ_*, $\mathfrak{c}^4 \Phi y \subset \Lambda_*$, $\mathfrak{c}^e(P_0 \cap V_0) \subset \Lambda_*$ and $\mathfrak{c} P_0 \subset P_0^{\#}$ give $\mathfrak{c}^{e+4} H_E(u_y, v_0) \subset \mathcal{O}_E$. Since ϖ kills v_0 and fixes y', the vector $u_y - \kappa u_y \in \Pi$ pairs with $v_0 + y'$ as u_y pairs with v_0, so $\mathfrak{c}^{e+5}(u_y - \kappa u_y) \subset P_0$. As $\varpi u_y = \varpi u_{y'} = 0$ and ϖ kills the span of the e_a, one has $H_E(u_y - \kappa u_y, u_{y'} - \kappa u_{y'}) = H_E(u_y, u_{y'})$, and pairing two such vectors gives $\mathfrak{c}^{2e+11} H_E(u_y, u_{y'}) \subset \mathcal{O}_E$. Now $\varpi\Phi y = y$, $H_E(y, y') = 0$ and $\vartheta + \bar{\vartheta} = 1$ give

$$H_E(s\Phi y - y, v_0 + y') = H_E(u_y, v_0) + \vartheta\big(H_E(\Phi y, \Phi y') - H_E(u_y, u_{y'})\big).$$

Each term is cleared by $\mathfrak{c}^{2e+11}$, so $\mathfrak{c}^{2e+12}(s\Phi y - y) \subset P_0$. Since $\varpi(s\Phi y - y) = 0$, these multiples lie in $P_0 \cap V_0$, which s fixes and $\mathfrak{c}^e$ carries into Λ_*. With $\mathfrak{c}^4 s\Phi y \subset s\Lambda_*$ this gives $\mathfrak{c}^{3e+12} y \subset s\Lambda_*$, and hence $\mathfrak{c}^{3e+12} P_0 \subset s\Lambda_* \cap \Pi$. Finally, for $v' \in s\Lambda_* \cap \Pi$, integrality of $s\Lambda_*$ gives $\mathfrak{c}^{3e+12} H_E(v', P_0) \subset \mathcal{O}_E$, so $\mathfrak{c}^{3e+13} v' \subset P_0$. Since $3e + 13 \le 6e + 30$, this proves Claim 4. In particular s_3 lies in the subgroup fixing both spaces, and the lattice $s_3\Lambda_2$ remains split.

Lean: `GlobalTheory.T.hermFramePair_ideal_kills_discriminantModules` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairIdealKillsDiscriminantModulesStatement`, lines 11–12)
Lean: `GlobalTheory.DegCnt.DualPos.exists_lift` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairSplitTermNearReferenceBySkew`, lines 319–375)
Lean: `GlobalTheory.DegCnt.DualPos.exists_skew` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairSplitTermNearReferenceBySkew`, lines 153–181)
Lean: `GlobalTheory.DegCnt.DualPos.core` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairSplitTermNearReferenceBySkew`, lines 183–293)
Lean: `GlobalTheory.HermLat.mem_dual_iff` (`Development.AbelianVarietyTheory.Definitions.SemilocalFrames.FlagPairDefinitePartLattice`, lines 38–42)
Lean: `GlobalTheory.DegCnt.DualPos.lift_bd` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairSplitTermNearReferenceBySkew`, lines 400–433)
Lean: `GlobalTheory.DegCnt.DualPos.near_of_skew` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairSplitTermNearReferenceBySkew`, lines 509–582)

As $6e_0 + 30 = 12r + 42 \le 12r + 50$, the same inclusions hold with $e = 12r + 50$. Thus every orbit has a split representative whose perpendicular lattice P satisfies $\mathfrak{c}^e P_0 \subset P \subset \mathfrak{c}^{-e} P_0$.

Lean: `GlobalTheory.T.normalForm_flag_lattices_near_under_stabiliser` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.NormalFormFlagLatticesNearUnderStabiliserStatement`, lines 11–12)
Lean: `GlobalTheory.T.normal_form_ideal_divisibility` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.NormalFormIdealDivisibilityStatement`, lines 11–12)
Lean: `GlobalTheory.DegCnt.NormalPos.P0_inf_V0_le_sup_of_radical` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairNormalFormExistsConjNearReference`, lines 89–127)
Lean: `GlobalTheory.T.hermFramePair_split_term_near_reference_by_skew` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairSplitTermNearReferenceBySkewStatement`, lines 11–12)
Lean: `GlobalTheory.DegCnt.NF.Near` (`Development.AbelianVarietyTheory.Definitions.SemilocalFrames.NormalFormDualOrthogonalLast`, lines 122–123)
Lean: `GlobalTheory.DegCnt.NormalPos.exists_near` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairNormalFormExistsConjNearReference`, lines 143–174)

The definite sublattice of index i_1 has at most i_1^{k+c} choices. For fixed intersections A with the definite space and P with its perpendicular, the number of unimodular lattices inducing these intersections is at most $|A^{\#}/A|^{2r+1}$. The definite intersection satisfies $|A^{\#}/A| \le (D_\delta N_T)^2$. Hence there are at most $(D_\delta N_T)^{2(2r+1)}$ gluings.

Lean: `GlobalTheory.T.encard_unimodular_hermGluings_le` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.EncardUnimodularHermGluingsLeStatement`, lines 11–12)
Lean: `GlobalTheory.DegCnt.OmCount.idx_dual_le` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairOmSumSameDatumLe`, lines 188–233)

The number of perpendicular lattices is at most $N(\mathfrak{c})^{2e(2r+1)^2}$, by counting submodules between $\mathfrak{c}^e P_0$ and $\mathfrak{c}^{-e} P_0$.

Claim 5. The relative inverse-stabilizer weight is invariant under the common unipotent action and has the three-factor decomposition below. For the weight, let $L = (X + X')^{\perp} \cap (\Lambda + X)$. The fixed normal form satisfies $e'_a = c_{\mathrm{nf}} e_a$ for $j \le a \le r$; in particular $z' = c_{\mathrm{nf}} z$, so z is orthogonal to X'. Its reference completion is exactly

$$\Lambda_{\mathrm{ref}} = P_0 + \big(L \cap (W_\theta + Ez)\big).$$

Lean: `GlobalTheory.DegCnt.NF.Ref` (`Development.AbelianVarietyTheory.Definitions.SemilocalFrames.NormalFormDualOrthogonalLast`, lines 77–77)

For any lattice A, write $\Gamma(A) = \{s \in R : sA = A\}$. The relative inverse covolume is normalized by

$$\omega(\Lambda) = \mathrm{iv}(\Gamma(\Lambda_{\mathrm{ref}}), \Gamma(\Lambda)), \qquad \mathrm{iv}(A, B) = \frac{[A : A \cap B]}{[B : A \cap B]}.$$

The two stabilizers are commensurable because both lattices are full. The same holds after intersection with a subgroup of R: its stabilizer acts on a finite set of lattices between two common scalar bounds. Conjugation by $s \in R$ preserves this weight. To see the index assertion, put $d = \mathrm{iv}(\Gamma(\Lambda), \Gamma(s\Lambda)) > 0$. Multiplicativity of relative indices gives $\mathrm{iv}(\Gamma(\Lambda), \Gamma(s^t\Lambda)) = d^t$. Since $(s-1)^{2r+1} = 0$, the binomial expansions of s^t and s^{-t} have a common denominator independent of $t \in \mathbf{N}$. The lattices $s^t\Lambda$ therefore lie between two fixed lattices, and two coincide. It follows that $d = 1$. The embedded datum, and hence its reference completion, is preserved by R.

Lean: `GlobalTheory.DegCnt.NF.om` (`Development.AbelianVarietyTheory.Definitions.SemilocalFrames.NormalFormDualOrthogonalLast`, lines 110–110)
Lean: `GlobalTheory.HL.invVol` (`Development.AbelianVarietyTheory.Definitions.Foundations.IsHermND`, lines 86–87)
Lean: `GlobalTheory.DegCnt.Unimod.invVol_smul` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.RelIndexNeZero`, lines 169–211)

Use

$$R_1 = \{s \in R : s|_{V_0} = 1\}, \qquad Z = \{s \in R_1 : s|_{W_\theta} = 1\}.$$

They are the normal subgroups of the common stabilizer's skew, middle, and top filtration.

Lean: `GlobalTheory.DegCnt.NF.R1` (`Development.AbelianVarietyTheory.Definitions.SemilocalFrames.NormalFormDualOrthogonalLast`, lines 93–93)
Lean: `GlobalTheory.DegCnt.NF.ZI` (`Development.AbelianVarietyTheory.Definitions.SemilocalFrames.NormalFormDualOrthogonalLast`, lines 95–95)

Put $A = \Gamma(\Lambda_{\mathrm{ref}})$, $B = \Gamma(\Lambda)$. Normality and multiplication of finite relative indices give

$$\mathrm{iv}(A, B) = \mathrm{iv}(A \cap Z, B \cap Z)\ \mathrm{iv}((A \cap R_1)Z, (B \cap R_1)Z)\ \mathrm{iv}(AR_1, BR_1).$$

Claim 6. Deep congruence and its lift give the following bounds for the two outer factors. Let Λ be a split term lattice at level $g \ge 1$. Let $\mathfrak{c} \ne 0$ be conjugation-stable, contained in (T_u), and satisfy both minor-clearing inclusions in Claim 4. Suppose $e \in \mathbb{N}$ and

$$\mathfrak{c}^e P_0 \subset \Lambda \cap \Pi, \qquad \mathfrak{c}^e(\Lambda \cap \Pi) \subset P_0.$$

First, every $s \in R_s$ with

$$(s-1)P_0 \subset \mathfrak{c}^{3e+10} P_0$$

preserves Λ. Second, for every conjugation-stable ideal $\mathfrak{n}$, including the zero ideal, every $s \in R$ satisfying

$$(s-1)(P_0 \cap V_0) \subset (T_u)\mathfrak{n}(P_0 \cap V_0)$$

has a lift $s' \in R_s$ with

$$s'|_{V_0} = s|_{V_0}, \qquad (s'-1)P_0 \subset \mathfrak{n}P_0.$$

The lifting assertion only uses the unramified CM field and the fixed normal form; it does not require Λ, g, or $\mathfrak{c}$. Consequently, with $n = N(\mathfrak{c})$,

$$\mathrm{iv}(A \cap Z, B \cap Z) \le n^{(3e+11)(2r+1)^2}, \qquad \mathrm{iv}(AR_1, BR_1) \le n^{(3e+11)(2r+1)^2}.$$

Lean: `GlobalTheory.T.stabilizer_hermitianLattice_of_deep_star_congruence` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.StabilizerHermitianLatticeOfDeepStarCongruenceStatement`, lines 11–12)

Lean: `GlobalTheory.T.hermFramePairNormalForm_congruence_lift_smallGroup` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairNormalFormCongruenceLiftSmallGroupStatement`, lines 11–12)

For the first assertion, write $x = w + p \in \Lambda$ with $w \in W_\theta$, $p \in \Pi$. The cleared discriminant inclusion and splitting give $\mathfrak{c}^3 p \subset \Lambda \cap \Pi$, and nearness gives $\mathfrak{c}^{e+3} p \subset P_0$. Apply the congruence assumption to these multiples of p. Cancel the nonzero ideal $\mathfrak{c}^{e+3}$ to obtain $sp - p \in \mathfrak{c}^e P_0 \subset \Lambda$. Since s fixes w, this proves $s\Lambda \subset \Lambda$. Hermitian unimodularity and the isometry property give the reverse inclusion. For the second assertion, extend the prescribed action on V_0 to its isotropic dual directions, fixing W_θ. Correct the Hermitian defect using an integral ϑ with $\vartheta + \bar{\vartheta} = 1$. The initial displacement is in $(T_u)\mathfrak{n}(P_0 \cap V_0)$, so clearing the final-vector norm leaves the extended displacement in $\mathfrak{n}P_0$. The correction preserves the action on V_0.

Lean: `GlobalTheory.DegCnt.stab_of_deep_congruence` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.StabilizerHermitianLatticeOfDeepStarCongruence`, lines 73–148)
Lean: `GlobalTheory.DegCnt.LiftS.lift_congruence` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairNormalFormCongruenceLiftSmallGroup`, lines 930–943)
Lean: `GlobalTheory.DegCnt.WtSkew.invVol_skew_le` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairNormalFormInvVolSkewLayerLe`, lines 74–126)
Lean: `GlobalTheory.DegCnt.invVol_top_layer_le` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairNormalFormInvVolTopLayerLe`, lines 70–142)

Both bounds arise from finite congruence actions. First $A \cap Z$ preserves $P_0 = \Lambda_{\mathrm{ref}} \cap \Pi$. An element acting trivially on $P_0/\mathfrak{c}^{3e+10}P_0$ preserves Λ, by the deep-congruence estimate. Thus it belongs to $B \cap Z$. For a projective lattice of rank d, the ring of endomorphisms modulo $\mathfrak{c}^t$ has cardinality $N(\mathfrak{c})^{td^2}$, as is seen after localization at the primes dividing $\mathfrak{c}$. Therefore the first factor, which is at most $[A \cap Z : A \cap B \cap Z]$, is bounded by $N(\mathfrak{c})^{(3e+10)(2r+1)^2}$.

For the top factor, AR_1 preserves $P_0 \cap V_0$. If $s \in AR_1$ is the identity modulo $\mathfrak{c}^{3e+11}$ on this lattice, the inclusion $\mathfrak{c} \subset (T_u)$ permits a lift $s' \in R_s$ agreeing with s on V_0 and acting trivially modulo $\mathfrak{c}^{3e+10}$ on P_0. The deep-congruence estimate puts s' in B; since $s'^{-1}s \in R_1$, it follows that $s \in BR_1$. The same finite-module count, with $\operatorname{rank}(P_0 \cap V_0) \le 2r + 1$, bounds the top factor. Hence both outer factors are at most

$$N(\mathfrak{c})^{(3e+11)(2r+1)^2}.$$

Lean: `GlobalTheory.DegCnt.WtSkew.invVol_skew_le` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairNormalFormInvVolSkewLayerLe`, lines 74–126)
Lean: `GlobalTheory.DegCnt.invVol_top_layer_le` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairNormalFormInvVolTopLayerLe`, lines 70–142)

For the middle factor use the additive rows

$$\rho_a(s) = s(e_a^*) - e_a^* \pmod{I}, \qquad j \le a < r, \quad s \in R_1.$$

Their common kernel is Z. Put $M = \Lambda \cap W_\theta$ and $L_\theta = L \cap W_\theta$. *Claim 7.* The index x of the fixed embedded datum and the definite and terminal indices i_1, i_2 satisfy

$$x \ge 1, \qquad i_1 \ge 1, \qquad x i_1^2 i_2^2 \le D_\delta, \qquad x i_1 \le D_\delta, \qquad [L_\theta : M] \le x i_1.$$

The last index is taken between lattices in the positive lift W_θ. The splitting identifies their quotient with the corresponding quotient modulo I.

Lean: `GlobalTheory.T.flagPairDefiniteThetaLattice_le_fixed_index_bound` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.FlagPairDefiniteThetaLatticeLeFixedIndexBoundStatement`, lines 11–12)
Lean: `GlobalTheory.DegCnt.MidLayer.invVol_mid_layer_le` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HaarInvVolumeMiddleLayerEichlerLe`, lines 1008–1183)

Filter the rows from the highest index downwards, and let

$$C_a = \{H_E(e_a^*, x) : x \in \Lambda \cap I,\ H_E(e_b^*, x) = 0 \text{ for } b > a\}.$$

The flag-lattice filtration gives

$$\prod_{a=j}^{r-1} [\mathfrak{a}_a : C_a \cap \mathfrak{a}_a] \le i_2.$$

Let Q_a be the possible a-th rows of $(B \cap R_1)Z$ with all higher rows zero, and put $P_a = Q_a + \mathfrak{a}_a L_\theta$, with both groups viewed in V/I and the products interpreted as additive spans of scalar products. The integral

Eichler transvection $E_{x,-v,-\vartheta H_E(v,v)}$, for $x \in \Lambda \cap I$ and $v \in M$, preserves Λ and gives row $H_E(e_a^*, x)v$. Thus Q_a contains $(C_a \cap \mathfrak{a}_a)M$. The rows of $(A \cap R_1)Z$ lie in P_a, and

$$[P_a : Q_a] \le [\mathfrak{a}_a : C_a \cap \mathfrak{a}_a]^k [L_\theta : M].$$

Successive passage to the row kernels bounds the relative inverse covolume by the product of these indices. Multiplying over the $r - j$ rows and using Claim 7 proves that the middle factor is at most $i_2^k (x i_1)^{r-j}$.
Lean: GlobalTheory.HermLat.flagPairDefinitePartLattice (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.FlagPairDefinitePartLattice, lines 151–151)
Lean: GlobalTheory.DegCnt.MidLayer.exists_eichler (Development.AbelianVarietyTheory.Proofs.FlagMeasures.HaarInvVolumeMiddleLayerEichlerLe, lines 780–853)
Lean: GlobalTheory.DegCnt.MidLayer.invVol_le_prod (Development.AbelianVarietyTheory.Proofs.FlagMeasures.HaarInvVolumeMiddleLayerEichlerLe, lines 19–148)
Lean: GlobalTheory.DegCnt.MidLayer.invVol_mid_layer_le (Development.AbelianVarietyTheory.Proofs.FlagMeasures.HaarInvVolumeMiddleLayerEichlerLe, lines 1008–1183)

Claim 7 gives $1 \le x i_1 \le D_\delta$, so this factor is at most $i_2^k D_\delta^r$. These estimates use the same nonzero conjugation-stable ideal $\mathfrak{c} \subset (T_u)$ and both minor-clearing inclusions of Lemma 33.17. Multiplying the factors proves

$$\omega(\Lambda) \le i_2^k D_\delta^r N(\mathfrak{c})^{2(3e+11)(2r+1)^2}.$$

Lean: GlobalTheory.DegCnt.Weight.invVol_three (Development.AbelianVarietyTheory.Proofs.FlagMeasures.SplitTermWeightLeNearReferenceLattice, lines 20–29)
Lean: GlobalTheory.T.haarInvVolume_middleLayer_eichler_le (Development.AbelianVarietyTheory.Proofs.FlagMeasures.HaarInvVolumeMiddleLayerEichlerLeStatement, lines 11–12)
Lean: GlobalTheory.T.hermFramePairNormalForm_invVol_skew_layer_le (Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairNormalFormInvVolSkewLayerLeStatement, lines 11–12)
Lean: GlobalTheory.T.hermFramePairNormalForm_invVol_top_layer_le (Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairNormalFormInvVolTopLayerLeStatement, lines 11–12)
Lean: GlobalTheory.T.flagPairDefiniteThetaLattice_le_fixed_index_bound (Development.AbelianVarietyTheory.Proofs.FlagMeasures.FlagPairDefiniteThetaLatticeLeFixedIndexBoundStatement, lines 11–12)
Lean: GlobalTheory.DegCnt.Weight.om_le (Development.AbelianVarietyTheory.Proofs.FlagMeasures.SplitTermWeightLeNearReferenceLattice, lines 35–76)

Claim 8. The representative counts and the three weight factors give the bound in the proposition. Put $e = 12r + 50$, $d_0 = 2r + 1$, $n = N(\mathfrak{c})$, and $M = D_b D_\delta N_T$. Multiplication of the number of representatives by the bound for each relative inverse stabilizer covolume gives

$$i_1^{k+c} n^{2e d_0^2} (D_\delta N_T)^{2d_0} \left(i_2^k D_\delta^r n^{2(3e+11)d_0^2} \right).$$

Since $D_b, D_\delta, N_T \ge 1$, one has $n \le M^2$, $D_\delta N_T \le M$, and $D_\delta \le M$. Thus this product is at most

$$(i_1 i_2)^k i_1^c M^{r + 4(4e+11)d_0^2 + 2d_0} \le (i_1 i_2)^k i_1^c M^{10000(r+1)^3}.$$

Indeed the first exponent equals $768r^3 + 4144r^2 + 3573r + 846$.
Lean: GlobalTheory.DegCnt.OmCount.numeric (Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairOmSumSameDatumLe, lines 122–147)

Now $(i_1 i_2)^k \le (D_\delta / x)^{k/2}$ and $i_1^c \le D_\delta^c$. Summing over the index pairs and inner lattices gives the bound for one matrix and scalar. Multiplying by their counts, with $8^j \le 8^r$, gives (33.4).
Lean: GlobalTheory.DegAsm.FixedForm.main (Development.AbelianVarietyTheory.Proofs.FlagMeasures.SumWeightsInequivalentHermFrameTermsLe, lines 44–95)
Lean: GlobalTheory.DegCnt.omSum_sameDat_le (Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFrameWeightSumSameDatumLe, lines 48–117)
Lean: GlobalTheory.DegCnt.OmCount.main (Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermFramePairOmSumSameDatumLe, lines 237–318)

The complete proof is contained in `GlobalTheory.DegAsm.sum_weights_per_datum`. □

Proof of 33.20

Proof. Take $T = T_u$, $D = D_0 = T^{2f}$, $X = X_*$, $Y = Y_*$, and $e = 3(2r + 1)$. By 33.7, T splits completely in E, $g \ge 2$ and $v_E \ge 1$. The row space is totally positive definite of rank $k \ge m + 3$. To identify its integral lattice, pass first to the perpendicular of the isotropic flag modulo that flag. The induced lattice is self-dual: restriction of integral functionals to the saturated isotropic lattice is surjective by projectivity, and dualizing

the resulting perpendicular quotient identifies it with its dual. The last frame vector spans a primitive copy of Oz in this quotient and satisfies $h(z,z) = -T$. Its perpendicular is the row lattice L_0. In a self-dual lattice the projections onto a primitive sublattice and its perpendicular have images their respective duals. The quotient by the direct sum thus projects isomorphically onto both discriminant modules. It follows that $L_0^{\#}/L_0 \simeq (Oz)^{\#}/Oz \simeq O/(T)$.

Lean: `GlobalTheory.DegCnt.PrimPerp.perp_prim` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.FlagPairDefinitePartLatticeDiscriminantEqQuotien`, lines 36–141)

Lean: `GlobalTheory.TypeAsm.lat0C` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.BaseLatticeConstructionHolds`, lines 163–183)

Lean: `GlobalTheory.PairsRow.main` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.GenusMeanEmbeddedPairsLeMassMulPairBound`, lines 13–36)

Count the triples of 33.13 by their fixed space U. For $0 < s = \operatorname{codim} U < k$, fix $a = d(L \cap U)$. By 33.14, a nonempty fibre has $1 \le a \le A_s \le DX$. Multiply its bound in 33.14 by the genus mean of $N_{k-s,a}$ from 33.12. The factors with exponent $k/2$ give

$$16^{skf}(a/D)^{k/2}(A_s/a)^{k/2} = X^{k/2}(32/g)^{skf}.$$

Since $aD \le Y$ and $X \le Y$, the remaining discriminant and graph factors satisfy

$$(aD)^{5s+5+2c} X^{2s+2c+1} Y^{3(2r+1)} \le Y^{\,7s+6r+9+4c}.$$

There are at most $DX \le Y$ values of a, and $(4sf)^{4sf} \le (4kf)^{4sf}$. Thus the contribution at s, per unit genus mass, is at most

$$X^{k/2} Y^{6r+10+4c} M_* \rho^f \left((32/g)^{kf} (4kf)^{4f} Y^7 \right)^s.$$

For $U = V$ there is one possible space, $a = D$, and $A_0/a = X$. The first bound of 33.14 is $X^{k/2} X^{2c+1} Y^e \le X^{k/2} Y^{6r+10+4c}$. For $U = 0$ the second bound of 33.14 applies, with $c + e \le 6r + 10 + 4c$. Adding these endpoints to the sum over $1 \le s < k$ gives (33.5). □

Proof of 33.21

Proof. Choose ξ with $A(x_1) \neq 0$. The bound on $a(x)$ gives absolute convergence of $B(x)$. Unitarity gives $\|A(x)\| \le \|\xi\| a(x)$. The invariance of a therefore gives

$$\|A(x)\|^2 \le \|\xi\|^2 \int_{U \times N} |m(anx)| a(anx)\, da\, dn.$$

Integrating and applying Tonelli's theorem and the quotient integral formula yields

$$\mathcal{N} \le c_S \|\xi\|^2 \int_G |m(y)| a(y)\, dy \le c_S C \|\xi\|^2 \int_G |m(y)| \|y\|_{\mathrm{op}}^{M}\, dy < \infty.$$

Choose $\epsilon > 0$ and a relatively compact open neighborhood O of x_1 on which $\|A\| > \epsilon$. Put

$$q_O(x) = \int_{U \times N} \mathbf{1}_O(anx)\, da\, dn.$$

This is a nonnegative measurable function on the quotient, and

$$\int q_O \, d\nu = c_S \operatorname{vol}_G(O) > 0.$$

Thus $P = \{x : q_O(x) > 0\}$ has positive ν-measure. For $x \in P$, some anx belongs to O. Equivariance and unitarity imply $\|A(x)\| = \|A(anx)\| > \epsilon$. Hence $\mathcal{N} \ge \epsilon^2 \nu(P) > 0$. Multiplication by $\mathcal{N}^{-1/2}$ proves the normalization and preserves equivariance and the convolution identity. □

Proof of 33.22

Proof. Write $\Phi(x) = \ell(\sigma(x)w)$. Fix $x_0 \in G$ and $\eta \in V$. Left translation in the reproducing integral, followed by the quotient integral formula, gives

$$\begin{aligned}\langle \eta, W(x_0)\rangle &= d\int_G \overline{\langle \sigma(x_0)w, \sigma(y)w\rangle}\langle \eta, W(y)\rangle\, dy \\ &= d\int_{S\backslash G} \big\langle \mathcal{P}(\sigma(x)w, \sigma(x_0)w)\eta, W(x)\big\rangle\, d\nu(x).\end{aligned}$$

Absolute convergence of the reproducing integral justifies the change of variables and Fubini's theorem. In the inner integral use the equivariance of W, the unitarity of π, and conjugate-linearity in the first argument of the inner product. The rank-one period identity then makes the last integrand

$$\langle \eta, \Phi(x_0)\rangle\langle \Phi(x), W(x)\rangle.$$

If Φ is zero, this formula gives $W = 0$, and $\kappa = 0$ works. If $\Phi(x_*) \neq 0$, take $\eta = \Phi(x_*)$ and $x_0 = x_*$. The first factor is $\|\Phi(x_*)\|^2 > 0$, so the preceding absolute convergence proves that $x \mapsto \langle \Phi(x), W(x)\rangle$ is integrable. Set

$$\kappa = d\int_{S\backslash G} \langle \Phi(x), W(x)\rangle\, d\nu(x).$$

The formula for arbitrary x_0, η gives $\langle \eta, W(x_0)\rangle = \langle \eta, \kappa\Phi(x_0)\rangle$, hence $W = \kappa\Phi$.

If W is nonzero, choose x_0 with $\Phi(x_0) \neq 0$ and put $\xi = \Phi(x_0)$. The period identity gives $A(x) = \|\xi\|^2\Phi(x)$ and $A(x_0) = \|\xi\|^2\xi \neq 0$. Apply the stated norm identity to $v = \sigma(x_0)w$ and divide by $\|\xi\|^2$. This proves $\int \|\Phi\|^2\, d\nu = 1/d$. The formulas for the norm of A and for W^0 follow. □

Proof of 33.23

Proof (outline). Claim 1. There are a smooth unit vector $w_0 \in \mathcal{H}$, an element $\tau \in G$, a function $\Phi : G \to V$, and scalars $c_1, c_2 \in \mathbf{C}$ such that

$$B_j(\sigma(\tau)\sigma(g)w_0) = c_j\Phi(g) \quad (g \in G,\ j = 1, 2).$$

The vector w_0, element τ, and function Φ can be chosen independently of B_1, B_2. Here τ centralizes U_z, preserves N by conjugation, and carries ψ_c to the standard Bessel character. To establish the uniform orbit bound

$$\sup_{g\in G} \|B_j(\sigma(g)w_0)\| < \infty \quad (j = 1, 2),$$

choose a frame matrix Γ with $\Gamma^* H_{k,r}\Gamma = H^{\mathrm{sp}}$, where H^{sp} has ones on the antidiagonal pairing positions $1, \dots, r+1$ with $n, \dots, n-r$, ones on the diagonal at positions $r+2, \dots, r+k$, and zeros elsewhere; Γ fixes the standard basis vectors at positions $1, \dots, r$ and $n+1-r, \dots, n$, and carries the difference of the standard basis vectors at positions $r+1$ and $r+k+1$ to a positive multiple of z. Call (a, b) an upper outer position if $a < b$ and at least one of a, b lies outside $\{r+2, \dots, r+k\}$. Choose finite families spanning $\mathrm{Lie}(U_z)$ and $\mathrm{Lie}(N)$, and a finite family (Q_i) spanning the real space of $X \in \mathrm{Lie}(G)$ with $(\Gamma^{-1}X\Gamma)_{ab} = 0$ at every upper outer position; together they span $\mathrm{Lie}(G)$. Factor any $g \in G$ in the Iwasawa–Levi–Cartan form $g = hm'g'$ with $h \in U_z$, $m' \in N$, $g' = \Gamma dR \in G$, R unitary, and $d = \mathrm{diag}(d_1, \dots, d_n)$ positive with $d_a d_{n+1-a} = 1$ for $a \leq r+1$, $d_a = 1$ for $r+2 \leq a \leq r+k$, and $d_{r+1} \leq 1$. By S_z-equivariance and unitarity of π and ψ_c, $\|B_j(\sigma(g)w_0)\| = \|B_j(\sigma(g')w_0)\|$. Continuity of B_j on $\mathcal{H}^\infty$ gives an integer $m \geq 0$ and a constant $C \geq 0$ with $\|B_j(v)\| \leq C\max_P \|d\sigma(P)v\|$ for smooth v, the maximum running over the words P of length at most m in the directions Q_i. The directions of $\mathrm{Lie}(U_z) \oplus \mathrm{Lie}(N)$ are absorbed by equivariance, since for such X the vector $B_j(\sigma(\exp tX)v)$ equals $\pi(\exp tX)B_j(v)$ or $\psi_c(\exp tX)B_j(v)$. Since w_0 is smooth, there is $C_w \geq 0$ with $\|d\sigma(X_1 \cdots X_p)w_0\| \leq C_w \prod_i \|X_i\|$ for all Lie words of length $p \leq m + nm$. Put $M = \max\big(1, \max_{1\leq a\leq r} d_a/d_{a+1}\big)$, so that $M = 1$ when $r = 0$. Then $d_a \leq Md_{a+1}$ for $1 \leq a < n$. For $a \leq r$ this is the definition of M. For $r+1 \leq a \leq r+k$ it holds because $d_a \leq 1 \leq d_{a+1}$, by $d_{r+1} \leq 1$ and $d_{r+k+1} = d_{r+1}^{-1}$. For $a \geq r+k+1$ it follows from $d_{n-a} \leq Md_{n+1-a}$ and the pairings

$d_a d_{n+1-a} = d_{a+1} d_{n-a} = 1$. Hence $d_b \le M^n d_a$ whenever $b \le a$, and $d_b = d_a$ when $a < b$ both lie in $\{r+2, \dots, r+k\}$. The (a,b) entry of $d^{-1}(\Gamma^{-1} Q_i \Gamma)d$ is $d_a^{-1} d_b (\Gamma^{-1} Q_i \Gamma)_{ab}$; it vanishes at upper outer positions, and at every other position $d_a^{-1} d_b \le M^n$. As R is unitary, $\|g'^{-1} Q_i g'\| \le C_0 M^n$, where C_0 depends only on Γ, the family (Q_i) and ψ_c. Put $A = \max(1, C_0)$. If $M = 1$, in particular whenever $r = 0$, these two bounds give $\|B_j(\sigma(g')w_0)\| \le C C_w A^{m+nm}$. If $M > 1$, choose a simple-root index a_0 with $d_{a_0}/d_{a_0+1} = M$ and a Lie vector Y with $\|Y\| \le C_0$, $g' \exp(\tau Y) g'^{-1} \in N$ and $\psi_c(g' \exp(\tau Y) g'^{-1}) = e^{iM\tau}$ for all real τ. Equivariance gives $B_j(\sigma(g')d\sigma(Y)^p w_0) = (iM)^p B_j(\sigma(g')w_0)$. Taking $p = nm$ and bounding the left side by the two estimates above,

$$M^{nm} \|B_j(\sigma(g')w_0)\| \le C C_w A^{m+nm} M^{nm}.$$

Hence $\|B_j(\sigma(g)w_0)\| \le C C_w A^{m+nm}$ for every $g \in G$.

Next conjugate by τ to normalize ψ_c to the standard Bessel character ψ_{st}, and put $B'_j(v) = B_j(\sigma(\tau)v)$. Then B'_j is continuous, satisfies $B'_j(\sigma(nh)v) = \psi_{\mathrm{st}}(n)\pi(h)B'_j(v)$ for $h \in U_z$ and $n \in N$, and is bounded on the orbit of w_0 by the previous paragraph. Write $\|y\|$ for the Hilbert–Schmidt norm on G. The opposite-sign gap $h \ge 120n^3$ gives $|\langle v, \sigma(y)w_0\rangle| \le C_v \|y\|^{-4n^3}$ for every smooth v, and $\|y\|^{-4n^3}$ is integrable on G because $4n^3 \ge 2(k+r)(r+1)+1$. Let V_{fl} be the space of $u \in \mathcal{H}$ with $|\langle w_0, \sigma(x)u\rangle| \le C_u \|x\|^{-4n^3}$ on G; it contains every $\sigma(g)w_0$. For $u, u' \in V_{\mathrm{fl}}$ the Bessel period operator

$$P(u,u') = \int_{U_z \times N} \langle u', \sigma(nh)u\rangle \, \overline{\psi_{\mathrm{st}}(n)} \, \pi(h)^{-1} \, dh \, dn$$

converges absolutely as an endomorphism of V, with the inner product conjugate-linear in its first argument.

Claim 1a. There is a linear map $l : V_{\mathrm{fl}} \to V$ with $l(\sigma(nh)u) = \psi_{\mathrm{st}}(n)\pi(h)l(u)$ and

$$P(u,u')\xi = \langle l(u'), \xi\rangle \, l(u) \qquad (u, u' \in V_{\mathrm{fl}},\ \xi \in V).$$

To prove Claim 1a, note that P is Hermitian, $\langle \xi, P(u,u')\eta\rangle = \langle P(u',u)\xi, \eta\rangle$, positive, $\langle \xi, P(u,u)\xi\rangle \ge 0$, equivariant, $P(\sigma(nh)u, u') = \psi_{\mathrm{st}}(n)\pi(h)P(u,u')$, and its two-by-two minors vanish:

$$\langle e_1, P(u_1,u_2)e_2\rangle \langle e_3, P(u_3,u_4)e_4\rangle = \langle e_1, P(u_1,u_4)e_4\rangle \langle e_3, P(u_3,u_2)e_2\rangle.$$

If $\kappa_0 = \langle \xi_0, P(v_0,v_0)\xi_0\rangle > 0$ for some v_0 and ξ_0, put $l(u) = \kappa_0^{-1/2} P(u, v_0)\xi_0$; the minor identity with $(u_3, u_4) = (v_0, v_0)$ gives the rank-one formula. Otherwise every diagonal value vanishes, the minor identity gives $|\langle \eta, P(u,u')\eta'\rangle|^2 = 0$, and $l = 0$. The minor identity is first proved for the vectors $\int_G \phi(g)\sigma(g)w_0 \, dg$ with ϕ smooth and compactly supported, by a Gelfand–Kazhdan argument. A frame of z gives an anti-involution θ of G, with $\theta(gg') = \theta(g')\theta(g)$ and $\theta^2 = 1$, which preserves N, U_z, the three Haar measures and ψ_{st} and fixes representatives of the relevant S_z-double cosets. Either the θ-twisted Schur pairings of σ or of π vanish, and then P vanishes on these vectors, or there are antiunitary operators C on $\mathcal{H}$ and C' on V with $C\sigma(g)C^{-1} = \sigma(\theta(g))^{-1}$ and $C'\pi(h)C'^{-1} = \pi(\theta(h))^{-1}$, so that $\langle y, \sigma(\theta(g))x\rangle = \langle Cx, \sigma(g)Cy\rangle$, and the resulting θ-symmetry of the Bessel distributions gives the minor identity. The identity extends to V_{fl} by the reproducing formula $P(u,v) = d_\sigma \int_G \langle \sigma(y)w_0, u\rangle P(\sigma(y)w_0, v) \, dy$ and the growth bound $\|P(\sigma(y)w_0, v)\| \le C_v \|y\|^{4n^3 - n^2 - 1}$, whose product with the decay of order $4n^3$ is integrable.

Discrete-series Schur orthogonality gives $\int_G \overline{\langle v, \sigma(y)w_0\rangle}\sigma(y)w_0 \, dy = d_\sigma^{-1} v$ for smooth v. The continuous map B'_j passes through this integral by truncation. Fix a smooth compactly supported function ρ on matrices with $\rho(0) = 1$, and for $R > 0$ let v_R be the same integral with the integrand multiplied by $\rho(R^{-1}y)$. Each v_R is smooth, and for $X \in \mathrm{Lie}(G)$ the vector $d\sigma(X)v_R$ is the truncated integral of $d\sigma(X)v$ plus the truncated integral of v with ρ replaced by $x \mapsto -d\rho(x)(Xx)$, a function vanishing at 0. Induction on the length of a word P in $\mathrm{Lie}(G)$ gives $d\sigma(P)v_R \to d_\sigma^{-1} d\sigma(P)v$ in $\mathcal{H}$ as $R \to \infty$, so $B'_j(v_R) \to d_\sigma^{-1} B'_j(v)$ by continuity. Continuity also writes B'_j on smooth vectors as a finite sum of bounded operators $\mathcal{H} \to V$ applied to derivatives $d\sigma(P)v$, and these commute with the truncated integrals, whose integrands have compact support. Hence $B'_j(v_R)$ is the truncated integral of $\overline{\langle v, \sigma(y)w_0\rangle} B'_j(\sigma(y)w_0)$, which converges to the untruncated integral by dominated convergence, using the orbit bound and the integrability of $y \mapsto \langle v, \sigma(y)w_0\rangle$. With $v = \sigma(x)w_0$ this gives

$$B'_j(\sigma(x)w_0) = d_\sigma \int_G \overline{\langle w_0, \sigma(y)w_0\rangle} \, B'_j(\sigma(xy)w_0) \, dy.$$

Fold this integral over $S_z = U_z \ltimes N$, using $\int_G f = \int_G b(y) \int_{U_z \times N} f(nhy)\, dh\, dn\, dy$ with a fixed weight b. By equivariance the inner integral equals $d_\sigma P(\sigma(x)w_0, \sigma(y)w_0) B_j'(\sigma(y)w_0)$, and Claim 1a turns it into $d_\sigma \langle l(\sigma(y)w_0), B_j'(\sigma(y)w_0)\rangle\, l(\sigma(x)w_0)$. Hence Claim 1 holds with $\Phi(g) = l(\sigma(g)w_0)$, which is independent of B_1, B_2, and

$$c_j = d_\sigma \int_G b(y)\, \langle l(\sigma(y)w_0), B_j'(\sigma(y)w_0)\rangle\, dy.$$

Lean: `CompanionTheory.Bmu.bounded` (`CompanionTheory.AutomorphicTheory.BesselOrbitBounds.BesselMapUniqueness.BesselMapUniqGapAlt`, lines 27–31)
Lean: `CompanionTheory.Bmu.uniq_of` (`CompanionTheory.AutomorphicTheory.BesselOrbitBounds.BesselMapUniqueness.OrbitMult`, lines 127–199)

Claim 2. The complex span of $\{\sigma(\tau)\sigma(g)w_0 : g \in G\}$ is dense in $\mathcal{H}^\infty$ for the smooth-vector topology. More explicitly, given $v \in \mathcal{H}^\infty$, finitely many elements $D_1, \dots, D_s$ of the complex enveloping algebra of the Lie algebra of G, and $\delta > 0$, there is a finite linear combination v_0 of these translates such that

$$\|d\sigma(D_i)(v - v_0)\| < \delta \qquad (1 \le i \le s).$$

Smooth cyclicity gives this assertion for the orbit of w_0. The action of $\sigma(\tau)$ and its inverse preserve the smooth-vector topology, so it gives the displayed assertion as well.
Lean: `CompanionTheory.D.Upq.Sm.smooth_cyclic_vectors` (`CompanionTheory.HarishChandraModules.BesselParameterIntegrals.SmoothCyclicVectors.SmoothCyclicVectors`, lines 6–6)
Lean: `CompanionTheory.Bmu.uniq_of` (`CompanionTheory.AutomorphicTheory.BesselOrbitBounds.BesselMapUniqueness.OrbitMult`, lines 127–199)

If $c_1 = 0$, take $(a, b) = (1, 0)$. Otherwise take $(a, b) = (c_2, -c_1)$. In either case $(a, b) \neq (0, 0)$ and $ac_1 + bc_2 = 0$. Claim 1 makes $aB_1 + bB_2$ zero on the orbit in Claim 2. Linearity, continuity, and Claim 2 make it zero on all of $\mathcal{H}^\infty$. The complete proof is contained in `CompanionTheory.Bmu.besselMapUniqGap`. □

Proof of 33.24

Proof (outline). Put $r = r_S$, $m = 2r + 2$, and write $v_i = \lambda_{v,i}$ for $1 \le i < k$. The Cayley parameter and signature are

$$\lambda = (v_1, \dots, v_{k-1}, -b_0, \dots, -b_{m-1}), \qquad \epsilon = (\underbrace{+1, \dots, +1}_{k-1}, +1, -1, \dots, +1, -1).$$

Thus the negative signature index is $r + 1$ and $N_v = 4n^2\sqrt{2(r+1)}$.
Lean: `Formalism.MainNotions.CayleySignature.lam48` (`Formalism.SharpRestriction.CuspSpanNormBounds.CayleySignature`, lines 18–20)
Lean: `Formalism.MainNotions.CayleySignature.eps48` (`Formalism.SharpRestriction.CuspSpanNormBounds.CayleySignature`, lines 22–23)

Claim 1. The precise gap and weighted coefficient integral have the following form. The gap condition used by the weighted coefficient theorem is the existence of an odd integer $t \ge n+5$, with $n+t \ge 1801$, such that

$$2(n + mt)^3 + t + 3 < b_{j+1} - b_j \qquad (0 \le j < m - 1).$$

For a regular Harish–Chandra parameter of the displayed form, a discrete-series realization with the displayed signature, and a unit vector w in its lowest compact type, this condition gives

$$\int_G |\langle w, \sigma(g)w\rangle|\, \|g\|_{\mathrm{op}}^{4n^2\sqrt{2(r+1)}}\, dg < \infty.$$

Here dg may be any Haar measure; the integral first follows for the fixed diagonal-signature Haar measure, and multiplication of that measure by a positive constant preserves finiteness. Because the first $k-1$ coordinates satisfy $v_i > -b_0$ with sign $+1$ and the tail coordinates $-b_0, \dots, -b_{m-1}$ carry alternating signs $+1, -1, \dots, +1, -1$, every pair of indices with $\epsilon_i \neq \epsilon_j$ satisfies $|\lambda_i - \lambda_j| \ge \min_j(b_{j+1} - b_j) > 2(n+mt)^3 + t + 3$, which exceeds the Cartan-decay threshold ensuring that the exponential matrix-coefficient bound in the Cartan decomposition dominates both the radial Weyl Jacobian and the weight $\|g\|_{\mathrm{op}}^{4n^2\sqrt{2(r+1)}}$.
Lean: `Formalism.MainNotions.CayleySignature.GapBig` (`Formalism.SharpRestriction.CuspSpanNormBounds.CayleySignature`, lines 35–38)

Lean: `Formalism.L2.SG.SplitPrimeNormLemmas.CoefWeightedLoc` (`Formalism.LocalToGlobal.SiegelLatticeTheory.HaarTransversals.UnitaryHaarProducts.SettingBlocks`, lines 323–333)
Lean: `ReciprocityTheory.ChartRestriction.DS.coefWeightedLoc` (`ReciprocityTheory.ReductiveGroups.DiscreteSeries.FormalDegree.CoefLoc`, lines 23–27)
Lean: `Formalism.L2.SG.Closed.BM.CoefWeightedLoc_holds` (`CompanionTheory.AutomorphicTheory.LocalBesselUniqueness.BesselModels.CoefWeightedLocHolds`, lines 17–17)

In the present application n is odd because k is even. Take $t = n + 6$. Then t is odd, $t \geq n + 5$, $n + t \geq 1801$, $m \leq n$, and $t \leq 2n$. Consequently

$$2(n + mt)^3 + t + 3 \leq 54n^6 + 2n + 3 < 16n^{19} \leq b_{j+1} - b_j.$$

Unitary Cayley conjugation preserves the operator norm and carries Haar measure to a positive multiple of Haar measure. Finally, for $x_0 \in G_v^0$, put $y = x_0 g$. Unitarity and submultiplicativity give

$$\int_{G_v^0} |\langle \sigma_v(x_0)w_v, \sigma_v(y)w_v\rangle| \|y\|_{\mathrm{op}}^{N_v}\, dg_v^0(y) \leq \|x_0\|_{\mathrm{op}}^{N_v} \int_{G_v^0} |\langle w_v, \sigma_v(g)w_v\rangle| \|g\|_{\mathrm{op}}^{N_v}\, dg_v^0(g) < \infty.$$

Lean: `PI.UpqRep.IsCayleyU` (`Formalism.PeriodIntegrals.UpqRep`, lines 343–345)

Claim 2. Under the numerical, discrete-series, lowest-type and Haar-measure hypotheses fixed above, for each $x_0 \in G_v^0$ there is $C_\phi(x_0) \geq 0$ such that

$$|\langle \sigma_v(x_0)w_v, \sigma_v(y)w_v\rangle| \|y\|_{\mathrm{op}}^{N_v} \leq C_\phi(x_0) \qquad (y \in G_v^0), \qquad I_N := \int_{N_{u,v}^0} \|n\|_{\mathrm{op}}^{-N_v}\, dn < \infty.$$

The first assertion follows by applying the compactly supported reproducing formula for the lowest-type vector to its coefficient. On the compact support the operator norm of the inverse is bounded. Moving the translate into the weighted coefficient integral of Claim 1 therefore bounds the displayed expression uniformly in y.
Lean: `WlocSk.PtD.ptDecay_of` (`Formalism.LocalToGlobal.SiegelLatticeTheory.HaarTransversals.UnitaryHaarProducts.LocalWeightCharts.NilradicalHalfSumNonneg`, lines 297–316)

For the second assertion, the unipotent coordinate map is a measurable bijection $\Phi : \mathbf{C}^M \times \mathbf{R}^r \longrightarrow N_{u,v}^0$, with measurable inverse, for some $M \leq 2r^2 + r(k + 1)$. Its coordinate Lebesgue measure is left invariant, and

$$1 + \|t\| \leq (r + 2)\|\Phi(t)\|_{\mathrm{op}}^2, \qquad 2(2M + r) < N_v.$$

Left multiplication is a triangular shear in these coordinates and preserves Lebesgue measure. Haar uniqueness therefore makes dn a finite positive multiple of $\Phi_* dt$. The displayed norm bound majorizes its inverse-norm integrand by a constant times $(1 + \|t\|)^{-N_v/2}$. Since $N_v/2 > 2M + r$, this function is integrable.
Lean: `WlocSk.nilpotent_chart_exists` (`Formalism.LocalToGlobal.SiegelLatticeTheory.HaarTransversals.UnitaryHaarProducts.LocalWeightCharts.NChartP`, lines 36–114)
Lean: `WlocSk.nIntDecay_of` (`Formalism.LocalToGlobal.SiegelLatticeTheory.HaarTransversals.UnitaryHaarProducts.LocalWeightCharts.OpnContinuous`, lines 90–132)

Let $K_U \geq 0$ bound $\|a^{-1}\|_{\mathrm{op}}$ on the compact group $U_{u,v}^0$, and write H for the Cayley Hermitian matrix. For $a \in U_{u,v}^0$, $n \in N_{u,v}^0$ and $y \in G_v^0$, submultiplicativity and the inverse formula in $\mathrm{U}(H)$ give

$$\|n\|_{\mathrm{op}} \leq K_U \|H\|_{\mathrm{op}}^2 \|y\|_{\mathrm{op}} \|any\|_{\mathrm{op}}.$$

Claim 2 consequently gives

$$|\langle \sigma_v(x_0)w_v, \sigma_v(any)w_v\rangle| \leq C_\phi(x_0)(K_U\|H\|_{\mathrm{op}}^2)^{N_v} \|y\|_{\mathrm{op}}^{N_v} \|n\|_{\mathrm{op}}^{-N_v}.$$

Integrating this bound proves the asserted estimate for $a_v(y; x_0)$, after increasing the constant to be positive. On each compact set of y's the same integrable function dominates the operator integrand, because the Bessel character has modulus one and $\pi_{u,v}(a)$ is unitary. Dominated convergence gives continuity. Right invariance on the unimodular semidirect product gives the stated equivariance. This establishes the pointwise bound as well as the integrability needed below.
Lean: `WlocSk.PtD.aBound_of` (`Formalism.LocalToGlobal.SiegelLatticeTheory.HaarTransversals.UnitaryHaarProducts.LocalWeightCharts.PsiGenNorm`, lines 103–173)
Lean: `WlocSk.aBound_of_chart` (`Formalism.LocalToGlobal.SiegelLatticeTheory.HaarTransversals.UnitaryHaarProducts.LocalWeightCharts.WavePacketL1L2OfNilpotentChart`, lines 41–43)

Claim 3. Nonvanishing holds for the specified lowest-type vector. Let $k \geq 2$ be even, $r \geq 0$, $n = 2r+k+1$, and $G = \mathrm{U}(H_{k,r})$. Let z lie in the middle $(k+1)$-dimensional block of the Cayley space and have $-\operatorname{Re} h(z,z) > 0$. Let $c_0, \dots, c_{r-1} \in \mathbf{C}^\times$ be the simple-root parameters of its generic Bessel character. Let $a = (a_1, \dots, a_k)$ and $\lambda = (v_1, \dots, v_{k-1}, -b_0, \dots, -b_{2r+1})$ be regular Harish–Chandra parameters for ranks k and n, respectively. Suppose

$$a_i > v_i > a_{i+1}\ (1 \leq i < k), \qquad a_k > -b_0, \qquad a_i - \lambda_s \in \mathbf{Z} + \tfrac{1}{2}\ (1 \leq i \leq k,\ 1 \leq s \leq n),$$

and suppose the gap condition of Claim 1 holds. Let σ be the discrete series on a complex Hilbert space with parameter λ and the displayed Cayley signature, and let π be the compact representation of U_z on a finite-dimensional complex Hilbert space V, of Harish–Chandra type a. For every unit lowest-type vector w and every choice of Haar measures da, dn on U_z, N, there are $x, x_0 \in G$ for which

$$\int_{U_z \times N} \langle \sigma(x_0)w, \sigma(an)\sigma(x)w \rangle \overline{\psi_{z,c}(n)}\, \pi(a)^{-1}\, da\, dn \neq 0 \quad \text{in } \operatorname{End}_{\mathbf{C}}(V).$$

Regularity here includes strict decrease and the lattices $a_i - (k-1)/2 \in \mathbf{Z}$ and $\lambda_s - (n-1)/2 \in \mathbf{Z}$.
Lean: `Formalism.MainNotions.CayleySignature.InterlacingCondition` (`Formalism.SharpRestriction.CuspSpanNormBounds.CayleySignature`, lines 25–28)
Lean: `Formalism.MainNotions.CayleySignature.HalfIntegralDifference` (`Formalism.SharpRestriction.CuspSpanNormBounds.CayleySignature`, lines 30–31)
Lean: `ReciprocityTheory.ChartRestriction.DS.Lap.TopNV.BesselNVG` (`ReciprocityTheory.ReductiveGroups.DiscreteSeries.LaplaceOperators.BesselVectors.BesselNV`, lines 39–57)

For the nonvanishing argument, let (τ_T, V_T) be a model of the lowest compact type of λ, let μ be the integral weight with $\mu_s = \lambda_s + (n+1)/2 - s$, let P be a Cayley matrix for the displayed signature, and let $\Psi_{T,\mu} : G \to \operatorname{End}_{\mathbf{C}}(V_T)$ be the τ_T-spherical function built from μ by averaging over the maximal compact subgroup. For $v \in V_T$ put $f_v(g) = \langle v, \Psi_{T,\mu}(PgP^{-1})v \rangle$. Under the hypotheses of Claim 3, with the gap witness of Claim 1, three statements hold. (i) Laplace nonvanishing. For every such T, μ, P, every nonzero $v \in V_T$, every π of type a and all Haar measures da, dn, there are $\xi \in V$ and $x_0, x \in G$ with

$$\int_{U_z \times N} f_v(x_0^{-1} anx)\, \overline{\psi_{z,c}(n)}\, \pi(a)^{-1}\xi\, da\, dn \neq 0.$$

At the base point the matrix-coefficient bounds justify the Gaussian limit of the rescaled period integral. The discrete-series lowering and raising operators then transfer nonvanishing along the admissible unit steps. At the base parameters $\lambda^{(t)}$, with $\lambda^{(t)}_s = a_s - \frac{1}{2}$ for $s \leq k$ and $\lambda^{(t)}_{k+j} = -b_j - jt$ for $1 \leq j \leq 2r+1$, the integral is nonzero for every sufficiently large integer t. A unit step $\lambda' \mapsto \lambda' \pm e_s$ between regular parameters whose opposite-sign gaps are at least a step gap h, with $h \leq b_{j+1} - b_j$ for all j and h supplied by the gap witness of Claim 1, preserves nonvanishing provided $\lambda'_s \pm \frac{1}{2}$ is not an entry of a. Interlacing and the half-integral differences give a walk of such steps from $\lambda^{(t)}$ to λ, and the nonvanishing is then transported to every model, every nonzero v and every Bessel datum. (ii) Model realization. There is a discrete series σ_0 with parameter λ and the displayed signature such that each lowest-type unit vector w_0 of σ_0 satisfies $\langle w_0, \sigma_0(g)w_0 \rangle = c_1 f_{v_1}(g)$ on G for some T, μ, P as above, some nonzero $v_1 \in V_T$ and some $c_1 \neq 0$. (iii) Lowest-type coefficients. The prescribed w has a lowest-type unit vector w_0 of σ_0 with $\langle w_0, \sigma_0(g)w_0 \rangle = \langle w, \sigma(g)w \rangle$ on G. By (iii) and (ii), $\langle w, \sigma(g)w \rangle = c_1 f_{v_1}(g)$, and by (i) with $v = v_1$ the corresponding shifted Bessel integral is nonzero on some $\xi \in V$.
Lean: `ReciprocityTheory.ChartRestriction.DS.Lap.nv_of` (`ReciprocityTheory.ReductiveGroups.DiscreteSeries.LaplaceOperators.ParitySignEqGapSign`, lines 18–28)
Lean: `ReciprocityTheory.ChartRestriction.DS.sigZero` (`ReciprocityTheory.ReductiveGroups.DiscreteSeries.DiscreteSeriesExistence.SigZeroTop`, lines 49–154)
Lean: `ReciprocityTheory.ChartRestriction.DS.Lap.dsUniv` (`ReciprocityTheory.ReductiveGroups.DiscreteSeries.LaplaceOperators.SpectralGap.DSUniv`, lines 94–106)

Transfer the coefficient identity to the common Cayley coordinates and use

$$\langle \sigma(x_0)w, \sigma(an)\sigma(x)w \rangle = \langle w, \sigma(x_0^{-1} anx)w \rangle.$$

The transferred operator integral evaluated on ξ is that nonzero scalar multiple of the nonzero vector integral. Hence the operator integral is nonzero. This argument applies to the prescribed w and the prescribed Haar measures.

Lean: ReciprocityTheory.ChartRestriction.DS.Lap.TopClosed.nvAll (ReciprocityTheory.ReductiveGroups.DiscreteSeries.LaplaceOperators.BesselVectors.Closed, lines 8–8)
Lean: ReciprocityTheory.ChartRestriction.DS.Lap.TopFinal.besselNVG_of_nv (ReciprocityTheory.ReductiveGroups.DiscreteSeries.LaplaceOperators.BesselVectors.BesselNVGOfNv, lines 15–16)
Lean: ReciprocityTheory.ChartRestriction.DS.Lap.TopNV.besselNVG_of (ReciprocityTheory.ReductiveGroups.DiscreteSeries.LaplaceOperators.BesselVectors.BesselNV, lines 118–171)

The local data in the proposition satisfy every hypothesis of Claim 3, with the gap witness verified in Claim 1. Apply Claim 3 with $w = w_v$, and call the resulting group elements x_1, x_0. This integral is $B_v(x_1, x_0)$. Choose $\xi \in V_{u,v}$ on which this nonzero operator is nonzero. This proves (2). The quotient integral formula here has $c_S = 1$ in Lemma 33.21. Right invariance gives $a_v(anx; x_0) = a_v(x; x_0)$. Apply that lemma with $G = G_v^0$, $S = S_{u,v}^0$, $M = N_v$, and $C = C_v(x_0)$, using the weighted coefficient integral in Claim 1 and the nonzero value just obtained. It gives

$$0 < \int_{S_{u,v}^0 \backslash G_v^0} \|A_v\|^2 \, d\nu_{u,v}^0 \le C_v(x_0)\|\xi\|^2 \int_{G_v^0} |\langle \sigma_v(x_0)w_v, \sigma_v(y)w_v\rangle| \|y\|_{\mathrm{op}}^{N_v} \, dg_v^0(y) < \infty.$$

Put $\phi_v(g) = d_v \overline{\langle w_v, \sigma_v(g)w_v\rangle}$. For each $x \in G_v^0$, the nilpotent-chart bound and Claim 1 give

$$\int_{G_v^0} |\phi_v(g)| a_v(xg; x_0) \, dg_v^0(g) \le C_v(x_0)\|x\|_{\mathrm{op}}^{N_v} \int_{G_v^0} |\phi_v(g)| \|g\|_{\mathrm{op}}^{N_v} \, dg_v^0(g) < \infty.$$

Thus Fubini's theorem applies to right convolution of A_v with ϕ_v. Schur orthogonality and $\|w_v\| = 1$ give

$$d_v \int_{G_v^0} \overline{\langle w_v, \sigma_v(g)w_v\rangle} \sigma_v(g)w_v \, dg_v^0(g) = w_v.$$

Applying $\sigma_v(anx)$, taking the inner product with $\sigma_v(x_0)w_v$, and integrating over a, n gives $\int_{G_v^0} \phi_v(g) A_v(xg) \, dg_v^0(g) = A_v(x)$. The same reproducing identity holds for $W_v^0 = A_v / \|A_v\|_{L^2(\nu_{u,v}^0)}$.
Lean: WlocSk.ANz.aNonzero_of (Formalism.LocalToGlobal.SiegelLatticeTheory.HaarTransversals.UnitaryHaarProducts.LocalWeightCharts.ANonzeroOf, lines 34–147)
Lean: WlocSk.AQ.wave_packet_l2_finite (Formalism.LocalToGlobal.SiegelLatticeTheory.HaarTransversals.UnitaryHaarProducts.LocalWeightCharts.WavePacketL1L2OfHypotheses, lines 397–478)
Lean: WlocSk.scWlocShape_of (Formalism.LocalToGlobal.SiegelLatticeTheory.HaarTransversals.UnitaryHaarProducts.LocalWeightCharts.L2SGWfEqOne, lines 87–124)

When W_u does not vanish identically, isolating the v-th tensor factor at a point where $W_u \neq 0$ and passing to G_v^0 shows that the transported local factor W_v is continuous, $(S_{u,v}^0, \pi_{u,v} \otimes \psi_{u,v}^0)$-equivariant, square-integrable on $S_{u,v}^0 \backslash G_v^0$, and fixed under right convolution by $d_v \overline{\langle w_v, \sigma_v(\cdot)w_v\rangle}$. The realization of this factor uses the following more precise statement.

Claim 4. An absolutely reproducing coefficient has the following smooth realization. Retain the group $G = G_v^0$, the discrete series $\sigma = \sigma_v$ on $\mathcal{H}_v$, its unit lowest-type vector $w = w_v$, the frame $z = z_{u,v}$, and the compact representation $\pi = \pi_{u,v}$ on $V = V_{u,v}$ from the proposition. Retain also its numerical parameters, so $k \ge 2$ is even, $m = 2r + 2$, $n = 2r + k + 1 \ge 1801$, the parameter and signature have the displayed form, and $b_{j+1} - b_j \ge 16n^{19}$. Fix a scalar $c \in \mathbf{C}^\times$, and let $\psi_{z,c}$ be the scalar-parameter Bessel character. Take Haar measures dg, da, dn on G, U_z, N, and a quotient measure $d\nu$ satisfying

$$\int_G F(g) \, dg = \int_{S\backslash G} \int_{U_z \times N} F(anx) \, da \, dn \, d\nu(x) \qquad (F \ge 0), \quad S = U_z N.$$

Put $\phi(g) = d_\sigma \overline{\langle w, \sigma(g)w\rangle}$, with formal degree relative to dg. If $W : G \to V$ is continuous and satisfies

$$W(anx) = \psi_{z,c}(n)\pi(a)W(x), \qquad 0 < \int_{S\backslash G} \|W(x)\|^2 \, d\nu(x) < \infty,$$

and, for every $x \in G$,

$$\int_G |\phi(g)| \, \|W(xg)\| \, dg < \infty, \qquad \int_G \phi(g)W(xg) \, dg = W(x),$$

then there is a complex-linear continuous map $\mathcal{B} : \sigma^\infty \to V$, continuous for the smooth Fréchet topology, with

$$\mathcal{B}(\sigma(an)v) = \psi_{z,c}(n)\pi(a)\mathcal{B}(v), \qquad W(x) = \mathcal{B}(\sigma(x)w).$$

Lean: Formalism.L2.SG.LocWSOf.BesselOfW (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.LocWSOf, lines 237–256)
Lean: Formalism.QuantumDeterminants.GapAtLeast (Formalism.FormalDegreeBessel.StatementDS, lines 16–17)
Lean: Formalism.FourierDecayAlt.ThreeDimData.GapOK (Formalism.HermitianCurvature.BesselNonvanishingReduction, lines 32–34)

Indeed, w is compact-finite, hence smooth, so $\mathcal{D} = \mathrm{span}_{\mathbf{C}}\{\sigma(x)w : x \in G\}$ consists of smooth vectors, and the low-type and infinite-range decay estimates give their coefficients the decay of order $4n^3$ used in Lemma 33.23. Claim 1a in the proof of that lemma, transported from the standard character to $\psi_{z,c}$ by a torus element that centralizes U_z and normalizes N, gives a smooth continuous Bessel map ℓ, with $y \mapsto \ell(\sigma(y)w)$ square-integrable on $S\backslash G$ by the coefficient-period norm identity of Claim 4 in the proof of Proposition 33.18, and, on $\mathcal{D}$, absolute convergence and the identity

$$\int_{U_z \times N} \langle v, \sigma(an)u\rangle \overline{\psi_{z,c}(n)}\, \pi(a)^{-1}\, da\, dn = \ell(u) \otimes \ell(v)^* \qquad (u, v \in \mathcal{D}).$$

Here the operator on the right sends η to $\langle \ell(v), \eta\rangle \ell(u)$. Substituting $y = xg$ in the reproducing integral $W(x) = \int_G \phi(g) W(xg)\, dg$, unfolding G over $S\backslash G$ via quotient disintegration, pulling $W(y)$ through the inner $U_z \times N$ integral by $(S, \pi \otimes \psi_{z,c})$-equivariance, and inserting the rank-one identity for $u = \sigma(x)w$ and $v = \sigma(y)w$ yields $W(x) = \kappa\, \ell(\sigma(x)w)$ with $\kappa = d_\sigma \int_{S\backslash G} \langle \ell(\sigma(y)w), W(y)\rangle\, d\nu(y)$ (absolutely convergent by Cauchy–Schwarz, since both W and $y \mapsto \ell(\sigma(y)w)$ are square-integrable on $S\backslash G$), as in Lemma 33.22. Thus $\mathcal{B} = \kappa\ell$. The two smooth Bessel-map formulations agree because they impose the same linearity, equivariance and smooth-topology continuity conditions.
Lean: CompanionTheory.Lime.BM.inputs (CompanionTheory.AutomorphicTheory.LocalBesselUniqueness.BesselModels.Inputs, lines 10–14)
Lean: CompanionTheory.D.Upq.BesselFunctionalConstruction.Inputs (CompanionTheory.HarishChandraModules.BesselParameterIntegrals.BesselOrbitConvolutions.BesselNonvanishingReduction, lines 12–12)
Lean: CompanionTheory.Lime.BM.dDFormOne0 (CompanionTheory.AutomorphicTheory.LocalBesselUniqueness.BesselModels.DoubleCosetBesselBoundOfGapConstantLe, lines 13–14)
Lean: CompanionTheory.D.Upq.BesselFunctionalConstruction.dDForm_scale (CompanionTheory.HarishChandraModules.BesselParameterIntegrals.BesselOrbitConvolutions.FinalC, lines 157–161)
Lean: Formalism.MainNotions.BridgeCS3dBOW.bow_core (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.BesselOfWPf, lines 132–221)
Lean: Formalism.MainNotions.BridgeCS3d.isBesselMapSm_iff (Formalism.SharpRestriction.CuspSpanNormBounds.FamilyDoubleCosetBesselBoundOfGap, lines 28–37)

To check the numerical input to this construction, take $h = 16n^{19}$. The alternating tail signs and the inequalities $v_i > -b_0$ give $|\lambda_i - \lambda_j| \geq h$ whenever $\epsilon_i \neq \epsilon_j$. The supplied dimension bounds make the following three inequalities sufficient for the construction:

$$h \geq 120n^3, \qquad m^2\big(2n^3 + 4(k+r)(r+1)\big) + 4(k+r)(r+1) + 3 \leq \frac{h}{8\sqrt{m}},$$

$$9(r+1)n^2 + 12(k+r)(r+1) + 3 \leq \frac{h}{8\sqrt{m}}.$$

Here $(k+r) + (r+1) = n$, so $4(k+r)(r+1) \leq n^2$, while $r + 1 = m/2 \leq n/2$. The two polynomial left sides are therefore at most $2n^5 + n^4 + n^2 + 3 \leq 13n^5$ and $9n^3/2 + 3n^2 + 3 \leq 20n^3$. The common right side is at least $2n^{18}$, and $16n^{19} \geq 120n^3$. Thus all three inequalities hold for the numerical setting already fixed in the proposition.
Lean: CompanionTheory.WaveGrid.GapSizes.consts (CompanionTheory.AutomorphicTheory.BesselWaveGrids.GapSizes, lines 27–52)

Claim 5. The tied local factor is eligible for Claim 4 with a scalar Bessel character. In the compatible Cayley frame, the arithmetic character has a common nonzero simple-root parameter c_*. The scalar and tuple character formulas agree,

$$\psi_{z,c_*} = \psi_{z,(c_*,\ldots,c_*)}.$$

This is an identity of the defining character functions. In the normalized frame used for the tied model the common parameter is $2\kappa_E/(g_0 g_1)$, where $\kappa_E = \pi/v_E > 0$. Thus no new choice of character is made when applying the scalar-parameter realization.

Lean: Formalism.L2.SG.Pf.WuBL.cS (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.WuBesselLoc, lines 182–182)
Lean: Formalism.L2.SG.LitS.psi_eq_psiGen (Formalism.LocalToGlobal.SiegelLatticeTheory.CasimirReductionTheory.GgExistsAddCharNeOne, lines 122–125)

If the transport uses $y \in G_v^0$ to place the chosen vector in the lowest type, apply Claim 4 to $\widetilde{W}(x) = W_v(xy)$ and $\widetilde{w} = \sigma_v(y)w_v$. Right translation preserves the quotient squared norm and transports the absolutely convergent reproducing identity. The local-factor identities supply continuity, equivariance, a positive finite squared norm, and that reproducing identity. Thus $\widetilde{W}(x) = \mathcal{B}(\sigma_v(x)\widetilde{w})$; replacing x by xy^{-1} yields $W_v(x) = \mathcal{B}(\sigma_v(x)w_v)$.
Lean: Formalism.L2.SG.Pf.WuBL.sideQ (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.WuBesselLoc, lines 201–261)
Lean: Formalism.L2.SG.Pf.WuBL.wuBesselOfLoc (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.WuBesselLoc, lines 265–340)

Applying this realization to W_v and to W_v^0 gives continuous Bessel intertwiners $\mathcal{B}, \mathcal{B}_0 : \sigma_v^\infty \to V_{u,v}$ with $\mathcal{B}_0 \neq 0$ since $\|W_v^0\|_{L^2(\nu_{u,v}^0)} = 1$. Lemma 33.23 applies to these two continuous Bessel maps. Its opposite-sign gap hypothesis follows from $16n^{19} \geq 120n^3$, the consecutive gaps of the $b_{\tau_v,j}$, and the fact that the first $k-1$ coordinates of λ_v lie above $-b_{\tau_v,0}$. It gives scalars a, b, not both zero, such that $a\mathcal{B} + b\mathcal{B}_0 = 0$ on σ_v^∞. Since $\mathcal{B}_0 \neq 0$, one has $a \neq 0$.
Lean: CompanionTheory.Lime.BM.UniqGap (CompanionTheory.AutomorphicTheory.LocalBesselUniqueness.BesselModels.AtFormAdmOneOfSteps, lines 15–26)
Lean: CompanionTheory.Bmu.besselMapUniqGap (CompanionTheory.AutomorphicTheory.BesselOrbitBounds.BesselMapUniqueness.BesselMapUniqGapAlt, lines 33–34)

Hence $\mathcal{B}$ is a scalar multiple of $\mathcal{B}_0$, and W_v is the same scalar multiple of W_v^0.
Lean: Formalism.MainNotions.BridgeCS3dBOW.besselOfW_holds (CompanionTheory.AutomorphicTheory.LocalBesselUniqueness.BesselModels.DoubleCosetBesselBoundOfGapConstantLe, lines 22–31)
Lean: Formalism.L2.SG.Pf.WuDec.BM.besselUniq_core (CompanionTheory.AutomorphicTheory.LocalBesselUniqueness.BesselModels.LocalUniquenessReduction, lines 18–66)

The complete nonvanishing and local proportionality proofs are contained in `ReciprocityTheory.ChartRestriction.DS.Lap.TopClosed.besselNVG` and `CompanionTheory.Lime.BM.wuDecayOfGap_holds'`. □

Proof of 33.25

Proof. At each infinite place v of F, let (p_v, q_v) be the signature of $G_v \simeq \mathrm{U}(p_v, q_v)$ ($p_v + q_v = n \geq 1801$), put $q = \min(p_v, q_v)$, and let $(\sigma_v, \mathcal{H}_v)$ be the local discrete series representation of formal degree $d_v > 0$ with maximal compact subgroup $K_v \leq G_v$ and unit lowest-K_v-type vector $w_v \in \mathcal{H}_v$. If $q = 0$, then G_v is compact and the continuous function $g \mapsto |\langle w_v, \sigma_v(g)w_v\rangle|\,\|g\|_{\mathrm{op}}^N$ is integrable on G_v. Suppose $q > 0$. Write $g = k_1 \exp(Y) k_2$ in the Cartan decomposition of G_v, with $k_1, k_2 \in K_v$, singular coordinates $a_1 \geq \cdots \geq a_q \geq 0$, and $d = \|Y\|_{\mathrm{HS}}$. Up to the fixed conjugation identifying G_v with $\mathrm{U}(p_v, q_v)$ in diagonal signature coordinates, k_1, k_2 are Euclidean unitary and $\|\exp(Y)\|_{\mathrm{op}} = e^{a_1} \leq e^d$, so $\|g\|_{\mathrm{op}}^N \leq C_{0,v} e^{Nd}$ for a constant $C_{0,v} > 0$ independent of g. By 37.16, for the spectral gap $h \geq 16n^{19}$ we have

$$|\langle w_v, \sigma_v(g)w_v\rangle| \leq \exp\bigl[-(h/8)\min(d^2, d/\sqrt{2q})\bigr].$$

In the Cartan integral formula the radial density is a constant multiple of

$$\prod_{1 \leq i < j \leq q} \sinh^2(a_i - a_j)\sinh^2(a_i + a_j) \prod_{i=1}^{q} \sinh^{2|p_v - q_v|}(a_i)\sinh(2a_i).$$

For $1 \leq i < j \leq q$ we have $a_i \geq a_j \geq 0$ and $2(a_i - a_j) + 2(a_i + a_j) = 4a_i$. Since $|\sinh t| \leq e^{|t|}$ and $|p_v - q_v| = n - 2q$, the i-th coordinate contributes at most $\exp\bigl((4(q-i) + 2(n-2q) + 2)a_i\bigr) = \exp\bigl((2n + 2 - 4i)a_i\bigr)$. Because $q \leq n/2$, each coefficient $2n + 2 - 4i \geq 2$ is positive, and $a_i \leq d$ gives $\sum_{i=1}^{q}(2n + 2 - 4i)a_i \leq 2q(n-q)d \leq 2nqd$. For $d > 1$, since $q \geq 1$ gives $\sqrt{2q} > 1$, we have $\min(d^2, d/\sqrt{2q}) = d/\sqrt{2q}$; moreover $1 \leq q \leq q_v \leq \max_{v'} q_{v'} \leq n$ and $n \geq 1801$ give

$$\frac{h}{8\sqrt{2q}} \geq \frac{16n^{19}}{8\sqrt{2n}} = \sqrt{2}\, n^{37/2} > 4\sqrt{2}\, n^{5/2} + 2n^2 \geq N + 2nq.$$

The region $d \le 1$ is compact, and on $d > 1$ the integral of $e^{-(h/(8\sqrt{2q})-N-2nq)d}$ over the q-dimensional Cartan subspace converges. Therefore

$$\int_{G_v} |\langle w_v, \sigma_v(g) w_v\rangle| \, \|g\|_{\mathrm{op}}^N \, dg_v < \infty. \tag{A.304}$$

For any $x_0 \in G_v$, substituting $g = x_0^{-1} y$ and using left Haar invariance with $\|y\|_{\mathrm{op}} \le \|x_0\|_{\mathrm{op}} \|g\|_{\mathrm{op}}$ gives $\int_{G_v} |\langle \sigma_v(x_0) w_v, \sigma_v(y) w_v\rangle| \, \|y\|_{\mathrm{op}}^N \, dg_v(y) < \infty$, and the same holds in Cayley coordinates G_v^0 for $\sigma_v^0(M) = \sigma_v(P_0 M P_0^{-1})$. Every $g \in G_v$ preserves a nondegenerate Hermitian form, so $1 = |\det g| \le \|g\|_{\mathrm{op}}^n$ and $\|g\|_{\mathrm{op}}^N \ge 1$. Thus $\phi_v(g) = d_v \overline{\langle w_v, \sigma_v(g) w_v\rangle}$ lies in $L^1(G_v)$ for each v, and Tonelli's theorem gives $\phi = \prod_v \phi_v \in L^1(G_\infty)$.
Lean: `Formalism.L2.SG.SplitPrimeNormLemmas.CoefWeightedLoc` (`Formalism.LocalToGlobal.SiegelLatticeTheory.HaarTransversals.UnitaryHaarProducts.SettingBlocks`, lines 323–333)
Lean: `Formalism.L2.SG.Pf.WuDec.phInt_of_gap` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.WuDecayAll`, lines 136–138)
Lean: `ReciprocityTheory.ChartRestriction.DS.coefWeightedN` (`ReciprocityTheory.ReductiveGroups.DiscreteSeries.FormalDegree.CoefWeightedG`, lines 14–62)

Fix a word u. If W_u vanishes identically on G_∞, then $\int \|W_u(x)\| L_u(x)^N \, d\nu_u(x) = 0 < \infty$, and $a_u = 0$ on the compact set $\{1\}$ satisfies all assertions. Assume that W_u does not vanish identically on G_∞. In the spectral realization X, we have $S_u = \prod_v S_{u,v}$ inside $G_\infty = \prod_v G_v$, and $W_u(x) = \bigotimes_v W_{u,v}(x_v)$ for continuous local functions $W_{u,v} : G_v \to V_{u,v}$ under a map $(y_v)_v \mapsto \bigotimes_v y_v$ whose inner products factor as $\langle \bigotimes_v y_v, \bigotimes_v y_v'\rangle = \prod_v \langle y_v, y_v'\rangle$ and whose image spans V_{π_u}. This inner-product identity makes $(y_v)_v \mapsto \bigotimes_v y_v$ multilinear and gives $\|W_u(x)\| = \prod_v \|W_{u,v}(x_v)\|$. Choose local Haar measures on the unimodular stabilizers $S_{u,v}$ and local quotient measures $\nu_{u,v}$ on $S_{u,v}\backslash G_v$ disintegrating dg_v. By uniqueness of Haar measure on S_u and Weil's quotient integral formula on $S_u\backslash G_\infty \simeq \prod_v (S_{u,v}\backslash G_v)$, we have $\nu_u = C_u \prod_v \nu_{u,v}$ for a finite constant $C_u \ge 0$.

Fix v and pass to $G_v^0 = \mathrm{U}(H_{k,r_S})$, $S_{u,v}^0 = U_{u,v}^0 \ltimes N_{u,v}^0$, and $\nu_{u,v}^0$ via the frame Cayley matrix $Q_{u,v} \in \mathrm{GL}_n(\mathbf{C})$ and the Euclidean-unitary Cayley matrix $P_0 \in \mathrm{GL}_n(\mathbf{C})$ of Proposition 33.24(3), under which the Bessel character has the common nonzero simple-root parameter $c_a = 2\pi/(v_E g_0 g_1)$ for $1 \le a \le r_S$. By Proposition 33.24(3), the transported local factor $W_v(m) = W_{u,v}(Q_{u,v} m P_0^{-1})$ satisfies $W_v = \kappa_v W_v^0$ for a scalar $\kappa_v \in \mathbf{C}$, where $W_v^0 = A_v / \|A_v\|_{L^2(\nu_{u,v}^0)}$ and $A_v(x) = B_v(x, x_0)\xi$ from (33.6) satisfies $\|A_v(x)\| \le \|\xi\| \, a_v(x; x_0)$. Put $L_{u,v}^0(x) = \inf_{s \in S_{u,v}^0} \|sx\|_{\mathrm{op}}$. Since $L_{u,v}^0(x) = L_{u,v}^0(anx) \le \|anx\|_{\mathrm{op}}$ for $(a,n) \in U_{u,v}^0 \times N_{u,v}^0$,

$$\|A_v(x)\| \, L_{u,v}^0(x)^N \le \|\xi\| \int_{U_{u,v}^0 \times N_{u,v}^0} |\langle \sigma_v^0(x_0) w_v, \sigma_v^0(anx) w_v\rangle| \, \|anx\|_{\mathrm{op}}^N \, da \, dn.$$

Integrating over $S_{u,v}^0\backslash G_v^0$ and unfolding to G_v^0 bounds $\int_{S_{u,v}^0\backslash G_v^0} \|A_v(x)\| \, L_{u,v}^0(x)^N \, d\nu_{u,v}^0(x)$ by a finite multiple of $\int_{G_v^0} |\langle \sigma_v^0(x_0) w_v, \sigma_v^0(y) w_v\rangle| \, \|y\|_{\mathrm{op}}^N \, dg_v^0(y)$, which is finite by (A.304).

Let $K_v^0 \le G_v^0$ be the maximal compact subgroup with probability Haar measure dk, and let d_τ be the dimension of the lowest K_v^0-type of σ_v^0. Because the lowest K_v^0-type has multiplicity one in σ_v^0, Schur orthogonality on K_v^0 gives $\int_{K_v^0} d_\tau \overline{\langle w_v, \sigma_v^0(k) w_v\rangle} \, \sigma_v^0(k) v' \, dk = \langle w_v, v'\rangle w_v$ for all $v' \in \mathcal{H}_v$. By strong continuity of σ_v^0, choose a compact neighborhood $U_1 \subset G_v^0$ of 1 on which $\|\sigma_v^0(y) w_v - w_v\| \le 1/2$. Writing $\mu_{G_v^0}$ for the Haar measure dg_v^0, we have $0 < \mu_{G_v^0}(U_1) < \infty$, and $v_1 = \mu_{G_v^0}(U_1)^{-1} \int_{U_1} \sigma_v^0(y) w_v \, dg_v^0(y)$ satisfies $|\langle w_v, v_1\rangle - 1| \le 1/2$, so $c_1 = \langle w_v, v_1\rangle \ne 0$. Set

$$\alpha_v^0(g) = c_1^{-1} \mu_{G_v^0}(U_1)^{-1} \int_{K_v^0} d_\tau \overline{\langle w_v, \sigma_v^0(k) w_v\rangle} \, \mathbf{1}_{U_1}(k^{-1} g) \, dk.$$

Then α_v^0 is measurable and bounded by $|c_1|^{-1} \mu_{G_v^0}(U_1)^{-1} d_\tau$, has support in the compact set $\Omega_v^0 = K_v^0 U_1$ containing 1, and satisfies $\int_{G_v^0} \alpha_v^0(z) \sigma_v^0(z) w_v \, dg_v^0(z) = w_v$ by Fubini's theorem. For every $x \in G_v^0$, Proposition 33.24(1) gives

$$\int_{G_v^0} |\alpha_v^0(z)| \, a_v(xz; x_0) \, dg_v^0(z) \le C_v(x_0) \|x\|_{\mathrm{op}}^{N_v} \int_{\Omega_v^0} |\alpha_v^0(z)| \, \|z\|_{\mathrm{op}}^{N_v} \, dg_v^0(z) < \infty.$$

By Tonelli's and Fubini's theorems applied to (33.6), $z \mapsto \alpha_v^0(z) A_v(xz)$ is Bochner integrable on G_v^0 with integral $A_v(x)$, and the same holds for $W_v = \kappa_v W_v^0$. Writing $L_{u,v}(y) = \inf_{s \in S_{u,v}} \|sy\|_{\mathrm{op}}$ for $y \in G_v$ and

setting $m = Q_{u,v}^{-1} y P_0 \in G_v^0$, every $s_0 \in S_{u,v}^0$ gives $s = Q_{u,v} s_0 Q_{u,v}^{-1} \in S_{u,v}$ with $sy = Q_{u,v}(s_0 m) P_0^{-1}$, so $L_{u,v}(y) \le \|Q_{u,v}\|_{\mathrm{op}} \|P_0^{-1}\|_{\mathrm{op}} L_{u,v}^0(m)$, while the pushforward of $\nu_{u,v}$ under $y \mapsto Q_{u,v}^{-1} y P_0$ is a positive multiple of $\nu_{u,v}^0$. Likewise, since $Q_{u,v}(mz)P_0^{-1} = y(P_0 z P_0^{-1})$ for $z \in G_v^0$, transporting α_v^0 to G_v by conjugation $z \mapsto P_0 z P_0^{-1}$ and scaling by the Haar-measure factor yields $\int_{S_{u,v}\backslash G_v} \|W_{u,v}(y)\|\, L_{u,v}(y)^N \, d\nu_{u,v}(y) < \infty$ together with a bounded measurable function α_v supported on a compact set $\Omega_v = P_0 \Omega_v^0 P_0^{-1} \subset G_v$ containing 1 such that $z_v \mapsto \alpha_v(z_v) W_{u,v}(y z_v)$ is Bochner integrable on G_v with integral $W_{u,v}(y)$ for every $y \in G_v$.

Since $L_u(x) = \inf_{s \in S_u} \prod_v \|s_v x_v\|_{\mathrm{op}} \le \prod_v L_{u,v}(x_v)$, Tonelli's theorem gives

$$\int_{S_u \backslash G_\infty} \|W_u(x)\|\, L_u(x)^N \, d\nu_u(x) \le C_u \prod_v \int_{S_{u,v}\backslash G_v} \|W_{u,v}(y)\|\, L_{u,v}(y)^N \, d\nu_{u,v}(y) < \infty.$$

Put $a_u(z) = \prod_v \alpha_v(z_v)$ for $z = (z_v)_v \in G_\infty$. Then a_u is bounded and measurable, with support in the compact set $\Omega = \prod_v \Omega_v$ containing 1. Because $a_u(z) W_u(xz) = \bigotimes_v \big(\alpha_v(z_v) W_{u,v}(x_v z_v)\big)$ and the image of $\bigotimes_v$ spans V_{π_u}, Fubini's theorem on $G_\infty = \prod_v G_v$ yields absolute Bochner integrability of $z \mapsto a_u(z) W_u(xz)$ and $\int_{G_\infty} a_u(z) W_u(xz)\, dz = \bigotimes_v W_{u,v}(x_v) = W_u(x)$ for every $x \in G_\infty$.

Lean: `Formalism.L2.SG.Pf.WuDec.QuotAll` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.WuDecayAll`, lines 431–431)
Lean: `Formalism.L2.SG.Pf.WuDec.wuWeighted_of_loc` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.WuDecayAll`, lines 467–489)
Lean: `Formalism.L2.SG.Pf.WuDec.wuReprod_of_loc` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.WuDecayAll`, lines 491–518)

□

Proof of 33.26

Proof. At the preliminary point on a host, 22.45 supplies two integer functions α, β in its boxes whose prime sum is at most x^{-9}. In the construction of 33.32 the same functions are prescribed at every translation of that host. By 33.27 they agree there with the member's top weights extended by zero, and the boxes and the prime sum agree with those at the preliminary point. In particular the member's own weights satisfy

$$\sum_{p \text{ candidate}} \frac{\log p}{p} \left(4\Sigma_p + 21_{\{\Sigma_p > x^{-\varepsilon}\}}\right) \le x^{-9},$$

where the candidate primes and shares are those of 22.45. This is $R_W(i)$ in (22.9). The final family cutoff in 33.32 already includes the preliminary choice cutoff, so the inequality holds at every indexed $x > x_1$. □

Proof of 33.27

Proof. The candidate tests of 14.8 agree by the common prime class, X', order of the Frobenius element h_p and band parameter ρ_B. At a candidate prime let $d_p = \mathrm{ord}(h_p)$. The frame identification preserves the least representative t_0 of each coset of $\langle h_p \rangle$, its marked positions $S_{t_0} = \{0 \le j < d_p : t_0 + j h_p \in TT\}$, and the arrays $(\alpha(t_0 + j h_p), \beta(t_0 + j h_p))_{0 \le j < d_p}$. These give the same window counts and good representatives. For $p < p_0 = k^2 x^{26}$ the share is $\min(1, K_{\kappa,II} |S_{t_0}| r_G^2 k x^{-26})$, with r_G the local rank parameter of 14.8. For $p \ge p_0$ the arrays identify the preceding even mark and its weights, so 14.10 gives equal D_I, c_6, t_I and equal constraints on the dominant base weights. The shares are the same infima because the nonnegative fractions passing the rank-one test at $(\lceil t_I \rceil_+, D_I, 8(D_I + 1))$ coincide for all the specified local rings. Summing over the good representatives and multiplying by m gives equal Σ_p. The common $s, n_H, k, \mathcal{L}$ give the same box endpoints on TT, and outside TT the box is $\{0\}$. The top-label identification transports the prescribed weights, hence also their extensions by zero. □

Proof of 33.28

Proof. Choose a compact open subgroup K' fixing e by left translation. Such a subgroup exists by local constancy and compactness of the support. Let $p = \phi \otimes \mathbf{1}_{K'} / \mu_f(K')$, where μ_f is finite-adelic Haar measure. The formal-degree orthogonality relation, applied at each place with the two middle vectors equal to w_v,

gives $\phi * \phi = \phi$. Hence $p * (\phi \otimes e) = \phi \otimes e$. If $h = R_e R_\phi h_0$ with $h_0 \in L^2(Y) \cap L^\infty(Y)$, convolution gives a bounded continuous representative of h: its change under right translation by u is at most $\|h_0\|_\infty$ times the L^1 change of the translated kernel. This representative is right K'-invariant and satisfies $h = R_p h = R_\phi h$.

We first prove the vanishing of the averages needed on the cusps. On a real component, let N be the unipotent radical of the stabilizer of a nonzero isotropic subspace. For a product of real groups take the product radical. We claim that $R_\phi F = 0$ for every bounded continuous left N-invariant function F on G_∞.

Choose a real factor with nonzero isotropic subspace W. In a Witt decomposition let $a(t)$ act by e^t on W, by e^{-t} on its dual, and by the identity on the middle space and the other factors. Choose $0 \neq v_W \in \bigwedge^{\dim W} W$. For a maximal-compact-invariant Euclidean norm put

$$H(x) = -\frac{\log(\|x^{-1} v_W\| / \|v_W\|)}{\dim W}.$$

Then $H(mx) = H(x)$ for $m \in N$, $H(a(t)x) = t + H(x)$, and $|H(xy) - H(x)| \leq c(y)$ for a continuous c, by the operator norms of y and y^{-1} on the exterior power. For every identity neighborhood U, all sufficiently small t have the following property, uniformly in x: some $m \in N$ satisfies $x^{-1} m a(t) x \in U$. Indeed write the chosen component of x as nlk in the parabolic decomposition, with l in the Levi subgroup and k in the maximal compact. Taking $m = n a(t) n^{-1} a(-t)$ reduces the conjugate to $k^{-1} a(t) k$.

On bounded continuous functions invariant on the left by both N and $a(\mathbf{R})$, define

$$Q_s F(x) = \int \phi(y) F(xy) e^{is(H(xy) - H(x))}\, dy.$$

The height identities preserve both invariances. The phases telescope in a double integral, so $Q_s^2 = Q_s$. The operators are norm continuous in s, since their differences are bounded by the integral of $|\phi(y)| \min(2, |s - s'| c(y))$. For $s > 0$ large take $t = \pi/s$ and $u = x^{-1} m a(t) x$ as above. Translation by u in the integration variable reverses the sign of the phase and preserves the value of F. Thus

$$2|Q_s F(x)| \leq \|\phi - \phi(u\,\cdot)\|_1 \|F\|_\infty.$$

Continuity of translations in L^1 gives $\|Q_s\| \to 0$. A nonzero idempotent has norm at least one, so the zero set of this continuous family of idempotents is open and closed in $\mathbf{R}$. Hence $Q_s = 0$ for all s.

For general bounded continuous left N-invariant F, the function $F_r(z) = F(a(r - H(z))z)$ has both invariances. Applying $Q_s = 0$ to F_r, and using Fourier inversion for $\rho \in C_c^\infty(\mathbf{R})$, gives

$$\int \phi(y) F_r(xy) \rho(r - H(xy))\, dy = 0.$$

Integrate in r and substitute $t = r - H(xy)$. Fubini applies with absolute integral at most $\|\phi\|_1 \|F\|_\infty \|\rho\|_1$. It follows that $\int \rho(t) (R_\phi F)(a(t)x)\, dt = 0$. The integrand without ρ is continuous, so its value at $t = 0$ vanishes. This proves the claim.

Lean: `CompanionTheory.D.gk.RF.selTw` (`CompanionTheory.HarishChandraModules.KFiniteAnalyticVectors.FundamentalDomainCusps.IntegrableMulBc`, lines 73–205)
Lean: `CompanionTheory.D.gk.RF.selP` (`CompanionTheory.HarishChandraModules.KFiniteAnalyticVectors.FundamentalDomainCusps.IntegrableMulBc`, lines 207–311)

Fix an identity neighborhood U. There are finitely many components at level K' by [97, Theorem 5.1], applied to $\mathrm{Res}_{F/\mathbf{Q}} G$; passing between commensurable compact open levels preserves finiteness. Each component is a quotient of G_∞ by an arithmetic subgroup Γ'. We describe the compact part and the cusp averages in one component. The group is reductive with anisotropic rational center, so the Siegel sets of [145, Theorem 13.1] have no unbounded split central coordinate. In left-quotient coordinates write a Siegel representative as $\omega a k$, up to one of finitely many rational translates, with ω in a fixed compact part of the minimal parabolic. Outside a compact set some simple root $\alpha(a)$ is arbitrarily large. For the corresponding maximal rational parabolic let N be its real unipotent radical. The intersection $\Gamma' \cap N$ is arithmetic in a unipotent group and has compact quotient by [145, Theorem 8.7]. Take bounded representatives for this quotient. Conjugation by ω^{-1} keeps these representatives in a bounded part of N. Each root in its Lie algebra contains α with positive coefficient, and all simple roots of a are bounded below in the Siegel set. Thus conjugation by a^{-1} contracts the representatives into any given neighborhood as $\alpha(a)$ increases.

Conjugation by k preserves this conclusion uniformly. The finitely many rational translates give the same assertion with the conjugated parabolics and arithmetic subgroups.

Consequently, outside a compact part, the probability measure on $(\Gamma' \cap N)\backslash N$ can be represented by a measure ν_x on elements m with $x^{-1}mx \in U$. For the bounded representative above, $F(y) = \int h(my)\, d\nu_x(m)$ is continuous and left N-invariant by right invariance of quotient Haar measure. Fubini and $h = R_\phi h$ give $F = R_\phi F$. The claim gives $F = 0$. On lifting to the adelic quotient, right K'-invariance gives

$$\int h(mxz)\, d\nu_x(m) = 0 \qquad (z \in G_\infty \times K').$$

This proves the zero average for the image of each bounded input.
Lean: `CompanionTheory.D.gk.RF.cuspidal_at_infinity_of` (`CompanionTheory.HarishChandraModules.KFiniteAnalyticVectors.FundamentalDomainCusps.CuspidalAtInfinityOf`, lines 14–67)
Lean: `CompanionTheory.D.gk.RF.adelic_cusp_of_infinite_cusp` (`CompanionTheory.HarishChandraModules.KFiniteAnalyticVectors.FundamentalDomainCusps.PBAdel`, lines 21–101)

Choose a continuous compactly supported kernel α in $G_\infty \times K'$ with $\|p - \alpha\|_1 \le 1/8$. For u in a sufficiently small identity neighborhood,

$$|\alpha(u^{-1}z) - \alpha(z)| \le \eta \mathbf{1}_B(z), \qquad \eta\mu(B) \le 1/8.$$

Here B is a fixed compact set containing the nearby translates of the support, and uniform continuity permits the indicated choice of η. On the cusp part just obtained for this neighborhood, the average of $R_\alpha h(mx)$ vanishes. Subtracting it from $R_\alpha h(x)$ bounds the latter in absolute value by $R_{\eta \mathbf{1}_B}|h|(x)$.

On the remaining compact part choose a compact set C of representatives. On C, the functions $R_\alpha h$ with $\|h\|_2 \le 1$ are uniformly equicontinuous. Indeed a difference at nearby points is bounded by the sup norm of the difference of the translated kernels times $\int_{CB} |h|$, after enlarging B if necessary. The latter integral is uniformly bounded by Cauchy–Schwarz because a compact set has bounded multiplicity over the quotient. Choose finitely many test points $y_1, \dots, y_d$ so that vanishing of all $R_\alpha h(y_j)$ bounds the integral of $|R_\alpha h|^2$ on the compact part by $\|h\|_2^2/8$.

Write $S = \|h\|_2^2$. Right translation is unitary on the quotient, hence $\|R_q h\|_2 \le \|q\|_1 \|h\|_2$ by the integral triangle inequality. The squared norms of the cusp bound and of $R_{p-\alpha}h$ are each at most $S/64$. If the tests vanish, $h = R_\alpha h + R_{p-\alpha}h$ therefore gives

$$S \le 2(S/8 + S/64 + S/64) = 5S/16.$$

We have proved

$$(R_\alpha h(y_j) = 0 \text{ for } 1 \le j \le d) \quad \Longrightarrow \quad h = 0 \text{ in } L^2(Y). \tag{A.305}$$

The image of the bounded inputs thus embeds linearly in $\mathbf{C}^d$ and is closed in $L^2(Y)$. Bounded inputs are dense, and $R_e R_\phi$ is continuous, so this image contains the image of every L^2 input.
Lean: `CompanionTheory.D.gk.RF.discrete_series_bound_of_inputs` (`CompanionTheory.HarishChandraModules.KFiniteAnalyticVectors.FundamentalDomainCusps.FiniteDimensionalFromLinearForms`, lines 227–304)
Lean: `CompanionTheory.D.gk.RF.rowsFinDim_of` (`CompanionTheory.HarishChandraModules.KFiniteAnalyticVectors.FundamentalDomainCusps.RowsFinDimOf`, lines 25–86)

□

Proof of 33.29

Proof. Apply 33.55 with the assumed rational tests and enlargement property to obtain the member and bundle supplies on the augmented extension. The nonendoscopic and geometric Galois hypotheses give the third-layer data. The series assertions for the member construction and for 36.107 are the conclusions of Theorem 36.106. Its remaining hypotheses are the norm input, depth-zero support, local finiteness, local Tate duality, the local Euler characteristic and the assumed linked bound at $2, 3$. Hence we have the linked implication

$$\mathcal{A} \longmapsto \mathrm{NL}(\mathcal{D}_3(\mathcal{A})). \tag{A.306}$$

The projection of the augmented extension preserves the member indices. Apply the transfer assertion of 33.74 to (A.306) and to its open rows.

In particular we now have the label realization on this extension. With the convergent Siegel–Weil formula of 34.17, it gives the counting row by 37.13. The finite dimension used there is 33.28. Apply 33.34 with the transferred open rows, (A.306), the assumed functional and covolume rows, the reference index input, and the level-one estimate of Lemma 33.75. For this last application the primitive and divisor comparisons have been transferred at the same indices, and the estimate holds for every C beyond a common point cutoff. This supplies all the member rows on the extension on which the bundles were chosen.

Apply 33.30. Its separate finite-dimension hypothesis follows from 33.28 for every kernel specified in 33.6. The analytic input is 37.1, the reproduction and integrability estimates are 33.25, and the minor-label and signed bounds are 33.18 and 33.99. The adjacent-rank mass formula is 37.5. The nonsingular Gram formula of 34.17 gives the volume comparison of 37.6, and 37.7 gives the genus mass. With the member and bundle supplies above, 33.30 gives one family class and selection satisfying 33.6. □

Proof of 33.30

Proof. The joint counting row of 33.34 first supplies an enlargement and then c_{30}, c_{Ic} with their sign and size conditions. Fix them before choosing a family or selection. Conjoin this row with the other 25 member rows and the T_u-row of 37.10, which uses item 3 with (33.7) and point condition $\top$. Intersection as in (A.308) gives a single enlargement for the conjunction H. Apply 33.32 to obtain

$$\mathcal{A}, \qquad \sigma, \qquad H(\mathcal{A}, \sigma). \tag{A.307}$$

Projection gives all these rows for the same σ. Choose a join at every selected point by the join condition of 33.74. The extra selection inequality follows from $x > x_1(F_m)$. The definitions of W_1, W_3 identify the working classes, so the closed implications retain their thresholds. The design supply and construction duty of 33.32 give the design supply, the seven family conditions and stability of the coefficient lattices, with the fifth family condition as in 33.26. Insert the hypotheses of items 2 and 3 in their fields of 33.6.
Lean: `GlobalTheory.Fc.AllRowsAlt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.AllRows`, lines 12–39)
Lean: `GlobalTheory.Fc.R1.CountsIrreducibilityConstants'` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CountsIrreducibilityConstants`, lines 113–119)

□

Proof of 33.31

Proof. The rank bounds (8.1) give $m \to \infty$, $m \le x$, and $k \ge x^{9/2+10\varepsilon}$ for large x. In particular $m^4/k \to 0$. Put $v_E = \mathrm{rd}(E)/2$. By 8.3, E contains $\mathbf{Q}(\sqrt{-d})$ and lies in $\mathbf{Q}(\zeta_N)$ with $N = Q_S q d$. The discriminant tower and the cyclotomic discriminant formula give $\sqrt{d} \le 2v_E \le N$. Thus $v_E \ge \pi$ and $\log(2v_E) \le L$ eventually, by the host modulus bound. The inertia groups in 5.3 omit complex conjugation: at d inertia is generated by the d-sign, and away from d inertia has trivial d-sign. Hence E/E^+ is unramified at finite places.

Let $F_j = \prod_{a=0}^{j-1} a!$. Apply 37.5 at $j = k+1, \dots, r$. In its Euler product each $\log L_\eta(E, j)$ is at least $-f \sum_{n \ge 2} n^{-2} \ge -f$, since there are at most f primes above a rational prime and $\log(1+y) \le y$. The factors $(v_E/\pi)^{\bar{j}}$ are at least one. Multiplication therefore gives

$$\log(M_r/M_k) \ge f \log(F_r/F_k) - f(m-1)\big(1 + \tfrac{1}{2}\log(2v_E)\big).$$

The number of free entries in the column group is $k(k-1)/2$, and in the row group it is $r(r-1)/2 - k$. Dividing $|\mathrm{GL}_j(\mathbf{F}_g)| = g^{j^2} \prod_{i=1}^{j}(1 - g^{-i})$ by the two subgroup orders gives

$$\frac{I_r(g)}{I_c(g)} = g^{km + m(m-1)/2} \prod_{i=k+1}^{r} (1 - g^{-i}) \;\ge\; \tfrac{1}{2} g^{km}.$$

Here $g \ge 3$ and the product is at least $1 - \sum_{i \ge 1} g^{-i} \ge 1/2$.
Lean: `GlobalTheory.EX.WinLow.idx_ratio` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.DimVRowPos`, lines 124–168)

It remains to bound the two Vandermondes before division by F_k, F_r. Put $a = xe^L$, $\delta = \lceil x^{c_D} \rceil$ and $S_c = 4k\delta$. The preliminary coordinates are formed by the formulas of 15.9: $v_i = 4\delta(k-2-i)$ and $U_{i+1} =$

$v_i - \frac{1}{2} - \xi_i\delta$, with $\xi_i = 0$ or 1. These assignments use the box weights and bits. At a mark the endpoints of U are $a_s + (k-1)/2$ and $-b_s - (k-1)/2$; elsewhere they are $v_0 + S_c + 1/2$ and $-1/2$. The box bounds give $a_s \le 2a$, $b_s \le 2a/\exp(x^{\varepsilon/3})$ eventually. Both k and S_c are polynomially bounded. Thus every gap meeting an endpoint is at most $4a$, uniformly. For $0 \le i < j \le k-3$ the interior gap $U_{i+1} - U_{j+1}$ is at most $v_i - v_{j+1}$. These pairs $(i, j+1)$ are distinct. All unused positive-row gaps are at least one. The negative row entries have successive gaps $2\lceil x^{c_h}\rceil$, and $b_{0,\rho} \ge 1$ in each of the three base cases in 15.9. Thus all cross gaps are at least one as well. Writing Δ for the numerator of W, we obtain

$$\frac{\Delta(\lambda_\rho)}{\Delta(U_{v,\rho})} \ge (4a)^{-(2k-3)}.$$

This uses no final member or word prime.
Lean: `GlobalTheory.EX.WinLow.places_ok` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.WindowLower`, lines 69–142)

The factorials in the mass bound cancel the Weyl denominators. Since $g_i \ge (ke^L)^8$ by 8.12 and $\log(4a) \le 2L$ eventually, the preceding inequalities give

$$\frac{1}{fk}\log(D^\star/d_v^\star) \ge 16mL - 4L - \frac{m-1}{k}(1 + L/2) - \frac{2\log 2}{k} \ge 15mL$$

eventually. Finally $2\log\Gamma/(fk) = (2C_\Gamma m^4/k)mL \le mL$ beyond a cutoff depending on C_Γ. All cutoffs preceded the choice of box weights and words.
Lean: `GlobalTheory.EX.WinLow.rank_consts` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.WindowLower`, lines 144–175)

□

Proof of 33.32

Proof. Conjoin point and bundle conditions by intersection. Take the minimum of the positive ε-cutoffs and the maximum of the lower thresholds in dependency order c_D, c_h, x. Take the unions of the required algebraic elements and of the corresponding lists of required or avoided primes. Intersect the two strengthenings R from bundle supply and (33.8). With $Q^* = Q' \wedge R$ this gives

$$\mathsf{B}(Q^*, L_k), \qquad \mathrm{Row}(Q^*, L_k, H). \tag{A.308}$$

In a parameter record $\kappa = (\varepsilon, C, c_D, c_h, C_5, c_6, C_b)$, where $C = (P_0, \mathbf{o}, \mathbf{u}, V)$ is the four-part constant record of Definition 36.10, call κ *linked* to a family F_m of parameter ε' if $\varepsilon = \varepsilon'$, $C_5, c_6, C_b > 0$, C is standard, and the preliminary, middle, last, and (after resetting entry 13 to $C_{Tu}(C)$) outer constant lists of F_m coincide with those of C. For the bundle condition E of the row, let $\mathcal{Q}_E(\kappa, D, O)$ be

$$(C_5, c_6, C_b > 0) \implies \forall c \in \mathbf{R}\ \exists \beta \in \mathrm{Bd}(D, O, c) \quad \begin{bmatrix} \beta \text{ comes from a linked } Q^*\text{-point with } \kappa \\ \wedge\ E(\kappa, D, O, c, \beta) \end{bmatrix}. \tag{A.309}$$

Beyond the larger cutoff in (A.308), bundle supply gives β and the row gives E on that bundle. Take $\mathcal{A}$ to consist of families in $\mathcal{C}(\mathcal{Q}_E(\kappa))$ for some linked κ. The linkage relation supplies $C_5, c_6, C_b > 0$ and identifies all point-construction constant lists up to entry 13 (C_{Tu}). Choose σ at $c = C_{\mathrm{sz}}(F_m)$ by (A.309). Since Q^* implies Q', this gives item 3, and the row gives item 4.

The construction class supplies the fixed parameters, windows, auxiliary-prime conditions, tame labels and pointwise weight test. Projection gives $q_0 < X'$ in clause 3 of 33.3. The two tame lists have the same entries $1, \chi_j, \chi_j^{-1}$. The square condition therefore transfers at the same cutoff as ordinary and prime-to-ℓ injectivity. For clause 5 use $x_1 + 1$, so that the pointwise test applies at every $x \ge x_1 + 1$ and gives the small-share inequality of 33.26. The defining point condition gives tuple use and the twist witnesses for every $x > x_1$. Use x_1 as the cutoff in clauses 6 and 7, which retain the strict inequality. The same point condition gives coefficient-lattice stability with cutoff $x_1 + 1$.
Lean: `GlobalTheory.EX.DQ.familyClass_sevenDuties_and_stableLattices` (`Development.AbelianVarietyTheory.Definitions.GlobalTwisting.FamilyClassSevenDutiesAndStableLattices`, lines 55–59)
Lean: `GlobalTheory.EX.DQ.duty_seven_library` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.DutySevenLibrary`, lines 15–23)

Lean: `GlobalTheory.Fc.copies_frozen` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.WorkingFamClassCopiesEqFrozen`, lines 9–16)

For item 1 assume Minkowski division. Fix $\lambda > 1$ and a requirement G as in item 4 of Definition 33.6 whose hosts exist on windows of exponent λ in the sense following Definition 8.19. For the target character ψ, this supply is Theorem 8.21 at $\lambda = 3/2$ under the cohomological window hypothesis of Definition 26.3. In Theorem 8.21 (via Theorems 8.20, 8.6, 8.3, 8.18, and 8.17), the good pairs (q, d) on $[Y, x]$ are supplied unconditionally by Theorem 7.25, the top set TT is chosen by Theorem 8.15, and the character bounds at $C_5 = 4$ are supplied as follows:

- the pointwise square-root bounds $|g_\chi(x)| \le x^{1/2+\varepsilon}$ and $|(L'/L)(1,\chi)| \le 4\log x$ for every primitive representative χ at level $8qd$ of $\chi_q = (\cdot/q)$, $\chi_d = (\cdot/d)$, and $\chi_q\chi_d$ are supplied in Theorem 8.17 by Lemma 8.16 from (35) of Definition 7.24 (with $\xi = 1$ modulo M);
- the bound $|\operatorname{Re}(L'/L)(1,\chi_d)| \le x^{\varepsilon}$ follows in Theorem 8.17 from the logarithmic-derivative bound of Lemma 8.16 for χ_d;
- the reality, conductor, and zero-free box conditions in $\mathcal{R}(8X_1(Y)X_2(Y)^j)$ for the six quadratic twists of $\chi_q, \chi_d, \chi_q\chi_d$ are supplied in Theorem 8.17 by Lemma 8.16 from (36) of Definition 7.24, while the real-character zero-free boxes at height $H_{Q_{\mathrm{odd}}}(j)$ and the thin-position bound (8.14) are supplied by Theorem 8.13 from (36);
- the bounds $|g_\chi(x)| \le x^{1/2+\varepsilon}$ and $|(L'/L)(1,\chi)| \le 4\log x$ for every $\theta \in \mathcal{B}_H(\mathcal{X})$ in the cyclic host requirement G of Definition 8.19 are supplied in Theorem 8.20 by Theorem 8.4 from (35) of Definition 7.24.

Take $\varepsilon > 0$ at most the minimum of $1/53$, the positive cutoff for point construction from item 4 of 33.30 and (A.308), and the cutoff of 22.45. Fix a requested threshold record $\mathcal{K}$. Host supply gives $C_{T_2}, C_8^0, C_5, c_6, C_b, x_H$ and $S = (J, \mathcal{W})$, with $C_8^0 \ge 1$, $C_5, c_6, C_b > 0$, and $J \ge 1$. The set $\mathcal{W}$ contains the intervals of (8.20). At every $x \ge x_H$ in $\mathcal{W}$ there are ranks and a completed host of Theorem 8.17 satisfying G at $s = (x, \varepsilon, J)$.

On such a host H, write $N = Q_S qd$, and denote its distinguished primes $p_{\mathrm{de}}, p'_{\mathrm{de}}$ by p, p'. The host conductor ideal $\mathfrak{c}_K \subseteq \mathcal{O}_K$ equals $\mathcal{O}_K$. Let r_0 be the least prime not dividing $N(p-1)(p'-1)$, and take $q_0(H)$ to be the least prime 1 modulo $r_0 N$. A prime factor of $N(p-1)(p'-1)+1$ bounds r_0 by $2N(p-1)(p'-1)$. Since $N \ge 2$ and $p, p' \le N^{C_8^0}$ by the completed-host bounds of 8.12 and 8.17, $r_0 N \le N^{3+2C_8^0}$. With the exponent $A_0 \ge 1$ of 8.7, we obtain

$$q_0(H) \le N^{C_8}, \qquad C_8 = (3 + 2C_8^0)A_0 \ge C_8^0. \tag{A.310}$$

The choice of r_0 excludes p, p', and the congruence modulo N gives splitting in the host field. Replace the level set by $\{g_0, g_1, p, p', q_0(H)\}$. This leaves q, d, Q_S, the frame, the arithmetic datum, TT, and the auxiliary rank unchanged, so the completed-host character bounds (including the bounds for $\chi_q, \chi_d, \chi_q\chi_d$ and the six quadratic twists from Lemma 8.16 and the real-character boxes and thin-position bound from Theorem 8.13) and the requirement G, which reads only the scale, the frame, the raw host, TT and the auxiliary rank, transfer verbatim. Thus this choice gives completed hosts with the specified $q_0(H)$ on the same S, with $J \le C_8$.

Lean: `UpperTheory.Top.hostQ` (`Development.AutomorphicArithmetic.CharacterBoxes.SmallestPrimePolynomialBound`, lines 64–102)

At the fixed $\varepsilon, C_8, C_5, c_6, C_b$, let $\mathcal{K}_O$ be the maximum of the thresholds for constructing preliminary points and for constructing points with the required member and bundle conditions, in both cases with prescribed box weights. Let $\mathcal{K}_W = \mathcal{K}_W(\varepsilon)$ be the numerical threshold of 22.38, used in 22.45. These records are chosen before either family. Put $t_* = c_T(\varepsilon, 9)$. Form $\mathcal{K}'$ by evaluating the middle and final functions of $\mathcal{K}$ with the C_{Tu}-entry of the outer list replaced by t_*. Take the maximum $\mathcal{K}_* = \mathcal{K}' \vee \mathcal{K}_O \vee \mathcal{K}_W$ as in 36.10. Write $\|$ for list concatenation. Use the construction constants C given by

$$\begin{aligned} P_0 &= \max(1000, \mathcal{K}_{*,\mathrm{pre}}),\\ \mathbf{o} &= (401, 2, C_{T_2}, 11\cdot 405, 405, 9/10, 9, C_8, 1, P_0, 1, 12, 377, 25, 6, C_8),\\ U &= \max(1, \mathcal{K}_{*,\mathrm{mid}}(\mathbf{o})), \qquad \mathbf{u} = (U, \dots, U) \in \mathbf{R}^{10},\\ V &= \max(1, \mathcal{K}_{*,\mathrm{last}}(\mathbf{o} \,\|\, \mathbf{u})). \end{aligned}$$

The preliminary list is (P_0), $\mathbf{o}$ is the 16-entry outer list in the canonical coordinate order of Definition 36.10 (so $\mathbf{o}[6] = C_{YL} = 9$ and $\mathbf{o}[13] = C_{Tu} = 25$ in zero-based indexing), the 10-entry middle list is $\mathbf{u}$ at

$CH = C_F = U$ (with $\mathbf{u}[1] = C_\Gamma = U$), and every final entry is V. With K the orbit parameter record of 14.8, $C_\Gamma = C_{\mathrm{sz}} = K_\Lambda = U$. The three inequalities of 36.10 follow from the maxima. For the point construction and its numerical size data, include 0 among the lower bounds for c_D and take their maximum. At that c_D, include 402 among the lower bounds for c_h and take their maximum. Fix these exponents and write

$$\kappa = (\varepsilon, C, c_D, c_h, C_5, c_6, C_b).$$

The two scale cutoffs x_O, x_N for the point construction and the numerical data now use the same κ.
Lean: `GlobalTheory.EX.stdConst` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsStdLarge`, lines 10–27)
Lean: `GlobalTheory.EX.exists_std_large` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsStdLarge`, lines 39–67)

At a completed host above x_O, prescribe any pair of integer weight functions in the intervals (22.27), extended by zero off the top set. Choose an injection of $M = \lceil 3\Gamma^2 x^{C_F} \rceil$ indices into binary words of length $k-2$ and a translation $0 \le u < n$, where n is the frame order. Here $L = x^\varepsilon$ and $b = e^L$. Put $\Gamma = \exp(C_\Gamma m^5 f L)$. For a coded word v let $d_v^\star$ be the mass of the standard column lattice times its pre-level indices and archimedean weight dimensions. Let $D^\star$ be the row mass times the two finite indices and the product of the row Weyl factors in Lemma 33.31. The completed host, the prescribed box weights and the coded word are the preliminary data of that lemma. Enlarge x_O by its uniform cutoff. It gives

$$S_v = \left(\frac{D^\star}{d_v^\star \Gamma^2} \right)^{1/(fk)}, \qquad \log S_v \ge mL.$$

At $x^{1/J} \le Y \le x$, the pair conditions of Lemma 8.16 and Theorem 8.17 give $d \equiv 7 \bmod 16$ and $\log N \le 2x^{\varepsilon/5}(\log x)^2$. They also give $d \ge 100$ eventually. By (8.1), eventually $m \ge x^{1/3}$ and $1 \le fk \le x^{12}$. The host bound $g_i \le 2(kb)^8$ of 8.12 gives $g_i \le b^9$ for large x since k is polynomial in x. Thus 37.11 with $c_s = 1$, $C_Y = 9$, $V = fk$ gives a norm prime in $(S_v, (1 + 1/(fk))S_v]$. It is at least $X' = \exp(x^{\varepsilon/3})$ for large x and satisfies the required norm congruence modulo $N(g_0 g_1)^2$.
Lean: `GlobalTheory.EX.WinLow.window_lower` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.WindowLower`, lines 177–295)

Fix these levels. Item 4 of 33.30 gives an untwisted Q-member on a KP-base containing the prescribed elements. Export its data and use them in the second-member clause, which preserves the frame and prime set. The resulting member satisfies Q and Q' on a KP-base. Since $Q \Rightarrow_{KP} R$, it satisfies $Q^* = Q' \wedge R$. Use the volume ratio on its tied datum and (A.309) on its linked member. The layer construction carries tuple use, compatible twists and coefficient-lattice stability to the point. It has scale s, the prescribed weights at the corresponding top labels, the prescribed code bits and translation u. Its orbit data and size parameters are those in 33.27. The cutoff x_O precedes the choices of host, weights, code and translation. The same supply gives preliminary points by retaining the scale, weight, code and translation data.
Lean: `GlobalTheory.Fc.memberTop_final_contextParameters` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.MemberTopFinalContextParameters`, lines 14–23)
Lean: `GlobalTheory.Fc.objCP_and_of_recExP` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsObjectsOfAdmissibleTwistDatum`, lines 70–119)
Lean: `UpperTheory.Top.objC_true` (`Development.AutomorphicArithmetic.CharacterBoxes.ObjCTrue`, lines 16–28)

Above x_N the numerical witnesses use the host arithmetic of 8.17 and the prescribed scale, sizes and ranks. Both lists of local parameters have the entries fixed by C. Put $\delta_1 = x^{-\varepsilon/8}$, $a = xb$, and $b' = a/X'$ as in 14.8. The second box ends at $b' + 2n\delta_1 x^\varepsilon \mathcal{L} b'/k$, where $\mathcal{L}$ is the orbit factor of 14.8. In (14.7), $u_1\delta_1 \le 2$ and $M_0 = O_\varepsilon(\log x)$, so $N_{\mathrm{win}} = O_\varepsilon(\log x)$. Hence $\mathcal{L} = O_{\varepsilon,C}(x^\varepsilon (\log x)^2)$. In the two terms of ρ_B in 14.8, substitute $n \le 2x^{2+8\varepsilon}$ and $k \ge x^{9/2+10\varepsilon}$ from (8.1). This gives

$$\rho_B = O_{\varepsilon,C}\big(x^{-5/2}(\log x)^2 + x^{-9/2-9\varepsilon}(\log x)^3\big) \le x^{-2} \tag{A.311}$$

after enlarging x_N.
Lean: `GlobalTheory.EX.szRow_numAt` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.SzRowNumAt`, lines 14–47)

Use $\lceil a \rceil, \lceil b' \rceil$ as preliminary weights on TT, extended by zero. Both box widths of 22.45 eventually exceed one since $k \le x^5$ by (8.1), $n, \delta_1 x^\varepsilon, \mathcal{L} \ge 1$ for large x, and b, b' exceed every fixed power of x. Let x_w be a common cutoff. By (8.1) and $f = 2n$,

$$m^5 f L \le 128 x^{9/2+9\varepsilon}, \qquad k \ge x^{9/2+10\varepsilon}.$$

Consequently $\log(3\Gamma^2 x^{C_F}) = o(k)$ uniformly in the ranks, and beyond a cutoff x_c we have $M \le 2^{k-3}$, as required by the point-size condition. Use the first M binary strings of length $k-2$ in lexicographic order. Put $\xi = \max(2, x_H, x_O, x_N, x_w, x_c)$.

For every $z \in \mathcal{W}$ with $z \ge \xi$, fix one host H_z and construct preliminary points $P_{z,u}$ at all $0 \le u < n_z$, where n_z is its frame order. Use the same preliminary weights and code for every translation. The common labels and weight arrays identify the remaining fibre data by 33.27. The code identifies the words by their indices. The offsets in 14.9 depend on the scale, layer and membership in TT, so they agree at matching labels.

Lean: `GlobalTheory.EX.exists_weights_in_windows` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsWeightsInWindows`, lines 18–78)
Lean: `GlobalTheory.EX.exists_injective_words` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.MPowFiveMulFMulLLe`, lines 40–94)
Lean: `GlobalTheory.EX.TieSame.same_of_from` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.SamePointOfSameScale`, lines 87–131)
Lean: `GlobalTheory.EX.DC.kit_of_points` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.KitOfPoints`, lines 16–28)

Choose $y \ge \xi$ in $\mathcal{W}$ by (8.20) and define

$$z(x) = \begin{cases} x, & x > \xi,\ x \in \mathcal{W}, \\ y, & \text{otherwise}, \end{cases} \qquad I_x = \{u \in \mathbf{Z} : 0 \le u < n_{z(x)},\ x \le \xi \text{ or } x \in \mathcal{W}\}. \tag{A.312}$$

Set $x_1(F^0) = \xi,\ J(F^0) = J,\ \mathrm{Win}(F^0) = \mathcal{W}$. At $u \in I_x$ use $P_{z(x),u}$, with the common top labels and box endpoint read at index 0 and the constants read from C. The index 0 exists at $x = 0$ and at each window point. At indexed $x > \xi$ the point scale is (x, ε, J). Numerical witnesses are available at every $z(x) \ge \xi$. The threshold lists agree with those of C since the window-dependent entry is $\max(C_8, J) = C_8$. Thus F^0 meets the thresholds $\mathcal{K}_W$ at C_F in the sense of Definition 36.10.

Lean: `GlobalTheory.EX.cutScale` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsFamPOfKits`, lines 15–16)
Lean: `GlobalTheory.EX.famOfKits` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsFamPOfKits`, lines 56–157)
Lean: `GlobalTheory.EX.exists_famP_of_kits` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsFamPOfKits`, lines 159–172)

The preliminary weights were chosen by rounding the lower box endpoints. For the resulting family F^0, the pre-list, outer list, middle list at C_F, and last list are the lists on which the construction constants passed $\mathcal{K}_W$. Thus 22.45 applies with $CH = C_F$ and gives a single point cutoff $x_W = x_W(F^0, C_F)$ for all its indices. The pair obtained at each subsequent scale is a trial pair on the preliminary point. It will be prescribed as the actual weights in the final construction. At each $x \in \mathcal{W}$ with $x \ge X_H = \max(\xi + 1, x_W)$, take the trial pair (α_x, β_x) furnished at $P_{x,0}$. By 33.27 it belongs to the host boxes, vanishes off TT, and satisfies

$$R_{H_x}(\alpha_x, \beta_x) = R_{P_{x,0}}(\alpha_x, \beta_x) \le x^{-9}. \tag{A.313}$$

Lean: `UpperTheory.Top.hostWW` (`Development.AutomorphicArithmetic.CharacterBoxes.HostWW`, lines 27–87)

For the design family take $\xi' \ge \max(2, X_H, x_O, x_N, x_c)$ large enough that $q_0(H_x) < X'$ on the window set. By (A.310) and the conductor bound of 8.17,

$$\log q_0(H_x) \le C_8 \log N \le 2C_8 x^{\varepsilon/5} (\log x)^2 < x^{\varepsilon/3}$$

for large x. At each such x, construct $P'_{x,u}$ for every $0 \le u < n_x$ with the single host pair (α_x, β_x) and one injective code at that scale. By 33.27, the pair gives the extended weights of each point, and (A.313) gives the small-share bound at each translation. Thus $t \mapsto \alpha_x(t)$ is fixed throughout the translation average in 6.12.

Assemble these fibres by (A.312) with cutoff ξ' and a reference $y' \in \mathcal{W}$ strictly above it. The numerical witnesses again give the constant lists. At every indexed $x > \xi'$ the points retain G and (A.309). Each constructed point $P'_{x,u}$ has the same primes q, d and conductor Q_S as H_x, so the completed-host pointwise bounds for χ_q, χ_d, and $\chi_q\chi_d$ with constant C_5 (supplied in Theorem 8.17 by Lemma 8.16 from (35) of Definition 7.24) transfer verbatim to item 4 of Definition 33.6 at every $x \ge \xi' + 1$ and every $i \in I_x$. The six quadratic-twist nonvanishing boxes and the real-character zero-free boxes (supplied in Theorem 8.17 by Lemma 8.16 and Theorem 8.13 from (36) of Definition 7.24) transfer verbatim to the family box and local

QD conditions. Likewise, the host thickness, bad-orbit, and G conditions (including the square-root bounds for every character in $\mathcal{B}_H(\mathcal{X})$ from Theorem 8.4 when G is the cyclic host requirement of Definition 8.19) transfer through the equal sets, prime data, and Frobenius labels. The prime q_0 retains its level membership and congruence. Since $X' \le b$, its logarithm is at most L, and the bounds $g_i \ge (kb)^8$ and $T_u \ge X'$ give the remaining exclusions. The tame-generator limit is

$$l_J = \frac{mbL}{k(m+t)\log(3(b+\mathrm{Sc}+\eta+t+1))},$$

where t is the auxiliary rank and Sc, η are the point sizes of 33.27. At fixed construction parameters $k, m, t, \mathrm{Sc}, \eta$ grow at most polynomially by (8.1) and 33.27. Thus $\log l_J = L - O_{\varepsilon,C,c_D,c_h}(\log x)$ and $X' \le l_J$ after enlarging ξ'. The word bijection gives M labels at the actual scale, with $M \le 3\Gamma^2 x^{C_F} + 1$. The full member construction gives the twisting and stability conditions. Minkowski division gives the word-lattice denominator divisibility by $M(2fN(m))$ of the working class in 33.6.

The family has $C_{YL} = 9$ and $CH = C_F \ge 1$. Modify only its C_{Tu} parameter to t_*, which replaces entry 13 of the zero-indexed outer list while leaving every point datum $(D_{x,i}, O_{x,i})$, the size constant C_{sz}, the cutoff x_1, the index sets I_x, membership in $\mathcal{C}(\mathcal{Q}_E(\kappa))$, and the bundle predicate $E(\kappa, D, O, c, \beta)$ unchanged. The middle and final tests of 36.10 for $\mathcal{K}$ now evaluate the functions used for $\mathcal{K}'$, and the preliminary test is unchanged. Resetting entry 13 back to 25 recovers $\mathbf{o}$ identically, so the modified family satisfies the linkage relation to κ. The family is therefore in $\mathcal{A}$, is large for $(\mathcal{K}, C_F)$, realizes G on $\mathcal{W}$, and has $C_{Tu} = c_T(\varepsilon, C_{YL})$. These choices give the design supply of Definition 33.6.

Lean: `UpperTheory.Top.kitS` (`Development.AutomorphicArithmetic.CharacterBoxes.KitS`, lines 14–62)
Lean: `GlobalTheory.EX.DC.has_withDuty_wall_and_choices` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.HasWithDutyWallAndChoices`, lines 20–60)
Lean: `UpperTheory.Top.designS` (`Development.AutomorphicArithmetic.CharacterBoxes.DesignS`, lines 21–98)
Lean: `UpperTheory.Top.recS` (`Development.AutomorphicArithmetic.CharacterBoxes.ObjCTrue`, lines 30–36)

□

Proof of 33.33

Proof. Fix μ. In the rational row coordinates of the construction each designated generator has a coordinate preimage. Choose one for each generator. The coordinate comparison realizes each such preimage by a rational row and preserves its ambient value. Conversely, an integral combination of the generators is realized by the same combination of their coordinate preimages, since the coordinate embedding is semilinear for ι_D. The coordinate lattice condition puts this vector in the integral lattice. Uniqueness of the rational form with its given complex realization identifies it with the original row. In the other direction the coordinate lattice condition sends every integral row into this span.

Now fix u. Take the specified reference identification when $\mu = \mu_0$ and the specified twisted identification otherwise. The row comparison says that pulling the ambient realization back by this identification gives the same form on every input as $r_{u,\mu}$. Applying the identification again gives the ambient value just computed. The generator preimages and the lattice equivalence therefore hold for this word. □

Proof of 33.34

Proof. The 26 member conditions of Definition 33.6 consist of the 25 individual rows covering items 6–11, 13, 17, 23–24, 26–28, 30–38, 42–43, and 46 together with the joint counting row unpacking into items 18–19, 25, and 39–41 (the remaining items 1–5, 12, 14–16, 20–22, 29, and 44–45 are supplied separately by the base record hypotheses and the family construction of Theorem 33.32). Use 33.77, 33.78, 37.21 and 37.6 for the arithmetic, spectral, split-prime and volume rows. For item 26, the standard-form comparison in the proof of Theorem 37.6 gives every admissible space the volume of the standard space of its signature. The rational forms are isometric, the finite lattice stabilizers are locally conjugate, and the real measures have the same normalization. Apply Theorem 37.6 with both unit signs to the standard signatures $(2,1)$ and $(1,2)$. Its rank-three ratio identifies the volumes of the three rank-two signatures. The ratio depends only on rank, so induction gives equality for all admissible constant signatures at each higher rank. In rank one negating the form preserves the normalized volume, and rank zero has one signature. In the definite case Borel finiteness gives finitely many classes, compactness at infinity gives finite stabilizers, and their reciprocal orders sum to

the genus mass of Proposition 37.7 in positive rank. The rank-zero mass is one. For the type-prime period bounds, take $R = \top$ in (33.9). From the good member take its built witness b and the genuine column component furnished by that witness. The row component of p is R_{row}, so the comparison supplied by b applies to these arithmetic row coordinates. By Lemma 33.33, the reference and twisted identifications carry the rational row forms to their prescribed period coordinates in every column. The designated generators have rational preimages, and the integral row lattice is the inverse image of their integral span. *Claim 1.* For each used μ, the same built witness gives the CM realization required in 17.22, simultaneously for all words.

Take its host and coefficient fields, the chosen split primes and their generators, the trivial auxiliary ideal, the discriminant twelfth root, the normalized theta family, the theta character and the differential generators. These are fixed before the word is chosen. Take the character chain, local constants, index sets and selected primes from the genuine column component. Its host and character identities identify these data with the member's fields, infinity weights and level primes. Changing the divided lattice preserves this component and its kernels.

Fix a word u, a type prime $p = g_i \neq T_u$, and a coefficient place $w \mid p$. Let S be the summation prime set for this word and tuple. For each adjacent pair of class lattices N, N', let $H = \mathrm{Hom}_S(N, N')$, c_A be the support denominator and c_B the period modulus. Use the normalized jet $J_{c,x,\alpha}$ of this theta family for each positive $c \mid c_A c_B$ and $x \in c_A^{-1} H$. The built witness's jet-field condition applies to every such c, x in the required multidegree; hence these are the rational jets of the same complex kernel. This uses the jet-field condition in addition to the row-coordinate comparison above.

At a prime above $\bar{\pi}_i$, form the ordinary finite test by multiplying the level-marking factor and the local unfolded entry factors in the chosen frames. At a type prime of residue cardinality q, use the selected enlarged reference lattices. In their scaled bases a matrix entry in row r, column s is multiplied by $q^{e_s - e'_r}$, where the exponents are zero on the original coordinates and $j + 1$ on the extra coordinate j. For every free entry $e_s - e'_r \geq 1$. A change of a free variable by an integral element therefore changes the completed matrix by q times an integral homomorphism of the enlarged lattices. The selected residue vector is invariant under these changes, also after transport by the two lattice stabilizers. The additive phase is unchanged, since its argument changes by an integral element and, on the support, has denominator at most q. The selected vector is supported on integral homomorphisms in the enlarged frames. Its original-to-original block is unscaled, so nonvanishing forces x to map the original local lattices into one another. Write P for this place, A for the denominator exponent of the free variables and B for the refinement depth. Each class in $q^{-A}\mathcal{O}_P/\mathcal{O}_P$ has q^B lifts at depth B in each free coordinate. Grouping by these classes cancels q^{-B} in each coordinate. The remaining summands are products of roots of unity or zero, and the inverse-volume factor is an integer stabilizer index. The same matrix comparison for a change of the original block by q times an integral homomorphism proves the required translation invariance. At the other row primes the denominators have residue characteristic different from g_i, so they are units at w. Multiplying the factors gives, with S_i the primes over $\bar{\pi}_i$,

$$|\Xi(x)|_w \leq 1, \qquad \Xi(x) \neq 0 \implies x \in \mathrm{Hom}_{S_i}(N, N'), \qquad \Xi(x + z) = \Xi(x)$$

for $z \in p\,\mathrm{Hom}_{S_i}(N, N') \cap c_B\,\mathrm{Hom}_{S \setminus S_i}(N, N')$. In the first coupled test, insertion of the admissible coset column into the selected residue vector gives a product of residue-field character values and an additive root of unity on its nonzero support. In the terminal test, the top-block indicator and the integral-row condition make the row pairing well defined modulo the conductor, and its phase is a root of unity. At the type primes neither coupled factor has an averaging denominator. At the other row primes the local denominators are units at w. The coset and row congruences preserve the same support and translation conditions.

Choose the penultimate index from the period realization of this word. At each intermediate representative, sum over its finite set of admissible cosets $Y = y + L_+$, using the same Y in both final kernels in each summand. If an intermediate representative is changed by g, change its coset to gY in both factors. The equivariance identities then preserve their product and the class sum. Thus the last two kernels have the common decoration required in 17.22.

Finally use the rational ramified-column isomorphism supplied by the same genuine component and compose its row-form map with the isomorphism in Lemma 33.33. The inverse-image description of the integral row lattice identifies its integral tests, and the genuine period identity identifies the resulting pairing with the member's period. This constructs the asserted common realization. Bundle linkage transports these identities together. Take $0 < \varepsilon \leq 1/100$ and middle size threshold 2, so the fixed-prime error coefficient

satisfies $c_g \geq 2$. Beyond the uniform degree cutoff, Proposition 17.22 gives the tuple bound with $C_1 = 1$. With the base terms of 15.10 and error data of 15.12, its budget bound holds for the same constant because $2C_1 \leq c_g$. These are the two eventual estimates required in 33.6.

Lean: `GlobalTheory.Fc.row_aAtG_at_built_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RowAAtGAtBuilt`, lines 10–18)

For the complementary bounds, use $\varepsilon_1 = 1/100$ and the zero threshold record in 36.10. Fix an eligible family in W_3 and $0 < \varepsilon \leq \varepsilon_1 < 1/52$. In the notation of 21.40, $B_0 = fmbL$ and $D_{\Gamma,u}$ is the stabilizer denominator. The level and stabilizer size bounds in the definition of W_3 give cutoffs x_T, x_Γ and a family constant $C_2 \geq 0$ such that at every indexed point above the respective cutoff,

$$\log T_u \leq C_{Tu} mL, \qquad \log D_{\Gamma,u} \leq C_2 f N_m \log(f N_m) \tag{A.314}$$

for all words. The point data exclude $T_u = g_i$. For the norm identity in 21.40, sum the valuation of each $\tau(\bar{\pi}_i)$ over the complete fibre $w \mid g_i$, where $\pi_i \bar{\pi}_i = g_i$. In the normalization of 15.12 and 20.7, the weighted sum for each τ is $\log g_i$. Multiplication by $A_{u,j,\tau}$ and summation in τ give $A_{uj} \log g_i$.

Lean: `GlobalTheory.TypeValNN.typePrime_valuationSum_eq_logGap_all` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.TypePrimeValuationSumEqLogGapFamily`, lines 13–24)

For $x \geq 2$ and $x > x_1(F_m)$ (where the point size inequalities of Definition 14.11 hold), put $W_j = \sum_\tau w_{j\tau}$ and let W_j^{off} be its sum over embeddings outside the top positions. At each embedding τ, split $A_{u,j,\tau}$ using 15.8–15.9 into the two endpoint rows $i = 0, k-1$ (whose difference from the Fock endpoints matches $S_{a,\tau}$ up to b'/x on top positions and $w_{j,\tau}$ plus $O(kS_c)$ off the top), the $k-2$ interior head rows (each within $S_c + N_m$ of $w_{j,\tau}$), and the $j(t_{\mathrm{ch}}+1)$ tail rows (each at most 2η in magnitude). Letting d_j^{Fock} be the sum over all embeddings of the absolute values of the entries in both Fock weight lists at layer j, summing the per-embedding discrepancies over all f embeddings yields

$$|A_{uj} - d_j^{\mathrm{Fock}} - S_a - W_j^{\mathrm{off}}| \leq 20\left[f(kS_c + N_m^2 + mt_{\mathrm{ch}}\eta) + |TT|b'/x\right]. \tag{A.315}$$

The size parameters give $N_m^2 \leq \eta$ and $mt_{\mathrm{ch}} \geq 1$. Since $0 \leq W_j^{\mathrm{off}} \leq W_j$, adding back W_j^{off} in (A.315) gives R_j with $C_{\deg} = 40$.

Lean: `GlobalTheory.FockDegreeFine.degree_fine` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.NuAbsPlaceFine`, lines 135–158)

Lean: `GlobalTheory.AdmissibleDegree.admissibleDegreeCount_along_family` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.AdmissibleDegreeCountAlongFamily`, lines 63–69)

Choose the type-prime constant after $C_{\deg}$. The point size bounds give $W_j \leq 3fb$ and $|TT|b'/x \leq fb$. They also give $kS_c, mt_{\mathrm{ch}}\eta, kN_m \leq b$. Therefore $R_j \leq (6 + 4C_{\deg})fb$. They also give $f\eta \leq b$, and hence $|H_g| \leq |c_g'|(1+m^2)b$. Since $m \leq f$ and $\log g_i \leq (C_{YL}+1)L$, these terms contribute at most $[(6+4C_{\deg}) + 4|c_g'|](C_{YL}+1)B_0$ to the type-prime bound. The denominator term satisfies

$$v_{g_i}(D_{\Gamma,u}) \log g_i \leq \log D_{\Gamma,u} \leq C_2 f N_m \log(f N_m).$$

For this term the size bound $kN_m \leq b$ gives $fN_m \leq b$ since $f \leq k$, so $fN_m \log(fN_m) \leq B_0$. Thus one may take

$$C_g = \max\{0, (6 + 4C_{\deg} + 4|c_g'|)(C_{YL}+1) + 2C_2\}. \tag{A.316}$$

For the stabilizer clause the family numerical sizes, using the level size bounds, also give $N_m L \leq b$ beyond a family cutoff. With $f \leq N_m \leq b$ this gives $fN_m \log(fN_m) \leq 2B_0$. The second inequality of (A.314) then gives $2 \log D_{\Gamma,u} \leq 4C_2 B_0$.

Lean: `GlobalTheory.TypeSize.typePrime_sizes_le_budget_single` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.TypePrimeSizesLeBudgetFamily`, lines 80–134)

Lean: `Formalism.MainNotions.J.gammaDiscriminant_budget_of_sizes` (`Formalism.SharpRestriction.AdmissibleParameterJoins.DIVfromGLbud`, lines 63–123)

On the complete level fibre over T_u, the nonnegative normalized losses of 15.12 sum to $E_u^\pi \log T_u$ because the local valuations of the level generator sum to $\log T_u$ with the weights of 20.7. This also bounds the sum on each finite subset of the fibre. With $W_{u,\tau}$ the archimedean level weight at τ, the full allowance of 21.40 is

$$E_u^\pi = 2\sum_\tau W_{u,\tau} + fN_m + \frac{|TT|(t_{\mathrm{ch}}+1)(m+1)}{4}.$$

For $x \geq 2$ and $x > x_1(F_m)$, each $W_{u,\tau} \leq 5b$, $fN_m \leq fb$, and $|TT|(t_{\mathrm{ch}}+1)(m+1) \leq fb$. Hence

$$E_u^{\pi} \leq 2(5fb) + fb + fb = 12fb. \tag{A.317}$$

By (A.314) and (A.317), take $C_T = \max(12C_{Tu}, 0)$. For the discriminant loss D_j^* of 15.12, at each top position the ramified base term is at most b, while at every embedding 15.9 gives $w_j(0,t) \leq \frac{9}{4}b'$ and the column correction across the k rows adds at most kb', yielding $|D_j^*| \leq |TT|b + fkb'$. Since $|TT| \leq f$, $kb' \leq b$ and $\log d \leq L$ in the point sizes for $x \geq 2$ and $x > x_1(F_m)$, we obtain $2\log d\,|D_j^*| \leq 4B_0$. Take $C_d = 4$.
Lean: `Formalism.MainNotions.product_first_zeroth_parts.ETpi` (`Formalism.SharpRestriction.GlobalSchurBounds.ProductFirstZerothParts`, lines 307–309)
Lean: `GlobalTheory.SeenHalf.seen_half` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.VAddNonnegOfIsIntegral`, lines 112–117)
Lean: `GlobalTheory.LevelBudget.admissible_off_type_level_budget` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.AdmissibleOffTypeFamilyBound`, lines 38–55)
Lean: `GlobalTheory.RamifiedOffD.offDiagonalRamified_bound_along_family` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.OffDiagonalRamifiedBoundAlongFamily`, lines 50–53)

For the last clause of 21.40, partition the finite set of eligible primes P into P_C, those in residue class zero with $h_p = 0$, and its complement P_I. On P_C the loss is $C_{j,p}^*$ of Definition 15.12. In 15.12 its digit contribution after multiplication by $\log p$ is $3fN_m^2 \log a$, and its other contribution is $W_j \log p/(p^2-1)$, with $a = xb$. The size condition $xN_m^2 \leq w''$ gives $X''N_m^2 \leq b$, where $X'' = a/w''$. With $W_j \leq 3fb$, $\log p \leq \log a$ and $\sum_{n\geq 2}(n^2-1)^{-1} \leq 1$, the sum is at most $6fb\log a$. Use $\log a \leq (1+1/\varepsilon)L$ and $m \geq 1$ to take $18(1+1/\varepsilon)B_0$ as an upper bound after the factor 2.

On P_I the loss is $I_{j,p}^*$ of Definition 15.12. Using the structural family normalizations $c_{\mathrm{dig}} = 3$, $c_6 = 12$, $K_{\delta,1} = 5$, $K_{\delta,2} = 4$ recorded in Section 5 of Chapter 16, together with the orbit-average bound of 15.1, the extra odd-order inert-class term, the digit and size bounds of 15.12, and the point size inequalities of Definition 14.11 ($N_m^2 \leq \eta$, $W_j \leq 3fb$, $m \geq 2$), we obtain

$$\sum_{p\in P_I} |I_{j,p}^*| \log p \leq 126fmb \sum_{p\in P_I} \frac{\log p}{p} + 9fbk \sum_{p\in P_I\cap \mathcal{C}_s} \frac{\log p}{p^3} + 70f\eta L\,|P_I|, \tag{A.318}$$

where $\mathcal{C}_s$ consists of the residue-class-zero primes for which h_p has odd order. The odd-order condition gives $p \equiv \pm 1 \pmod{2^{e+2}}$ on this set, where 2^e is the frame's two-power parameter. Put $A = 2^e$. Each such prime is in one of the two progressions $4Aj \pm 1$ with $j \geq 1$. For $p = 4Aj \pm 1$, we have $p \geq 3Aj$ and $\log p \leq 2(p/A)\log A$, by $\log(p/A) \leq p/A - 1$ and $\log A \geq 1/2$. Thus

$$\frac{\log p}{p^3} \leq \frac{\log A}{A^3}\frac{2A^2}{p^2} \leq \frac{\log A}{4A^3j^2}.$$

Summing the two progressions with $\sum_{j\geq 1} j^{-2} \leq 2$ gives $\log A/A^3$. The frame bound $n \leq oA$, the size lower bound $n \geq x^{2+\varepsilon/100}$, and $o \leq x^{\varepsilon/100}$ give $x^2 \leq A$. Hence $k \leq x^5 \leq A^{5/2}$, and $\log A \leq 2\sqrt{A}$ gives $k\sum_{p\in P_I\cap\mathcal{C}_s} \log p/p^3 \leq 2$.
Lean: `UpperTheory.sum_log_div_cube_le_of_two_pow_dvd` (`Development.AutomorphicArithmetic.OrbitLabellings.SumLogDivCubeLeOfTwoPowDvd`, lines 48–123)

Since every prime $p \in P_I$ satisfies $p \leq X''$ in 21.40, we have $|P_I| \leq X''$, and the point size inequalities $X''\eta \leq b$ and $X'' \leq b$ give $\eta|P_I| \leq b$. By Rosser and Schoenfeld [34, (3.24)], $\sum_{p\leq b} \log p/p < \log b = L \leq 3L$ for $L \geq 1$. With $m \geq 2$ and $L \geq 1$, the right side of (A.318) is at most $(378 + 9 + 35)B_0 \leq 424B_0$. The two prime sets contribute disjoint sums, so take $C_I = 18(1+1/\varepsilon) + 848$, with the larger of their cutoffs.
Lean: `GlobalTheory.ConjLossPrimes.loss_sum_primes` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LossSumPrimes`, lines 103–111)
Lean: `GlobalTheory.OtherLoss.sum_cube_le` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.OtherLossSum`, lines 438–445)
Lean: `GlobalTheory.OtherLoss.sum_eatI` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.OtherLossSum`, lines 447–497)
Lean: `GlobalTheory.InertLossAdm.inertPrimes_loss_log_sum_admissible` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.InertPrimesLossLogSumAdmissible`, lines 12–47)

Take the maximum of the six cutoffs. The constants depend on the fixed family and work at every point above that cutoff with $x > x_1(F_m)$. The bounds in 21.40 hold for every word and layer and for every finite set specified there. Thus the complementary bounds satisfy (33.9) with $R = \top$.

Lean: GlobalTheory.familySecondComplementInputs_of_tuSize_of_dGamSize (Development.AbelianVarietyTheory.Proofs.DivisorProofs.AdmissibleComplementTwoAlongFamily, lines 18–40)
Lean: GlobalTheory.admissibleConstants_closed_under_sizeBounds (Development.AbelianVarietyTheory.Proofs.DivisorProofs.AdmissibleConstantsClosedUnderSizeBounds, lines 10–18)

Use the pointwise rows of 33.74. Primitivity (item 8) gives a K_D-linear right inverse to the level projection at the base tuple and each lift layer (their composition is the identity), proving surjectivity of the reference embedding and of each layer projection (item 42). The comparison of the two realizations identifies the selected twists at the places in the reference twisting set. Use 37.24 for the extremal vectors and one-map identities, and 37.26 with reciprocity and linkage for the local lifts and the kernel conditions. For the counting row use the label realization among the assumed open conditions of 33.74. On its original enlargement, 37.13 applies to that realization by the assumed convergent formula of 34.17. It supplies the two counting constants before the member rows are intersected. The assumed norm condition gives $NN(W_3(\mathcal{A}))$. For the simultaneous row (33.3), impose the primitivity, divisor, finite-type goodness, geometry, multiplicity, κ and definite-side conditions of 33.74 together with the covolume and functional bounds. Apply 36.7 on their common strengthening with the compatible summands. The paired-index and functional bounds use the family division estimate (36.18). The lower weight bound uses the prescribed leaf map at every layer, as in 36.7. □

Proof of 33.35

Proof. Write $K \subset \mathbf{C}$ for the embedded coefficient field and $K_0 \subset K$ for its specified fixed subfield. First choose the places on which to retain the row data. At the reference tuple retain v if, at v or its conjugate, some layer has nonzero Iwasawa index, nonzero lattice index in the row pairing, or nonzero prescribed content. The prescribed content at layer j depends on v through its residue characteristic.

In the untwisted case the choice is made uniformly as follows. For the fixed preliminary datum and frame, group the initial choices by the generators at the two distinguished primes and at the word primes. Their defining norm conditions give a finite set of generator tuples. For each tuple admitting adelic data choose one such realization. Include generators of its coefficient field among the finite requirements on K_0, and include the residue characteristics of its nonzero reference indices among the required primes. The index supports are finite because they come from the elementary divisors of the finitely many reference lattices. Take the unions over the generator tuples and include the primes supporting the prescribed content. These choices precede the choice of w.
Lean: GlobalTheory.Mem.R3.Rw.Resize.finite_gens (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.FiniteNonzeroIndexPlacesReferenceTuple, lines 114–119)
Lean: GlobalTheory.Mem.R3.Rw.Resize.seedW_bound (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.FiniteNonzeroIndexPlacesReferenceTuple, lines 166–174)
Lean: GlobalTheory.Mem.R3.Rw.Resize.seeds_uniform_exists (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ExistsUniformNormalisedAdelicSeeds, lines 12–41)

Extend the chosen reference realization to the coefficient data of w, with its frame and generator tuple identified. Under this extension the support of the first two indices pulls back from the reference field. Its image in $\mathbf{C}$ lies in K_0. Thus two places of K with the same restriction to K_0 lie over the same reference place. They also have the same residue characteristic, so they agree on membership in the content support. This proves the fibre assertion. All retained places have residue characteristic in the required prime set. In the twisted case the eligibility of w supplies adelic data with this support condition and trivial twist off the selected set. Restriction to the selected places preserves the indicated indices. The multiplicity identity and the rational tests of Theorem 34.41 give the row lifting data there. Denote the resulting set of places by W.
Lean: GlobalTheory.Mem.R3.Dat.mem_auxiliaryPlaces_iff_three_parts (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.AuxiliaryPlacesEqUnionThreeParts, lines 43–45)
Lean: GlobalTheory.Mem.R3.Rw.UntwSat.dat_saturated_of_untw (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.UntwistedAdelicDataSaturatedOverFixedField, lines 47–97)
Lean: GlobalTheory.Mem.R3.Rw.exists_pre2_of_fine_of_resChar_mem (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ConstructionStartingChoicesOfFineAdelicData, lines 195–240)

We now specify the integral state over these rows. In the row construction the parameters $\eta_{j,v}$ range over nonnegative rationals. Take the function $(j, v) \mapsto 0$. Let $b_1, \dots, b_a$ be the chosen complex row basis, let $\iota : K \hookrightarrow \mathbf{C}$ be the coefficient embedding, and put $e(x) = \sum_i \iota(x_i) b_i$ on K^a. Write $R_{\mu(v)}$ for the prescribed row set at the tuple selected by v. In the common row coordinates take the module

$$\left\{ x \in K^a : \ \forall v \ \exists d_v \in \mathcal{O}_K \setminus \mathfrak{p}_v, \ d_v x \in \operatorname{span}_{\mathcal{O}_K} \{ y \in K^a : e(y) \in R_{\mu(v)} \} \right\}.$$

It is closed under addition since two admissible denominators at v can be multiplied. For each word take the inverse image of its level-zero lattice under the inclusion of its ramified subspace. Give this subspace the restricted norm. On the row space choose a frame for the given norm at the distinguished embedding and keep the prescribed frames at the other infinite places. Take all word heights equal to zero and the separate real error parameter equal to $6x^{-C_F}$. For a row pairing whose values on these coordinates lie in $\iota(K)$, pull the values back by ι^{-1}. Injectivity of ι gives bilinearity and the required pairing identity. These are the division data for the chosen zero function.
Lean: `GlobalTheory.Mem.R3.Rw.prescribedRowLat` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ExistsDivStageOfPk`, lines 54–71)
Lean: `GlobalTheory.Mem.R3.Rw.exists_divStage_of_pk` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ExistsDivStageOfPk`, lines 76–130)

For the Galois data retain the given length data and take the set indexing local contributions to be $\varnothing$. At g_0, g_1 use the actual places above the split primes, with the tame generators and series of the frame. Identify the resulting finite group with the original one by its place coordinates. Pull the two actions of the centre of the group algebra back along this identification, on the ambient rows and on their multiplicity spaces. The distinguished idempotents have equal coefficients under the identification, so their operators agree in both actions. This checks the conditions that still involve the group when the local index set is empty. The equation at every (j, v) now reads

$$\eta_{j,v} = \sum_{\alpha \in \varnothing} 0 = 0.$$

The conditions on an individual local index have empty domain. In particular this construction keeps the tame group data at the two distinguished primes.
Lean: `GlobalTheory.MemN.emptyLayer` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ExistsUnramifiedDataCharacter`, lines 17–104)
Lean: `GlobalTheory.Mem.Top.trueLayer` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.TypeIdempotentCoeffAcrossLayers`, lines 196–209)
Lean: `GlobalTheory.Mem.R3.Rw.Bare.exists_nStage_act` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.LocalizationFreeStageTrueGroupColumnChoiceStep`, lines 72–143)
Lean: `GlobalTheory.Mem.R3.Rw.Bare.exists_bare` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.LocalizationFreeStageTrueGroupColumnChoiceStep`, lines 167–193)

Use the spectral stage for the same rows and the original Haar datum. Its hypotheses are positivity of the starting columns and the inequality between the row dimension and the ambient dimension, supplied by the column and row frames. Keep the original geometric stage. The division coordinates just constructed, the group identification and the two central actions now fit these stages simultaneously. Transport the row lifts along these identifications and read their maps in the transported bases. This adds the row realization to the later state with the chosen division and Galois components. This gives the initial state over w with empty local index set and zero length coordinates. The localization at the last finite level uses the twisted row spaces and their integral generators as in Lemma 33.54.
Lean: `GlobalTheory.Mem.SProv.exists_sStage_true` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ExistsSStageTrue`, lines 17–25)
Lean: `GlobalTheory.Mem.R3.Rw.st_of_fits` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.FittingLaterStagesRowIsConstructionState`, lines 22–27)

□

Proof of 33.36

Proof. We first choose weights $w_j = \ell^{e_j}$ for the substitution $Z_j = (1+T)^{w_j} - 1$. For each s and each monomial Z^α, record the order of its coefficient in $F_{M,s}$, capped at $\mu_s + 1$, assigning this last value to a zero coefficient. There are countably many entries, each with finitely many possible values. Successive extraction and a diagonal subsequence give an unbounded sequence of M's on which each entry is eventually constant. Among the finitely many monomials of degree at most D, at least one has limiting order at most μ_s. Let h_s be the least limiting order among all monomials. Among those with order h_s, choose α_s lexicographically least, comparing the last coordinate first.

Choose successive powers w_j so that

$$w_j > \sum_{i<j} (\alpha_s)_i w_i \quad \text{for every } s.$$

Additional lower bounds on any w_j are permitted. Then α_s uniquely minimizes the weight $\sum_i \alpha_i w_i$ among the monomials of limiting order h_s. Indeed, at the last differing coordinate any other such monomial has a larger exponent, and its gain there exceeds the entire earlier weight of α_s.

Choose the first R weights by this rule. Put $m_* = b\sum_{i=1}^{R} w_i$, $V = \max(\{0\}\cup\{v_y\})$, and $L = (V+1)(m_*+1)$. Choose the last R weights sufficiently large that every coefficient of degrees $1,\dots,L$ in $(1+T)^{w_j}-1$ belongs to the $(V+1)$-st power of the maximal ideal of O. This is possible since

$$\binom{\ell^n}{d} = \frac{\ell^n}{d}\binom{\ell^n-1}{d-1} \qquad (1 \le d \le \ell^n).$$

For sufficiently large M the same divisibility holds for $(1+T)^{w_i\ell^M}-1$ at every first-half index i.

Write $P^*(T) = P(((1+T)^{w_j}-1)_j)$. Set $W_s = \sum_j (\alpha_s)_j w_j$. Terms of total degree greater than W_s cannot contribute to $[T^{W_s}]F^*_{M,s}$. For large M on our subsequence, all coefficients through total degree W_s have their limiting capped orders. Divide this truncation by a uniformizer of O_0 to the power h_s and reduce. In residue characteristic ℓ, the substitution becomes $Z_j = T^{w_j}$. By uniqueness of the least weight, the coefficient of T^{W_s} in the reduction is the nonzero coefficient contributed by α_s. Thus $F^*_{M,s}$ has a coefficient of order h_s at degree W_s.

For the second family, take M large enough that $\ell^M > m_*$. The first nonzero residue coefficient of A^* is at degree $\sum_i n_i w_i \le m_*$. For a nonzero polynomial $H = \sum a_j T^j$ over O put

$$\omega(H) = \min_j\bigl(\operatorname{ord}_O(a_j) + j/(m_*+1)\bigr).$$

This quantity adds under multiplication. In each factor choose the least degree realizing the minimum. At the sum of these degrees the product of the two chosen coefficients is the unique contribution of minimum weight. Hence

$$\omega(A^*g^*) \le m_*/(m_*+1) + v_y < v_y + 1.$$

It follows that A^*g^* has a coefficient of order at most v_y at a degree at most L. Through degree L, the coefficients of $f^* - A^*g^*$ are divisible by the $(V+1)$-st power of the maximal ideal, by the choice of the last weights and of M. The same coefficient bound therefore holds for f^*.

It remains to choose the order of the root of unity. Let W bound L and all W_s. Consider any one of the resulting univariate polynomials $H = \sum a_j T^j$ over its coefficient ring A. It has a coefficient at some degree $d \le W$ of order at most its prescribed bound v. For $t > 0$, consider the minimum of the lines $\operatorname{ord}_A(a_j) + jt$ with $a_j \ne 0$. As t decreases, the degrees of the successive minimizing lines increase. At most $W+1$ of those degrees are at most W. A minimizing degree greater than W must have intercept strictly less than $\operatorname{ord}_A(a_d)$, by comparison with the line of degree d. The successive intercepts strictly decrease. Thus at most $W+v+1$ positive values of t can give a tie. This bound is independent of M.

Choose $k_0 > \max(\{0\}\cup\{e_j\})$ so large that $W/\varphi(\ell^{k_0}) \le \delta$ and $w_j/\varphi(\ell^{k_0}) \le \delta$ for all j. Take M on the subsequence large enough that the interval $[k_0, M]$ contains more integers than the sum of the tie bounds for all the polynomials. For a primitive ℓ^k-th root, $v_\ell(\zeta-1) = 1/\varphi(\ell^k)$. Indeed, all primitive powers give differences from one of the same valuation, since their ratios to $\zeta-1$ are units, and their product up to sign is $\Phi_{\ell^k}(1) = \ell$. The numbers $d_A/\varphi(\ell^k)$ are distinct as k runs through the interval. Choose k avoiding the ties in every polynomial at its corresponding value of t. Each $H(\zeta-1)$ now has a unique term of least valuation. It is nonzero and its valuation is at most

$$v/d_A + d/\varphi(\ell^k) \le v/d_A + \delta.$$

The substitution gives $(1+(\zeta-1))^{w_j} - 1 = z_j$, proving both assertions for every primitive root of this order. □

Proof of 33.37

Proof. Apply Lemma 33.36 to all the indicated polynomials with this δ. In the coefficient ring at level k, take its designated primitive ℓ^k-th root ζ and set

$$\xi(g) = \prod_{j=1}^{2R} \bigl(\zeta^{\ell^{e_j}}\bigr)^{d_j(g)}.$$

Each exponent is well defined modulo ℓ^M since $k \le M$. The character is continuous, anticyclotomic and trivial above ℓ by the corresponding properties of the d_j. At a place of the pair of P_j, choose an inertia element on which $d_j = 1$. Every other coordinate is zero there. Thus the inertia image has order exactly $\ell^{k-e_j} > 1$, and all inertia images outside Q are trivial. The character factors through the finite abelian Artin quotient unramified outside Q with the indicated anticyclotomic and above-ℓ conditions. At a place in the pair of P_j, the chosen inertia element bounds the tame valuation by

$$v_\ell(\zeta^{\ell^{e_j}} - 1) = \ell^{e_j}/\varphi(\ell^k) \le \delta.$$

The same computation applies after raising to h. Since $|Q| \le 4R$, the tame contribution is at most $4Rt\delta \le e$.

Fix s, h and use $u_j = (\zeta^{h\sigma_s})^{\ell^{e_j}}$ in the presentation. The resulting determinant is $F_{M,s}((u_j - 1)_j)$. The root $\zeta^{h\sigma_s}$ is primitive of order ℓ^k, so Lemma 33.36 bounds the valuation of this nonzero determinant by $\nu_a + \delta$. To compute the length, diagonalize the specialized matrix over A by elementary divisors. A diagonal entry of uniformizer order n has kernel $\mathfrak{m}_A^{-n}/A$ on $\mathrm{Frac}(A)/A$, of length n. Adding the lengths and dividing by d_A gives the normalized determinant valuation. Thus every indicated Selmer group has finite length, and $|\mathcal{S}|\delta \le e$. Taking ζ^h or ζ^{-h} in the second conclusion of Lemma 33.36 proves the assertion for the theta polynomials on the same Q. □

Proof of 33.39

Proof. At a split unramified prime of norm q outside the exceptional set, write the unshifted parameters as α_j. At the conjugate prime they are α_j^{-1}. Thus the eigenvalues of T at the first prime are $q^{(1-t)/2}\alpha_j$, and those of its Tate dual at the conjugate prime are

$$q\big(q^{(1-t)/2}\alpha_j^{-1}\big)^{-1} = q^t q^{(1-t)/2}\alpha_j.$$

These are the eigenvalues of $T(t)$. The traces agree on G_E. Indeed, modulo each power of the maximal ideal the two actions factor through a finite quotient. Apply Lemma 36.24 to that quotient, excluding the exceptional primes and those over rational primes ramified in the host. A degree-one prime so obtained splits over the maximal real subfield, since its local degree over $\mathbf{Q}$ is one and it is unramified. Equality modulo every power gives equality in O_0.

For the integral conclusion, let R_1, R_2 denote the two representations being compared. The matrices in each image span $M_r(O_0)$. For $T(t)$ this follows by reduction and Nakayama, since twisting multiplies each matrix by a unit. Inversion in the group, transposition and scalar multiplication give the same span for the conjugate Tate dual. Extend both representations linearly to $O_0[G_E]$. If $R_1(x) = 0$, then $\mathrm{tr}(R_2(x)R_2(y)) = \mathrm{tr}(R_1(xy)) = 0$ for every y. Testing against the matrix units gives $R_2(x) = 0$. Interchanging the two actions gives equality of their kernels. The resulting automorphism of $M_r(O_0)$ is inner. To see this, the images of the diagonal matrix units decompose O_0^r into r isomorphic summands. Each is free of rank one over the local ring. Transport a generator of the first summand by the images of the units E_{i1}. In the resulting basis the images of all the matrix units act as the original units. This gives the stated integral isomorphism for T. Since $\eta^c = \eta^{-1}$, tensoring it with η^{-1} gives the isomorphism for V.

An isomorphism $V \simeq V(t)$ would give $\varepsilon_\ell^{tr} = 1$ by determinants. Here $tr > 0$ by the rank inequalities. The cyclotomic image of a number field is infinite, since its intersections with the fields of ℓ-power roots of unity have bounded degree and the degrees of those fields tend to infinity. The same determinant calculation in the residue field gives $\bar{\varepsilon}_\ell^{tr} = 1$ if the reductions are equivalent. This is precisely the condition that every $\varepsilon_\ell(g)^{tr} - 1$ be divisible by ℓ in $\mathbf{Z}_\ell$, which gives $\nu_a > 0$. □

Proof of 33.40

Proof. Choose a finite set S, disjoint from T_0, containing the primes above ℓ and prime ideal representatives generating the class group. The representatives may be chosen away from T_0 by weak approximation applied to fractional ideals. Then the ring of S-integers has trivial class group. Adjoin μ_n and the n-th roots of a finite set of generators of its unit group, and take H to be the group fixing the resulting splitting field. Finite generation follows from the unit theorem and the valuation map from S-units to $\mathbf{Z}^S$. At every place

in T_0, n is a unit and the elements whose roots are being adjoined are units. Their roots lie in the maximal unramified extension by reduction and Hensel's lemma.

For P as in the statement, its residue field contains μ_n and every S-unit reduces to an n-th power. Take a surjection from the cyclic multiplicative group of the residue field to $\mathbf{Z}/n$. For an ideal prime to P, principalize it over the S-integers and apply the surjection to the reduction of a generator. A change of generator multiplies by an S-unit. This defines an ideal character, trivial on principal generators congruent to one at P and on the primes in S. By the Artin reciprocity assertion of Definition 1.20, it gives a Galois character unramified outside P, trivial on decomposition groups above ℓ, and surjective on inertia at P. The last assertion follows from the surjection on the residue units at P. Subtract the conjugate character to obtain d_P. Since $P \neq cP$, the subtracted character is unramified at P. At cP the roles reverse.

Put $W = U \cap H \cap G_{E(\mu_{2n})}$. The element z belongs to W except possibly for the condition $z \in U$: it belongs to H by construction and fixes μ_{2n} since the prime below T_0 is prime to $2n$. Apply Lemma 36.24 in G_E/W to its class, successively excluding previously chosen pairs, the original exclusions and their conjugates, and all primes over ramified rational primes. This gives the first primes with the required splitting and norm. Inertia at each lies in W. On this inertia d_{P_j} is surjective for its own index and all the other first characters vanish. Hence their product maps W onto $(\mathbf{Z}/n)^R$. Multiply h_i by an element of W so that its vector of first character values is the i-th coordinate vector. Apply Lemma 36.24 to the corrected element modulo $W \cap \bigcap_j \ker d_{P_j}$, with the same finite exclusions enlarged at each choice. Its images modulo U, modulo H and on μ_{2n} have been preserved. This gives the second primes after all the first ones have been fixed. □

Proof of 33.41

Proof. Write k_0 for the residue field of O_0. The ring Λ is finite free over O_0, and the base coefficient ring O_0 is finite over $\mathbf{Z}_\ell$, so Λ is noetherian and ℓ-adically complete. It is local, hence complete for its maximal ideal, whose powers define the ℓ-adic topology. Indeed, modulo a uniformizer its augmentation ideal is generated by the nilpotent elements $[b_j]-1$, where b_j are the coordinate generators. Its residue field is k_0, and κ reduces to one. Put $W = T \otimes k_0$. The full residual matrix algebra and $r \geq 2$ give $H^0(E, W) = H^0(E, W^\vee(1)) = 0$. The same invariant vanishing holds for any simple coefficient subquotient, and hence for every discrete coefficient module by a finite-length filtration and passage to unions.

Here are the local counts entering the proof. At $v \mid \ell$, let $d_v = \dim H^1_f(E_v, T \otimes \mathrm{Frac}(O_0))$. The space of rational finite cocycles has dimension $r + d_v$, since rational coboundaries have dimension r. Its integral-valued elements form a saturated lattice of this rank. Indeed, finitely many coordinate evaluations separate a basis of the rational space, so the integral-valued elements lie in a lattice. Continuity on the compact decomposition group lets one clear denominators in each basis vector. Saturation follows by testing the values. Reduction of this lattice modulo the uniformizer injects into the functions with values in W: if all values are divisible by the uniformizer, divide the cocycle by it. Subtracting the dimension $r - \dim H^0(E_v, W)$ of the residual coboundaries gives the order of the residual local condition L_v:

$$|L_v| = |H^0(E_v, W)|\,|k_0|^{d_v}.$$

The local dimension formula and the conjugate-embedding hypothesis give $\sum_{v|\ell} d_v = r[E : \mathbf{Q}]/2$. Thus the product of the local ratios is $|W|^{[E:\mathbf{Q}]/2}$.

The functor S is additive and left exact. Global injectivity in degree one for an inclusion of coefficients follows from invariant vanishing in the quotient. Above ℓ, choose finitely many coordinate evaluations which give full column rank on a basis of the lattice of finite cocycles modulo the uniformizer. A square minor is a unit. Evaluating at these coordinates recovers the coefficients of a finite cocycle over any B. Hence finite cocycles pull back under inclusions and lift under surjections. Coboundaries belong to the finite condition. Outside the ramification set, inertia acts trivially and the unramified test is vanishing on inertia. At places with unrestricted condition there is no further pullback test. These observations give left exactness with the local conditions imposed. The functor commutes with increasing unions of finite submodules: a continuous cocycle on the compact group has finite image, and its finitely many local coefficient witnesses above ℓ lie in a finite submodule.

For a simple extension $0 \to k_0 \to B \to B'' \to 0$ we claim

$$|S(B'')/\operatorname{im} S(B)| \leq |S(k_0)|.$$

Let Σ contain the primes above ℓ and the finite places where ramification is allowed, including Q. Work on the Galois group unramified outside Σ. Write H^i_G for its cohomology with coefficients W, and put

$$D_2 = \ker\left(H^2_G \longrightarrow \prod_{v|\ell} H^2(E_v, W) \right), \qquad C_1 = \operatorname{coker}\left(H^1_G \longrightarrow \prod_{v|\ell} H^1(E_v, W)/L_v \right).$$

The connecting class of any element of $S(B'')$ belongs to D_2, since its finite local representatives lift above ℓ. In a fibre of the connecting map, subtract one fixed class and choose a global lift of the difference. Compare it at the places above ℓ with finite local lifts. Their differences come from local classes for W. Choosing such differences gives an element of C_1. Two choices of global classes in the fibre that give the same element of C_1 differ by a Selmer lift: subtract a global W-class realizing the difference of the local corrections modulo L_v. Consequently the number of cosets is at most $|D_2|\,|C_1|$.

The map from H^2_G onto the displayed local product is surjective by Theorem 27.2, applied to the finite ℓ-torsion module W over the totally complex field E; its global Tate dual has no invariants. The local and global Euler characteristic formulas of Definition 1.20 give

$$|H^1_G| = |H^2_G|\,|W|^{[E:\mathbf{Q}]/2}, \qquad |H^1(E_v, W)| = |H^0(E_v, W)|\,|H^2(E_v, W)|\,|W|^{[E_v:\mathbf{Q}_\ell]}.$$

Inserting these identities and the product of the local ratios in the definition of C_1 gives $|S(k_0)| = |D_2|\,|C_1|$. This proves the claim.

For completeness, let I be the continuous additive dual of Λ with values in $\mathbf{Q}_\ell/\mathbf{Z}_\ell$. The pairing $(u, f) \mapsto f(u)$ identifies $\mathrm{End}_\Lambda(I)$ with Λ by duality. Put $M = S(I)^\vee$, using the same additive dual. Its quotient modulo the maximal ideal $\mathfrak{m}$ is dual to $S(k_0)$: the socle of I is one copy of k_0, and its image under S is the simultaneous kernel of multiplication by generators of $\mathfrak{m}$. Thus M has $a = \dim_{k_0} S(k_0)$ minimal generators. To see that lifts generate, pass to finite quotients of the compact module M and apply Nakayama. The image of the resulting map $\Lambda^a \to M$ is compact, hence closed.

For a finite B, present $B^\vee$ by finite free Λ-modules and dualize. This exhibits B as a kernel between finite powers of I. Applying S and using left exactness identifies $S(B)$ with $\mathrm{Hom}_{\mathrm{cont}}(M, B)$, where homomorphisms are Λ-linear. The identification is evaluation: a map $B \to I$ acts on $S(B)$ by functoriality. It is therefore natural and extends to discrete torsion modules by the union property.

Write $N = \ker(\Lambda^a \to M)$, so $N \subset \mathfrak{m}\Lambda^a$. Embed k_0 as the socle of I. A map $N \to k_0$ extends to a continuous map $\Lambda^a \to I$ after this embedding. Indeed continuous Λ-maps to I correspond by evaluation at 1 to continuous additive characters. Such a character on N factors through a finite quotient induced from Λ^a, on which it extends by divisibility of $\mathbf{Q}_\ell/\mathbf{Z}_\ell$. Reduction modulo k_0 now gives a map $M \to I/k_0$. Its class modulo maps lifting to I is independent of the extension. This embeds $\mathrm{Hom}_\Lambda(N, k_0)$ into $S(I/k_0)/\operatorname{im} S(I)$. In the kernel, the original map $N \to k_0$ extends to $\Lambda^a \to k_0$ and hence vanishes on N. The simple-extension inequality gives $\dim N/\mathfrak{m}N \le a$. The ring is noetherian, so Nakayama gives at most a generators of N. Use them as rows and append zero rows to make a square matrix. A tuple in B^a lies in its kernel precisely when the map $\Lambda^a \to B$ that it defines factors through M. □

Proof of 33.42

Proof. Let V be the residual Selmer space with the conditions relaxed at Q, obtained by augmentation of the matrix. Choose inertia elements γ_i at P'_i with $d_{P'_i}(\gamma_i) = 1$. At cP'_i use their conjugates. Evaluation at these elements and at the indicated Frobenius elements is well defined on classes, since coboundaries vanish on the two decomposition groups. The map

$$I : V \longrightarrow (k_0^r \oplus k_0^r)^R, \qquad I(x)_i = (x(\gamma_i), x(c\gamma_i c^{-1}))$$

has kernel S_0. At a second prime the maximal pro-ℓ quotient of tame inertia is procyclic, and the specified element generates it. At the first primes there is no residual local cohomology.

By the sum of the local invariants in Theorem 27.2, there are $b_i \in k_0^\times$ such that, for $x \in V,\ y \in S'_0$,

$$\sum_i b_i\{x(\gamma_i)\cdot y(\phi_i) - x(c\gamma_i c^{-1})\cdot y(c\phi_i c^{-1})\} = 0.$$

Indeed the local conditions away from Q are orthogonal. At P_i' the dual class is unramified. The tame cup invariant therefore pairs the inertia value of x with the Frobenius value of y. Its coefficient is nonzero by the uniformizer normalization in Theorem 27.2. Using the conjugate inertia and Frobenius elements at the other prime changes the coefficient by -1, since c inverts the roots of unity. The formula over k_0 follows from the prime-field formula by testing all scalar multiples and using the nondegenerate finite-field trace pairing.

Define $J : V \to (k_0^r \oplus k_0^r)^R$ by evaluating on the two Frobenius elements and multiplying the i-th pair by a_i/b_i. Then

$$W = \{v \in \ker I : Jv \in I(V)\}$$

has dimension at most b. For $Iv' = Jv$, substitute v' for x in the reciprocity identity. It puts v in the left kernel from the hypothesis. Applying J from $\ker I$ to the quotient by $I(V)$ also gives $\dim V \le 2rR + b$.

Work over $K = k_0((Z))$, where Z is an indeterminate. Put $\beta_i = Za_i/b_i$. In the polynomial variables for the second primes take the direction β_i; in those for the first primes take the direction

$$1 - \sum_j \beta_j \, \overline{d_{P_j'}(\phi_i)}.$$

The bar means reduction from $\mathbf{Z}/\ell^M$ to k_0. Write these directions as v_j. On substituting $X_j = v_j D$ with $D^2 = 0$, the universal inverse character becomes $1 + D\lambda$, with $\lambda(\phi_i) = -1$ and $\lambda(\gamma_i) = -\beta_i$. This uses the Kronecker delta in Lemma 33.40. The values at the conjugate elements have the opposite signs.

A class lifting in this direction has residual value $x \in V \otimes K$. Scalar extension here commutes with the residual Selmer condition: a continuous cocycle into a discrete vector space has finite image, and expanding its coordinates in a k_0-basis of K gives the old cocycle and local linear equations coordinate by coordinate. The tame relation $\phi_i \gamma_i \phi_i^{-1} = \gamma_i^{q_i}$ in the pro-ℓ inertia quotient, $q_i = \mathrm{N}P_i'$, gives

$$\lambda(\phi_i)x(\gamma_i) = \lambda(\gamma_i)x(\phi_i).$$

Indeed in a cocycle with action $1 + D\lambda$, the first-order term on the q_i-th power contributes $\binom{q_i}{2}\lambda(\gamma_i)x(\gamma_i)$ in addition to q_i times the first-order value. Here $q_i = 1$ and $\binom{q_i}{2} = 0$ in k_0, since $q_i \equiv 1 \bmod 2\ell^M$. The same computation at the conjugate prime gives $Ix = ZJx$.

The kernel of $I - ZJ$ over K has dimension at most b. Its intersection with $V[[Z]]$ is a saturated submodule of a finite free $k_0[[Z]]$-module. Reduction of that intersection therefore has dimension equal to its rank. If $v_0 + Zv_1 + \cdots$ belongs to it, then $Iv_0 = 0$ and $Iv_1 = Jv_0$, so its reduction lies in W.

Lift the matrix entries to polynomials and substitute $X_j = v_j D$, now in $K[[D]]$. Its constant matrix has nullity $d = \dim V$. In elementary-divisor form there are d entries divisible by D, counting zero entries. The number of entries divisible by D^2, again including zero entries, equals the dimension of the subspace of its residual kernel lifting modulo D^2. This follows by solving each diagonal equation modulo D^2. That number is at most b by the preceding paragraph and naturality in Lemma 33.41. For $b = 0$, the determinant has order $d \le 2rR$. For $b = 1$, discard an entry of largest order. The remaining product has order at most $\max(0, d-1) \le 2rR$. The ideal of cofactors is invariant under invertible row and column operations, so some cofactor has at most this order. A matrix of size one has cofactor 1.

Finally, in $K[D]/(D^{4rR+1})$ the substitutions $[b_j] \mapsto 1 + v_j D$ respect the group relations because $(1 + v_j D)^{\ell^M} = 1$. Thus every polynomial representative of the detected entry has the same nonzero image in this quotient. If all coefficients through total degree $4rR$ lay in $\mathfrak{m}_0$, that image would be zero. □

Proof of 33.43

Proof. On L the cocycles are additive, and $x(gug^{-1}) = U(g)x(u)$, with the corresponding identity for y. The scalar actions of z on U, U' are d, d^{-1}; those of $\bar{z}$ are d^{-1}, d, by the displayed polarization. Apply the diagonal identity to uv and subtract its values at u, v. Do this a second time with u replaced by zuz^{-1}. The two cross terms then acquire the factors d, d^{-1}, interchanged on the conjugate side. Since $d - d^{-1}$ is a unit, solving the two equations gives

$$x(u) \cdot y(v) = x(\bar{v}) \cdot y(\bar{u}).$$

Replace u by gug^{-1}. With column-vector notation put

$$B = x(u)y(v)^{\mathsf{t}}, \qquad B' = J^{-1}y(\bar{u})x(\bar{v})^{\mathsf{t}}J, \qquad \theta = \varepsilon_\ell^t.$$

The cross-term identity reads

$$\operatorname{tr}(U(g)B) = \operatorname{tr}\big(\theta(g)U(g)B'\big).$$

Let $f, f' : A[G_E] \to M_r(A)$ be induced by $U, \theta U$. Both are surjective by the span hypothesis. Put $I = \{a \in A : aB = 0\}$. For $v_0 \in \ker f'$, apply the trace identity to pv_0q with $f(p), f(q)$ arbitrary matrix units. It gives $f(v_0)_{ij}B_{kl} = 0$ for all indices. Hence f modulo I factors through f'. This factor map kills $IM_r(A)$ and gives an A/I-algebra endomorphism of $M_r(A/I)$ carrying $f'(g)$ to $f(g)$ modulo I. If $I = A$ the assertion already holds. Otherwise A/I is local. The matrix-unit argument in Lemma 33.39 applies to a unital endomorphism as well: the images of the diagonal units give r isomorphic projective summands of total rank r. Consequently this endomorphism preserves determinants. Since $\det U(g)$ is a unit,

$$\theta(g)^r - 1 \in I.$$

Choose g attaining the minimum in Definition 33.3(7). The minimum is finite by the cyclotomic argument in Lemma 33.39. In $\mathbf{Z}_\ell$ one has $\theta(g)^r - 1 = \ell^{\nu_a} u_0$ with u_0 a unit. Its image annihilates B, proving the formula entry by entry. □

Proof of 33.44

Proof. Set the variables in J to zero and reduce modulo ϖ. Some homogeneous part of degree at most w is nonzero. Over the algebraic closure of the residue field choose nonzero values for the remaining variables on which that part is nonzero. Such a point exists since the product of the homogeneous polynomial and the remaining coordinate functions is a nonzero polynomial over an infinite field. All the coordinates lie in a finite extension k' of the residue field.

Take a monic lift of the minimal polynomial of a generator of k'. Adjoining a root to $O_0/(\varpi^{d+1})$ gives a finite free algebra C with reduction k'. Its ideal ϖC is nilpotent, so C is local. Lift the chosen coordinates to units of C, and put $A = C[u]/(u^{2L+2})$. Reduction of $G((a_i u)_i)$ modulo ϖ is u^j times a unit for some $j \le w$. Modulo $G((a_i u)_i)$ this gives $u^j \in \varpi A/(G((a_i u)_i))$. Raising to the power $d+1$ gives the divisibility. Finally $1, u, \dots, u^{2L+1}$ form a free C-basis and ℓ^{ν_a} is a unit times ϖ^d, nonzero in C. □

Proof of 33.45

Proof. Use the parameter of the finite-type row occurring at s, its coefficient embedding at the selected place, and the Galois realization of this parameter. These are taken from the same column. Its type projector sums the series projectors over tuples good at both type primes. On an occurring row in its image some good component is nonzero. Local type compatibility identifies the row labels with the dual of that tuple, which also satisfies the good-row condition. In the Galois comparison there is a scalar character between the row realization and ρ_s. Reducing their matrix identity gives the identity of residual characteristic polynomials with this same scalar. At a type prime different from ℓ the shift character is unramified. The inertia value of the block character there is the prescribed tame value or its inverse, according to the embedding. At the conjugate place these values are interchanged, by Artin reciprocity and the triviality of the anticyclotomic character on the real idèles.

We now give the residual image argument. At a designated place above a type prime g, the label has degree r, the dimension of the row parameter. The label counts the tame characters on the wild-invariant part. Equality of the degrees exhausts the dimension, so wild inertia has characteristic polynomial $(X-1)^r$. In the residual realization its image is a finite g-group if $g \neq \ell$. A unipotent element of such a group in characteristic ℓ is the identity. The eigenvalues of the tame generator are consequently the readings of the prime-to-ℓ parts of the roots of the label. For the exceptional sets $B(g_i, r)$ of Lemma 32.58, the constructed type data satisfy

$$B(g_0, r) \cap B(g_1, r) = \varnothing.$$

Thus ℓ lies outside at least one of these two sets. Choose such an index i and put $g = g_i$. Since $g_i \in B(g_i, r)$, this choice also gives $\ell \neq g$.

Lean: `GlobalTheory.TripleAdm.irredOn_rbar_and_inertia_of_residual (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.SGalRead`, lines 199–246)

The eigenvalues are distinct. Over an algebraic closure of k_0, a globally invariant subspace selects a subset of the eigenlines at every designated place above g. Frobenius carries the tame generator to its g-th

power modulo wild inertia. The subset therefore gives a divisor over $\mathbf{F}_g$ of the label at every such place, of the dimension of the subspace. Its complement gives the complementary degree. The good-row condition in Lemma 32.60 excludes a common degree between 1 and $r/2$. This proves absolute irreducibility. This is the factor-degree argument of Lemma 36.25 applied to the row's own residual realization. Scalars preserve its invariant subspaces. By the density theorem for a simple module, the algebra generated by the image over $\overline{k}_0$ is the full matrix algebra. The linear span is already an algebra since the image is a group. Descent of dimension gives the full span over k_0.

At *either* type prime different from ℓ, the other root exclusion of the good row says that the tame generator has no eigenvalue equal to the block-character value. Thus T has no residual inertia invariants. The same holds for T', since the cyclotomic character is trivial there on inertia and inverse transpose preserves the exclusion of 1.

At D_0, ρ_s is unramified and the inertia character in T_s is the prescribed order-fifteen character or its inverse. Choose $p = 3$ if $\ell \neq 3$, and $p = 5$ otherwise. At a place in D_0 take an element mapping to order p under that character. In the procyclic closure of this element retain its prime-to-$\ell\,\mathrm{N}v$ component. The order-p image survives since p is prime to $\ell\,\mathrm{N}v$. This gives z. Its character value and its square have nontrivial reduction, since reduction is injective on roots of unity of order prime to ℓ. The cyclotomic character is unramified at v. This proves the scalar assertion. The same construction at each place in D_0 proves the invariant exclusion there. At all the exceptional places in question local Tate duality now gives $H^2 = 0$ on the residual modules. The local Euler characteristic away from ℓ gives $H^1 = 0$. These are the finite-module formulas of Definition 1.20.

Lean: `GlobalTheory.WildFull.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.FullDegreeLabelForcesTrivialWildInertia`, lines 14–20)
Lean: `GlobalTheory.TwSys.exists_tameInertiaElementAtAuxiliaryPrime` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsTameInertiaElementAtAuxiliaryPrime`, lines 85–143)
Lean: `GlobalTheory.TwSys.TameScalar.exists_scalar_tmat_of_tame_inertia` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TameInertiaElementActsByScalar`, lines 29–61)

For the characteristic-zero local comparison, use the cuspidal row first with the inverse of the frame character and then with the layer character. The Galois shifts for ρ_s and T_s are respectively $(1 - r - t)/2$ and $(1 - t)/2$. The split Hecke polynomial identities identify these semisimple representations with their attachments. Indeed, the scalar twists preserve semisimplicity, and split Frobenius elements are dense by Lemma 36.24. At a spherical place the local parameter is unramified. At a type place its full degree and separated labels give zero monodromy and the tame parameter, with the same properties after the prescribed tame character twist. Thus the local clauses of 39.6 and 39.8 apply also when the coefficient prime is a type prime. They give de Rham comparison above ℓ, and unramifiedness of the corresponding integral actions at spherical places away from ℓ. Passing from one prolongation of a place to another conjugates the inertia group.

We compute the filtration. At a CM embedding, take the row

$$y \in \{v_0, \dots, v_{k-2}, -b_0, \dots, -b_{m-1}\}$$

with multiplicities, in the notation of Lemma 15.8. For layer $j+1$, the two conjugate infinitesimal parameters are $y + e_j$ and $-(y + e_j)$. Put $h = (t + 3)/2$. The inequalities of Lemma 15.7 give $v_i > -b_0$, $e_j - b_i \geq h$ for $i \leq j$, and $e_j - b_i \leq -h$ for $i > j$. In particular $|y + e_j| \geq h$. If z_τ is one of the two infinitesimal parameters, its filtration degree on T is

$$a_\tau = -z_\tau + (t - 1)/2.$$

These degrees are integers by regular algebraicity and the odd ranks. The attachment identifies their multiplicities with the dimensions of the graded pieces. Every degree is at most -2 or at least $t + 1$, and the conjugate multiset is the image under $a \mapsto t - 1 - a$. In each matched pair exactly one degree is negative. Since the multiset has cardinality r, the two dimensions of $D_{\mathrm{dR}}/\mathrm{Fil}^0$ sum to r. A rational invariant would supply a degree-zero line. For the Tate dual the degrees are $-1 - a_\tau$, which also exclude zero. Thus both rational invariant spaces vanish above ℓ. The local Bloch–Kato formula in 36.15 applies to these de Rham representations.

Lean: `GlobalTheory.TripleLocal.rowFacts` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.RankOneCuspidalUnitaryCharEquivApplySymm`, lines 260–302)
Lean: `GlobalTheory.TripleGal.Jumps.exists_jumpsAre_goodJ_n` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.StringRepresentations.FrobDualOfTripleAttN`, lines 85–133)

Lean: GlobalTheory.TripleGal.negCount_of_goodJ (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ConjugateNegativeJumpCountOfGoodHodgeJumps, lines 26–35)

At a remaining place of norm q, every eigenvalue of $T(\mathrm{Frob})$ is $q^{(1-t)/2}\alpha$. The unramified bound of Definition 1.20 for the unitary cuspidal row implies $|\alpha| \le q^{(r+3)/2}$. Since $t \ge r+5$, the resulting absolute value is less than one. An invariant of T would require eigenvalue 1, and an invariant of T' would require eigenvalue q on T. At the exceptional places away from ℓ the residual inertia exclusion already implies the rational exclusion, by scaling a fixed vector to be primitive. This proves the local invariant assertions at all places. Lemma 33.39 applies to the conjugate self-dual row with the shift just computed and gives its integral polarization.

Finally an abelian label gives a positive integer u whose double is its assigned parameter. The range in the row data is $1 \le u \le k + m(t + b_r + 1) + 1$, with b_r as in Definition 33.38. The empty-conductor coefficient sum is a nonzero common scalar times the reading of the toric sum at u. Unfolding the toric sum of Definition 1.15 at empty auxiliary level gives $L(u, \chi)$. The character is unitary and has angular exponents $2(u + \kappa_\tau)$ on the CM type, with $\kappa_\tau \ge 0$. Lemma 34.1 gives nonvanishing, including $u = 1$. The member's rank and type-prime conditions give $r \ge 100$ and $2r \le g_i$. Together with the polarization and filtration computation these prove the three index conditions.

Lean: GlobalTheory.TripleGal.pure_of_tripleAtt_n (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.StringRepresentations.FrobDualOfTripleAttN, lines 256–281)

Lean: GlobalTheory.LamTie.sum_th_empty_ne_zero (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SumThEmptyNeZero, lines 10–23)

□

Proof of 33.46

Proof. Write $W = T \otimes k_0$, $W' = T' \otimes k_0$, and use a generator of the residual Tate line to write the pairing as a dot product. At empty auxiliary conductor the Selmer spaces S, S' for these two representations are finite and of equal dimension. To see which dual occurs, let ϖ be a uniformizer. At $v \mid \ell$, embed W' into $T' \otimes (\mathrm{Frac}(O_0)/O_0)$ by division by ϖ. A class annihilating the reductions of all integral finite cocycles maps into the annihilator of the integral finite condition. Integral and divisible Bloch–Kato duality in 36.15 identifies this annihilator with the image of the rational finite condition. Choose a rational finite cocycle giving the image and adjust it by a rational coboundary lifting the difference of representatives in the divisible module. Its values now agree modulo the lattice with those of the residual cocycle divided by ϖ. Multiplying by ϖ gives an integral finite lift. The reverse inclusion follows by the same pairing and its compatibility with division by ϖ. This proves residual orthogonality, using the finite-field trace pairing to pass between additive and k_0-valued tests. At exceptional places away from ℓ both local H^1's vanish by Lemma 33.45. Elsewhere the unramified conditions are orthogonal. The two global H^0's vanish by absolute irreducibility and $r \ge 2$. In the finite Selmer order formula of 27.2 and 1.20, the product of the local ratios above ℓ is $|W|^{[E:\mathbb{Q}]/2}$ by Lemma 33.41. This cancels the factor at the complex places. The remaining ratios are one. Hence $|S| = |S'|$, which gives equality of dimensions over the finite field.

Lean: GlobalTheory.ResDual.card_eq (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ResidualSelmerFinrankEqDual, lines 155–249)

Lean: GlobalTheory.ResDual.Perp.mem_bkOn_of_orthogonal (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.MemDualBlochKatoOfOrthogonalResidual, lines 57–98)

Write $\bar{g} = cgc^{-1}$, where c denotes complex conjugation. Put $I = (\ell^{\nu_a})\mathfrak{m}_0$. Inside the ray subgroup H of Lemma 33.40, intersect the kernels of the T_s mod I, the residual cyclotomic kernel and the subgroup fixing $\mu_{2\ell^M}$. Intersect also with the conjugate of the resulting subgroup, obtaining L. It is open, normal and stable under c. It contains every commutator $[g, z]$. Indeed, H contains the designated inertia and is normal, the cyclotomic quotients are abelian, and both z and $\bar{z}$ act by scalars in the indicated matrix quotients. On L, W and W' are trivial. If a cocycle x of either vanishes on L, and the action of z is the scalar d, comparison of $x(gz)$ with $x([g, z]zg)$ gives

$$(d-1)x(g) = (g-1)x(z).$$

Here $d-1$ is invertible by Lemma 33.45, also for the dual scalar. Thus x is a coboundary. Independent classes restrict to linearly independent vector-valued functions on L. On this subgroup we have $x(gug^{-1}) = gx(u)$.

Lean: GlobalTheory.ResSel.SmallSub.exists_small (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsOpenNormalSubgroupUnderInertia, lines 132–184)

For $u \in L$ put $B_u(x,y) = x(u)\cdot y(u) - x(\bar u)\cdot y(\bar u)$. First suppose W and $g \mapsto W'(\bar g)$ are inequivalent over k_0. Take cocycles x_i, y_j lifting bases of S, S'. Suppose $C = (C_{ij})$ annihilates every matrix $(B_u(x_i, y_j))$ under entrywise pairing. Denote $\sum_{ij} C_{ij} X_i \cdot Y_j$ by $D(X,Y)$. Polarize the equality at uv and subtract the equalities at u, v. Replace u by gug^{-1}, first with g, then with gz. In the second equality the terms containing $x(u)$ and $y(u)$ acquire respectively the scalars d_s, d_s^{-1}, interchanged on the conjugate side. Subtraction and division by $d_s - d_s^{-1}$ give

$$D(W(g)x(u), y(v)) = D(x(\bar v), W'(\bar g)y(\bar u)).$$

The two sides are linear functionals of the respective representation matrices. A nonzero common matrix coefficient of this form would identify the two representations. Indeed, both group-algebra maps onto $M_r(k_0)$ are surjective. If an element of the kernel of the second has nonzero image under the first, the ideal generated by this image contains all matrix units. Applying the equality to left and right multiples forces the first functional to vanish on every unit matrix E_{ij}. Otherwise the first map factors through the second and the resulting automorphism of $M_r(k_0)$ is inner by the matrix-unit argument of Lemma 33.39. Thus $D(Ax(u), y(v)) = 0$ for every matrix A. Taking $A = E_{pq}$ gives

$$\sum_{ij} C_{ij} x_i(u)_q y_j(v)_p = 0.$$

Independence on L, first for the y_j and then for the x_i, gives $C = 0$. The matrices of the B_u span all matrices, in particular the identity in the two equal-dimensional bases.

In the equivalent case choose an invertible matrix J with $W'(\bar g)J = JW(g)$. Then

$$W(\bar g)^{\mathrm t} J W(g) = \bar\varepsilon_\ell(g) J,$$

where $\bar\varepsilon_\ell$ denotes reduction of the cyclotomic character. Replacing g by $\bar g$ and transposing gives this equation for $J^{\mathrm t}$ too. Hence $J^{-1}J^{\mathrm t}$ centralizes the full matrix algebra. Write $J^{\mathrm t} = bJ$. Transposing twice and taking determinants give $b^2 = b^r = 1$. Since r is odd, $b = 1$, in every residue characteristic.

The map $x \mapsto (g \mapsto Jx(\bar g))$ takes cocycles and coboundaries for W to those for W'. Transport S' back by its inverse to a space S_2, and choose cocycles x_i lifting a basis of $S + S_2$. Use $y_i(g) = Jx_i(\bar g)$. For an annihilating matrix C, put

$$P(u,v) = \sum_{ij} (C_{ij} - C_{ji}) x_i(u)^{\mathrm t} J x_j(\bar v).$$

Symmetry of J gives $P(u,u) = 0$. Additivity on L gives $P(u,v) + P(v,u) = 0$. Conjugating v by z gives $d_s^{-1}P(u,v) + d_s P(v,u) = 0$, hence $P(u,v) = 0$. Conjugate u by arbitrary g. Full matrix span again permits every matrix unit on $x_i(u)$. Independence on L for x_i, invertibility of J, and independence for $x_j \circ c$ then give $C_{ij} = C_{ji}$. Every alternating matrix annihilates every symmetric matrix under entrywise pairing, also in characteristic two since its diagonal is zero. By finite-dimensional duality the B_u therefore span all alternating forms on $S + S_2$.

There is such a form with left kernel of dimension at most one on $S \times S_2$. When $S = S_2$, pair consecutive basis vectors in alternating planes. Otherwise take $v \in S \setminus S_2$, $w \in S_2 \setminus S$. Choose α, β with $\alpha(v) = \beta(w) = 1$, $\alpha|_{S_2} = \beta|_S = 0$. Pair v, w by $\alpha \wedge \beta$ and apply induction to $S \cap \ker\alpha$, $S_2 \cap \ker\beta$ in $\ker\alpha \cap \ker\beta$. The added pair has zero cross terms with that subspace. This proves the assertion, including dimension zero. By Lemma 33.39, the equivalent case requires $\nu_a > 0$.

Lean: `GlobalTheory.ResSel.SpanOddPf.span_abstract` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CocycleFormsSpanOddResidueChar`, lines 96–174)
Lean: `GlobalTheory.ResSel.AltSpan.symm_of_forms` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SpanTwoOfGaloisHypothesis`, lines 110–171)
Lean: `GlobalTheory.AltCorank.exists_alt_corank_le_one` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsAlternatingFormLeftCorankLeOne`, lines 108–113)

For each signature express the required form as a linear combination of a basis selected from the B_u. In the first case at most $(\dim S)^2$ elements suffice, and in the second at most $(\dim(S+S_2))^2$. Pad each list to that size with identities. These sizes are fixed before L or M. Concatenate the lists into $h_1, \dots, h_{R_0} \in L$ and call their total size R_0. In L intersect the kernels on L of all the chosen basis cocycles and their conjugates. This subgroup $U \subset L$ is open by continuity and normal in G_E by the conjugation formula. If $g\phi g^{-1} \equiv u \bmod U$, then $B_u = \bar\varepsilon_\ell(g) B_\phi$ on the associated bases. The scalar is a unit, so the span is preserved. Given $R \ge R_0$,

further elements $h_{R_0+1}, \dots, h_R \in L$, and an open normal subgroup $H' \triangleleft G_E$, apply Lemma 33.40 to $U \cap H'$ and $h_1, \dots, h_R$. Since $U \cap H' \subset L \subset H$ and every h_i fixes $\mu_{2\ell^M}$, inertia at each first prime lies in L and Frobenius there is conjugate to z modulo $L \cap H'$. At every second prime inertia and Frobenius belong to L, hence so does the decomposition group. At each first prime residual Frobenius on W is d_s or d_s^{-1}, and its norm is 1 mod ℓ. Inertia is trivial, and neither W nor W' has a fixed vector. The local Euler formula gives $H^1 = 0$. This verifies all hypotheses of Lemma 33.42 on the first R_0 pairs.

Lean: `GlobalTheory.ChooseForms.single_field` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsElementsSelmerFormsSmallCorank`, lines 427–465)
Lean: `GlobalTheory.ChooseForms.chooseH` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsElementsSelmerFormsSmallCorank`, lines 557–577)

□

Proof of 33.47

Proof. Set $Y_n = 0$ in the new group ring and reduce modulo I, obtaining the old group ring over O_0/I, denoted by A. For an old Frobenius reading at the new prime, let b be the vector of its old ray coordinates. Put

$$p = \prod_j (1 + Z_j)^{b_j} - 1,$$

where Z lists the old variables and the exponents are their least nonnegative representatives. Its constant term is zero and its X_n-coefficient is one. Let Φ, Φ' be an old and a new presentation over A, respectively, and let $M_\Phi, M_{\Phi'}$ be the free modules on their column indices modulo their row relations. For an A-module B, homomorphisms from these modules to B are the respective Selmer classes. Here A is finite and killed by $\ell^{\nu+1}$, so all the coefficient modules in this description may carry the discrete topology.

At the new pair the old character is unramified, and T is trivial over A on both decomposition groups. Inertia values of a relaxed class are therefore well defined on classes and are determined by one pro-ℓ tame generator at each place. They give $2r$ elements of the representing module $M_{\Phi'}$, by applying evaluation to its universal class. Killing these elements imposes precisely the old unramified condition, so the quotient is M_Φ. Each element is killed by p. Indeed, if σ, γ are Frobenius and tame inertia, the tame relation on a cocycle gives $\kappa(\sigma)x(\gamma) = (\mathrm{N}v)x(\gamma) = x(\gamma)$. At the conjugate place the old Frobenius character is inverted. The two differences from one generate the same ideal (p).

For $i = 0, 1$ this gives

$$p^{2r} \operatorname{Fitt}_i(M_\Phi) \subseteq \operatorname{Fitt}_i(M_{\Phi'}).$$

To see the inclusion, use generators for M_Φ, lifted to $M_{\Phi'}$, and adjoin the $2r$ inertia generators. An old relation lifts after adding a linear combination of the latter. Include the relations p times each inertia generator. Taking a minor with all $2r$ of these last rows and columns multiplies the corresponding old minor by p^{2r}. Fitting ideals can be computed in these generators since the modules are finitely presented over the commutative ring A. In the original square presentations the two ideals are generated by the determinant and the cofactors respectively. An old presentation of positive size can always be used, by adjoining a generator with relation 1 if necessary.

For the determinant calculation quotient the polynomial ring by I and all monomials of degree $\geq c_0 + 2r$. For the cofactor calculation quotient instead by $\mathfrak{m}_0$, those monomials, and X_j for $j > n$. Both quotients receive the indicated maps from A: for $0 < k < \ell^{M-\nu-1}$,

$$v_\ell\binom{\ell^M}{k} \;\geq\; M - v_\ell(k) \;\geq\; \nu + 1.$$

This follows by multiplying the binomial coefficient by k. Hence the group relations $(1 + Z_j)^{\ell^M} = 1$ hold in both quotients.

If the new determinant failed its test, a representative with all the indicated coefficients in I would give zero in the first quotient after setting $Y_n = 0$. If the new positive-size matrix failed its cofactor test, one could choose a failing representative for each cofactor. All would give zero in the second quotient. In either case the inclusion would kill $p^{2r}F$ for an old representative F passing the relevant test. After the indicated variables are set to zero, F has a nonzero homogeneous term of least degree $< c_0$. The linear part of p is

monic in X_n, so multiplication by it is injective on the polynomial ring even over O_0/I. The least nonzero degree in $p^{2r}F$ is therefore $< c_0 + 2r$, a contradiction. The new size-zero case of the cofactor test holds by definition. □

Proof of 33.48

Proof. Choose a failing presentation and determinant representative. Its size is positive, since augmentation of a representative of the empty determinant is 1. Choose a representative of the cofactor specified by the old test. Apply Lemma 33.44 with the unused first variables set to zero. All other variables specialize to $a_i u$ with a_i a unit. These substitutions define a group-ring map: the binomial divisibility in Lemma 33.47 handles the low powers, and the high powers vanish by nilpotence. Every term of the failing determinant vanishes, by its coefficient in degrees $< 2l + 2$ and by its degree thereafter. This gives all the displayed assertions. Write $C = (\operatorname{adj} \Psi)_{ij}$, $\lambda = \ell^\nu u^l$ and $b = \lambda u$. Since $C \mid u^l$, the inequality $\ell^\nu u^{2l+1} \neq 0$ gives $bC \neq 0$.

Let $Y = \operatorname{Hom}_{\mathbf{Z}}(A, \mathbf{Z}/\ell^{\nu+1})$, with A acting on the argument. Evaluation is a perfect balanced pairing since A is finite and killed by $\ell^{\nu+1}$. Denote by $S(Y)$ the old Selmer group, and by $D(A)$ its group of dual classes. The dual condition is unramified away from the indicated ramification set, zero at the unrestricted places, and the cup-product annihilator above ℓ. For this pair, and for the induced perfect pair $Y/bY,\ A[b]$, the finite order formula gives

$$|S(Y)| = |D(A)| \prod_{v \in Q} |H^0(E_v, T'\kappa^{-1} \otimes A)|.$$

Here κ is the specialized inverse ray character and the tensor notation includes the original O_0-action on T'. For the second formula replace Y, A by $Y/bY, A[b]$. Indeed, the dot product with evaluation is perfect with values in the Tate line. Global invariants vanish on either side by the scalar at z. At $v \mid \ell$ the free lattice of integral finite cocycles in Lemma 33.41 evaluates injectively after tensoring with either coefficient module, by its unit evaluation minor. Dividing by coboundaries gives local ratio $|Y|^{d_v}$, or $|Y/bY|^{d_v}$; their products cancel the complex-place factors. At a relaxed place away from ℓ the Euler formula leaves the order of the dual invariants. At the fixed exceptional places these are zero by Lemma 33.45 and a composition series. These observations give the displayed formula by 27.2.

Lean: `GlobalTheory.SelOrder.orderFormula` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SelmerOrderFormulaDualSelmer`, lines 150–185)

Multiplication by u kills the dual local invariants at all the unrestricted places. At the fixed exceptional places this follows from their vanishing. At the first primes the residual Frobenius scalar differs from one, so Frobenius minus one is invertible over the local ring, also after an ℓ-power character twist. At an old second prime take inertia of ray coordinate 1. All other coordinates vanish on it, and the untwisted dual is unramified. The scalar on the dual is $1 + a_i u$, with a_i a unit. This proves the assertion, also at the conjugate prime by inversion. In particular the local products in the two order formulas agree.

Lean: `GlobalTheory.NewPrime.LocInv.local_invariants_killed` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalInvariantsKilled`, lines 36–85)

Suppose that λ kills every dual class. For a representative y write $\lambda y = \partial m$, where $\partial m(g) = gm - m$. The image of m modulo λA is globally invariant and hence zero, by the scalar at z. Write $m = \lambda m'$. Then $y - \partial m'$ takes values in $A[\lambda]^r \subset A[b]^r$. At an unrestricted place let m_v be a local primitive for y. The vector $\lambda(m_v - m')$ is locally invariant, so $u\lambda(m_v - m') = 0$. The corrected cocycle has a primitive in $A[b]^r$ there. Away from the exceptional set inertia acts trivially. Above ℓ every finite test with coefficients in Y/bY lifts to one with coefficients in Y, by lifting the coefficients of its integral finite cocycles. Compatibility of the pairings now proves orthogonality for the corrected cocycle. Thus the dual classes with coefficients in $A[b]$ map onto $D(A)$. The order formulas give

$$|\ker(\Psi : Y^a \to Y^a)| \leq |\ker(\Psi : (Y/bY)^a \to (Y/bY)^a)|,$$

where a is the size of Ψ. For a square endomorphism of a finite module its kernel and cokernel have the same order. The map on cokernels here is surjective, so the inequality makes it an isomorphism. In particular $bY^a \subset \Psi Y^a$. Row i of the adjugate kills ΨY^a. Testing it on b times the j-th coordinate copy of Y and using perfectness gives $bC = 0$, a contradiction. This constructs the required y.

The column defining x lies in $\ker \Psi$ by the adjugate identity. On L_1 both specialized actions are trivial. This subgroup is normal and stable under c, and contains $[g, z]$ since the extra kernels are abelian ray kernels. The twist is trivial at z. Full residual span gives full matrix span over A by Nakayama. The polarization of Lemma 33.39 extends to the specialized anticyclotomic character by tensoring its integral intertwiner. If $B_h(x, y) = 0$ throughout L_1, Lemma 33.43 gives $\ell^{\nu} J\, y(L_1) = 0$, where $J \subset A$ is the ideal generated by the coordinates of $x(L_1)$. Modulo J, x vanishes on L_1. The commutator computation of Lemma 33.46 works over A/J, since $d_s - 1$ is a unit. Thus x mod J is a coboundary. Naturality and injectivity of the presentation on A/J put the whole adjugate column in J. In particular $C \in J$, so $u^{l} \in J$. It follows that λy vanishes on L_1; the same commutator computation for the dual makes it a coboundary. This contradicts the choice of y. □

Proof of 33.49

Proof. Write $l = (d+1)w$. If the signature already passes the determinant test at $2l+2$, take the next second prime by the coordinate prescription of Lemma 33.40 with target element 1, keeping all existing primes. The unused first inertia groups lie in L and allow the next coordinate vector to be prescribed. Lemma 33.47 proves the result in this case.

Otherwise use A, Ψ, x, y, h, L_1 from Lemma 33.48. In L_1 take the common kernel U of x, y and of their conjugate restrictions. This is open and normal by the restriction and conjugation identities of Lemma 33.46. Every unused first inertia group lies in U. Indeed, the active ray coordinates vanish there and the two actions are unramified there. For a cocycle, the tame relation forces its inertia value into the kernel of Frobenius minus the residue norm. The reduction of this difference is a nonzero scalar since the norm is 1 mod ℓ and the Frobenius scalar differs from one. This applies at both members of the pair and to both cocycles.

The active old coordinates of h vanish. Multiply h by elements of unused first inertia groups to give it first-coordinate vector e_n. Each such inertia group surjects onto its own coordinate and vanishes on the others. The correction lies in U. Apply Lemma 36.24 modulo U intersected with all first-coordinate kernels. Exclude the previous pairs and their conjugates, the exceptional sets of Lemma 33.40 and the given exclusions. The resulting second prime P has decomposition group in L_1, since inertia lies there and Frobenius does also. It is split of degree and ramification index one, and $q = \mathrm{N}P \equiv 1 \bmod 2\ell^{M}$. Choose a tame inertia generator γ with new coordinate 1. Adjust the Frobenius lift σ by inertia to give it new coordinate zero. Its old coordinates and its class up to conjugacy modulo U are unchanged. On conjugating, both terms in B_h are multiplied by the same cyclotomic unit. Consequently $B_{\sigma}(x, y) \neq 0$.
Lean: `GlobalTheory.NewPrimePick.exists_newPlace` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsNewPlaceFrobeniusConjugateModOpenNormalSub`, lines 12–126)

Put $m_1 = 2r$ and $B = A[X, Y]/(X^2, Y^{m_1})$, with augmentation $\pi : B \to A$. Give the new coordinate value X, the first coordinate numbered n value Y, and the other old coordinates their previous values. These are values of the polynomial variables, so a group generator maps to one plus its value. The binomial calculation of Lemma 33.47 defines the specialization. Its inverse character reduces to the old one. Its values at γ, σ are $(1+X)^{-1}, (1+Y)^{-1}$, and at their conjugates the inverses of these. Choose *any* new presentation and denote its matrix over B by Φ'.

For a square presentation P_0 over a finite coefficient ring, a universal cocycle is obtained by taking a cocycle representing the identity of its representing module M_{P_0}. Lifting its values to the free generators gives a matrix of values $Z(g)$; multiplying it by any column in the kernel over a coefficient module gives the corresponding cocycle up to coboundary. Use Z, Z' for the old and new presentations. Set $M' = B^{a'}/\operatorname{row}(\Phi')$, with a' the size of Φ'. Denote the m_1 rows of Z' at $\gamma, \bar{\gamma}$ by t_{α}, and those at $\sigma, \bar{\sigma}$ by v_{α}, in matching order. In M' these rows satisfy

$$X v_{\alpha} = Y \eta_{\alpha} t_{\alpha}, \qquad \eta_{\alpha} = \begin{cases} (1+Y)^{-1} & \text{at } P, \\ 1 & \text{at } cP. \end{cases}$$

Indeed the cocycle formula on $\sigma\gamma\sigma^{-1} = \gamma^{q}$ in the pro-ℓ tame quotient gives the relation. In computing a value on γ^{q}, the sum of the q scalar powers is q plus a multiple of $\binom{q}{2} X$, since $X^2 = 0$. In B, $q = 1$ and $\binom{q}{2} = 0$ by $q \equiv 1 \bmod 2\ell^{M}$. Thus at P the equation is $(1 - (1+X)^{-1}) v_{\alpha} = (1 - (1+Y)^{-1}) t_{\alpha}$; at cP both scalar actions are inverted.

Lean: GlobalTheory.NewPrimeRel.rel (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.UniversalCocycleRowsRelationNewPlace, lines 37–122)

In $M' \otimes_B A$ the quotient by the rows πt_α represents the old condition. Indeed, the old specialized action is trivial on the new decomposition groups, and vanishing at the two tame generators is exactly unramifiedness. Choose lifts E_b of the old free generators to $M' \otimes A$. Modulo the inertia rows, the old relations on the E_b are Ψ, and πv_α has old coordinates $V_{\alpha b}$, the corresponding row of $Z(\sigma)$ or $Z(\bar\sigma)$. Evaluation here is independent of the choice of cocycle, because the action at these Frobenius elements is trivial over A.

There is one more relation in $M' \otimes A$. Put $k_\alpha = y(\sigma)_i$ at P, and $k_\alpha = -y(\bar\sigma)_i$ at cP, where i is the corresponding representation coordinate. Then

$$\sum_\alpha k_\alpha \, \pi t_\alpha = 0.$$

To prove this test the row on every relaxed class with coefficients in the additive dual of A. Its cup product with y has vanishing local invariant away from the new pair by the old dual conditions. At each new place y is unramified. As in Lemma 33.42, the normalized invariant pairs the inertia value with the dual Frobenius value up to a unit, and using the conjugate generators reverses its sign on the Tate line. The local sum in 27.2 is zero. Testing also all scalar multiples of each class, the perfect evaluation pairing gives the asserted relation. Indeed, the additive dual of A separates points of every finite A-module, by extending additive characters from a cyclic subgroup.

Lean: GlobalTheory.NewPrime.comparison_cocycles (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ComparisonCocycles, lines 24–117)

Lean: GlobalTheory.NewRecip.reciprocity (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ReciprocityRelaxedSelmerDualSelmerNewPlace, lines 190–315)

Lift the E_b to M' and adjoin $\eta_\alpha t_\alpha$ as generators. They generate after augmentation, hence generate M' since (X, Y) is nilpotent. A relation modulo augmentation lifts with its specified reduced coefficients: lift the coefficients arbitrarily, express the error in $(X, Y)M'$ in these generators with coefficients in (X, Y), and subtract. In the same way express v_α in the generators by a row W_α whose first a coefficients reduce to $V_{\alpha b}$, where a is the size of Ψ.

Form a top relation rows D. In reduction their first block is Ψ with row j replaced by zero, where j is the column index of the old adjugate in Lemma 33.48. For the other rows choose the inertia coefficients that express the old relation in $M' \otimes A$. In row j take the inertia coefficients k_α, using the reciprocity relation. The bottom m_1 relation rows are $Y(0, 1_{m_1}) - XW$. Call this square relation matrix D'. Its determinant belongs to $(\det \Phi')$: the rows are relations on a generating list for M', and its zeroth Fitting ideal in the original presentation is $(\det \Phi')$.

Expand $\det D'$ by multilinearity in the bottom rows. The term with no W-row contains Y^{m_1}, and any term with two W-rows contains X^2. Thus $\det D' = XY^{m_1-1} f$. To compute πf, put the identity rows $(0, 1_{m_1})$ in the bottom block and in turn replace bottom row α by $-\pi W_\alpha$. Swap this row with top row j. The sign of the swap cancels the minus sign on the new top row. The lower-left block is zero, and the lower-right block is the identity with row α replaced by k. Its determinant is k_α. Expansion in row j of the upper-left block therefore gives

$$\pi f = \sum_\alpha k_\alpha \sum_b V_{\alpha b} (\operatorname{adj} \Psi)_{bj} = B_\sigma(x, y) \neq 0.$$

In the free A-basis $X^i Y^j$, $0 \le i < 2$, $0 \le j < m_1$, multiplication by XY^{m_1-1} retains exactly the augmentation coefficient. Hence $\det D' \neq 0$, and $\det \Phi' \neq 0$.

Every specialized variable is a multiple of u, or is X or Y. A surviving monomial has degree at most $(2l+1)+1+(m_1-1) = 2l+m_1+1$. If a polynomial representative of the new determinant had all coefficients in I below $2l + 2 + m_1$, its value would vanish in B. This proves the requested test for every representative of every new presentation, on the single new pair chosen above. □

Proof of 33.50

Proof. Use R_0 of Lemma 33.46. If $\nu_a = 0$, take $R = R_0$, $D = 4rR_0$ and $M_0 = D + 2$. For $M \ge M_0$ we have $4rR_0 + 2 < \ell^M$. The primes of Lemma 33.46 meet the hypotheses of Lemma 33.42 with left kernel zero. That lemma gives a unit coefficient through degree D.

Now let $\nu = \nu_a > 0$, $d = \mathrm{ord}_{O_0}(\ell^\nu)$, and let t_0 be the number of signatures. Set

$$R = R_0 + t_0, \qquad w_0 = 4rR_0, \qquad w_{j+1} = 2(d+1)w_j + 2 + 2r, \qquad D = w_{t_0}, \qquad M_0 = D + \nu + 3.$$

For $M \geq M_0$ we have $w_0 + 2 < \ell^M$, $D \leq \ell^{M-\nu-1}$ and $M \geq \nu + 2$. In Lemma 33.40 pad the residual list with t_0 identities. Keep all R first primes and initially keep the first R_0 second primes. For every signature the cofactor test holds at $w_0 + 1$. Indeed, set the unused first variables to zero. At their conductor pairs the remaining character is unramified. The possible inertia value of a cocycle is killed by Frobenius minus the norm; this operator is invertible modulo the maximal ideal by Lemma 33.46, hence is invertible over the remaining group ring. Thus unrestricted classes there are already unramified. The specialized matrix presents the problem on the first R_0 matched pairs in each half. Lemma 33.42 applies with $b = 1$. Specializing an adjugate commutes with taking it, and a monomial surviving this substitution uses none of the unused first variables.

Enumerate the signatures. After j additions, require every signature to pass the cofactor test at $w_j + 1$ and the first j signatures to pass the determinant test at $w_j + 1$. For signature $j + 1$, Lemma 33.49 gives the next prime and a determinant coefficient in degree $< w_{j+1}$. For the same extension, Lemma 33.47 carries all the old tests to $w_j + 1 + 2r \leq w_{j+1}$. Its numerical hypothesis follows from $w_{j+1} \leq D$. Increasing the cutoff to $w_{j+1} + 1$ gives the induction assertion. At the last stage the two halves both have size R, and every determinant has a coefficient outside I of degree at most D. In a discrete valuation ring, $I = (\varpi^{d+1})$, so this is the asserted coefficient bound.

For the chosen primes use the matrix of Lemma 33.41, whose hypotheses hold by Lemma 33.45. Substituting the character values and taking the divisible coefficient module at the designated level gives the kernel required in Lemma 33.37, with its local conditions at this Q. The polynomial-to-group-ring map is onto since its images include all $[b_j] - 1$. Thus polynomial representatives exist at each M. □

Proof of 33.51

Proof. Write H for the definite unitary group, $\Gamma = H(\mathbf{Q})$ in restriction-of-scalars notation, $J \subset H(\mathbb{A}_f)$ for the level of the twist, and λ for its level character. Let ρ denote the action of Γ on coefficient polynomials. A class form satisfies

$$F(\gamma x k) = \lambda(k)\rho(\gamma)F(x) \qquad (\gamma \in \Gamma,\ k \in J).$$

All the values of λ are in $O_t^\times$. Indeed the fixed twist and the chosen twisting character take their values in the designated coefficient rings. Their local unit characters are obtained by evaluating those same characters on idèles supported at the specified places. The finite type character is defined in the coefficient field and has finite order. At a type prime above g_i the residue field has order g_i, so a residue-unit character has order dividing $g_i - 1$. The member field contains the roots of unity of order dividing $g_i(g_i - 1)$. Each of these character values and its inverse is integral, by the equation $X^d = 1$ for its order d. Taking the product gives the assertion about λ.

Lean: `GlobalTheory.TwInputs.levLam_mem_eOk` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmT4OfRootsInField`, lines 91–108)

Let h be the finite point of a generator, including its allowed translation at the auxiliary split places, and let P be its polynomial. Then P belongs to the O_t-coefficient lattice at h. To see the effect of the translation, let B be the product of the norms of its supporting primes. The original lattice and its translate coincide away from B. They are full lattices, so a power of B takes either one into the other. The same assertion holds for their coefficient polynomials. Indeed, in lattice coordinates their change of basis and its inverse have denominators supported on B. Substituting in each minor of a generating product and expanding by minors gives coefficients in $\mathcal{O}_E[1/B]$. The determinant character, including a negative determinant power, has the same property by use of the inverse matrix. Clearing these finitely many denominators proves the assertion for each generator and hence for its span. The auxiliary places avoid the places over ℓ, so B is a unit in O_t.

Lean: `GlobalTheory.LatTranslate.lat_translate` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CoefLatLeTranslateSplitPlacesPrimeToL`, lines 49–69)

Define a class form by

$$F_{h,P}(x) = \sum_{\substack{\gamma \in \Gamma \\ k_\gamma = h^{-1}\gamma^{-1}x \in J}} \lambda(k_\gamma)\rho(\gamma)P.$$

When nonempty, the indexing set is a torsor under $\Gamma_x = \Gamma \cap xJx^{-1}$, acting on the left. This stabilizer is finite since the unitary group is definite at infinity and J is compact. Reindexing the sum at $\gamma_0 x k_0$ by $\gamma = \gamma_0\gamma'$ gives the class-form law. In each summand $x = \gamma h k_\gamma$. The rational change of variables carries the coefficient lattice at h to the one at γh, and k_γ preserves the level lattice. Multiplication by $\lambda(k_\gamma)$ preserves integrality. Thus $F_{h,P} \in M_t$.

Lean: `GlobalTheory.TwTraceCx.traceForm_mem_mtLat` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TwistedThetaGeneratorEqClassSumIntegralForm`, lines 115–135)

Write $\Theta_t(x, P)$ for the kernel with the twisted splitting and the finite vector of the twist. Its transformation law is

$$\Theta_t(\gamma h k, \rho(\gamma)P) = \lambda(k)^{-1}\Theta_t(h, P).$$

Here the finite vector is in the inverse level-character space. The finite part of k acts on it by $\lambda(k)^{-1}$. The infinite part of γ acts on the coefficient polynomial by substitution, and left multiplication by the rational element γ reindexes the theta sum. These give the displayed identity with the same J and λ as in the class form.

Lean: `GlobalTheory.TwTraceCx.dualLaw_kern` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TwistedThetaGeneratorEqClassSumIntegralForm`, lines 147–196)

Choose representatives x for $\Gamma\backslash H(\mathbb{A}_f)/J$. In

$$A_t(F_{h,P}) = \sum_{[x]} |\Gamma_x|^{-1}\Theta_t(x, F_{h,P}(x))$$

the only contributing class is that of h. For its representative there are exactly $|\Gamma_x|$ terms, each equal to $\Theta_t(h, P)$. Thus $A_t(F_{h,P}) = \Theta_t(h, P)$, including when ℓ divides $|\Gamma_x|$.

Finally the member specifies its generator using the reference splitting and the determinant twist on the target. For either unitary group U, put $d_{t,U}(g) = t(z)$, where $z/\bar{z} = \det g$. The character is trivial on the fixed-field idèles, so this value is independent of z. Changing both splitting characters by t multiplies the action at (h, g) by $d_{t,U_{\mathrm{target}}}(g)d_{t,H}(h)^{-1}$, the quotient of their determinant characters in the Weil action. The member's generator is therefore $d_{t,H}(h)\Theta_t(h, P)$. Its preimage is $d_{t,H}(h)F_{h,P}$, still in M_t since the twisting value lies in $O_t^\times$. The identification of the member's class-form lattice with M_t respects the class-sum map. Injectivity of its coefficient realization gives the asserted containment in the member coordinates.

Lean: `GlobalTheory.TwTraceCx.gen_eq_smul` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TwistedThetaGeneratorEqClassSumIntegralForm`, lines 78–105)

Lean: `GlobalTheory.ClassTrace.classSum_traceForm` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.FiniteStabOfReps`, lines 155–182)

□

Proof of 33.52

Proof. If every $x_\alpha = 0$, take $N = 0$. Otherwise take one nonzero x_α and write $\lambda r(x_\alpha) = r(y)$ with $y \in B$. In the given basis let $s_j, t_j \in K$ be the coordinates of x_α, y. Complex linear independence gives $t_j = \lambda s_j$ for every j. Choosing $s_j \neq 0$ gives $\lambda = t_j/s_j \in K^\times$. For every index β, injectivity now gives $\lambda x_\beta \in B$. Choose $N \geq 0$ with $N \operatorname{ord}_O(\ell) \geq \operatorname{ord}_O(\lambda)$. Multiplication by $\ell^N/\lambda \in O$ gives the result. The coefficient at an image ray class is the sum over its fibre, so it belongs to the same lattice after multiplication by ℓ^N. □

Proof of 33.53

Proof. Keep the real row, series realization and attached Galois realization of the starting column used in Lemma 33.35. Use the chosen weak local correspondence h_L on this realization. Lemma 33.45 verifies the index conditions at its eligible indices. A selected coefficient place has eligible index by restriction of its place to the fixed coefficient subfield.

First fix the integral data that precede the choice of a character. At an abelian label let Δ_Q be the finite Galois ray quotient, and let $\pi_Q : \mathrm{Pic}(O_Q) \to \Delta_Q$ be the map induced by geometric reciprocity. At a prime P outside the base exclusions and the auxiliary conductor, $\pi_Q([\varpi_P])$ is the image of inverse arithmetic Frobenius. Use the fixed coefficient embedding and write $T(c)$ for the resulting reading of the toric class term at c. For the fixed nonzero scalar u_0 of the label, the ray coefficient at δ is $u_0 \sum_{\pi_Q(c^{-1})=\delta} T(c)$. Indeed

evaluation at a ray character ρ weights the term at c by $\rho(\pi_Q(c^{-1})) = (\rho \circ \pi_Q)(c)^{-1}$, exactly the weight of that Picard twist in Lemma 36.94.

Lean: `GlobalTheory.ToricDenom.IsRecMap` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.PlacesBelowSplitCoprimeConductorOfAdmissible`, lines 71–73)
Lean: `GlobalTheory.ToricDistribution.thEval_eq` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.ExistsIsHeckeLOfHasInf`, lines 213–246)

The host is abelian CM, unramified over its real subfield, and has the distinguished imaginary quadratic subfield and induced type of Definition 1.15. The base conductor has the prescribed split prime factors, and the auxiliary pairs avoid it and ℓ. The parameter u is positive and its angular weights are those of the label. The class-term bound (1.2), proved in Lemma 1.19, is uniform in these auxiliary pairs. Taking the twelfth root in its valuation inequality bounds the denominators of the class terms relative to the fixed period. Clear the remaining common period and scalar in the coefficient reading by a power of ℓ. Finite sums of the resulting integral elements are integral, and a discrete valuation ring is integrally closed in its fraction field. This gives an integer for each label. Take their maximum, using zero for an empty set, to obtain $N_0(a)$. Then all $\ell^{N_0(a)}$-scaled ray coefficients are integral, at every admissible auxiliary conductor.

Lean: `GlobalTheory.ToricDenom.exists_bound_at` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.BoundedDenominatorsOfBeta`, lines 133–152)
Lean: `GlobalTheory.ToricDenom.exists_gen_bound` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.BoundedDenominatorsOfBeta`, lines 156–177)

For each layer and sign, project the allowed untwisted vectors onto the prescribed finite-type part. Equivalently one can multiply the projector by the order of the finite type group, using its integral central group-algebra element. Multiplication by this nonzero scalar identifies the two image lattices. Apply Lemma 33.51 with empty auxiliary conductor and the trivial character. Under the map $x \mapsto 1 \otimes x$, each allowed untwisted generator belongs to the theta image of the definite class-form lattice at that level. The coefficient extension at this level is finite and injective; tensoring with it is faithfully flat over the base discrete valuation ring. Thus the projected integral span is finitely generated and torsion free. Reduce the generating set modulo the maximal ideal, select a basis of the reduction from it, and lift the corresponding allowed entries. Nakayama gives generation; equality of the number of lifts with the rank gives independence. Take these finite sets as $S(a, j, \sigma)$, and take the empty set at unused indices or layers.

At a selected place these spans have full rank. In the real realization each vector in the prescribed good summand is read as evaluation of its model vector at the fixed frame, multiplied by the nonzero comparison scalar. The column corresponding to the place has the base tuple of this realization. Its good projected lifts and their right translates span the model values: apply the theta lift in the prescribed range, with the level projections, and take the class sum at base level. In this operation averaging a right translate over the base level is a finite sum on its smooth vector, and averaging fixes the target base-level vector. Thus evaluating the projected lifts gives the span of the allowed class sums. The range here is that of the constructed theta model, with the member's level characters and weights; cuspidal rapid decrease in 1.22 permits its integrations. The separating rational readings of 34.41 apply to the same row and identify its coefficient form. In a rational basis full rank is witnessed by a nonzero minor, which remains nonzero in the prescribed coefficient embedding. This proves full rank at the selected place.

Lean: `GlobalTheory.RankOnto.sumOnto_of_goodRange` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.GoodFormsClassSumsOfProjectedLiftRange`, lines 19–29)
Lean: `GlobalTheory.TieRank.rank_of_ties` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.FullRankThetaSeriesFormsOfRationalScalars`, lines 82–89)
Lean: `GlobalTheory.TwistChoice.exists_basisOf` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsEquivMapLattice`, lines 188–210)

We prove the uniform assertion of Definition 33.38. Fix an eligible index satisfying its conditions, suppose the projected spans have full rank, and give any allowed integral bases, $e > 0$, and a finite avoidance set. Let O denote the coefficient ring for the theta polynomials in Lemma 33.36. We describe the polynomials on the ray coordinates of the member. Use in Lemma 33.40 the admissible single-pair coordinates for its Artin maps. The coordinate on a retained pair is the same after restriction from a larger conductor: both maps on G_E are the chosen single-pair character. Surjectivity of the Artin quotient then gives the same identity there. Proposition 33.50 works for this family of coordinates.

For an abelian label λ, use as f a polynomial reading of its scaled ray sum at all pairs, and as g its reading at the first pairs alone. Explicitly, push its finitely many integral coefficients into the product of the cyclic ray groups and replace a group element with coordinates c_i by $\prod_i (1 + Z_i)^{c_i}$, using least nonnegative exponents. For a split pair of norm q on which the testing character is unramified, passage from the smaller

to the larger order multiplies the sum by

$$(1 - aq^{-u})(1 - a^{-1}q^{-u}) + (q - 1)q^{-2u},$$

where a is the product of the label and testing characters evaluated at arithmetic Frobenius at the chosen prime P. In the common coefficient field the label factor is $\chi_f(\varpi_P)^{-1}$, and the testing factor is $\rho(\pi_Q([\varpi_P])^{-1})$ at the smaller conductor. Thus both factors use the same Frobenius. Replacing the testing character by its inverse inverts the testing factor. Indeed, locally the smaller order is the product of the two valuation rings and the new order requires equal residues. Among units this leaves the diagonal residue units. If both valuations are positive the congruence is automatic, with unit index $q-1$. Summing the two valuation series and dividing by the unrestricted Euler series gives the displayed factor. This computation in the convergence half-plane extends to u by the toric class-sum formula of Definition 1.15 and 34.1. The expression is invariant under inversion of a.

Require first Frobenius to agree up to conjugacy with z on the residual label characters. Their values at z differ from one: on designated inertia their ratio to the prescribed order-fifteen value or its inverse has residue-characteristic-power order, whereas the chosen component of z has order prime to that residue characteristic in every finite quotient and retains its nontrivial order-p value. Since $q \equiv 1 \bmod \ell$, the first-pair factors at the trivial testing character are units. Consequently

$$\operatorname{ord}_O g(0) = \operatorname{ord}_O \left(\ell^{N_0(a)} \sum_{\delta} \theta_{\varnothing,\lambda,\delta} \right).$$

Here $\theta_{\varnothing,\lambda,\delta}$ denotes an empty-conductor ray coefficient. The sum is nonzero by Lemma 33.45. Set the second variables to zero. The Frobenius reading at second prime i of a character on the first pairs is $1+X_i$. Modulo the first cyclic relations use $(1+X_i)^{\ell^M-1}$ for its inverse. The product A_λ of the displayed local factors gives $f = A_\lambda g$ in that quotient. One may check this at all tuples of ℓ^M-th roots in the designated extension: a polynomial reduced to degree $< \ell^M$ separately in each variable and vanishing at every tuple is zero, by successive univariate interpolation in the fraction field. In the residue quotient each factor is

$$(1 - \alpha(1 + X_i))(1 - \alpha^{-1}(1 + X_i)^{-1}),$$

with α the residual label value. It has a unit constant term if $\alpha \neq 1$; for $\alpha = 1$ it is $-X_i^2/(1 + X_i)$. This gives the second hypothesis of Lemma 33.36 with exponent bound two.

Lean: `GlobalTheory.ToricDistribution.gen_relation` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.ExistsIsHeckeLOfHasInf`, lines 314–393)
Lean: `GlobalTheory.ToricFactor.propE` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.PropEOfAdmissible`, lines 420–561)

For a projected basis of size b use the lattice obtained by intersecting the full integral type lattice with the good fraction-field space. It is a full lattice there, since denominators of each vector in the subspace can be cleared in the full lattice. The integral multiple of the projector preserves the full lattice, takes it into this intersection and commutes with the Hecke operators. Indeed, integral finite-group translates preserve the lattice, and the Hecke operators in question are at split places outside the type levels. They preserve the full lattice by the integral Hecke comparison of 36.106. Fix a basis of the intersection lattice.

There is one power of ℓ clearing the coordinates of the projected universal series for the given b entries, for all ray levels disjoint from their levels. Here one uses the integral series assertion on the pinned frame in 36.106. For each small-field generator apply Proposition 36.100 to its cyclic additive lattice at the chosen cover place. To obtain a prescribed universal coefficient, multiply the readings by the indicators of the fibres of the determinant ray coordinate and sum. The fibres partition the components, so precisely one indicator contributes at each point. These indicators are integral component functions on the ray cover. Right translation by a base level element permutes the local labels in this sum: its reduction preserves the test lattice and its upper left block at each type prime is the identity. The inverse change of the determinant label carries the fibre indicator to the indicator of the translated test. Reindexing gives invariance under the base compact subgroup.

Claim 1. The descent exponent for these covers is independent of M, Q. Fix the twist index, layer and sign. Use its original coefficient DVR O, before adjoining the testing roots of unity, and identify its fraction field F_O with its prescribed complex reading. This ring is complete with algebraically closed residue

field. Put $e_O = \mathrm{ord}_O(\ell)$. The fixed small coefficient field lies in F_O. For each allowed ray level choose the squarefree auxiliary covering level prime to ℓ, containing the required primes and the fixed determinant-twist level. Write the required ray modulus as m_0N, where m_0 is the fixed modulus divisible by g_0g_1, N is prime to ℓ, and $m_0 = m'\ell^a$ with m' prime to ℓ. The fixed small field contains the ray field of modulus m_0. The ray field of modulus $m'N$ is unramified above ℓ over the quadratic host and embeds in F_O over the chosen host embedding. Some odd $g_i \neq \ell$ divides m'. Thus both moduli m_0 and $m'N$ contain this common odd prime. The host units are $\{\pm 1\}$, so the ray-class compositum formula for these two moduli gives the ray field of their least common multiple m_0N inside F_O. Adjoin within F_O the finitely many type and determinant-character values supplied by the coefficient readings. This gives a finite cover coefficient field $L \subset F_O$ containing the base field K, with the required markings and ray functions. This construction applies for every Q, M while O remains fixed. The restriction of its valuation to L is a positive integral multiple of the normalized valuation at the selected place w, so $\mathrm{ord}_w(\ell) \leq e_O$.

The remaining hypotheses of Lemma 36.101 hold on these layers. The prescribed weight is dominant, and the last entry of each nonempty positive block is at least one by the adjacent weight-gap inequality. Its theta degree is $n_V \geq 8fr + 4 = 4[E : \mathbf{Q}]r + 4$. At this degree, pull a bounded holomorphic cusp section to the principal-level uniformization and apply the doubled theta reproducing formula. Unfolding over the lattice classes expresses it as a finite linear combination of harmonic theta series. The degree bound gives convergence of the reproducing integrals. Express the harmonic coefficients in the span of integral harmonic polynomials; their short theta readings are readings of base algebraic sections. Dominance and the strictly positive last entries make these sections cuspidal. The resulting equality holds on every uniformization component and vanishes off the domains. Thus the holomorphic cusp section space at the designated embedding is contained in the complex span of readings of base algebraic cusp sections lying in that space. Finally the level-character comparison identifies the multiplier on a base compact element with the prescribed tame character on finite units congruent to one. That character is trivial, giving $\mu(gk) = \mu(g)$ for every base compact element k.

The preceding fibre-indicator calculation gives invariance of the cover reading itself. Lemma 36.101 therefore applies at the selected place, including when it is ramified, and multiplication by $\ell^{e_O^2}$ gives a combination of base integral sections with integral coefficients. Both this exponent and the integral-series exponent were chosen before Q, M, proving the claim.

An allowed harmonic coefficient function is a finite integral linear combination of these generators in the coefficient lattice. The series is linear in that function, so clearing its finitely many coefficient denominators works for every ray class and every cover. Write the member's reading as $r = \nu^{-1}\,\mathrm{ev}$, where ev is extremal evaluation and $\nu \neq 0$ is the fixed comparison scalar. If $\ell^c u\, r(x)$ is the evaluation of an element of the integral section lattice, then $(\ell^c u/\nu)r(x)$ belongs to the reading of the member's lattice. The rational cusp frame stays linearly independent in its complex realization. Lemma 33.52 now gives a single clearing power for this datum, including after adding coefficients in each fibre of the ray-coordinate map. Taking the maximum over the finite basis gives the asserted common power. Exclude its test levels and the places over the host primes q, d.

Lean: `GlobalTheory.XiIntComb.xi_int_comb` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.XiIntComb`, lines 78–108)
Lean: `GlobalTheory.XiBound.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.UniversalSeriesCoeffBoundedInSupOfTwistSeriesSid`, lines 40–94)
Lean: `GlobalTheory.TwInputs.univSup_of_xi` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.UnivSupOfXi`, lines 13–44)

Form the b-column coordinate matrix of these scaled projected vectors over the ray group ring. Write $m_\varnothing, m_1, m_Q$ for the matrices at empty conductor, first pairs, and all pairs, using polynomial representatives. The determinant of $m_\varnothing$ is nonzero by the basis property. At a split place P of norm q write $H_j(P)$ for the matrix of the order-j normalized Hecke operator on the intersection lattice. Removing a pair on which the testing character is unramified expresses the larger-conductor columns as the smaller-conductor columns acted on by

$$C_P(y) = 1 + \sum_{j=1}^{r} (-1)^j q^{j(j-1)/2} H_j(P) y^j,$$

where y is the signed Frobenius value at the chosen member of the pair. Write O_P for the local integers, ϖ for a uniformizer, and N for the number of columns of the local test matrix. For $Z \in M_{r,N}(O_P)$, summing

the determinant test over its label counts the $(N-r)$-planes complementary to the rows of $\bar{Z}$, with weight $q^{-r(N-r)}$ each. At rank r, fix one complement. All complements are the graphs of the linear maps from it to the row space, giving $q^{r(N-r)}$ choices. At smaller rank the two subspaces cannot span $\mathbf{F}_q^N$. Thus summing out the label gives the indicator of row rank r.

For a right coset representative g of the order-j spherical double coset, $L = gO_P^r$ runs once through the intermediate lattices $\varpi O_P^r \subset L \subset O_P^r$ of index q^j. Indeed the inverse image of any codimension-j residue subspace has these elementary divisors, and two representatives give the same lattice exactly when they differ on the right by $\mathrm{GL}_r(O_P)$. The translated integral test requires $g^{-1}Z$ integral. This says that every column of $\bar{Z}$ lies in $L/\varpi O_P^r$. Taking annihilators identifies the admissible lattices with the j-planes in $\ker(\bar{Z}^{\mathrm{t}})$. If this kernel has dimension h, the alternating weighted count is $\sum_j \begin{bmatrix} h \\ j \end{bmatrix}_q (-1)^j q^{\binom{j}{2}} = \prod_{a=0}^{h-1}(1-q^a)$. The polynomial identity before evaluation at one follows by induction from $\begin{bmatrix} h \\ j \end{bmatrix}_q = \begin{bmatrix} h-1 \\ j \end{bmatrix}_q + q^{h-j}\begin{bmatrix} h-1 \\ j-1 \end{bmatrix}_q$, obtained by intersecting with a fixed hyperplane. The product equals the full-rank indicator, including at $h=0$. For a nonintegral matrix every lattice test vanishes since $L \subset O_P^r$.

Let x_P be the uniformizer idèle on the chosen side, and let μ be the splitting character of the theta frame. On this double coset the normalized Hecke sum of right translates has prefactor $\mu(x_P)^{-j} q^{-j\lfloor N/2 \rfloor}$. The determinant has valuation j, and unit factors act trivially on the unramified character. This is the inverse of the splitting-character and modulus factor in the Weil action on the integral test. The theta series is linear in that test, so the preceding indicator identity becomes the Hecke identity with coefficient $(-1)^j q^{\binom{j}{2}}$. To track its ray value, write c for the remaining additive ray coordinates and δ for their reading on x_P. The smaller-level coefficient on the translated term is evaluated at $c + j\delta$. The testing character factors through the smaller level. Reindexing its sum therefore multiplies the term by $\chi(-j\delta)$. With the Artin convention of the ray coordinates, δ is minus the coordinate of arithmetic Frobenius, so $\chi(-j\delta) = y^j$ on the chosen side. The other member of the pair gives the inverse reading. Finally the projection onto the good type commutes with all these translates because its type places are disjoint from P. Applying the identity to each basis vector gives left multiplication of the smaller-level coordinate matrix by $C_P(y)$.

Lean: `GlobalTheory.DistRel.xi_rel` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.IsDetRootOfDetAt`, lines 190–312)
Lean: `GlobalTheory.DistRel.thURel_of_tie` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.UniversalTwistedSeriesDistributionOfCompatibilit`, lines 152–363)
Lean: `GlobalTheory.TwistDet.det_factor` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AevalOneSubOne`, lines 142–208)

Impose also the first-prime congruences for all residual Hecke readings and their conjugates. These are open normal congruences: take the kernels of the finitely many residual Galois representations giving the reverse Hecke polynomials, and their conjugates. On the split spherical primes the reverse polynomial is that of $T_s(t)$ at inverse arithmetic Frobenius; coefficient j is $(-1)^j q^{j(j-1)/2}$ times the order-j Hecke eigenvalue. This follows by taking elementary symmetric functions of the attached Satake parameters with the shift of Lemma 33.45. For a good signature s, Lemma 33.45 gives $T_s(z) = d_s 1_r$, with $d_s^2 - 1$ a unit and with cyclotomic character one at z. Hence the extra Tate twist in the Hecke reading has value one there. In the finite quotient just chosen, arithmetic Frobenius at a first prime is conjugate to the image of z, with conjugation of the place allowed. Thus its inverse in the residual reading is a scalar $c_s 1_r$ with $c_s \in \{\bar{d}_s, \bar{d}_s^{-1}\}$. Evaluating the reverse polynomial at one gives $\det(1 - c_s 1_r) = (1-c_s)^r \neq 0$ for both choices.

Extend the coefficient discrete valuation ring to split the finitely many characteristic-zero Hecke eigensystems on the good type space. A common invariant flag exists by simultaneous triangularization. Its intersections with the extended type lattice are saturated, since the flag lies in the fraction-field space. Their successive quotients are free of rank one and reduce to a full residual flag. On each quotient the diagonal of $C_P(1)$ is the reading just computed for the corresponding signature in the row. The reduction of its determinant is the product of these nonzero readings, counted with their multiplicities. A nonunit in the original ring remains a nonunit in the extension. Hence $\det C_P(1)$ is a unit in the original ring as well.

Lean: `GlobalTheory.TripleGal.HkFrobProof.hkFrob_of_tripleAtt_n` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.StringRepresentations.FrobDualOfTripleAttN`, lines 189–250)
Lean: `GlobalTheory.TwistDet.exists_subgroup_isUnit` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsSubgroupIsUnit`, lines 75–185)

Use $f = \det m_Q$, $g = \det m_1$. At the trivial character, successive removal of the first pairs gives $\mathrm{ord}_O\, g(0) = \mathrm{ord}_O \det m_\varnothing$. Removing the second pairs with $Y_i = 0$ gives $f = Ag$ modulo the first cyclic relations, where A is the product of the determinants of $C_{P_i'}((1+X_i)^{\pm 1})$. This follows by the same interpo-

lation argument as for the abelian labels. Over the residue field $\det C_{P_i'}(y)$ is a polynomial of degree at most rb with value 1 at $y = 0$. Its order at $y = 1$ is at most rb. Substituting $y = (1 + X_i)^{\pm 1}$ preserves this order since $y - 1$ is X_i times a unit in the local power series ring. This yields the factor required in Lemma 33.36, modulo $X_i^{\ell^M}$, with exponent at most rb. For $b = 0$ the determinant is 1.
Lean: `GlobalTheory.TwistDet.propABCD_of_rel` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AevalOneSubOne`, lines 293–465)

Intersect the open normal subgroups for these finitely many tests and unite the finite exclusion sets with the requested one. Proposition 33.50 supplies the Selmer matrices on primes satisfying precisely these extra restrictions. The constants just obtained for the theta polynomials depend on the fixed bases and labels and are independent of M. Apply Lemma 33.37 to all these polynomials, taking

$$\delta = e/(1 + |\mathcal{S}| + 2m + 4Rt).$$

It gives one triple Q, k', ξ and the Selmer and tame inequalities for every required signed coprime power. For an abelian label, undoing $\ell^{N_0(a)}$ gives lower valuation bound $-N_0(a)$ and upper bound $c_0(a, \lambda) + \delta$, since the empty-conductor reference is nonzero and was scaled by the same power.

For a layer and sign, the specialized determinant is nonzero. Its normalized valuation is at most the normalized valuation of $\det m_\varnothing$ plus δ. The two clearing powers contribute the same b-fold scalar to the two determinants. Thus the twisted basis columns span a full lattice with signed index at most the untwisted index plus δ. Each column is a specialized allowed theta vector because Q avoids its level. By Lemma 33.51, the full projected twisted span lies in the projection of the theta image of the definite class-form lattice for this twist. That image is finitely generated. Its submodule is finitely generated over the coefficient discrete valuation ring and is torsion free in the coefficient space. It lies in the prescribed good space by the type of its generators. It contains the full-rank sublattice supplied by the specialized columns, so it is a lattice there and has signed index at most that of the sublattice. Removing the common multiple of the projector cancels the same scalar on the two sides. Over a finite injective extension of the coefficient discrete valuation rings, determinant orders and $\operatorname{ord}(\ell)$ multiply by the same ramification index. A change to any test lattice shifts both signed indices by the same determinant valuation. This proves the required inequality for every such extension and test lattice. There are $2m$ layer-sign tests, so their errors sum to at most e.
Lean: `GlobalTheory.ExportFromInputs.export_at` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.IsLatInOfIndexLe`, lines 371–450)

The same triple works for all coprime powers because the specialization estimates hold at every primitive root of the chosen order, including its inverse. The integers $N_0(a)$ and the selected basis sets were fixed before the exclusions or powers. This proves Definition 33.38. □

Proof of 33.54

Proof. The input to localization is the rational realization of the multiplicity spaces with its chosen tuple and conductor at w. The character at the last finite level is the product of the reference character and the chosen twist, and its conductor set is the one used to select the split operators. Thus the integral Hecke action at that level applies to the twisted space W. More explicitly, for $s \in S$ and an allowed generator H, the double-coset action on the embedded vector is a linear combination of the embedded generators with coefficients integral at w. The rational map into the space of forms is injective and intertwines H. Pulling the equality back gives the same combination for Hs. It follows that $\Lambda = \sum_{s \in S} O(1 \otimes s)$ is stable under $\mathbb{T}$. These generators span W over K_0 by the layer lattice condition. Thus Λ is a full lattice and each H has integral eigenvalues.
Lean: `GlobalTheory.HeightCounting.Fin3.mtInput_of_levelC` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RowsAtStart`, lines 31–33)
Lean: `GlobalTheory.HeckeLocN.heckeIntTw_base` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.HeckeOperatorIntegralOnLatticeAtTwistedPlaces`, lines 23–32)
Lean: `Formalism.L2.Fam.NPC.LatticeFinitenessFree.span_latL` (`Formalism.LocalToGlobal.NumberFieldFamilies.IsLocalizedOfSpan`, lines 105–111)

Operators at different places commute. At one split spherical place transpose fixes each double coset by the elementary divisor decomposition and reverses convolution. Thus the spherical algebra is commutative. To choose a splitting field at w, take finitely many of the operators spanning their K_0-linear span in $\operatorname{End}_{K_0}(W)$. Adjoin the roots of their characteristic polynomials to the completion at w. Commuting split operators admit a common triangularization, obtained by taking a common eigenvector and passing to the

quotient. Each remaining operator is a linear combination of the chosen ones, hence also splits there. Taking the compositum with the finite local fields required for the coefficient and character embeddings gives the K' used above.

Lean: `GlobalTheory.HeckeLocN.ndSpec_exists_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.NdSpecExists`, lines 21–103)

Decompose V successively into the primary spaces of the chosen commuting operators. Each decomposition preserves the preceding ones. The resulting nonzero joint spaces have characters $\lambda : \mathbb{T} \to O$, since the diagonal values of all generators lie in O. Write $J_\lambda = \ker \lambda$ and $\mathfrak{m}_\lambda = \ker(\mathbb{T} \xrightarrow{\lambda} O \to k)$, where k is the residue field of O. Every maximal ideal is some $\mathfrak{m}_\lambda$. Indeed, on a common triangular flag the product of the kernels of the successive diagonal characters annihilates V. The action of $\mathbb{T}$ on V is faithful, so this product is zero in $\mathbb{T}$. A maximal ideal must contain one J_λ. Since λ is the identity on scalars, $\mathbb{T}/J_\lambda = O$, which has just one maximal ideal. This proves the assertion. For $V = 0$ the endomorphism algebra is zero and the index set is empty.

Collect the joint spaces with the same $\mathfrak{m}_\lambda$. This gives $V = \bigoplus_{\mathfrak{m}} V_{\mathfrak{m}}$. In the construction a local index consists of a layer, a place with nonreference tuple, and a maximal ideal of this integral algebra. The fibre at (j, w) is taken by equality of the first two entries. The map to maximal ideals forgets those entries and has inverse $\mathfrak{m} \mapsto (j, w, \mathfrak{m})$. This also explains why a reference place has empty fibre.

Lean: `Formalism.L2.Fam.NPC.locsC_mem` (`Formalism.LocalToGlobal.NumberFieldFamilies.NPartCNd`, lines 153–155)
Lean: `Formalism.L2.Fam.NPC.mxC_inj` (`Formalism.LocalToGlobal.NumberFieldFamilies.NPartCNdTie`, lines 56–67)
Lean: `Formalism.L2.Fam.NPC.mxC_all` (`Formalism.LocalToGlobal.NumberFieldFamilies.NPartCNdTie`, lines 69–71)

For each $\mathfrak{m}$, the projector $p_{\mathfrak{m}}$ preserves Λ. To see this, choose finitely many Hecke generators distinguishing all the distinct residual characters. For one chosen operator, group the linear factors of its characteristic polynomial according to their roots modulo the maximal ideal of O, retaining their multiplicities. Factors belonging to different groups have unit resultant. The Chinese remainder theorem gives polynomials over O acting as the primary projectors for these groups. Multiply the projectors for the residues prescribed by $\mathfrak{m}$ over the chosen operators. On each joint space the product is the identity or zero according to its residual character. Hence the product is $p_{\mathfrak{m}}$, and belongs to $\mathbb{T}$.

The construction takes $M_i = p_{\mathfrak{m}}\Lambda$. It is generated by the indicated projected vectors, is torsion free over the discrete valuation ring O, and spans $V_{\mathfrak{m}}$. It is therefore finite free. Localization of its inclusion gives the displayed isomorphism. The local comparison sends a row vector to its projection followed by the inverse of this isomorphism. In particular its image in V differs from the original row vector by an element of $\bigoplus_{\mathfrak{n}\neq\mathfrak{m}} V_{\mathfrak{n}}$. The comparison intertwines every Hecke generator because each commutes with $p_{\mathfrak{m}}$. Taking dimensions, and using $\dim_{K'} V = \dim_{K_0} W$, gives the rank formula.

Lean: `Formalism.L2.Fam.NPC.MtSub` (`Formalism.LocalToGlobal.NumberFieldFamilies.NPartCNd`, lines 185–186)
Lean: `Formalism.L2.Fam.NPC.prMtC` (`Formalism.LocalToGlobal.NumberFieldFamilies.NPartCNd`, lines 196–199)
Lean: `Formalism.L2.Fam.NPC.PsiC_pr` (`Formalism.LocalToGlobal.NumberFieldFamilies.NPartCNdTie`, lines 88–96)

For the length assertion, use the good spectral points of Definition 36.11 on each of these modules. Write G_i for the sum of their eigenspaces in $K' \otimes_O M_i$ and H_i for the sum for the remaining points. The local spectral realization is split semisimple on these spaces. Under the comparison just constructed, $1 \otimes M_i$ has image $p_{\mathfrak{m}}\Lambda$, and the eigenspaces have the same Hecke characters on both sides. Thus the local quotient in Definition 36.11 is carried isomorphically to

$$\frac{G_i \cap (p_{\mathfrak{m}}\Lambda + H_i)}{p_{\mathfrak{m}}\Lambda \cap G_i},$$

where we use G_i, H_i also for their images in $V_{\mathfrak{m}}$. In particular, if $G = \bigoplus_i G_i$ and $H = \bigoplus_i H_i$, then

$$\frac{G \cap (\Lambda + H)}{\Lambda \cap G} \;\simeq\; \bigoplus_i \frac{G_i \cap (p_{\mathfrak{m}}\Lambda + H_i)}{p_{\mathfrak{m}}\Lambda \cap G_i}.$$

Indeed $\Lambda = \bigoplus_{\mathfrak{m}} p_{\mathfrak{m}}\Lambda$, so both intersections can be taken componentwise. The quotients have finite length because the upper and lower lattices each span the good subspace. Division of the length by $\mathrm{length}_O(O/\ell O)$ gives exactly the sum of normalized lengths. This normalization also agrees before and after finite extension of the local coefficient field. An elementary divisor of order a acquires order ea under an extension of ramification index e, as does the order of ℓ with the same factor.

At the last finite level the construction first forms the local modules and then assigns the good spectral points on those very modules. For $1 \le j \le m$ it sets the row coordinate equal to the sum of their normalized

quotient lengths. It puts the new local data and this coordinate into the same member over the column. The subsequent spectral comparison uses its maximal-ideal map and its maps from the localized modules. Thus the comparison maps used for the ranks also carry the lattices and Hecke subspaces used for the lengths.
Lean: Formalism.BasicNotions.NData.etaLoc (Formalism.SharpRestriction.LevelLetterCocycles.DataN, lines 219–220)
Lean: GlobalTheory.HeightCounting.NLay0.fixZ (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.NStageTrivialCharacterCoeffEqAlt, lines 62–64)
Lean: GlobalTheory.HeightCounting.setN (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.NStageTrivialCharacterCoeffEqAlt, lines 125–126)
Lean: GlobalTheory.HeightCounting.Fin3.oL (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.TwistDataAgreeAcrossCoefficientsLocalized, lines 50–51)
Lean: GlobalTheory.EtaMem.exists_etaJ_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.NonGaloisEtaJMem, lines 29–92)

□

Proof of 33.55

Proof. For the first assertion apply Theorem 37.5 to each field and rank in the mass assertion. Its field hypothesis is the prescribed unramified CM condition, and the adjacent-rank formula is required for $M \geq 3$.

First work on the origin extension of (33.1). The base assertion chooses a preliminary datum and frame before the two finite coefficient requirements are specified. Use the coordinate preliminary datum of the host and a frame on it. The first requirement puts its algebraic elements in the fixed coefficient subfield, and the second puts its elements in the full coefficient field. The frame is transported from the preliminary datum in both choices. On subsequent admissible enlargements the preliminary datum and the prescribed field of frame values are identified, so the base condition persists.
Lean: GlobalTheory.Mem.Top.Bl.KPxCond.baseCond_KPx (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ExistsPrePin, lines 30–36)

On an eligible untwisted base containing the required primes, choose a starting column with the given frame. The untwisted character supplies its support and conjugacy conditions. In P, the twisted Siegel–Weil identity gives the comparison for the theta realization, and twisted local multiplicity gives equality of the row multiplicities. Use the separating functionals of Theorem 34.41 on this same realization. The row values are rational at its specified embedding. These are the multiplicity and rationality inputs of Lemma 33.35; apply it to this column.
Lean: GlobalTheory.Mem.member_hodgeHyp_of_localInputs_rationalWeights (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.MemberHodgeHypOfLocalInputsRationalWeights, lines 12–15)

Starting from its state with empty local index set, put the local correspondence for h_L on the later stage. Use its multiplicity spaces, rational row realization and cuspidal realization together as the intermediate level. Beyond the host cutoffs choose $v_0 \mid q_0$ outside S_1 and the prescribed split lifts. The wild characters above S_1 and the tame character above v_0 give the layer characters of Definition 34.47. Together with the lattice vertices they give an initial grid at $(1, c_{\mathrm{Dep}}, b)$. For the fixed preliminary datum, one number field works at every stage. At each stage choose a grid for which this field contains the values of each layer character at its unramified finite primes and the required roots of unity. Include generators of that field in the first coefficient requirement. At the first local level the toric characters in the twist coefficient rings give agreement with the signed lists. Retain this state as the origin through the two later levels. Localization preserves the grid condition.
Lean: GlobalTheory.T.exists_character_grid_data (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ExistsCharacterGridDataStatement, lines 7–8)
Lean: GlobalTheory.T.settingsFormsExistList_of_localLanglands (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ReductionSeriesB0Statement, lines 7–8)

At the last level retain the selected places and their characters. For each twisted tuple apply Lemma 33.54 to its own multiplicity space. The tuple is the reference character times the chosen twist, and its conductor is the one defining the allowed double cosets. The integral action on the layer generators thus applies at precisely those places. Install the local modules, their Hecke actions and their projection maps together. The row coordinate at each lift layer is the sum of the normalized lengths on this local family. The resulting member lies over the original column. The theta data for its rows and columns have the same indexing, so adjoining the ambient coordinates and the division data gives a constructed point. Apply Proposition 33.53 to this point, using the real row and the series of the column just constructed and its Galois realization for h_L. This gives the twisting data of Definition 33.38 for the point's own theta vectors.

Lean: GlobalTheory.Mem.Top.Bl.has3_neo_of_localLanglands_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.Has3NeoOfSteps, lines 19–35)
Lean: GlobalTheory.Mem.Top.Bl.exists_baseLattice_of_pair_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ExistsBaseLatticeOfPrime, lines 36–56)
Lean: GlobalTheory.Mem.Top.Bl.memberE_of_steps2 (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ExistsMemberOfConstructionStepsSecondPass, lines 38–61)

For the enlargement assertion, start with the given untwisted member and its export data. For each residual characteristic unite the requested avoidance set with the avoidance set for transporting the column. In the local character choice exclude every host place over these primes, as well as the places already excluded by the host. Unpack Definition 33.38 for this member and keep its integers $N_0(a)$ and its basis sets. If w is selected, its index is eligible by restriction to the fixed coefficient subfield. The last condition at w gives full rank and the integral basis property for these particular sets, and the three index conditions hold there. Apply the uniform assertion with those sets, the enlarged exclusions and $e = \ell_a^{-2}$. This is where the triple (Q_a, k_a, ξ_a) is chosen. In the simultaneous specialization of Lemma 33.37, its Selmer test is evaluated on $(\zeta^{h\sigma_s \ell^{e_j}} - 1)_j$ for the sign σ_s of the signature. The theta tests use the same exponents and the same Q_a. To make the choices simultaneously, choose one selected representative for each pair of conjugate restrictions to the fixed coefficient subfield, and use its index throughout that pair. Enumerate the distinct indices so obtained; there are finitely many selected places. At each index add all previously chosen conductor sets to the exclusions. The new conductor is disjoint from their union, and each test with the original exclusions still holds, since the added exclusions only strengthen the disjointness requirement. In particular the global Selmer inequalities and the theta-lattice inequalities hold for the same ξ_a, with the bases fixed at the start.
Lean: GlobalTheory.Mem.Top.ExportAt (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.AmbientBaseHeq, lines 36–40)
Lean: GlobalTheory.HeightCounting.Up.nonempty_seed_of_kept (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.TwistDataAgreeAcrossCoefficientsSymm, lines 133–210)

On the selected places use the character, conductor set and exponent so obtained. Off the selected places take the trivial character, the empty conductor set and exponent zero. Lemma 33.35 ensures that over a fixed coefficient place the two prescriptions agree: the whole fibre is selected or the whole fibre is omitted. In the selected case the character choice is constant on this fibre.
Lean: GlobalTheory.Mem.Top.TwChar.exists_twDat_stagesOn (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.NotMemSqOfRamificationIdx, lines 414–460)

The twist has now been specified, so its character set can be used as the argument of the requested finite coefficient set. Adjoin that set together with the algebraic elements required for transport and the roots needed for the chosen exponents. Transport the frame by the identity of preliminary data and pull the twist back to the larger coefficient field. The set of rational primes of the twist is unchanged. At each new place its character data are read at the place below it, so it satisfies the original avoidance condition. The new character set identifies with the one used in choosing the coefficient elements. The enlarged coefficient field therefore contains precisely the requested elements in the sense of item 4 of Theorem 33.30.
Lean: GlobalTheory.HeightCounting.Up.exists_twisted_cohomology_class_lift_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ExistsTwistedCohomologyClassLift, lines 15–53)

Use the same column generators and place selections on this enlargement. Pull back the selected set by restriction of places. Transport the row realization and its character readings to the new column. Fix a selected place of the enlarged coefficient field, lying over v, and write ℓ, k, Q, ξ for the residue prime, exponent, conductor set and ray character chosen below it. The enlarged coefficient field contains all ℓ^k-th roots of unity. The two readings of their cyclic group therefore differ by raising to one power b prime to ℓ, determined by the image of a primitive root. To identify the characters on the entire ray quotient, take any element of that quotient and lift it to the absolute Galois group of E. Apply 36.24 with the open normal kernel of the ray map and with the finite exclusion set consisting of Q and the places above ℓ. The resulting Frobenius represents the chosen element. Both readings give the inverse of the value of the same complex Hecke character at this prime, after their respective root identifications. Thus the enlarged member reads ξ^b on every element of the quotient. Its signed h-th power, for h prime to ℓ, reads ξ^{bh} or ξ^{-bh}. Both exponents are prime to ℓ. The transported signature representations and local conditions agree at these readings, as do the theta generators and their projections. The bounds furnished before enlargement apply to these very powers and to the transported bases.
Lean: GlobalTheory.HeightCounting.Rigid.pow_of_tied (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.RawPackEqPow, lines 23–46)
Lean: GlobalTheory.HeightCounting.Up.GoodPrimeSelection.goodO_of_seed (Development.AbelianVarietyTheory.Proofs.

IwasawaDecomposition.GoodTwistOfSeedCharacterPowers, lines 157–227)

The resulting member on the geometric extension projects to a constructed member on the origin extension, with both the goodness and twisting conditions. This proves the second member assertion with the frame and prime set held fixed throughout.

Lean: GlobalTheory.Mem.Top.ReW.extend_pinned (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.GensJAlgebraic, lines 184–255)

Lean: GlobalTheory.HeightCounting.Up.secondNo_of_sMt_sRat_sCusp_sFinl (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.FiniteLevelAndSecond, lines 72–115)

Lean: GlobalTheory.Mem.Top.Closing.goodE_of_builtE_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.GoodEOfBuiltE, lines 16–31)

Write Q_o, Q'_o for the origin constructed and good member conditions. For a scale context K, a point X on base B, frame f, twist T and auxiliary datum c, augmentation gives

$$Q_A(K, X, f, T, c) \iff Q_o(K, X, f, T, c) \wedge A(K, B, f, T). \tag{A.319}$$

The same identity holds for Q'_A, Q'_o. For the first member take the common cutoffs for its construction and for A. For export use the first conjunct. For the second member include the finite requirements for A on the enlarged base in the unions just used. The base condition persists there, so the eventual hypothesis on A applies there as well. Thus the same enlarged member satisfies both augmented conditions.

Lean: GlobalTheory.Mem.Top.GroupRetwisting.pass_step (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.PassStepCongr, lines 16–47)

Lean: GlobalTheory.Mem.LiteralAutomorphy.memberE_extensionWithAtProperty_of_memberE (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.IsBuiltEPrmAIff, lines 35–53)

It remains to construct the linked bundle. In the definition preceding Theorem 33.32 take the additional base condition to be the constant true condition. Work above the common cutoffs for the spectral data and the row multiplicities. Fix a good augmented point β and then an arbitrary $c \in \mathbf{R}$. Unpacking its good-member witness gives a weighted point q with the same host datum and arithmetic object as β, and local place matrices giving its kernel entries. The augmentation retains this witness and its projection to the list extension at the same point indices.

For q one may compute row multiplicities on the stage with the same group and prescribed tame series and with empty local index set. Copy the row realization by the group identification. The character labels, levels and central idempotent actions are preserved. Twisted multiplicity one on the copied realization gives equality of the two multiplicities. Transport the equality back along the same identification. This construction applies on the augmented extension since it uses the group and rows of q.

Lean: GlobalTheory.Mem.Top.Bl.witExE_of_witMv (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ExistsIsEOfHeq, lines 33–36)

Lean: GlobalTheory.Mem.Top.Bl.MultQClose.multiplicative_closure_prime_mem_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.MultiplicativeClosurePrimeMem, lines 18–39)

The preliminary host has level set $\{g_0, g_1, p_{\mathrm{de}}, p'_{\mathrm{de}}, q_0\}$. The member level also includes the word primes T_u. The linked points q and β have the same arithmetic datum, so take the spectral comparison on q with this full level record. Its local indices, integral modules and maps from the twisted layers belong to this datum. Adjoining the kernel to the spectral comparison preserves its projection to the same spectral comparison, so these local data are also those used by the recursion. Combine them with the kernel of q using the common column frames: the coordinate changes in the kernel and spectral description are the identity at the pinned entries. The lower row maps in these coordinates agree on the reference and twisted rows. The Artin and ray-field inputs give the host data for the same place selections. Together these choices give a recursion realization R with the prescribed spectral and kernel components.

Lean: GlobalTheory.Mem.Top.Bl.exists_kernel_spectral_data_at_witnesses (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.TiedSKAtWitnesses, lines 23–70)

Lean: GlobalTheory.Mem.Top.Bl.WPt.exists_recPinE_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ExistsRecPinE, lines 21–31)

Use the saturated spectral spaces already attached to q. They apply to R because its column datum is the column datum of q. Their columns lie in the specified Hilbert space. The multiplicity equality above gives the nonzero row pieces at each required place. Match the square-integrable columns of the prescribed kernel to these pieces, using the row and column realizations of q. The matching supplies a definite face, its spectral identification and the isometry on the indefinite output with its prescribed norm normalization. Use the same matched data for the arithmetic pair of row sides. Its lower maps are the ones just identified. On the indefinite side take the transported row coordinates and the same place embeddings and finite indices

as on q. The character idempotents and the identified row multiplicities give the labels on each nonzero eigenspace. Thus the arithmetic and indefinite realizations have the same face and spectral component.
Lean: GlobalTheory.Mem.R3.SelectionPairing.exists_selection_pair_local_langlands_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ExistsSelectionPairLocalLanglands, lines 69–87)
Lean: GlobalTheory.Mem.R3.SelRealIn.realIn_exists_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RealInExists, lines 54–92)
Lean: GlobalTheory.Mem.R3.RecInAt.recIn_parts_exists_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RecInPartsExists, lines 30–69)

Finally take the integers d_0, h_0 from the numerical data of β, so that $\Delta = d_0$ and the size coordinate is mh_0. At the scale prescribed by d_0, h_0, c, form the bundle of Definition 33.6 from R, the saturated spectral realization, the arithmetic row sides with their lower maps, these integers, the square-integrable columns and the matched indefinite realization. The last realization uses exactly the face, spectral space and scale of the preceding components. This gives a bundle over q with recursion component R. Transport it across the equality of the host and arithmetic object with those of β. The point q, that bundle and this equality constitute the required linkage. The cutoffs were chosen before c.
Lean: GlobalTheory.Mem.R3.SelAll.bwit_exists_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.BwitExists, lines 53–78)
Lean: GlobalTheory.Mem.Top.Bl.exBL_isLinked_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ExistsLinkedOfEq, lines 31–51)

□

Proof of 33.56

Proof. Fix the word. Write F for the maximal totally real subfield of the host E, and ξ_t for the finite-order character by which tuple t twists the reference tuple t_0. Thus $\xi_{t_0} = 1$. Denote the level-measure scalar and the column-kernel scalar at step j by $I_{t,j}$ and $K_{t,j}$. Write $V_{t,j}$ for the vector scalar of the double step beginning at even j, and P_t for the scalar of the terminal period. The scalar that commutes the column change through the last lift is a single $s_u \neq 0$ for this word. With these conventions the comparison uses

$$c_t = s_u P_t \prod_{\substack{0 \le j < m \\ j \text{ even}}} \frac{V_{t,j} K_{t,j} K_{t,j+1}}{I_{t,j} I_{t,j+1}}.$$

Indeed, a double step contributes its two kernel scalars and its vector scalar on the column side, and its two level measures on the row side. The terminal period occurs once.

First consider the vector and level factors. For each finite translating element b_j of the row chain choose an idele x_j with $x_j/\bar{x}_j = \det b_j$. At the endpoint use the translating element of the terminal group for b_m. In the Weil action the source determinant occurs with the inverse splitting character and the target determinant with the splitting character. Twisting both characters by ξ_t therefore multiplies the step-j vector by

$$z_{t,j} = \xi_t(x_j)^{-1} \xi_t(x_{j+1}).$$

The real changes of frame have determinant representatives with finite part one. The finite-order character is trivial on the connected infinite idele group of E, so these changes give the same formula for every real test vector. In particular $z_{t,j}^{\operatorname{ord}(\xi_t)} = 1$. The reference vector scalar is common to the tuples and nonzero: at the type primes it is the scalar of the fixed reference vacuum, and the twist operators are supported away from those primes. The two sets of local operators act on distinct tensor factors.
Lean: GlobalTheory.TupRoot.weil_tuple_root_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.WeilTupleRoot, lines 28–83)
Lean: GlobalTheory.StepVacForm.main_alt_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ImportsBis, lines 25–75)

For the torus factor, let a_i be the idele of the specified local torus element at marked index i. Its contribution at a double step is

$$M_{t,j} = \prod_i r_{ji}\, \mu_{t,j+1}(a_i)^{e_{ji}}, \qquad r_{ji} > 0, \quad e_{ji} \in \mathbf{Z},$$

where r_{ji} and e_{ji} are the real multiplier and the character exponent of the fixed torus test. The tuple character at this step is $\mu_{t_0,j+1}\xi_t$. Thus $M_{t,j}/M_{t_0,j} = \prod_i \xi_t(a_i)^{e_{ji}}$, also killed by $\operatorname{ord}(\xi_t)$. Let $J_{t,j}$ be

the product of the local indices of the new step level in the reference level, in the common level frames. Subdivision into cosets gives $I_{t,j}J_{t,j} = I_{t_0,j}$. The two local averages in $V_{t,j}$ contribute $(J_{t,j}J_{t,j+1})^{-1}$. Hence their contribution after division by the level measures is exactly

$$\frac{(J_{t,j}J_{t,j+1})^{-1}}{I_{t,j}I_{t,j+1}} = \frac{1}{I_{t_0,j}I_{t_0,j+1}}.$$

The remaining factor of $V_{t,j}$, apart from the two vector scalars and $M_{t,j}$, is the common local integral with the fixed type-prime tests. The intermediate-step multiplier in that integral is 1 on the last pair. We have thus computed the relative contribution of $V_{t,j}/(I_{t,j}I_{t,j+1})$.

Lean: `GlobalTheory.SeeSaw.mono_pow_eq` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.ListProdMapNeZero`, lines 27–40)

Lean: `GlobalTheory.Fam.lawsLinks_of_geo_rows_cols_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LawsLinksOfGeoRowsCols`, lines 36–189)

We compute the normalization of the column kernels. At a twisting place, a chart at level j is a choice of local coset representative in the finite level average. For source and target charts α, γ, write the splitting multiplier of the local kernel as a source factor times a target factor. Before the last two steps these are signs. More explicitly, if $N_j = k + j(t_{\mathrm{ch}} + 1)$ is the column rank, put $h_j = N_j + m - 1 - j$ for $j < m$, where t_{ch} is the auxiliary rank. The source sign is the sign of the source chart raised to h_{j+1}, and the target sign is the sign of the target chart raised to h_j. These exponents come from the determinant on the other space, including its $m - 1 - j$ extra coordinates at level j. At an intermediate chart the two signs cancel since

$$h_j + h_{j+2} = 2(k + m - 1 + (j + 1)t_{\mathrm{ch}}).$$

At the join of the last two steps, write the reduction of the change of parahoric representative as a block triangular matrix with diagonal blocks A, D. Let δ be the tame character of the twisting place, whose residue field has order q. The parahoric line contributes $\delta(\det D)$. The determinant in the splitting contributes $\delta(\det g)^{-1}$, since the unramified part is one on the unit $\det g$. Reduction of the determinant gives

$$\delta(\det D)\delta(\det g)^{-1} = \delta(\det A)^{-1}.$$

This has order dividing $q - 1$. Moving the same representative from one side of the join to the other gives its inverse. Consequently at each place the local multiplier can be written $\sigma_j(\alpha)\tau_j(\gamma)$ with both factors roots of unity and $\tau_j(\gamma)\sigma_{j+1}(\gamma) = 1$. The ordinary joins use the sign calculation above. The last join uses the block determinant calculation. Multiplying over the twisting places gives factors $p_j(\alpha), q_j(\gamma)$ with the same cancellation.

Lean: `GlobalTheory.LevelLaw.last_steps_law` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.ParahoricChainLastTwoStepsRootOfUnity`, lines 127–197)

Lean: `GlobalTheory.TwPlaces.exists_roots_of_unity_signs_facAt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RatDatumAddTwistPlaces`, lines 176–227)

There is also a determinant character in the reading of a chart. Write $x_{j,\alpha}/\bar{x}_{j,\alpha} = \det g_{j,\alpha}$ for its determinant representative and χ_{j+1} for the step character of the tuple under consideration. Set

$$A_j(\alpha) = p_j(\alpha)\chi_{j+1}(x_{j,\alpha}), \qquad B_j(\gamma) = q_j(\gamma)\chi_{j+1}(x_{j+1,\gamma})^{-1}.$$

These are roots of unity. Indeed $d = \det g_{j,\alpha}$ has infinite part one and finite part an integral unit: the finite matrix and its inverse preserve the standard lattice. A continuous circle character on the compact group of finite integral units has finite image, since this group is profinite and the circle has a neighborhood of one containing no nontrivial subgroup. Some positive power of the step character is trivial on the ideles of F, by its prescribed restriction and the finite order of the twist. Now $x_{j,\alpha}^2 = d(x_{j,\alpha}\bar{x}_{j,\alpha})$, with the second factor an idele of F. Taking a common multiple of the two character orders proves the assertion.

Lean: `GlobalTheory.CharRoot.pow_eq_one_of_isDetRoot` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.TupleCharacterAtBasePointDetRootRootOfUnity`, lines 41–56)

The value $B_j(\gamma)A_{j+1}(\gamma)$ is a constant k_j on the intermediate charts. The local factors $q_j p_{j+1}$ cancel. The remaining character is χ_{j+2}/χ_{j+1} on $x_{j+1,\gamma}$. Its restriction to the ideles of F is trivial, so the choice of determinant representative does not affect it. Before the penultimate level the charts give the same standard finite matrix. At the penultimate level their additional determinants are local units at the twisting places. The two characters have the same tame restriction there, so their quotient is one on these units. This proves

constancy, and k_j is a root of unity. The initial chart is unique. Let d_0 be its value of A_0, put $d_j = d_0 \prod_{i<j} k_i$, and rescale the chart vectors by

$$s_0(\alpha) = 1, \qquad s_{j+1}(\gamma) = d_j B_j(\gamma).$$

Induction gives $s_j(\alpha)A_j(\alpha) = d_j$. The residual multiplier of each rescaled kernel entry is therefore

$$\frac{s_j(\alpha)}{s_{j+1}(\gamma)} A_j(\alpha)B_j(\gamma) = 1.$$

In this normalization $K_{t,j}$ is the product of the common step volume, the common normalization of the reference kernel and its initial column scalar. The terminal scalar is the common terminal volume times $\sum_\beta s_m(\beta)$. The terminal level has one local label at each twisting place, so the product of the local label sets is a singleton, also for the empty twisting set. Its one summand is a root of unity.
Lean: `GlobalTheory.ChartQuot.chr_quot_chart_free_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ChrQuotChartFree`, lines 24–71)
Lean: `GlobalTheory.Fam.QuadraticExtensionData.dat_exists` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.DatExists`, lines 22–90)
Lean: `GlobalTheory.Fam.colSide_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ColSide`, lines 35–222)

In the quotient c_t/c_{t_0} the common nonzero factors cancel. The product of the orders of the finitely many remaining roots of unity gives an admissible $N > 0$, with $r = 1$ for these choices. Complex conjugation preserves the power identity.

Finally let p be the residue characteristic of the coefficient place w. Off the selected twisting places of the coefficient field, $\mu_w = t_0$ and Q_{t_0} is empty. At a selected w, the character is the product of the auxiliary character for p and the chosen ray twist, which is unramified above p. The auxiliary character uses the other exceptional prime when p equals its first exceptional prime. Its local restriction above p is therefore unramified as well. Every place in Q_{μ_w} is ramified for the product. Hence none lies above p. □

Proof of 33.57

Proof. First work before the good projector. Let σ be the product at infinity of the prescribed discrete series, w its product of unit lowest-type vectors, and P the convolution by its normalized matrix coefficient. Use the finite translation carrying the reference level and its character to the spectral level and its character. In the linked realization this translation is obtained by carrying the two local level frames to their specified frames and taking the identity at the places where they already agree. Right translation is invertible and preserves the quotient measure.

Evaluation at w identifies the space of level intertwiners with H. In detail, formal-degree orthogonality gives

$$PR(g)P = \langle w, \sigma(g)w\rangle P.$$

For $z \in H$, this identity shows that $\sigma(g)w \mapsto R(g)z$ preserves the Gram matrix up to the scalar $\|z\|^2$. It therefore extends from the cyclic span of w to a bounded intertwiner. Its image has the required level since the finite action commutes with G_∞. Evaluation recovers z; cyclicity gives uniqueness. This construction takes values in the cuspidal space: 33.25 gives integrability at the rank and weight gaps of the construction, and the constant-term calculation in 33.28 annihilates each proper constant term on the projected space. The latter proof identifies the projected image of bounded inputs with the entire image, so its vanishing applies to every z. On smooth vectors the intertwiner gives smooth automorphic functions by differentiation under right translation and local Sobolev embedding. Conjugation and the inverse finite translation give precisely the reference multiplicity maps. At infinity conjugation identifies the prescribed contragredient models by their discrete-series parameters in P. Thus translated evaluation gives an isomorphism J from the full space of reference multiplicity maps onto $\overline{H}$, the complex conjugate of H in L^2.
Lean: `GlobalTheory.Mem.SRe.exists_multR_eval_of_star_mem` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ExistsRowLevelTranslatedEvaluationOfConjRow`, lines 28–51)
Lean: `GlobalTheory.Mem.SRe.evalOn_all` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.LowestVectorEvaluationIdentifiesLevelRowsConj`, lines 19–86)

Write e_0 for the good idempotent acting on the values of a reference multiplicity map S. In the identified level frames, its coefficients are the complex conjugates of those of G. Evaluation commutes with each finite

translate, with the two finite actions identified by the change of frame. Summing over the finite group gives

$$J(e_0 S) = \overline{G\,\overline{J(S)}}.$$

If $z \in GH$, take $S = J^{-1}(\bar{z})$. The identity and injectivity of J give $e_0 S = S$. Conversely $e_0 S = S$ gives $\overline{J(S)} \in GH$. Thus the fixed subspace of e_0 maps onto $\overline{GH}$.
Lean: `GlobalTheory.Mem.SRe.evalOn_good` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.EvaluationGoodRowsToConjugateGoodRows`, lines 58–215)

We identify the arithmetic space in this calculation. In the row coordinates of the linked point, let j be the reference coordinate isomorphism from E_{ar} to the reference row of the chain in tensor coordinates. The latter row is the image of the e_0-fixed multiplicity maps under linearization of a multilinear map on the archimedean factors. Denote this linearization by L, and the realization of an arithmetic coordinate x as a multiplicity map by S_x. The compatibility of the reference row coordinates in (33.1) reads $j(x) = L(S_x)$. For an e_0-fixed map S, the vector $x = j^{-1}(L(S))$ therefore satisfies $S_x = S$, since linearization on a tensor product is injective. Conversely $j(x)$ lies in that reference row, so S_x is fixed. This gives equality of ranges before taking dimensions. The two row norms are the same Petersson pairing multiplied by their respective positive measure normalizations. Dividing the composition by its positive norm factor yields the specified isometry onto $\overline{GH}$. Conjugating gives the spectral embedding onto GH.
Lean: `GlobalTheory.Mem.R3.RowsLvl.range_eE_and_norm` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.RangeEEAndNorm`, lines 68–118)
Lean: `GlobalTheory.Mem.SRe.exists_rowJ_of_evalOn` (`Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.ConcreteLevelAdelicSubset`, lines 88–152)

The finite-type assignment places each transported basis vector in its assigned type space. Apply the single-label idempotent to a basis expansion of any vector in the space for a good t. Orthogonality of the idempotents in 32.44 deletes exactly the terms whose labels differ from t. The retained vectors therefore span that space, and they are independent by the isometry.
Lean: `Formalism.L2.SG.AllRowsOf.rowsLab_of` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.AllRowsOf`, lines 864–881)

□

Proof of 33.58

Proof. Write a, b, c, d for the lengths of A, B, C, D. The centered lists are regular Harish–Chandra parameters for discrete series on both sides. We apply Atobe's unconditional discrete-series criterion. Its branching input is He's theorem for pairs of discrete series, as Atobe explains immediately after the statement [148, Definition 1.6 and Theorem 1.7]. For the smaller source the centered entries are integers and none is zero. Thus the integer k_λ of that criterion is -1, and its two signature indices are $a + d, b + c$. The desired target has signature

$$(a + d + t, b + c).$$

In part (1) of the criterion its parameters are $h = (t - 1)/2$ and $l = 0$, where the target signature in that part is $(a + d + 2h + l + 1, b + c + l)$. The two exceptional sets counted there are empty. Indeed any entry α they count would have $0 \le \pm\alpha - 1 < h$, contrary to the bound on $|\alpha|$. The lift to this signature is nonzero. Removing $i > 0$ planes keeps h fixed and gives $l = -i$, violating the necessary inequality $l \ge 0$.

In the reverse direction the centered list contains I_t in one sign. Neither neighbor $(t + 1)/2, -(t + 1)/2$ occurs, so this is the maximal central string and $k_\lambda = t$. The signature indices after deleting the string are $a + b, c + d$. The smaller target has exactly this signature. The equal-increment clause of part (2) gives nonvanishing at $(a + b + l, c + d + l)$ precisely for $l \ge 0$. Again every lower member in this Witt tower has $l = -i < 0$.

For the global assertion, a nonzero lift to a lower member would give a nonzero local theta pairing at the chosen place. Indeed, on pure tensors fix all the other local inputs and evaluate the lifted form. Varying the two inputs at the chosen place gives a pairing invariant under the source action, by the change of integration variable in the theta integral. It factors through the local theta quotient there. The vanishing just proved kills this pairing for every choice of the other inputs. Linearity and continuity give vanishing for the allowed tests. Theorem 9.5 identifies the constant term along the parabolic of an isotropic i-plane with the lower lift. Thus all proper constant terms vanish. On an anisotropic target there are no such parabolics. □

Proof of 33.59

Proof. On a right half-plane multiply the Euler factors at each w. The resulting function is

$$L^S(B \otimes \mu_{j+1}^{-1}, s+1/2) \prod_{i=1}^{j} \prod_{l=0}^{t-1} L^S(\mu_i/\mu_{j+1}, s+t/2-l).$$

The first factor is holomorphic and nonzero for $\operatorname{Re} s > (r+1)/2$. To see this, put $\sigma = \operatorname{Re} s + 1/2 - r/2 > 1$. The absolute logarithmic series on any closed half-plane with this lower bound is at most

$$\frac{r[E:\mathbf{Q}]}{1-2^{-\sigma}} \sum_{n=2}^{\infty} n^{-\sigma}.$$

This uses unitarity of the twist and the fact that at most $[E:\mathbf{Q}]$ primes lie above a rational prime. Exponentiating the locally uniformly convergent logarithm gives the claim.

At $s = t/2$ the remaining arguments are exactly $t, t-1, \dots, 1$. The ratio of two distinct indexed characters has nonzero angular exponent at the specified embedding. Lemma 34.1 applies to each ratio, including at the argument one and with the finite Euler factors in S removed. All these finitely many functions are holomorphic and nonzero at their indicated arguments. Since $t/2 > (r+1)/2$, their product with the bottom factor gives the required continuation and value. For $j = 0$ the product over i is empty. □

Proof of 33.60

Proof. Fix the levels away from T. The two type primes split completely, and $g_0 \ge (kb)^8$ gives $g_0 - 1 > 2fk$. The stabilizer argument of Theorem 37.21 already applies at this level. Indeed, a nontrivial finite stabilizer would contain an element of order g_0, whereas its rational representation has dimension $2fk < g_0 - 1$. Thus each class contributes a whole copy of its coefficient space. The product of the number of these classes and the coefficient dimension is precisely the mass, type indices and Weyl product defining $d_u^\star$ in Lemma 33.31.

Over each class the new level specifies one nonzero vector of $\mathbf{F}_T^k$ at each of the f split places. The absence of a stabilizer gives $(T^k-1)^f$ distinct choices. Coordinatewise scalar multiplication acts freely on these choices. The diamond translations act on the corresponding coefficient fibres by isomorphisms. On each orbit choose a basis over one point and transport it by translation. Freeness makes this a well-defined basis over the whole orbit. Taking one copy of the orbit for each basis vector gives the asserted function model. The level norm at a place is the sum of its diamond translations, which identifies its kernel in this model.

The level window gives $T_* \le T \le (1 + 1/(fk))T_*$. Since $T \ge 2$ and $f \le k$, we have $T^k \ge 2^k \ge 2f$. Consequently

$$(T^k-1)^f = T^{fk}(1-T^{-k})^f \ge T^{fk}(1 - fT^{-k}) \ge \tfrac{1}{2} T_*^{fk}.$$

The upper bound is $(T^k-1)^f \le T^{fk} \le \exp(1) T_*^{fk}$. Multiply by $d_u^\star$ and use the definition of T_*. □

Proof of 33.61

Proof. Throughout the proof $C' m^5 f L \le k$. In particular $a_2 \le k$, since $m^2 + 2m^4 \le 3m^5 fL$. Also $1 + \log_2(2f) \le m$ by $\log f \le \sqrt{r}$ and $m = 2r + 2 \ge 40$. To bound the ceiling defining q, put $t = r + 1 = m/2$. We have $t \ge 1$, $k \ge 2t + 3$. The contribution before the ceiling that contains c is at most

$$4ct\{k + (k+2t)2t + 2t + (t-1)^2\} \le 36ckt^2.$$

The remaining contribution including the ceiling is at most $tk/2 + 12t^3 + 1 \le 12kt^2$. This gives $q \le (9c+3)km^2$.

Put $P = b^{Am^3 f}$. Comparing the cutoffs in each of the $f(r+1)$ factors of S_c and S_{C_h} gives $h_Z \le P S_Z^4$. For a finite set of occurring labels we claim

$$\sum_Z S_Z^{-12(r+1)^2} \le (41^f P^3)^{(r+1)^2}.$$

Indeed, in a single entry lattice the number of vectors with $|y_\tau| \le R_\tau$, $R_\tau \ge 1$, is at most $9^f \prod_\tau R_\tau^2$. The products of open disks of radius $1/2$ about such vectors are disjoint by the product lower bound for

differences. Compare their volumes with the product of disks of radii $R_\tau+1/2$. Integrating this count against $\prod_\tau 3R_\tau^{-4}\,dR_\tau$ bounds the sum of $\prod_\tau \max(1,|y_\tau|)^{-3}$ by 27^f. Each normalized entry has product height at most PS_Z^4. Multiply the resulting inequalities over the $(r+1)^2$ entries and enlarge the set of entry tuples to the product of its coordinate projections. The entry tuple determines Z, so $27^f \le 41^f$ proves the claim.
Lean: `Formalism.MainNotions.BridgeLemma₂.rows_sizeSum` (`Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsSiz`, lines 129–179)

Fix s, i and two labels. Apply Lemma 33.100 with $d=i$ in that lemma, using the stated count and the two amplitude inequalities. The factors $b_i(Z)^k b_i(Z')^k$ cancel. Dividing by $\sqrt{n_Z n_{Z'}}$ cancels $(\sqrt{G_Z G_{Z'}}/N)^k$ and leaves $(kv/(\exp(1)\pi))^{kf}$. The two square roots contribute one factor ΓF_i. For the endpoint masses use

$$\Xi_s(h_Z h_{Z'}) \le \Xi_s(P^2)(S_Z S_{Z'})^{4ck}.$$

This also holds at interior ranks, where the mass is independent of its argument. After bounding $h_Z, h_{Z'}$, the remaining exponent on each size is at most

$$4ck + 4e_s + 4cr^2 + a_i/2 - q/(r+1) \le -12(r+1)^2.$$

Apply the label sum once for each label. Per unit of d, the resulting bound is

$$(2/g)^{(s-i)kf}\left(\frac{kv}{\exp(1)\pi}\right)^{kf}\Xi_s(P^2)(csf)^{6sf}\Gamma F_i P^{2e_s+2cr^2}(41^f P^3)^{2(r+1)^2}.$$

Write $H = b^{Cm^2 f}$, and put $\Xi_s = M_s M_{k-s}/M_k$ for $s \le k-2$, $\Xi_s = (Ck)^{kf}(H^{Cm}N)^{Ck}$ at the endpoints. Since $2A \le C^2$, we have $P^2 \le H^{Cm}$ and $\Xi_s(P^2) \le \Xi_s$. Also

$$2e_s + 2cr^2 + 6(r+1)^2 \le (6c+2)(s+m)m.$$

Since $b \ge 2$, the factors $2\cdot 41^{2f(r+1)^2}$ cost at most $b^{1+12f(r+1)^2}$. The choice $C^2 \ge 4 + A(6c+2)$ thus absorbs these factors and all the powers of P in $H^{C(s+m)m^2}$. Define

$$F_s(C,\Gamma) = (kv)^{kf}\Xi_s C^{kf} H^{C(s+m)m^2}(Csf)^{6sf}\Gamma.$$

The two finite sums of corank s, including the factor d, are consequently bounded by

$$dR\,(2/g)^{(s-1)kf}F_s(C,\Gamma),$$
$$d\,(2/g)^{(s-2)kf}F_s(C,\Gamma)e^{Cm^6 fL}(C/k)^{kf}.$$

In particular the degenerate part has one factor of d, including at $s=2$, where its power of $2/g$ is zero.

We give the numerical summation. The inequalities $8C_\Gamma \le C'$ and $C'm^5 fL \le k$ give $\Gamma \le e^{k/8}$. Hence $C^{kf}\Gamma \le (\exp(1)C)^{kf}$. Set $D = \exp(1)C$. Increasing C to D in the other positive factors and evaluating with $\Gamma = 1$ gives an upper bound for either row. We have $k \ge 16D^{32}m^5 fL$, $1 \le v \le b$, $1 \le N \le b^{Dmf}$, and $g \ge (kb)^8$.

For $2 \le s \le k-2$ we need

$$(kv)^{kf}\Xi_s \le b^{D^2m^2 f}D^{skf}.$$

To see this, put $R_j = M_j/M_{j-1}$ and expand $\Xi_s = M_2\prod_{j=3}^{s}R_j/\prod_{j=k-s+1}^{k}R_j$ by Theorem 37.5. Pair each numerator index j with $j+k-s-2$. All paired factorial quotients are at most one. The two remaining denominator factorials have product $(k-2)!(k-1)! \ge (k/\exp(1))^k$ for $k \ge 6$, by integration of $\log x$. The net power of π/v, before raising to f, is $2k-1+(s-2)(k-s-2)$, between k and $(s+1)k$. The Euler factors and their reciprocals are at most 2^f at each index by the Euler product estimate in Lemma 33.11. Thus the factors apart from $M_2(2v)^f$ are absorbed in $D^{skf}(kv)^{-kf}$. The rank-two bound in the same lemma gives $M_2 \le (20f(2v)^2)^{16f}$. Since $2v \le b$ and $\log f \le \sqrt{r}$, this gives $M_2(2v)^f \le b^{D^2m^2 f}$.

Put $h = \log k$ and $\ell = \log g$. For the interior ranks this mass bound gives

$$\frac{\log F_s(D,1)}{kf} \le (s+1)\log D + \frac{D^2[m^2+(s+m)m^4]L}{k} + \frac{6s}{k}\log(Dsf) \le \frac{sh}{8}.$$

Indeed $\log D \le h/32$, the middle term is at most $(s+2)/16$, and the last is at most $sh/32$ since $s, f \le k$ and $k \ge 400$. For $s = k-1, k$, the contribution of $(kv)^{kf}\Xi_s D^{kf}$ to the left side is at most

$$2h + 2\log D + (1 + D^3m^3 + D^2m)L.$$

Here we used the endpoint formula and $N \le b^{Dmf}$. The last term is at most $k/64$ by the rank threshold. Since $s \ge k-1$, these three terms together are at most $sh/16$. The remaining height and counting terms $D^2(s+m)m^4L/k$ and $6s\log(Dsf)/k$ are each at most $sh/32$. This proves the same bound at both endpoints.

Now $\ell \ge 8h$ and $Dm^6L/k \le 1/16$, since $m \le f$. For $s \ge 2$,

$$-(s-1)(\ell - \log 2) + sh/8 \le -\ell/2 - h/(kf),$$
$$-(s-2)(\ell - \log 2) + sh/8 + Dm^6L/k + \log D - h \le -h/2 - h/(kf).$$

Both left sides decrease with s, so it suffices to substitute $s = 2$. The two corank sums therefore cost at most $k^{-1}g^{-kf/2}$ and $k^{-1}e^{-B/2}$ per rank, apart from dR and d, respectively. There are fewer than k ranks. Summing and taking the supremum over finite subsets proves the assertion. □

Proof of 33.62

Proof. Absolute summability gives $\sum_{t,t'} |\langle v_t, v_{t'}\rangle| \le (\sum_t \|v_t\|)^2 < \infty$. Hence $P(U)$ and $|P|(U)$ are finite for every U, and $|P(U)| \le |P|(U)$.

Fix $E_1 \in \{E_{\mathrm{gen}}, E_{\mathrm{ng}}\}$, and let $\mathcal{A}(E_1)$ be the set of same-row ordered pairs in E_1. Expand the squared norms as absolutely convergent double series. Then

$$\sum_i \|w_i(E_1)\|^2 = \sum_i \sum_{\substack{t,t' \in E_1 \\ \iota(\mathrm{row}(t)) = \iota(\mathrm{row}(t')) = i}} \langle v_t, v_{t'}\rangle = \sum_{(t,t') \in \mathcal{A}(E_1)} \langle v_t, v_{t'}\rangle.$$

The second equality uses (iii). The left side is real, so the right side equals its real part. Every term of E_1 has nonzero Gram invariant. By (iv), $\mathcal{A}(E_1)$ is therefore the disjoint union of its pairs of corank one and its pairs of corank at least two. The latter pairs form $\mathcal{B} \cap (E_1 \times E_1)$. Write $\mathcal{U}(E_1)$ and $c(E_1)$ for $\mathcal{U}_{\mathrm{gen}}, c_{\mathrm{gen}}$ or $\mathcal{U}_{\mathrm{ng}}, c_{\mathrm{ng}}$. By (ii),

$$\|\operatorname{part}(E_1)\|_{\mathrm{HS}}^2 \le P_g\big(P(\mathcal{B} \cap (E_1 \times E_1)) + P(\mathcal{U}(E_1))\big) \le P_g\big(|P|(\mathcal{B} \cap (E_1 \times E_1)) + c(E_1)\big).$$

The sets E_{gen} and E_{ng} are disjoint. So the two sets $\mathcal{B} \cap (E_1 \times E_1)$ are disjoint subsets of $\mathcal{B}$. Since the summands of $|P|$ are nonnegative, their two values of $|P|$ add up to at most $|P|(\mathcal{B}) \le c_{\mathrm{col}}$. The inequality $\|x+y\|^2 \le 2\|x\|^2 + 2\|y\|^2$, applied to $X = \operatorname{part}(E_{\mathrm{gen}}) + \operatorname{part}(E_{\mathrm{ng}})$, proves the first assertion.

For the numerical part, use $1 + x \le e^x$, $f \le k$, $k \ge 1$ and $b \ge 2$:

$$\left(1 + \frac{3}{g_j}\right)^f \le \exp\frac{3f}{g_j} \le \exp\frac{3k}{k^8 b^8} \le \exp\frac{3}{256} \qquad (j = 0, 1).$$

Thus $P_g \le e^{3/128} < 6/5$. Put $C_* = \max(C_{\mathrm{gen}}, C_{\mathrm{ng}})$, so $C_1 = C_* + 1 \ge 1$. Since $Y \ge 1$, we have $2 \le e^Y$, and hence

$$e^{C_{\mathrm{gen}}Y} + e^{C_{\mathrm{ng}}Y} \le 2e^{C_*Y} \le e^{(C_*+1)Y} = e^a.$$

Also $(g_0 g_1)^{-kf/2} \le 1$. Substituting these bounds and $2P_g \le 12/5$ into the first assertion gives the displayed estimate. □

Proof of 33.63

Proof. The inner product in the Hilbert direct sum is the sum of the inner products in its coordinates $\rho \in \mathcal{R}_u$. The supports of v_t and $v_{t'}$ are contained in $\{\rho_t\}$ and $\{\rho_{t'}\}$, respectively. If $\rho_t \ne \rho_{t'}$, at least one of the two vectors vanishes in each coordinate, so every coordinate inner product is zero and $\langle v_t, v_{t'}\rangle = 0$. If $\mathrm{row}(t) \ne \mathrm{row}(t')$ and $\iota(\mathrm{row}(t)) = \iota(\mathrm{row}(t'))$, then $\iota_t = \iota_{t'}$ and hence $\rho_t \ne \rho_{t'}$, which gives $\langle v_t, v_{t'}\rangle = 0$. □

Proof of 33.64

Proof. For $t \in E_{\mathrm{gen}}$ its class is $c(t) = \{t' \in E_{\mathrm{gen}} : \mathrm{row}(t') = \mathrm{row}(t),\ s(t,t') = 1\}$. If $t \sim t'$, symmetry and transitivity show $c(t) = c(t')$. Conversely, reflexivity gives $t' \in c(t')$, so equality of the classes implies $t \sim t'$. Hence $\mathcal{U}_{\mathrm{gen}} = \coprod_{c \in \mathcal{C}} (c \times c)$. The Cauchy–Schwarz inequality for each inner product gives

$$\sum_{t,t'} |\langle v_t, v_{t'} \rangle| \le \Big(\sum_{t} \|v_t\| \Big)^2 < \infty.$$

We may therefore regroup the signed pair sum by classes. For each class, expanding the squared norm gives $\sum_{t,t' \in c} \mathrm{Re}\langle v_t, v_{t'} \rangle = \| \sum_{t \in c} v_t \|^2$. The subdivisions by point and normal labels preserve absolute summability, and their subtotals sum to $\sum_{t \in c} v_t$. This proves the first identity.

If $J_2(\gamma_0) > 0$, the second identity is the definition of a_2. If $J_2(\gamma_0) = 0$, every term in the defining subtotal vanishes by hypothesis, so both sides are zero. The definition also gives $a_2 \ge 0$. Every nonzero term belongs to its unique class and carries its assigned normal label; conversely, a nonzero term in such a fibre represents its point label. This proves the assertion about $\mathcal{P}_*$.

For the finite amplitude estimate, group S_0 by its assigned tuples. The factorization and the fibre bound give

$$\sum_{(c,\nu_0) \in S} a_2(c, \gamma_0, \nu_0) = \sum_{\mathbf{e}} \Big(\sum_{\substack{(c,\nu_0) \in S_0 \\ \mathbf{d}(c,\nu_0) = \mathbf{e}}} \rho_c \Big) \prod_{p \in \mathcal{V}} |a_p(e_p)|^2 \le V_0 \prod_{p \in \mathcal{V}} \Big(\sum_{e_p \in D_p} w_p(e_p) |a_p(e_p)|^2 \Big).$$

Put $q_p = 8 H_p^{c_0 (r+1)^2 + 2m}$ for $p \in \Omega$ and $q_p = 1$ for $p \notin \Omega$. The local assumptions bound the last expression by $V_0 \prod_p H_p^{c_1 (r+1)^2} q_p \prod_p n_p^{\mathrm{gen}}$. For $p \in \Omega$ the inequality $32 \le H_p^5$ gives $H_p^{c_1 (r+1)^2} q_p \le H_p^{e_7}$, and for $p \notin \Omega$ the same inequality follows from $H_p \ge 1$. Multiplying over p gives $\prod_p H_p^{c_1 (r+1)^2} q_p \le H_{\mathrm{tot}}^{e_7}$, which proves the bound since $M_{\mathrm{gen}} \ge 0$. □

Proof of 33.65

Proof. At each $v \in \mathcal{V}_{\mathrm{typ}}$, let $U_v \le \mathrm{GL}_k(\mathbf{F}_{q_v})$ be the upper unitriangular subgroup. For every $y \in \mathrm{GL}_k(\mathbf{F}_{q_v})$, the conjugate $y \cdot 1 \cdot y^{-1} = 1$ lies in U_v, where the Whittaker character takes the value 1. The local induced factor of Lemma 33.112 at 1 is therefore $|\mathrm{GL}_k(\mathbf{F}_{q_v})|/|U_v|$. Taking the product over $v \in \mathcal{V}_{\mathrm{typ}}$ gives $\Gamma_1 = |\mathcal{I}|/|\mathcal{U}| = |\mathcal{I}/\mathcal{U}| \ge 1$.

Let $n = \dim_E V$, fix the model's E-basis identifying V with E^n, let A_f be the finite adele ring of F, and embed V coordinatewise into $(E \otimes_F A_f)^n$ via $E \hookrightarrow E \otimes_F A_f$. Let $L_f \subset (E \otimes_F A_f)^n$ be the standard finite-adelic lattice, G_f the finite-adelic unitary group, $K_{\max} \le G_f$ the stabilizer of L_f, $K_2 \le K_{\max}$ the g-adic mirabolic level subgroup, and $x_u \in K_{\max}$ the word shift. The flag F_u has isotropic basis $e_1, \dots, e_r$ spanning X_u, dual isotropic vectors $f_1, \dots, f_r$ satisfying $h_V(e_a, f_b) = \delta_{ab}$ and $h_V(f_a, f_b) = 0$, and final vector $z_u \in X_u^{\perp}$ orthogonal to $f_1, \dots, f_r$, with x_u fixing $e_1, \dots, e_r, f_1, \dots, f_r$. Let $H_{\mathrm{fin}}(u) \le G_f$ be the subgroup fixing the embedded vectors $e_1, \dots, e_r, f_1, \dots, f_r, z_u$, put

$$K_u = \{h \in H_{\mathrm{fin}}(u) : x_u^{-1} h x_u \in K_2\},$$

and let Γ_u be the rational subgroup of $G(F)$ whose finite-adelic projection $\iota_f : \Gamma_u \to G_f$ lands in $H_{\mathrm{fin}}(u)$. Neatness of K_u on the tied model identifies C_u with the direct sum of copies of π_u indexed by the finite class set Cl_u of representatives $h_\rho \in H_{\mathrm{fin}}(u)$ for $\iota_f(\Gamma_u) \backslash H_{\mathrm{fin}}(u) / K_u$, so $d = |\mathrm{Cl}_u| n_\pi$. Partition $\mathrm{Cl}_u = \coprod_{\Lambda \in \mathcal{R}_0(u)} \mathrm{Cl}_u(\Lambda)$ according to the assigned representative lattice $\Lambda_\rho \in \mathcal{R}_0(u)$. For $h \in G_f$, set

$$\mathcal{L}_u(h) = \{y \in V : (h x_u)^{-1} y \in L_f\},$$

so that $\mathcal{L}_u(h_\rho) = \Lambda_\rho$ for $\rho \in \mathrm{Cl}_u$. If $\kappa \in K_u$, then $(x_u^{-1} \kappa x_u)^{-1} \in K_2 \subseteq K_{\max}$ preserves L_f; writing $(h\kappa x_u)^{-1} = (x_u^{-1} \kappa x_u)^{-1} (h x_u)^{-1}$ gives $\mathcal{L}_u(h\kappa) = \mathcal{L}_u(h)$. If $\gamma \in \Gamma_u$, its rational matrix $M \in \mathrm{GL}_n(E)$ preserves the Hermitian Gram matrix and has finite-adelic image $\iota_f(\gamma)$. Because $\iota_f(\gamma) \in H_{\mathrm{fin}}(u)$ fixes the embedded coordinates of e_a, f_a, z_u and $E \to E \otimes_F A_f$ is injective, M defines an E-linear Hermitian isometry $g_\gamma \in \mathrm{Stab}(F_u)$ fixing e_a, f_a, z_u and satisfying $((\iota_f(\gamma) h x_u)^{-1}) y = (h x_u)^{-1} (g_\gamma^{-1} y)$, whence $\mathcal{L}_u(\iota_f(\gamma) h) =$

$g_\gamma(\mathcal{L}_u(h))$. Whenever $\mathcal{L}_u(h)=\Lambda\in\mathcal{R}_0(u)$ and $h=\iota_f(\gamma)h_\rho\kappa$ with $\gamma\in\Gamma_u$, $\rho\in\mathrm{Cl}_u$, and $\kappa\in K_u$, it follows that $\Lambda=g_\gamma(\Lambda_\rho)$, and uniqueness of orbit representatives in $\mathcal{R}_0(u)$ forces $\Lambda_\rho=\Lambda$, so $\rho\in\mathrm{Cl}_u(\Lambda)$.

Fix $\Lambda\in\mathcal{R}_0(u)$ with $\mathrm{Aut}_{F_u}(\Lambda)$ finite, and let $A_\Lambda\subseteq\iota_f(\Gamma_u)$ be the set of $\eta=\iota_f(\gamma)$ for which there exist $h,h'\in H_{\mathrm{fin}}(u)$ with $\mathcal{L}_u(h)=\mathcal{L}_u(h')=\Lambda$ and $\kappa\in K_u$ satisfying $h'=\eta h\kappa$. For $\eta=\iota_f(\gamma)\in A_\Lambda$, the invariance formulas give $\Lambda=\mathcal{L}_u(h')=g_\gamma(\mathcal{L}_u(h))=g_\gamma(\Lambda)$. Because $E\to E\otimes_F A_f$ is injective, $\eta=\iota_f(\gamma)$ uniquely determines M and hence g_γ. Since g_γ is an isometry fixing $e_1,\dots,e_r$, it fixes X_u pointwise and preserves $X_u^\perp$; passing to $X_u^\perp/X_u$ gives a well-defined map $\varphi(\eta)\in\mathrm{Aut}_{F_u}(\Lambda)$. Suppose $\varphi(\eta_1)=\varphi(\eta_2)$, and write $g_i=g_{\gamma_i}$. For $x\in X_u^\perp$, the difference $g_1(x)-g_2(x)$ lies in X_u, so $g_1(x)-g_2(x)=\sum_{a=1}^r c_a e_a$ with $c_a\in E$. Because g_1,g_2 are isometries fixing every f_b and $h_V(f_b,e_a)=\delta_{ab}$,

$$c_b=h_V\big(f_b,g_1(x)-g_2(x)\big)=h_V(g_1(f_b),g_1(x))-h_V(g_2(f_b),g_2(x))=0$$

for $1\le b\le r$. Thus $g_1=g_2$ on $X_u^\perp$. Every $v\in V$ decomposes as $\big(v-\sum_{a=1}^r h_V(e_a,v)f_a\big)+\sum_{a=1}^r h_V(e_a,v)f_a$, whose first summand pairs to zero with every e_b and therefore lies in $X_u^\perp$. Since g_1,g_2 agree on $X_u^\perp$ and fix each f_a, we obtain $g_1=g_2$ on V; hence their rational matrices coincide, so their finite-adelic images satisfy $\eta_1=\eta_2$. Hence $|A_\Lambda|\le|\mathrm{Aut}_{F_u}(\Lambda)|$.

Set $c_u(s)=x_u^{-1}sx_u$ for $s\in H_{\mathrm{fin}}(u)$, and put $H_{\mathrm{int}}(u)=\{s\in H_{\mathrm{fin}}(u):c_u(s)\in K_{\max}\}$. Every $c_u(s)$ with $s\in H_{\mathrm{int}}(u)$ preserves L_f and fixes the finite set W_u consisting of the embedded vectors $e_1,\dots,e_r,f_1,\dots,f_r$ and $x_u^{-1}z_u$. By the parabolic block structure modulo g, the g-adic Levi reduction $\bar\ell(s)=\mathrm{levi}(c_u(s))\in\mathcal{I}$ is a group homomorphism on $H_{\mathrm{int}}(u)$ that maps K_u into $\mathcal{U}$. To see that $\bar\ell$ maps $H_{\mathrm{int}}(u)$ onto $\mathcal{I}$, let $e\in A_f$ be the characteristic idempotent of $\mathcal{V}_{\mathrm{typ}}$, let $\mathcal{O}_{E,f}$ be the integral subring of $E\otimes_F A_f$ with CM involution σ_A, and let $e_w\in\mathcal{O}_{E,f}$ be the model's split-place idempotent above g, with $e_w^2=e_w$ and $e_w\sigma_A(e_w)=0$. Put $\varepsilon=(1\otimes e)e_w\in\mathcal{O}_{E,f}$, so $\varepsilon^2=\varepsilon$ and $\varepsilon\sigma_A(\varepsilon)=0$. The model supplies vectors $\beta_1,\dots,\beta_k$ on e_w and $\beta_1^*,\dots,\beta_k^*$ on $\sigma_A(e_w)$ in L_f, orthogonal modulo g to W_u and satisfying $h_A(\beta_i^*,\beta_j)\equiv\delta_{ij}e_w\pmod g$ for the adelic Hermitian form h_A. Since e_a,f_a form hyperbolic pairs and $h_A(x_u^{-1}z_u,x_u^{-1}z_u)=-T_u$ is a v-adic unit at every $v\in\mathcal{V}_{\mathrm{typ}}$ by $(T_u,g)=1$, orthogonal projection away from W_u on $\mathcal{V}_{\mathrm{typ}}$ replaces $\varepsilon\beta_j$ and $\sigma_A(\varepsilon)\beta_l^*$ by vectors $b_j,c_l\in L_f$ on $\varepsilon,\sigma_A(\varepsilon)$ that are strictly orthogonal to W_u and satisfy $G_{lj}=h_A(c_l,b_j)\equiv\delta_{lj}\varepsilon\pmod g$. On $\varepsilon\mathcal{O}_{E,f}$, elements of $1+g\mathcal{O}_{E,f}$ have g-adic unit norms and are invertible, so $\varepsilon\det G$ has an inverse εy with $y\in\mathcal{O}_{E,f}$. Setting $M=(\varepsilon y)\,\mathrm{adj}(G)$ and $b_i^*=\sum_{l=1}^k\sigma_A(M_{il})c_l\in L_f$ gives dual isotropic tuples $b_1,\dots,b_k$ on ε and $b_1^*,\dots,b_k^*$ on $\sigma_A(\varepsilon)$, orthogonal to W_u, with $h_A(b_i^*,b_j)=\delta_{ij}\varepsilon$ and modulo-g pairings $\delta_{ij}e_w$ against β_j^*,β_i. Given $A\in\mathcal{I}$, lift A and A^{-1} to $M_k(\varepsilon\mathcal{O}_{E,f})$ and invert their product on $\varepsilon\mathcal{O}_{E,f}$ to obtain $C_0,C_0'\in M_k(\varepsilon\mathcal{O}_{E,f})$ with $C_0'C_0=C_0C_0'=\varepsilon I_k$. At each $v\in\mathcal{V}_{\mathrm{typ}}$ their reductions are A_v and A_v^{-1}, respectively. Set $C=C_0+(1-\varepsilon)I_k$ and $C'=C_0'+(1-\varepsilon)I_k$, so $C'C=CC'=I_k$. Writing $P_{x,y}(N)v=\sum_{i,j=1}^k x_iN_{ij}h_A(y_j,v)$, the endomorphism

$$s'=I_n+P_{b,b^*}(C-I_k)+P_{b^*,b}\big(\sigma_A(C')^T-I_k\big)$$

preserves h_A and L_f, with inverse obtained by interchanging C and C'; it also fixes W_u pointwise and has Levi reduction $A\in\mathcal{I}$. Thus $s_F=x_us'x_u^{-1}\in H_{\mathrm{int}}(u)$ satisfies $\bar\ell(s_F)=A$.

Choose lifts $a_1,\dots,a_{\Gamma_1}\in H_{\mathrm{int}}(u)$ of the Γ_1 left cosets in $\mathcal{I}/\mathcal{U}$. Fix $\Lambda\in\mathcal{R}_0(u)$. If $\mathrm{Aut}_{F_u}(\Lambda)$ is infinite, $w(\Lambda)=0\le|\mathrm{Cl}_u(\Lambda)|/\Gamma_1$. Assume $\mathrm{Aut}_{F_u}(\Lambda)$ is finite, and let $h_b=h_{\rho_{\mathrm{base}}(\Lambda)}$ be the base representative of Λ, with $\mathcal{L}_u(h_b)=\Lambda$. Put $t_q=h_ba_q\in H_{\mathrm{fin}}(u)$ for $1\le q\le\Gamma_1$. Because $c_u(a_q)\in K_{\max}$ preserves L_f, we have $\mathcal{L}_u(t_q)=\mathcal{L}_u(h_b)=\Lambda$. If $t_p^{-1}t_q=a_p^{-1}a_q\in K_u$, then $\bar\ell(a_p)^{-1}\bar\ell(a_q)=\bar\ell(a_p^{-1}a_q)\in\mathcal{U}$, so $p=q$. Each double coset $\iota_f(\Gamma_u)t_qK_u$ has representative h_ρ with $\rho\in\mathrm{Cl}_u(\Lambda)$; hence at most $|\mathrm{Cl}_u(\Lambda)|$ distinct double cosets occur as q varies. In a single double coset $\iota_f(\Gamma_u)t_{q_0}K_u$, each $t_q=\eta_qt_{q_0}\kappa_q$ ($\eta_q\in\iota_f(\Gamma_u)$, $\kappa_q\in K_u$) satisfies $\eta_q\in A_\Lambda$; if $\eta_q=\eta_{q'}$, then $t_q^{-1}t_{q'}=\kappa_q^{-1}\kappa_{q'}\in K_u$ and $q=q'$. Each fibre therefore has cardinality at most $|A_\Lambda|\le|\mathrm{Aut}_{F_u}(\Lambda)|$, whence $\Gamma_1\le|\mathrm{Aut}_{F_u}(\Lambda)|\,|\mathrm{Cl}_u(\Lambda)|$ and $\Gamma_1w(\Lambda)\le|\mathrm{Cl}_u(\Lambda)|$. Summing over $\Lambda\in\mathcal{R}_0(u)$ and multiplying by n_π gives

$$\Gamma_1\mu_un_\pi=\Big(\sum_{\Lambda\in\mathcal{R}_0(u)}\Gamma_1w(\Lambda)\Big)n_\pi\le\Big(\sum_{\Lambda\in\mathcal{R}_0(u)}|\mathrm{Cl}_u(\Lambda)|\Big)n_\pi=|\mathrm{Cl}_u|n_\pi=d.$$

□

Proof of 33.66

Proof. Let G_∞ be the archimedean unitary group of the model with Haar measure μ_∞, let V_{π_u} be the space of π_u with complex inner product $\langle\cdot,\cdot\rangle$ conjugate-linear in its first variable, and let $W_u : G_\infty \to V_{\pi_u}$ be the continuous Bessel–Whittaker function of u. Let β_t be the rational Hermitian isometry representing t (carrying F_u to F_t'), let $\gamma_{t,\infty} \in G_\infty$ be its archimedean component, and let $\Gamma_{\mathrm{orb}}(t) \le G_\infty$ be the archimedean image of the rational common unipotent lattice stabilizer of t. For $x \in G_\infty$, let $f_t(x)$ be the continuous linear operator $v \mapsto \langle W_u(\gamma_{t,\infty}^{-1}x), v\rangle W_u(x)$ on V_{π_u}. By Lemma 33.63 and the definition of the normalized orbital amplitude, the single nonzero row coordinate of v_t is a scalar multiple of the Hilbert–Schmidt image of the Bochner orbital integral

$$\mathcal{O}_t = \int_{\Omega_t} f_t(x)\, d\mu_\infty(x)$$

over a chosen measurable fundamental domain $\Omega_t \subseteq G_\infty$ for left multiplication by $\Gamma_{\mathrm{orb}}(t)$, with $\mathcal{O}_t = 0$ whenever no such fundamental domain exists or f_t is not Bochner integrable on Ω_t. Assume therefore that Ω_t exists and f_t is integrable on Ω_t.

As a finite product of real unitary groups, G_∞ is unimodular. Let $N_\infty(u) \le G_\infty$ be the closed unipotent radical stabilizing the archimedean image of F_u. On $N_\infty(u)$, W_u transforms on the left by a continuous unitary character $\psi_{\infty,u} : N_\infty(u) \to \mathbf{C}^\times$. Set

$$R_t = \{r \in N_\infty(u) : \gamma_{t,\infty}^{-1} r \gamma_{t,\infty} \in N_\infty(u)\},$$

a closed subgroup of G_∞. On R_t, define the continuous character

$$\chi_t(r) = \psi_{\infty,u}(r)\overline{\psi_{\infty,u}(\gamma_{t,\infty}^{-1} r \gamma_{t,\infty})}.$$

For $r \in R_t$ and $x \in G_\infty$, the equivariance $W_u(rx) = \psi_{\infty,u}(r)W_u(x)$ and $W_u(\gamma_{t,\infty}^{-1}rx) = \psi_{\infty,u}(\gamma_{t,\infty}^{-1}r\gamma_{t,\infty})W_u(\gamma_{t,\infty}^{-1}x)$ gives $f_t(rx) = \chi_t(r)f_t(x)$. The rational unipotent elements defining $\Gamma_{\mathrm{orb}}(t)$ preserve the flag filtrations of both F_u and F_t', so $\Gamma_{\mathrm{orb}}(t)$ is a discrete subgroup of R_t. With $r = \dim_E X_u$ as in Lemma 33.65, filtering the skew-Hermitian intersection Lie algebra $\mathfrak{n}(F_u) \cap \mathfrak{n}(F_t')$ in $2r+1$ graded steps and applying rational lattice approximation and Cayley transforms on each step produces a compact subset of R_t whose left $\Gamma_{\mathrm{orb}}(t)$-translates cover R_t, so $\Gamma_{\mathrm{orb}}(t)\backslash R_t$ is compact. Moreover, for rational elements in $\Gamma_{\mathrm{orb}}(t)$, the product formula for the adelic additive character and level integrality of the finite-adelic unipotent conjugates give $\chi_t|_{\Gamma_{\mathrm{orb}}(t)} = 1$.

Suppose t is not relevant. Choose $Y \in \mathfrak{n}(F_u) \cap \mathfrak{n}(F_t')$ with $\mathrm{Tr}_{E/F}(\ell_{F_u}(Y)) \neq \mathrm{Tr}_{E/F}(\ell_{F_t'}(Y))$. Write $\ell = \ell_{F_u}(Y)$, $\ell' = \ell_{F_t'}(Y)$, and $\delta = (\ell - \ell') + \overline{\ell - \ell'} \in E$. Since $\delta = 0$ would force $\mathrm{Tr}_{E/F}(\ell - \ell') = 0$, we have $\delta \neq 0$, and $\bar\delta = \delta$. Let $N = [F : \mathbf{Q}]$ be the number of infinite places of F, let $g = g_0 g_1$, put $c = g/(2N) \in \mathbf{Q}^\times$, and set $s = c\delta^{-1} \in E^\times$, so $\bar s = s$ and $s\delta = c$. Because Y is nilpotent and skew-Hermitian and $\bar s = s$, the truncated exponential $n = \exp(sY)$ on V is a rational Hermitian isometry in the unipotent stabilizer of F_u, while $\beta_t^{-1} n \beta_t = \exp(s\,\beta_t^{-1} Y \beta_t)$ also lies in the unipotent stabilizer of F_u because $Y \in \mathfrak{n}(F_t')$. Let $r_0 \in G_\infty$ be the archimedean point of n; then $r_0 \in R_t$. In the root pairings defining $\psi_{\infty,u}(r_0)$ and $\psi_{\infty,u}(\gamma_{t,\infty}^{-1} r_0 \gamma_{t,\infty})$, all quadratic and higher powers of Y map the flag steps into lower filtration subspaces orthogonal to the dual isotropic vectors, so only the linear term sY contributes. At each infinite place v of F, the local additive functional $\Phi_v : E \to \mathbf{R}$ satisfies $\Phi_v(q) = q$ for $q \in \mathbf{Q}$, and the local phase exponent is $(2\pi i/g)\Phi_v(s(\ell + \bar\ell))$ for F_u and $(2\pi i/g)\Phi_v(s(\ell' + \bar\ell'))$ for F_t'. Their difference at each v is

$$\frac{2\pi i}{g}\Phi_v\big(s(\ell + \bar\ell - \ell' - \bar\ell')\big) = \frac{2\pi i}{g}\Phi_v(s\delta) = \frac{2\pi i}{g}\cdot\frac{g}{2N} = \frac{\pi i}{N}.$$

Summing over the N infinite places of F gives $\chi_t(r_0) = \exp(\pi i) = -1 \neq 1$.

Unimodularity of G_∞ and the triviality $\chi_t|_{\Gamma_{\mathrm{orb}}(t)} = 1$ allow the integral $\mathcal{O}_t$ over Ω_t to be averaged on the left over the compact quotient $\Gamma_{\mathrm{orb}}(t)\backslash R_t$ against its right-R_t-invariant Haar measure. Because χ_t is continuous on R_t and $\chi_t(r_0) \neq 1$, its integral over $\Gamma_{\mathrm{orb}}(t)\backslash R_t$ vanishes, so $\mathcal{O}_t = 0$ and $v_t = 0$. Specializing to $t \in \mathcal{T}_0 \setminus \mathcal{T}_{\mathrm{rel}}$ gives the displayed assertion. □

Proof of 33.67

Proof. Use the notation $H_{\mathrm{fin}}(u)$, $K_{\max}$, K_2, K_u, x_u, Cl_u, h_ρ, $\mathcal{R}_0(u)$, Λ_ρ, $\rho_{\mathrm{base}}(\Lambda)$, and $w(\Lambda)$ from Lemma 33.65 and its proof. Under the orthogonal decomposition $C_u \simeq \bigoplus_{\rho\in\mathrm{Cl}_u} V_{\pi_u}$, each continuous linear operator A on C_u has matrix blocks $B_{\rho,\rho'}(A) \in \mathrm{End}_{\mathbf{C}}(V_{\pi_u})$ defined by $B_{\rho,\rho'}(A)v = (A\delta_{\rho'}v)_\rho$. Taking the coordinate-singleton orthonormal basis built from an orthonormal basis of V_{π_u} and interchanging finite sums gives the block Parseval identity

$$\langle A, B\rangle_{\mathrm{HS}} = \sum_{\rho,\rho'\in\mathrm{Cl}_u} \big\langle B_{\rho,\rho'}(A), B_{\rho,\rho'}(B)\big\rangle_{\mathrm{HS}}.$$

The level character χ_2 on K_2 factors as $\chi_2(\kappa) = \chi_N(\kappa)\chi_L(\kappa)$, where χ_N is the unipotent root-pairing character modulo g and $\chi_L(\kappa) = \psi_{\mathcal{I}}(\mathrm{levi}(\kappa))$ is the Levi Whittaker phase. If $\kappa, k_1, k_2 \in K_{\max}$ satisfy $k_1^{-1}f_{0,a} \equiv f_{0,a} \pmod g$ and $k_2 v_{0,a} \equiv v_{0,a} \pmod g$ for the adelic dual isotropic and next-step flag vectors $f_{0,a}, v_{0,a} \in L_f$ $(1 \le a \le r)$, then expanding $h_A(f_{0,a}, k_1\kappa k_2 v_{0,a}) = h_A(k_1^{-1}f_{0,a}, \kappa(k_2 v_{0,a}))$ shows that the sum of root pairings differs from that of κ by g times an element of $\mathcal{O}_{E,f}$; dividing by g and applying the finite-adelic additive character, which is trivial on $\mathcal{O}_{E,f}$, gives $\chi_N(k_1\kappa k_2) = \chi_N(\kappa)$. For $\rho \in \mathrm{Cl}_u$, write $\delta(x,y) = (xx_u)^{-1}(yx_u)$ for $x, y \in H_{\mathrm{fin}}(u)$ and put $s_u(\rho) = \mathrm{levi}(\delta(h_{\rho_{\mathrm{base}}(\Lambda_\rho)}, h_\rho)) \in \mathcal{I}$. For a term t with lattice pair $q_t = (\Lambda_t, \Lambda_t') \in \mathcal{R}_0(u)^2$ and normalized Hilbert–Schmidt amplitude a_t, the transition factors between ρ, ρ' and the base representatives of Λ_t, Λ_t' satisfy these modulo-g congruences whenever $\Lambda_\rho = \Lambda_t$ and $\Lambda_{\rho'} = \Lambda_t'$. Thus the χ_N-factor is independent of ρ, ρ' in those fibres, while the relative finite-adelic element $k(\rho, \rho', t)$ lies in K_2 if and only if $(\Lambda_\rho, \Lambda_{\rho'}) = q_t$ and $\mathrm{levi}(k(\rho,\rho',t)) = s_u(\rho)^{-1}\iota_t s_u(\rho') \in \mathcal{U}$. Consequently,

$$B_{\rho,\rho'}(O_t) = \begin{cases} \psi_{\mathcal{I}}\big(s_u(\rho)^{-1}\iota_t s_u(\rho')\big)\, a_t, & (\Lambda_\rho, \Lambda_{\rho'}) = q_t, \\ 0, & (\Lambda_\rho, \Lambda_{\rho'}) \ne q_t, \end{cases}$$

and the q_t-coordinate of v_t is $w_{q_t} a_t$ with $w_{(\Lambda,\Lambda')} = \sqrt{w(\Lambda)w(\Lambda')}$.

For $\Lambda \in \mathcal{R}_0(u)$ with base representative $h_b = h_{\rho_{\mathrm{base}}(\Lambda)}$, define

$$A_u(\Lambda) = \big\{\mathrm{levi}(\delta(h_b, \iota_f(\gamma)h_b)) : \gamma \in \Gamma_u,\ \delta(h_b, \iota_f(\gamma)h_b) \in K_{\max}\big\} \subseteq \mathcal{I}.$$

We claim that the multiplication map

$$\Phi_\Lambda : A_u(\Lambda) \times \mathrm{Cl}_u(\Lambda) \times \mathcal{U} \longrightarrow \mathcal{I}, \qquad (a, \rho, w) \longmapsto a\, s_u(\rho)\, w,$$

is bijective and that $w(\Lambda) = |A_u(\Lambda)|^{-1}$. Suppose $a_1 s_u(\rho_1) w_1 = a_2 s_u(\rho_2) w_2$, with $a_m = \mathrm{levi}(\delta(h_b, \iota_f(\gamma_m)h_b))$ $(m = 1, 2)$. Put $p_m = \iota_f(\gamma_m)h_{\rho_m} \in H_{\mathrm{fin}}(u)$. Since $\mathcal{L}_u(h_{\rho_m}) = \Lambda = \mathcal{L}_u(h_b)$, we have $\delta(h_b, h_{\rho_m}) \in K_{\max}$, and hence $\delta(h_b, p_m) = \delta(h_b, \iota_f(\gamma_m)h_b)\delta(h_b, h_{\rho_m}) \in K_{\max}$. Multiplicativity of levi gives $\mathrm{levi}(\delta(h_b, p_m)) = a_m s_u(\rho_m)$, so $\delta(p_1, p_2) = \delta(h_b, p_1)^{-1}\delta(h_b, p_2) \in K_{\max}$ has Levi image $w_1 w_2^{-1} \in \mathcal{U}$. An element of $K_{\max}$ satisfying the word-level flag congruences whose Levi reduction lies in $\mathcal{U}$ belongs to K_2, so $p_1^{-1}p_2 \in K_u$. Thus $h_{\rho_2} = \iota_f(\gamma_2^{-1}\gamma_1)h_{\rho_1}(p_1^{-1}p_2)$; uniqueness of double-coset representatives in Cl_u gives $\rho_1 = \rho_2$, and neatness of K_u on $h_{\rho_1}^{-1}\iota_f(\gamma_1^{-1}\gamma_2)h_{\rho_1} = p_1^{-1}p_2 \in K_u$ gives $\gamma_1 = \gamma_2$, whence $a_1 = a_2$ and $w_1 = w_2$. Conversely, given $s \in \mathcal{I}$, the split-frame lift in the proof of Lemma 33.65 gives $s_F \in H_{\mathrm{int}}(u)$ with $c_u(s_F) \in K_{\max}$ and $\mathrm{levi}(c_u(s_F)) = s$. Put $h = h_b s_F \in H_{\mathrm{fin}}(u)$, so $\delta(h_b, h) = c_u(s_F) \in K_{\max}$, $\delta(h, h_b) = c_u(s_F)^{-1} \in K_{\max}$, and $\mathrm{levi}(\delta(h_b, h)) = s$. Decompose $h = \iota_f(\gamma)h_\rho\kappa$ $(\gamma \in \Gamma_u,\ \rho \in \mathrm{Cl}_u,\ \kappa \in K_u)$. Since $\delta(h_b, h) \in K_{\max}$, we have $\mathcal{L}_u(h) = \mathcal{L}_u(h_b) = \Lambda$, which as in Lemma 33.65 forces $\Lambda_\rho = \Lambda$ and $\rho \in \mathrm{Cl}_u(\Lambda)$. Then $\delta(h_b, \iota_f(\gamma)h_b) = \delta(h_b, h)\,\delta(1,\kappa)^{-1}\delta(h_\rho, h_b) \in K_{\max}$ defines $a \in A_u(\Lambda)$, while $w = \mathrm{levi}(\delta(1,\kappa)) \in \mathcal{U}$ satisfies $a\, s_u(\rho)\, w = \mathrm{levi}(\delta(h_b, h)) = s$. Thus Φ_Λ is bijective and $|A_u(\Lambda)| \ge 1$. Set $\Gamma_u(\Lambda) = \{\gamma \in \Gamma_u : \delta(h_b, \iota_f(\gamma)h_b) \in K_{\max}\} = \{\gamma \in \Gamma_u : g_\gamma(\Lambda) = \Lambda\}$. The injectivity argument above with $\rho_1 = \rho_2 = \rho_{\mathrm{base}}(\Lambda)$ shows via neatness of K_u that $\gamma \mapsto \mathrm{levi}(\delta(h_b, \iota_f(\gamma)h_b))$ is a bijection $\Gamma_u(\Lambda) \xrightarrow{\sim} A_u(\Lambda)$. Moreover, the map $\gamma \mapsto \varphi(\iota_f(\gamma))$ from $\Gamma_u(\Lambda)$ to $\mathrm{Aut}_{F_u}(\Lambda)$ is injective by the proof of Lemma 33.65, and it is surjective: the standard lattice L_f splits orthogonally across $(X_u \oplus Y_u) \oplus (X_u \oplus Y_u)^\perp$ (where $Y_u = \mathrm{span}(f_1, \dots, f_r)$), and $h_b \in H_{\mathrm{fin}}(u)$ fixes $X_u \oplus Y_u$ while $x_u \in K_{\max}$ preserves L_f, so $\Lambda = \mathcal{L}_u(h_b)$ splits orthogonally as $(\Lambda \cap (X_u \oplus Y_u)) \oplus (\Lambda \cap (X_u \oplus Y_u)^\perp)$; hence any $g \in \mathrm{Stab}(F_u)$ preserving Λ (which fixes $e_1, \dots, e_r$ and $z_u \bmod X_u$) has a block-diagonal lift $\tilde{g} \in \Gamma_u(\Lambda)$ fixing e_a, f_a, z_u and inducing the same automorphism on $X_u^\perp/X_u$. Therefore $\mathrm{Aut}_{F_u}(\Lambda)$ is finite of cardinality $|A_u(\Lambda)|$, and $w(\Lambda) = |A_u(\Lambda)|^{-1} > 0$.

Fix $S \in \{E_{\mathrm{gen}}, E_{\mathrm{ng}}, \mathcal{T}_0\}$. Because $\sum_t \|v_t\| < \infty$ and $w_q > 0$ for all $q \in \mathcal{R}_0(u)^2$, the subseries

$$Z_q(i) = \sum_{\substack{t \in S \\ q_t = q,\ \iota_t = i}} a_t \qquad \big(q \in \mathcal{R}_0(u)^2,\ i \in \mathcal{I}\big)$$

converge absolutely in Hilbert–Schmidt norm. Put $G_q(i,j) = \langle Z_q(i), Z_q(j)\rangle_{\mathrm{HS}}$. Both the nonzero-Gram condition and the generic condition are invariant under rational Hermitian isometries of V, and the identity cell is preserved by left and right multiplication by elements of $\Gamma_u \le \mathrm{Stab}(F_u)$, so each of $E_{\mathrm{gen}}, E_{\mathrm{ng}}, \mathcal{T}_0$ is preserved by rational row-stabilizer transport on either side. By the χ_N-invariance above and independence of the normalized orbital amplitudes from choices of representatives and fundamental domains, left action by $b \in A_u(\Lambda)$ at $q = (\Lambda, \Lambda')$ gives a bijection φ_L of $\{t \in S : q_t = q\}$ onto itself and a unitary operator U_L on Hilbert–Schmidt space such that $\iota_{\varphi_L(t)} = b\,\iota_t$ and $a_{\varphi_L(t)} = U_L(a_t)$. Right action by $b \in A_u(\Lambda')$ similarly gives a bijection φ_R of $\{t \in S : q_t = q\}$ onto itself and a unitary operator U_R such that $\iota_{\varphi_R(t)} = \iota_t b^{-1}$ and $a_{\varphi_R(t)} = U_R(a_t)$. Reindexing Z_q along φ_L^{-1} and φ_R^{-1} gives $Z_q(bi) = U_L(Z_q(i))$ and $Z_q(ib^{-1}) = U_R(Z_q(i))$, and unitarity of U_L, U_R yields

$$G_q(bi, bj) = G_q(i,j) \quad \big(b \in A_u(\Lambda)\big), \qquad G_q(ib^{-1}, jb^{-1}) = G_q(i,j) \quad \big(b \in A_u(\Lambda')\big).$$

Since $\sum_{t \in S} \|O_t\|_{\mathrm{HS}} < \infty$ and $B_{\rho,\rho'}$ is continuous and linear,

$$B_{\rho,\rho'}\big(\mathrm{part}(S)\big) = \sum_{i \in \mathcal{I}} \psi_{\mathcal{I}}\big(s_u(\rho)^{-1} i\, s_u(\rho')\big)\, Z_{(\Lambda_\rho, \Lambda_{\rho'})}(i).$$

For $q = (\Lambda, \Lambda') \in \mathcal{R}_0(u)^2$ and $x, y \in \mathcal{I}$, put

$$F_q(x,y) = \sum_{i,j \in \mathcal{I}} \overline{\psi_{\mathcal{I}}(x^{-1}iy)}\, \psi_{\mathcal{I}}(x^{-1}jy)\, G_q(i,j),$$

so $\|\mathrm{part}(S)\|_{\mathrm{HS}}^2 = \mathrm{Re} \sum_{\rho,\rho' \in \mathrm{Cl}_u} F_{(\Lambda_\rho, \Lambda_{\rho'})}(s_u(\rho), s_u(\rho'))$. For $b \in A_u(\Lambda)$ and $w \in \mathcal{U}$, the relation $\psi_{\mathcal{I}}(w^{-1}z) = \psi_{\mathcal{I}}(w^{-1})\psi_{\mathcal{I}}(z)$ with $|\psi_{\mathcal{I}}(w^{-1})| = 1$, followed by the bijection $i \mapsto bi$ and $G_q(bi, bj) = G_q(i,j)$, gives $F_q(bxw, y) = F_q(x,y)$. Similarly, for $b' \in A_u(\Lambda')$ and $w' \in \mathcal{U}$, the relation $\psi_{\mathcal{I}}(zw') = \psi_{\mathcal{I}}(z)\psi_{\mathcal{I}}(w')$ with $|\psi_{\mathcal{I}}(w')| = 1$ and the bijection $i \mapsto i(b')^{-1}$ with $G_q(i(b')^{-1}, j(b')^{-1}) = G_q(i,j)$ give $F_q(x, b'yw') = F_q(x,y)$. By the bijections Φ_Λ and $\Phi_{\Lambda'}$, each fibre over $(\rho, \rho') \in \mathrm{Cl}_u(\Lambda) \times \mathrm{Cl}_u(\Lambda')$ has size $|A_u(\Lambda)|\,|\mathcal{U}| \cdot |A_u(\Lambda')|\,|\mathcal{U}|$, whence

$$|A_u(\Lambda)|\,|A_u(\Lambda')|\,|\mathcal{U}|^2 \sum_{\substack{\rho \in \mathrm{Cl}_u(\Lambda) \\ \rho' \in \mathrm{Cl}_u(\Lambda')}} F_q\big(s_u(\rho), s_u(\rho')\big) = \sum_{x,y \in \mathcal{I}} F_q(x,y).$$

For fixed $i, j \in \mathcal{I}$, $\psi_{\mathcal{I}}(x^{-1}jy)$ vanishes unless $u_0 = x^{-1}jy \in \mathcal{U}$, that is, $x = jyu_0^{-1}$ with $u_0 \in \mathcal{U}$. Substituting $x = jyu_0^{-1}$ gives $x^{-1}iy = u_0 y^{-1} j^{-1} iy$ and

$$\sum_{x,y \in \mathcal{I}} \overline{\psi_{\mathcal{I}}(x^{-1}iy)}\, \psi_{\mathcal{I}}(x^{-1}jy) = \sum_{y \in \mathcal{I}} \sum_{u_0 \in \mathcal{U}} \overline{\psi_{\mathcal{I}}(u_0)\psi_{\mathcal{I}}(y^{-1}j^{-1}iy)}\, \psi_{\mathcal{I}}(u_0) = |\mathcal{U}| \sum_{y \in \mathcal{I}} \overline{\psi_{\mathcal{I}}(y^{-1}j^{-1}iy)} = |\mathcal{U}|^2 \mathcal{G}_{ij}.$$

Dividing by $|A_u(\Lambda)|\,|A_u(\Lambda')|\,|\mathcal{U}|^2$ and using $w_q^2 = w(\Lambda)w(\Lambda') = (|A_u(\Lambda)|\,|A_u(\Lambda')|)^{-1}$ gives

$$\sum_{\substack{\rho \in \mathrm{Cl}_u(\Lambda) \\ \rho' \in \mathrm{Cl}_u(\Lambda')}} F_q\big(s_u(\rho), s_u(\rho')\big) = w_q^2 \sum_{i,j \in \mathcal{I}} \mathcal{G}_{ij} G_q(i,j).$$

Projecting the absolutely convergent sum $w_i(S)$ to coordinate q gives $w_i(S)_q = w_q Z_q(i)$, so $\langle w_i(S), w_j(S)\rangle = \sum_{q \in \mathcal{R}_0(u)^2} w_q^2 G_q(i,j)$. Summing the previous identity over $q \in \mathcal{R}_0(u)^2$ and interchanging the finite sums over q and i, j gives $\sum_{\rho,\rho' \in \mathrm{Cl}_u} F_{(\Lambda_\rho, \Lambda_{\rho'})}(s_u(\rho), s_u(\rho')) = \sum_{i,j \in \mathcal{I}} \mathcal{G}_{ij} \langle w_i(S), w_j(S)\rangle$; taking real parts yields

$$\|\mathrm{part}(S)\|_{\mathrm{HS}}^2 = \mathrm{Re} \sum_{i,j \in \mathcal{I}} \mathcal{G}_{ij} \big\langle w_i(S), w_j(S)\big\rangle.$$

Taking $S = \mathcal{T}_0$ gives the final assertion. □

Proof of 33.68

Proof. Put

$$C_c = 21 + 4|C_H|, \qquad K_* = 1 + C + C^2 + C_h^2 + C_c^2,$$

and define

$$M_* = \max\{16, K_*^8\}, \qquad \eta_* = \max\{16K_*, M_* \log M_*\}, \qquad C_{\ell,\mathrm{pair}} = \max\{16, \eta_*\}.$$

These choices precede the choice of the model and the word. We prove the assertion for this threshold.

The normal form established in the proof of Proposition 33.18 gives $1 \le j \le r$ and $\delta_j \neq 0$ for every type (j, δ) represented in $\mathcal{T}_{\mathrm{rel}}$. If $r = 0$, this set is empty, and (33.11) makes every summand zero. Hence assume $r \ge 1$.

We first sum the minor labels. Put

$$q_l = \sum_{a=1}^{l} (r + 1 - a), \qquad H_1 = b^{C_H m}, \qquad \mathfrak{d} = \mathfrak{D}_{F/\mathbf{Q}} \mathcal{O}_E.$$

The integrality of the normalized nonzero minor ideals used in the proof of Proposition 33.18 gives $(\delta_l)\mathfrak{d}^{-2q_l} \subseteq \mathcal{O}_E$ for a nonzero minor with $1 \le l \le j$. Since $q_l \ge 0$ and $\mathfrak{d}$ is integral, it follows that

$$(\delta_l) \subseteq \mathfrak{d}^{2q_l} \subseteq \mathcal{O}_E.$$

Thus every coordinate δ_l is an algebraic integer. The coordinates $\delta_1, \dots, \delta_j$ determine the label, since $\delta_0 = 1$ and $\delta_l = 0$ for $l > j$.

Let τ range over the fixed CM type, which has f embeddings. For distinct $x, y \in \mathcal{O}_E$,

$$\prod_{\tau} |\tau(x - y)|^2 = |N_{E/\mathbf{Q}}(x - y)| \ge 1.$$

The products of open complex disks of radius $1/2$ about the embedded algebraic integers are therefore disjoint. If $R_\tau \ge 1$, the disks about points satisfying $|\tau(x)| \le R_\tau$ lie in the product of disks of radii $R_\tau + 1/2$. Comparing volumes for each finite subset gives

$$\#\{x \in \mathcal{O}_E : |\tau(x)| \le R_\tau \text{ for all } \tau\} \le \prod_{\tau} (2R_\tau + 1)^2 \le 9^f \prod_{\tau} R_\tau^2. \tag{A.320}$$

The same finite-subset bound proves that the set on the left is finite.

Fix $R \ge 1$. Partition $\mathcal{O}_E$ by the integers $n_\tau \ge 0$ satisfying

$$2^{n_\tau} \le \max\{1, |\tau(x)|/R\} < 2^{n_\tau + 1}.$$

By (A.320), a shell contains at most $36^f R^{2f} 2^{2\sum_\tau n_\tau}$ points. Its weight $\prod_\tau \max\{1, |\tau(x)|/R\}^{-3}$ is at most $2^{-3\sum_\tau n_\tau}$. Summing the resulting geometric series gives

$$\begin{aligned} \sum_{x \in \mathcal{O}_E} \prod_{\tau} \max\{1, |\tau(x)|/R\}^{-3} &\le 36^f R^{2f} \sum_{(n_\tau) \in \mathbf{N}^f} 2^{-\sum_\tau n_\tau} \\ &= 72^f R^{2f} \le (64R)^{2f}. \end{aligned} \tag{A.321}$$

For $1 \le l \le j$, set

$$R_l = \max\{1, \rho^{2q_l} (2H_1)^{2r}\}.$$

Let $s_\tau = \max\bigl(1, \max_{\delta_l \neq 0,\, 1 \le l \le j} d_{\tau l}/(2H_1)^{2r}\bigr)$ be the factor of S_δ at τ in Proposition 33.18. For each $l \le j$,

$$\max\{1, |\tau(\delta_l)|/R_l\} \le s_\tau.$$

Using $j \le r$ and $s_\tau \ge 1$, we obtain

$$S_\delta^{-3r} \le \prod_{l=1}^{j} \prod_{\tau} \max\{1, |\tau(\delta_l)|/R_l\}^{-3}.$$

Let T_j be any finite set of represented labels of rank j. Enlarge its set of coordinate tuples to the product of its coordinate projections and apply (A.321) to each coordinate. All summands are nonnegative, so

$$\sum_{\delta\in T_j} S_\delta^{-3r} \le \prod_{l=1}^{j}(64R_l)^{2f} \le 2^j \prod_{l=1}^{j}(64R_l)^{2f}. \tag{A.322}$$

Put

$$B_0 = \rho^{2r^2}\max\{1,2H_1\}^{2r}, \qquad c = 17 + 4/C_\ell + 4|C_H|.$$

Since $q_l \le r^2$, $\rho \ge 2$, and $H_1 > 0$, we have $R_l \le B_0$. Also $\max\{1,2H_1\} \le 2b^{|C_H|m}$. Hence $j \le r$, $f \ge 1$, $L > 1$, and $\log\rho \le L/C_\ell$ give

$$\begin{aligned}\log\bigl(2^j(64B_0)^{2fj}\bigr) &\le frL\left(13 + 4r + \frac{4r^2}{C_\ell} + 4|C_H|rm\right)\\ &\le c\,m^2frL \le c^2m^2frL \le C_c^2m^2frL.\end{aligned}$$

For the second inequality use $r \le m$ and $13 + 4r \le 17m^2$; for the last two use $1 \le c \le C_c$, which follows from $C_\ell \ge 1$. Exponentiation in (A.322) proves the label bound

$$\sum_{\delta\in T_j} S_\delta^{-3r} \le H(C_c)^{C_c r}. \tag{A.323}$$

We now sum the fixed-type estimates. Since $\delta_j \ne 0$ and $\operatorname{disc} E = (\operatorname{disc} F)^2$ (from Definition 33.7), we have $N(\mathfrak{d}) = \operatorname{rd}(E)^{2f}$, so the product formula for the normalized terminal minor gives

$$0 < D_\delta = \prod_\tau d_{\tau j}^2 \le (2H_1)^{4rf} S_\delta^2 \le H(C)S_\delta^2.$$

The last inequality follows from $2H_1 \le b^{C_H m+1}$ and

$$4rf(C_H m + 1) \le 2(C_H + 1)m^2 f \le Cm^2 f.$$

Define

$$Q_j = (C/k)^{j(k-1-j)f} H(C)^{Cr^4} H(C_h)^{C_h r^4}.$$

The factors $A(j,\delta)$ and $M(j,\delta)$ of Proposition 33.18 satisfy the exact identity

$$(AM)(j,\delta) = Q_j D_\delta^{(j+1)/2} S_\delta^{-4k},$$

because $C_s = C_p + 4$. Since $k \ge 2r + 5$, $j \le r$, and $S_\delta \ge 1$,

$$\begin{aligned}(AM)(j,\delta) &\le Q_j H(C)^{(j+1)/2} S_\delta^{j+1-4k}\\ &\le Q_j H(C)^{(j+1)/2} S_\delta^{-3r}.\end{aligned}$$

Consequently, for every finite set T_j as above,

$$\sum_{\delta\in T_j}(AM)(j,\delta) \le A_j, \qquad A_j = Q_j H(C)^{(j+1)/2} H(C_c)^{C_c r}.$$

Let U be a finite subset of $\mathcal{Q}_0$. By (33.11), its pairs with an irrelevant entry contribute zero. Partition the remaining pairs by the ranks and labels of both entries. For each j, let T_j and T_j' be the labels of rank j occurring as first and second entries. The fixed-type estimate applied to every fibre, followed by enlargement to the product of these two finite sets of types, gives

$$\begin{aligned}\sum_{(t,t')\in U}\|v_t\|\|v_{t'}\| &\le n_\pi\mu_u\left(\sum_{j=1}^{r}\sum_{\delta\in T_j}(AM)(j,\delta)\right)\left(\sum_{j=1}^{r}\sum_{\delta'\in T_j'}(AM)(j,\delta')\right)\\ &\le n_\pi\mu_u\left(\sum_{j=1}^{r}A_j\right)^2.\end{aligned}$$

Taking the supremum over finite U proves summability and

$$\sum_{(t,t')\in\mathcal{Q}_0} \|v_t\|\|v_{t'}\| \le n_\pi \mu_u \left(\sum_{j=1}^{r} A_j\right)^2. \tag{A.324}$$

The logarithm of the height factor in A_j is

$$m^2 fL\left((C^2+C_h^2)r^4 + C\frac{j+1}{2} + C_c^2 r\right) \le K_* m^6 fL.$$

Here $r^4 \le m^4$, $(j+1)/2 \le m^4$, and $r \le m^4$. As $C \le K_*$ and $j(k-1-j)f \ge 0$, it follows that

$$A_j \le \exp(K_* m^6 fL)(K_*/k)^{j(k-1-j)f}. \tag{A.325}$$

Write

$$S_* = \sum_{j=1}^{r} \exp(K_* m^6 fL)(K_*/k)^{j(k-1-j)f}.$$

The threshold and the source sizes give

$$\eta_* m^6 L \le C_\ell m^6 L \le k\log k.$$

Since $m^6 L \ge 1$, we have $k\log k \ge M_* \log M_*$. The function $x\log x$ is increasing for $x \ge 1$, so $k \ge M_*$. Therefore

$$k \ge 16, \qquad \log K_* \le \tfrac{1}{8}\log k, \qquad K_*/k \le 1, \qquad k-2 \ge \tfrac{7}{8}k.$$

For $1 \le j \le r$,

$$j(k-1-j) - (k-2) = (j-1)(k-j-2) \ge 0,$$

and hence

$$S_* \le r\exp(K_* m^6 fL)(K_*/k)^{(k-2)f}.$$

Set $B = kf\log k$. Since $r \le k$, $f \ge 1$, and $\eta_* \ge 16K_*$,

$$\begin{aligned}\log(2S_*^2) &\le \log 2 + 2\log r + 2K_* m^6 fL - 2(k-2)f(\log k - \log K_*)\\ &\le 3\log k + \tfrac{1}{8}B - \tfrac{49}{32}B\\ &\le \left(\tfrac{3}{16} + \tfrac{1}{8} - \tfrac{49}{32}\right)B = -\tfrac{39}{32}B \le -B.\end{aligned}$$

Thus

$$2\left(\sum_{j=1}^{r} A_j\right)^2 \le 2S_*^2 \le e^{-kf\log k}. \tag{A.326}$$

Finally, $g_i \ge (kb)^8$, $b \ge 2$, and $f \le k$ imply $g_i \ge 12f$. Hence

$$0 < P_g \le \exp(3f/g_0 + 3f/g_1) \le e^{1/2} \le 2.$$

Multiplying (A.324) by $P_g\Gamma_1$, and then using (33.11) and (A.326), gives

$$\begin{aligned}P_g\Gamma_1 \sum_{\mathcal{Q}_0} \|v_t\|\|v_{t'}\| &\le P_g\Gamma_1 n_\pi \mu_u \left(\sum_{j=1}^{r} A_j\right)^2\\ &\le 2d\left(\sum_{j=1}^{r} A_j\right)^2 \le de^{-kf\log k}.\end{aligned}$$

This proves (33.12). □

Proof of 33.69

Proof. The operator subseries is absolutely convergent by comparison with the full series. The vector subseries defining the w_i are also absolutely convergent. Moreover, Cauchy–Schwarz gives

$$\sum_{t,t'} |\langle v_t, v_{t'}\rangle| \le \left(\sum_t \|v_t\|\right)^2 < \infty.$$

Thus the inner-product series can be regrouped by finite-type indices and row labels.

By Lemma 33.112, the finite-type Gram matrix $\mathcal{G}$ satisfies $\mathcal{G} \preceq P_g\Gamma_1 I$ in positive semidefinite order. Expanding the finite family $(w_i)_{i\in\mathcal{I}}$ in an orthonormal basis of its finite-dimensional span and applying this matrix inequality to each coordinate vector in (33.13) (equivalently, applying the second assertion of Lemma 33.112 to the cell $\mathcal{T}_0$) gives

$$\|Z\|_{\mathrm{HS}}^2 \le P_g\Gamma_1 \sum_i \|w_i\|^2.$$

Expand each squared norm into its absolutely convergent double series. At a fixed finite-type index, distinct complete row labels have distinct lattice-pair coordinates (Λ_t, Λ'_t). Their orbital vectors are orthogonal by Lemma 33.63. Consequently

$$\sum_i \|w_i\|^2 = \sum_{(t,t')\in\mathcal{Q}_0} \mathrm{Re}\langle v_t, v_{t'}\rangle.$$

Since $P_g\Gamma_1 \ge 0$, another application of Cauchy–Schwarz gives

$$\begin{aligned}\|Z\|_{\mathrm{HS}}^2 &\le P_g\Gamma_1 \sum_{(t,t')\in\mathcal{Q}_0} \mathrm{Re}\langle v_t, v_{t'}\rangle \\ &\le P_g\Gamma_1 \sum_{(t,t')\in\mathcal{Q}_0} |\langle v_t, v_{t'}\rangle| \\ &\le P_g\Gamma_1 \sum_{(t,t')\in\mathcal{Q}_0} \|v_t\|\|v_{t'}\| \le d e^{-kf\log k}.\end{aligned}$$

□

Proof of 33.70

Proof. Apply Proposition 33.18 to the fixed constants to obtain $C_H, C_h, C_p \ge 0$. First choose

$$C_s = C_p + 4.$$

Let $C_{\min}$ be the amplitude threshold supplied for this value of C_s, and choose

$$C = \max\{C_{\min}, 2(C_H + 1)\}.$$

At this value of C, Proposition 33.18 supplies thresholds C'_{type} and $C_{\ell,\mathrm{type}}$. Proposition 33.68 supplies $C_{\ell,\mathrm{pair}}$ for the same constants. Set

$$C'_{\mathrm{req}} = \max\{C'_{\mathrm{type}}, 1\}, \qquad C_{\ell,\mathrm{req}} = \max\{C_{\ell,\mathrm{type}}, C_{\ell,\mathrm{pair}}, 16\}. \tag{A.327}$$

Fix parameters above these thresholds and a model as in the statement. Write $r = (m-2)/2$, $L = \log b$, and $\rho = \mathrm{rd}(E)$. By (A.327) and Definition 33.7, $m \ge C_\ell \ge 16$, so $r = (m-2)/2 \ge 1$ and $k \ge f \ge m^4 \ge 16^4 > e$, whence $\log k \ge 1$. Since $C' \ge 1$, $C_\ell \le m$, and $m^2 \le m^4 \le f$, the source inequalities of Definition 33.7 give

$$C_\ell m^6 L \le m^7 L \le m^5 f L \le k \le k\log k.$$

Combining $\log f \le \sqrt{r}$ from Definition 33.7 with $r \ge 1$ gives

$$\log f \le \sqrt{r} \le r^2.$$

Multiplying $\log \rho \le L/C_\ell$ by $C_\ell > 0$ gives $C_\ell \log \rho \le L$. The adjacent Harish–Chandra gaps in Definition 33.7 satisfy $\lambda_i - \lambda_{i+1} > C_\ell k m^4 + n m^{10} > C_\ell k m^3$; summing adjacent gaps gives the required bound between every two distinct parameters. The auxiliary-prime bounds and the upper bound $T_w \le b^{C_T m}$ for every word w are source inequalities. The distinct level and auxiliary primes give $(T_w, g_0 g_1) = 1$ for every word w. The model has the stated unramified CM property. Thus the parameter conditions of Proposition 33.18 hold.

By (A.327), the fixed-type estimate applies at $C_s = C_p + 4$ and the chosen C. Lemmas 33.65 and 33.66 give (33.11), so Proposition 33.68 gives the summable pair-amplitude family and (33.12). Under the assumed absolute convergence, Proposition 33.67 gives (33.13). The tied local data have nontrivial selected additive characters, the primes g_0, g_1 split completely in E, and $g_i \ge (kb)^8 \ge 3$; hence the selected local hypotheses of Theorem 33.78 used in Lemma 33.112 hold. Since $\Gamma_1 \ge 0$ by Lemma 33.65, Lemma 33.69 gives both conclusions of (33.14). Every constant was chosen before the datum, the tied model, and the word, which proves the stated uniformity. □

Proof of 33.71

Proof. The level window and $(1 + 1/(fk))^{fk} \le \exp(1)$ give

$$0 < d \le d^\star T^{fk} \le \exp(1) D^\star \Gamma^{-2} \le D^\star e^{-2a}.$$

For the last inequality use $2(C_\Gamma - C_1)Y \ge 1$. Cauchy–Schwarz gives $\|X + Z\|^2 \le \frac{3}{2}\|X\|^2 + 3\|Z\|^2$ in Hilbert–Schmidt norm. Since $B/2 \ge 5$, we have $e^{-B} \le (2/5)e^{-B/2}$. Substituting the two equal-word bounds gives

$$\|K - I\|_{\mathrm{HS}}^2 \le 4(1 + e^a)\frac{d^2}{D^\star} + (4e^{-B/2} + 2e^{-B})d.$$

Division by d and the bound $d/D^\star \le e^{-2a}$ give the stated e.

On a free orbit each character of $(\mathbf{F}_T^\times)^f$ occurs once. Summation in coordinate i acts by $T - 1$ when the i-th character is trivial and by zero otherwise. Exactly $(T-2)^f$ of the $(T-1)^f$ characters survive in the common kernel. Summing over orbits gives

$$\dim U = d\left(1 - \frac{1}{T-1}\right)^f \ge d\left(1 - \frac{f}{T-1}\right).$$

Since $T > 1$, its congruence gives $T - 1 \ge (g_0 g_1)^2 \ge 4f$. Thus $\delta \le 1/4$. Both a and $B/2$ are at least 5, so $2e \le 28e^{-5} < 3/4$, as required. □

Proof of 33.72

Proof. Extend an orthonormal basis of $\ker P_u$ to one of C_u. On each of the first $\dim \ker P_u$ basis vectors $K_u - 1$ acts as -1. Hence

$$\dim \ker P_u \le \|K_u - 1\|_{\mathrm{HS}}^2 \le e d_0 < (1 - \delta) d_0 \le \dim U_u^{\mathrm{ram}}.$$

Choose $\phi \in U_u^{\mathrm{ram}} \setminus \ker P_u$. At the linked realization let C represent this column in the model. The isometry identifying the unrestricted targets takes $P_u\phi$ to its model period p, so $p \ne 0$.

Write J_z for evaluation of a full reference multiplicity map at the prescribed unit lowest-type tensor, followed by the prescribed finite translation, the central translation z, and complex conjugation in the row-function convention. The full-level evaluation in the proof of Lemma 33.57 is onto H. The central translation acts bijectively on H: it commutes with the level projector and its inverse is translation by z^{-1}. Thus $T_0 = J_z^{-1} p$ is a full reference multiplicity map. Its model pairing with C is

$$\langle J_z T_0, p \rangle = \|p\|^2 > 0.$$

□

Proof of 33.73

Proof. Write $I_t = \prod_{j<m} I_{t,j}$ and

$$W_{t,j} = V_{t,j} K_{t,j} K_{t,j+1} \quad (j \text{ even}), \qquad H_t = P_t \prod_{j \text{ even}, j<m} W_{t,j}.$$

Thus $c_t = s_u H_t / I_t$. We describe how the descending chain acts on a linear relation. Allow a finite list of top rows, each with its own tuple and column. At even height $j < m$, denote by $D_{i,j}(y, v)$ the Bessel period for entry i, with column evaluated in frame y and row evaluated at smooth vector v. The row is descended from the top to j. The column has been carried up from zero by successive pairs of kernels, with the prescribed column operator before the first kernel of each pair. At height m write $D_{i,m}$ for the scalar terminal period of this column and the given top row.

At an interior double step the following implication holds for arbitrary complex coefficients β_i, where t_i is the tuple of entry i:

$$\sum_i \beta_i W_{t_i,j} D_{i,j+2}(y', v') = 0 \quad \text{for all } y', v'$$
$$\Longrightarrow \sum_i \beta_i I_{t_i,j} I_{t_i,j+1} D_{i,j}(y, v) = 0 \quad \text{for all } y, v.$$

We now give the calculation explaining the common tests in this implication. Evaluate the lower row on a vector obtained by two polynomial theta tests applied to v'. The two defining row-lift integrals absorb $I_{t_i,j} I_{t_i,j+1}$. Unfold the resulting Bessel period first along the intermediate Fourier–Jacobi subgroup and then along the next Bessel subgroup. At fixed real translates, the finite average of the resulting expression has the form

$$\sum_{y'} a(y')\, W_{t_i,j} D_{i,j+2}(y', v''),$$

where v'' is the real translate of v'. To see the independence of a, first do one definite class sum. On the fixed finite-dimensional polynomial coefficient space, the compact average with the prescribed polynomial test is a linear combination of frame evaluations: the evaluations on invertible frames span the dual, since a polynomial vanishing there vanishes identically. Choose this combination before inserting the class form. In the class sum its coefficients are multiplied by the source inverse determinant power and the target determinant power of the chosen frames. Both powers are fixed by the weights. Applying this calculation twice and multiplying the two finite matrices of coefficients gives a. The finite chart coefficients of entry i are inside its two column kernels. Their scalar normalizations are $K_{t_i,j} K_{t_i,j+1}$; the change of the test vector contributes $V_{t_i,j}$. Real translation acts on the argument of each intertwiner by the same representation operator, giving the same v''.

Sum over i before taking the quotient integrals. For the finite integration take the intersection of the finitely many level stabilizers and the union of the finite support sets. Subdivision into cosets computes each original integral on this common level. The weighted integrand after finite averaging vanishes by the upper relation for each y', v''. The quotient integrations preserve the relation. The interchanges use cusp decay on the row and its intermediate lift, with the polynomial growth of the fixed theta kernels as in the proof of Lemma 34.23. The polynomial theta tests span densely in the smooth row models: at each real place their translates generate the prescribed irreducible Harish–Chandra module, and its compact-finite vectors are dense in its smooth globalization. The period against fixed column data is continuous in each smooth argument by the same decay estimate. This extends the relation to every v.

Lean: `GlobalTheory.SeeSaw.kerX_of_clsPinD` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.SeeSawKernelLiftCombinationOfDefiniteClassData`, lines 14–49)
Lean: `GlobalTheory.SeeSawStep.stepX_of_laws` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.SeeSawDoubleStepLinearRelationsPeriods`, lines 82–149)

For the terminal pair the upper expression is $W_{t_i,m-2} P_{t_i} D_{i,m}$. Indeed the terminal compact average at y', v' is a scalar multiple of the fixed branching map. Its multiplicity is one for the interlacing terminal weights. The scalar depends on y', v' and the weights. The finite class sum multiplying this branching map is $P_{t_i} D_{i,m}$. Thus the extra scalar multiplies the entire upper relation at each test. Iterating the interior

implication, beginning with this terminal pair, gives

$$\sum_i \beta_i H_{t_i} D_{i,m} = 0 \quad \Longrightarrow \quad \sum_i \beta_i I_{t_i} D_{i,0}(y,v) = 0 \quad \text{for all } y, v.$$

At zero the chart set is a singleton. Its finite operator has coefficient one and acts by the single prescribed right translation. Hence $D_{i,0}$ is the bottom Bessel period of the descended row with that translation.
Lean: GlobalTheory.SeeSaw.perX_of_clsPinD (Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.SeeSawLastPeriodFactorizationOfClassData, lines 72–144)
Lean: GlobalTheory.SeeSaw.lastX_of_laws (Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.SeeSawLastTwoStepsTransportLinearRelations, lines 93–137)
Lean: GlobalTheory.SeeSaw.mainX_of_stepX_lastX (Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.MainXOfStepXLastX, lines 87–114)
Lean: GlobalTheory.MiTop.linkZero_of_chartZero_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LinkZeroOfChartZeroAlt, lines 22–36)

A relation valid for all y, v gives the same relation between translated Petersson pairings. In the bottom Bessel integral, change variables by the finite column translation. The column becomes the model column after the bottom isometry. The row is translated by the prescribed row element and by a level element. The latter acts on every full reference multiplicity map by the same nonzero level character value, which cancels from the relation. The column coefficient vectors read in the frames y span the coefficient space. By linearity the relation holds at every coefficient vector. In the unfolding of the pairing with the period series, at a real group element g insert its column coefficient vector and translate v by g. This gives the relation pointwise in the remaining real integral. Integration against its common measure proves the assertion for the Petersson pairings on the full level space.
Lean: GlobalTheory.MiTop.botRel_of_pin_of_mem_center (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.BotRelTwo, lines 32–136)

Choose one reference top row descending to T_0. Write S^{-1} for the column change in descent and R for the bottom isometry. Since S^{-1} multiplies norms by a positive constant, it is injective and hence onto the finite-dimensional permitted space. Choose ϕ_0 so that $RS^{-1}\phi_0$ represents the column giving b_0. For this reference top row and the column started at $S^{-1}\phi_0$, put $p = H_{t_0} D_{0,m} / I_{t_0}$.

For the second entry take the top row of F at t. Its descent is the bottom row of $\Theta_t j_t e_t F$: at every arrow the row map is the defining row lift, and the final row projection fixes its image. Start its column at $S^{-1}\phi$. Commuting the column change through the terminal lift gives $D_{1,m} = s_u A_t(\phi, F)$. Apply the preceding implication with coefficients λ / I_{t_0} and λ' / I_t. We have proved

$$\lambda p + \lambda' c_t A_t(\phi, F) = 0 \quad \Longrightarrow \quad \lambda b_0 + \lambda' B_t(\phi, F) = 0.$$

If $p = 0$, take $(\lambda, \lambda') = (1, 0)$ to get $b_0 = 0$. Hence $p \neq 0$. Taking $(\lambda, \lambda') = (c_t A_t(\phi, F), -p)$ proves the identity, including when the second pairing vanishes.
Lean: GlobalTheory.MiTop.linkDown_of_ok (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.GeometricDatumOfFamily, lines 77–86)

□

Proof of 33.74

Proof. Fix h_L, c_{Dep} and use the list extension in (33.1). Its projection to the law keeps the point indices. The primitive row data and the definite evaluation are supplied by its parameters. The Weil action on those rows and the discrete-series realization give the recursion identification. Combining it with the definite evaluation gives the joins of Definition 33.6.

Fix a linked bundle in the common range of the row comparison. The matching and doubling identities at the prescribed levels give the surjections used in the downward seesaw. Use the row choices with common level data for the different tuples. The bottom comparison uses a single central finite translate for all words. Right translation by this element preserves the good subspace, since the translate is central, and preserves its norm by invariance of the Petersson measure. Denote its action on the output by V.
Lean: GlobalTheory.Fam.miStar2_of_allTrue_alt_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.MainTheoremFromPublishedResultsAltAltAltAlt, lines 23–28)
Lean: GlobalTheory.MiTop.famAllQ_of_allTrue_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.FamAllQOfAllTrue, lines 38–54)

Lean: GlobalTheory.MiTop.exists_isShift_of_center_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.PostOnMemEEatOfCenter, lines 49–94)

For a fixed word u, use the column map from the seesaw comparison. It multiplies norms by a positive constant. Divide it by this constant, compose with the bottom column isometry and express the result in the prescribed column coordinates. Its matrix U_u preserves the column norm. Choose the reference row in the full space of level-zero multiplicity maps. The reference descent has this entire space as target. Indeed, its arrows follow the reference row spaces at levels $0, \dots, m-1$, followed by the terminal space. Write L for one descent arrow and use its Petersson adjoint L^*. Start with any nonzero vector in a bottom Hecke eigenspace, including its full multiplicity space. At the current stage let v be its image under the preceding adjoint arrows, with $v \neq 0$ and the transferred Hecke parameters as induction hypotheses. At each real place choose a polynomial in the prescribed joint theta type and a unit highest-weight vector. The real theta pairing gives $r_\tau > 0$; if $i_0 > 0$ is the squared norm of the fixed Fock vacuum and $c_\infty > 0$ is the Haar comparison factor, the real zeta integral is

$$Z_\infty = i_0 c_\infty \prod_\tau (r_\tau / i_0) \neq 0.$$

The finite tests remain those of the chain. On the integration domain of elements integral outside the exceptional finite places, their coefficient vanishes off the prescribed finite level; on the product of the real group and that level, the two level characters cancel since $\chi(\kappa)\overline{\chi(\kappa)} = 1$. Haar disintegration therefore gives the local doubling integral $\overline{\langle v, v\rangle} Z_\infty \neq 0$. Lemma 33.59 gives the nonzero global numerator, and the normalizing denominator is nonzero. The doubling identity makes the corresponding upward theta lift nonzero. It is cuspidal by Lemma 33.58; the adjoint identity and its expansion in multiplicity-map evaluations identify it with an evaluation of $L^* v$. Thus $L^* v \neq 0$, and spherical transfer adjoins the fixed parameters of this step. Induction gives a nonzero image under the full adjoint chain for every nonzero vector of the bottom eigenspace. Distinct systems of unramified parameters stay distinct after adjoining the fixed chain parameters, since those parameters can be cancelled in the multisets at each place. The adjoint images of the distinct eigenspaces are therefore independent. The bottom space is their finite-dimensional span, so the adjoint has zero kernel and the descent is onto. This applies before imposing the good-type idempotent.

Lean: GlobalTheory.ChainOnto.ontoAll_of_chainUp (Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.ThetaLiftChainSurjectiveOfAdjointNonvanishing, lines 14–38)

Lean: GlobalTheory.BaseChain.linkOnto_of_ontoAll_third_place_series (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LinkOntoOfOntoAll, lines 46–54)

For the diagonal estimate, put $Y = m^5 f \log b$, $B = kf \log k$, and $d = \dim C_u$. The generic corank-one pairs are the disjoint union of the scalar pairs U_{sc} and the nonscalar pairs S_{nsc}. Lemmas 33.99 and 33.106 give

$$P(\mathcal{U}_{\mathrm{gen}}) \le P(U_{\mathrm{sc}}) + |P|(S_{\mathrm{nsc}}) \le \left(e^{C_B Y} + e^{-B/3}\right) \frac{d^2}{D^\star} \le e^{(C_B + \log 2) Y} \frac{d^2}{D^\star},$$

using $C_B \ge 0$, $B \ge 0$, and $Y \ge 1$. Fix the counting parameter of Lemma 33.99 before its thresholds. Take $C_{\mathrm{gen}} = C_B + \log 2$ and $C_{\mathrm{ng}} = C_4$ from its nongeneric estimate. Choose $C_1 = \max(C_{\mathrm{gen}}, C_{\mathrm{ng}}) + 1$, then $C_\Gamma \ge C_1 + 1$, and take the size thresholds for these choices. For the higher-corank sum use the tied term and loop models for the reference word paired with itself. Specialize the mean flag-pair count to width three and to this same word. For each represented Gram class choose the integral equivalent representative supplied by the bounded-minor reduction. The order-one minors of this representative give the entry height bound h_Z of Lemma 33.61. The position lattices of the tied model have product of absolute values at least one on nonzero vectors. Group the actual terms by the two reduced labels and the rank of the exceptional lattice. The weighted flag-pair count uses inverse stabilizer orders. Combining it with the exceptional-lattice determinant and the two gluing indices gives the two bounds for W_i in Lemma 33.61, with the mass ratio for interior ranks and the index bound of Proposition 33.103 at the endpoints. For the degenerate class the determinant comparison gives the extra factor $T^{-f/2} K(Z)$ on each side. Use the squared-integral estimate with loss parameter one and the absolute-amplitude estimate in Claim 2 of Proposition 33.103 on these same terms. With the row weight and the coefficient dimension removed, these bound the squared amplitudes by $h_Z^{2cr^2} \Gamma F_i S_Z^{a_i - 2q/(r+1)} / (n_Z b_i(Z)^{2k})$ for the two classes. Choose the integer q of Lemma 33.61, which satisfies $q \le (9c+3)km^2$, the range of those estimates. Enlarge the size thresholds jointly for the count, reduction, and amplitude estimates. Multiplying the fibre count by the two square-root bounds gives the required fibre majorant, with the factor d from the row-mass normalization. Lemma 33.61 now gives the remaining pair

sum in Lemma 33.62. With the row orthogonality of 33.63 and the norm expansion of 33.67, this gives the required bound on the nonzero-Gram part X. Theorem 33.70 gives $\|Z\|_{\mathrm{HS}}^2 \le e^{-B}d$ for the remaining nonidentity part. Apply Lemma 33.71 with the level window and the free level model of Lemma 33.60 to obtain $\|K - I\|_{\mathrm{HS}}^2 \le ed$ and $\dim U_u^{\mathrm{ram}} \ge (1-\delta)d$, with $2e < 1 - \delta$. By Lemma 33.72, this diagonal Siegel–Weil estimate and the dimension bound for the ramified subspace give a column with nonzero period, and surjectivity of translated bottom-row evaluation gives a row pairing nontrivially with it. Use this full reference row as T_0 in Lemma 33.73. Reference descent supplies its top preimage. For the second row take the prescribed tuple and rational coordinate F. Its image belongs to the good subspace by the row identification of Lemma 33.57 and the theta-row comparison in (33.1). Thus every arithmetic input uses the same reference pair for this word. In the complex coordinates the resulting identity reads

$$c_{u,t}\,\iota(\mathrm{Pt}_{u,t}(\phi,F))\,b_u = p_u\,\langle s_{EE}(V\Theta_t j_t e_t F),\, P_u(U_u\iota(\phi))\rangle\,, \qquad p_u b_u \neq 0.$$

More explicitly, let $r_u > 0$ be the norm multiplier of the column change and let $\kappa \neq 0$ convert the model pairing into the spectral pairing in the fixed row normalization. In the notation of Lemma 33.73, the column isometry is $r_u^{-1}RS^{-1}$. Substitution of $r_u^{-1}\phi$ in that lemma gives $p_u = p$, $b_u = \kappa r_u^{-1} b_0$ and $c_{u,t} = c_t$ in the direct complex coordinates. In the conjugate coordinates conjugate all three scalars and conjugate the column matrix entrywise. The model pairing is linear in its column, so the same displayed equality holds. Both normalization scalars are nonzero.

Lean: `GlobalTheory.MiTop.AtReal.anchor_word_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.MiStar2AtRealisation`, lines 48–79)
Lean: `GlobalTheory.MiTop.miStar2_at_realisation_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.MiStar2AtRealisation`, lines 84–196)

Set

$$\zeta_{u,t} = \frac{c_{u,t}}{c_{u,\mu_0}}, \qquad a'_u = \frac{p_u}{b_u c_{u,\mu_0}}.$$

Division gives (33.16), with $\zeta_{u,\mu_0} = 1$. Fix a finite place w and $z \in K$ such that $\iota(z) = \zeta_{u,\mu_w}$. Lemma 33.56 gives $N > 0$ and $r \in \mathbf{Q}^\times$ with $z^N = r$, by injectivity of ι. Every prime dividing the reduced numerator or denominator of r belongs to S_{μ_w}. The residue characteristic of w belongs to neither support. Hence both numerator and denominator are units in $\mathcal{O}_{K,w}$, and

$$N v_w(z) = v_w(r) = 0.$$

This proves the finite-place clause with the same scalars as the period identity.

Lean: `GlobalTheory.MiTop.miStar2_of_anchored` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.MiStar2OfAnchored`, lines 15–29)
Lean: `GlobalTheory.Fam.vw_eq_zero_of_unitRatio` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.ValuationEqZeroOfRootOfSUnitRatio`, lines 58–93)

For the divisor row use the rational row projections and lifts of the law on this realization. The comparison then has the form $\zeta_{u,t}\iota(B) = \iota(B')$ for the two rational pairing values on a given input. If some $B \neq 0$, its ratio B'/B is the unique field element representing $\zeta_{u,t}$, and the comparison holds on every input. If $\zeta_{u,t} \notin \iota(K)$, the displayed equation forces $B = B' = 0$ on every input; choose the field factor 1 there. This gives the rational factors for the divisor row, with valuation zero at the specified places.

Lean: `Formalism.IndefR.Cx.CxData.zetaOK_of_miUnit2` (`Formalism.LocalToGlobal.IndefiniteReal.IsometricPeriodsOfUnitHypotheses`, lines 42–57)
Lean: `Formalism.IndefR.Cx.CxData.matrix_identity_of_unit_hypotheses` (`Formalism.LocalToGlobal.IndefiniteReal.IsometricPeriodsOfUnitHypotheses`, lines 102–118)

The real-group decomposition gives the isotypic row spaces and their multiplicities. The Harish–Chandra parameters in that decomposition give the asserted infinite-place equality. For the finite types use the projection to the law, which preserves the assigned types and tame generators. The Green character formula in Theorem 32.37 and the Zelevinsky input in P give the finite-type conditions on this assignment. Apply local Bessel uniqueness with these same types. These local identifications put each basis vector in its assigned single-label space. Apply Lemma 33.57 to the linked reference realization to obtain the isometry onto the full good subspace. In particular the number of selected basis indices of each good label is its global multiplicity. The recursion and divisor rows with the ray-class-field input give the geometric row. Theorem 36.5 supplies the definite-side row with its cutoff for each $C \ge C_0$.

Lean: `GlobalTheory.T.linked_bundles_good_of_local_langlands_alt_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LinkedBundlesGoodOfLocalLanglands`, lines 96–97)

Lean: GlobalTheory.Fc.LabLinked.ties_of_nAtData_real (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RowLabOfLayer, lines 44–56)

All pointwise identifications were made at the indices of the linked bundle. Taking their eventual forms therefore gives the indicated rows on W_3. For the transfer assertion, project a witness of the stronger extension to a witness of the old extension at the given indices. The linked bundle and the family selection then satisfy the old premise with its original cutoffs, which gives the claimed conclusion. □

Proof of 33.75

Proof. Take the middle threshold of Definition 36.10 to be at least 8. The family size constant C_{sz} is a middle entry, so $C_{\mathrm{sz}} \ge 8$. Fix F_m, C_H in this range. We first check the sizes in 21.16 uniformly in the member and layer. Write $n = |A|$, $r = k+m-1$, $n_V = r+t$, and use b', w'', η, S_c from Definition 34.19. The rank ranges (8.1) give $k \ge 5$ and $m \le k$ eventually. Thus $r \le 2k$. The rank identity of 38.3 gives $n_V \ge 8fr+4$, hence $n_V \ge 8$ and $t \ge r+5$.

For the widths, $b'/b = x/\exp(x^{\varepsilon/3})$ tends to zero and $w'' = b'/(xn)$. The width and spacing conditions in Definition 34.19 give $\eta \le w''$ and $S_c \le w''$. The top-weight inequalities there give $b_s \le (1+x^{-2}/4)b'$ at a top position s. By the three intervals of 14.2, the layer shift is positive and is bounded above by $b_s + b'/x + \eta$, $b+\eta+1$, or $w''+\eta$, according to the position. All three are at most $2b$ beyond one family cutoff. The cloud entries lie between zero and S_c, and the auxiliary entries differ from the shift by at most η, by the weight-box conditions. This gives the two bounds by b in 21.16. The type-prime condition in Definition 34.19 gives $g_\nu \ge (kb)^8$ for $\nu = 0, 1$. These conditions concern the member before the point realization or C is chosen.
Lean: GlobalTheory.LowRow.weightSizes_along (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ThetaLiftWeightSizesEventuallyAlongFamily, lines 16–59)

Fix a point in $\mathcal{P}$ at such a member. Its primitive row has separated types and the level-one Hecke decomposition of 33.74. Its normalized realization supplies the standard definite lattice, base phase one, the extremal vectors, and the class expansion with its level action as in 37.24. The twist set of the reference tuple is empty. Each marked prime lies over one of g_0, g_1. The host level conditions make these primes coprime to $Q_S qd$ and to the conductor ideal. Thus the prime restriction in 21.16 holds. The analytic inputs for that lemma are supplied by P: rapid decay, the unitary Siegel–Weil formulas, the local doubling formula and its unramified evaluation, finite length, and the compact-dual formal-degree formula.
Lean: GlobalTheory.LowRow.primeToHost_base (Development.AbelianVarietyTheory.Proofs.DivisorProofs.BadPlacesPrimeToHostBaseTuple, lines 13–18)
Lean: GlobalTheory.LowRowAdel.adelC_of_siegelWeil_of_doubling_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.FinitenessInputsRealData, lines 13–27)

Put $H_j = \sum_\rho \log \mathrm{Hook}(a^+_{j\rho})$, where $a^+_{j\rho}$ is the positive partition of 21.16, and put $D_j = H_j - 8fn_V^2 L'$, with $L' = \log(b\ \mathrm{rd}(E) g_0 g_1 n_V)$. The coefficient of C in the sixth entry is zero at this scale. The identification of the positive partitions in 21.21 therefore gives, for every C,

$$L_{6,j} = H_j - C_{\mathrm{sz}} f n_V^2 L' \le D_j.$$

Indeed $L' \ge 0$, since $b \ge 1$, $\mathrm{rd}(E) \ge 1$ and the remaining factors are positive integers.

At the distinguished place the primitive row prescribes the scalar of the projected lift times its adjoint on each occurring block. At every other place it prescribes the scalar in the conjugate pairing on rational layer coordinates. These are the two scalar identities in 21.21. The one-map comparison and the phase normalization at our point give the comparison of the rational lift in 21.19. Thus 21.21 applies, including its convention for an absent block. It gives the asserted bound at every place, with the cutoff already chosen.
Lean: GlobalTheory.LowRow.lamLow_closing_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.FinitenessInputsRealDataThree, lines 14–23)

□

Proof of 33.76

Proof. Subtract $\sum_j 2e_w F_{0,j,w} + c_w$ from the first inequality and take the positive part. Since all the u_{jw} are nonnegative, the result R_w satisfies

$$0 \le R_w \le \sum_j u_{jw}.$$

The index error prescribed at the point is $e'_{jw} = R_w/m$. Its sum over the layers is R_w, which proves the first two reference inequalities. For each layer h, nonnegativity of $J_{0,h,w}$ gives $\sum_w \ell_w u_{hw} \le aU$. Hence

$$\sum_w \ell_w e'_{jw} \le \frac{1}{m} \sum_{h=1}^{m} \sum_w \ell_w u_{hw} \le aU.$$

The hypothesis for layer j also gives $\sum_w \ell_w e_w J_{0,j,w} \le aU$. Addition proves the third inequality for $C = 2a$, and $U \ge 0$ gives it for every larger C. Thus the same x_0 works after C is quantified. The construction uses the thresholds of the hypotheses without increasing them. □

Proof of 33.77

Proof. Take $R = \top$ in (33.9). Good members supply the symmetric column system with finite-order terminal and level-zero characters valued in $\iota_D(K_D)$, the reference forms map with its label injectivity, stabilizer bijections and norm bound (33.17), the integral interlacing lowering maps on the reference and used twisted chains, and the Petersson class decomposition. Lemma 33.33 supplies rational row lifts of $\mathcal{O}_{K_D}$-combinations of the row generators, whose class polynomials lie in the local Weyl duals at all allowed places. Lemma 17.3 gives integrality of the rational point maps in items 1 and 2 and represents each period by $y = \sum_c b_c/h_c \in K_D$, where $h_c \mid D_{\Gamma,u}$ and b_c is integral at every allowed w. This gives (33.18) simultaneously at these places. The real comparison gives item 5. Apply 17.20 to the tied level chain for the reference scaling and the localized twisted representation. □

Proof of 33.78

Proof. On W_3 the level primes of the twisting selection eventually satisfy $b^{m/2} \le T_w$ and $\log T_w \le C_{Tu} m \log b$, uniformly in the point and word. Apply 37.6 to a unit hyperplane in the strict range of 34.17 for one represented vector and ambient rank at least three. For the level-frame row put $g = g_0 g_1$, $a = \pi/v_E$, $\xi = a/g$, and $d_E = \mathrm{rd}(E)$. The critical level is the positive number defined by

$$T_{\mathrm{crit},\tau}(w)^k = a^{-[(m-1)k+m(m-1)/2]} d_E^{-(m-1)/2} g^{km+m(m-1)/2} \frac{VV_\tau}{V(w_\tau)}, \qquad VV_\tau = d_{\sigma,\tau} \prod_{j=0}^{k+m-2} j!,$$

where V is the Vandermonde product and $d_{\sigma,\tau}$ is the local formal-degree factor in the column normalization.
Lean: `Formalism.MainNotions.S.Tcrit` (`Formalism.SharpRestriction.CuspSpanNormBounds.StandingNumericalData`, lines 39–43)

Raise the last threshold of Definition 36.10 to at least $\max(40, 8C_\Gamma)$. Each of the six entries defining C_ℓ belongs to the last family list, so $C_\ell \ge 40$, and likewise $C' = C_k \ge 8C_\Gamma$. Use $C_3 = 1$, and cutoffs with $2C_3 \le m$, $C_u \log b \le m-1$, $C_0 e^8 \le k$, where $C_u = 3$ is the weight bound parameter and $C_0 = 3$ is the dimension bound parameter. By the mass and Euler-factor product identity of 21.36, we have $\prod_\tau (T_{\mathrm{crit},\tau}(w)/T_w)^k \le e^{f 2^{3-k}} D^\star/d_w^S$. To bound $T_{\mathrm{crit},\tau}(w)$ from below, write $B_j = b_{\tau,j}$ $(1 \le j \le m = 2r_S + 2)$ and let a_{eff} be the edge denominator ($a_{\mathrm{eff}} = (u_1 + 1/2) - v_1$ off the top and $a_{\mathrm{eff}} = (u_1 + 1/2)B_*/(k\mathcal{D})$ on the top). Lemma 37.8 gives the formal-degree lower bound $VV_\tau \ge \frac{1}{2}\Delta(\lambda_\tau)$; comparing the resulting Vandermonde ratio against $V(w_\tau)$ and using the quadratic prefactor bound $\frac{1}{2} a^{-m(m-1)/2} d_E^{-(m-1)/2} g^{m(m-1)/2} \ge (g/(2\pi))^{m(m-1)/2} \ge b^{2m} \ge \prod_{j=1}^m B_j$ to absorb $\prod_{j=1}^m B_j$ into the k-th power comparison yields $T_{\mathrm{crit},\tau}(w) \ge g^m a^{-(m-1)} k^{-C_3} (\prod_{j=1}^m B_j)/a_{\mathrm{eff}}$. Since $C_\ell \ge 40$, we have $a_{\mathrm{eff}} \le B_1 b^{C_u}$ and $B_j \ge k^4$ for all $2 \le j \le m$, giving $\prod_{j=2}^m B_j \ge k^{4(2r_S+1)} = k^{8r_S+4}$. Together with $k \ge 5 > \pi$ (so $\pi^{2r_S+1} \le k^{2r_S+1}$), $C_u \log b \le m-1 = 2r_S+1$ and $\log k \ge 1$ (so $b^{C_u} \le k^{2r_S+1}$), and $2C_3 \le m$ (so $k^{C_3} \le k^{r_S+1}$), this gives $\pi^{2r_S+1} k^{3r_S} b^{C_u} k^{C_3} \le k^{8r_S+3} \le \prod_{j=2}^m B_j$; cancelling B_1 yields $T_{\mathrm{crit},\tau}(w) \ge g^m a^{-(m-1)} \pi^{m-1} k^{3r_S}$ at each place τ. Furthermore, $f \le k \le 2^k$ gives $f 2^{3-k} = 8f/2^k \le 8$, while $8C_\Gamma \le C'$ and $C' m^5 f \log b \le k$ give $2C_\Gamma m^5 f \log b \le k/4$ and hence $\Gamma_P^2 = \exp(2C_\Gamma m^5 f \log b) \le e^{k/4}$; combined with the source dimension comparison $D^\star/d_w^S \le C_0 \Gamma_P^2$ and $a \le \pi$, these estimates yield

$$g^2 \le \xi^{2r_S} T_{\mathrm{crit},\tau}(w), \qquad \prod_\tau (T_{\mathrm{crit},\tau}(w)/T_w)^k \le e^{f 2^{3-k}} D^\star/d_w^S \le e^8 C_0 e^{k/4}. \tag{A.328}$$

If $T_w\xi^{2r_S} < 1$, each factor of the product is at least $(g^2)^k$. The upper bound is strictly smaller by $C_0e^8 \le k < g^k$ and $e^{k/4} \le g^k$. This proves (33.23).
Lean: `Formalism.MainNotions.BridgeLemma.TFrame_of_sizes` (`Formalism.SharpRestriction.CuspSpanNormBounds.BruhatCellsKronecker.CriticalLevelLowOfRefined`, lines 408–427)
Lean: `GlobalTheory.bothRowsFromSiegelWeil_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.BothRowsFromSiegelWeil`, lines 11–15)

Let $L_* = \mathbf{o}(F_m) \,\|\, \mathbf{m}(F_m, CH)$ in the canonical coordinate schema of Definition 36.10, where $\|$ denotes list concatenation. Evaluate the loop thresholds at $C_{YL}, C_{Tu}, a_F(\mathbf{o}(F_m))$ and the row-bound threshold computed from a_F, L_*. Impose these together with the size, volume and raised selection cutoffs. Then 33.106 gives (33.21). For (33.22), use 21.36 with $C_0 = 3$ and the constant function $a_* = 3e^{7/4}$. It bounds the level ratio in either order by a_* and the Vandermonde ratio by a_*^k. Use the loop row with $C \ge 4\sqrt{a_*}$. Entry 17 of L_*, counted from zero (i.e., $\mathbf{m}(F_m, CH)[1]$ in Definition 36.10), is $C_\Gamma > 0$. The list determines $k_0, C_n, C_0', C_{\ell,0}$ for 33.113. Raise the last threshold to $\max(C_0', 29, C_{\ell,0}, 40)$. The entry C_k and all six entries in the minimum belong to the last list. Thus $C' \ge \max(C_0', 29)$ and $C_\ell \ge \max(C_{\ell,0}, 40)$. The source and selection supply the counting and gap hypotheses, ties and convergence. Apply 33.113 with the nongeneric estimate of 33.99. Take the maximum of the point cutoffs and $R = \top$ in (33.9). □

Proof of 33.79

Proof. Write $r = r_S,\ g = g_0g_1$. At a represented point w, let $D_i(w)$ be the leading minors. Its kind $\kappa(w)$ is D_{r+1}/D_r for $D_r \ne 0$ and a separate symbol for $D_r = 0$. Set

$$G(w) = D_{r+1}\overline{D_{r+1}} - TT'D_r\overline{D_r}, \qquad n(w) = c_{\mathrm{count}}\alpha|N_{E/\mathbf{Q}}(G(w)/g^m)|^{(k-1)/2}.$$

The count constant c_{count} is specified below. At a finite place v, partition integral matrices in the lower and upper unipotent orbit of w into integral double orbits O. Let $h_v(w, O)$ be the orbital measure with its character and terminal phase, and $\theta(O)$ the integral presentation class after the last diagonal shift. Put $B_v(w, \theta) = \sum_O h_v(w, O)\#\mathrm{Iso}(\theta, \theta(O))$. An isomorphism set is an automorphism torsor for equal classes and empty otherwise. The orbital sum for θ is therefore $B_v(w, \theta)/|\operatorname{Aut}(\theta)|$.

Index the coefficients by $j = (\kappa, \boldsymbol{\theta})$ with the standard class outside finitely many places. Put $v_j(w) = \mathbf{1}_{\kappa(w)=\kappa}\prod_v B_v(w, \theta_v)$ and $a_j = \prod_v |\operatorname{Aut}(\theta_v)|^{-1}$. Expansion counts the local isomorphisms between the two presentations with conjugate phase at the second point. The local square sums converge and the reference factor is one away from the exceptional places. Thus on each finite point list

$$K_{\mathrm{pt}}(w_x, w_y) = \sum_j a_j v_j(w_x)\overline{v_j(w_y)}. \tag{A.329}$$

Removing finitely many summands leaves a positive semidefinite matrix.
Lean: `Formalism.L2.SG.Pf.ktop_sub_posSemidef` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.SupportProposition`, lines 533–595)

Let $\mathcal{E}_x$ be the generic terms with $G \ne 0$ at w_x. For t write $\rho(t)$ for its scalar-partner class, m_t for its row mass divided by total row mass, $h_t(x)$ for its orbital coefficient and $j(t)$ for its kernel index. With d_j the density divided by α, the count identities at positive normalizers are

$$\sum_{t' \in \mathcal{E}_y:\, \rho(t') = \rho(t)} h_{t'}(y) = v_{j(t)}(w_y), \qquad \sum_{t \in \mathcal{E}_x:\, j(t) = j} m_t h_t(x) = n(w_x)d_j a_j v_j(w_x). \tag{A.330}$$

For the first identity, fix a tuple of admissible double orbits at w_y. At its nonstandard places and the exceptional reference places choose local unitriangular parameters and congruence depths realizing it. Approximate the parameters entrywise by a rational lower unitriangular matrix L and an upper one U, with denominators supported on those places. For a point w with block data (t_i, s_i), let $M(w)$ be the block diagonal matrix whose i-th block is t_i times the order-reversing permutation matrix of size s_i. Then $LM(w_y)US_g$, with S_g the standard involution in point coordinates, realizes the tuple. At all other places the matrices and their inverses are integral and preserve the reference orbit. Intersect the local partner lattices with the rational space to obtain the partner frame lattice. Localization recovers the lattices with their identifications, so the partner count is the restricted product of local isomorphism counts. An inadmissible local orbit contributes zero. Weight by the local phase measures and sum over orbit tuples.

For the second identity, the perpendicular space of the rank-m frame is definite of rank $k-1$. Unimodular extensions correspond to complement lattices with their discriminant modules glued to that of the frame. Count the gluings with inverse stabilizer weights. Nonintegral frames have count zero. At an integral place with c_v nonunit elementary divisors, put $R = k+m-1$ and $\alpha_v(a) = \prod_{i=1}^{a}(1-\eta_v^i q_v^{-i})$, where q_v is the residue cardinality and $\eta_v = 1, -1$ in the split and inert cases. The local complement ratio is

$$\frac{\alpha_v(k-1)}{\alpha_v(k-1-c_v)} = \frac{\alpha_v(k-1)}{\alpha_v(R)}\,\frac{\alpha_v(R)}{\alpha_v(k-1-c_v)}.$$

The two factors give the product of $L(i,\eta^i)$ for $k \le i \le R$ and the represented densities, respectively. The determinant factor is $|\operatorname{disc} F|^{-2r(r+1)(k-1)}|N_{E/\mathbf{Q}}(G/g^m)|^{(k-1)/2}$, where F is the maximal real subfield of E. Take c_{count} to be the rank-$(k-1)$ mass constant times the field factor and the L-values divided by total row mass. Dividing the local orbital sums by the automorphism orders gives the second identity.

Lean: `Formalism.L2.SG.Pf.formulaExplicit_of` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.DualFamilyCounting.Assemble`, lines 158–247)
Lean: `Formalism.L2.SG.LsDatumB.partnerSumC_of` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.LsDatumBPf`, lines 27–81)
Lean: `Formalism.L2.SG.LsDatumB.fiberCntC_of` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.LsDatumBPf`, lines 83–172)

The point-coordinate row is $C_\rho(x) = n(w_x)^{-1/2}\sum_{\rho(t)=\rho}\sqrt{m_t}h_t(x)$ for $n(w_x) > 0$, and zero otherwise. Scalar partners have equal row mass. By (A.330), on positive support the Gram entry is

$$\sqrt{n(w_x)/n(w_y)}\sum_j d_j a_j v_j(w_x)\overline{v_j(w_y)}.$$

The sum uses the indices of the represented classes. On the support of v_j, the shifted determinants are locally associated. If $D_r = 0$, this fixes the absolute norm of D_{r+1}. If $D_r \neq 0$, the common kind fixes the shift factor and the associations fix the absolute norm of D_r. In that case $G = D_r\overline{D_r}(\kappa\bar{\kappa} - TT')$. Hence n is constant on each support.

Lean: `Formalism.L2.SG.Pf.supportVec_of_normProd` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.SupportProposition`, lines 484–503)
Lean: `Formalism.L2.SG.DataCut.rows_pair` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.DataCut`, lines 674–752)

Each rank-one summand of (A.329) therefore enters the row Gram matrix with a multiplier in $[0, s_d]$, with zero multiplier on zero support. Subtract the row Gram matrix from $s_d K_{\mathrm{pt}}$ term by term. The difference is positive semidefinite. Pull back to the flag points and take the diagonal congruence by the inverse positive flag normalizers, with zero for flags without points. Restricting the rows to S adds positive Gram matrices to the difference. Evaluate at z. □

Proof of 33.80

Proof. Put $M = \max(u_l, u_l')$ and $m_l = \min(u_l, u_l')$. The displacement bound gives $2-M \le y, y+1 \le -2-m_l$ for $y \in Y$. For $i < l$ both weights have i-th coordinate at least $M+1$, and for $i > l$ both have i-th coordinate at most $m_l - 1$. Thus the two products in $q(y)$ have opposite signs from coordinate l, and each denominator factor has absolute value at least one. In particular $q(y) > 0$.

Lean: `CompanionTheory.WaveGrid.nq_onCut_pair` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.NqMovedExample`, lines 85–106)

For $x = -y$ pair $\lambda_i > x$ with the upper neighbour and $\lambda_i \le x$ with the lower one. Monotonicity of λ_i makes the two lists of neighbours disjoint. For the paired factor the half-unit inequalities give $|y+1/2+u_j| \ge |y+\lambda_i|$, and likewise for u'. For $|y+\lambda_i+e|+c$, $e = 0, 1$, the nearest paired factor on either side of x costs at most $2+c$. The h-th further factor has distance at least $h(1+c)$ and costs at most $1+1/h$. These factors cost at most k on each side. There are at most $2r+3$ unpaired numerator factors, including the exceptional index, each at most $3b$. Using the unit lower bounds on the remaining denominators gives a cost of at most $(2+c)^2k^2(3b)^{2r+3} \le b^{4r+10}$. Use u at $e = 0$ and u' at $e = 1$ to obtain

$$P_*(y)P_*(y+1) \le b^{8r+20}q(y)/(T_pT_q). \tag{A.331}$$

Lean: `GroundTheory.BlockD.FourierDecayAlt.WaveGridLiftGrid.pst_mul_le` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.InterlacingGrids.PstMul`, lines 23–55)

There are at least $3(d+1)$ unused nodes since $d \le 2r$ and $|w| \le N-3$. Spacing five permits the positive and negative extensions. At an unused node each existing letter contributes a positive factor and costs at most $4/3$ in $q(y)/q_w(y)$. Thus the square root in (33.24) is bounded by Λ by (A.331). The degree-d coefficient of the j-th Lagrange basis polynomial in row a is L_{aj}, with $\sum_j |L_{aj}| \le (d+1)5^{-d} \le 1$. Factor the three row sums.

Lean: `CompanionTheory.WaveGrid.gridOK_of_free` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.Bridge`, lines 58–112)

□

Proof of 33.81

Proof. If $I_0 = 0$, the zero function gives the expansion. Otherwise discard zero-energy children and put $V_\mu = (I_0/I_\mu)^{1/2}\mathcal{J}_\mu$ at each surviving index. These functions have energy I_0 and $V_0 = \mathcal{J}_0$. The coefficient from μ to $\mu + \mathbf{1}_I$ becomes $\tau_{\mu I}(I_{\mu+\mathbf{1}_I}/I_\mu)^{1/2}$. Summing over the at most n^s children gives

$$\sum_I |\tau_{\mu I}|(I_{\mu+\mathbf{1}_I}/I_\mu)^{1/2} \le (2n^{3/2}H)^s.$$

Here $\sqrt{\exp(1)} < 2$. We have $2n^{3/2} \le b^{5/2}$. Since $m = 2s \ge 2$,

$$s\big((C_h+10)m + 5/2\big) \le (C_h/2+6)m^2.$$

Expand successively along paths of length a. At each step the sum of absolute coefficients is multiplied by at most this one-step bound. Combining paths ending at the same index can only decrease that sum. □

Proof of 33.83

Proof (outline). Claim 1. The regular compact parameter admits integer coordinates and a spanning family of interlacing Capelli spaces. Choose $\beta \in \mathbf{C}$ and strictly decreasing integers $L_0, \dots, L_{k-1}$ such that

$$u_{j+1} - k/2 + 1/2 = \beta + L_j \quad (0 \le j < k).$$

Then $L_i - L_j = u_{i+1} - u_{j+1}$, so $L_0 - L_{k-1} \le 2b$. Choose an orthonormal basis $e_0, \dots, e_{k-1}$ of the positive Hermitian space E_z, and put $e = e_{k-1}$. Under the subgroup fixing e, the Capelli eigenspaces V_M, with shifted parameters $\beta + M_j$, span V, where

$$M \in \mathbf{Z}^{k-1}, \qquad L_{j+1} < M_j \le L_j \quad (0 \le j < k-1).$$

Lean: `GlobalTheory.exists_int_coords_of_regular_param` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.ExistsIntCoordsOfRegularParam`, lines 5–42)
Lean: `Formalism.FourierDecayAlt.P.branching_decomposition` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.IrreducibleCapelliModules`, lines 25–27)

Define

$$w_+(L,M,h) = \frac{\prod_{j=0}^{k-2}(M_j - 1 - L_h)}{\prod_{i\ne h}(L_i - L_h)}, \qquad w_-(L,M,h) = \frac{\prod_{j=0}^{k-2}(M_j - L_h)}{\prod_{i\ne h}(L_i - L_h)}. \tag{A.332}$$

Lean: `GlobalTheory.WigSh.wP` (`Development.AbelianVarietyTheory.Definitions.Jets.Imports`, lines 9–10)
Lean: `GlobalTheory.WigSh.wM` (`Development.AbelianVarietyTheory.Definitions.Jets.Imports`, lines 12–13)

Claim 2. For a strictly decreasing list L with $L_j \ne L_h + 1$ for every j, the one-step raising map is built as follows. Identify $V \otimes E_z$ with V^k through the frame $e_0, \dots, e_{k-1}$, with U_z acting by $(a \cdot y)_c = \sum_d h(ae_d, e_c)\pi(a)y_d$, where h is linear in its first argument. Let E_{cb} be the infinitesimal operators of π in this frame and let Ω be the split Casimir operator $(\Omega y)_b = \sum_c E_{cb}y_c$. It is self-adjoint, its eigenvalues lie among $\beta + L_0, \dots, \beta + L_{k-1}$, and its eigenspace V_1 for $\beta + L_h$ is U_z-stable and irreducible with shifted parameters $\beta + L + \mathbf{1}_h$, that is, of type $u + \mathbf{1}_{h+1}$. For $y \in V_1$ put $A(\omega)y = \sum_c h(e_c, \omega)y_c$. This map is equivariant, of degree one in $\overline{\omega}$, and $\|A(\omega)\|^2 \le \operatorname{Re} h(\omega,\omega)$ on E_z, since $\sum_c \|y_c\|^2 = \|y\|^2$. As $A(e)y = y_{k-1}$, the vector $A(e)^*v$ is the orthogonal projection of $v \otimes e$ to V_1, and $A(e)A(e)^*v$ is its e-coordinate. The last row of the comatrix of the Capelli matrix gives linear maps $K(t)$ with $K(t)(tx + \Omega x) = \prod_j (t + \beta + L_j)\, x_{k-1}$ and $K(t)(v \otimes e) = C'(t-1)v$, where C' is the Capelli determinant of the subgroup fixing e, so $C'(t-1) = \prod_{j=0}^{k-2}(t - 1 + \beta + M_j)$ on V_M.

Taking the residue at the simple root $t = -\beta - L_h$ gives $A(e)A(e)^* v = w_+(L, M, h)v$ on V_M. In particular $V_1 \neq 0$, since this coefficient is nonzero for $M_j = L_j$. The lowering map is built in the same way from $V \otimes E_z^*$ and the dual split Casimir operator when $L_j \neq L_h - 1$ for every j. It has type $u - \mathbf{1}_{h+1}$ and coefficient $w_-(L, M, h)$. Composing these maps gives the chains. More precisely, if $L_h + d < L_j$ for every $j < h$, there is a degree-d raising contraction with

$$A(e)A(e)^* \big|_{V_M} = \prod_{a=0}^{d-1} w_+(L + a\mathbf{1}_h, M, h)\ \mathrm{id}_{V_M}\,.$$

If $L_j < L_h - d$ for every $j > h$, the lowering chain has eigenvalue

$$\prod_{a=0}^{d-1} w_-(L - a\mathbf{1}_h, M, h).$$

For the induction, the degree-zero map is the identity. If A is a chain and B the next one-step map, their composition has the stated polynomial degree, equivariance and real homogeneity. Its operator norm is bounded by the product of the two operator norms. The operator $A(e)^*$ maps into the eigenspace with the same Capelli parameter in the intermediate representation because it intertwines the subgroup fixing e. Thus $(AB)(AB)^* = A(BB^*)A^*$ multiplies the preceding scalar by the next factor. The strict gap conditions ensure that every intermediate type and every one-step map is defined.

Lean: `GlobalTheory.WigSh.chain_raise` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.ExistsWignerContractionChainRaising`, lines 15–97)
Lean: `GlobalTheory.WigSh.exists_wigMap_chain_lowering` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.ExistsWignerContractionChainLowering`, lines 17–103)

Claim 3. The scalar estimate used at each step is as follows. Suppose $D \geq 1$, $L_0 - L_h \leq D$, and $L_h - L_{k-1} \leq D$. Then

$$w_+(L, M, h) \geq D^{-2} \quad \text{if every } M_j \neq L_h + 1, \qquad w_-(L, M, h) \geq D^{-2} \quad \text{if every } M_j \neq L_h.$$

To prove this, set $d_i = \max(1, |L_i - L_h|)$, so $d_h = 1$, and put

$$Q = \prod_{j<h} d_{j+1} \prod_{j\geq h} d_j,$$

where both products use $0 \leq j < k-1$. Integrality and interlacing show that each absolute numerator factor is at least d_{j+1} when $j < h$, and at least d_j when $j \geq h$. The stated nonzero-factor conditions handle the adjacent factor. The denominator has absolute value Qd_0d_{k-1}. Numerator and denominator have the same sign, since each has $k - 1 - h$ negative factors. Consequently each quotient is at least $(d_0d_{k-1})^{-1} \geq D^{-2}$.

Lean: `GlobalTheory.WigSh.inv_sq_le_wP` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.InvSqLeWignerCoeffSqRaising`, lines 35–95)
Lean: `GlobalTheory.WigSh.inv_sq_le_wM` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.InvSqLeWignerCoeffSqLowering`, lines 60–102)

Claim 4. On each interlacing space, one of the two degree-d chains has eigenvalue at least $(3b)^{-2d}$. Put $i = l - 1$. If $M_i \geq L_{i+1} + d + 1$, use the chain raising coordinate $i + 1$; otherwise use the chain lowering coordinate i. For every $0 \leq a < d$, the interlacing inequalities persist and the required numerator factors are nonzero. This follows for the distinguished coordinate from $L_i - L_{i+1} \geq 2d + 1$; the other coordinates lie strictly above L_i or at most L_{i+1}. The endpoint distances of the shifted list are at most $2b + a \leq 3b$. Hence one of the two chain eigenvalues is at least $(3b)^{-2d}$ on each V_M.

Claim 5. The chain eigenvalue bound implies the claimed norm inequality on all of E_z. Let $X = A_+(e)A_+(e)^* + A_-(e)A_-(e)^*$. Both summands are positive and self-adjoint. On every nonzero V_M, X acts by the sum of the two nonnegative chain eigenvalues, hence by a scalar at least $(3b)^{-2d}$. Since these spaces span V, the spectral theorem gives

$$(3b)^{-2d}\|v\|^2 \leq \mathrm{Re}\langle v, Xv\rangle = \|A_+(e)^*v\|^2 + \|A_-(e)^*v\|^2.$$

Lean: `GlobalTheory.frame_exists_and_vector_eq_smul_translate` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.FrameExistsAndVectorEqSmulTranslate`, lines 189–259)

Finally write $\omega = tae$, with $a \in U_z$, $t \in \mathbf{R}$, and $t^2 = \operatorname{Re} h(\omega, \omega)$. Equivariance and unitarity transfer the unit-vector inequality to ae; homogeneity multiplies it by t^{2d}. This proves the asserted bound.
Lean: `GlobalTheory.WigSh.chain_lower_bound` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.ChainLowerBound`, lines 22–115)
Lean: `GlobalTheory.norm_sq_adjoint_add_ge_of_joint_eigenspaces` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.NormSqAdjointAddGeOfJointEigenspaces`, lines 7–82)
Lean: `GlobalTheory.WigSh.lower_bound_spreads` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.NormAdjointApplyOfNormPreserving`, lines 46–58)
The complete proof is contained in `GlobalTheory.WigSh.exists_shifted_types_lower_bound`. □

Proof of 33.84

Proof. For (a), expand the squared norm of $A_{p,x}$ on a compact annulus and use $D = \omega^{-1} z^s$. The Jacobian is $s^2|z|^{2s-2}$. The equal-root terms give the integral of $\|b_p\|^2$ on the inverse image of that annulus. Every unequal-root term is the Fourier transform of an integrable function at a frequency tending to infinity. Its limit is zero. The assumed error tends to zero in the annular L^2 norm, so the bound on $U_{p,x}$ bounds this integral by M_p. Exhausting $\mathbf{C}^\times$ proves $\|b_p\|_2^2 \le M_p$.

The function $a_p(D) = \sum_{z^s = \omega D} \|B_p(z)\|$ satisfies $\|a_p\|_2^2 \le s\|b_p\|_2^2$. It dominates $A_{p,x}$, uniformly in x. The maps T_x are isometries. This proves the stated diagonal bound; the block matrix is the Gram matrix of the vectors $u_{p,x}, a_{p,x}$.

To compute the boundary limit put $x' = x + \sigma$ and choose $\rho_x \to 1$ with $\rho_x^s = (x/x')^{s-1}$. In the paired root sums the phases are

$$\chi\big(s(\zeta \rho_x x' - x)z\big), \qquad \zeta^s = 1.$$

Their densities converge in L^1, because normalized dilations and rotations are continuous in L^2. The expansion

$$\rho_x = 1 - \frac{s-1}{s}\frac{\sigma}{x'} + O(|x|^{-2})$$

gives limiting frequency σ when $\zeta = 1$; all other frequencies tend to infinity. Fourier continuity and decay prove the boundary-boundary limit. For each mixed limit, first restrict to an annulus and use the assumed approximation. On its complement use Cauchy–Schwarz, the uniform L^2 bound on the original vectors, and the uniformly small tails of the dilated boundary dominators. This proves (33.25).

For (b), choose a lattice basis and periodize D. Its Fourier coefficients are $\widehat{D}(\sigma)/\operatorname{covol}(L)$. Take product Fejér means in the lattice coordinates. The real part of the Fejér polynomial formed from A is nonnegative: its weights are autocorrelation coefficients of a finite lattice box, and the assumed quadratic-form inequality applies also after multiplying those box coefficients by the character at the evaluation point. Adding this real part bounds each Fejér mean of the periodization by

$$\frac{J}{\operatorname{covol}(L)} \sum_{\sigma \in L^\vee} F(\sigma).$$

For the finitely many translates corresponding to S, their sum is nonnegative and lower semicontinuous and is bounded almost everywhere by the periodization. The approximate-identity property therefore bounds its value at c_0 by the lower limit of its Fejér means. This gives (33.26).

For (c), $L_R^\vee = ((2R)^{-1}(\mathbf{Z} + i\mathbf{Z}))^f$, with a harmless sign change in the imaginary lattice coordinate. In one coordinate put $a = 2Rz/g$. The lattice points in the disc of radius $\max(1, a)$ contribute at most $C_0(1 + a^2)$. In the annulus of radii $2^j \max(1, a)$ and $2^{j+1}\max(1, a)$, the number of points is at most $C_1 2^{2j} \max(1, a)^2$, and the summand is at most

$$\left(\frac{a}{2^j \max(1, a)}\right)^K.$$

Summing the geometric series $\sum_j 2^{(2-K)j}$ bounds the tail by $C_2(1 + a^2)$. This also proves convergence. Taking products, absorbing the factor 2 into C, and dividing by R^{2f} proves the inequality and its limit superior. □

Proof of 33.85

Proof. For a polynomial $P(t) = \sum_{q=0}^{d} t^q c_q$ with coefficients in V, the coefficient of t^d in its Lagrange interpolation formula is

$$c_d = \sum_{j=0}^{d} \left(\prod_{h \neq j} (y_j - y_h)^{-1} \right) P(y_j).$$

Apply this identity to coordinate $a = 0$, with all other coordinates initially zero. In each resulting summand apply it to coordinate $a = 1$, and continue. At step a, the leading coefficient is precisely $\sigma_a W_a$ by hypothesis. After K steps all coordinates have been evaluated on their grids, and the products of the row coefficients give the displayed formula. □

Proof of 33.86

Proof. The two vectors in the one-slot contraction are

$$\widetilde{W}_\omega I^+(x) = (W_\omega x, \omega W_\omega x), \qquad I^-(y) = (y, 0).$$

Consequently,

$$\mathscr{J}_\Xi\big(\widetilde{W}_\omega I^+(x), I^-(y)\big) = \Xi(\omega W_\omega x, y) - \Xi(W_\omega x, 0) = \omega \Xi(W_\omega x, y).$$

The scalar is ω, because the first argument of Ξ is linear.

For the iterated formula, each contraction chooses either the first term of the one-slot rule or its negative second term. This gives the displayed signed sum. Every term vanishes except the term with $\epsilon = (1, \dots, 1)$, since all other terms have $B_{\mathbf{1}-\epsilon} = 0$. The surviving term has sign 1 and value

$$\Xi(\omega^K W_\omega x, y) = \omega^K \Xi(W_\omega x, y).$$

This also proves the assertion at zero frequency. □

Proof of 33.87

Proof (outline). We first spell out the word frame $(\mathcal{F}_w, d_w)$, its positivity on the cut, the integral, and the realization in the statement. Starting from $\mathcal{F}_\varnothing = 1$, $c_\varnothing = 1$, $x_\varnothing(Y) = x(Y)$, $e_\varnothing(Y) = e(Y)$, $q_\varnothing(Y) = q(Y)$, and $d_\varnothing = 1$, at a word w of length N set $\ell_w(Y) = \ell_0(Y) + \overline{\zeta}\zeta^{-1}\mathcal{G}^{-1} e_w(Y)$, $d_w = (|\zeta|^2 \mathcal{G}^{-1})^N c_w$, $m_w(Y) = d_w^{-1} \ell_w(Y-1)^* d_w$, $\Delta_w(Y, Y_a) = (Y - Y_a)^{-1}(e_w(Y) - e_w(Y_a))$, and $t_\zeta = T_p T_q \overline{\zeta}^{-1}$. Prepending a positive letter $a = (+, s, Y_a)$ updates

$$\mathcal{F}_{a::w} = \mathrm{ins}_s\left(\begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \overline{\zeta}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mathcal{G}\ell_w(Y_a) & \mathcal{G} \end{pmatrix} \right) \mathcal{F}_w, \qquad q_{a::w}(Y) = q_w(Y) \frac{Y - Y_a + 1}{Y - Y_a},$$

$$x_{a::w}(Y) = x_w(Y)\, \mathrm{ins}_s\, \mathrm{diag}\Big(\frac{Y - Y_a}{Y - Y_a - 1}, 1 \Big), \qquad c_{a::w} = c_w\, \mathrm{ins}_s\, \mathrm{diag}\big(q_w(Y_a) x_w(Y_a + 1)^{-1}, x_w(Y_a) \big),$$

and $e_{a::w}(Y) = \mathrm{ins}_s\left(\begin{smallmatrix} e_w(Y) - \Delta_w(Y, Y_a) + t_\zeta & -(Y - Y_a)^{-1} \\ 0 & x_w(Y_a)^{-1} e_w(Y) x_w(Y_a) \end{smallmatrix} \right)$; prepending a negative letter $b = (-, s, Y_b)$ updates

$$\mathcal{F}_{b::w} = \mathrm{ins}_s\left(\begin{pmatrix} \overline{\zeta}^{-1} & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mathcal{G} m_w(Y_b) & \mathcal{G} \end{pmatrix} \right) \mathcal{F}_w, \qquad q_{b::w}(Y) = q_w(Y) \frac{Y_b - Y}{Y_b - Y - 1},$$

$$x_{b::w}(Y) = x_w(Y)\, \mathrm{ins}_s\, \mathrm{diag}\Big(\frac{Y_b - Y}{Y_b - Y - 1}, 1 \Big), \qquad c_{b::w} = c_w\, \mathrm{ins}_s\, \mathrm{diag}\big(q_w(Y_b - 1) x_w(Y_b - 1)^{-1}, x_w(Y_b) \big),$$

and $e_{b::w}(Y) = \mathrm{ins}_s\left(\begin{smallmatrix} e_w(Y) & 0 \\ -\frac{q_w(Y_b - 1)}{Y_b - Y - 1} x_w(Y_b)^{-1} x_w(Y_b - 1)^{-1} & x_w(Y_b)^{-1} e_w(Y) x_w(Y_b) - \Delta_w(Y, Y_b) + t_\zeta \end{smallmatrix} \right)$, with $d_{a::w} = (|\zeta|^2 \mathcal{G}^{-1})^{N+1} c_{a::w}$ in both cases. Gauss elimination on the 2×2 Yangian transfer relations of Definition 33.82 proves by induction on w the exact top-level factorization $R_r(w) = \mathcal{F}_w^* d_w \mathcal{F}_w$. Moreover, the normal form

of the actual level supplies an algebra homomorphism κ from the complex functions on $\mathcal{T}$ to $\mathscr{A}$, compatible with complex conjugation and commuting with the terminal multiplication and differentiation operators, such that $x(Y)=\kappa(\rho\mapsto x_{\rho,l}(Y))$ with $x_{\rho,l}(Y)=(-1)^{k-l}\prod_{i=1}^{k-1}(Y+a_i+\rho_i)$. If $f(\rho)>0$ for every $\rho\in\mathcal{T}$, then $\langle((\kappa f)\phi)(c,g),\phi(c,g)\rangle_{\mathrm{HS}}\geq 0$ for every $\phi\in V$ and every point (c,g). Indeed, for a real Y_0 at which the values $x_\rho(Y_0)$ are distinct, $\mathcal{G}^{-1}Q_{\mathrm{act}}(Y_0)_{11}$ acts at each point of $\mathcal{D}$ by right composition with a symmetric quantum determinant on W_q, and κf is a polynomial in this operator that is positive on the spectrum of that determinant. For every admissible cut word w on cut $1\leq l\leq k$, interlacing of $a_i+\rho_i$ with u_i,u_i' makes exactly $k-l$ factors of $x_\rho(Y)$ (and of $x_\rho(Y\pm1)$) negative, so $x_{\rho,l}(Y),x_{\rho,l}(Y\pm1)>0$; simultaneously the l-th pair of $q(Y)=-T_pT_q\prod_{i=1}^{k}(Y+\frac12+u_i)(Y+\frac12+u_i')$ is strictly negative while all other pairs are positive (and likewise at $Y-1$ by the cut margin 2), so $q(Y),q(Y-1)>0$; and every cross-letter rational factor $\frac{Y-Y_a+1}{Y-Y_a}$, $\frac{Y-Y_a}{Y-Y_a-1}$, $\frac{Y_b-Y}{Y_b-Y-1}$ is strictly positive because $|Y-Y'|>2$ for distinct letters. Since $|\zeta|^2\mathcal{G}^{-1}>0$ on $\mathcal{D}$, the e-th diagonal entry of d_w is $(|\zeta|^2\mathcal{G}^{-1})^N\kappa(f_e)$, where $f_e(\rho)>0$ for every $\rho\in\mathcal{T}$ is obtained from the scalar recursions above with x replaced by $x_{\rho,l}$. By the positivity of κ on positive functions, for $x,y\in V^{\{0,1\}^S}$ the top density

$$\delta_w(x,y;c,g)=\sum_{a\in\{0,1\}^S}\big\langle\big(d_w\mathcal{F}_wx\big)_a(c,g),\big(\mathcal{F}_wy\big)_a(c,g)\big\rangle_{\mathrm{HS}}$$

(with the Hilbert–Schmidt pairing linear in its first argument) satisfies $\delta_w(x,x;c,g)\geq 0$ pointwise on $\mathcal{D}$. Here operator matrices act by their first components as in Definition 33.82, and the top word operator factors as $R_r(w)=\mathcal{F}_w^*d_w\mathcal{F}_w$. Put

$$B_{[0,r)}(w)=B_{r-1}(w)\cdots B_0(w).$$

The empty product is the identity. For $x,y\in\mathcal{K}_0^{\{0,1\}^S}$ and $c\in\Omega$, the asserted composite formula is

$$\Xi_{0,w}(x,y;c)=\int_{\mathbb{C}^r\times G_r}\left(\prod_{j=0}^{r-1}|t_j|^{-2}\right)\delta_w\big(B_{[0,r)}(w)x,B_{[0,r)}(w)y;t,g\big)\,dt\,d\mu(g).\tag{A.333}$$

Its integrand is absolutely integrable. The left-hand side is independent of c, since at level zero every radial coordinate is integrated.

Lean: `Formalism.FourierDecayAlt.P5.bTall` (`Formalism.HermitianCurvature.CompositeFormulaLevelZero`, lines 229–233)
Lean: `Formalism.FourierDecayAlt.P5.topDens` (`Formalism.HermitianCurvature.CompositeFormulaLevelZero`, lines 91–94)
Lean: `Formalism.FourierDecayAlt.P5.CompositeFormulaLevelZero` (`Formalism.HermitianCurvature.CompositeFormulaLevelZero`, lines 235–246)
Lean: `Formalism.FourierDecayAlt.P5.ofWordRR` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.TopIdentFromParts`, lines 11–22)

The group size used here is $g_N(g)=(\sum_{a,b}|g_{ab}|^2)^{1/2}$. The weighted differentiability condition means that for each real a and each integer $q\geq 0$, the coefficient is q times differentiable on the group and there is a constant $C_{a,q}$ such that

$$g_N(g)^a\,\|d^L_{X_1}\cdots d^L_{X_u}d^R_{Y_1}\cdots d^R_{Y_v}F(g)\|\leq C_{a,q}\quad(u+v\leq q)$$

for the normalized Lie directions in the coefficient bounds. For every such F with compact support in the ambient matrix space and the stated Gram lower bound, the realization is a vector $\phi_F\in\mathcal{K}_0$ satisfying

$$\phi_F(c,g)=\mathrm{res}_r(J_DF)(c,g)\qquad((c,g)\text{ in the radial domain}).$$

The averaging J_D, the restriction res_r, and the radial domain use the fixed real-place datum; no change of measure or representation is made. In subsequent applications to wave-grid and distinct-word coefficient bounds, this realization clause is invoked exclusively on smooth compactly supported cutoffs (such as $H_{\varepsilon,M}$ or $F_{v,\varepsilon,R}$), which satisfy the weighted differentiability bound for every a and q simultaneously by Lemma 33.110, whereas finite decay thresholds such as $(A,5)$ or (A_0,m_*) govern only passage to the cutoff limits. Explicitly, for $0\leq i\leq r$ put $m_i=2(r-i)+k+1$, let $\mathrm{pos}_i=(i,i+1,\dots,i+m_i-1)$ index the central $m_i\times m_i$ residual coordinates inside $\{0,\dots,n-1\}$, and set $u_i(Y)=Y+(m_i-1)/2$. Let $E_{a,b}$ be the left-invariant $\mathfrak{u}(H_{k,r})$ differential operators acting on smooth matrix functions f, and for equal-length index tuples α,β write $\mathrm{qmin}_{\alpha,\beta}(u)=\sum_{\sigma\in S_t}\mathrm{sgn}(\sigma)\big(u\delta_{\alpha_{\sigma(1)}\beta_1}+E_{\alpha_{\sigma(1)}\beta_1}\big)\cdots\big((u-t+1)\delta_{\alpha_{\sigma(t)}\beta_t}+E_{\alpha_{\sigma(t)}\beta_t}\big)$ for the column-determinant quantum minor on rows α and columns β of length t; the factors are composed from left to

right, and the j-th column carries the spectral parameter $u-j+1$. Define the level-i Capelli (unbordered) quantum minor $\mathrm{cap}_i(u)=\mathrm{qmin}_{\mathrm{pos}_i,\mathrm{pos}_i}(u)$ and, writing $\mathrm{pos}_i(p)$ for the p-th entry of pos_i ($1\le p\le m_i$) and keeping the order of the entries that remain after a removal, the singly bordered quantum minor

$$\widehat{c}_i(u;a,\beta)f(g)=\sum_{p,q=1}^{m_i}(-1)^{p+q}\beta_p(g)a_q(g)\ \mathrm{qmin}_{\mathrm{pos}_i\setminus\{\mathrm{pos}_i(p)\},\,\mathrm{pos}_i\setminus\{\mathrm{pos}_i(q)\}}(u)f(g),$$

and the doubly bordered quantum minor

$$\widehat{c}_{2,i}(u;a,a',\beta,\beta')f(g)=\sum_{\substack{1\le p<p'\le m_i\\ 1\le q<q'\le m_i}}(-1)^{p+p'+q+q'}(\beta_p\beta'_{p'}-\beta_{p'}\beta'_p)(a_qa'_{q'}-a_{q'}a'_q)\cdot\mathrm{qmin}_{\mathrm{pos}_i\setminus\{\mathrm{pos}_i(p),\mathrm{pos}_i(p')\},\,\mathrm{pos}_i\setminus\{\mathrm{pos}_i(q),\mathrm{pos}_i(q')\}}(u)f(g).$$

From the level-i residual frame vectors z_i,z'_i (projections of z,z' onto the level-i residual subspace), form the coordinate vectors $x_i(j)=(z'_i)_{\mathrm{pos}_i(j)}$, $\bar{x}_i(j)=h_{H_{k,r}}(e_{\mathrm{pos}_i(j)},z'_i)$, $y_i(g,j)=(g^{-1}z_i)_{\mathrm{pos}_i(j)}$, and $\bar{y}_i(g,j)=h_{H_{k,r}}(ge_{\mathrm{pos}_i(j)},z_i)$. The actual level-$i$ quantum-minor 2×2 matrix $Q_i^{\mathrm{act}}(Y)$ is then

$$Q_i^{\mathrm{act}}(Y)=\begin{pmatrix}\mathrm{cap}_i(u_i(Y)) & -\widehat{c}_i(u_i(Y)-1;x_i,\bar{y}_i)\\ -\widehat{c}_i(u_i(Y);y_i,\bar{x}_i) & -\widehat{c}_{2,i}(u_i(Y)-1;y_i,x_i,\bar{y}_i,\bar{x}_i)\end{pmatrix},$$

and at $i=r$ this formula gives the action on the radial domain of the paired entries from Definition 33.82, namely $(Q_{\mathrm{act}}(Y))_{ab}=(Q_r^{\mathrm{act}}(Y)_{ab},Q_r^{\mathrm{act}}(\overline{Y})_{ba})$, while the level-zero positive and negative coefficient insertions act by the corresponding ordered-minor differential operators on the ambient matrix space.

Lean: `Formalism.FourierDecayAlt.B.gN` (`Formalism.HermitianCurvature.StatementB`, lines 230–230)
Lean: `Formalism.FourierDecayAlt.B.CCle` (`Formalism.HermitianCurvature.StatementB`, lines 236–239)
Lean: `Formalism.FourierDecayAlt.B.InCC` (`Formalism.HermitianCurvature.StatementB`, lines 241–241)
Lean: `Formalism.FourierDecayAlt.P5.IsRadJF` (`Formalism.HermitianCurvature.TopDensNonnegOfKappa`, lines 25–26)

Claim 1. Suppose $r>0$. The spectral data of the actual level carry terminal operators $D_r,\overline{D}_r,D_r^{-1},\overline{D}_r^{-1},\mathcal{G}^{-1}$ and operator-valued functions $A_r,B_r,A_r^\vee,B_r^\vee$, with the specified radial-action identities on the equivariant coefficient domain. In this claim these are the fixed operators of that spectral datum, with those identities as part of the data. Put $M_r=D_r^{-1}\overline{D}_r^{-1}\mathcal{G}^{-1}$. For every real Y with $Y-1,Y,Y+1$ in the invertibility set of the start data, the two radial operators

$$\begin{aligned}R_+(Y)&=\overline{D}_r\big(Q_{\mathrm{st}}(Y)_{10}A_r(Y)+Q_{\mathrm{st}}(Y)_{11}B_r(Y)\big),\\ R_-(Y)&=D_r\big(Q_{\mathrm{st}}(Y)_{01}A_r^\vee(Y)+Q_{\mathrm{st}}(Y)_{11}B_r^\vee(Y)\big),\\ T_+(Y)&=M_r\,\xi(Y)R_+(Y),\qquad T_-(Y)=M_r\,\xi(Y+1)R_-(Y+1)\end{aligned}$$

have group differential-operator lifts $\Theta_+(Y),\Theta_-(Y)$. All the displayed factors are the specified terminal operators of the level and start data, in the displayed order. For either sign, whenever J belongs to the specified equivariant coefficient domain and ϕ is its radial realization,

$$\mathrm{res}_r(\Theta_\pm(Y)J)=T_\pm(Y)\phi$$

on the radial domain. Each lift carries the smooth-coefficient and equivariance conditions on the positive Gram locus, polynomial coefficient bounds through order $\mathrm{ord}(\Theta_\pm)+5$ on each locus $G_{\mathrm{loc}}\ge\varepsilon>0$, and preservation of the designated smooth test class by both the operator and its formal adjoint. These are properties of the lifted differential-operator data with its given test domain.

Lean: `Formalism.FourierDecayAlt.P.CutSE.GiR` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.RadialIdentityOfQuadratic`, lines 106–106)
Lean: `Formalism.FourierDecayAlt.P.CutSE.RhR` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.RadialIdentityOfQuadratic`, lines 100–101)
Lean: `Formalism.FourierDecayAlt.P.CutSE.RhvR` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.RadialIdentityOfQuadratic`, lines 103–104)
Lean: `Formalism.FourierDecayAlt.P.CutSE.RadApply1` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.CutSERDIdentSpec`, lines 34–39)
Lean: `Formalism.FourierDecayAlt.P.CutSE.OpProp1` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.CutSERDIdentSpec`, lines 48–51)

Lean: `Formalism.FourierDecayAlt.P.CutSE.RDSOpR` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.CutSERDIdentSpec`, lines 76–85)
Lean: `Formalism.FourierDecayAlt.P.CutSE.RDSOpRv` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.CutSERDIdentSpec`, lines 87–96)

For the positive lift, first realize the action of $R_+(Y)$ by the right group differential action and realize $M_r\xi(Y)$ by the order-zero spectral operator. Compose these operators in that order. The radial equality follows by applying the order-zero realization to the radial equality for the right group action. The negative lift has the same construction with the adjoint right group action and the argument $Y+1$. The composition theorem preserves all the stated coefficient, equivariance and test-class properties.
Lean: `Formalism.FourierDecayAlt.P.CutSE.rDSOpR_of_parts` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.CutSERDSOpR`, lines 107–119)
Lean: `Formalism.FourierDecayAlt.P.CutSE.rDSOpRv_of_parts` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.CutSERDSOpR`, lines 121–134)
Lean: `Formalism.FourierDecayAlt.P.CutSE.spectral_composition_zero` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.SpectralCompositionZero`, lines 279–325)
Lean: `Formalism.FourierDecayAlt.P.CutSE.spectral_operator_R` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.CutSETopFinal`, lines 10–10)
Lean: `Formalism.FourierDecayAlt.P.CutSE.spectral_operator_R_adjoint` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.CutSETopFinal`, lines 12–12)

Claim 2. For every compatible actual datum, the two uniform lifting assertions of Claim 1 imply the existence of subspaces with the four conclusions of the lemma, including the compact-support realization clause. The assertion at $r=0$ is included, so no positive-rank hypothesis is imposed on the lemma. Explicitly, define $\mathcal{K}_i \subset V$ $(0 \le i \le r)$ as the $\mathbf{C}$-linear span of the smallest family of radial functions generated by: (i) at $i=0$, all radial realizations $\phi_F = \mathrm{res}_r(J_D F)$ of smooth compactly supported coefficients F satisfying the weighted differentiability bounds for all (a,q) and $G_{\mathrm{loc}}(g) \ge \varepsilon > 0$ on $\mathrm{supp}(F)$; (ii) for each $0 \le i < r$ and admissible word w, all slice-matrix entry images $B_i(w)_{ab}\phi \in \mathcal{K}_{i+1}$ of $\phi \in \mathcal{K}_i$ (giving slice closure $B_i(w) : \mathcal{K}_i^{I_S} \to \mathcal{K}_{i+1}^{I_S}$ by construction); (iii) for each $0 \le i < r$, $s \in \mathbf{C}$, and admissible w, all entries of the coordinate plane-wave twist operator multiplying by $e^{sc_i - \bar{s}\bar{c}_i}$ preserving $\mathcal{K}_i$; and (iv) at $i=r$, all entries of the terminal plane-wave twist operator multiplying by $e^{s\zeta(g) - \bar{s}\overline{\zeta(g)}}$ preserving $\mathcal{K}_r$. For every $0 \le i \le r$, $\phi \in \mathcal{K}_i$ and $c \in \Omega$ there are an open neighbourhood U of c and a compact set $K \subset \{g \in G_r : \mathcal{G}(g) > 0\}$ such that $\phi(c', g) = 0$ for $c' \in U$ and $g \in G_r \setminus K$. For a generator ϕ_F this survives the average over the noncompact unipotent group in J_D for the following reason. The frame minors D_j are invariant under the two-sided action of the groups averaged over in J_D, and at the group point x at which res_r evaluates $J_D F$ for (c', g) in the radial domain one has $D_r(x) = p(c')$ and $D_{r+1}(x) = \zeta(g)p(c')$, where $p(c') = c'_0 \cdots c'_{r-1}$. Hence $G_{\mathrm{loc}} = |p(c')|^2 \mathcal{G}(g)$ on the whole double coset of x. If $J_D F(x) \neq 0$, then F is nonzero at some point of this double coset, so $|p(c')|^2 \mathcal{G}(g) \ge \varepsilon$ and $|\zeta(g)p(c')| \le R_F$, where R_F bounds $|D_{r+1}|$ on the support of F. For $U = \{c' : |p(c)|/2 < |p(c')| < 2|p(c)|\}$ this puts g in the compact set $K = \{g \in G_r : \mathcal{G}(g) \ge \varepsilon/(4|p(c)|^2),\ |\zeta(g)| \le 2R_F/|p(c)|\}$. The slice matrices and the plane-wave twists are built from radial coordinate differential operators, which are local in c, from group differential operators, which are local in g on the open set $\{\mathcal{G} > 0\}$, and from multiplication operators. All three keep the vanishing on $U \times (G_r \setminus K)$, and finite sums are handled by intersecting the sets U and taking the union of the sets K. By Haar invariance of μ on G_r, each $\mathfrak{u}(H_{k,r})$ Lie derivative is skew-Hermitian on smooth compactly supported functions, so integration by parts with zero boundary terms, in which the derivatives of the bordering coordinates are absorbed by the column shifts of the quantum minors, yields the integral adjoint identity $\int_{G_r} \langle Q_{\mathrm{act}}(Y)_{ab}\phi, \psi\rangle_{\mathrm{HS}}\, d\mu = \int_{G_r} \langle \phi, Q_{\mathrm{act}}(\overline{Y})_{ba}\psi\rangle_{\mathrm{HS}}\, d\mu$; inducting along $R_r(w) = \mathcal{F}_w^* d_w \mathcal{F}_w$ yields $\Xi_{r,w}(x,y;c) = \int_{G_r} \delta_w(x,y;c,g)\, d\mu(g)$, which is absolutely convergent and nonnegative on the diagonal since $\delta_w(x,x;c,g) \ge 0$. For the downward slice steps when $r > 0$, combining the two lifts $\Theta_\pm(Y)$ of Claim 1 with the coordinate, quadratic-form, and pairing adjoint identities yields four polynomial-coefficient group differential operators $\Theta_0, \Theta_1, \Theta_2, \Theta_3$, each of which, with its formal adjoint, maps into itself the class of smooth coefficients with compact support in some locus $G_{\mathrm{loc}} \ge \varepsilon > 0$ and all weighted differentiability bounds. Fix the first i nonzero radial coordinates. Level transport sends $\mathcal{K}_i$ into the level-zero family of a residual datum of rank $(k, r-i)$. Write D for this datum and retain c_i for its first radial coordinate. Its first frame minor D_1 equals c_i on the radial domain and is invariant under the averaging in J_D. Thus for $\mathrm{wv}_s F = e^{sD_1 - \bar{s}\overline{D_1}} F$ one has $\mathrm{res}_{r-i}(J_D \mathrm{wv}_s F) = e^{sc_i - \bar{s}\bar{c}_i}\, \mathrm{res}_{r-i}(J_D F)$.
Lean: `Formalism.FourierDecayAlt.P.level_transport_hermitian` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.CutSHChk`, lines 23–23)

For F in the class the operators satisfy the radial spectral identity $s\, J_D(\mathrm{wv}_s F) = J_D(\Theta_0^* \mathrm{wv}_s \Theta_1 F) -$

$J_D(\Theta_2^*\mathrm{wv}_s\Theta_3F)$ at the Bessel points, and hence, for F, F' in the class and the Bessel pairing β,

$$s\,\beta(\mathrm{wv}_sF,F') = \beta(\mathrm{wv}_s\Theta_1F,\Theta_0F') - \beta(\mathrm{wv}_s\Theta_3F,\Theta_2F').$$

Since the class is preserved, the identity can be iterated: $s^{m+3}\beta(\mathrm{wv}_sF,F')$ is a finite signed sum of pairings $\beta(\mathrm{wv}_sG,G')$ with G,G' in the class. Each of these is bounded uniformly in s, because $J_D\mathrm{wv}_sG = e^{sD_1-\bar{s}\overline{D_1}}J_DG$ has the norm of J_DG at every point and the Bessel pairing of two coefficients of the class converges absolutely. For the residual realizations of F, F', integrating the first-step density against $e^{st-\bar{s}\bar{t}}$ gives the level-zero pairing with its first vector twisted, hence a fixed nonzero multiple of $\beta(\mathrm{wv}_sF,F')$. The same decay holds on the span of the plane-wave twists of these realizations, which contains the residual level-zero family. This gives, for every integer $m\ge 0$, the bound $O((1+|s|)^{-m-3})$ for the planar Fourier transforms in $t=c_i$ of the weighted step density Dens_i below and of $(1+|t|^2)^2\mathrm{Dens}_i$. Both transforms are absolutely convergent integrals. Indeed $\mathrm{Dens}_i(u,v;c,\cdot)$ is integrable on $\mathbf{C}$, with integral $\Phi_i(u,v;c)$, by Fubini's theorem and Cauchy–Schwarz, because the vectors entering it are square-integrable for $c\in\Omega$. At level zero of the residual datum, the weighted pairing of $\phi_F,\phi_{F'}$ is a fixed positive multiple of the Bessel-coordinate integral with pairing $\langle J_DF, J_DF'\rangle_{\mathrm{HS}}$. The weighted differentiability bounds give absolute convergence. Plane-wave twists have modulus one, and each component of the operator for an admissible word applied to an array in the residual level-zero family is square-integrable. Moreover $(1+|t|^2)^2\mathrm{Dens}_i(u,v;c,t) = \mathrm{Dens}_i(u,v';c,t)$, where v' is v multiplied by $(1+|c_i|^2)^2$. For $v=\phi_{F'}$, invariance of D_1 under the averaging gives $v'=\phi_{(1+|D_1|^2)^2F'}$, the realization of a coefficient with the same support, so v' is again square-integrable at residual level zero. This multiplication commutes with the plane-wave twists.

Lean: `Formalism.FourierDecayAlt.P.JCx2.rb_pairAt` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.LevelFormSesquilinearFiniteClass`, lines 2356–2380)

Moreover Dens_i is continuous in $t\in\mathbf{C}^\times$. Fix m. The two transforms are integrable against $(1+|s|)^m$, so Fourier inversion represents both functions almost everywhere by C^m functions with bounded derivatives of order at most m, and continuity on $\mathbf{C}^\times$ makes the representation exact at every $t\neq 0$. Hence Dens_i, including its factor $|t|^{-2}$, agrees on $\mathbf{C}^\times$ with a C^m function $\mathrm{Dens}_i^\sharp$ on $\mathbf{C}$, and dividing the representative of $(1+|t|^2)^2\mathrm{Dens}_i$ by $(1+|t|^2)^2$ shows that $\mathrm{Dens}_i^\sharp$ and its derivatives of order at most m are $O((1+|t|)^{-4})$. At each step $i\mapsto i+1$, put $\mathrm{Dens}_i(u,v;c,t) = |t|^{-2}\Phi_{i+1}(u,v;c\#_it)$. For every real-point single letter $\lambda=(\pm,s_0,Y)$, let $\check{B}_i(\lambda)$ be the action of the slot insertion at s_0 of $\left(\begin{smallmatrix}0 & -(c_i^{-1})^*\\ c_i^* & Y+1+p_i^*\end{smallmatrix}\right)$ when $\epsilon_\lambda=+$, and of $\left(\begin{smallmatrix}0 & -c_i^{-1}\\ c_i & Y-1+p_i\end{smallmatrix}\right)$ when $\epsilon_\lambda=-$. For $t\in\mathbf{C}^\times$, Leibniz's rule on the coordinate differential pairs $(p_i,\bar{p}_i,c_i,c_i^{-1})$ yields the one-letter Wirtinger divergence identity $\mathrm{Dens}_i(\check{B}_i(\lambda)u,B_i(\lambda)v;c,t) = \mathrm{Dens}_i(u,v;c,t) - \partial_{\bar{t}}\mathrm{Dens}_i(u,e^{10}_{s_0}v;c,t)$ for $\epsilon_\lambda=+$, and the same identity with $+\partial_t$ in place of $-\partial_{\bar{t}}$ for $\epsilon_\lambda=-$. Fix m at least the length of w. For $1\le j\le m$, every derivative of order j of $\mathrm{Dens}_i^\sharp$ is integrable on $\mathbf{C}$ and has integral zero, by Stokes' theorem on the discs $|t|\le R$, whose boundary integrals are $O(R^{-3})$ by the decay of the derivatives of order $j-1$. Hence the derivative terms of the one-letter identities integrate to zero over $\mathbf{C}$. Expanding w letter by letter and integrating gives the identity for the first residual step. Level transport intertwines the slice and word operators and scales both adjacent pairings by the same positive factor. Cancelling it gives $\Xi_{i,w}(x,y;c) = \int_{\mathbf{C}}|t|^{-2}\Xi_{i+1,w}(B_i(w)x,B_i(w)y;c\#_it)\,dt$ with absolutely integrable integrand; descending from $i=r$ to $i=0$ yields sesquilinearity, positive semidefiniteness of every $\Xi_{i,w}$ on $\mathcal{K}_i^{I_S}$, and the absolutely convergent composite integral (A.333).

Lean: `Formalism.FourierDecayAlt.P5.HerD.stepDensCptAll_of_levT` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.TheoremPCompactOfStepDensity`, lines 133–243)

Lean: `Formalism.FourierDecayAlt.P.CutSE.RDIdent` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.CutSERD`, lines 65–71)

Lean: `Formalism.FourierDecayAlt.P.CutSE.RDBeta` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.CutSERD`, lines 38–46)

Lean: `Formalism.FourierDecayAlt.P.CutSE.RDBdd` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.CutSERD`, lines 48–52)

Lean: `Formalism.FourierDecayAlt.P.CutSE.ClsDecayJ` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.CutSE2`, lines 394–405)

Lean: `Formalism.FourierDecayAlt.P5.HerD.herKc0_le_waveJ` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.LevelOneSquareIntegrableGeneral`, lines 76–111)

Lean: `Formalism.FourierDecayAlt.P5.HerC.radial_transform_square_integrable` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.RadialTransformSquareIntegrable`, lines 160–171)

Lean: `Formalism.FourierDecayAlt.P.CutSE.herKc0_le_dp` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.DecayPairEstimate`, lines 188–226)

Lean: Formalism.FourierDecayAlt.P.CutSE.decay_pair_estimate (Formalism.HermitianCurvature.HermitianCurvatureProofs.DecayPairEstimate, lines 230–259)
Lean: Formalism.FourierDecayAlt.P.CutSE.rDIdentSpec_of_R (Formalism.HermitianCurvature.HermitianCurvatureProofs.CutSETopR, lines 11–12)
Lean: Formalism.FourierDecayAlt.P.CutSE.radDecay_of_spec (Formalism.HermitianCurvature.HermitianCurvatureProofs.RadialIdentityOfSpectralData, lines 25–26)
Lean: Formalism.FourierDecayAlt.P.CutSE.one_letter_density_estimate (Formalism.HermitianCurvature.HermitianCurvatureProofs.CutSEOneLetterDens, lines 644–659)
Lean: Formalism.FourierDecayAlt.P.CutSE.density_smooth_class (Formalism.HermitianCurvature.HermitianCurvatureProofs.CutSE_TopSpec, lines 11–11)
Lean: Formalism.FourierDecayAlt.P.CutSE.thmPActualCpt_of_leaves (Formalism.HermitianCurvature.HermitianCurvatureProofs.CutSETopHer, lines 13–15)
Lean: Formalism.FourierDecayAlt.P.CutSE.thmPActualCpt_of_spec (Formalism.HermitianCurvature.HermitianCurvatureProofs.CutSE_TopSpec, lines 13–15)
Lean: Formalism.FourierDecayAlt.P.CutSE.thmPActualCpt_of_R (Formalism.HermitianCurvature.HermitianCurvatureProofs.CutSETopR, lines 14–15)

The operator assertions of Claim 1 hold uniformly for the compatible actual data. Applying Claim 2 gives one family $\mathcal{K}_i$ for the fixed position set and all its admissible words. In particular, positivity is asserted only on the product subspace with all components in $\mathcal{K}_i$; the realization conclusion retains its compact-support hypothesis. The complete proof is `Formalism.FourierDecayAlt.P.CutSE.actual_compact_theorem`. □

Proof of 33.88

Proof (outline). Claim 1. The slice coefficients in (a) have the stated support and regularity. On a valid slice the complete block-end minor used for that slice is nonzero; no condition is imposed on the intermediate principal minors inside a block. The coefficient insertions there are represented by differential operators applied to the original compact coefficient, followed by multiplication by the fixed minor inverses. Local differentiation preserves support. The slice integral has bounded integration variables on the support, and the Gram identity for the slice transfers the positive lower bound to the residual Gram polynomial. Differentiation under this integral gives smoothness on the residual group. At level zero the insertions have only their zero component, so the same conclusion holds there.
Lean: Formalism.FourierDecayAlt.P.PBE.slice_pos_cpt (Formalism.HermitianCurvature.HermitianCurvatureProofs.SliceCompactSupportCoefficient, lines 14–232)
Lean: Formalism.FourierDecayAlt.P.PBE.slice_level_zero_compact_support (Formalism.HermitianCurvature.HermitianCurvatureProofs.SliceCompactSupportCoefficient, lines 234–262)

For finitely many compact components choose a common positive Gram lower bound and a common support bound. Smoothness on the compact part of the group gives every weighted derivative bound. Their values away from the group may be replaced by zero, producing ambient compact support without changing the Bessel averages. Apply Lemma 33.87 to the residual parameter of rank $(k, r-j)$, using the actual residual operators, the residual start, and the Haar measure transported from its terminal group. This supplies the representatives in $\mathcal{K}_0^{(j)}$. The radial coefficient-action identity and the Bessel-pairing measure identity give the displayed comparison, with the same positive scalar for all words and all component arrays. The positioned cut word satisfies the start conditions by the actual compact interlacing and cut conditions. The point $\mathbf{1}$ belongs to the residual nonzero-coordinate domain. Positivity follows from the radial form, and sesquilinearity follows from the coefficient-pairing identities.
Lean: Formalism.FourierDecayAlt.P.PBF.common_eps_R (Formalism.HermitianCurvature.HermitianCurvatureProofs.SizeStandardCompact, lines 17–38)
Lean: Formalism.FourierDecayAlt.P.PBF.pic_of_cptClause (Formalism.HermitianCurvature.HermitianCurvatureProofs.MemLevelZeroOfMemGroup, lines 88–111)
Lean: Formalism.FourierDecayAlt.P.JCTx5.wordOK_of (Formalism.HermitianCurvature.HermitianCurvatureProofs.LevelsOfBuiltSetting, lines 559–606)

Claim 2. The mixed estimate uses the subspace $\mathcal{U}_{\mathrm{cls}} \subseteq \mathcal{K}_0^{(j)}$ of radial representatives whose images under every finite product of the residual two-by-two matrix entries at the cut evaluation points remain in $\mathcal{K}_0^{(j)}$. Each such entry has a coefficient differential-operator realization preserving the common support and Gram lower bound, so induction on products places every compact coefficient representative in $\mathcal{U}_{\mathrm{cls}}$. Position w^+ as Lw, where $L = (a_1, \dots, a_K)$ consists of the positive companion letters $a_b = (+, s_b, y_b)$ in the first K positions, and let $L^\vee w$ be the positioned word obtained by replacing each a_b with $(-, s_b, y_b + 1)$; up to the

relabelling of positions it is $\mathrm{pos}(w^{-})$. Put $L_b = (a_{b+1}, \dots, a_K)$, $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $J_L = \prod_{b=1}^{K} \mathrm{ins}_{s_b} J$, and

$$\Psi = \prod_{b=1}^{K} \mathrm{ins}_{s_b}\Big(-q_{L_b w}(y_b)^{-1} J\, \mathsf{T}_0[L_b w; y_b + 1]^{\mathsf{T}}\Big),$$

the product ordered with $b = 1$ on the left. Under the hypotheses of (b), the start relations of Definition 33.82, namely the Yangian RTT relation and the quantum-determinant relation, give the paired-word contraction identity: Ψ is invertible and

$$\Psi^{*} R_0(Lw) = J_L R_0(w), \qquad \Psi^{*} R_0(Lw) \Psi = \kappa^{-1} R_0(L^{\vee} w), \qquad \kappa = \prod_{b=1}^{K} q_w(y_b),$$

where the value of κ uses that all letters of L are positive. The entries of Ψ lie in the algebra generated by the residual matrix entries at the points $y_b + 1$, so Ψ and Ψ^{*} preserve $\mathcal{U}_{\mathrm{cls}}$, on which the integral adjoint identities hold. Let β be the level-zero radial pairing summed over components, and let Φ, Φ' be the representatives of x, y given by (a). Applying (a) at each value of the new binary indices and expanding each inserted J as $\Xi(x_1, y_0) - \Xi(x_0, y_1)$ gives $\mathcal{J}^{\mathrm{coef}}_{\Xi_{j,w}}(x, y) = c_j \beta(\mathrm{act}(J_L R_0(w))\Phi, \Phi')$, and the first identity turns this into $c_j \beta(\mathrm{act}\, R_0(Lw)\Phi, \mathrm{act}\, \Psi\Phi')$. The form $\beta(\mathrm{act}\, R_0(Lw)\cdot, \cdot)$ is positive semidefinite on $\mathcal{U}_{\mathrm{cls}}$ by Lemma 33.87, so Cauchy–Schwarz and the second identity give

$$\big|\beta(\mathrm{act}(J_L R_0(w))\Phi, \Phi')\big|^2 \le \kappa^{-1} \beta(\mathrm{act}\, R_0(Lw)\Phi, \Phi)\, \beta(\mathrm{act}\, R_0(L^{\vee} w)\Phi', \Phi').$$

The cut conditions make every factor $q_w(y_b)$ strictly positive, so $\kappa > 0$. Multiplying by $c_j^2 \kappa$ and using (a) for w^{+} and w^{-} yields the inequality in (b).

Lean: `Formalism.FourierDecayAlt.P.PBF.transportC` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.PicMemUclsC`, lines 16–67)

Lean: `Formalism.FourierDecayAlt.P.PBF.pic_mem_UclsC` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.PicMemUclsC`, lines 101–124)

Lean: `Formalism.FourierDecayAlt.P.PBF.pb_sizeStdCpt_main` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.SizeStandardCompact`, lines 40–181)

The triangular substitutions are finite linear combinations and $|f_s| = 1$, so the wave array has the same support bounds. Its coefficient-to-radial intertwining identifies the wave with multiplication by the first remaining radial coordinate, or by the terminal coordinate when $r - j = 0$, together with its prescribed word action. The radial wave identity gives the form equality in (b). Finally, expand the actual terminal definition. Its empty-word branch is the Hilbert–Schmidt pairing of the orbital vectors, and its nonempty branch is the sum of the displayed projected pairings with real multipliers in both arguments. This proves (c). The complete compact comparison and mixed estimates are proved in `Formalism.FourierDecayAlt.P.module_form_pos_compact_support`, `Formalism.FourierDecayAlt.P.size_standard_compact`, and `Formalism.FourierDecayAlt.P.vector_isomorphism_compact_support`; the terminal expansion is proved in `Formalism.FourierDecayAlt.P.PBASM.rr_top_rep`. □

Proof of 33.89

Proof (outline). Put $s = r + 1$, $p_0 = 2s$, $K = 3$, $N = 3s$, and

$$q_0 = se + 40s^3 + s(s+1), \qquad c = (C_a + 5)m^4.$$

We use c as the displacement bound in P_{*} and n_c of Definition 33.82. Thus every normalizing factor is strictly positive. If the orbital supremum at γ is zero, the conclusion is immediate. Otherwise γ has a representative, so $G_{\mathrm{loc}}(\gamma) > 0$. The actual-place construction gives the normalized coefficient tower with the prescribed frames, compact representations and measures, together with its base frame, compatible actual terminal lift and compact-type projections. At any chosen cut it also gives the differential-operator bounds for its terminal lift, with a constant independent of the local datum. In the distinct-weight case choose a cut l_0 for which

$$|u_{l_0} - u'_{l_0}| \ge 30s^2 + 6.$$

The gap hypothesis supplies such a cut once the fixed threshold for C_g is enlarged. The equal-weight case will use the two actual places constructed in Claim 4 below.
Lean: `CompanionTheory.WaveGrid.TypeHC.exists_place` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.TypeHC`, lines 114–155)

For this tower let $\mathscr{R}^{\mathrm{act}}_{j,w}$ be its unweighted partial word kernel, and set

$$\widetilde{E}_j(w;Q;\gamma)=p_{k,r}(T_p,T_q;\gamma)\,\mathrm{Re}\,\mathscr{R}^{\mathrm{act}}_{j,w}(Q;\gamma,\gamma),\qquad \widetilde{E}(Q;\gamma)=\widetilde{E}_{r+1}([];Q;\gamma).$$

At $j\le r$ this kernel is the coefficient form on the two actual slice arrays; at $j=r+1$ it is the terminal pairing in Definition 33.82. All these kernels are zero at invalid block lists. We use these actual tower energies in the interpolation and cutoff arguments. Their comparison with the orbital energy $E(Q;\gamma)$ is made at the end of the proof. Put

$$\widetilde{I}(Q)=\int_{\mathbf{C}^{r+1}}\widetilde{E}(Q;c_{\mathrm{diag}})\,dc_{\mathrm{diag}},$$

where c_{diag} denotes the singleton block point and the integral is a nonnegative integral with respect to product planar Lebesgue measure.
Lean: `Formalism.FourierDecayAlt.TBB.NP` (`Formalism.HermitianCurvature.StatementTBB`, lines 38–40)
Lean: `Formalism.FourierDecayAlt.TBB.Ew` (`Formalism.HermitianCurvature.StatementTBB`, lines 42–44)
Lean: `Formalism.FourierDecayAlt.TBB.jN2` (`Formalism.HermitianCurvature.StatementTBB`, lines 46–47)

Claim 1. The endpoint eigenfunction decomposition exists for the coefficients used in this proof. More precisely, let F belong to the coefficient span of a discrete series with parameter $\lambda+\nu_0$, where $\nu_0\in\mathbf{Z}^n$ and $\|\nu_0\|_1\le se$, and put $H=G^{p_0}_{\mathrm{loc}}F$. There are finitely many distinct parameters μ, smooth coefficients H_μ, and a number $A\ge 2(k+r)s+1$ such that

$$H|_G=\sum_{\mu}H_\mu|_G,\qquad |\mu_a-\lambda_a|\le se+mp_0\le c,$$

$$\mathcal{C}(y)H_\mu|_G=\Big(\prod_{a=1}^{n}(y+\mu_a)\Big)H_\mu|_G,\qquad \beta(H_\mu,H_{\mu'})=0\quad(\mu\ne\mu').$$

Every H_μ has the invariant-derivative decay $\mathscr{C}(A,5)$: all mixed left and right invariant derivatives of total order at most five are bounded after multiplication by $\|g\|_F^A$. In particular, for every real y,

$$\left|\prod_{a=1}^{n}(y+\mu_a)\right|\le\prod_{a=1}^{n}(|y+\lambda_a|+c)=P_*(y).$$

We now give the construction and its quantitative margin. Multiplication by $G^{p_0}_{\mathrm{loc}}$ belongs to the degree-mp_0 coordinate-polynomial space. The tensor and dual-tensor decompositions place H in the sum of the discrete-series coefficient spaces whose parameters satisfy $|\mu_a-(\lambda_a+\nu_{0,a})|\le mp_0$. Membership in this algebraic sum gives a finite sum, with one summand for each distinct parameter. The quantum determinant acts by the displayed scalar on each summand. Its adjoint identity and infinitesimal-character orthogonality give the displayed β-orthogonality, with the stated decay ensuring the integrations by parts. The K-finite coefficient in the proposition belongs to the initial coefficient span.

To check the decay conditions, put

$$X=s^4(3sn^2+n^3),\qquad \Delta=m^{9/2}(3sn^2+n^3),\qquad d_{\max}=mp_0+(30s^2+6)s\le 5m^3.$$

The tensor threshold is at most C_tn^3, with an absolute C_t. Choose a fixed $C_A\ge 46$ larger than the constants in the invariant-derivative, cutoff-convergence and terminal-lift estimates. Since $X,m^4,n^3\le\Delta$, increasing $C_{g,1}$, independently of k,r,e, gives

$$C_tn^3+C_AX+3d_{\max}+1<C_g\Delta-2se.$$

The parameter $\lambda+\nu_0$ retains opposite-sign gap at least $C_g\Delta-2se$. The discrete-series decay estimate therefore gives the required decay for all summands, and gives the higher derivative bounds used below after subtracting the polynomial degree. This also proves the same assertion with $G^{p_0}_{\mathrm{loc}}FA_\pm$ in place of H, when

$A_{\pm}$ are the polynomial contractions in Claim 4, whose group-coordinate degree is $(30s^2+6)s$. In that case the parameter displacement is at most $se + d_{\max} \le c$.
Lean: CompanionTheory.WaveGrid.Coef.kfin_wsp (CompanionTheory.AutomorphicTheory.BesselWaveGrids.CoefFacts, lines 18–31)
Lean: CompanionTheory.WaveGrid.mul_mem_box (CompanionTheory.AutomorphicTheory.BesselWaveGrids.EigStrOfMem, lines 30–50)
Lean: CompanionTheory.WaveGrid.eigStr_of_mem (CompanionTheory.AutomorphicTheory.BesselWaveGrids.EigStrOfMem, lines 69–87)
Lean: CompanionTheory.WaveGrid.eigStr_fpmW (CompanionTheory.AutomorphicTheory.BesselWaveGrids.EigStrOfMem, lines 89–101)
Lean: CompanionTheory.WaveGrid.eigStr_fpmShW (CompanionTheory.AutomorphicTheory.BesselWaveGrids.EigStrOfMem, lines 103–124)
Lean: CompanionTheory.WaveGrid.kDecay_of_gap (CompanionTheory.AutomorphicTheory.BesselWaveGrids.KDecayOfGap, lines 25–29)

For an admissible grid word w with actual points $t_1, \dots, t_q$, let $\sigma_{l_0} = (-1)^{k-l_0}$ and $e_\mu(y) = \prod_a (y + \mu_a)$. The level-zero word identity, applied in the prescribed order, now gives

$$\begin{aligned} \widetilde{E}_0(w; H; []) &= \operatorname{Re} \beta\big((\sigma_{l_0}\mathcal{C}(t_1)) \cdots (\sigma_{l_0}\mathcal{C}(t_q))H, H\big) \\ &= \sum_{\mu} \operatorname{Re}\Big(\prod_{a=1}^{q} \sigma_{l_0} e_\mu(t_a)\Big)\beta(H_\mu, H_\mu) \le n_c(w)\beta(H,H). \end{aligned} \tag{A.334}$$

Indeed, the compact grid words have distinct positions, real cut points and distances greater than two. They therefore satisfy the positioned-word and letter-regularity hypotheses of the level-zero identity. Each $\beta(H_\mu, H_\mu)$ is real and nonnegative, and the eigenvalue product has absolute value at most $n_c(w)$. The empty-word case uses the empty product one. This proves the endpoint bound for the actual H; the cutoff coefficients below need no eigenfunction decomposition. Under the decay just proved the diagonal Bessel pairing is finite, and the singleton-section identity gives

$$\operatorname{Re} \beta(H,H) = \widetilde{I}(H) < \infty.$$

Thus the endpoint normalized by $n_c(w)$ is bounded by the actual tower section integral of H.
Lean: CompanionTheory.WaveGrid.WAt.normSq_eq (CompanionTheory.AutomorphicTheory.BesselWaveGrids.CoefOfOfWsp, lines 86–128)
Lean: GlobalTheory.T.jN2_eq_normSq_base (Development.AbelianVarietyTheory.Proofs.JetHarmonics.BesselNormSqEqBigCellNormSqStatement, lines 9–10)
Lean: Formalism.FourierDecayAlt.P.CutBP.good_positioned_of_cut (Formalism.HermitianCurvature.HermitianCurvatureProofs.CutLimE, lines 11–39)
Lean: Formalism.FourierDecayAlt.P.CutBP.limBd_of_eigStr (Formalism.HermitianCurvature.HermitianCurvatureProofs.CutLimBdPf, lines 179–222)

Claim 2. Every coefficient array to which positivity or Cauchy–Schwarz is applied is a compact coefficient array in the residual datum of Lemma 33.88. Choose a smooth χ vanishing on $(-\infty, \delta)$, for some $\delta > 0$, and equal to one on $[1, \infty)$. Let θ be the smooth transition from zero on $(-\infty, 0]$ to one on $[1, \infty)$, and set

$$H_{\varepsilon,M}(g) = \theta\left(\frac{4 - \|g\|_F^2/(M+1)^2}{3}\right) \chi\left(\frac{G_{\mathrm{loc}}(g)}{\varepsilon}\right) H(g), \qquad \varepsilon > 0, \quad M \in \mathbf{N}.$$

This coefficient is smooth on G. Its nonzero locus on G satisfies

$$\|g\|_F \le 2(M+1), \qquad G_{\mathrm{loc}}(g) \ge \delta\varepsilon > 0.$$

Thus it satisfies the compact-coefficient condition of Lemma 33.88. For every $j \le r$, every valid level-j point γ, and every word w, define

$$x_{j,w,\gamma}[H_{\varepsilon,M}] = \mathrm{slice}_{j,\gamma}(\mathcal{V}_{j,w} H_{\varepsilon,M}),$$

where the iterated insertions and their order are those of Definition 33.82. The compact-slice assertion in Lemma 33.88 applies to every component of this array, including the arrays for all positive and negative grid extensions. In particular it supplies a common positive residual Gram bound and a common bounded support for each finite family of components. It also supplies their radial representatives in the level-zero positive space for the residual rank $r - j$, and the positive scalar relating the residual radial form to $\Xi^{\mathrm{coef}}_{j,w}$.

Lean: `Formalism.FourierDecayAlt.P.CutBP.cptCoef_apx` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.CutDec`, lines 19–85)
Lean: `Formalism.FourierDecayAlt.P.slice_compact_support_coefficient` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.SliceCompactSupportCoefficient`, lines 268–274)

For brevity write $Q = H_{\varepsilon,M}$ and $x_{w,\gamma} = x_{j,w,\gamma}[Q]$. At every level define

$$\mathscr{N}^{Q}_{j,w}(\gamma,\gamma') = \frac{\sqrt{p_{k,r}(T_p,T_q;\gamma)}\sqrt{p_{k,r}(T_p,T_q;\gamma')}}{n_c(w)}\mathscr{R}^{\mathrm{act}}_{j,w}(Q;\gamma,\gamma'), \qquad \mathscr{E}^{Q}_{j,w}(\gamma) = \operatorname{Re}\mathscr{N}^{Q}_{j,w}(\gamma,\gamma).$$

At $j \le r$ the normalized kernel and its diagonal are exactly

$$\mathscr{N}^{Q}_{j,w}(\gamma,\gamma') = \frac{\sqrt{p_{k,r}(T_p,T_q;\gamma)}\sqrt{p_{k,r}(T_p,T_q;\gamma')}}{n_c(w)}\Xi^{\mathrm{coef}}_{j,w}(x_{w,\gamma},x_{w,\gamma'}),$$
$$\mathscr{E}^{Q}_{j,w}(\gamma) = \frac{\widetilde{E}_j(w;Q;\gamma)}{n_c(w)} = \mathscr{N}^{Q}_{j,w}(\gamma,\gamma) \ge 0.$$

The unweighted partial kernel is the coefficient form on the displayed slices, by the definition of the actual tower. Multiplying by the two square-root point weights and dividing by $n_c(w)$ gives these identities; $p_{k,r}$ is the complete point weight of Definition 33.82. Extend the kernels by zero at invalid block lists. Consequently

$$|\mathscr{N}^{Q}_{j,w}(\gamma,\gamma')| \le \sqrt{\mathscr{E}^{Q}_{j,w}(\gamma)\mathscr{E}^{Q}_{j,w}(\gamma')} \le \tfrac{1}{2}\big(\mathscr{E}^{Q}_{j,w}(\gamma) + \mathscr{E}^{Q}_{j,w}(\gamma')\big).$$

Lean: `Formalism.FourierDecayAlt.P.PBL.rr_eq_form` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.AeNeZeroComplex`, lines 15–20)
Lean: `Formalism.FourierDecayAlt.P.TBBx5.li_np` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.OnePointBasisFundamentalDomainVolume`, lines 513–519)
Lean: `Formalism.FourierDecayAlt.P.TBBx5.li_ew` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.OnePointBasisFundamentalDomainVolume`, lines 506–511)

For $j = r+1$, use the terminal projected Gram formula of that lemma; it gives the same positivity and Cauchy–Schwarz assertions. A finite nonnegative weighted sum of these inequalities is valid term by term, with each word's own normalizer.

Let $W_{\omega,w}$ be the coefficient wave action, consisting of the triangular action on the binary indices followed by multiplication by

$$\phi_\omega(g) = \exp\big(\omega D_1(g) - \overline{\omega}\,\overline{D_1(g)}\big)$$

in the residual datum. For componentwise compact arrays x, y and a cut word w, it satisfies

$$W_{\omega,w}x \text{ is componentwise compact}, \qquad \Xi^{\mathrm{coef}}_{j,w}(W_{\omega,w}x, W_{\omega,w}y) = \Xi^{\mathrm{coef}}_{j,w}(x,y).$$

To obtain this identity, realize the original and transformed compact arrays in the residual level-zero space. Their radial representatives are related by the actual radial wave intertwiner. Its integral adjoint identity gives the equality, and the common positive comparison scalar cancels. Compact realization of the transformed arrays proves their membership in the positive space. In residual rank zero the phase is the terminal coordinate; in positive residual rank it is the first radial coordinate.
Lean: `Formalism.FourierDecayAlt.P.PBFV.vIsoCpt_main` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.VectorIsomorphismCompactSupport`, lines 12–111)
Lean: `Formalism.FourierDecayAlt.P.vector_isomorphism_compact_support` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.VectorIsomorphismCompactSupport`, lines 117–117)

The one-node identity fixes the wave convention. When $r = j = 0$ with compact coefficient Q and $w = []$, both signs have insertion $Q \mapsto (Q, 0)$, while on a positive extension

$$W_{\omega,+}(x_0,x_1) = (\phi_\omega x_0, \phi_\omega(x_1 + \omega x_0)).$$

Since the pairing is linear in its first argument,

$$\mathcal{J}_\beta\big(W_{\omega,+}(Q,0),(Q,0)\big) = \beta(\omega\phi_\omega Q, Q) - \beta(\phi_\omega Q, 0) = \omega\beta(\phi_\omega Q, Q).$$

Both sides vanish at $\omega = 0$. This is the degree-zero case of the wave interpolation identity.

Lean: `Formalism.FourierDecayAlt.P.PBG.letter_level_zero_eq` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.PBGWaveMain`, lines 13–26)
Lean: `Formalism.FourierDecayAlt.KB.uInv` (`Formalism.HermitianCurvature.RealInputsClosedAdmissibleSettings`, lines 37–43)
Lean: `Formalism.FourierDecayAlt.KB.waveG` (`Formalism.HermitianCurvature.RealInputsClosedAdmissibleSettings`, lines 60–62)

Claim 3. The wave and lift interpolation bounds hold for these normalized kernels, with all their arrays and words on the domains of Claim 2. For a distinct-weight cut take

$$Y = \{2 - \max(u_{l_0}, u'_{l_0}) + 5a : 0 \le a < N(2r+1)\}, \qquad \Lambda = \max\left(1, \sqrt{\frac{(4/3)^N b^{8r+20}}{T_p T_q}}\right).$$

The coordinate, interlacing and gap assumptions imply

$$1 \le c, \quad c+2 \le b, \quad \exp(1)k \le b, \quad 2c \le C_g m^4, \quad 5N(2r+1) \le 30s^2 + 6 \le |u_{l_0} - u'_{l_0}|.$$

The grid therefore lies in the cut with the required margin, including the actual negative points $y+1$. Its distinct base points have spacing five. The grid selection in Lemma 33.80, for every old word of length at most $N-K$, gives K rows of unused nodes such that both extended words are cut words and within each extended word the actual evaluation points have pairwise distance greater than two. Applying the actual start construction to these cut words verifies the start conditions of Definition 33.82. Moreover $1 \le \Lambda \le b^{6m}$, since $T_p T_q \ge 1$.
Lean: `CompanionTheory.WaveGrid.grids_diff` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.GridsText`, lines 70–76)
Lean: `CompanionTheory.WaveGrid.gridΛ_bounds` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.GridsText`, lines 65–68)

For K selected nodes $\mathbf{y} = (y_1, \dots, y_K)$, let w^+ prepend positive letters at y_a, and let w^- prepend negative letters at $y_a + 1$, in their prescribed order. The mixed arrays in the wave identity, at a valid level-j point γ, are

$$W_{\omega,w^+} x_{j,w^+,\gamma}[Q], \qquad x_{j,w^-,\gamma}[Q].$$

For a valid prefix η and $t, t' \in \mathbf{C}^\times$, the arrays in the lift identity at $\gamma_t = \eta +\!\!+ [(t,h)]$ and $\gamma_{t'} = \eta +\!\!+ [(t',h)]$ are

$$x_{j,w^+,\gamma_t}[Q], \qquad x_{j,w^-,\gamma_{t'}}[Q].$$

Here $h \ge 1$ is the block length and $j = \mathrm{level}(\eta) + h \le r$. All four arrays are componentwise compact by Claim 2. The mixed contraction inequality of Lemma 33.88, and the wave isometry, bound each normalized node term by

$$\left(\frac{\prod_a P_*(y_a) P_*(y_a+1)}{\prod_a q_w(y_a)}\right)^{1/2} \left(\mathscr{E}^Q_{j,w^+}(\gamma)\mathscr{E}^Q_{j,w^-}(\gamma')\right)^{1/2}.$$

Here $q_w(y_a) > 0$ is the real dressed scalar from that lemma, $\gamma' = \gamma$ in the wave case, and $(\gamma, \gamma') = (\gamma_t, \gamma_{t'})$ in the lift case. Thus these mixed terms use the two extended-word forms and their own positive normalizers.

Define

$$\mathcal{W}_w(\omega) = \Xi^{\mathrm{coef}}_{j,w}(W_{\omega,w} x_{j,w,\gamma}[Q], x_{j,w,\gamma}[Q]), \qquad \mathcal{R}_w(t,t') = \Xi^{\mathrm{coef}}_{j,w}(x_{j,w,\gamma_t}[Q], x_{j,w,\gamma_{t'}}[Q]).$$

The superscripts $+,-$ below denote the alternating contraction $\mathcal{J}_{\Xi^{\mathrm{coef}}_{j,w}}$ of the two displayed extended arrays, for the selected node list $\mathbf{y}$. The leading-coefficient extraction of Lemma 33.85 has degree $2j$ and multiplier ω for the wave, and degree $2(j-h)+h$ and multiplier $(-1)^{h-1}(t^{-1} - t'^{-1})$ for the lift. Indeed, on the positive-Gram locus each level-j positive or negative letter insertion acts as a differential operator polynomial of degree j in the evaluation variable Y with leading term proportional to the top block minor D_j: for the wave, pairing a positive insertion with a negative insertion yields a polynomial of degree $2j$ in each node variable whose leading coefficient pulls out the phase factor ω from $W_{\omega,+}$ (extending the degree-zero identity above); for a lift appending a block of length h after a prefix η of length $j-h$, the preceding $j-h$ positions contribute degree $2(j-h)$ while the shared final block of length h undergoes cancellation of its top h degrees

and contributes degree h with leading multiplier $(-1)^{h-1}(t^{-1}-t'^{-1})$. It gives

$$\omega^K \mathcal{W}_w(\omega) = \sum_{j_1,\dots,j_K} \Big(\prod_a L_{aj_a}\Big)\mathcal{W}^{+,-}_{w,\mathbf{y}}(\omega),$$
$$[(-1)^{h-1}(t^{-1}-t'^{-1})]^K \mathcal{R}_w(t,t') = \sum_{j_1,\dots,j_K} \Big(\prod_a L_{aj_a}\Big)\mathcal{R}^{+,-}_{w,\mathbf{y}}(t,t').$$

The positive point-weight factor $p_{k,r}(T_p,T_q;\gamma)/n_c(w)$ converts the wave coefficient form into the Fourier transform of the normalized singleton kernel; its diagonal converts into $\mathscr{E}^{Q}_{j,w}(\gamma)$. For the lift the factor is $\sqrt{p_{k,r}(T_p,T_q;\gamma_t)p_{k,r}(T_p,T_q;\gamma_{t'})}/n_c(w)$. The grid bounds the sum of the absolute Lagrange coefficients times the displayed square-root factor by Λ^K. Combining the mixed bound with the unextended Cauchy–Schwarz bound gives an admissible extension of length at most K for the wave estimate, and two such extensions for the lift estimate. Their multipliers are

$$M_\Lambda(a) = \begin{cases} 1, & a = 0, \\ \min\{1,(\Lambda/a)^3\}, & a > 0, \end{cases} \qquad a = |\omega| \text{ or } |t^{-1}-t'^{-1}|. \tag{A.335}$$

In the lift estimate the upper bound is $M_\Lambda(a)$ times the arithmetic mean of the two extended normalized diagonal energies. In the wave estimate it is $M_\Lambda(|\omega|)$ times one extended normalized diagonal energy.
Lean: `Formalism.FourierDecayAlt.P.degree_of_plane_wave` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.PBGWaveMain`, lines 55–55)
Lean: `Formalism.FourierDecayAlt.P.degree_of_lift_pairing` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.PBGDegLift`, lines 4–5)
Lean: `Formalism.FourierDecayAlt.P.TBBx5.integral_three_at_place_of_partsCpt` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.CutCptManual`, lines 265–414)
Lean: `Formalism.FourierDecayAlt.P.TBBx5.integral_four_at_place_of_partsCpt` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.CutCptManual`, lines 93–256)

The compact coefficients also satisfy the block-boundary hypothesis needed by Lemma 33.84. In its lattice formulation, take a block of length $h \ge 2$, a valid prefix η, $j = \mathrm{level}(\eta)+h \le r+1$, and an admissible word w with $|w|+K \le N$. For linearly independent $\omega_1,\omega_2 \in \mathbf{C}$, let $\mathcal{L} = \mathbf{Z}\omega_1+\mathbf{Z}\omega_2$ and $\mathcal{L}^\vee = \{\sigma : f_\sigma(v) = 1\ (v \in \mathcal{L})\}$, where $f_\sigma(v) = \exp(\sigma v - \overline{\sigma}\,\overline{v})$. For every $z_0 \in \mathbf{C}$, the normalized energy on the points $\eta +\!\!+ [(z_0 + a\omega_1 + b\omega_2, h)]$ has lattice sum, multiplied by $|\operatorname{Im}(\overline{\omega}_1\omega_2)|/\pi$, bounded by $\sum_{\sigma\in\mathcal{L}^\vee}|k(\sigma)|$. For each dual-lattice point σ,

$$|k(\sigma)| \le \limsup_{\substack{t\to 0\\ t\neq 0}} \Big|\mathscr{N}^{Q}_{j-1,w}\big(\eta +\!\!+ [(t,h-1)], \eta +\!\!+ [((t^{-1}+\sigma)^{-1},h-1)]\big)\Big|.$$

These are nonnegative sums in $[0,\infty]$. As in the block-boundary assertion, its hypothesis is boundedness of the shortened diagonal energies for all admissible K-letter extensions; these are the induction hypotheses in block shortening. The actual compact-coefficient boundary theorem supplies this assertion for Q. Its hypotheses are precisely the smooth compact coefficient, positive normalizer, actual compact theorem and quantitative terminal-lift bound already verified. The boundary kernels use the same slice arrays and normalized pairings as above. Their positive semidefiniteness follows from Claim 2.
Lean: `Formalism.FourierDecayAlt.TBB.BlockGrowthBound` (`Formalism.HermitianCurvature.StatementTBB`, lines 75–88)
Lean: `Formalism.FourierDecayAlt.P.CutBP.CompactFormNormalFormDomination.blockGrowthBound_apx_cutF` (`CompanionTheory.AutomorphicTheory.CompactTopologicalForms.FinalS`, lines 55–61)

Since $\int_0^\infty M_\Lambda(t)t\,dt = 3\Lambda^2/2$, singleton shortening and the block estimates of Lemma 33.84 now apply. Head continuity follows for the same compact slices from the actual singleton-pairing identity. After at most $r+1$ shortening steps there is an admissible word w, of length at most N, such that

$$\mathscr{E}^{H_{\varepsilon,M}}_{r+1,[]}(\gamma) \le (C_2\Lambda^2)^{r+1}\mathscr{E}^{H_{\varepsilon,M}}_{0,w}([]),$$

where C_2 is absolute. Every use of positivity in this induction is supplied by Claim 2, including the terminal level.
Lean: `Formalism.FourierDecayAlt.Final.tbb_stepOne` (`Formalism.HermitianCurvature.HeisenbergConjugationMeasurePreserving`, lines 2159–2159)
Lean: `Formalism.FourierDecayAlt.Final.tbb_stepUp` (`Formalism.HermitianCurvature.HeisenbergConjugationMeasurePreserving`, lines 2160–2160)

Lean: Formalism.FourierDecayAlt.Final.tbb_lemmaSOfSteps (Formalism.HermitianCurvature.HeisenbergConjugationMeasurePreserving, lines 2161–2161)
Lean: Formalism.FourierDecayAlt.P.CutBP.corS_of_cptCpt (Formalism.HermitianCurvature.HermitianCurvatureProofs.CutMainCpt, lines 188–207)

First let $M \to \infty$ with ε fixed. The invariant-derivative decay in Claim 1 gives convergence of the terminal and level-zero energies of the ambient truncations. The possible words form a finite set, because their length is at most N and all letters come from $Y \cup (Y+1)$; hence a subsequence uses one word. Then let $\varepsilon \to 0^+$. In the diagonal chart, the terminal normal-form expansion gives an ε-independent integrable majorant

$$M_0 \sum_D |G_{\mathrm{loc}}(c_{\mathrm{diag}})|^{2p_0-N}\, \widetilde{E}(DF; c_{\mathrm{diag}}), \tag{A.336}$$

with finitely many prescribed differential coefficient operators D. The Euler derivatives of $\chi(G/\varepsilon)$ are uniformly bounded. The invariant-derivative margin $C_A X$ covers the differential operators in this expansion and the section-integrability estimates. Since $N \le 2p_0$, this gives dominated convergence; the level-zero cutoff forms converge by the same decay and the endpoint word identity. A second constant-word subsequence and (A.334) yield

$$\widetilde{E}(H;\gamma) \le (C_2\Lambda^2)^{r+1}\widetilde{I}(H).$$

The endpoint is applied only after both cutoffs have been removed. At a represented positive-Gram point, multiplication of the coefficient by $G_{\mathrm{loc}}^{p_0}$ multiplies its actual orbital vector by $G_{\mathrm{loc}}(\gamma)^{p_0}$; hence $\widetilde{E}(H;\gamma) = G_{\mathrm{loc}}(\gamma)^{2p_0}\widetilde{E}(F;\gamma)$. The differential expansion supplied below bounds the corresponding section integral.
Lean: Formalism.FourierDecayAlt.P.TBBx4.vec_Gpow (Formalism.HermitianCurvature.HermitianCurvatureProofs.RegularityAtPlaceOfSubadditive, lines 122–138)
Lean: Formalism.FourierDecayAlt.P.CutBP.limTopN_cut_of_inCC (Formalism.HermitianCurvature.HermitianCurvatureProofs.CutLimTop, lines 42–53)
Lean: Formalism.FourierDecayAlt.P.CutBP.limZeroN_of_formLimit (Formalism.HermitianCurvature.HermitianCurvatureProofs.LimZeroNPf, lines 13–33)
Lean: Formalism.FourierDecayAlt.P.CutBP.ptBound_dec_allCpt (Formalism.HermitianCurvature.HermitianCurvatureProofs.CutMainCpt, lines 218–238)
Lean: Formalism.FourierDecayAlt.P.CutBP.corS_of_apx (Formalism.HermitianCurvature.HermitianCurvatureProofs.CutDec, lines 123–166)
Lean: Formalism.FourierDecayAlt.P.CutBP.nfDomPtwS_of_cut (Formalism.HermitianCurvature.HermitianCurvatureProofs.CutNFDomS, lines 15–143)
Lean: Formalism.FourierDecayAlt.P.CutBP.dominated_L2_operator_bound_holds (Formalism.HermitianCurvature.HermitianCurvatureProofs.DominatedL2OperatorBoundHolds, lines 228–298)
Lean: Formalism.FourierDecayAlt.P.CutBP.CompactFormNormalFormDomination.thmBpAt_top (CompanionTheory.AutomorphicTheory.CompactTopologicalForms.NormalFormTopDomination, lines 14–35)

Claim 4. In the equal-weight case the same construction applies at two specified shifted actual places. Put $n_0 = 30s^2+6$, and choose $1 \le l_0 < k$ with $u_{l_0} - u_{l_0+1} \ge 2n_0+1$. The shifted right compact parameters and cuts are

$$u'_+ = u + n_0\mathbf{1}_{l_0+1}, \quad l_+ = l_0+1, \qquad u'_- = u - n_0\mathbf{1}_{l_0}, \quad l_- = l_0;$$

the left compact parameter remains u. Lemma 33.83 gives the two polynomial contractions. The actual-place construction for these compact types gives normalized towers with the same frames and Haar measures, the displayed cuts, and the required terminal-lift bounds at both places. On the open set

$$\Omega_+ = \{g \in G : G_{\mathrm{loc}}(g) > 0,\ D_a(g) \ne 0\ (1 \le a \le r)\},$$

the Bessel average of F has local smooth ambient extensions, the prescribed left and right unipotent-character covariance, and the prescribed two compact-type equivariance laws. The coefficient quantum-determinant eigenvalue equation and the derivative margin of Claim 1 imply the smoothness assertion, and change of variables in the convergent averages gives the covariance laws. The shifted averages equal the original average composed with the smooth polynomial contractions. These identities prove the same regularity and covariance assertions for FA_+ and FA_- at their respective places. The coefficient $FA_\pm$ has group-coordinate polynomial degree loss at most $n_0 s$.
Lean: CompanionTheory.WaveGrid.w_ffkb (CompanionTheory.AutomorphicTheory.BesselWaveGrids.EigStrOfMem, lines 60–67)
Lean: CompanionTheory.WaveGrid.Coef.smooth_coefficient_bessel (CompanionTheory.AutomorphicTheory.BesselWaveGrids.SmoothCoefficientBessel, lines 12–34)
Lean: CompanionTheory.WaveGrid.EqShift.ffkb_swap (CompanionTheory.AutomorphicTheory.BesselWaveGrids.EqShift, lines 112–118)

Lean: `CompanionTheory.WaveGrid.EqShift.shiftAtR_full` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.EqShift`, lines 122–181)

At each place use its own progression $Y_{\pm}$, formed by the formula in Claim 3 with $u, u'_{\pm}, l_{\pm}$. The cut width is exactly n_0, so the same grid construction gives both wave and lift grids. Replace $H_{\varepsilon,M}$ by the identical double-cutoff construction applied to $G_{\mathrm{loc}}^{p_0} F A_{\pm}$. Every component is compact and is realized in the residual level-zero positive space of its own shifted datum, by Lemma 33.88. Its normalized kernels, mixed contractions and wave isometries are therefore those proved in Claims 2 and 3. Claim 1 gives its finite eigenfunction decomposition and P_*-bound, with degree $mp_0 + n_0 s$, and its signed endpoint uses $\sigma_{l_{\pm}} = (-1)^{k-l_{\pm}}$. Thus both shifted applications have their eigenvalue, positivity, grid, differential-lift and cutoff-limit hypotheses fulfilled.

Lean: `CompanionTheory.WaveGrid.grids_plus` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.GridsText`, lines 78–94)
Lean: `CompanionTheory.WaveGrid.grids_minus` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.GridsText`, lines 96–112)
Lean: `CompanionTheory.WaveGrid.endEqW_of` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.EigStrOfMem`, lines 188–231)

The flag vector ω_γ perpendicular to the right frame has squared norm $G_{\mathrm{loc}}(\gamma)/T_q$. Since $n_0 \le b$, the contraction bounds give

$$\|v_1\|^2 \le (3b)^{2n_0} (T_q/G_{\mathrm{loc}}(\gamma))^{n_0} \big(\|A_+(\omega_\gamma)^* v_1\|^2 + \|A_-(\omega_\gamma)^* v_1\|^2 \big), \tag{A.337}$$

and

$$v_{FA_\pm}(\gamma) = v_F(\gamma) A_\pm(\omega_\gamma), \qquad \|A_\pm(\omega_\gamma)\|^2 \le (G_{\mathrm{loc}}(\gamma)/T_q)^{n_0}.$$

Apply (A.337) to $v_F(\gamma)^* e_i$ and sum over an orthonormal target basis. Adding the two shifted nonnegative energies gives the desired comparison. The differential expansion is needed through order $2p_0$ in the distinct-weight case and $2p_0 + n_0$ in the equal-weight case. Both are at most $40s^2$.

Lean: `Formalism.FourierDecayAlt.P.CutBP2.AmapsSh` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.NormSqLeShifted`, lines 39–51)
Lean: `Formalism.FourierDecayAlt.P.CutBPEq.CompactFormNormalFormDomination.thmBpAt_eqR_top` (`CompanionTheory.AutomorphicTheory.CompactTopologicalForms.NormalFormTopDomination`, lines 50–76)

It remains to supply the differential and minor-power expansions for the preceding estimates. Recall $s = r + 1$, $n_0 = 30s^2 + 6$, $m = 2s$, $p_0 = 2s$, and $q_0 = se + 40s^3 + s(s+1)$. For a coefficient with the original compact parameters u, u', spectral displacement $\|\nu_0\|_1 \le se$, and derivative order $a_1 \le 2p_0 + n_0$, one has

$$2p_0 + n_0 = 4s + 30s^2 + 6 \le 40s^2, \qquad \|\nu_0\|_1 + sa_1 + s(s+1) \le q_0.$$

Also $e \le C_a m^3$ gives $q_0 + 1 \le (C_a/2 + 3)m^4$. Choose the fixed threshold at least $\max(10^6, 100(C_a/2+3))$. Put $\Delta_\lambda = C_g m^4$ and $\Delta_\pm = C_g m^{9/2}(3sn^2 + n^3)$. Both gap parameters are at least $100(q_0+1)$, and the splitting scale is at least $10^6 n^7$. The quarter-gap estimates for the Bessel period construction follow from

$$\Delta_\pm/4 \ge 120 n^3, \qquad \frac{\Delta_\pm}{32\sqrt{m}} = \frac{C_g m^4 (3sn^2 + n^3)}{32} \ge 125 m^2 n^3,$$

which bound the growth losses in that construction. The coordinate λ_k gives

$$b \ge |\lambda_k| \ge b_v \ge 10^6 n^7 \ge \max(2, n).$$

The interlacing, parity, splitting and gap assumptions therefore give the numerical family conditions for both u and u'.

Lean: `CompanionTheory.WaveGrid.GapSizes.gapOK_of` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.GapSizes`, lines 54–121)

Fix one such coefficient F, at spectral displacement ν_0. A member of the coefficient span can be expressed as a finite linear combination of K-finite coefficients of any chosen discrete series R_0 with parameter $\lambda + \nu_0$. Indeed, uniqueness at this parameter supplies a unitary intertwiner from each representation occurring in the finite sum to R_0, and the intertwiner preserves compact-finite vectors. The retained gap is greater than $n^3 + 1$, as required for this uniqueness statement. For the original coefficient choose its given representation as R_0.

Lean: `CompanionTheory.WaveGrid.WAt.coefOf_of_wsp` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.CoefOfOfWsp`, lines 40–73)

For every $\nu \in \mathbf{Z}^n$ with $\|\nu\|_1 \le q_0$, choose a discrete-series representation R_ν of parameter $\lambda + \nu$, with $R_{\nu_0} = R_0$. The numerical family conditions give regular integral parameters, the required Cayley signature and the retained opposite-sign gap at every such ν. The discrete-series existence input supplies the other members. At indices outside this ball choose R_0. Thus the starting representation is preserved exactly.
Lean: `Gp91.WG.family_through_discreteSeries` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.WaveBallEstimateWeightedDualAlt`, lines 70–95)

On the index ball $\|\nu\|_1 \le q_0$, choose the period-normalized Bessel maps ℓ_ν, ℓ'_ν and positive formal degrees d_ν. The coefficient-period identity assigns a scalar α to a coefficient of R_ν; its finite squared big-cell tower energy is

$$\widetilde{I}(F) = c_1 \frac{|\alpha|^2}{d_\nu}, \qquad c_1 > 0,$$

with c_1 depending on the local Bessel datum and its measures, uniformly in ν. The actual tower orbital vector of that coefficient is α times its period-normalized Bessel orbital vector. Hence $\widetilde{I}(F) = 0$ implies that the orbital vector vanishes at every represented positive-Gram big-cell point, and the required expansion is zero in this case.

Suppose $\widetilde{I}(F) > 0$, and let $\alpha_0 \neq 0$ be its period scalar. Set

$$t = \frac{d_{\nu_0}}{|\alpha_0|} > 0, \qquad F_{\nu_0} = tF.$$

For each remaining index in this ball where both Bessel maps are nonzero on smooth vectors, choose F_ν with period scalar α_ν of modulus d_ν. Set $F_\nu = 0$ when either map is zero or $\|\nu\|_1 > q_0$. At the starting index both maps are nonzero because $\alpha_0 \neq 0$. Each coefficient is continuous, and the big-cell energies are measurable. For $\|\nu\|_1 \le q_0$, the formal-degree formula for the fixed Haar measure is

$$d_\nu = c_2 \mathcal{D}(\lambda + \nu), \qquad \mathcal{D}(\mu) = \frac{\prod_{1 \le i < j \le n} |\mu_i - \mu_j|}{\prod_{1 \le i < j \le n} (j - i)}, \qquad c_2 > 0.$$

Consequently, for $\|\nu\|_1 \le q_0$, either $F_\nu = 0$ and $\widetilde{I}_\nu = 0$, or

$$\widetilde{I}_\nu := \widetilde{I}(F_\nu) = c_1 c_2 \mathcal{D}(\lambda + \nu) > 0.$$

Lean: `Gp91.WG.coef_norm_finite_and_zeroW` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.WaveBallEstimateWeightedDualAlt`, lines 376–404)
Lean: `Gp91.WG.exists_coef_family` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.WaveBallEstimateWeightedDualAlt`, lines 97–137)
Lean: `Gp91.WG.suppliedData_existsW` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.WaveBallEstimateWeightedDualAlt`, lines 351–374)
Lean: `Gp91.WG.sup_norm` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.WaveBallEstimateWeightedDualAlt`, lines 215–258)
Lean: `Gp91.WG.sup_meas` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.WaveBallEstimateWeightedDualAlt`, lines 260–266)

For $\mu = \lambda + \nu$ with $\|\nu\|_1 \le q_0$, every difference $\mu_i - \mu_j$, $i < j$, is an integer at least two. Therefore

$$\mathcal{D}(\mu) > 0, \qquad \mathcal{D}(\mu + \mathbf{1}_A) \le n^{|A|} \mathcal{D}(\mu),$$

and cancellation of the factors not involving i gives

$$\frac{\mathcal{D}(\mu + \mathbf{1}_i)}{\mathcal{D}(\mu)} = \prod_{j \neq i} \frac{|\mu_i + 1 - \mu_j|}{|\mu_i - \mu_j|}.$$

For the upper bound, add the coordinates in A one at a time. At each step the factors involving $j < i$ decrease. The product of the factors involving $j > i$ is at most $\prod_{a=1}^{n-i} (1 + 1/a) = n - i + 1 \le n$, because the intermediate coordinates remain strictly decreasing with integral differences. Multiplication proves the bound. Thus, whenever $\widetilde{I}_\nu > 0$ and both indices lie in the ball,

$$\widetilde{I}_{\nu + \mathbf{1}_A} \le n^{|A|} \widetilde{I}_\nu \le (\exp(1) n)^s \widetilde{I}_\nu \qquad (|A| = s),$$

including the case $F_{\nu+\mathbf{1}_A}=0$.

Lean: `GlobalTheory.FamilyDegreeRatio.degree_ratios` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.AbsCoordLeOfLOneNormLe`, lines 60–107)

Write $\mathcal{V}_\nu$ for the actual tower big-cell orbital vector of F_ν, and use the common point weight in its squared integral $\widetilde{I}_\nu$. At each index with $\|\nu\|_1+s(s+1)\le q_0$, the exterior-power decomposition of $\bigwedge^s \mathbf{C}^n\otimes R_\nu$ has components $R_{\nu+\mathbf{1}_A}$, $|A|=s$, and signs $\epsilon_{\nu A}\in\{1,-1\}$. Let $q_z,q_{z'}$ be the exterior products of the two Bessel frames in their determinant order. In a branch where both F_ν and $F_{\nu+\mathbf{1}_A}$ are nonzero, the projection pr_A to the indicated component satisfies

$$\ell_{\nu+\mathbf{1}_A}(\mathrm{pr}_A(q_z\otimes v))=\kappa_{\nu A}\ell_\nu(v),\qquad \ell'_{\nu+\mathbf{1}_A}(\mathrm{pr}_A(q_{z'}\otimes v))=\kappa'_{\nu A}\ell'_\nu(v)$$

on smooth vectors. When both coefficients are nonzero, put

$$\tau_{\nu A}=\epsilon_{\nu A}\kappa'_{\nu A}\overline{\kappa_{\nu A}}\frac{\alpha_\nu}{\alpha_{\nu+\mathbf{1}_A}};$$

put $\tau_{\nu A}=0$ if either coefficient is zero. The tensor identity and the two Bessel moment bounds give, at represented positive-Gram points,

$$D_s\mathcal{V}_\nu=\sum_{|A|=s}\tau_{\nu A}\mathcal{V}_{\nu+\mathbf{1}_A},\qquad |\tau_{\nu A}|\le H^s,\qquad H=\sqrt{H_\square(T_p)H_\square(T_q)},\qquad H_\square(T)=\max(1,T)b^{20s}.$$

These identities include the zero-coefficient branches. For $C_h^*=2|C_h|$, the norm assumptions imply

$$H_\square(T_p),H_\square(T_q)\le b^{(C_h^*+10)m},\qquad H\le b^{(C_h^*+10)m}.$$

Lean: `Gp91.WG.exists_tauL` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.WaveBallEstimateWeightedDualAlt`, lines 686–764)

Lean: `Gp91.WG.supplied_pieriL` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.WaveBallEstimateWeightedDualAlt`, lines 801–848)

Lean: `Gp91.WG.tau_boundL` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.WaveBallEstimateWeightedDualAlt`, lines 766–799)

Lean: `GlobalTheory.TopNum.top_num` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.TopNum`, lines 25–35)

Every index reached in $a'\le a_1$ steps has distance at most sa' from ν_0. The inequality $\|\nu_0\|_1+sa_1+s(s+1)\le q_0$ therefore supplies the projection reserve at every stage. Apply Lemma 33.81 with origin ν_0, norm exponent C_h^*, and the family just constructed. Undo the common positive scalar t at the starting coefficient. The resulting functions have energy at most $\widetilde{I}(F)$, and their absolute coefficient sum for $D_s^{a'}$ is at most

$$b^{(C_h^*/2+6)m^2a'}=b^{(|C_h|+6)m^2a'}.$$

Lean: `Gp91.WG.familyDerivativeExpansion_of_isDS_of_coefOf` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.WaveBallEstimateWeightedDualAlt`, lines 406–510)

The coefficient-space tensor expansion of D_l^eF uses parameters $\lambda+\nu_0$ with $\|\nu_0\|_1\le se$. The preceding argument therefore applies to every term, with its differential expansion through order $2p_0+n_0$. Combining the $(2n+1)^{2(r+1)e}$ tensor summands, the grid estimates, the shifted comparisons and the family moment bounds gives

$$|D_l(\gamma)|^{2e}\widetilde{E}(F;\gamma)\le b^{C_5m^5}\max(1,G_{\mathrm{loc}}(\gamma)^{-1})^{40s^2}\widetilde{I}(F).$$

Here $2n+1\le b^2$, because $b\ge b_v\ge C_gn^7$. The actual terminal vector and the orbital supremum have the exact comparison

$$E(F;\gamma)=\pi^{2\,\mathrm{pc}(\gamma)}\widetilde{E}(F;\gamma),\qquad \mathrm{pc}(\gamma)=\sum_{a=1}^{d}\frac{s_a(s_a-1)}{2}\le s^2,$$

for the block lengths $s_1,\dots,s_d$ of γ. Here $\mathrm{pc}(\gamma)$ counts pairs of positions $a<b$ in the same block. Its hypotheses are the original K-finite coefficient, the invariant-derivative decay, positive Gram value and existence of a representative, all verified above. The point-weight comparison is exact. The section-measure comparison gives

$$\widetilde{I}(F)\le\pi^{2m^2}I_{\mathrm{std}}.$$

Each of the two powers of π is at most b^{4m^5}. Taking $C = C_5 + 8$ proves (33.28).
Lean: Formalism.FourierDecayAlt.P.CutBP.BigCellDerivativeExpansion (Formalism.HermitianCurvature.HermitianCurvatureProofs.CutBP, lines 130–138)
Lean: Gp91.WG.bigCellDerivativeExpansion_of_isDS_of_coefOf (CompanionTheory.AutomorphicTheory.BesselWaveGrids.WaveBallEstimateWeightedDualAlt, lines 512–524)
Lean: Formalism.FourierDecayAlt.P.CutBP.CompactFormNormalFormDomination.thmBpAt_top (CompanionTheory.AutomorphicTheory.CompactTopologicalForms.NormalFormTopDomination, lines 14–35)
Lean: Formalism.FourierDecayAlt.P.CutBPEq.CompactFormNormalFormDomination.thmBpAt_eqR_top (CompanionTheory.AutomorphicTheory.CompactTopologicalForms.NormalFormTopDomination, lines 50–76)
Lean: CompanionTheory.WaveGrid.WGlue.wLoc1N_of (CompanionTheory.AutomorphicTheory.BesselWaveGrids.WeightedUniformRotatedOfExpansion, lines 30–159)

The coefficient estimate used here is valid for every actual place with $k \geq 2$, the stated signature, the two compact-type conditions, the splitting and gap inequalities, and either the selected distinct-weight cut with its terminal-lift bound or the equal-weight consecutive gap. These are precisely the hypotheses verified above. The orbital-vector and section-measure identities also hold for arbitrary $k \geq 2$, so this argument applies to the local datum in the statement.
Lean: CompanionTheory.WaveGrid.WEnd.wBp_of (CompanionTheory.AutomorphicTheory.BesselWaveGrids.PowFourLeRpowNineHalves, lines 106–302)
Lean: Formalism.L2.SG.Pf.BesselExpansion.weightTie (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.TBBTie, lines 198–204)
Lean: Formalism.MainNotions.BigCellDensity.pc (Formalism.SharpRestriction.CuspSpanNormBounds.BigCellDensity, lines 592–593)
Lean: Formalism.L2.SG.Pf.BesselExpansion.pc_le (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.TBBTie, lines 56–59)
Lean: Formalism.L2.SG.Pf.BesselExpansion.tieTopXk (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.TBBTie, lines 43–54)
Lean: CompanionTheory.WaveGrid.Ties.energy_identity (CompanionTheory.AutomorphicTheory.BesselWaveGrids.NormalizedMeasureScaling, lines 73–89)

The complete coefficient estimate and its orbital translation are contained in `CompanionTheory.WaveGrid.WEnd.wBp_of` and `CompanionTheory.WaveGrid.WGlue.wLoc1N_of`. □

Proof of 33.90

Proof (outline). The rank-one torus comparison holds at every threshold $C_{gf} \geq 100$; take $C_{gf} = 100$, so that $C_g \geq C_{g,1} \geq 1000 \geq C_{gf}$ supplies its opposite-sign gap hypothesis.
Lean: Formalism.MainStatements.RankOne.one_bessel_torus_bound (CompanionTheory.AutomorphicTheory.RankOneTori.FrameBesselBoundOfLargeConstants, lines 223–224)

Claim 1. Let L be the local datum in the statement, with $\epsilon = \epsilon_{k,r}$, character $-2i\kappa_0$, $\kappa_0 > 0$, and unipotent Haar scale $0 < \kappa_N < \infty$. Assume its opposite-sign gap is at least $100m^{9/2}(3(r+1)n^2 + n^3)$. For any positive torus parameters κ, τ and any compact-finite coefficient F, one can choose a compact-finite coefficient F_1 and a scalar $a \neq 0$ so that the torus-translated section operator and every torus-translated orbital operator of F_1 equal a times the respective operator of F. The orbital assertion holds for every representative and quotient measure meeting the stated compatibility conditions at a block point of nonzero Gram factor.

We now give the simultaneous choice. Write $F(g) = P\sigma(g)Q$. The rank-one construction gives common section and orbital functions Φ, Ψ, a scalar $\lambda_0(P, Q)$, and translated scalars $\lambda(P, Q; \kappa, \tau)$. Both original operators are their common function multiplied by $\lambda_0(P, Q)$, and both translated operators use $\lambda(P, Q; \kappa, \tau)$. If $\lambda_0(P, Q) = 0$, take $F_1 = 0$ and $a = 1$. Both identities then have zero on each side. If $\lambda_0(P, Q) \neq 0$, the rank-one construction gives a compact-finite pair (P_1, Q_1) with $\lambda(P_1, Q_1; \kappa, \tau) \neq 0$. Take $F_1(g) = P_1\sigma(g)Q_1$ and

$$a = \frac{\lambda(P_1, Q_1; \kappa, \tau)}{\lambda_0(P, Q)}.$$

The two operator identities follow from the same scalar division. The required nonzero translated pair is obtained from density of compact-finite vectors and the two nonzero Bessel functionals.
Lean: Formalism.MainStatements.RankOne.rank_one_torus_bound (CompanionTheory.AutomorphicTheory.RankOneTori.FrameBesselBoundOfLargeConstants, lines 221–221)
Lean: Formalism.L2.SG.Pf.OBT.oneBesselTor_of_rankOne (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.OneBesselTor, lines 67–93)

For a local datum L, write

$$\mathcal{E}_L(F; \gamma) = p_{k,r}(T_p, T_q; \gamma)\|J(F; \gamma)\|^2_{\mathrm{HS}},$$

with the supremum convention of Definition 33.82. *Claim 2.* Changing the unipotent Haar scale preserves the normalized inequality. Compare two choices related by $dn = s\,dn_{\mathrm{std}}$, where $0 < s < \infty$. The unipotent measure occurs twice in the product measure defining the Bessel operator, so that product measure changes by s^2. The section operator changes by s^2, and its squared integral satisfies

$$I = s^4 I_{\mathrm{std}}.$$

For every admissible orbital quotient measure, multiplication by s^{-2} gives an admissible quotient measure for the standardized datum. The corresponding orbital operator changes by s^2. The point weight is nonnegative on represented points, and both orbital suprema vanish on unrepresented points. Taking suprema and multiplying by the point weight gives

$$\mathcal{E}_L(F;\gamma) \le s^4 \mathcal{E}_{L_{\mathrm{std}}}(F;\gamma).$$

These two relations transfer the standardized inequality to arbitrary finite positive Haar scale.
Lean: `CompanionTheory.WaveGrid.Ties.pairMeas_smul` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.NormalizedMeasureScaling`, lines 97–106)
Lean: `CompanionTheory.WaveGrid.Ties.orbSq_le` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.NormalizedMeasureScaling`, lines 108–126)
Lean: `CompanionTheory.WaveGrid.Ties.jNorm2_eq` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.NormalizedMeasureScaling`, lines 128–141)

Set $\kappa = \kappa_0^{-1}$ and $\tau = \kappa_0^{r}$. For $0 \le a \le r$, put $t_a = \tau\kappa^{r-a}$, and let $D_{\kappa,\tau}$ be the diagonal matrix with entries $t_0, \dots, t_r$.

Claim 3. For the local datum and coefficient in the statement, there are a local datum L_1 and a compact-finite coefficient F_1, with the same signature, Harish–Chandra parameter and two compact weights as L, character parameter $-2i$, finite positive unipotent Haar scale, and final norms

$$T_{p,1} = \tau^2 T_p, \qquad T_{q,1} = \tau^2 T_q.$$

For every big-cell section for L, there are a big-cell section for L_1 and a real constant $K_1 > 0$, independent of the block point, such that their squared section integrals satisfy $I_1 = K_1 I$. For each block point γ with $G_{\mathrm{loc}}(\gamma) \neq 0$, let γ_1 be the block point whose matrix is

$$\mathrm{Mat}(\gamma_1) = D_{\kappa,\tau}\,\mathrm{Mat}(\gamma) D_{\kappa,\tau}.$$

Here $\mathrm{Mat}(\gamma)$ is the block matrix with blocks tJ_s used to define the leading minors. The transported point has the same block lengths as γ, and

$$G_{\mathrm{loc},1}(\gamma_1) = \kappa^{2r(r+1)} \tau^{4(r+1)} G_{\mathrm{loc}}(\gamma),$$

$$\mathcal{E}_{L_1}(F_1;\gamma_1) \le K_1 F_*(\kappa_0) \mathcal{E}_L(F;\gamma), \qquad K_1 \mathcal{E}_L(F;\gamma) \le F_*(\kappa_0) \mathcal{E}_{L_1}(F_1;\gamma_1).$$

The inputs to this comparison are the prescribed signature, the opposite-sign gap at threshold $C_{gf} = 100$, compact-finiteness of F, positivity of κ_0, κ, τ, and the finite positive Haar scale. They hold by the hypotheses and $C_g \ge C_{g,1} \ge 1000$. Claim 1 supplies a common nonzero scalar for the section and all orbital operators, including its zero-operator case. The scalar K_1 is the product of the positive Jacobian of the section-coordinate change, the positive multiplier of its density, and the squared modulus of the nonzero scalar transporting both the section operator and every admissible orbital operator. Thus the section identity and both orbital inequalities use the same K_1.
Lean: `Formalism.MainNotions.OnePlaceEstimates.FrameBesselBound` (`Formalism.SharpRestriction.CuspSpanNormBounds.OnePlaceEstimates`, lines 262–289)
Lean: `Formalism.MainNotions.OnePlaceExpansion.frame_bound_of_bessel_torus` (`Formalism.SharpRestriction.CuspSpanNormBounds.OnePlaceExpansion`, lines 71–115)
Lean: `Formalism.MainNotions.FrameBoundFromBesselTorus.leadMinor_ptTr` (`Formalism.SharpRestriction.CuspSpanNormBounds.RatioInvLeFrameFactor`, lines 101–114)
Lean: `Formalism.MainNotions.FrameBoundFromBesselTorus.tF` (`Formalism.SharpRestriction.CuspSpanNormBounds.ContinuousCjIT`, lines 12–12)

Claim 4. The transformed datum obeys the size and minor bounds required for the standardized estimate. The hypotheses give $T_{p,1}, T_{q,1} \ge 1$, and

$$T_{p,1}, T_{q,1} \le b^{|C_h|(m+2r)} \le b^{2|C_h|m}.$$

The leading minors satisfy

$$|D_l(\gamma)| \le b^{|C_h|m^2}|D_l(\gamma_1)| \quad (1 \le l \le r+1).$$

Lean: CompanionTheory.WaveGrid.WGlue.dpt_le (CompanionTheory.AutomorphicTheory.BesselWaveGrids.WeightedUniformRotatedOfExpansion, lines 188–216)

Moreover,

$$g_*(\kappa_0) = (\kappa_0^{-1})^{2r(r+1)}(\kappa_0^{r})^{4(r+1)} = \kappa_0^{2r(r+1)}.$$

If $Q = \max(\kappa_0, \kappa_0^{-1})$, then

$$F_*(\kappa_0) = Q^{2m^2+2rm} \le b^{|C_h|(2m^2+2rm)} \le b^{4|C_h|m^2}.$$

Lean: Formalism.MainNotions.TheoremBConstants.GsC (Formalism.SharpRestriction.CuspSpanNormBounds.TheoremBConstants, lines 133–133)
Lean: Formalism.MainNotions.TheoremBConstants.num_Fr (Formalism.SharpRestriction.CuspSpanNormBounds.TheoremBConstants, lines 796–826)

Apply Proposition 33.89 to L_1, F_1, γ_1, using the Haar-scale transfer above. Multiply its bound by $b^{2|C_h|m^2 e}F_*(\kappa_0)$, substitute the minor comparison and $I_1 = K_1 I$, and cancel $K_1 > 0$. This gives the asserted inequality with $g_*(\kappa_0) = \kappa_0^{2r(r+1)}$. The complete proof is contained in `CompanionTheory.WaveGrid.WGlue.w_in_frame`. □

Proof of 33.91

Proof (outline). Apply Proposition 33.89 with $C_a = 2(1 + A_w + 10)$ to obtain $C \ge 0$ and $C_{g,1}$, and set $C_{g,0} = C_{g,1} \ge 1000$, which serves for both Proposition 33.89 and Lemma 33.90. Fix $C_g \ge C_{g,0}$. Put $a = 10m^2$, $p = A_w(r+1)^2$, and at each infinite place v put $d_v(\gamma) = |D_{r+1}(\gamma)|_v^2$ and γ_v for the rotated local block point of γ.

Claim 1. Every represented nonzero local factor obeys $0 < |G(\gamma)|_v \le d_v(\gamma)$ and $p_v(\gamma) \ge 0$. Fix a rational block point γ in the sum and a place v. If $\|J_v(\gamma)\|_{\mathrm{HS}}^2 = 0$, the v-th factor vanishes. Otherwise γ_v has a unitary representative in the local Bessel datum of character parameter $-2i\xi$, so $G_{\mathrm{loc}}(\gamma_v) = \big(\prod_{i=0}^{r-1}\mathrm{rd}(E)^{-(r-i)}\big)^4 x_{G,v} \ge 0$, where $x_{G,v} = |D_{r+1}(\gamma)|_v^2 - TT'|D_r(\gamma)|_v^2 = v(G(\gamma))$. Since $G(\gamma) \ne 0$, we have $0 < |G(\gamma)|_v = x_{G,v} \le d_v(\gamma)$ and $p_v(\gamma) \ge 0$.

Claim 2. The weighted local factor satisfies (A.338) below, with a constant independent of the block point and the place. For $1 \le l \le r+1$, set $s_{l-1} = \big(\prod_{i=0}^{l-1}\mathrm{rd}(E)^{-(r-i)}\big)^{-4}$, so $1 + s_{l-1}|D_l(\gamma_v)|^2 = 1 + |D_l(\gamma)|_v^2$. From the coordinate and splitting-scale bounds we have $b \ge b_v \ge C_g n^7 \ge 2$, and then $\xi = \pi/(v_E g_0 g_1) > 0$ with $\xi^{\pm 1} \le b^{C_h}$ gives $C_h \ge 0$; the root-discriminant bound $\mathrm{rd}(E) = 2\pi/(\xi g_0 g_1) \le b^{3+C_h}$ then gives $s_{l-1} \le b^{(3+C_h)m^2}$. Expanding $w_v(\gamma)^2(1 + d_v(\gamma))^a W_v(\gamma)^p$ in the block-end minors assigns exponents $N_l \in \mathbf{N}$ with $\sum_{l=1}^{r+1} N_l \le (1 + A_w + 10)m^3$ such that

$$w_v(\gamma)^2(1 + d_v(\gamma))^a W_v(\gamma)^p \le \Big(\prod_{l=1}^{r+1}\big(1 + s_{l-1}|D_l(\gamma_v)|^2\big)^{N_l}\Big)^2.$$

Choose l_* maximizing $|D_l(\gamma_v)|^2$, and set $e = 0$ when $|D_{l_*}(\gamma_v)| \le 1$ and $e = 2\sum_l N_l \le C_a m^3$ otherwise; this bounds the right-hand side by $\big(2b^{(3+C_h)m^2}\big)^{2\sum_l N_l}|D_{l_*}(\gamma_v)|^{2e} \le b^{(4+C_h)C_a m^5}|D_{l_*}(\gamma_v)|^{2e}$. Applying Lemma 33.90 at $\kappa_0 = \xi$ and using $\max\big(1, |g_*(\xi)G_{\mathrm{loc}}(\gamma_v)|^{-1}\big)^{40(r+1)^2} \le b^{10(3+3C_h)m^5}\max(1, |G(\gamma)|_v^{-1})^a$ give

$$w_v(\gamma)^2(1 + d_v(\gamma))^a p_v(\gamma) W_v(\gamma)^p \|J_v(\gamma)\|_{\mathrm{HS}}^2 \le b^{C_* m^5}\max(1, |G(\gamma)|_v^{-1})^a I_v \tag{A.338}$$

with $C_* = (4+C_h)C_a + 2C_hC_a + 4C_h + C + 10(3+3C_h)$. *Claim 3.* The product of the weighted local factors is at most $b^{C_* m^5 f}\prod_v I_v$. Since $(1 + d_v(\gamma))^{-1}\max(1, |G(\gamma)|_v^{-1}) \le |G(\gamma)|_v^{-1}$ and $\prod_v |G(\gamma)|_v = |N_{F/\mathbf{Q}}(G(\gamma))| \ge 1$ for nonzero integral $G(\gamma)$, taking the product of (A.338) over v gives $w(\gamma)^2 p(\gamma) W(\gamma)^p \prod_v \|J_v(\gamma)\|_{\mathrm{HS}}^2 \le b^{C_* m^5 f}\prod_v I_v$.

Claim 4. The reciprocal weight sum satisfies (A.339) below. For fixed block lengths (s_i) and block-end minors, the ratios D_j/D_{j-s_i} determine $t_i^{s_i}$ up to at most $\prod_i s_i \le e^{(r+1)/e}$ root choices, across 2^r compositions of $r+1$. In each nonzero algebraic-integer box $|v(t)| \le R_v$ with $U = \prod_v R_v \ge 1$, dilating the v-th plane by

$U^{1/f}/R_v$ separates distinct points by at least 1 in supremum norm; disjoint polydisks of radius $1/2$ fit inside radius $U^{1/f}+1/2$ and bound the count by $9^f U^2$. Integrating against $\prod_v 4R_v(1+R_v^2)^{-3}\,dR_v$ gives at most 16^f per block-end minor, and hence

$$\sum_{\gamma} w(\gamma)^{-1} \le 2^r e^{(r+1)/e} 16^{f(r+1)} \le 64^{f(r+1)} \le b^{6m^5 f}. \tag{A.339}$$

Multiplying the product bound by $w(\gamma)^{-1}$ and summing over γ with $C_b = C_* + 6$ proves the inequality.
Lean: `CompanionTheory.WaveGrid.WGlue.weight_le` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.WeightedUniformRotatedOfExpansion`, lines 275–291)
Lean: `Formalism.MainNotions.TheoremBConstants.weights_le` (`Formalism.SharpRestriction.CuspSpanNormBounds.TheoremBConstants`, lines 636–664)
Lean: `Formalism.MainNotions.TheoremBConstants.sum_weighted_point_counts_le_cube` (`Formalism.SharpRestriction.CuspSpanNormBounds.TheoremBConstants`, lines 957–975)
Lean: `Formalism.MainNotions.TheoremBConstants.num_G` (`Formalism.SharpRestriction.CuspSpanNormBounds.TheoremBConstants`, lines 854–904)
Lean: `Formalism.MainNotions.BesselShellMomentBounds.theorem_B_weighted_bound_one_place` (`Formalism.SharpRestriction.CuspSpanNormBounds.BesselShellMomentBounds`, lines 992–996)
Lean: `Formalism.MainNotions.BesselShellMomentBounds.theorem_B_second_bound_64` (`Formalism.SharpRestriction.CuspSpanNormBounds.BesselShellMomentBounds`, lines 265–286)

The complete proof is contained in `CompanionTheory.WaveGrid.weighted_bound_rotated_uniform`. □

Proof of 33.92

Proof. Projection by the ρ pairings with the orthonormal vectors splits U as O_E^{ρ} and its orthogonal complement. Expand $q^a \mathbf{1}_Z$ in the additive characters of $O/\varpi^a O$ evaluated on $h(u,u)$. The trivial character contributes $\mathbb{E}\chi = \mathbf{1}_{\chi=1}$.

Consider a character of conductor ϖ^i, $1 \le i \le a$. In one orthonormal coordinate its normalized sum has the form

$$S = \mathbb{E}_x\, \psi(x\bar{x})\chi_0(x), \qquad x \in O_E/\varpi^N O_E,$$

where χ_0 is the linear character in that coordinate. Expand $|S|^2$ using the difference y of the two summation variables. The inner average in x is the character average for $\psi(\mathrm{Tr}_{E/F}(x\bar{y}))$. The trace pairing on O_E is perfect over O, also at residue characteristic two. The average vanishes unless $y \in \varpi^i O_E$. The proportion of such y is q^{-2i}, so $|S| \le q^{-i}$. Taking the product over the ρ coordinates and bounding the average on the complement by one gives $q^{-i\rho}$. There are $q^i - q^{i-1}$ characters of conductor ϖ^i. Their total contribution is at most

$$\sum_{i \ge 1} q^{-i(\rho-1)} \le \frac{8}{7} q^{1-\rho},$$

since $q^{1-\rho} \le 1/8$.

If $h(u,W)$ is proper, all the ρ coordinates belong to one maximal ideal πO_E. In the field case take $\pi = \varpi$, with $\pi\bar{\pi} = \varpi^2$. In the split case the possibilities are $(\varpi, 1)$ and $(1, \varpi)$, each with norm ϖ. Write $j \in \{1,2\}$ for the valuation of this norm. Restriction to the specified ideal in all coordinates costs $q^{-j\rho}$. Conditional on that restriction, write $x = \pi z$ with z uniform. The norm character of conductor i then has conductor exponent $(i-j)_+$. The same square calculation bounds the conditional product by $q^{-(i-j)_+\rho}$. Including the trivial character, the sum of these bounds with multiplicities is at most

$$q^j + \sum_{i > j} q^i q^{-(i-j)\rho} \le (8/7) q^j.$$

After the restriction cost, each ideal contributes at most $(8/7)q^{j(1-\rho)}$ to q^a times the probability. There are at most two ideals. □

Proof of 33.93

Proof. If a solution exists, associate to it the integral ideal $I = (x)J^{-1}$. Then

$$I\bar{I} = (a)(J\bar{J})^{-1} =: \mathfrak{b}.$$

In particular $\mathfrak{b}$ is integral and I divides $\mathfrak{b}$. If $\mathfrak{p}^h \| \mathfrak{b}$, there are at most $h+1 \le (N\mathfrak{p})^h$ possibilities for the exponent in I. Thus the number of possible I's is at most $N\mathfrak{b} = |N_{E/\mathbf{Q}} a|/(NJ)^2$.

Two solutions giving the same I have quotient an integral unit u with $u\bar{u} = 1$. Every conjugate of u has absolute value one. The characteristic polynomials over $\mathbf{Q}$ of its positive powers have integral coefficients bounded by the binomial coefficients of order $2f$. They therefore have only finitely many roots in total. Two powers coincide, so u is a root of unity. If w is the number of roots of unity of E, then $\varphi(w) \le 2f$, since a finite subgroup of the multiplicative group of a field is cyclic. For $p^e \| w$ the ratio $\varphi(p^e)^2/p^e = p^{e-2}(p-1)^2$ is at least one except for $p^e = 2$, when it equals $1/2$. Hence $w \le 2\varphi(w)^2 \le 8f^2$. This proves the first estimate, also for fractional J.

Apply it to each coordinate with $a = 1$. Injectivity gives the upper bound $64f^4(TT')^{2f}P^4$. Since $f \ge 1$, induction gives $8f^2 \le 81^f$: it holds at $f = 1$, and $(f+1)^2/f^2 \le 4$. Thus $64f^4 \le 9^{4f} \le (TT')^{4f}$. Together with $P^4 \le P^6$ this proves the second estimate. □

Proof of 33.95

Proof. Continuity is preserved by multiplication by a_w. Scaling the two measures gives

$$\int_{q_w^{-1}(K)} \Phi_w \, d\mu_w = c_w \int_K f_w \, d\nu_w^0 = \int_K (c_w/e_w) f_w \, d(e_w \nu_w^0).$$

For each finite set S of finite places,

$$\prod_{v \in S} g_v(T_v) = \left(\prod_{v \in S} a_v \right) \left(\prod_{v \in S} f_v(T_v) \right).$$

Taking the limit proves convergence and identifies the product. Multiplication by $g_\infty(T_\infty) = a_\infty f_\infty(T_\infty)$, followed by the assumed product-one identity, proves the last formula. □

Proof of 33.96

Proof (outline). Proposition 33.24 supplies the local uniqueness and unit-scale archimedean convergence. Five claims establish the Siegel–Weil density construction, the admissible mass-data normalization, and the local and global mass bounds.

Claim 1 (measure form of the convergent Siegel–Weil formula). Let E/F be CM with conjugation ι, and let J be a nonsingular Hermitian matrix of rank M_0. Let T be a nonsingular Hermitian matrix of rank n_0, with $n_0 = 0$ or $n_0 + 2 \le M_0$. Fix an F-basis of E, field Haar measures at infinity and every finite place normalized so that the ring of integers of F has covolume 1 in $F \otimes_{\mathbf{Q}} \mathbf{R}$ and the ring of integers of each completion F_p has volume 1, Haar measure μ on $\mathrm{U}(J)(\mathbb{A}_F)$, and a measurable fundamental domain $\mathcal{F}$ for $\mathrm{U}(J)(F)$. For the rational Gram fibre $\mathcal{X}_T = \{X \mid \mathrm{Gram}_J(X) = T\}$, define its averaged counting measure by

$$I_T = \mu(\mathcal{F})^{-1} \sum_{X \in \mathcal{X}_T} (h \mapsto h^{-1}X)_*(\mu|_{\mathcal{F}}).$$

Choose at infinity and at each finite place Gram-fibre measures $m_{w,S}$, for nonsingular Hermitian S, obtained by disintegrating the coordinate tuple Haar measure on the tuples of nonsingular Gram matrix against the coordinate Hermitian-matrix Haar measure, with $m_{w,S}$ carried by the fibre of S and with $S \mapsto \int \Phi \, dm_{w,S}$ continuous on the nonsingular locus for every continuous compactly supported Φ supported on tuples of nonsingular Gram matrix. Two such families agree at every S, since their fibre integrals agree almost everywhere and are continuous on an open set on which the Haar measure has full support. Write $m_{\infty,T}, m_{p,T}$ for their values at T, and Λ_p for the integral coordinate lattice determined by the fixed basis. Then I_T is their Euler-product measure with constant 1, meaning that the numbers $m_{p,T}(\Lambda_p)$ are finite with $\sum_p |m_{p,T}(\Lambda_p) - 1| < \infty$, and there is a finite set S_0 such that, whenever $S \supseteq S_0$ is finite, the integral of every nonnegative measurable cylinder function with factors F_∞, F_p for $p \in S$ and factor 1_{Λ_p} outside S equals

$$\left(\int F_\infty \, dm_{\infty,T} \right) \prod_{p \in S} \left(\int F_p \, dm_{p,T} \right) \prod_{p \notin S} m_{p,T}(\Lambda_p).$$

The rank-zero case is the base case. In positive rank, split a nonzero represented scalar from T. If that scalar is represented by J, split the same scalar from J and apply induction to the complements, where the rank inequality persists. If it is not represented by J, the corresponding block fibre has zero measure. Congruence invariance transports these conclusions back to J, T.
Lean: `GlobalTheory.CompanionTheory.SW.SWMeas` (`CompanionTheory.SpectralGeometry.CountingDefinitions.AdelicEulerProducts.IsEulerProd`, lines 111–121)
Lean: `GlobalTheory.CompanionTheory.SW.IsEulerProd` (`CompanionTheory.SpectralGeometry.CountingDefinitions.AdelicEulerProducts.IsEulerProd`, lines 99–109)
Lean: `GlobalTheory.T.siegel_weil_measure_of_rank_small_or_zero` (`CompanionTheory.SpectralGeometry.LocalGlobalProofs.AdelicAnalysis.SiegelWeilMeasureOfRankSmallOrZeroStatement`, lines 7–8)
Lean: `GlobalTheory.CompanionTheory.SW.AllFrames.swMeas_all` (`CompanionTheory.SpectralGeometry.LocalGlobalProofs.AdelicAnalysis.SiegelWeilMeasureOfRankSmallOrZero`, lines 16–48)

Claim 2 (integration of a pure tensor). Let I be any Euler-product measure of constant 1 for the chosen basis, with locally finite local factors m_∞, m_p. Let Φ_∞ be continuous and compactly supported, let every Φ_p be locally constant and compactly supported with $\Phi_p = 1_{\Lambda_p}$ for all but finitely many p, and set $\Phi = \Phi_\infty \Phi_f$, where Φ_f is their restricted product. Then Φ is measurable and integrable, the product of the finite-place integrals converges, and

$$\int \Phi \, dI = \left(\int \Phi_\infty \, dm_\infty \right) \prod_p \left(\int \Phi_p \, dm_p \right).$$

To prove this, enlarge S_0 to include every exceptional test factor. On tuples integral outside that finite set, the test function is a finite product, and the cylinder identity identifies its pushforward measure with the finite product of the local measures multiplied by the convergent tail of the lattice masses. Local finiteness and compact support give integrability, and the finite-product integral formula gives the displayed identity.
Lean: `GlobalTheory.T.integral_pure_tensor_eq_prod_of_isEulerProd` (`CompanionTheory.SpectralGeometry.LocalGlobalProofs.AdelicAnalysis.EulerProductLocallyFiniteStatement`, lines 7–8)
Lean: `GlobalTheory.CompanionTheory.SW.EulerInt.euler_int` (`CompanionTheory.SpectralGeometry.LocalGlobalProofs.AdelicAnalysis.EulerProductLocallyFinite`, lines 114–269)

Claim 3 (normalized theta mean). For any set $\mathcal{X}$ of rational tuples, define $I_{\mathcal{X}}$ by the averaged-counting-measure formula of Claim 1 with $\mathcal{X}$ in place of $\mathcal{X}_T$, and put $\Theta_{\Phi,\mathcal{X}}(h) = \sum_{X \in \mathcal{X}} \Phi(h^{-1}X)$. If Φ is measurable and integrable for $I_{\mathcal{X}}$, then

$$\mu(\mathcal{F})^{-1} \int_{\mathcal{F}} \Theta_{\Phi,\mathcal{X}}(h) \, d\mu(h) = \int \Phi \, dI_{\mathcal{X}}.$$

This identity uses the same CM extension, nonsingular form, Haar measure, and fundamental domain without imposing an extra rank condition, and follows by integrating the sum of pushforward measures in the definition of $I_{\mathcal{X}}$.
Lean: `GlobalTheory.CompanionTheory.SW.thetaMeas` (`CompanionTheory.SpectralGeometry.CountingDefinitions.AdelicEulerProducts.IsEulerProd`, lines 84–87)
Lean: `GlobalTheory.T.siegel_mean_value_integrable` (`CompanionTheory.SpectralGeometry.LocalGlobalProofs.AdelicAnalysis.SiegelMeanValueIntegrableStatement`, lines 7–8)

Apply Claims 1–3 with $2n_0 < M_0$, which implies $n_0 = 0$ or $n_0 + 2 \le M_0$. For a continuous compactly supported local test function, its fibre integral $f_w(S) = \int \Phi_w \, dm_{w,S}$ is a continuous density on the nonsingular locus. Fixing the same admissible global additive character for all self-dual source and target measures, whose ratios c_w and e_w against the coordinate measures satisfy $a_w = c_w/e_w > 0$ and $a_\infty \prod_p a_p = 1$, Lemma 33.95 produces the normalized densities $g_w = a_w f_w$ satisfying the global theta-mean identity.
Lean: `GlobalTheory.CompanionTheory.SW.TopDens.siegelDensities` (`CompanionTheory.SpectralGeometry.LocalGlobalProofs.AdelicAnalysis.SiegelDensitiesOfHermitianForm`, lines 47–106)

Claim 4 (admissible mass data and frame passage). For the tied realization, the finite-volume and Siegel–Weil density assertions give the stabilizer-volume input, while the global-to-local measure identifications, local character compatibilities, local period ties, and nonsingular reference Gram tuple give convergent positive products for the ordinary-place volume units, central and prime-power local factors, ordinary-place covolumes, and Jacobian factors. If s_0, t_0 are the thresholds from the size estimates, enlarging to $C' \ge \max(s_0, 29)$ and $C_\ell \ge \max(t_0, 16)$ supplies the large-gap hypothesis of Proposition 33.24 at infinity. Combining unit-scale archimedean formal-degree convergence with the adelic volume, level, mass, column, and index relations produces positive constants $A_0, c_0, c_0', V_1, V_{ZZ} > 0$ and real integrals I_C, I_N (rescaled simultaneously by the positive unipotent scale factor $\kappa > 0$, which leaves $A_0, c_0, c_0', V_1, V_{ZZ}$ unchanged) satisfying $I_N = (V_{ZZ}/V_1) I_C$ and the unscaled mass-normalization bound $\Gamma_1 A_0 (c_0 c_0')^{-1} I_N \le \Delta_{u,v}$. With

$d_0 = \mathrm{rd}(E) > 0$ and $g = g_0 g_1 > 0$, the frame-volume ratio is

$$\frac{V_1}{V_{ZZ}} = d_0^{f(r+1)(8r^2+10r+3)/3}\, g^{f(r+1)(2r+3)},$$

and defining I_{rat} and I_{re} by $(c_0 c_0')^{-1} I_C = 2^{f(r+1)} I_{\mathrm{rat}}$ and $I_{\mathrm{rat}} = d_0^{2r(r+1)(k-1)f + fr(r+1)(8r+10)/3} I_{\mathrm{re}}$ gives the frame-passage identity

$$d_0^{2r(r+1)(k-1)f} I_{\mathrm{re}} = (d_0/2)^{f(r+1)} g^{f(r+1)(2r+3)} (c_0 c_0')^{-1} I_N,$$

since $(8r^2 + 10r + 3) - r(8r + 10) = 3$. Setting $A_{\mathrm{re}} = d_0^{2r(r+1)(k-1)f} A_0 > 0$ and multiplying the unscaled mass-normalization bound by $(d_0/2)^{f(r+1)} g^{f(r+1)(2r+3)}$ yields

$$\Gamma_1 A_{\mathrm{re}} I_{\mathrm{re}} \le (d_0/2)^{f(r+1)} g^{f(r+1)(2r+3)} \Delta_{u,v}.$$

Lean: `GlobalTheory.St.swxs_of_siegelWeil` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.SwxsOfSiegelWeil`, lines 8–10)
Lean: `Formalism.L2.SG.LocVol.ProdConv` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.LocVol`, lines 222–230)
Lean: `Formalism.L2.SG.Pf.ProdConvCut2.prodConv_onTied` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.ProdConvHoldsGen`, lines 23–26)
Lean: `Formalism.L2.SG.NGAnalytic.StepChoiceConstantsC` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.NGAnalytic`, lines 161–171)
Lean: `Formalism.L2.SG.Pf.choice_massData` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.MassDataChoiceLeaves`, lines 91–96)
Lean: `Formalism.L2.SG.Pf.stepChoiceDataC_of_leavesG` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.MassDataChoiceLeaves`, lines 57–89)
Lean: `Formalism.L2.SG.NGAnalytic.MassDataLemmas.massDataValid_bk_of` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.CanonicalAnalyticData`, lines 53–116)
Lean: `Formalism.L2.SG.NGAnalytic.massDataValid_unscale` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.NGAnalytic`, lines 62–67)
Lean: `Formalism.FourierDecayAlt.PN.NormInputs` (`Formalism.HermitianCurvature.StatementPN`, lines 16–18)
Lean: `Formalism.FourierDecayAlt.PN.FrameInputs` (`Formalism.HermitianCurvature.StatementPN`, lines 30–35)
Lean: `Formalism.FourierDecayAlt.PN.FramePassage` (`Formalism.HermitianCurvature.StatementPN`, lines 37–41)
Lean: `Formalism.FourierDecayAlt.PN.NormRe` (`Formalism.HermitianCurvature.StatementPN`, lines 43–48)
Lean: `Formalism.FourierDecayAlt.P.HermitianFormNormLemmas.A.t_NormOfLemmas` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.UnipotentBlockMeasure`, lines 51–85)
Lean: `Formalism.FourierDecayAlt.P.HermitianFormNormLemmas.A.t_FramePassage` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.UnipotentBlockMeasure`, lines 87–108)
Lean: `Formalism.FourierDecayAlt.P.HermitianFormNormLemmas.A.t_NormRe` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.UnipotentBlockMeasure`, lines 110–119)

Claim 5 (local and global generic mass bounds). Since $k \ge 4r + 7$ and $k \ge m + f + 5$ above the rank threshold $C' \ge 2$, the local comparison on Hermitian frames gives $C_0' = 9$. In frame-pair coordinates at p the total and generic masses become orbit sums over the same representatives with one common orbit constant. There is $\kappa_p > 0$ such that the mass of every integral overlattice of the sublattice attached to a frame class lies between $\kappa_p/2$ and $2\kappa_p$ times the inverse $(k-1)$-st power of its index. Each frame class has at most $H_p(\gamma)^{2m}$ such overlattices, so its term mass is at most $2\kappa_p H_p(\gamma)^{2m}$, and there are at most $H_p(\gamma)^{r(r+1)}$ frame classes. The generic mass, the term mass with $\beta_p = 1$ in the class of a fixed term, is at least $\kappa_p/2$. Hence $0 < n_p^{\mathrm{gen}} < \infty$ and $n_p \le 4H_p(\gamma)^{r(r+1)+2m} n_p^{\mathrm{gen}} \le 8H_p(\gamma)^{9(r+1)^2+2m} n_p^{\mathrm{gen}}$ for $p \in \Omega$. For $p \notin \Omega$ the local height is one, every term has $\beta_p = 1$, a single frame class occurs, and $n_p = n_p^{\mathrm{gen}}$. The matched term set is closed under local orbits at every place. The matching supplies a term over the base datum at each finite place. Outside the finite family these terms can be chosen with lattices obtained by localizing one global lattice. Hence, for a local tuple $\mathbf{d}$ attained by a class and a normal label, extended by the base datum at the places outside the finite family, the class weights of the pairs lying over $\mathbf{d}$ sum to the mass of the genus with local data $\mathbf{d}$. The genus mass formula for the fixed base frames gives one $V_0 > 0$ such that this mass equals $V_0 \prod_p w_p(d_p)$ for every such $\mathbf{d}$, where $w_p(d_p)$ is the local weight of d_p relative to the base datum. This gives the mass law on each full attained fibre. Restricting the sum of nonnegative weights to a finite subset gives the bound in Lemma 33.64.

Lean: `GlobalTheory.Unfold.Tied.unfold_tied` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.UnfoldingIdentityMatchedFamilyClosedTerms`, lines 119–182)

Unfolding $n_{\mathrm{gen}}(\gamma)$ over definite rank-$(k-1)$ complement lattices glued along $W_t^{\perp}$ bounds it by $H(\gamma)^{3(r+1)^2}$ times the weighted count of glued lattices. That count is at most $e_1 = 2$ times the product of the explicit count constant, the row mass and the local factors, times the $(k-1)$-st power of the square root of the absolute norm of the scaled Gram matrix. The stabilizer-volume identity of Claim 4 identifies this product of constants with A_0 times a frame normalizing factor, and that factor times the Gram power equals $d_0^{2r(r+1)(k-1)f}\widehat{N}_G(\gamma)^{k-1}$, where $\widehat{N}_G(\gamma) = d_0^{-2r(r+1)f}N_G(\gamma)$. Since $A_{\mathrm{re}} = d_0^{2r(r+1)(k-1)f}A_0$, this gives $0 < n_{\mathrm{gen}}(\gamma) \le A_{\mathrm{re}}e_1H(\gamma)^{3(r+1)^2}\widehat{N}_G(\gamma)^{k-1}$. Finally, telescoping the normalized block-end minor products (each at least 1; their product $\widehat{N}_D(\gamma) = \prod_\tau \prod_j d_0^{-2j(2r+1-j)}|\tau(D_j(\gamma))|^2$ over the block ends j satisfies $\widehat{N}_D(\gamma) \le H(\gamma)$, because $H(\gamma) = g^{-(2r+1)f}\widehat{N}_G(\gamma)\widehat{N}_D(\gamma)$ and $\widehat{N}_G(\gamma) \ge g^{(2r+2)f}$) in the block formula for the real Haar point weight $p(\gamma)$ of Lemma 33.91 gives $\widehat{N}_G(\gamma)^{k-1} \le H(\gamma)^{6m}p(\gamma)$; since $H(\gamma) \ge 1$ and $3(r+1)^2 + 6m \le 10(r+1)^2 + 6m + 1 = e_N$, this yields (33.29).

Lean: `GlobalTheory.T.rowWeights_localOrbits_sum_eq_genusMass` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.RowWeightsLocalOrbitsSumEqGenusMassStatement`, lines 10–11)
Lean: `GlobalTheory.T.mass_genus_local_data_product_local_weights` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.MassGenusLocalDataProductLocalWeightsStatement`, lines 10–11)
Lean: `GlobalTheory.localMass_le_genericMass_nine` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.LocalMassLeGenericMassConstNine`, lines 7–9)
Lean: `GlobalTheory.GenMassBound.genericMass_le_heightPow_mul_glueSum` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.GenericMassLeHeightPowMulGluedCount`, lines 109–239)
Lean: `GlobalTheory.GlueCount.glueSum_le` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.GluedLatticeWeightedCountLeExplicit`, lines 75–144)
Lean: `GlobalTheory.CountConst.countConst_eq_unscaledConst` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.MeanCountConstantEqUnscaledMainConstant`, lines 39–85)
Lean: `GlobalTheory.HaarWeight.gram_pow_le_haar` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.GramInvariantPowLeHaarWeight`, lines 151–205)
Lean: `GlobalTheory.WeightsFamPf.weightsFam_of` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.FamilyGenericMassPositiveBoundedProof`, lines 21–144)

The complete proof is contained in `GlobalTheory.St.stepChoiceDataC_of_siegelWeil_inInfinity`, `GlobalTheory.localMass_le_genericMass_nine`, and `GlobalTheory.T.familyGenericMassPositiveBounded_proof`. □

Proof of 33.97

Proof. Both $P(\gamma)$ and $Q(\gamma)$ are positive, so $\vartheta > 0$. The factors ϑ and ϑ^{-1} cancel in the disintegration identity, which proves the quotient-measure assertion. Linearity of the orbital integral in its measure gives

$$\mathcal{A}_\tau(\nu_{\tau,\vartheta}) = \vartheta^{-1}\mathcal{A}_\tau(\nu_{\tau,1}), \qquad \|\mathcal{A}_\tau(\nu_{\tau,\vartheta})\|_{\mathrm{HS}}^2 = \vartheta^{-2}\|\mathcal{A}_\tau(\nu_{\tau,1})\|_{\mathrm{HS}}^2.$$

Since $\vartheta^{2f} = P(\gamma)/Q(\gamma)$, multiplying over τ gives (33.30). Multiplication by $a_2(c,\gamma,\nu)$ proves the last identity. □

Proof of 33.98

Proof (outline). Claim 1. The uniform constants can be chosen in the following order. Fix the six standing constants. The uniform restriction theorem first chooses C_1, C_n and thresholds c_0, d_0. For every tied realization satisfying the base and pair standing conditions, every choice of auxiliary data for the mass data constructed in the proof of Lemma 33.96, with an admissible tuple formed from them, and every pinned quotient measure, it supplies $I_{\mathrm{re}} \ge 0$, a positive base scale θ in

$$b^{-C_1m^2} \le \theta \le b^{C_1m^2},$$

local frame and Bessel data, and the section-product bound

$$\prod_\tau M_\tau \le b^{C_nm^2f}I_{\mathrm{re}}.$$

The same choice gives positive unit-root local amplitude product $P(\gamma)$ whenever $Q(\gamma) > 0$, and places $\vartheta = (P(\gamma)/Q(\gamma))^{1/(2f)}$ in this same interval. To construct the Bessel data from the local row functions,

enlarge the restriction thresholds to $\max(c_0, 29), \max(d_0, 16)$, apply Proposition 33.24 to the transported discrete series and compact types, and choose their formal-degree normalizations. The two mass identities and the quotient-volume formula express the section loss as a power of $\mathrm{rd}(E) \le b$, giving the factor $b^{C_n m^2 f}$. The uniform restriction estimate supplies a positive normalization factor K with $b^{-C_1 m^2} \le K^{1/(2f)} \le b^{C_1 m^2}$. Quotient-measure covariance identifies this factor in the pointwise identity $P(\gamma) = KQ(\gamma)$ when $Q(\gamma) > 0$.
Lean: `Formalism.L2.SG.Pf.ThmBChain6.AmplitudeBoundsAllChoicesUniform` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.AmplitudeBoundsAllChoicesUniform`, lines 36–47)
Lean: `Formalism.MainStatements.RowRest.uniform_restriction_bound` (`CompanionTheory.AutomorphicTheory.UniformLocalFormulas.Closed`, lines 18–18)
Lean: `Formalism.MainStatements.RowRest.rowDataUnifN_of_6` (`CompanionTheory.AutomorphicTheory.UniformLocalFormulas.StepThmBPinnedCOfN`, lines 124–143)
Lean: `Formalism.MainStatements.RowRest.goodS_deg` (`CompanionTheory.AutomorphicTheory.UniformLocalFormulas.StepThmBPinnedCOfN`, lines 26–57)

After fixing C_1, the uniform local Bessel comparison chooses $C_a \ge 0$ and thresholds a_0, b_0. These depend on the six standing constants and on this C_1. The point-coordinate identifications for the pinned nongeneric and equal-word generic data are both uniform with thresholds $0, 0$.
Lean: `Formalism.MainNotions.S.Pv.LemmaJ.AmpUnif` (`Formalism.SharpRestriction.CuspSpanNormBounds.FrobeniusCoefficientMemLp.AmpUnif`, lines 12–14)
Lean: `CompanionTheory.Lime.BM.ampUnif_holds` (`CompanionTheory.AutomorphicTheory.LocalBesselUniqueness.BesselModels.MultiplicityOneUnitaryBranchingOfGapHyp`, lines 10–10)
Lean: `Formalism.L2.SG.PointUniformity.ptUnif'` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.PointUniformity`, lines 131–136)
Lean: `Formalism.L2.SG.PointUniformity.ptUnifG` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.PointUniformity`, lines 138–143)

Claim 2. One choice of local constants gives the two comparisons. For the local frames and normalized discrete-series data selected in Claim 1, the amplitude construction chooses compact-finite coefficient functions f_τ and finite positive real numbers K_τ. The construction uses the base standing conditions, the pair-prime and congruence conditions, positivity of the two frame norms, and the base-scale window for θ. Its conclusions are simultaneous: for every compatible big-cell section s,

$$K_\tau I_\tau(s) \le M_\tau(s),$$

and for every group element γ_τ representing a block point with nonzero Gram factor, every $b^{-C_1 m^2} \le \theta' \le b^{C_1 m^2}$, and every measure ν_R disintegrating the chosen group Haar measure along $\theta' \mu_R$, with μ_R the root-coordinate stabilizer measure,

$$\|\mathcal{A}_\tau(\nu_R)\|_{\mathrm{HS}}^2 \le K_\tau b^{C_a m^2} \sup_{\text{compatible representatives and quotients}} \|J(f_\tau)\|_{\mathrm{HS}}^2.$$

Here $I_\tau(s)$ is the Bessel section integral of f_τ, and $M_\tau(s)$ is the row-function section integral. The numbers K_τ are chosen once for the local data, before the section, the represented point, θ', and ν_R. Thus the same constants occur in both inequalities.
Lean: `Formalism.L2.SG.Pf.BDat.Inputs` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.BDataOf`, lines 90–98)
Lean: `Formalism.L2.SG.Pf.BDat.AmpIn` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.BDataOf`, lines 167–177)
Lean: `Formalism.MainStatements.RowRest.ties_coreN` (`CompanionTheory.AutomorphicTheory.UniformLocalFormulas.StepThmBPinnedCOfN`, lines 59–100)

Claim 3. The orbital comparison holds at every represented point, and the subtotal comparison holds on $Q > 0$. Suppose now $Q(\gamma) > 0$. Claim 1 supplies $P(\gamma) > 0$ and the required window for ϑ. Lemma 33.97 gives the factorization with the selected measures $\vartheta^{-1}\nu_{\tau,1}$. Claim 2, applied with $\theta' = \vartheta$, gives

$$\|\mathcal{A}_\tau(\nu_{\tau,\vartheta})\|_{\mathrm{HS}}^2 \le K_\tau b^{C_a m^2} \|J_\tau(\gamma)\|_{\mathrm{HS}}^2.$$

Multiplication and Lemma 33.97 give

$$Q(\gamma) \le C_J \prod_\tau \|J_\tau(\gamma)\|_{\mathrm{HS}}^2.$$

Since the same pinned measure defines $a_2 = \|V\|^2/Q$, multiplication by $a_2 \ge 0$ gives the subtotal estimate on $Q > 0$. If $Q = 0$, the displayed analytic inequality follows from the nonnegativity of its right-hand side.

Lean: Formalism.L2.SG.Pf.BDatR.jointAmplitude_bound_rotated (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.BDataOfR, lines 46–78)

Claim 4. The section comparisons give the stated global bound with the following uniform thresholds. For every compatible family of big-cell sections, the two section bounds are $(\prod_\tau K_\tau)\prod_\tau I_\tau \le \prod_\tau M_\tau$ and $\prod_\tau M_\tau \le b^{C_n m^2 f} I_{\mathrm{re}}$. The constants K_τ therefore cancel the same factors in C_J. More precisely, if C_b^0 is the constant from Lemma 33.91 at scale b^{C_β}, with $C_\beta = \max(2, C_w)$ and C_w the standing coordinate-size constant, the remaining factor is bounded by

$$\exp\big((\max(0, C_b^0)C_\beta + |C_a| + |C_n|)m^5 f \log b\big)\, I_{\mathrm{re}}.$$

This follows by multiplying the factors $b^{C_a m^2 f}$, $b^{C_b^0 C_\beta m^5 f}$, and $b^{C_n m^2 f}$, and using $m^2 \le m^5$. For a real $A_w \ge 0$, take $A = \lceil A_w \rceil$. The integer exponent $A(r+1)^2$ gives the required bound because $W(\gamma) \ge 1$. More explicitly, put $C_h = |C_T| + 1/8 + 2|C_g|$, and let C_b^0, C_g^0 be the local-bound constants at weight A and size parameter C_h. After including all earlier thresholds, one may take

$$C_b = \max(0, C_b^0)\max(2, C_w) + |C_a| + |C_n|,$$

$$C' \ge \max(a_0, \max(c_0, 0)), \qquad C_\ell \ge \max\big(8, \max(128\max(C_g^0, 0), \max(b_0, \max(d_0, 0)))\big),$$

where c_0, d_0 now include the enlargements in Claim 1. In each case the final point-sum conclusion applies to its admissible mass data, the pinned measure used to define Q, and arbitrary auxiliary quotient data. Its bound is $\exp(C_b m^5 f \log b) I_{\mathrm{re}}$: one conclusion is for the pinned nongeneric point sum of (u, v), the other for the generic point sum of (u, u).

Lean: Formalism.L2.SG.NGAnalytic.StepThmBPinnedC (Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.NGAnalytic, lines 317–330)

Lean: Formalism.L2.SG.Pf.BDat.StepThmBG (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.BDataOf, lines 641–654)

Lean: Formalism.MainStatements.RowRest.stepThmBPinnedC_ofN (CompanionTheory.AutomorphicTheory.UniformLocalFormulas.StepThmBPinnedCOfN, lines 145–201)

Lean: Formalism.MainStatements.RowRest.stepThmBG_ofN (CompanionTheory.AutomorphicTheory.UniformLocalFormulas.StepThmBPinnedCOfN, lines 203–259)

The complete proof is contained in `Formalism.MainStatements.RowRest.analytic_pinnedTheoremB_uniform` and `Formalism.MainStatements.RowRest.bDat_theoremB_global`. □

Proof of 33.99

Proof (outline). Put $r = r_S$ and $\kappa = k - 2r - 5$, with $k \ge 4r + 7$ from the rank thresholds. Use the partner classes c, block points γ, normal labels ν, subtotals $V(c, \gamma, \nu)$ and local defect data of Definition 33.94. *Claim 1.* The signed sums have the following class-square identities. By Lemma 33.64 and absolute summability,

$$P(U_{\mathrm{ng}}) = \Gamma_1 \sum_c \left\| \sum_{\gamma,\nu} V(c, \gamma, \nu) \right\|^2, \tag{A.340}$$

and the corresponding class-square identity holds for $P(U_{\mathrm{sc}})$ over the equal-word generic scalar-partner classes.

Claim 2. Under $k \ge 4r+7$, the positive-defect local amplitude satisfies the first bound in (A.341), and its zero-defect counterpart satisfies the second. At a finite place p with height $H_p = H_p(\gamma)$, residue cardinality q_p and unsigned reference mass n_p, the local defect sums of Definition 33.94 satisfy

$$\sum_{d \in D_p} w_d |a_N(d)|^2 \le H_p^{39(r+1)^2} q_p^{-\kappa} n_p, \qquad \sum_{d \in D_p} w_d |a_Z(d)|^2 \le H_p^{39(r+1)^2} n_p. \tag{A.341}$$

For the first bound set $A_0 = H_p^{9(r+1)^2 + 12(r+1)}$ and $A_1 = 4H_p^{4(r+1)}$. At defect $\beta = q_p^i > 1$ the total mass is at most $A_1 \beta^{-(k-1)} n_p$, while image cardinality q_p^l at fixed data costs at most $A_0 q_p^{l(k-1)}$; summing the proper-image geometric progression gives at most $2A_0(\beta/q_p)^{k-1}$ and saves $2A_0 A_1 q_p^{-(k-1)}$.

In a full-image family at $\beta = q_p^i > 1$, fix the last occupied layer J, its coset count β_J, and d, so the image mass is at most $2A_0\beta^{k-1}$. Write Λ for a starting lattice in the family, X_J, X_J' for the two flag prefix

spans, $Y = X_J + X'_J$, $\theta = \theta_p$, ϖ for a uniformizer of F_p, $a \geq 0$ for the norm-congruence exponent, and h for the local Hermitian pairing linear in its first variable. For $b \in X_J$, $b' \in X'_J$ with $b + b' \in \theta\Lambda$ of exact level J modulo the frame lattice, $v \in U = \Lambda \cap Y^{\perp}$ with $h(v,v) \in \varpi^a O$, and trace-zero $t_0 \in E_p$ such that $c_0 = h(v,v)/2 + t_0 \in \varpi^a O_E$, the map $T_{b,v,c_0}(z) = z + h(z,v)b - h(z,b)(v + c_0 b)$ is a Hermitian isometry with inverse $T_{b,-v,\bar{c}_0}$. Giving probability measure to the full parameter domain and multiplying by $q_p^a \beta/\beta_J$ on targets $\Lambda' = T_{b,v,c_0}^{-1}(\Lambda)$ remaining in the family defines a symmetric stabilizer-invariant kernel $m(x,y)$ satisfying

$$\left|1 - \sum_y m(x,y)\right| \leq a_p, \qquad \left|\sum_y m(x,y)\omega_d(y)\right| \leq a_p, \qquad a_p = \tfrac{24}{7} q_p^{1-(k-2r-3)},$$

for the orbital phase ω_d, defined by $\omega_d(\Lambda') = \psi(n)\overline{\psi'(n')}$ for a term Λ' over d with witnesses n, n' in the unipotent radicals of the fixed and moving frames (f_j) and (f'_j). They carry the two standard lattices of d to Λ' and satisfy $n' f'_j - n f_j \in \theta\Lambda'$ for every index j. The congruence graph of Λ' is the graph prescribed by d. The two frame Bessel characters ψ, ψ' give a phase of modulus one independent of the witnesses.
Lean: `GlobalTheory.HL.Frames.IsTermW` (`Development.AbelianVarietyTheory.Definitions.SemilocalFrames.ComplementReferenceCompactSubgroup`, lines 121–124)

A move multiplies the phase by a character of the move vector: $\omega_d(\Lambda') = \omega_d(\Lambda)\chi_{b,b'}(v)$ for $\Lambda' = T_{b,v,c_0}^{-1}(\Lambda)$, where $\chi_{b,b'}(v) = \psi_0(-\theta^{-1}h(y_{b,b'}, v))$ for the fixed additive character ψ_0 and the vector $y_{b,b'} = \alpha_J e_{J+1} + \alpha'_J e'_{J+1}$, where $b = \sum_i \alpha_i e_i$ and $b' = \sum_i \alpha'_i e'_i$ in the two frames transported by the witnesses at Λ, indexed from 1. If $\chi_{b,b'}$ were trivial on U, testing all integral multiples of each $u \in U$ and using lattice duality would give $\theta^{-1} y_{b,b'} = v_\Lambda + y$ with $v_\Lambda \in \Lambda$ and $y \in Y$. Then $v_\Lambda \in F_u + F'_t$. The full-image level bound on v_Λ forces the last coefficient pair (α_J, α'_J) to satisfy the frame congruence excluded by exact level J. Thus $\chi_{b,b'}$ is nontrivial. For the norm average, $W = \Lambda \cap (F_u + F'_t)^{\perp} \subseteq U$ is saturated of codimension at most $2r+2$ in rank $k+2r+1$, so its modulo-ϖ reduction has radical of dimension at most $2r+2$ and retains a nonsingular subspace of dimension at least $\rho = k - 2r - 3 \geq 4$, which lifts by Hensel's lemma and surjective unramified unit norms (or dual split bases) to ρ orthonormal vectors in W. Moves satisfying $h(v, W) = O_E$ remain in the family, the (b,b')-constraint has relative measure β_J/β, the t_0-coset has measure 1, and Lemma 33.92 bounds the absolute error in the q_p^a-normalized average over the norm congruence by $(8/7)q_p^{1-\rho}$, with centre 1 for the trivial character and 0 for $\chi_{b,b'}$. The q_p^a-normalized mass of vectors satisfying both $h(v,v) \in \varpi^a O$ and $h(v,W) \neq O_E$ is at most $(16/7)q_p^{1-\rho}$. Summing against the relative inverse stabilizer volumes $\mu(x)$ over stabilizer orbits gives $\sum_{[x]} \mu(x) \sum_y m(x,y)\omega_d(y) = \sum_{[x]} \mu(x)\omega_d(x) \sum_y m(x,y)$, whence $|\sum_{[x]} \mu(x)\omega_d(x)| \leq 2a_p \sum_{[x]} \mu(x) \leq 7 q_p^{1-(k-2r-3)} \sum_{[x]} \mu(x)$. The squared phase bound and the two mass bounds give at most $98 A_0 A_1 q_p^{2(1-\rho)} n_p$ per full-image family. Both this and the proper-image bound are at most $98 A_0 A_1 q_p^{-\kappa} n_p$. There are at most H_p^{r+2} families. Cauchy–Schwarz bounds the squared total amplitude by the number of families times the sum of their squared amplitudes. If the positive-defect term set is empty, the left-hand side of the first inequality in (A.341) is zero. Otherwise a positive-defect term gives $H_p \geq q_p \geq 2$, and the combined factor satisfies $98 A_0 A_1 H_p^{2(r+2)} \leq H_p^{39(r+1)^2}$, giving that inequality.
Lean: `GlobalTheory.HL.TypesCS.ampSq_le_of_types` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.AmpSqLeOfHermitianTermTypesCauchySchwarz`, lines 27–134)
Lean: `GlobalTheory.T.hermitianDoubleCoset_move_term_weight_character` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermitianDoubleCosetMoveTermWeightCharacterStatement`, lines 9–10)
Lean: `GlobalTheory.T.frameMove_character_ne_one` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.FrameMoveCharacterNeOneStatement`, lines 9–10)
Lean: `GlobalTheory.T.hermitianDoubleCoset_kernel_bound_by_weight` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermitianDoubleCosetKernelBoundByWeightStatement`, lines 11–12)

Claim 3. For the local frame family of Definition 33.94, if $k \geq 4r+7$, the orbital sums over presentations of defect $\beta_p = 1$ satisfy

$$\sum_{d \in D_p} w_d |a_Z(d)|^2 \leq H_p^{39(r+1)^2} n_p.$$

Indeed, when $\beta_p = 1$ every admissible term (d, Λ) in the local family has trivial image quotient cardinality $\mathrm{card}_I(d, \Lambda) = 1$; applying the local Hermitian double-coset mass bound at image cardinality 1 and summing over $d \in D_p$ yields the displayed bound directly without requiring phase cancellation, establishing the second inequality of (A.341).
Lean: `GlobalTheory.HL.NoCancellation.ampSq_le_height_mul_mass` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.AmpSqLeHeightMulMass`, lines 112–159)

Lean: GlobalTheory.HL.exists_symm_invariant_kernel (Development.AbelianVarietyTheory.Proofs.FlagMeasures.ExistsSymmInvariantKernelHermitianFrames, lines 17–31)
Lean: GlobalTheory.HL.FirstType.first_type_bound (Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermitianFramesFirstTypeAmpSqBound, lines 44–111)
Lean: GlobalTheory.HL.AmpCancel.amp_le_mass (Development.AbelianVarietyTheory.Proofs.FlagMeasures.AmplitudeLeMassSecondTypeFamily, lines 31–104)
Lean: GlobalTheory.HL.FamilyBound.family_bound (Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermitianDoubleCosetSecondTypeFamilyAmpSqLe, lines 59–124)
Lean: GlobalTheory.HL.NonzeroDefect.ampSq_nonzero_defect (Development.AbelianVarietyTheory.Proofs.FlagMeasures.NonzeroDefectHeightPowerAbsorbsConstants, lines 49–80)
Lean: GlobalTheory.HL.NoCancellation.ampSq_le_height_mul_mass (Development.AbelianVarietyTheory.Proofs.FlagMeasures.AmpSqLeHeightMulMass, lines 112–159)

Claim 4. Under the size, split-prime, character and measure ties of this lemma, choose compatible admissible mass data from the assumed finite-volume and Gram-density formula. For every represented block point and every matched local family of the partner classes whose local frame pairs are relevant and admit integral terms at each place in the finite family, the orbit mass bound of Lemma 33.64 is an equality on each full attained fibre. There is $V_0 > 0$ such that, for every local tuple attained by a class and a normal label, the class weights ρ_c of the pairs attaining it sum to $V_0 \prod_p w_p$. The matching supplies a term over the base datum at each finite place, with lattices obtained from one global lattice outside the finite family. Since the term set is closed under local orbits at every place, the class weights of the pairs lying over a fixed attained local tuple, extended by the base datum outside the finite family, sum to the mass of the genus with those local data; the genus mass formula for the fixed base frames writes this mass as $V_0 \prod_p w_p$ with one $V_0 > 0$ for all such tuples. For the selected quotient measures the same family factors the normalized squared amplitude as $\rho_c |\prod_p a_p|^2$ on the attained pairs and as zero on the others, which is the factorization hypothesis of that lemma.
Lean: GlobalTheory.SnFam.familyRowWeightSumUnfoldingIdentity_proof (Development.AbelianVarietyTheory.Proofs.FlagMeasures.FamilyRowWeightSumUnfoldingIdentityProof, lines 46–53)
Lean: GlobalTheory.T.unfolding_identity_matched_family_closed_terms (Development.AbelianVarietyTheory.Proofs.FlagMeasures.UnfoldingIdentityMatchedFamilyClosedTermsStatement, lines 10–11)
Lean: GlobalTheory.T.familyGroupSumFactorsRowWeightAmplitude_proof (Development.AbelianVarietyTheory.Proofs.FlagMeasures.FamilyGroupSumFactorsRowWeightAmplitudeProofStatement, lines 5–6)

Its local and global generic masses satisfy

$$\begin{aligned} n_p &\le 8 H_p^{9(r+1)^2+2m} n_p^{\mathrm{gen}} && (p \in \Omega),\\ n_p &= n_p^{\mathrm{gen}} && (p \notin \Omega),\\ 0 < n_{\mathrm{gen}}(\gamma) &\le 2 A_{\mathrm{re}} H(\gamma)^{e_N} p(\gamma), && e_N = 10(r+1)^2 + 6m + 1. \end{aligned}$$

Let $A_{\mathrm{mass}} > 0$ denote the unscaled mass constant of these same admissible mass data. The mass-to-section normalization in the displayed bound is precisely

$$A_{\mathrm{re}} = (2v_E)^{2r(r+1)(k-1)f} A_{\mathrm{mass}} = \rho^{2r(r+1)(k-1)f} A_{\mathrm{mass}}, \qquad \rho = \mathrm{rd}(E) = 2v_E.$$

Thus the archimedean discriminant factor is included in A_{re}. The local estimate compares the total self-dual-lattice mass with the generic mass in the represented frame class; when the local height is one, every term is generic and the masses agree.
Lean: Formalism.FourierDecayAlt.PN.abelian₂ (Formalism.HermitianCurvature.StatementPN, lines 28–28)
Lean: Formalism.BasicNotions.Host.v (Formalism.LocalRingCardinality.DataHost, lines 143–143)
Lean: GlobalTheory.HL.GenMass.total_le_generic (Development.AbelianVarietyTheory.Proofs.FlagMeasures.TotalLeGeneric, lines 97–169)
Lean: Formalism.L2.SG.FamDefs.famOKS_of (Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.FamDefs, lines 261–322)

For the global bound, put $\widehat{N}_G(\gamma) = \rho^{-2r(r+1)f} N_G(\gamma)$. Indeed, the normalized Gram factor is the product of $\rho^{-2r(r+1)} |G(\gamma)|_\tau$ over the f infinite places. Its product formula gives this equality with the unnormalized N_G of Definition 33.94. The block formula for the normalized point weight agrees with $p(\gamma)$ of Lemma 33.91. Multiply the Gram factor to the power $k-1$ into the powers of the block entries, using the separate final-singleton factor when present.
Lean: Formalism.L2.SG.NGCut.NormalInvariants.ngPt_eq (Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.NormalInvariants, lines 71–87)
Lean: Formalism.L2.SG.RowsConc.nPt (Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RowsConc, lines 371–376)

The matching orbit mass law, the weighted count of glued unimodular lattices, and the convergent mass formula give

$$n_{\mathrm{gen}}(\gamma) \le 2A_{\mathrm{re}} H(\gamma)^{3(r+1)^2} \widehat{N}_G(\gamma)^{k-1}.$$

The discriminant-normalized block-end minor norms are at least one, and their product $\widehat{N}_D(\gamma)$ is at most $H(\gamma)$, since $H(\gamma) = g^{-(2r+1)f} \widehat{N}_G(\gamma)\widehat{N}_D(\gamma)$ and $\widehat{N}_G(\gamma) \ge g^{(2r+2)f}$. Telescoping their partial products in the block formula for the point weight gives

$$\widehat{N}_G(\gamma)^{k-1} \le H(\gamma)^{6m} p(\gamma).$$

Since $H(\gamma) \ge 1$ and $3(r+1)^2 + 6m \le e_N$, these two inequalities give the global assertion. The rank condition for the glued-lattice count is $m + f + 5 \le k$, supplied by the size hypotheses after the uniform threshold choice. All constants are selected before the individual point.

Lean: `GlobalTheory.WeightsFamPf.weightsFam_of` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.FamilyGenericMassPositiveBoundedProof`, lines 21–144)
Lean: `GlobalTheory.T.meanCountConstant_eq_unscaledMainConstant` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.MeanCountConstantEqUnscaledMainConstantStatement`, lines 11–12)
Lean: `GlobalTheory.HaarWeight.gram_pow_le_haar` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.GramInvariantPowLeHaarWeight`, lines 151–205)

Set $C_0' = 9$ and $e_1 = 2$. *Claim 5.* The finite-place squared amplitude sum has the bound (A.342) below. Expanding the nongeneric finite-place amplitude over nonempty subsets of Ω (with a_N on the subset and a_Z outside), applying Cauchy–Schwarz at cost $4^{|\Omega|}$, and inserting (A.341) together with the local mass comparison $n_p \le 8H_p^{C_0'(r+1)^2+2m} n_p^{\mathrm{gen}}$ of Claim 4 (with $4 \cdot 8 \le H_p^5$ for $p \in \Omega$) bounds the finite-place squared amplitude sum at γ by $n_{\mathrm{gen}}(\gamma)$ times

$$2^{-\kappa} H(\gamma)^{e_7}, \qquad e_7 = (39 + C_0')(r+1)^2 + 2m + 5, \tag{A.342}$$

while for equal-word generic scalar classes Lemma 33.64 gives $H(\gamma)^{e_7} n_{\mathrm{gen}}(\gamma)$ with saving factor 1.

Lean: `Formalism.FourierDecayAlt.P.HermitianFormNormLemmas.B.t_MeanAmpJoint` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.UnipotentBlockMeasure`, lines 326–367)

Claim 6. The point sum and mass normalization give the two stated global estimates. At each (c, γ), Lemma 33.93 applied to the two norm-one coordinates of ν bounds the number of normal labels by $P_c N_G(\gamma)^6$, where the class factor P_c is absorbed into the class mass weight with global factor $(T_u T_v)^{6f} d_0^{12r(r+1)f}$ ($d_0 = \mathrm{rd}(E)$). Apply Cauchy–Schwarz to the inner sum over (γ, ν) in (A.340) with weights $w(\gamma) P_c N_G(\gamma)^6$; by (A.339) the reciprocal sum is at most $64^{f(r+1)}$. Choose $A_w = 39 + C_0' + 38$, so $A_w(r+1)^2 \ge e_7 + e_N + 6$ and $H(\gamma)^{e_7+e_N} N_G(\gamma)^6 \le g^{fA_w(r+1)^2} W(\gamma)^{A_w(r+1)^2}$. Let $c_{\mathrm{st}} > 0$ be the local Jacobian-measure comparison constant from the measure input, satisfying $J_{2,\mathrm{loc}}(\gamma) \le c_{\mathrm{st}}\, J_{\mathrm{bnd}}(\gamma)$ for the local squared Jacobian bound and bounded uniformly by $c_{\mathrm{st}} \le b^{C_{\mathrm{st}} m^3 f}$. Combining (A.342), the global bound of Claim 4 and Proposition 33.98 bounds $P(U_{\mathrm{ng}})$ and $P(U_{\mathrm{sc}})$ by $2^{-\kappa}\Delta_{u,v}$ and $\Delta_{u,u}$, respectively, times

$$64^{f(r+1)} c_{\mathrm{st}} e_1 d_0^{12r(r+1)f} T_*^{6f} g^{fA_w(r+1)^2} b^{C_b m^5 f} (d_0/2)^{f(r+1)} g^{f(r+1)(2r+3)}, \tag{A.343}$$

where $T_* = T_u T_v$. Since $e_1 = 2$, $d_0 \le b$, $g \le b^{2C_g}$, $T_* \le b^{2C_T m}$, $c_{\mathrm{st}} \le b^{C_{\mathrm{st}} m^3 f}$, and $b \ge 16$, the product (A.343) is bounded by $e^{C_4 m^5 f \log b}$ in the first case and $e^{C_B m^5 f \log b}$ in the second.

Lean: `GlobalTheory.rowNGP_of_siegelWeil_of_stepChoice` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.RowPositivityOfSiegelWeilConstants`, lines 11–18)
Lean: `GlobalTheory.rowMainEqP_of_siegelWeil_of_stepChoice` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.RowMainEqPOfSiegelWeilConstants`, lines 14–22)
Lean: `Formalism.FourierDecayAlt.P.HermitianFormNormLemmas.F.t_RowNGJoint` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.UnipotentBlockMeasure`, lines 1032–1074)
Lean: `Formalism.L2.SG.Pf.MainEqCut.rowGenJoint_of` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.MainEqCut`, lines 259–298)

The complete proof is contained in `GlobalTheory.rowNGP_of_siegelWeil` and `GlobalTheory.rowMainEqP_of_siegelWeil`. □

Proof of 33.101

Proof. Put $z_{iv} = \min(1, s_{iv})$. For $s > 0$ one has $\min(1,s)/s \le 1$ and $\sqrt{1+s^2}\,\min(1,s)/s \le \sqrt{2}$, by considering $s \le 1$ and $s \ge 1$. Therefore

$$z_{1v} z_{2v} \frac{c_{1v} + c_{2v}}{s_{1v} s_{2v}} \le 2\sqrt{2} \le 4.$$

Multiplication over v gives $(\prod_v z_{1v})(\prod_v z_{2v}) \le (4g/\sqrt{TT'})^f$. The smaller of the two products has square at most this bound. Multiply by T'^f and use $T' z_{iv}^2 = \min(T' s_{iv}^2, T')$. For a single side, $\min(1,s)^2 \le s$, so the hypothesized product of $1/s_v$ gives the conclusion even with 4 replaced by 1.

Write $q = (C/k)^{2r(1-\eta)}$. From $k^{-1} \le C/k \le k^{-2/3}$ and $3r/2 \le 2r(1-\eta) \le 2r$ we obtain $k^{-2r} \le q \le k^{-r}$. In particular $T' a_v g^{-m} q \ge k^{3r} k^{-2r} \ge 1$. Consequently

$$\min(1, \nu_v a_v g^{-m} q) \le \min(T', \nu_v)\, a_v g^{-m} q.$$

Take the product, use the first conclusion and raise to the power k. The product bound on a_v^k gives the displayed estimate. Both terms belong to the ordered word pair (u, u'), so $T = T_u$, $T' = T_{u'}$ in either application. □

Proof of 33.103

Proof (outline). Claim 1. For the tied row data and standing pair hypotheses of this proposition, the finite-fibre counts have the two bounds specified below. Group each row by the reduced Gram labels Z, Z' and the rank-s exceptional lattice. Counting the lattice fibre with inverse stabilizer weights splits orthogonally into ranks s and $k-s$ with mass factor $\Xi_s = M_s M_{k-s}/M_k$ for $2 \le s \le k-2$, and uses the index bound (33.31) at $s = k-1, k$; index and generator choices in the rank-s quotient contribute $(Csf)^{6sf}$. For the modulo-g congruences, recall from Lemma 33.61 that a pair lies in $\mathcal{P}_{\mathrm{nd},s}$ precisely when $b_\beta(x', x'') = h(x', x'') - h(\beta x', x'')$ is nondegenerate on $(H'/\Theta) \times (H''/\Theta'')$. For a represented pair with both Gram factors nonzero, compatibility of its isometry with the congruence graphs gives the lattice-gluing bound on the exceptional lattice determinant times its two index factors. The bound is Δ_1 in the nondegenerate case and Δ_2 in the degenerate case, as displayed below. Their k-th powers enter the weighted counts.
Lean: `Formalism.MainNotions.LatticeGluingRadicals.theorem_A_section` (`Formalism.SharpRestriction.CuspSpanNormBounds.LatticeGluingRadicals`, lines 9243–9243)

The count and amplitude inputs can be combined quantitatively as follows. Choose the count constant $c \ge \max(1, C_{Tu})$ and the amplitude constants C_a, C_2, C_h, with $C_h \ge 0$, before the point threshold.
Lean: `GlobalTheory.T.rowCounts_uniform_constants_families` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.RowCountsUniformConstantsFamiliesStatement`, lines 15–16)
Lean: `GlobalTheory.RowCounts.uniform_counts` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.RowCountsUniformConstantsFamilies`, lines 25–45)
Lean: `GlobalTheory.T.amplitude_bounds_uniform_large_gap_families` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.AmplitudeBoundsUniformLargeGapFamiliesStatement`, lines 16–17)
Lean: `GlobalTheory.GapAmpUniform.amplitude_bounds_uniform_gap` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.AmplitudeBoundsUniformLargeGapFamilies`, lines 33–54)

Put

$$\sigma_Z = S_{C_h}(Z), \qquad e_s = c(s+m)m + c(1+\log_2(2f)), \qquad P_h = b^{(c^2+2\max(C_h-c,0))m^3 f}, \qquad \zeta = 12(r+1)^2.$$

For $x > 0$, define

$$\widehat{\Xi}_s(x) = \begin{cases} M_s M_{k-s}/M_k, & 2 \le s \le k-2, \\ (ck)^{kf}(xN)^{ck}, & s = k-1, k. \end{cases}$$

Index the nondegenerate and degenerate cases by $i = 1, 2$, respectively, and set

$$\begin{aligned} b_1(Z) &= 1, \qquad b_2(Z) = T_u^{-f/2} K(Z), \\ \Delta_i &= (2/g)^{(s-i)f} \sqrt{G_Z G_{Z'}}\, b_i(Z) b_i(Z'), \\ \mathcal{B}_i(Z, Z') &= (\Delta_i/N)^k \widehat{\Xi}_s(h_Z h_{Z'})(h_Z h_{Z'})^{e_s} (csf)^{6sf}, \\ a_1 &= C_2 m^2, \qquad a_2 = C_2(m^2 + 2m^4), \\ \eta_1 &= 2R', \qquad \eta_2 = e^{2C_2 m^6 f L_b} (|C_a|/k)^{kf}. \end{aligned}$$

Thus the two determinant cutoffs are

$$\Delta_1 = (2/g)^{(s-1)f} \sqrt{G_Z G_{Z'}}, \qquad \Delta_2 = (2/g)^{(s-2)f} \sqrt{G_Z G_{Z'}}\, T_u^{-f} K(Z) K(Z').$$

The mean-count estimate is used in the range $1 \le \Delta_i \le (h_Z h_{Z'})^3$. For a nonempty fibre, choose one represented pair. Its determinant factor and its two index factors are each at least one, and their product is at most Δ_i; hence $1 \le \Delta_i$. For its upper endpoint, $g \ge 2$, $T_u \ge 1$ and $0 < G_Z \le h_Z$ give $\Delta_1 \le (h_Z h_{Z'})^{1/2}$. In the degenerate case the additional input $K(Z)^2 \le 9^f h_Z^3$, together with $9^f \le h_Z h_{Z'}$, gives

$$\Delta_2 \le 9^f (h_Z h_{Z'})^2 \le (h_Z h_{Z'})^3.$$

For a row family of mass M, the finite-fibre input is the weighted count bound $M\mathcal{B}_i(Z, Z')$.
Lean: `Formalism.MainNotions.BridgeLemma₂.rows_nd_count` (`Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsND`, lines 157–189)
Lean: `Formalism.MainNotions.BridgeLemma₂.rows_KBound` (`Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsDeg`, lines 39–41)
Lean: `Formalism.MainNotions.BridgeLemma₂.rows_deg_count` (`Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsDeg`, lines 185–231)

Claim 2. Fix an integer $q \ge 0$ with $q \le (9c+3)km^2$, and retain the standing and tied hypotheses, the count and volume constants, and the amplitude thresholds fixed above. There are nonnegative majorants $A_i(Z)$ for the orbital amplitudes after the row weight and coefficient dimension have been removed, satisfying

$$n_Z^{(2)} A_i(Z)^2 b_i(Z)^{2k} \le h_Z^{2cr^2} \Gamma \eta_i \sigma_Z^{a_i - 2q/(r+1)}.$$

For a represented label, $n_Z^{(2)} > 0$ and $b_i(Z) > 0$. One can take

$$A_i(Z) = \left(\frac{h_Z^{2cr^2} \Gamma \eta_i \sigma_Z^{a_i - 2q/(r+1)}}{n_Z^{(2)} b_i(Z)^{2k}} \right)^{1/2}.$$

For $i = 1$, apply the squared-integral amplitude estimate with loss parameter $\delta = 1$; its factor $(1+\delta)R'$ is $2R' = \eta_1$. For $i = 2$, apply the absolute amplitude estimate to each relevant term with nonzero Gram value and reduced label Z. In squared form it inserts $K(Z)^{2k} T_u^{-kf} = b_2(Z)^{2k}$ in the denominator and yields the minor saving $(|C_a|/k)^{kf}$ inside η_2, together with the exponential loss $e^{2C_2 m^6 f L_b}$ and exponent a_2. Positivity of the denominators and monotonicity of square roots give the two majorants. Thus the displayed estimates are supplied for the actual terms, for every q in the permitted range.
Lean: `Formalism.MainNotions.BridgeLemma₂.rows_nd_term` (`Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsND`, lines 134–149)
Lean: `Formalism.MainNotions.BridgeLemma₂.rows_deg_term` (`Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsDeg`, lines 111–130)

Claim 3. The count and amplitude factors cancel to the following fibre bound. These inputs give a fibre contribution at most $M \dim(\pi_u) \mathcal{B}_i(Z, Z') A_i(Z) A_i(Z')$. Lemma 33.100, or direct cancellation of

$$\sqrt{n_Z^{(2)} n_{Z'}^{(2)}} = \left(\frac{\exp(1)\pi}{kv} \right)^{kf} \left(\frac{\sqrt{G_Z G_{Z'}}}{N} \right)^{k},$$

gives

$$\mathcal{B}_i(Z, Z') A_i(Z) A_i(Z') \le (2/g)^{(s-i)kf} \left(\frac{kv}{\exp(1)\pi} \right)^{kf} \widehat{\Xi}_s(h_Z h_{Z'}) (h_Z h_{Z'})^{e_s + cr^2} \cdot (csf)^{6sf} \Gamma \eta_i (\sigma_Z \sigma_{Z'})^{a_i/2 - q/(r+1)}.$$

In particular, raising the determinant cutoff to the k-th power gives exactly the exponents $(s-1)kf$ and $(s-2)kf$.

Each factor defining S_c is at most $b^{\max(C_h - c, 0) m^2}$ times the corresponding factor defining S_{C_h}. There are $f(r+1)$ factors, so $m = 2r+2$ gives

$$1 \le h_Z \le P_h \sigma_Z^4, \qquad \sigma_Z \ge 1.$$

The definition of $\widehat{\Xi}_s$ then gives

$$\widehat{\Xi}_s(h_Z h_{Z'}) \le \widehat{\Xi}_s(P_h^2) (\sigma_Z \sigma_{Z'})^{4ck}.$$

Consequently, whenever

$$4ck + 4e_s + 4cr^2 + \tfrac{1}{2}a_i + \zeta \le q/(r+1),$$

the preceding fibre product is at most

$$\mathcal{R}_i(s)\sigma_Z^{-\zeta}\sigma_{Z'}^{-\zeta}, \qquad \mathcal{R}_i(s) = (2/g)^{(s-i)kf}\left(\frac{kv}{\exp(1)\pi}\right)^{kf}\widehat{\Xi}_s(P_h^2) \cdot P_h^{2e_s+2cr^2}(csf)^{6sf}\Gamma\eta_i.$$

Indeed the remaining exponent of $\sigma_Z\sigma_{Z'}$ is $4ck + 4e_s + 4cr^2 + a_i/2 - q/(r+1)$, which is at most $-\zeta$. *Claim 4.* A single permitted moment order works in both cases. The choice

$$q = \left\lceil (r+1)\left(4ck + 4e_k + 4cr^2 + \tfrac{1}{2}|C_2|(m^2+2m^4) + \zeta\right)\right\rceil$$

works for both cases and every $s \le k$. The numerical conditions used to bound this moment order are

$$r \ge 1, \quad m = 2r+2, \quad m+3 \le k, \quad \log f \le \sqrt{r}, \quad |C_2|(m^2+2m^4) \le k.$$

The last inequality follows from the point threshold $3|C_2|m^5 f L_b \le k$, since $m \ge 2$, $f \ge 1$, and $L_b \ge \log 2 > 1/2$. Put $c_f = 1 + \log_2(2f)$. Since $\log 2 > 1/2$,

$$c_f = 2 + \frac{\log f}{\log 2} \le 2 + 2\sqrt{r} \le 2 + 2r = m.$$

Let Q_0 be the expression inside the ceiling defining q, and put $\theta = r+1 = m/2$. Using $c_f \le 2\theta$, $r^2 \le \theta^2$, $|C_2|(m^2+2m^4) \le k$, $\theta \ge 1$, $2\theta \le k$ and $5 \le k$, termwise estimation gives

$$q \le Q_0 + 1 \le (24c+7)k\theta^2 \le (36c+12)k\theta^2 = (9c+3)km^2.$$

Thus this choice lies in the allowed moment range.

Claim 5. Summing the fibre bound over represented labels gives the bounds in the proposition. The label-sum input is

$$\sum_{Z\in\mathcal{Z}} \sigma_Z^{-\zeta} \le \Phi_0, \qquad \Phi_0 = (41^f P_h^3)^{(r+1)^2},$$

for each finite set $\mathcal{Z}$ of represented reduced labels.

Lean: `Formalism.MainNotions.BridgeLemma₂.rows_nd_size` (`Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsND`, lines 278–290)

Grouping a finite pair sum by its two labels and applying the fibre estimate therefore gives $M\dim(\pi_u)\mathcal{R}_i(s)\Phi_0^2$. The numerical comparison with the constants in the statement uses

$$c \le C, \quad 1 \le C, \quad 4 + (c^2 + 2\max(C_h - c, 0))(6c+2) \le C^2, \quad 2C_2 \le C, \quad |C_a| \le C,$$

and yields $\mathcal{R}_1(s)\Phi_0^2 \le R'U_s$ and $\mathcal{R}_2(s)\Phi_0^2 \le V_s$.

Lean: `Formalism.MainNotions.BridgeLemma₂.rows_RowsND_record` (`Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsNDn`, lines 157–169)

Lean: `Formalism.MainNotions.BridgeLemma₂.rows_RowsDeg_record` (`Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsDegF`, lines 117–135)

Multiplying by $\Gamma_1\dim(\pi_u)\,\mathrm{mass}_u \le f_u$ from Lemma 33.65 gives (33.32). The complete proof is contained in `GlobalTheory.RowsFive.rowsND_of` and `GlobalTheory.RowsFive.rowsDeg_of`. □

Proof of 33.105

Proof (outline). For any real a, write $h_a(Z) = b^{a^2m^3f}S_a(Z)^4$. Let c_{sw} and c_{par} be the constants furnished by the represented-frame and partner-count bounds. Enlarge the common count constant so that

$$c \ge \max\{1, C_{Tu}, |c_{\mathrm{sw}}| + 2, |c_{\mathrm{par}}| + 2\}.$$

Claim 1. The two counting constants can be enlarged simultaneously. Either bound at a real constant c' remains valid at c whenever $|c'| + 2 \le c$. In particular both bounds hold with this common constant. Indeed

$c' \le c$, $c + c' \ge 2$, and the height attached to a represented label is at least one. The height comparison gives $h_{c'}(Z) \le h_c(Z)$; increasing its exponent from $c'r^2$ to cr^2 can only increase the bound. The local represented-frame factors have nonnegative product, so the represented-frame inequality is preserved. For the partner bound, also use

$$(c')^f \le |c'|^f \le c^f.$$

Apply the height comparison in both labels and the same exponent monotonicity. These comparisons preserve the point and family quantifiers because the enlargement is made from the fixed counting constants.
Lean: `Formalism.MainNotions.BridgeLemma₂.switch_bound_mono_constant` (`Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsOneF`, lines 59–70)
Lean: `Formalism.MainNotions.BridgeLemma₂.partner_count_bound_mono_constant` (`Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsOneF`, lines 72–91)

Claim 2. In the tied term construction of this proposition, two terms with the same row lattice are equal whenever their moving frames differ by an element of the integral fixed-frame unipotent stabilizer followed by an upper-unitriangular integral change of frame basis.

For a moving frame in a row lattice, every upper-unitriangular integral matrix ξ is realized on the frame by an element of its integral unipotent stabilizer. The row-lattice and frame-lifting conditions supply this realization. Given the equivalence in the claim, absorb ξ into that moving-frame stabilizer. The resulting two-sided integral frame equivalence identifies the two tied terms.
Lean: `Formalism.L2.SG.Pf.LoopBFrame.frameChangeAt_holds` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.LoopBFrame`, lines 25–28)
Lean: `Formalism.L2.SG.LoopsFinal.frameEqvInj_of` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.LoopsFinal`, lines 77–85)

Claim 3. Let Λ be a unimodular row lattice with fixed lifted frame F satisfying the fixed level congruences. Suppose F' and F'' are generic integral terms relative to F, and β_1, β_2 are lattice isometries carrying each flag subspace of F' to the corresponding one of F'', carrying the final vector of F' to that of F'', and preserving their congruence graphs relative to F. If both associated pairings have corank one, then

$$\beta_2 x - \beta_1 x \in X_{F''} \qquad (x \in X_{F'}^{\perp}),$$

where $X_{F'}$ and $X_{F''}$ are the respective isotropic subspaces. The lattice-isometry uniqueness theorem gives this statement by comparing the isometries on the common definite part and its final line. In the pair equivalence relation, the displayed containment is precisely the remaining condition after the two terms have been identified. Therefore generic corank-one pairs with nonzero Gram factors and the same two terms are equal.
Lean: `beta_l_unique` (`Formalism.PeriodIntegrals.OrbitalBoundsConclusion.BetaLUniq`, lines 6–8)
Lean: `Formalism.MainNotions.BridgeLemma₂.wordPairBetaLattice` (`Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsCnt`, lines 607–612)
Lean: `Formalism.MainNotions.BridgeLemma₂.rows_BetaUnique_of` (`Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsCnt`, lines 614–646)

Claim 4. The two ordered weighted counts obey (33.33). For a represented frame (F_u, F_t') of label Z, row-lattice extensions correspond to definite rank-$(k-1)$ complement lattices in $W_t^{\perp}$ glued along the frame discriminant module as in Lemma 33.79; summing inverse stabilizer weights over the genus bounds the weighted count of first frames of label Z by $Mh_Z^{cr^2} n_Z$. Fix t. All corank-one partners counted here lie over the same rank-$(k-1)$ sublattice determined by t. Put $H_{Z,Z'} = h_Z h_{Z'}$. Both $N = T_u^f$ and the determinant of this sublattice are at most $H_{Z,Z'}$. Choosing the partner-count constant at least 37, the fixed-sublattice Gram-class count is at most $9^f H_{Z,Z'}^{2r+r+9+9r+16r^2} \le c^f H_{Z,Z'}^{cr^2}$. Its hypotheses are that the first and partner frames are generic integral terms in the same row, have the prescribed labels satisfying the block-point condition in (2), and admit a compatible pair isometry of corank one.
Lean: `Formalism.MainNotions.S.UnitaryVolumeMass.PartnerCountBound` (`Formalism.SharpRestriction.CuspSpanNormBounds.CongruenceAtSplitFrobeniusHolds`, lines 569–577)

A finite fibre of pairs maps into these classes by taking the second moving frame. Claim 2 identifies two second terms whose frames have the same class. Claim 3 then identifies their pair isometries in the pair equivalence relation. Thus this map is injective. Multiplying the first-frame bound by this partner count gives the first expression in (33.33), reversing the roles of t and t' gives the second, and taking their square-root product gives $Mc^f (h_Z h_{Z'})^{3cr^2/2} \sqrt{n_Z n_{Z'}}$.
Lean: `Formalism.MainNotions.BridgeLemma₂.rows_countOne` (`Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsCnt`, lines 273–325)

Lean: Formalism.MainNotions.BridgeLemma₂.rows_OneFibre_of (Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsCnt, lines 403–439)
Lean: Formalism.MainNotions.BridgeLemma₂.rows_OneFibre'_of' (Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsCnt, lines 604–605)

Claim 5. The nonscalar subtotal satisfies (33.34). For this analytic estimate, choose an auxiliary constant $c_* \ge c+2$ that also dominates the determinant-height threshold and the absolute values plus two of the height-comparison and volume constants. Claim 1 preserves both counting bounds with the auxiliary height h_{c_*}. Choose the amplitude constants C_a, C_2, C_h, with $C_a, C_h \ge 0$, for moment range $(9c_*+3)km^2$, before choosing the point threshold. The latter ensures

$$|C_2|m^5 \le k, \qquad (\max(C_a,1))^3 \le k.$$

Put

$$q = \left\lceil (r+1)\left(12c_*r^2 + |C_2|m^2/2 + |C_2|m^4 + 12(r+1)^2\right)\right\rceil.$$

The inequalities $m = 2r+2$, $r \ge 0$ and $m+3 \le k$ imply $0 \le q \le (9c_*+3)km^2$, so this moment is permitted. It suffices to take

$$C \ge \max\left\{4\sqrt{a_F},\ 2\sqrt{2C_0}\,c_*,\ C_\Gamma + C_2, 4 + 2\big(c_*^2 + 2\max(C_h - c_*, 0)\big)(c_*+1)\right\},$$

where C_Γ is the fixed exponent bounding the family factor Γ. These choices are all made before the point, the row and the pair subset.
Lean: Formalism.MainNotions.BridgeLemma₂.decay_length_sufficient (Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsOneF, lines 12–37)

A pair containing an irrelevant term contributes zero. Restrict the sum to the remaining pairs, whose two terms are relevant and have nonzero Gram factors.
Lean: Formalism.MainNotions.BridgeLemma.rowBound_relevant (Formalism.SharpRestriction.CuspSpanNormBounds.BruhatCellsKronecker.FamilyPacketOfEquivalentStatement, lines 860–876)

For such a term with $D_r \ne 0$ put $c_\tau = |\tau(D_{r+1}/D_r)|/\sqrt{T_uT_{u'}}$, which exceeds 1 on the positive Gram locus, and $s_\tau = \sqrt{c_\tau^2 - 1}$. Its local normalized scale is $\nu_\tau = G(\gamma_\tau)/(T_u|D_r(\gamma_\tau)|^2)$, where $G = |D_{r+1}|^2 - T_uT_{u'}|D_r|^2$ is the frame Gram polynomial, so $\nu_\tau = T_{u'}s_\tau^2$. Put $\chi_\tau = c_\tau/s_\tau$ and $\iota_\tau = 1/s_\tau$ for such a term, and $\chi_\tau = 1$, $\iota_\tau = 0$ for a term with $D_r = 0$. For a generic corank-one pair (t,t') with nonzero Gram factors, the lattice-gluing theorem for the two moving frames relative to the fixed frame shows that the pair is scalar or

$$\left(\frac{\sqrt{T_uT_{u'}}}{g}\right)^f \le \prod_\tau \big(\chi_\tau(t)\iota_\tau(t') + \iota_\tau(t)\chi_\tau(t')\big).$$

On S_{nsc} the second alternative holds. Its right side vanishes when $D_r = 0$ on both sides, so this case does not occur. When $D_r \ne 0$ on both sides it is the product hypothesis of Lemma 33.101, and when $D_r \ne 0$ on one side only it is the single-side hypothesis of that lemma for that side. Hence at least one of t, t' satisfies the small-side condition $D_r \ne 0$ and $\prod_\tau \min(\nu_\tau, T_{u'}) \le (4\sqrt{T_{u'}/T_u}\,g)^f$.
Lean: Formalism.MainNotions.LatticeGluingRadicals.theorem_B_section (Formalism.SharpRestriction.CuspSpanNormBounds.LatticeGluingRadicals, lines 6762–6762)
Lean: Formalism.MainNotions.BridgeLemma₂.small_term_of_corank_one_pair (Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsOneD, lines 195–211)
Lean: Formalism.MainNotions.BridgeLemma₂.rowsT_small (Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsTrm, lines 945–1037)

Split the remaining pairs of U into the subset where the first term satisfies this small-side product inequality and the complementary subset where the second term satisfies the symmetric inequality. On each subset, Lemma 33.101 multiplies the small-side squared amplitude bound by $2C_0\Gamma^2(4\sqrt{T_{u'}/T_u}\,g^{1-m}k^{-r})^{kf}$ (with $4\sqrt{T_{u'}/T_u} \le C$ in both orders from $\max(T_u/T_{u'}, T_{u'}/T_u) \le a_F$ and $C \ge 4\sqrt{a_F}$), while $\sqrt{n_Z n_{Z'}}$ from (33.33) cancels $n_Z^{-1/2}n_{Z'}^{-1/2}$ from the two amplitude bounds. Taking the square root gives $(Cg^{-(2r+1)}k^{-r})^{kf/2}$. The chosen length moment absorbs $(h_{c_*}(Z)h_{c_*}(Z'))^{3c_*r^2/2}$ and leaves $S_{C_h}(Z)^{-12(r+1)^2}S_{C_h}(Z')^{-12(r+1)^2}$ (for the cutoff constant $C_h \ge 0$ of Lemma 33.61); summing over (Z,Z') costs at most $C^fH^{Cm^3}$ with decay and dimension losses bounded by $e^{Cm^6fL_b}$, and summing over row classes via $\Gamma_1 \dim(\pi_u)\,\mathrm{mass}_u \le f_u$ yields (33.34).
Lean: Formalism.L2.SG.RowsU.rowsOneU (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.DegenerateConstant, lines 297–421)

Lean: Formalism.L2.SG.RowsU.rowsFU (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.DegenerateConstant, lines 425–441)

The complete proof is contained in `Formalism.L2.SG.RowsU.rowsOneU`, which proves the row bound `Formalism.MainNotions.BridgeLemma.RowsOne` and enters the loop estimate through `Formalism.L2.SG.RowsU.rowsFU`. □

Proof of 33.106

Proof (outline). Use the notation of Propositions 33.103 and 33.105. Put $\Delta = f_u R' = \Delta_{u,u'}$, using the tied identities $d_w^C = d_w^S$ and $D_C^\star = D^\star$. Choose $C \geq 1$ above the count and amplitude thresholds, C_{Tu}, and $408e\pi$. Set $A_0 = \max(1, C_0)$. Enlarge the subsequent thresholds so that

$$C' \geq 16C^{32}, \qquad C_\ell \geq \max\{4, 3 + 6\log(3A_0)\}.$$

These choices depend only on the previously fixed constants.

Claim 1. The actual pair sum satisfies

$$\mathrm{PairAbs}_{u,u'}(S_{\mathrm{big}} \cup S_{\mathrm{nsc}}) \leq A_0\Delta\left[\sum_{s=2}^{k}(U_s + \Gamma^2 V_s) + J\right]. \tag{A.344}$$

The spectral factor in Definition 33.7 is the common size factor $\Gamma = e^{C_\Gamma m^5 f L_b}$. Indeed the size data give $\log\Gamma_w^{\mathrm{sp}} = C_\Gamma m^5 f L_b$, and $\Gamma_w^{\mathrm{sp}} > 0$, for every word w. Thus the lower dimension bound there gives $1 \leq C_0 R' \Gamma^2$.
Lean: Formalism.MainNotions.BridgeLemma.d_GamP_eq (Formalism.SharpRestriction.CuspSpanNormBounds.BruhatCellsKronecker.FamilyPacketOfEquivalentStatement, lines 220–222)

Partition the pairs by corank and by the nondegenerate and degenerate alternatives. For a finite subset, Proposition 33.103 bounds its contributions by $f_u R' U_s$ and $f_u V_s$. The dimension inequality bounds the second by $A_0\Delta\Gamma^2 V_s$. Proposition 33.105 bounds the nonscalar corank-one contribution by $A_0\Delta J$. Summation and then the supremum over finite subsets prove the claim. All summands are nonnegative; absolute convergence is among the hypotheses.
Lean: Formalism.L2.SG.LoopsCut.loopsAmp_of_classes (Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.LoopsCut, lines 87–110)

Claim 2. Set

$$\gamma = \exp\bigl(\min(C, C_\Gamma) m^5 f L_b\bigr), \quad \rho = \Gamma/\gamma, \quad A = A_0\rho^3.$$

Then $1 \leq \gamma \leq \Gamma$, $\rho \geq 1$, $A \geq 1$, and

$$6\log(3A) \leq P, \qquad \mathrm{PairAbs}_{u,u'}(S_{\mathrm{big}} \cup S_{\mathrm{nsc}}) \leq A\Delta\left[\sum_{s=2}^{k}\bigl(U_s(\gamma) + \gamma^2 V_s(\gamma)\bigr) + J(\gamma)\right].$$

Here a parenthesized argument replaces only the common factor Γ. The three expressions U_s, $\Gamma^2 V_s$, and J have degrees one, three, and one in Γ, respectively. Their remaining factors are nonnegative, so replacing $\Gamma = \rho\gamma$ costs at most ρ^3. Moreover,

$$0 \leq \log\rho \leq C_\Gamma m^5 f L_b \leq k/8, \qquad f\log k \geq C_\ell L_b \geq 3 + 6\log(3A_0).$$

Since $k \geq 1$, these inequalities give $6\log(3A) = 6\log(3A_0) + 18\log\rho \leq kf\log k$.
Lean: Formalism.L2.SG.LoopsCut.log_size (Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.LoopsCut, lines 237–251)
Lean: Formalism.L2.SG.LoopsEv.loopsAmp_at (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.LoopsEv, lines 18–89)

Claim 3. For the definite genus factors and endpoint factors Ξ_s of Proposition 33.103, the standing CM, split-prime, rank and size hypotheses imply

$$\begin{aligned} 0 \leq \Xi_s, \quad (kv)^{kf}\Xi_s &\leq H^C C^{skf} && (2 \leq s \leq k-2), \\ 0 \leq \Xi_s, \quad \Xi_s &\leq (Ck)^{kf}(H^{Cm}N)^{Ck} && (s = k-1, k). \end{aligned}$$

The first estimate follows from the explicit mass ratio of Theorem 37.5, as in the interior-rank computation in the proof of Lemma 33.61. After the pairing of numerator indices, the two unpaired denominator factorials

$(k-2)!(k-1)! \ge (k/\exp(1))^k$ give the factor $(\exp(1)\pi/(kv))^{kf}$. The rank-two mass bound and the Euler factor estimate of Lemma 33.11, with the rank and discriminant bounds of Definition 33.7, give $M_2(2v)^f \le H^C$, and $C \ge 408e\pi$ absorbs the remaining numerical factors. The second is the endpoint index estimate (33.31). These estimates use the same level norm $N = T_u^f$.
Lean: `Formalism.L2.SG.LoopsCut.xiSizesC_of` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.LoopsCut`, lines 335–351)

The remaining scalar hypotheses, after the replacement of Claim 2, are

$$4 \le m \le f \le k, \quad b \ge 2, \quad (kb)^8 \le g, \quad 16C^{32}m^5 f \log b \le k,$$
$$1 \le v \le b, \quad 1 \le N \le b^{Cmf}, \quad 1 \le \gamma \le e^{Cm^5 f \log b}.$$

They follow from Definition 33.7 and the threshold choices above. In particular $v \le b$ follows from $\log \mathrm{rd}(E) \le L_b/C_\ell$ and $C_\ell \ge 1$, and the bound on N follows by raising $T_u \le b^{C_{T_u} m}$ to the power f.
Lean: `Formalism.L2.SG.LoopsCut.params_at` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.LoopsCut`, lines 140–181)

Claim 4. For any scalar data satisfying the inequalities in Claim 3, with $m = 2r+2$, the functions U_s, V_s, J defined in Propositions 33.103 and 33.105 obey

$$\sum_{s=2}^{k} U_s(\gamma) \le g^{-kf/2}, \qquad \sum_{s=2}^{k} V_s(\gamma) \le e^{-P/2}\gamma^{-2}, \qquad J(\gamma) \le e^{-P}. \tag{A.345}$$

For each sum, separate $2 \le s \le k-2$ from $s = k-1, k$, and use the corresponding bound of Claim 3. The scalar exponent inequalities give the pointwise bounds

$$U_s(\gamma) \le \exp\bigl(-\tfrac{1}{2}kf\log g - \log k\bigr), \quad V_s(\gamma) \le \exp\bigl(-P/2 - 2\log\gamma - \log k\bigr).$$

For $s = 2$, the factor $(C/k)^{kf}$ supplies the negative term in the second estimate. There are at most k terms in each sum, which proves the first two assertions. The same parameter inequalities applied to the square-root expression defining J give the third assertion.
Lean: `Formalism.HermitianLatticeCounting.P.sumNondegenerateRowsLe` (`Formalism.FlagUnipotentLattices.MinkowskiFactorialBounds.ExponentInequalityNondegenerateMid`, lines 609–669)
Lean: `Formalism.HermitianLatticeCounting.P.sumDegenerateRowsLe` (`Formalism.FlagUnipotentLattices.MinkowskiFactorialBounds.ExponentInequalityNondegenerateMid`, lines 671–733)
Lean: `Formalism.HermitianLatticeCounting.P.singleRowBound` (`Formalism.FlagUnipotentLattices.MinkowskiFactorialBounds.ExponentInequalityNondegenerateMid`, lines 735–764)

Since $g \ge (kb)^8 \ge k^8$, the first bound in (A.345) is at most e^{-4P}. Multiplying the second by γ^2 and using Claim 2 now gives

$$\mathrm{PairAbs}_{u,u'}(S_{\mathrm{big}} \cup S_{\mathrm{nsc}}) \le A\bigl(e^{-4P} + e^{-P/2} + e^{-P}\bigr)\Delta \le 3Ae^{-P/2}\Delta \le e^{-P/3}\Delta.$$

Here $P \ge 0$ and the last inequality is exactly $6\log(3A) \le P$. The complete proof is contained in `Formalism.FourierDecayAlt.P.GrowthPolynomialOperator.t_LoopsJoint`. □

Proof of 33.107

Proof. For $0 \ne l \in J$, the integral ideal $(l)J^{-1}$ has norm at least one, so $\prod_v |v(l)| \ge \sqrt{N(J)} =: D$. If some $z_v = 0$ the sum vanishes. Otherwise partition the points by $n_v = \max(0, \lceil \log_2(|v(l)|/z_v)\rceil)$. Put $h = \sum_v n_v$, $R_v = 2^{n_v} z_v$. A nonempty box has $U = \prod_v R_v/D \ge 1$. Subdivide the coordinate intervals $[-R_v, R_v]$ in each complex plane into intervals of length at most $R_v/(2U^{1/f})$. Each product box contains at most one point since a nonzero difference in it would have product of absolute values strictly below D. There are at most $25^f U^2$ boxes. Each summand is at most 2^{5f-5h} and $U \le 2^h(a/\Theta)^f$. Since $2^h \ge (\Theta/a)^f$, the shell contributes at most $800^f(a/\Theta)^{5f/2}2^{-h/2}$. Sum over (n_v) using $\sum_{(n_v)} 2^{-h/2} = (1-2^{-1/2})^{-f}$.

The Fourier character on $E \otimes \mathbf{R}$ is $\exp(2\pi i \sum_v 2\,\mathrm{Re}(v(l)c_v)/g)$, which gives the stated dual. Since $g^2\mathfrak{d}^\nu \subset L_{b_0 s}$, trace duality gives $L_{b_0 s}^* \subset g^{-1}\mathfrak{d}^{-\nu-1}$. The inverse-difference lattice is contained in $g^{-1}\mathfrak{d}^{-\nu}$ by definition. □

Proof of 33.108

Proof. Make the changes $X = tV$, $B = tb$, $\rho = taJ_n(1+V)$, and denote the last matrix entry by σ. The resulting matrix is

$$M = \begin{pmatrix} J_n(tI + X) & J_n B \\ \rho & \sigma \end{pmatrix}.$$

Write $x_1, \dots, x_n$ for the first superdiagonal entries of $(X\ B)$, including $x_n = B_n$. The character is $\psi_\beta((\rho_1 + \sum_i x_i)/t)$. Indeed $\rho_1 = ta_n$. The Jacobian from the three changes before introducing σ is $|t|^{-n(n+3)}$ in the integral.

Put $D' = (-1)^n D$. The matrix $J_s M$ has first subdiagonal t, zeros below that subdiagonal, diagonal $(\rho_1, x_1, \dots, x_n)$, and determinant D'. Use its strictly upper entries y and the x_i as independent variables. In replacing ρ_1 by σ, the derivative has modulus $|t|^n/|C_t|$, where C_t is the cofactor of ρ_1 in $J_s M$. This follows by differentiating the determinant in these two entries. The cofactor of σ, up to sign, is t^n. Thus the remaining Jacobian, after multiplication by $|t|^{n(n+1)}$, is $|C_t|^{-2}$. At $t = 0$, $C_0 = \prod_i x_i$.

Every independent matrix entry is bounded on the support. Expansion of the determinant at $t = 0$ gives $D' = \rho_1 \prod_i x_i + O(|t|)$ there. Since $|D'|$ has a positive lower bound, each $|x_i|$ and $|C_t|$ has a positive lower bound for small t. We can therefore solve for ρ_1 throughout the support. Put

$$p(x) = \sum_i x_i + D' / \prod_i x_i.$$

After removal of $\psi_\beta(p(x)/t)$, the amplitude is

$$|C_t|^{-2}\, \psi_\beta\!\left(\frac{\rho_1 - D'/\prod_i x_i}{t} \right) \Phi(M).$$

The determinant equation is polynomial in t and affine in ρ_1. Division by C_t shows that the divided difference in this amplitude extends smoothly to $t = 0$. A fixed smooth cutoff on a compact neighborhood of the support gives uniform bounds for all its derivatives in x, y.

The critical equations are $x_i = D'/\prod_j x_j$ for every i. Thus $x = (z, \dots, z)$ with $z^s = D'$, and $p(x) = sz$. The complex Hessian there is $z^{-1}(I + \mathbf{1}\mathbf{1}^{\mathsf{T}})$, of determinant sz^{-n}. The roots stay separated on the compact parameter set. Outside fixed small neighborhoods of them the gradient is bounded below, so $n+1$ integrations by parts give $O(|t|^{n+1})$ in the transformed integral. At a root apply stationary phase in $2n$ real dimensions to $\mathrm{Re}(\beta|t|p/t)$. Its Hessian has signature zero and absolute determinant $|\beta|^{2n}|sz^{-n}|^2$. The amplitude is compactly supported with uniformly bounded derivatives, and the inverse Hessians are uniformly bounded. The leading multiplier in the measure $d\mu_\beta^n$ is $|t|^n|z|^n/s$, with error $O(|t|^{n+1})$. The scalar bound is uniform on compositions of Φ with linear functionals of norm at most one, which gives the Banach-space estimate. [147, Theorem 7.7.5]

At the critical point write $J_s M|_{t=0} = zI + Y_0$, with Y_0 strictly upper triangular. Differentiate the determinant with x, y, D' fixed. The subdiagonal has derivative one, and the first diagonal entry has derivative $\partial_t \rho_1$. The trace identity for the derivative, using the first superdiagonal of $(zI + Y_0)^{-1}$, gives $\partial_t \rho_1 = \ell(Y_0)/z$. There the cofactor contributes $|z|^{-2n}$. Substitution $Y_0 = zY$ contributes $|z|^{n(n+1)}$ and turns the residual integral into $L_\Phi(z)$. The powers of $|z|$ give $n - 2n + n(n+1) = n^2$. Finally $|t|^{n(n+1)}/|t|^n = |t|^{n^2}$, and the transformed remainder becomes $O(|t|)$. □

Proof of 33.109

Proof. For a compact annulus K, set $K' = \{z : \omega^{-1} z^s \in K\}$. Expand $\int_K \|A_{p,x}\|^2$ and change variables by $D = \omega^{-1} z^s$. The pairs of roots are indexed by $\zeta^s = 1$. The diagonal contribution is exactly $\int_{K'} \|b_p\|^2$. Each other contribution is the Fourier integral of an L^1 density evaluated at frequency $s(\zeta - 1)x$. Cauchy–Schwarz gives integrability on K', and the Riemann–Lebesgue lemma makes the contribution tend to zero. The annular approximation and the bound on $U_{p,x}$ now give $\int_{K'} \|b_p\|^2 \le M_p$. Exhaust $\mathbf{C}^\times$ by these annuli.

The function $a_p(D) = \sum_{z^s = \omega D} \|B_p(z)\|$ satisfies $\|a_p\|_2^2 \le s\|b_p\|_2^2$ by the same Jacobian and the finite Cauchy–Schwarz inequality. It dominates every $A_{p,x}$. In the D-coordinates of the first fibre the second boundary vector is dominated by $|\lambda| a_{p'}(\lambda D)$, where $\lambda = (x/x')^{s-1} \to 1$. These dominators converge to $a_{p'}$

in L^2. This follows first for continuous compactly supported functions by change of variables, then by density and the isometry of normalized dilation. Given $\epsilon > 0$, choose a compact annulus on whose complement the dominators have L^2-norm below ϵ for all sufficiently large $|x|$. On the annulus the error in replacing $U_{p,x}$ by $A_{p,x}$ tends to zero. On its complement the mixed error is at most $(\sqrt{M_p} + \|a_p\|_2)\epsilon$. Interchanging the prefixes proves the other mixed comparison.

For the boundary-boundary pairing choose the root $\rho_x^s = \lambda$ tending to one. The second roots are $\zeta\rho_x z$. The change of variables just used gives densities

$$\langle b_p(z), |\rho_x| b_{p'}(\zeta\rho_x z)\rangle$$

and phases $\chi(s(\zeta\rho_x x' - x)z)$. The densities converge in L^1 by L^2-continuity of dilations and rotations. Since $\rho_x = 1 - (s-1)\sigma/(sx') + O(|x|^{-2})$, the frequency tends to σ for $\zeta = 1$. For the other roots its modulus tends to infinity. Fourier continuity and decay give the displayed integral. The block positivity follows by taking all the $T_x U_{p,x}, T_x A_{p,x}$ as vectors in the same $L^2(dc)$. □

Proof of 33.110

Proof. The argument of θ is at least one on the smaller ball and nonpositive outside the larger one. The equation $g^* \operatorname{diag}(1_p, -1_q) g = \operatorname{diag}(1_p, -1_q)$ defines a closed subset of matrix space and forces g to be invertible. Its intersection with the larger closed ball is compact and avoids zero. For fixed left and right word lengths the differentiated coefficient depends continuously on g and multilinearly on the Lie algebra directions. Take the maximum on this compact set times the corresponding closed unit balls, and then the maximum over the finitely many pairs of word lengths. Outside the support every derivative vanishes. The factor $\|g\|_F^A$ is bounded on the same compact set. □

Proof of 33.111

Proof (outline). Put $r = r_S,\ g = g_0 g_1,\ v_E = \operatorname{rd}(E)/2,\ \kappa = \pi/v_E$, and choose a differing interior position l of the two binary words. At each place let v_l be the corresponding interlacing parameter of 33.7, and write u' for the weight of word v. Use $N = 20(r+1) = 10m$ letters and a cut of width $I = \mathcal{D}/2 - 5/2$, where $\mathcal{D}$ is the choice spacing. Set

$$P(y) = \prod_j (y + \lambda_j), \qquad q(y) = -T_u T_v \prod_i (y + 1/2 + u_i)(y + 1/2 + u_i').$$

Write B for the supremum of $g^m \kappa^{1-m} |P(y)P(y+1)/q(y)|^{1/2}$ over $5/2 \le y + v_l \le \mathcal{D}/2$, and B_1 for the supremum on the part of this interval with $y + v_l \le N^3$. The cut avoids all roots of q. For a threshold constant C_A put $d = C_A \kappa N/(gI)$. Define Θ^{-f} as the maximum of the products over places of the following factors, with the indices common to all places:

$$\begin{aligned}
&\widehat{X}_1 = C_A B_1, && (i = 1,\ s = 1),\\
&\widehat{X}_i = C_A B d^{2i-2}, && (2 \le i \le r+1,\ s = 1),\\
&\widehat{\Lambda}_{is} = C_A B d^{2(i-1)+s-1}, && (i \ge 1,\ s \ge 2,\ i + s \le r + 2),\\
&\widehat{\Lambda}'_i = (C_A B \kappa d^{2(i-1)}/\sqrt{T_u T_v})^{1/2}, && (1 \le i \le r+1).
\end{aligned} \tag{A.346}$$

For use in the arithmetic coordinates define $J_{is} = g^{-1}\mathfrak{d}^{-\nu-1}$, $\nu = 2r + 3 - 2i - s$. Divide the first three factors of (A.346) by $g\,\operatorname{rd}(E)^{\nu+1}$ to obtain the scales $z_{is,\tau}$, with i now the block start and τ indexing the real place. With $s = r + 2 - i$, divide the last factor by $g\,\operatorname{rd}(E)^{\nu}$ to obtain the terminal scale $z'_{i,\tau}$. The product of these divisors is the inverse square root of the norm of the corresponding ideal J_{is} or $g^{-1}\mathfrak{d}^{-\nu}$.

With $d_E = \operatorname{rd}(E)$, $g = g_0 g_1$, and $\kappa = \pi/(d_E/2)$, define the critical level $T_{\mathrm{crit},\tau}(w) > 0$ for each word w and place τ by

$$T_{\mathrm{crit},\tau}(w)^k = \kappa^{-[(m-1)k + m(m-1)/2]} d_E^{-(m-1)/2} g^{km + m(m-1)/2} \frac{VV_\tau}{V(w_\tau)}, \qquad VV_\tau = d_{\sigma,\tau} \prod_{j=0}^{k+m-2} j!,$$

where V is the Vandermonde product and $d_{\sigma,\tau}$ is the local formal-degree factor, and put $t_\tau = \sqrt{T_{\mathrm{crit},\tau}(u)T_{\mathrm{crit},\tau}(v)/(T_uT_v)}$. For the cut estimate put $x = -y$ and let δ_i, δ_i' be the binary choices at i, so that the applicable weight coordinates are $v_i - 1/2 - \delta_i\mathcal{D}$ and $v_i - 1/2 - \delta_i'\mathcal{D}$. Pair the interlacing spectral roots with the weight roots by writing the interior factors as

$$R_i(x) = \frac{\sqrt{|(v_i - x)(v_i + 1 - x)|}}{\sqrt{|(v_i - x - \delta_i\mathcal{D})(v_i - x - \delta_i'\mathcal{D})|}}.$$

The top-place last weight uses $B_* + x - 1/2$ in the denominator. For $i < l$, the denominators are bounded below by the corresponding distances $v_i - x - \mathcal{D}$, and the product of these factors is at most $3.3\sqrt{k}$ by the spacing of the v_i. For $i > l$ in the interior, the ratios are at most one. At $i = l$, exactly one denominator is shifted. With $h = v_l + y$, this gives $R_l(x)^2 = (h + 1)/(\mathcal{D} - h) \le (14/5)h/\mathcal{D}$ from $5/2 \le h \le \mathcal{D}/2$.

The remaining spectral factors satisfy $\prod_j \sqrt{(B_j + x)(B_j + x - 1)} \le e^\beta \prod_j B_j$, where β can be taken as the sum of the upper bounds x/B_j on the cut. The size bound $B_j \ge C_\ell mk\mathcal{D}$ makes β bounded. In the edge comparison the first weight factor $1/(u_1 + 1/2 - x)$ times the last ratio is at most $8/a_{\mathrm{eff}}$, where a_{eff} is the edge denominator: off the top $a_{\mathrm{eff}} = (u_1 + 1/2) - v_1$ (giving $(1/a_{\mathrm{eff}}) \cdot 1 \le 8/a_{\mathrm{eff}}$), and on the top $a_{\mathrm{eff}} = (u_1 + 1/2)B_*/(k\mathcal{D})$ (avoiding any division by v_{k-1}, since the top edge factor yields $(2/(u_1+1/2))(4k\mathcal{D}/B_*) = 8/a_{\mathrm{eff}}$). Lemma 37.8 gives $VV_\tau \ge \frac{1}{2}\Delta(\lambda_\tau)$; comparing the Vandermonde factors in $T_{\mathrm{crit},\tau}(w)$ and using the quadratic prefactor bound $\frac{1}{2}\kappa^{-m(m-1)/2}d_E^{-(m-1)/2}g^{m(m-1)/2} \ge (g/(2\pi))^{m(m-1)/2} \ge b^{2m} \ge \prod_{j=1}^m B_j$ to absorb $\prod_{j=1}^m B_j$ into the k-th power comparison yields the sharp lower bound $T_{\mathrm{crit},\tau}(w) \ge c_1 g^m \kappa^{1-m}(\prod_j B_j)/a_{\mathrm{eff}}$ for a fixed constant $c_1 > 0$ and each $w \in \{u, v\}$. Multiply the two comparisons and take the square root, then divide by $\sqrt{T_uT_v}$. Consequently on the cut

$$g^m\kappa^{1-m}|P(y)P(y+1)/q(y)|^{1/2} \le C_2 t_\tau\sqrt{k}\sqrt{(v_l + y)/\mathcal{D}}.$$

Taking suprema gives $B \le C_2\sqrt{k}\, t_\tau$ and $B_1 \le C_2\sqrt{kN^3/\mathcal{D}}\, t_\tau$.
Lean: `Formalism.FourierDecayAlt.P.Sz7.betaOnCut_of_tcritLower` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.BetaBoundOnBoundary`, lines 163–254)

Write $T_* = T_uT_v$ and $D = C_A\pi N/I$. Take the rank cutoff so that $0 < d \le D \le 1$. Let $K \ge 1$ absorb the fixed constants in the two cut bounds above, including C_A and $\kappa \le \pi$. Substitution in (A.346) bounds the four local entries, in their displayed order, by

$$Kt_\tau\sqrt{kN^3/\mathcal{D}}, \qquad Kt_\tau\sqrt{k}\, D^2, \qquad Kt_\tau\sqrt{k}\, D, \qquad K\sqrt{t_\tau}(k/T_*)^{1/4}.$$

Indeed $2i - 2 \ge 2$ for a later singleton, $2(i - 1) + s - 1 \ge 1$ for a longer block, and the power of d inside the terminal square root is nonnegative. The Euler-factor product identity of 21.36 together with the level, dimension and mass ties for u, v yields $\prod_\tau t_\tau \le C_t^f$. Thus taking the geometric mean over places replaces t_τ by at most C_t, and $\sqrt{t_\tau}$ by at most $\sqrt{C_t}$. Since Θ^{-1} is the maximum of these geometric means and $D^2 \le D$, taking reciprocals gives

$$\Theta \gg \min\{\sqrt{\mathcal{D}/(kN^3)},\ I/(\sqrt{k}N),\ (T_uT_v/k)^{1/4}\}. \tag{A.347}$$

Choose C_A above the constants for the decay and recursion before taking the rank cutoff. With $\mathcal{D} \ge (kf\log b)^5$, $m \le f \le k$, $b \ge k^{3/2}$, and $T_u, T_v \ge b^{m/2}$, the three terms grow at least as $k^{1/2}, k^{1/2}, k^{1/8}$. Thus the flag-size exponents $(5, \frac{1}{2}, 3)$ allow $\Theta \ge k^{1/16}$.
Lean: `Formalism.L2.SG.ThmSOwn.SizesRowS` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.ThmSOwn`, lines 1008–1013)
Lean: `Formalism.FourierDecayAlt.P.threshold_products` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.CmRingIdempotentUnitRatioIdentity`, lines 241–333)
Lean: `Formalism.FourierDecayAlt.P.theta_lower_bound` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.CmRingIdempotentUnitRatioIdentity`, lines 335–392)

Claim 1. The source data supply an actual realization with the prescribed coefficient, measure, point-weight, and coordinate identifications with the scalar row. It has finite-dimensional coefficient spaces, the normalized towers of Definition 33.82, the base frame and actual terminal-lift compatibility required by Lemma 33.88, the compact-type projections, and the coefficient eigenvalue identities

$$\mathcal{C}_\tau(Y)F_\tau = P_\tau(Y)F_\tau \qquad (Y \in \mathbf{R}).$$

For each cut word $\mathbf{w}$ and binary component, the actual terminal lift is represented on the positive-Gram group locus by a coefficient differential operator. Its total derivative order o and numerator polynomial degree $\kappa_{\deg}$ satisfy

$$o \le C_b|\mathbf{w}|n, \qquad \kappa_{\deg} \le C_b(r+1)o, \qquad n = k + 2r + 1.$$

Its coefficient denominators are powers of the terminal minor, its conjugate, and the Gram polynomial. For every positive Gram lower bound ϵ_0 and derivative order d, its fibre factors have derivatives bounded by $C_{\epsilon_0,d}\|g\|^{3(r+1)d}$ on that locus. These are the terminal-lift bounds used in the cutoff limit below.
Lean: `Formalism.FourierDecayAlt.KB.InDDplus2` (`Formalism.HermitianCurvature.RealInputsClosedAdmissibleSettings`, lines 90–95)
Lean: `Formalism.FourierDecayAlt.C6.FibBound` (`Formalism.HermitianCurvature.BigCellVanishingOfCommute`, lines 92–94)
Lean: `Formalism.FourierDecayAlt.C6.InDDPlus` (`Formalism.HermitianCurvature.BigCellVanishingOfCommute`, lines 96–103)

The binary position is indexed by l, and the cut in the compact interlacing datum is indexed by $l+1$. In particular $1 \le l$, $l+2 \le k$, and the choices at l differ. The coefficient is smooth on the group and has the required bounds for all derivatives with exponent A_0. With $n = k + 2r + 1$, the constants are chosen before the rank cutoff. The number I is a common lower bound for the actual cut widths, and C_c is at least the endpoint-width constant $C_6 > 0$ and every constant required for the levels, slices, compact bounds, and model transfer. The construction gives

$$\begin{aligned} I &\ge C_c\{kN(r+1)^2 + N^2(r+1)^2 + N^3\}, \\ A_0 &\ge C_c(r+1)^4(Nn^2 + n^3). \end{aligned}$$

The fixed constants may also be chosen so that $I \ge 6N^3 + 6N$, which is the grid-richness bound. To record the slice budget explicitly, put

$$A_* = 127(r+1)^3(Nn^2 + n^3), \qquad m_* = 5 + 3(r+1)(N+2)n + Nr.$$

The construction takes $A_0 = (25000 + 21C_*)n^7$ for a fixed $C_* \ge |C_c|$ large enough for its coefficient and lift bounds. The source spectral-gap estimate supplies this exponent, and for every $0 \le j \le r$,

$$9(j+1)(A_*+1) + N\{1 + 3(r+1)\}j + \{6j + 3(r+1)\}m_* \le A_0.$$

This verifies the slice budget at each place. Since $C_c \ge C_6$, the displayed cut bound also implies $C_6\{kN(r+1)^2 + N^2(r+1)^2\} \le I$, the endpoint-width hypothesis. The coordinate transport identifies the tensor product of the actual terminal orbital vectors and its energy with the row vectors and I_N, including its measure factors.

Here the realization is taken from the built-data witness in the source construction at the chosen distinct words, before that witness is packaged into its final arithmetic conclusion. Its frame and representation identifications fix the original and residual data, and its coefficient construction supplies the displayed eigenvalue identity. The cutoff and rank constants are enlarged to the maximum of the constants for the slices, the endpoint model, and the compact estimates. The source construction then gives the same arithmetic carrier, including its terminal tensor and energy identifications. These choices precede the finite prefix sets and the cut parameter.
Lean: `GlobalTheory.CarrierGap.carrierP` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.OffK0Closed`, lines 665–671)
Lean: `Formalism.L2.SG.Pf.CarrierXAll.carrierDataBuilt_of_hyp` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.CarrierXAll`, lines 666–756)
Lean: `Formalism.L2.SG.Pf.CarrierT.sliceBudget_decayThreshold` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.PolynomialLeDecayThreshold`, lines 35–57)
Lean: `Formalism.L2.SG.Pf.CarrierT.poly_le_decayThreshold` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.PolynomialLeDecayThreshold`, lines 59–93)
Lean: `Formalism.L2.SG.Pf.CarrierT.isBuiltPlace_of_compact_test` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.IsBuiltPlaceOfCompactTest`, lines 62–105)
Lean: `Formalism.L2.SG.SpectralTiesReal.carrierP_of_compact_model` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.CarrierInputExistsCompact`, lines 27–54)

Use the finite grid system of this realization. An admitted word has at most N letters at each place. A positive letter uses a base node y; its negative companion uses the actual evaluation point $y+1$. The grids put both points on the cut, make P_τ nonzero there, and impose

$$2 < |y_a - y_b + \delta| \qquad (a \ne b,\ \delta \in \mathbf{Z},\ -2 \le \delta \le 2)$$

on the evaluation points of an admitted word. Consequently every such word, with the positioning of Definition 33.82, satisfies the cut and regularity conditions of Lemma 33.88. Set

$$n_\tau(\mathbf{w}) = \prod_{a\in\mathbf{w}} |P_\tau(y_a)| > 0.$$

For every allowed enlargement the grid construction supplies distinct new positions, both admitted companion words, and nonnegative interpolation weights of total mass one. For a row of degree $d \le 2r$, its sum of absolute Lagrange coefficients is at most $(28N/I)^d$. Thus the word budget is accompanied by admissibility and positive normalization at every step.
Lean: `Formalism.FourierDecayAlt.P.PBCut.cutInside_of_built` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.PBCutInside`, lines 7–36)
Lean: `Formalism.FourierDecayAlt.P.new_letters_admissible` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.AdmissibleAtOfAdmissibleList`, lines 413–413)
Lean: `Formalism.FourierDecayAlt.P.new_letters_admissible_level_one` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.AdmissibleAtOfAdmissibleList`, lines 414–414)
Lean: `Formalism.FourierDecayAlt.P.PBL.sizeAt_pos` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.SizeAtMinusW`, lines 64–83)

Claim 2. Every local normalized kernel used in the finite recursion is positive semidefinite, and the coefficient arrays used to prove this assertion are the actual inserted and sliced arrays.

First fix $\varepsilon > 0$, and put

$$F_{\tau,\varepsilon,R}(g) = \eta_R(g)\chi(G_{\mathrm{loc}}(g)/\varepsilon)F_\tau(g),$$

where $R = 1, 2, \dots$ and η_R is the smooth matrix-norm cutoff equal to one on the radius-R ball and zero outside the radius-$2R$ ball. This coefficient is compact in the sense of Lemma 33.88. For a block point $p = ((s_1, t_1), \dots, (s_d, t_d))$, put

$$\widehat{t}_a = (g/(2\pi))^{\nu_{b_a s_a}} i^{s_a - 1} t_a, \qquad b_a = 1 + \sum_{h<a} s_h, \qquad \nu_{bs} = 2r + 3 - 2b - s,$$

and let $\widehat{p}$ be the resulting standard block point. Write $n_j(p)$ for the prescribed arithmetic point weight of p before standardization, with the convention $\sqrt{n_j(p)} = \sqrt{\max(n_j(p), 0)}$, and write $b(p) > 0$ for the real section factor in the coordinate change. More explicitly, with $\kappa_g = 2\pi/(g\,\mathrm{rd}(E))$,

$$b(p) = \prod_{(s,t)\in p} \prod_{a=1}^{s-1} (\kappa_g^{1-2a}/\pi)^{s-a}.$$

The level measure factor of this same coordinate change, denoted separately from the row factor Γ_1, is

$$D_g = \frac{2}{\mathrm{rd}(E) g^3}, \qquad \Gamma_j^{\mathrm{lev}} = \prod_{\ell=j+1}^{r+1} \frac{\kappa_g^{-4(r+1-\ell)(M_\ell - 1)}}{D_g} > 0, \qquad M_\ell = k + 1 + 2(r + 1 - \ell).$$

For $j \le r$ define

$$U_{\tau,j,\mathbf{w},p} = \frac{\sqrt{n_j(p)}}{b(p)} x_{\widehat{p},\mathbf{w}}[F_{\tau,\varepsilon,R}]$$

at a valid point, and define it to be zero otherwise. The exact normalized local kernel is

$$G^R_{\tau,j,\mathbf{w}}(p, q) = \frac{1}{\Gamma_j^{\mathrm{lev}} n_\tau(\mathbf{w})} \Xi^{\mathrm{coef}}_{j,\mathbf{w}}(U_{\tau,j,\mathbf{w},p}, U_{\tau,j,\mathbf{w},q}). \tag{A.348}$$

It is the partial orbital kernel with its point-weight square roots, its two section factors, and its level measure factor, divided by $n_\tau(\mathbf{w})$. The square-root convention is relevant outside represented terminal support; on its positive-Gram locus it is the usual square root.

Lemma 33.88(a) puts every component of these actual arrays, and of every finite linear combination of them, in the compact coefficient class on the residual group. It realizes them in the residual subspace $\mathcal{K}_0^{(j)}$, with the exact positive-scalar comparison of forms. Equation (A.348) therefore proves positivity on any

finite set of prefix points. Extending the finite family by zero handles invalid points and points outside that set. At $j = r + 1$, use the actual finite projected Gram representation in Lemma 33.88(c), scaling its vector at $\widehat{p}$ by $\sqrt{n_{r+1}(p)}/(b(p)\sqrt{\Gamma^{\mathrm{lev}}_{r+1} n_\tau(\mathbf{w})})$. This proves the same assertion for all terminal words, including the empty word.

Lean: `Formalism.FourierDecayAlt.D.toStd` (`Formalism.HermitianCurvature.StatementD`, lines 65–68)
Lean: `Formalism.FourierDecayAlt.D.bSt` (`Formalism.HermitianCurvature.StatementD`, lines 243–244)
Lean: `Formalism.FourierDecayAlt.D.GamLev` (`Formalism.HermitianCurvature.StatementD`, lines 246–247)
Lean: `Formalism.FourierDecayAlt.P.PBL.nnS` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.AeNeZeroComplex`, lines 30–30)
Lean: `Formalism.FourierDecayAlt.P.PBL.uB` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.FormSmulL`, lines 52–58)
Lean: `Formalism.FourierDecayAlt.P.PBL.gN_uB` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.FormSmulL`, lines 97–129)
Lean: `Formalism.FourierDecayAlt.P.PBASM.gN_psdC` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.PositivityBoundOfActual`, lines 89–123)
Lean: `Formalism.FourierDecayAlt.P.PBASM.gramLoc_top_psd` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.PositivityBoundOfActual`, lines 125–150)

Claim 3. The singleton waves and the mixed lift contractions have positive block kernels with the normalizations of Claim 2, and their interpolation averages are supported on the admitted words of Claim 1.

For a singleton let $j = i - 1$, put $\lambda_j = (g/(2\pi))^{\nu_{j+1,1}}$, and use the wave parameter $s_j(\xi) = 2\pi i\xi/(g\lambda_j)$. The normalized singleton Fourier transform is exactly

$$\widehat{G}^R_{\tau,j+1,1,\mathbf{w}}(\xi; p, q) = \frac{1}{\Gamma^{\mathrm{lev}}_j n_\tau(\mathbf{w})} \Xi^{\mathrm{coef}}_{j,\mathbf{w}}\big(\mathscr{W}_{j,\mathbf{w}}(s_j(\xi)) U_{\tau,j,\mathbf{w},p}, U_{\tau,j,\mathbf{w},q}\big).$$

Lemma 33.88(b) proves compactness and isometry for this wave array. In the Hilbert space obtained from the positive coefficient form, the original arrays and their wave images therefore have a joint Gram matrix. Its two diagonal blocks are $G^R_{\tau,j,\mathbf{w}}$, and its mixed block is the displayed Fourier transform. The singleton integral identity has the same measure normalization and gives $\widehat{G}^R_{\tau,j+1,1,\mathbf{w}}(0; p, q) = G^R_{\tau,j,\mathbf{w}}(p, q)$.

Lean: `Formalism.FourierDecayAlt.P.PBL.hatB_uB_C` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.ExistsNormBoundThreeFamilies`, lines 182–226)
Lean: `Formalism.FourierDecayAlt.P.PBL.block0C` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.GNPsdC`, lines 20–53)

For five new nodes, the singleton-wave contraction uses exactly

$$\mathcal{J}_{\Xi^{\mathrm{coef}}_{j,\mathbf{w}}}\big(\mathscr{W}_{j,\mathbf{w}^+}(s_j(\xi)) U_{\tau,j,\mathbf{w}^+,p}, U_{\tau,j,\mathbf{w}^-,q}\big).$$

The lift contraction instead uses the two untransformed arrays

$$\mathcal{J}_{\Xi^{\mathrm{coef}}_{j,\mathbf{w}}}\big(U_{\tau,j,\mathbf{w}^+,p(s,t)}, U_{\tau,j,\mathbf{w}^-,q(s,t')}\big),$$

where $p(s,t)$ denotes appending the indicated block. Their new binary indices are restricted as in $\mathcal{J}$ of Definition 33.82. The source polynomial identity identifies the first contraction with the leading coefficient of the singleton wave polynomial and the second with the lift polynomial. For a lift whose last block has length s and whose full level is $j = b_0 + s - 1 \le r$, the degree in each evaluation variable is $2(j - s) + s$, and the unscaled leading-coefficient multiplier is

$$\Big((-1)^{s-1}\lambda^{-1}_{b_0 s}(t^{-1} - t'^{-1})\Big)^5, \qquad \lambda_{b_0 s} = (g/(2\pi))^{\nu_{b_0 s}} i^{s-1}.$$

Each mixed pairing is bounded by Lemma 33.88(b). Its factor is $\prod_a q_{\mathbf{w}}(y_a) > 0$, and its two squared norms are those of the actual positive and negative companion arrays. The normalization factors are

$$n_\tau(\mathbf{w}^+) = n_\tau(\mathbf{w}) \prod_a |P_\tau(y_a)|, \qquad n_\tau(\mathbf{w}^-) = n_\tau(\mathbf{w}) \prod_a |P_\tau(y_a + 1)|.$$

After taking a square root and dividing by the original word weight, the ratio contributed by these normalizations is exactly

$$\prod_a \sqrt{\frac{|P_\tau(y_a) P_\tau(y_a + 1)|}{q_{\mathbf{w}}(y_a)}}.$$

The cut estimate bounds this ratio uniformly, before averaging. If $\ell_a(n_a)$ are the Lagrange coefficients in the five rows, the probability weights are exactly

$$\theta(n) = \frac{\prod_a |\ell_a(n_a)|}{\prod_a \sum_h |\ell_a(h)|}.$$

Their denominators are positive, and summing the numerator over all choices gives the denominator. No point-weight or word-weight factor enters θ; those factors are accounted for in the displayed ratio. The argument applies to arbitrary two finite linear combinations of prefixes, so it proves a positive block matrix. More explicitly, let M^+, M^- denote the two θ-averaged companion kernels. In the wave case they are on the original prefixes; in the lift case they use the appended entries t and t', respectively. There are nonnegative scales $X_{j+1,\tau} \le z_{j+1,1,\tau}$ and $\Lambda_{b_0 s,\tau} \le z_{b_0,s+1,\tau}$ for which the two positive block matrices have diagonal blocks $X^5_{j+1,\tau}M^+, X^5_{j+1,\tau}M^-$ and $\Lambda^5_{b_0 s,\tau}M^+, \Lambda^5_{b_0 s,\tau}M^-$, respectively. Their mixed blocks are

$$i^5\xi^5 \widehat{G}^R_{\tau,j+1,1,\mathbf{w}}(\xi;p,q), \qquad \left(\frac{|\lambda_{b_0 s}|(-1)^{s-1}}{\lambda_{b_0 s}}\right)^5 (t^{-1}-t'^{-1})^5 G^R_{\tau,j,\mathbf{w}}(p(s,t),q(s,t')).$$

The factors i^5 and $(|\lambda_{b_0 s}|(-1)^{s-1}/\lambda_{b_0 s})^5$ have modulus one. The scales come from the preceding ratio and Lagrange bounds with the coordinate factors $g\lambda_j/(2\pi)$ in the wave case and $|\lambda_{b_0 s}|$ in the lift case.
Lean: `Formalism.FourierDecayAlt.P.PBL.blockK_waveC` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.BlockKWaveC`, lines 20–176)
Lean: `Formalism.FourierDecayAlt.P.PBL.blockK_liftC` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.LiftLocalPositiveCompactSupport`, lines 21–183)
Lean: `Formalism.FourierDecayAlt.P.wave_local_compact_support_route` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.WaveLocalPositiveCompactSupport`, lines 40–76)
Lean: `Formalism.FourierDecayAlt.P.lift_local_compact_support_route` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.LiftLocalPositiveCompactSupport`, lines 221–267)
Lean: `Formalism.FourierDecayAlt.P.PBL.sizeAt_plusW` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.SizeAtMinusW`, lines 52–56)
Lean: `Formalism.FourierDecayAlt.P.PBL.sizeAt_minusW` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.SizeAtMinusW`, lines 58–62)

At the terminal level and $s \ge 2$, apply the degree-$r+1-s$ interpolation to each actual projected terminal vector. The positive-letter inclusion preserves the old binary components, and the real squared multipliers satisfy the positive one-letter weight comparison. Weighted Cauchy–Schwarz in their finite direct sum gives a positive-semidefinite-order bound of the original terminal diagonal by $(|t|z'_{b_0,\tau})^{10}$ times a probability average of its positive extensions. If $t^{-1}-t'^{-1} \ne 0$, one of $|t|, |t'|$ is at most $2/|t^{-1}-t'^{-1}|$. Applying this bound on that side and ordinary Gram Cauchy–Schwarz gives the terminal inverse-difference scale $2z'_{b_0,\tau}$. Every terminal vector and every extension here is the vector of Lemma 33.88(c).
Lean: `Formalism.FourierDecayAlt.P.PBL.topLocB` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.TopLocalBuiltPlaces`, lines 12–154)

Claim 4. The boundary kernels for longer blocks are Gram kernels of the same normalized local kernel, with all the continuity, integrability, and annular estimates needed for the lattice step.

For $b_0 + s - 1 \le r+1$, $s \ge 2$, set

$$D_g = \frac{2}{\mathrm{rd}(E) g^3}, \qquad j_{b_0 s} = \frac{\mathrm{rd}(E)^{2\nu_{b_0 s}}}{D_g} > 0.$$

For this boundary construction use inner products linear in the second argument, as in Lemma 33.84. By Claim 2 the kernel $G^R_{\tau,b_0+s-1,\mathbf{w}}$ on all local block points has a Hilbert-space realization Z with $G^R_{\tau,b_0+s-1,\mathbf{w}}(P,Q) = \langle Z(P), Z(Q)\rangle$ in that convention. This realization may be taken in the conjugate of the coefficient-form Hilbert space; it is distinct from the actual terminal row tensor vectors. The norm-square difference formula for $Z(P(c)) - Z(P(c_0))$, expressed through four entries of this kernel, shows continuity on each punctured coordinate chart. Define the singleton fibre vector and the root vector by

$$V_{p,x}(c) = j^{-1/2}_{b_0+s-1,1} Z(p(s-1,x^{-1})(1,c)), \qquad B_p(z) = \frac{\phi(z)}{s|z|^{s-1}\sqrt{j_{b_0 s}}} Z(p(s,z)), \quad z \ne 0,$$

with $|\phi(z)| = 1$, and set the root vector to zero at zero. Here $p(s,t)$ means appending the indicated block. These choices give exactly the normalized density and fibre-pairing identities, since the square of each displayed real denominator is the corresponding measure factor.

For the compact coefficients of Claim 2, put

$$\mathcal{U}_{p,x}(D) = |x|^{s-1} V_{p,x}(x^{s-1} D), \qquad \chi_\tau(z) = e^{2\pi i\, 2\,\mathrm{Re}\, z/g}.$$

The character is nontrivial because $g > 0$. The joint-continuity theorem and the lower-level and terminal germ-rate estimates give the phase ϕ above and a constant ω_τ of modulus one such that, on each compact $K \subset \mathbf{C}^\times$,

$$\left\| \mathcal{U}_{p,x}(D) - \sum_{z^s = \omega_\tau D} \chi_\tau(szx) B_p(z) \right\| \le C_{p,K} |x|^{-1}$$

for almost every $D \in K$ and every $x \neq 0$. Expanding the squared norm in kernel values gives the real part of the coefficient form on the weighted inserted and sliced arrays at lower levels, and the finite Gram sum of Lemma 33.88(c) on the weighted projected orbital vectors at the terminal level. In both cases the concrete vectors are combined with the conjugates of the displayed coefficients. The block germ expansion gives the sum over $z^s = \omega_\tau D$ after rescaling. Its remainder is bounded uniformly on the rescaled compact determinant set by a constant times the absolute value of the short-block parameter, a fixed multiple of x^{-1}. Applying the lower-level form estimate or the terminal Gram estimate gives the displayed bound.

Lean: `Formalism.FourierDecayAlt.P.FormNormSumIdentities.norm_sq_sum_eq_form` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.NormSumLeOfForm`, lines 37–44)
Lean: `Formalism.FourierDecayAlt.P.GermRateSplit.germ_core` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.IsCompactRoots`, lines 30–261)

Measurability follows from continuity off zero, which also gives local square-integrability of $s|z|^{s-1} B_p(z)$ on compact subsets of $\mathbf{C}^\times$. At the step of induction on block length, the preceding length has a uniform bound for its diagonal at every nonzero last entry. The singleton integral identity gives $V_{p,x} \in L^2$ and turns that bound into a uniform squared fibre-norm bound. The displayed rescaling preserves this norm, so $\mathcal{U}_{p,x}$ satisfies the same bound. The initial uniform bound follows from singleton density decay. The inductive step uses the preceding bound, the lift estimate, and this boundary construction; the letter budget decreases by five. Thus the hypotheses of Lemma 33.84(a) hold for the actual normalized kernel at each step. Its root-map Jacobian argument gives global square-integrability of $s|z|^{s-1} B_p$. It gives vectors whose pairings are U, Q, P_b, with joint block kernel positive semidefinite, and

$$\lim_{|x|\to\infty} P_b(p, x; q, x+\sigma) = \lim_{|x|\to\infty} Q(p, x; q, x+\sigma) = \lim_{|x|\to\infty} Q^*(p, x; q, x+\sigma) = \widehat{D}_{pq}(\sigma), \tag{A.349}$$

where

$$D_{pq}(c) = G^R_{\tau, b_0+s-1, \mathbf{w}}(p(s,c), q(s,c)), \qquad \widehat{D}_{pq}(\sigma) = j_{b_0 s}^{-1} \int_{\mathbf{C}} D_{pq}(c) e^{2\pi i\, 2\,\mathrm{Re}(\sigma c)/g}\, dc.$$

The fibre construction and induction prove these assertions simultaneously for every finite prefix family; the common Hilbert space is obtained from the kernel on all local points.

Lean: `Formalism.FourierDecayAlt.P.GermFibreKernels.fibres_of` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.GermPartsCompactSupport`, lines 111–224)
Lean: `Formalism.FourierDecayAlt.P.kernel_continuity_built` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.LevelFormsJointContinuity`, lines 450–450)
Lean: `Formalism.FourierDecayAlt.P.germ_rate_bound` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.GermRateBound`, lines 4–5)
Lean: `Formalism.FourierDecayAlt.P.GermFibreKernels.germPartsC_of` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.GermPartsCompactSupport`, lines 267–283)
Lean: `Formalism.FourierDecayAlt.P.PBC.locSup_all` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.LocSupStep`, lines 224–242)
Lean: `Formalism.FourierDecayAlt.P.PBC.germ_global` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.ImageProdLoc`, lines 55–128)

Tensor these vectors over the places and with Gram vectors of a positive semidefinite prefix majorant. A finite sum over prefixes gives the shorter vectors $u(x)$ and boundary vectors $b(x)$. Their joint block kernel is positive semidefinite, and the contracted density $D(c)$ is nonnegative, integrable, and continuous where every coordinate is nonzero. Extend it by zero on the coordinate hyperplanes; positivity makes this extension lower semicontinuous. This construction introduces no new word. For the ordinary dual lattice $L^\vee$, the Fourier transform satisfies

$$\widehat{D}(\sigma) = \Big(\prod_\tau j_{b_0 s} \Big) \widehat{D}_0(g\sigma).$$

The inverse-difference bounds of Claim 3 give a summable bound $F(\sigma)$ on $|\langle u(x), u(x+g\sigma)\rangle|$. The invariant mean of the residual Gram pairing of $u-b$ is positive definite. Equation (A.349) expresses the sum of this positive-definite function and $\widehat{D}_0$ as the mean of the shorter pairing. Lemma 33.84(b) therefore gives, for every finite $Y_0 \subset c_0 + L$,

$$\operatorname{covol}(L) \sum_{c\in Y_0} D(c) \le \Big(\prod_{\tau} j_{b_0 s}\Big) \sum_{\sigma\in L^\vee} F(\sigma). \tag{A.350}$$

Lean: `Formalism.FourierDecayAlt.P.PBC.herglotz_germ` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.PBCStep`, lines 97–161)
Lean: `Formalism.FourierDecayAlt.P.PBC.germ_step` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.GermStep`, lines 16–156)

On square lattices of spacing R_0, the degree-five majorant satisfies

$$R_0^{-2f} \sum_{\sigma \text{ dual}} M_z(g\sigma) \le R_0^{-2f} \prod_{\tau} C\{1 + (R_0 z_\tau/g)^2\}, \qquad M_z(l) = \prod_{\tau} \min\{1, (z_\tau/|\tau(l)|)^5\}. \tag{A.351}$$

The factor in M_z is defined as one when $\tau(l) = 0$. Lemma 33.84(c) supplies this estimate. Restoring the section factors gives

$$\mathrm{vol}_{b_0,a} = \frac{(\prod_\tau z_{b_0,a,\tau})^2}{N(g^{-1}\mathfrak{d}^{-(2r+3-2b_0-a)-1})} \le \Theta^{-2f}. \tag{A.352}$$

The singleton integral starts the induction. At each longer length use (A.350) on square lattices and then let R_0 increase. At its last step use the arithmetic period lattice $L_{b_0 s}$; its covolume is $\prod_\tau j_{b_0 s}$ by the coordinate and ideal normalization of Lemma 33.107. Thus a length-s coset sum has multiplier at most

$$C_4^{fs} \prod_{a=1}^{s-1} \mathrm{vol}_{b_0,a}$$

times a probability average of preceding diagonal energies. The positivity required in this induction is Claim 2, and the positive joint boundary kernel is the one just constructed.
Lean: `Formalism.FourierDecayAlt.P.PBC.sumFormConcrete_of` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.PBCFinal`, lines 107–217)

Claim 5. The normalized kernel positivity, mixed block inequalities, singleton integral identity, and coset-sum estimate of Claims 2–4 hold for the original cut coefficient $F_{\tau,\varepsilon} = \chi(G_{\mathrm{loc}}/\varepsilon)F_\tau$.

At fixed $\varepsilon > 0$, the compact coefficients $F_{\tau,\varepsilon,R}$ converge in every partial word pairing used here as $R \to \infty$. The source bounds on all coefficient derivatives, the terminal-lift degree bounds, and the fixed letter budget imply this convergence at every pair of block points. The uniform decay bound supplies an integrable majorant for the singleton density integrand, unchanged by a Fourier phase of modulus one. Hence the singleton integral and normalized Fourier identities pass to the limit. For each finite matrix, its quadratic forms are limits of the nonnegative quadratic forms in Claim 2, so its limit is positive semidefinite. The same observation applies to each positive block matrix in Claim 3. The interpolation distributions and their bounds are independent of R. The coset-sum estimate concerns finite sums and passes to the limit with these distributions and constants. This proves the assertion for every admitted word and every finite prefix set. In particular the passage from compact representatives to the original cut kernels uses closedness of positive semidefiniteness under entrywise limits.
Lean: `Formalism.FourierDecayAlt.P.limit_of_module_forms` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.LimitOfModuleForms`, lines 10–32)
Lean: `Formalism.FourierDecayAlt.P.integral_formula_passes_to_limits` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.IntegralFormulaPassesToLimits`, lines 8–83)
Lean: `Formalism.FourierDecayAlt.P.PBASM.psd_of_tendsto` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.BuiltBoundsCompactOfParts`, lines 119–134)
Lean: `Formalism.FourierDecayAlt.P.PBASM.core` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.BuiltBoundsCompactOfParts`, lines 136–249)
Lean: `Formalism.FourierDecayAlt.P.built_bounds_compact` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.BuiltBoundsCompact`, lines 6–6)

Fix a finite set S of scalar-partner rows. For each block start b_0 and length s, choose a finite $S_{b_0 s} \subset E^\times$. Let X_i consist of the lists $p = ((s_1, t_1), \dots, (s_d, t_d))$ with $\sum_a s_a = i$ and $t_a \in S_{1+\sum_{h<a} s_h, s_a}$. Every finite terminal set belongs to such a tower. For an admitted global word $\mathbf{w} = (\mathbf{w}_\tau)_\tau$, let

$$H_i(\mathbf{w}; p, q) = \prod_\tau G_{\tau,i,\mathbf{w}_\tau}(p_\tau, q_\tau).$$

The product uses the actual normalized cut kernels of Claim 5. For a positive semidefinite arithmetic kernel B_0 on a finite prefix set X, put

$$[B_0, H]_X = \mathrm{Re} \sum_{p,q \in X} B_0(p,q) H(p,q).$$

Claim 6. This contraction is nonnegative and monotone in B_0 for positive-semidefinite order. Its square root satisfies the triangle inequality when a vector sum is split by last-block length.

Each local kernel is positive semidefinite by Claim 5. Tensor its Gram vectors over the places, and tensor once more with Gram vectors for B_0. The contraction is the squared norm of their sum. This proves nonnegativity and the triangle inequality. Apply the same argument to $B_1 - B_0$ to prove monotonicity whenever $B_0 \preceq B_1$. It applies to every companion word in the interpolation distributions, since Claim 1 proves its admissibility and positive word weight, and Claims 2–5 prove its kernel identity and positivity.

Use the arithmetic flag kernels K_i, with $K_0 = 1$, and scale the terminal kernel to $K_{r+1} = e_1 K_{\mathrm{top}}$, where $e_1 = \max(1, s_d)$ and s_d is the represented-density supremum of Lemma 33.79. For a fixed last block (s,t), their local coset majorants satisfy $C_{\tau,\bar{t}} \succeq A_{\tau,t} \succeq 0$ and $\sum_{\bar{t}} C_{\tau,\bar{t}} \preceq B_\tau$. These are the orbital Gram kernels obtained by summing presentations over a full residue class; restricting the presentations gives the diagonal kernel, and summing distinct residue classes gives a sub-sum of the preceding kernel. Their products, with the prescribed positive arithmetic normalizations, give majorants Θ_t with

$$\Theta_t \succeq K_i|_{t,t}, \qquad \sum_{t \bmod L_{b_0 s}} \Theta_t \preceq \eta_i K_{i-s}, \quad \eta_i = 1 \ (i \le r), \quad \eta_{r+1} = e_1.$$

Here the sums use any finite set of distinct cosets, and the multiplier e_1 is the same one used in the terminal kernel scaling. Positive-semidefinite order survives the products by telescoping their difference. Distinct global cosets have distinct tuples of local cosets, so completing a finite set of tuples to the corresponding product can only increase its Gram kernel. Claim 6 now contracts these actual arithmetic comparisons with the actual analytic kernel.

Lean: `Formalism.FourierDecayAlt.P.DensitySumsAndDomination.domAfterOfLocal` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.DensitySumsAndDomination`, lines 112–142)

For a singleton, the zero Fourier coefficient is the preceding energy by Claim 3. The nonzero-frequency estimate and Lemma 33.107 cost at most $2^f e$, where

$$e = (C_1 \Theta^{-5/2})^f. \tag{A.353}$$

For distinct t, t', the inverse-difference estimate costs at most $3^f e$; the three choices at a place are the original word and the two companion words. The arithmetic support is in the inverse-difference ideal, and $t' \mapsto t^{-1} - t'^{-1}$ is injective for fixed t. Thus the singleton multiplier is at most $\eta_i(1+3^f e)(1+2^f e)$. The top singleton has only diagonal arithmetic support. For $s \ge 2$, Claim 4 and the same off-diagonal estimate give

$$\eta_i (1+3^f e) C_4^{fs} \prod_{a=1}^{s-1} \mathrm{vol}_{b_0,a}.$$

All the diagonal energies averaged here are nonnegative, all averaging coefficients are nonnegative, and their total mass is one. These facts follow from Claims 1, 5, and 6 at each reachable word.

Lean: `Formalism.FourierDecayAlt.P.singleBlockStep` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.FormDomination`, lines 115–195)

Lean: `Formalism.FourierDecayAlt.P.schurRecursion_true` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.BlockStartPoints`, lines 171–326)

Claim 7. At the signed level-zero endpoint every normalized admitted word has the required cut limit, and the empty terminal word agrees with the row vectors on a fixed finite tower for all sufficiently small positive cuts.

The actual coefficient realization of Claim 1 supplies its compact-type scalar model, including the intertwining of the coefficient word fold with its cutoff-energy form. The model has the decay and adjoint estimates through order $2N+2$, the spectral polynomial P_τ, and the same admitted words and word weights

$n_\tau(\mathbf{w})$. Its cutoff scale is $\sigma_\tau\varepsilon$ with $\sigma_\tau > 0$, and its energy comparison scalar is $C_\tau > 0$. The cut-width inequality retained in Claim 1 gives, for every admitted local word,

$$\lim_{\varepsilon\downarrow 0}\frac{\mathscr{R}^{\mathrm{arith}}_{\tau,0,\mathbf{w}}(F_{\tau,\varepsilon};[],[])}{n_\tau(\mathbf{w})}=J_\tau,\qquad J_\tau=\frac{C_\tau}{\Gamma_0^{\mathrm{lev}}}\int\sum_\rho|j_{\tau,\rho}(q)|^2\,d\mu_\tau(q)\ge 0.$$

The numerator here is the actual coordinate-normalized partial word pairing before multiplying by its point-weight factors. At the empty point these factors are one. The equality follows from the scalar-model cutoff theorem and the actual level-zero residue identity. In particular the signs in the word fold and the evaluation at $y+1$ for negative letters are retained. The energy identification in Claim 1 is $\prod_\tau J_\tau = I_N$. The admitted global word set is finite, so these convergences hold uniformly over the finite set needed by the recursion.
Lean: `Formalism.FourierDecayAlt.P.CutMP.SmoothTransferRegularity.model_place_built_holds` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.TransferSmoothAtBuiltPlace`, lines 9–9)
Lean: `Formalism.FourierDecayAlt.P.lemma_E_of_model` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.TheoremPIntegralOfPlaces`, lines 166–197)
Lean: `Formalism.FourierDecayAlt.P.lemma_E_of_residue` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.ComponentFamilySettingOfSetting`, lines 1776–1777)
Lean: `Formalism.FourierDecayAlt.P.RealInputsCompactSupport.real_inputs_closed_compact_support_of` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.RealInputsClosedCompactSupportOf`, lines 75–116)

For the terminal empty word the finite Gram vectors of Lemma 33.88(c), after the point and section normalization of Claim 2 and the tensor product over the places, are the row vectors V_p under the prescribed tensor isometry. On a represented terminal fibre its Gram value is constant. If it is positive, $\chi(G/\varepsilon)=1$ for all sufficiently small $\varepsilon>0$; if the point is invalid or has no positive represented fibre, both vectors are zero. For a finite terminal set one cut works for every entry. Therefore

$$H_{r+1}([];p,q)=\langle V_q,V_p\rangle\qquad(p,q\in X_{r+1})$$

for all sufficiently small cuts, with the row inner product linear in the second argument. The actual terminal tensor identification in Claim 1 gives this equality with the row vectors, including their coordinate and measure factors.
Lean: `Formalism.FourierDecayAlt.P.gram_at_built_place` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.TheoremPIntegralOfPlaces`, lines 703–730)
Lean: `Formalism.L2.SG.Pf.Carrier.propGram_of_vtTensor` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.CarrierPropGram`, lines 11–27)

Apply Lemma 33.79 to these terminal Gram vectors and the finite rows in S, using the already scaled terminal kernel $K_{r+1}=e_1K_{\mathrm{top}}$. Take square roots in the preceding block estimates, sum over last-block lengths by Claim 6, and divide by $\prod_{j\le i}\sqrt{\eta_j}$. The skipped factors are at least one. If R_i denotes the resulting uniform bound over admitted words in the remaining budget, increasing a fixed constant C gives

$$R_i\le(1+h)R_{i-1}+\sum_{s=2}^{i}h^{s-1}R_{i-s},\qquad h=(C/\Theta)^f.\tag{A.354}$$

A singleton uses at most ten letters in the combined estimates, and a block of length s uses at most $5(s+1)$; the total budget is $N=20(r+1)$. The grid construction of Claim 1 applies at every extension. Claim 7 bounds the starting energies when the cut tends to zero on the fixed tower. The finite recursion has total error $\delta_s=C(r+1)(C/\Theta)^f$, after a fixed enlargement of C. The size estimates and the rank cutoff give $0\le\delta_s\le b^{-7}$. Now enlarge the finite tower. Absolute convergence of each row bounds its omitted vector tail by $\sum_{p\notin X_{r+1}}|c_{\rho p}|\|V_p\|$, which tends to zero. Consequently

$$0\le\delta_s\le b^{-7},\qquad\sum_{\rho\in S}\Big\|\sum_p c_{\rho p}V_p\Big\|^2\le(1+\delta_s)e_1I_N.\tag{A.355}$$

Lean: `Formalism.FourierDecayAlt.P.recursion_with_densities` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.RecursionWithDensities`, lines 1162–1162)
Lean: `Formalism.FourierDecayAlt.P.GrowthPolynomialOperator.t_MainRow` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.GrowthOfBasic`, lines 1070–1183)

At a finite place of residue cardinality q the logarithm of a local represented density is at most $2q^{-(k-m)}$, and the normalizer is at least $\exp(-2q^{-(k-m)})$. There are at most f places above a rational prime. For $j = k - m \geq 2$, the integral comparison gives $\sum_{n\geq 2} n^{-j} \leq 3 \cdot 2^{-j}$. Thus

$$\log e_1 \leq 12 f\, 2^{-(k-m)}. \tag{A.356}$$

Lean: `Formalism.L2.SG.Pf.DensLoc.densLocBound_at` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.DensLoc`, lines 134–202)

For the mass conversion let K_V be the orbital quadratic form, c, c' the individual unfolding constants, and ρ_0 the normalized row mass. The contribution is $\Gamma_1 \rho_0 K_V$. Unfolding by scalar-partner classes and using (A.355) gives $cc' K_V \leq n_0 A I_N$, with n_0 the count normalizer and $A = (1+\delta_s)e_1$. Write V_{ZZ} for the real coordinate volume, V_{loc} for the unit-level volume with its real factor, C_∞ for the Jacobian, V_B for the stabilizer volume factor, M for the product mass and μ_K for the initial level measure. The measure and row identities are

$$I_N = \frac{V_{ZZ} I_{\mathrm{orb}}}{V_{\mathrm{loc}} C_\infty}, \qquad I_{\mathrm{orb}} \leq \deg^{-1}, \qquad n_0 \rho_0 = \frac{V_B C_\infty M}{V_{ZZ} \mu_K}. \tag{A.357}$$

Here I_{orb} is the orbital integral in isometric column coordinates, and deg the formal degree. The middle bound is the squared coefficient orthogonality with this degree. Ordinary places contribute $\mu_K^2/(\mu_S \mu_{S'})$ to M/V_{loc}, where $\mu_S, \mu_{S'}$ are the two stabilizer measures. At the level places the natural measures differ from the typed ones by the finite ratio $|\mathrm{GL}_k(\mathbf{F}_q)|/|U_k(\mathbf{F}_q)|$ for the level and its stabilizers. With double primes indicating the typed measures this gives

$$\frac{M}{V_{\mathrm{loc}} \mu_K} = \frac{\mu_{K''}}{\Gamma_1 \mu_{S''} \mu'_{S''}}, \qquad c^{-1} = \frac{d_u^S \mu_{S''}}{V_{H,\infty}}, \qquad (c')^{-1} = \frac{d_v^S \mu'_{S''}}{V'_{H,\infty}}.$$

The two real column volumes satisfy $V_{H,\infty} V'_{H,\infty} = V_B V_G$, where V_G is the real group volume factor. Substitution into (A.357) gives

$$\Gamma_1 \rho_0 K_V \leq \frac{\Gamma_1 A V_B}{cc' \deg} \frac{M}{V_{\mathrm{loc}} \mu_K} = \frac{A d_u^S d_v^S \mu_{K''}}{\deg V_G} = A \Delta_{u,v}, \qquad D^\star = \deg V_G / \mu_{K''}.$$

Lean: `Formalism.FourierDecayAlt.P.GrowthPolynomialOperator.t_MainRow` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.GrowthOfBasic`, lines 1070–1183)

The compact analytic closure is proved in `Formalism.FourierDecayAlt.P.RealInputsCompactSupport.real_inputs_closed_compact_support_of`, and the complete row estimate, including the arithmetic recursion and mass conversion, is proved in `Formalism.FourierDecayAlt.P.GrowthPolynomialOperator.t_MainRow`. □

Proof of 33.112

Proof. At a place over g_j put $q = g_j$, $G = \mathrm{GL}_k(\mathbf{F}_q)$, and let U be its upper unitriangular subgroup. The primes split completely, so the residue fields are prime fields. Let $\xi(u) = \psi(\sum_{i<k} u_{i,i+1})$, where the selected additive character ψ is nontrivial. Extend ξ by zero off U. The local factor in the finite Gram kernel is the induced character

$$\gamma(x) = |U|^{-1} \sum_{y \in G} \xi(yxy^{-1}), \qquad \gamma(1) = |G|/|U|.$$

On a free level-group orbit there is exactly one group element carrying one specified index to another. In orbit coordinates the matrix is therefore convolution by γ, up to complex conjugation. Entries between different orbits vanish.

On an irreducible summand with character θ, central convolution by $\overline{\gamma}/\gamma(1)$, using counting measure, has scalar

$$\frac{\sum_{x\in G} \overline{\gamma(x)} \theta(x)}{\gamma(1)\theta(1)} = \frac{|U|}{\theta(1)} \dim \mathrm{Hom}_U(\xi, \theta|_U).$$

Complex conjugation of the convolution kernel permutes the irreducibles and preserves this list of real eigenvalues. To compute the multiplicity, factor each monic degree-k polynomial with nonzero constant

term. By 32.23 its irreducible factors with repetitions specify a multiset of cuspidals. By 32.24 its induced product contains a generic constituent, and 32.25 gives different constituents for different multisets. By 32.29 this accounts for $q^k - q^{k-1}$ positive multiplicities. Their squares, summed over all irreducibles, total exactly $q^k - q^{k-1}$ by 32.26. Every positive multiplicity is one.

For the dimension bound, take a representation containing the specified Whittaker vector. In rank $a+1$ restrict to the abelian last-column subgroup $\mathbf{F}_q^a$. The restriction of the Whittaker character to this subgroup is nonzero. The $q^a - 1$ nonzero characters form one $\mathrm{GL}_a(\mathbf{F}_q)$-orbit, and their eigenspaces have the same dimension. The stabilizer of the specified character contains the rank-a mirabolic subgroup. Its upper unitriangular subgroup acts on the original vector by the rank-a Whittaker character. Repeat the last-column decomposition inside this eigenspace. At rank one the dimension is at least one. Multiplication gives $\theta(1) \ge \prod_{a=1}^{k-1}(q^a - 1)$. This argument also applies in the dual realization of the Whittaker functional.
Lean: `PI.GGMirBoundW.mirabolicBound_of` (`Formalism.PeriodIntegrals.UnitaryGroupMassFormula.MirabolicBound.Development`, lines 112–127)

Since $|U| = q^{k(k-1)/2}$, the normalized local eigenvalues are bounded by

$$\prod_{a=1}^{k-1}(1-q^{-a})^{-1} \le \left(1 - \sum_{a\ge 1} q^{-a}\right)^{-1} = 1 + \frac{1}{q-2} \le 1 + \frac{3}{q}.$$

Here $q \ge 3$ by the level hypotheses. Take the tensor product over the f places above each distinguished prime. All eigenvalues are nonnegative. Restoring the product of the identity values, Γ_1, gives $0 \preceq G_{\mathrm{fin}} \preceq P_g \Gamma_1 I$.

For a finite term set the norm expansion contracts G_{fin} against the vectors $w_i(E_1)$. In an orthonormal basis of their span, apply the matrix inequality to the coefficient vector for each basis coordinate and sum. The result is $P_g \Gamma_1 \sum_i \|w_i(E_1)\|^2$. Absolute summability gives convergence of both the orbital vector sums and the operator sum over finite subsets of E_1. Continuity of the squared norms gives the same bound on E_1. □

Proof of 33.113

Proof. Let G be the unitary group of the selected row, F the real base field, and $\Gamma = G(F)$. Write $S, S' \subset \Gamma$ for the rational stabilizers. The kernel on column vectors is the pairing of their Poincaré series divided by $\sqrt{c_u c_v}$, where c_u, c_v are the unfolding constants.

For compact $\Omega, C' \subset G(\mathbb{A})$, the translate multiplicity into Ω at $c(g,1)$ is at most $C\|g\|_\infty^N$ for $c \in C'$, with N as in 33.25. The level charts bound the rational translation parameters by archimedean norm balls with finite components in a fixed compact set. The stabilizer fundamental domain and the level ties express the resulting absolute integral as a finite-level integral of the real section. Shifting the representative through its stabilizer gives the quotient weight of 33.25, which gives finiteness. The compact reproducer of that lemma bounds the absolute section by a constant times its local average. Folding this bound over the compact neighbourhood gives L^2 bounds for both absolute Poincaré series. Their product is integrable by Cauchy–Schwarz.

Unfold the first series over $S\backslash G(\mathbb{A})$ and partition the second by double cosets in $S\backslash\Gamma/S'$, indexed after inversion. The translates for a tile a fundamental domain for $S \cap aS'a^{-1}$. The integral pairs the first section at x with the second at $a^{-1}x$. Tonelli gives absolute unfolding. Partition the finite components by the row lattices and stabilizers. The finite-character matrix and real orbital integral give $\mathrm{op}(t)$. The sharp coordinates span the column spaces, so the term pairings recover the kernel form and absolute convergence permits grouping by cells.
Lean: `Formalism.L2.SG.Pf.Slab.dcSingleE` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.DoubleCosetSingleUnfolding`, lines 31–75)
Lean: `Formalism.L2.SG.TermOp.KExpCut.kExpansion_of_steps` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.KExpCut`, lines 773–789)
Lean: `Formalism.L2.SG.Closed.KExpansionC_holds` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedVolumeStatements.UnipotentInLevelSubgroup`, lines 63–66)

In the identity cell conjugate the stabilizers into common coordinates and lift to the compatible level. The local type pairing is zero for distinct words. Summing in the lifted coordinates gives that the sum over the identity cell is zero in the continuous Hilbert–Schmidt embedding. Other terms with zero Gram value have zero orbital integral if irrelevant. For a relevant zero Gram term, the orbital integral is covariant for

the real common pointwise stabilizer of the first frame and the translated second frame. Let W be the span of these two frames at a real place v_0. By Claim 1 of Proposition 33.18 (whose zero-Gram normal form holds verbatim for distinct words $u \neq v$ upon normalizing the two frame vectors), we may subtract vectors of the respective isotropic flag spaces from the two orthonormal definite frames, so that the resulting frames (e_a) and (e'_a) lie in $W^{\perp}$. They remain orthonormal because the subtracted vectors are isotropic and perpendicular to the corresponding original definite frame.

Put $K_{ab} = \langle e_a, e'_b \rangle$ in Hermitian coordinates. To see $K^*K = I$, use the first frame's hyperbolic partners to subtract from each e'_b an isotropic first-flag vector and project to the span of the first definite frame. The coefficients of the projection are K_{ab}. Replacing the first definite vectors by e_a changes the residual vector within the first isotropic flag. The residual $e'_b - \sum_a K_{ab} e_a$ lies in the first isotropic flag space X by this projection, and in $W^{\perp}$ because both corrected frames lie there. Claim 1 of Proposition 33.18 gives $X \cap W^{\perp} = X \cap X'$, where X' is the second flag space. Its pairing with e'_a vanishes. Taking inner products with e'_a gives $K^*K = I$.

For each $M \in U(k)$, extend its action on the positive span of (e_a) by the identity on the orthogonal complement. This fixes W pointwise and so gives an element h of the common stabilizer at v_0; because $h|_W = \mathrm{id}_W$ fixes every Bessel flag vector and its anisotropic character partner on both sides, its unipotent components inside both Bessel stabilizers have trivial Bessel character values $\psi_{z_1}(n_1) = \psi_{z_2}(n_2) = 1$. Take identity components at the other real places. The two actions on definite type coordinates are M and K^*MK, since h fixes the isotropic corrections and acts on the second corrected frame by its matrix coefficients against the first. Covariance now yields an intertwiner after taking matrix entries on the tensor factors away from v_0. Conjugation by K identifies the second action with its original $U(k)$-type. The two highest weights at v_0 differ, so every such matrix entry vanishes by Schur's lemma. The orbital integral is zero, and the full operator equals K_0.

Lean: `Formalism.L2.SG.FixedDecomposition.OOt_eq_zero0` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.FixedDecomposition`, lines 744–753)
Lean: `PI.SchurPlaces.Geo.geomB_of` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.SubOneMulVecFix`, lines 207–505)
Lean: `PI.SchurPlaces.schurNFC_of` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.SchurNFCOf`, lines 20–30)

Apply 33.112 in the generic and nongeneric cells. Within each finite-type fibre, different rows are orthogonal. In a same-row pair with first term of nonzero Gram value the $2r+2$ frame vectors of F_0, F_t, $r = r_S$, are linearly independent. The pair space lies in their perpendicular in dimension $k + 2r + 1$, so it has dimension at most $k - 1$. The corank is positive even for identical terms.

Lean: `PI.PairCorankPos.W.corankPosG` (`Formalism.PeriodIntegrals.UnitaryGroupMassFormula.PairCorankPositivity.FinrankOrthAddLe`, lines 122–131)
Lean: `Formalism.L2.SG.Pf.SatGram.corankPos_tO` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.SatGram`, lines 431–443)

Put $\ell = e^{-kf \log k/3}$ and $n = 2^{-(k-2r-5)} e^{C_n m^5 f \log b}$, with $C_n \geq C_4$ of 33.99. The generic scalar contribution is bounded by $A\Delta_{u,v}$ by 33.111. Each loop contribution is bounded by $\ell\Delta_{u,v}$ by 33.106, since its absolute sum is over a subset of the union in that lemma. In the nongeneric cell the higher-corank pairs are in S_{big}. Apply 33.99 to the signed corank-one sum. The source hypotheses give the counting and pair standing conditions, with ties and convergence from the selection. The triangle inequality gives

$$\|K_0\|_{\mathrm{HS}}^2 \leq \left(\sqrt{P_g(A + 2\ell)} + \sqrt{P_g(\ell + n)}\right)^2 \Delta_{u,v}. \tag{A.358}$$

Lean: `GlobalTheory.NGRow.rowNG_of_cut2P` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.NonGenericCorankOnePairsBoundOfSplitFiniteRealRo`, lines 20–36)

Set $z = b^{-7}$. The thresholds give $k \geq \max(32, 4r + 12, 70 \log b)$, $(kb)^8 \leq g_j$, $f \leq k$, $8C_n m^5 f \log b \leq k$. Then $P_g \leq e^{6z}$. By (A.356), $\log e_1 \leq 12k\, 2^{-k/2-4} \leq e^{-k/5}$. Also $\log n \leq -(k/2+1)\log 2 + k/8 \leq -k/5$ and $\ell \leq e^{-k/5}$. Thus $\log e_1, n, \ell \leq z^2$ and $A \leq (1+z)e^{z^2}$. For $b \geq 16$ the radicands in (A.358) are at most $1 + 11z$ and $3z^2$. Their roots sum to at most $1 + (11/2 + 7/4)z$, whose square is at most $1 + 16z \leq 1 + b^{-6}$. □

Proofs for Chapter 34: Theta correspondence, test vectors and evaluation

Proof of 34.1

Proof. For a finite place w, put $q_w = Nw$. Let $c_w = \chi_w(\varpi_w)$ if χ_w is unramified, and let $c_w = 0$ otherwise. Unitarity gives $|c_w| \le 1$. For real $\sigma > 1$,

$$\sum_w \sum_{h\ge 1} \frac{|c_w|^h}{h q_w^{h\sigma}} \le \frac{[E:\mathbf{Q}]}{1-2^{-\sigma}} \sum_{n\ge 2} n^{-\sigma} < \infty.$$

Indeed there are at most $[E:\mathbf{Q}]$ places over each rational prime p, and their norms are at least p. Exponentiating the logarithmic Euler series gives $L(s,\chi) \neq 0$ for $\operatorname{Re} s > 1$.

The angular character at a place with $m_v \neq 0$ is nontrivial on the unit circle. Thus neither χ nor χ^{-1} is a power of the idelic norm. By Proposition 26.29(5), Tate's global functional equation and local factors give entire completed functions for both characters. Dividing out their archimedean factors gives entire finite L-functions, since $1/\Gamma$ is entire. For the trivial character the same theorem gives a simple pole of ζ_E at one. The pole of its completion at zero is cancelled in the finite function by the reciprocal gamma factor at a complex place [12, Th. 4.4.1 and §4.5].

Lean: `GlobalTheory.CompanionTheory.HeckeFE.heckeTateFunctionalEquation` (`Development.AutomorphicSpectrum.Proofs.GlobalProofs.HeckeTateFunctionalEquation`, lines 254–262)

Suppose $L(1,\chi) = 0$. Then

$$F(s) = \zeta_E(s)^2 L(s,\chi)\overline{L(\bar{s},\chi)}$$

is entire. For $\operatorname{Re} s > 1$ its Euler products give

$$F(s) = \exp\left(\sum_w \sum_{h\ge 1} a_{wh} q_w^{-hs}\right), \qquad a_{wh} = \frac{2 + 2\operatorname{Re}(c_w^h)}{h} \ge 0.$$

Expand the exponent at $s = 2 - t$. For $|t| < 1$ its series converges locally absolutely. Its Taylor coefficients in t are nonnegative, since $q_w^{-h(2-t)} = q_w^{-2h}\exp(th\log q_w)$. The Taylor coefficients of $F(2-t)$ dominate those of the exponent term by term, because in the exponential series the linear term gives the exponent and all other terms have nonnegative coefficients. The Taylor series of the entire function $F(2-t)$ converges at $t = 2$. Summing the dominated coefficients there and interchanging nonnegative sums gives

$$\sum_{w,h} a_{wh} = \sum_{w,h} a_{wh} q_w^{-2h} \sum_{n\ge 0} \frac{(2h\log q_w)^n}{n!} < \infty.$$

Fix any w. For $z = c_w^h$ and $u = \operatorname{Re} z$,

$$h(a_{wh} + a_{w,2h}) = 3 + 2u + \operatorname{Re}(z^2) \ge 2 + 2u + 2u^2 \ge \tfrac{3}{2}.$$

The two series on the left after division by h are summable, whereas $\sum_{h\ge 1} 3/(2h)$ diverges. This proves nonvanishing at one and applies also to χ^{-1}.

Lean: `GlobalTheory.landau_exp_summable` (`Development.AbelianVarietyTheory.Proofs.AdelicLatticeProofs.LandauSummableOfExpDirichletSeriesEntire`, lines 84–293)

Lean: `GlobalTheory.eulerProduct_ne_zero_one_of_entire` (`Development.AbelianVarietyTheory.Proofs.AdelicLatticeProofs.EulerProductNeZeroOneOfEntire`, lines 95–128)

Lean: `GlobalTheory.CMVal.IsHeckeL.ne_zero_nat` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.HeckeLNeZeroNat`, lines 10–21)

Now let $k \le 0$ satisfy the stated inequalities. At $1 - k \ge 1$ the finite value for χ^{-1} is nonzero. Its archimedean arguments $1-k+|m_v|/2$ are positive. By Proposition 26.29(5), the functional equation therefore gives a nonzero completed value for χ at k, since the conductor factor and the root number are nonzero. Each $\Gamma_{\mathbf{C}}(k + |m_v|/2)$ is finite and nonzero by the strict inequality at that place. Division gives $L(k,\chi) \neq 0$, including $k = 0$.

Lean: `GlobalTheory.ThK.Dbl.heckeL_ne_zero` (`Development.WeilBound.ThetaLifts.LatticeBasics.ExistsIsHeckeLOfCompleted`, lines 134–160)

Finally, omitting a finite set S multiplies the continued finite L-function by $\prod_{w\in S}(1-c_w q_w^{-s})$. This identity follows first from the Euler product and then by continuation. At $s=k\geq 1$ every factor is nonzero because $|c_w|q_w^{-k}<1$. □

Proof of 34.2

Proof. Split the variables into the R positive columns and the S negative columns. The source action on their tensor product is the sum of the two actions, which we denote by D_+ and D_-. We use the following polynomials x, y in the respective sets of columns. A determinant of size zero means 1.

	x	y
T	det(positive rows, first P positive columns)^n	det(negative rows, first Q negative columns)^m
T'	det(negative rows, last Q' positive columns)^n	det(positive rows, last P' negative columns)^m

The identities $R=P+Q'$, $S=Q+P'$ ensure that all four minors exist.

We show that xy survives the source relations. Use $h=c$ in the first row of the table and $h=-c$ in the second. Let J be the diagonal source signature matrix. On the negative-column polynomials introduce the action

$$\widetilde{D}_-(Z)=h\,\mathrm{tr}\,Z-D_-(JZ^tJ).$$

For either row, x under D_+ and y under $\widetilde{D}_-$ have the same character on the block diagonal source algebra. Indeed, on the positive and negative row blocks the vacuum weights for D_+ are $R/2,-R/2$, and those for D_- are $-S/2,S/2$. In the first row the minors add $(n,0)$ and $(0,m)$ to these pairs. In the second they add $(0,-n)$ and $(-m,0)$. In both cases the sums are (h,h). Off-diagonal units within either block act by zero on the determinants. The lower left source units kill x under D_+ and y under $\widetilde{D}_-$. In each case the relevant operator on the indicated half of the variables is a sum of second derivatives, one in each row sign, and the minor uses one row sign.

Give the polynomial spaces their positive Hermitian Fischer forms, with orthogonal monomials of squared norm $\alpha!$ in multi-degree α. For both source actions the adjoint of E_{ij} is $J_{ii}J_{jj}E_{ji}$, by the formulas for A_l, B_l. Consequently the map $Ux\mapsto Uy$, for words U in source matrices with the respective actions, defines an intertwiner of the cyclic spaces. To see that it respects every relation, normalize the two starting vectors to have norm one and compute the Gram entry of Ux, Vx by commuting the word U^*V into upper right, block diagonal, lower left order. Terms with a lower left factor on the right vanish. Terms with an upper right factor on the left have zero pairing with x, by adjunction. The remaining terms evaluate in the common block diagonal character. This gives identical Gram matrices for the two cyclic spaces. Rescaling the intertwiner sends the original x to y.

The cyclic space of x is graded by polynomial degree. On the positive columns each off-block source matrix unit is homogeneous of degree 2 or -2, and block diagonal units preserve degree. Orthogonal projection onto the cyclic space therefore exists on polynomials, degree by degree. Its orthogonal complement is invariant by the adjoint identity. Composing this projection with the intertwiner gives a map L on all positive-column polynomials. For the bilinear Fischer form $[\,,\,]$, with the same diagonal monomial entries $\alpha!$, the functional

$$u\otimes v\longmapsto [Lu,v]$$

has source character $h\,\mathrm{tr}$. Indeed the transpose of $D_-(Z)$ for this bilinear form is $D_-(JZ^tJ)$. The functional takes $x\otimes y$ to $[y,y]>0$, since y is nonzero with real coefficients. This proves nonvanishing in both quotients.

We compute the generating compact type in the untwisted polynomial models. For the pair $(\mathrm{U}(b,e),\mathrm{U}(R,S))$ the source vacuum shift is $((R-S)/2,-(R-S)/2)$ on the two blocks. The target shift is $((b-e)/2,-(b-e)/2)$. Subtracting the source shift from the constant weight h gives $(n^{[P]};m^{[Q]})$ on the first side and $((-m)^{[P']};(-n)^{[Q']})$ on the second. The joint-harmonic correspondence gives the following target weights after subtraction of their respective vacuum shifts:

$$\big(n^{[P]},0^{[Q']};m^{[Q]},0^{[P']}\big),\qquad \big(0^{[P]},(-n)^{[Q']};0^{[Q]},(-m)^{[P']}\big).$$

These are the two rectangular cases of [157, Lemma 2.5(1)]. Its length conditions here are $P,Q'\leq R$ and $Q,P'\leq S$. The displayed minors give the joint highest vectors since the upper compact units annihilate

them, and every quadratic contraction differentiates a pair of variables of which at least one is absent. Their products have degrees $Pn+Qm$ and $Q'n+P'm$, respectively, the minimum degrees in [157, Lemma 2.5(2)]. For each pair the source character has a single compact type. The generation statement of Proposition 1.22, or [6, §4, equation (4.5)], therefore gives generation by the image of this joint type. We have proved that its highest vector has nonzero image.

Finally, put $d'=(P'-Q')/2$. Since $d-d'=(R-S)/2$, one has $c+d'=d+n$ and $c-d'=m-d$. Adding the vacuum shifts and the primed trace twist now gives the asserted equality at every coordinate. □

Proof of 34.3

Proof. If $S=0$, then $\mathfrak{p}=0$. Otherwise fix one side. Let d_0 be its minimum source-character degree, computed in Lemma 34.2. All polynomials of degree less than d_0 map to zero because their finite-dimensional source compact modules contain no copy of the source character, and the quotient map transforms by that character. Elements of L have representatives of degree d_0. An off-block target unit acts as the sum of a quadratic multiplication and a quadratic differentiation. Thus $\mathfrak{p}L$ is represented in degree d_0+2.

In that degree take the orthogonal complement, for the Fischer form, of the images of the degree-raising parts of the off-block source units. Those images map to zero. Indeed, on a polynomial of degree d_0 the full off-block source unit acts by zero in the quotient and its lowering part has degree d_0-2 or vanishes. The complement is the joint kernel of the adjoint contractions. Explicitly these contract a positive and a negative source row over the R columns, or over the S columns separately. The complement is target compact stable, and its image contains $\mathfrak{p}L$. Hence each irreducible type of $\mathfrak{p}L$ has a representative of that type in this kernel, by complete reducibility in finite degree.

For such a type of weight μ, subtract the target vacuum shift and, on the primed side, the trace twist. Write $\widehat{\mu}$ for the resulting pair of strings. The harmonic condition on the R columns and on the S columns gives

$$\sum_i |\widehat{\mu}_i| = d_0+2.$$

More explicitly, in one block with t columns and with b_+, b_- copies of the defining and dual column representations, the kernel of the row contractions is the joint-harmonic space for the auxiliary pair with compact member $\mathrm{U}(t)$ and other member $\mathrm{U}(b_+,b_-)$, in this polarization. A shifted column weight there has at most b_+ positive and b_- negative entries. Its homogeneous degree is the sum of their absolute values. To see the degree directly in the joint-harmonic formula, form highest minors of sizes given by the column lengths of the positive partition, and dual highest minors for the negative partition. They use disjoint column indices, at the beginning and end of the ordered column basis, so every contraction vanishes. Their degree is the asserted sum. The compact-type correspondence and multiplicity one exhaust the joint harmonics of that column type. This applies separately in both column blocks, whose degrees add. The bounds on the numbers of entries and the multiplicity assertion are precisely [157, Theorem 2.2(1) and Lemma 2.5(1)], applied with one member compact. For a source of signature (b,e), the two pairs of bounds are (b,e) and (e,b).

Now $\mathfrak{p}L$ is an image of $\mathfrak{p}\otimes L$. The two summands of $\mathfrak{p}$ are a defining representation of one block tensored with the dual defining representation of the other. Thus $\mu=\lambda+e_i-e_j$ is block dominant, with i,j in opposite blocks. This one-box rule also follows by multiplying the Weyl character numerator in each block by $\sum x_i$ or $\sum x_j^{-1}$, since the alternating determinants with two equal exponents vanish, and the others give the indicated dominant changes.

For λ the sum of absolute shifted entries is d_0. To increase it by two, each of the two changed entries must increase its absolute value by one. On the unprimed side all allowed positive entries are already present in each block (P in block 1 and Q in block 2), so any allowed addition is to a positive entry and any allowed subtraction is from a zero entry. On the primed side all allowed negative entries are present (Q' in block 1 and P' in block 2), so any allowed subtraction is from a negative entry and any allowed addition is to a zero entry. In the boundary case $Q=0$, no positive entries are allowed in block 2 on the unprimed side (and no zero entries lie in block 2 on the primed side), eliminating the move adding to block 2 and subtracting from block 1 and leaving only the move adding to block 1 and subtracting from block 2 (with $P>Q=0$ and $P'=S>0$, giving $\lambda_i-\lambda_j=n+2d>0$ directly). Similarly, in the boundary case $P'=0$ (so $Q=S>0$), block 2 has no zero entries on the unprimed side and no negative entries on the primed side, eliminating the move adding to block 1 and subtracting from block 2 and leaving only the opposite move (which occurs

when $Q' > 0$ and gives $\lambda_i - \lambda_j = m - 2d > 0$, and is vacuous if also $Q' = 0$). In every surviving case, both rules add to the initial constant string of one block and subtract from the final constant string of the other. Dominance forces i to be the first position of its block and j the last of its block. The two possible values of $\lambda_i - \lambda_j$ are $n + 2d$ and $m - 2d$. Both are positive, since $P > Q$ by (34.1), and $m > R \geq P$.

On a block of size t, C_K has eigenvalue $\sum_{k=1}^{t} \lambda_k(\lambda_k + t + 1 - 2k)$. This follows by commuting the positive matrix units past the negative ones on a highest vector. Adding at the first position of one block and subtracting at the last of the other changes the sum by $D + 2(\lambda_i - \lambda_j) > D$. □

Proof of 34.4

Proof. Identify the two generating compact types by their highest weight and let $L \subset T \oplus T'$ be the graph of this identification. Write $M = U(\mathfrak{gl}_D)L$ with the componentwise action. Both projections are surjective by Lemma 34.2. We shall prove that the b_0-eigenspace of C_K in M is exactly L.

Filter M by the spaces M_k spanned by words of at most k off-block matrices applied to L. They are finite-dimensional compact modules since $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. Their union is M. For $k \geq 1$ multiplication gives a compact-equivariant surjection

$$\mathfrak{p}^{\otimes k} \otimes L \longrightarrow M_k / M_{k-1}.$$

It is symmetric in the k slots because exchanging two adjacent factors introduces their commutator in $\mathfrak{k}$, which can be moved to the right and gives terms of smaller filtration degree. Choose compact-invariant positive Hermitian forms on the factors and identify the quotient with the orthogonal complement V_k of the kernel.

Let B be the kernel of the action $\mathfrak{p} \otimes L \to \mathfrak{p}L$ inside M. A tensor with its last slot and its L-factor in B maps to zero, and symmetry gives the same conclusion in every slot. Thus V_k belongs to $B^{\perp}$ in each such pair of slots, with the other slots as tensor factors. The diagonal Casimir on that pair, denoted C_{jL} for slot j, has eigenvalues greater than $b_0 + D$ on $B^{\perp}$. Indeed $B^{\perp} \simeq \mathfrak{p}L$, and the latter embeds in the sum of the two first images from Lemma 34.3.

The Casimir on $\mathfrak{p}$ equals D since the trace-form Casimir on the defining or dual defining representation of a block of size t is t. Expanding the diagonal Casimir on the whole tensor gives

$$C_{\mathrm{total}} - b_0 = C_{\mathfrak{p}^{\otimes k}} + \sum_{j=1}^{k} \bigl(C_{jL} - D - b_0\bigr).$$

The first term has nonnegative quadratic form, since $E_{ab}^{*} = E_{ba}$ for the compact action and the Casimir is the sum of $E_{ab}E_{ab}^{*}$. Every summand in parentheses has strictly positive quadratic form on a nonzero vector of V_k, by the preceding slot calculation. Hence M_k / M_{k-1} has no b_0-eigenvector. Since $M_0 = L$, this proves the eigenspace assertion.

A proper submodule U of M intersects L in zero, since L is compact irreducible and generates M. For $u \in U \cap M_k$, projection to the b_0-eigenspace is a polynomial in $C_K|_{M_k}$, by finite-dimensional compact complete reducibility. The projection therefore belongs to $U \cap L$ and vanishes. The sum J of all proper submodules still has zero projection to L, as each of its elements is a finite sum. It follows that J is proper and M/J is simple. The two coordinate kernels are proper submodules of M, so both projections induce surjections $T \to M/J$, $T' \to M/J$. Their equivariance follows from the componentwise action used to define M. □

Proof of 34.5

Proof. Put $d = a + a'$ and take $(n, \chi) = (a, \nu^{-1})$ or (a', ν). On each side average a real polynomial of nonzero quotient image from Lemma 34.2 over the compact source. Use the vacua off T. At $v \in T$ take the indicator of $U = Y_0 + \varpi_v^{e} \operatorname{Mat}_{n,d}(\mathcal{O}_v)$, with $Y_0 = (I_n \ 0)$. If ${}^{t}gU \cap U \neq \varnothing$, comparison of the first n columns gives

$${}^{t}g = (1 + \varpi_v^{e} A)(1 + \varpi_v^{e} B)^{-1}$$

with A, B integral. Such a matrix stabilizes U. Choose e large enough that the determinant character is trivial on this subgroup. The coefficient is zero off the stabilizer by disjoint supports and equals its value at one on

the stabilizer. On the subgroup of elements integral at all finite places outside T ($G_\infty \times \prod_{v\in T} G_v \times \prod_{v\notin T} K_v$), the support is compact and includes an identity neighborhood, and its doubling height equals one. Write $H \neq 0$ for its measure times the coefficient at one.

Double on the source against the constant function. Multiplying the unramified local doubling integrals outside T for the trivial representation [150, §4.2.1] yields the partial Euler product $L_E^T(\chi, z + n/2 - i)$ omitting places above T. Since $\chi = \nu^{\mp 1}$ is ramified at both split places of E above every $v \in T$, every local Euler factor of the primitive Hecke L-function $L_E(\chi, s)$ above T equals 1, so $L_E^T(\chi, s) = L_E(\chi, s)$ identically. This gives

$$Z(z)D_n^T(z) = H \prod_{i=0}^{n-1} L_E(\chi, z + n/2 - i), \qquad D_n^T(z) = \prod_{v\notin T} \prod_{j=1}^{n} (1 - \eta_v(\varpi_v)^{n-j}(Nv)^{-(2z+j)})^{-1}. \tag{A.359}$$

The normalizer converges without zeros for $\operatorname{Re} z > 0$. Continue to $z = (d-n)/2$ by Theorem 34.16. Write m_w for the angular exponent of χ at an infinite place w, so $|m_w| = m_{\alpha,w} - m_{\beta,w} > d$. The arguments $k = d/2 - i$ are integers with $2k \geq -d$, so $k + |m_w|/2 > 0$ at every infinite place. Because (A.359) contains the primitive L-function $L_E(\chi, k)$ with no nontrivial Euler factor omitted, Lemma 34.1 gives a nonzero value at every such integer k, including $k \leq 0$. The definite Siegel–Weil value (34.8) and the doubled product identify this value with a nonzero multiple of the squared norm of the theta integral. This gives a witness on each side. For the primed witness use Lemma 34.9 at v_1 to take a full-rank representative. Absolute convergence for the definite source and source invariance preserve its integral. □

Proof of 34.6

Proof. Enlarge E_{exc} to include all finite places where P is not a single pure-tensor factor in the split model. Fix $v \notin E_{\mathrm{exc}}$ above a rational prime split completely in E, fix all local coordinates of x and P away from v, and fix one map U_j. Realize $\Pi_v = i_{P_{a,a'}}^{\mathrm{GL}_d}(\alpha_v \boxtimes \beta_v)$ on the source opposite to S by its unramified split matching quotient together with the full-rank realization of Lemma 34.9; for $S = V'$ use the unprimed block to read Π_v, and for $S = V$ use the primed block. Since the entire image of U_j lies in R_v, pairing U_j on the opposite local Schwartz space against $\overline{\theta_{\omega_S}}$ in the v-slot defines a distribution in the two local Schwartz arguments. With $d = a + a'$, in the split models the two local arguments of the pairing occupy complementary row blocks of a square d-by-d matrix. Complex conjugation in the reverse lift gives the right covariance $\zeta_v(\det h)|\det h|_v^{-d/2}$ under simultaneous right translation by $h \in \mathrm{GL}_d(F_v)$, where ζ_v is ν_v or ν_v^{-1} according to the chosen side and split embedding. On invertible matrices the distribution is twisted Haar measure and has the same left covariance. For a fixed left translate the defect of this covariance is supported on singular matrices and has the original right covariance. On a singular right orbit a stabilizer can scale a kernel coordinate by ϖ_v. A covariant distribution there pulls back locally to twisted Haar measure on the group. The stabilizer twist must equal the positive quotient modulus. Here $\zeta_v(\varpi_v)|\varpi_v|_v^{-d/2} \notin \mathbf{R}_{>0}$ by Lemma 34.9. Apply this on the row-space charts of each stratum, in descending rank, to kill the defect. On the reverse source block the left character cancels the Weil multiplier. Summing over j and over finite sums of pure tensors for P proves invariance of $T_P u$ under $\mathrm{U}(S)(F_v)$ for every completely split $v \notin E_{\mathrm{exc}}$.

By (34.1), $\dim_E S \geq 2$. Strong approximation for the special unitary group at one good split place reduces constancy to density in the norm-one determinant torus [151]. A character of a proper quotient by the rational points and good local groups pulls back under $z \mapsto z/\bar{z}$ to a nontrivial Hecke character of E, by Hilbert 90. It is trivial at almost all absolute degree-one primes by Galoisness. The quotient of its L-function by ζ_E is regular and nonzero at one, by absolute convergence of the degree-at-least-two factors near one. This contradicts holomorphy of its L-function there [12]. The argument allows exclusion of any fixed finite set. □

Proof of 34.7

Proof. At a ramified split place $v \in T$, we first verify that the finite-support Cartan sum in $P(\phi)(h)$ and the integral in $P'(\phi')(h)$ of (34.2) are well defined and invariant under the source actions. Because ϕ is locally constant and compactly supported, shifting the period exponent of ϕ by $\max(0, -\min_{r,s} v((h^{-t})_{rs}))$ gives integers $N(h) \leq M(h)$ such that $\phi(gX_0h^{-t}) = 0$ whenever $g \in K_a\varpi^{\mathbf{n}}K_a$ has some exponent $n_i <$

$N(h)$, and $\phi((g+\Delta g)X_0h^{-t}) = \phi(gX_0h^{-t})$ whenever $\Delta g \in \varpi^{M(h)}\operatorname{Mat}_a(\mathcal{O}_v)$. Since ν_v is ramified at $v \in T$, pick a unit $u \in \mathcal{O}_v^\times$ with $\nu_v(u) \neq 1$. On any Cartan double coset $K_a\varpi^{\mathbf{n}}K_a$ having some exponent $n_i \geq M(h)$, let $d_i(u) = \operatorname{diag}(1,\dots,u,\dots,1) \in K_a$ have u in position i. Pull the integral over $K_a\varpi^{\mathbf{n}}K_a$ back along $(x,y) \mapsto x\varpi^{\mathbf{n}}y$ to $K_a \times K_a$ with measure $\operatorname{vol}(K_a\varpi^{\mathbf{n}}K_a)\,dx\,dy$ (where dx, dy are normalized Haar measures on K_a). Under right translation $x \mapsto xd_i(u)$ on the first factor, $g = x\varpi^{\mathbf{n}}y$ changes by $\Delta g = x\operatorname{diag}(0,\dots,(u-1)\varpi^{n_i},\dots,0)y \in \varpi^{M(h)}\operatorname{Mat}_a(\mathcal{O}_v)$, leaving $\phi(gX_0h^{-t})$ unchanged while multiplying $w(g) = \nu_v(\det g)|\det g|_v^{D/2}$ by $\nu_v(u)$. Right K_a-invariance of dx therefore gives $\int_{K_a\varpi^{\mathbf{n}}K_a} = \nu_v(u)\int_{K_a\varpi^{\mathbf{n}}K_a}$, forcing the integral to vanish since $\nu_v(u) \neq 1$. Hence only finitely many Cartan double cosets (those with $N(h) \leq n_i < M(h)$ for all i) contribute to $\sum_{\mathbf{n}}^{\mathrm{fin}}$. Group the integrals by $m = v(\det g) = \sum_i n_i$. On the support in a fixed shell each n_i lies between $N(h)$ and $m-(a-1)N(h)$, so the shell integral is finite. It vanishes for $m < aN(h)$ by support and for $m \geq aM(h)$ by double-coset cancellation. Left or right multiplication by a fixed matrix shifts the shell index by the valuation of its determinant. Haar change of variables on each shell and reindexing the finite sum therefore give the transformation factor prescribed by w. The identical support and u-character cancellation argument on $\mathrm{GL}_{a'}(F_v)$ proves finite Cartan support and invariance under $g \mapsto g_0^{-1}g$ and $g \mapsto gD_1$ for $P'(\phi')$. Thus both operators in (34.2) take values in the same normalized parabolic induction: for an upper block triangular p with diagonal blocks A, D_1 of sizes a, a', substituting gA^{-t} on the first side and gD_1 on the second yields the common left multiplier

$$\alpha_v(\det A)\beta_v(\det D_1)|\det A|_v^{a'/2}|\det D_1|_v^{-a/2}.$$

An indicator of a small ball about each standard row block gives at $h=1$ the positive volume of a principal congruence subgroup on which $w=1$. Thus both maps are nonzero. Bernstein's irreducibility theorem [15] gives surjectivity onto the common induction.

At an unramified split place $v \notin T$ (covering every unramified unitary character ν_v, both $\nu_v \neq 1$ and $\nu_v = 1$), we construct the pole-free matching maps p_v, p_v' explicitly by finite convolution against the inverse Euler factor. For $n \in \{a, a'\}$ and $\varphi \in \mathcal{S}(\operatorname{Mat}_n(F_v))$, define the determinant-shell sequence

$$z_m(\varphi) = \nu_v(\varpi_v)^m q_v^{-mD/2}\int_{v(\det g)=m}\varphi(g)\,dg \qquad (m \in \mathbf{Z}),$$

and form the finite inverse Godement–Jacquet Euler polynomial $P_{n,D}(T) = \prod_{j=0}^{n-1}(1-\nu_v(\varpi_v)q_v^{j-D/2}T)$. Because φ is locally constant and compactly supported, $z_m(\varphi) = 0$ for all sufficiently negative m; moreover, shifting φ into $\operatorname{Mat}_n(\mathcal{O}_v)$ and applying the Hermite-normal-form / Hecke recurrence on $\operatorname{Mat}_n(\mathcal{O}_v)$ shows that the discrete convolution $(P_{n,D} * z(\varphi))_m$ vanishes for all sufficiently large m. Hence the finite sum

$$\zeta_{n,D}^{\mathrm{unr}}(\varphi) = \sum_{m\in\mathbf{Z}}(P_{n,D} * z(\varphi))_m$$

is well defined for every unramified ν_v (including $\nu_v = 1$, with no pole at $s = D/2$), satisfies $\zeta_{n,D}^{\mathrm{unr}}(\mathbf{1}_{\operatorname{Mat}_n(\mathcal{O}_v)}) = 1$ by direct evaluation on the integral-matrix vacuum, and on the primed side (where $D/2 > a'-1$ gives absolute convergence) equals $P_{a',D}(1)$ times the convergent Godement–Jacquet zeta integral. Define

$$p_v(\phi)(h) = \beta_v(\det h)|\det h|_v^{-a/2}\zeta_{a,D}^{\mathrm{unr}}(g \mapsto \phi(gX_0h^{-t})), p_v'(\phi')(h) = \alpha_v(\det h)|\det h|_v^{a'/2}\zeta_{a',D}^{\mathrm{unr}}(g \mapsto \phi'(gY_0h)).$$

By left and right $\mathrm{GL}_n(F_v)$-equivariance of $\zeta_{n,D}^{\mathrm{unr}}$, both maps kill their respective local source differences and take values in the same normalized parabolic induction $\Pi_v = i_{P_{a,a'}}^{\mathrm{GL}_D}(\alpha_v \boxtimes \beta_v)$, which is irreducible for all unitary unramified α_v, β_v (including $\nu_v = 1$) by Bernstein's unitary irreducibility theorem [15]. For every $h \in K_v = \mathrm{GL}_D(\mathcal{O}_v)$, right multiplication by h preserves the integral-matrix indicators s_v, s_v', giving $p_v(s_v)(h) = 1 = p_v'(s_v')(h)$; hence $p_v(s_v) = p_v'(s_v') \neq 0$, and irreducibility makes both p_v and p_v' surjective onto Π_v.

Lean: `PositivityTheory.ThK.SplitAsm.finite_ramified_of_bernstein` (`Development.WeilBound.CoreStatements.CompactAveragingSplitting.EigenvectorLifting.SplitTop`, lines 14–16)

Lean: `PositivityTheory.ThK.SplitAsm.finite_split_holds` (`Development.WeilBound.CoreStatements.CompactAveragingSplitting.EigenvectorLifting.SplitTop`, lines 18–21)

At an inert place take functionals l, l' killing the two source difference spaces, invariant under the target maximal compact K_v and normalized to take value one on the vacua s, s'. They are obtained by averaging

the zero evaluations over K_v. For the Iwasawa parabolic P_v stabilizing a maximal isotropic space, the unaveraged normalized vacuum coefficients are right K_v-invariant and left P_v-equivariant. Their characters are $e(p) = \beta(z)R^{a/2}$ and $e'(p) = \alpha(z)R^{a'/2}$, where $R = |N_{E_v/F_v}(\det t)|_v$ for the Levi block t, and z is the corresponding local determinant idele. The unramified splitting characters give $\alpha(z)\beta(z) = 1$. Write $J = P_v \cap K_v$, $n_+(p) = [J : J \cap pJp^{-1}]$ and $n_-(p) = n_+(p^{-1})$. The parabolic modulus gives $n_-(p) = R^{D/2}n_+(p)$. Thus

$$n_+(p)e(p) = n_-(p)e'(p^{-1})$$

by $a/2 + a'/2 = D/2$. In the finite average on K_vhK_v/K_v, group the Iwasawa representatives into J-orbits on P_v/J. Inversion exchanges the J-double cosets of p and p^{-1}, and the Cartan decomposition gives $K_vh^{-1}K_v = K_vhK_v$. The displayed identity pairs the contributions, giving $l(hs) = l'(hs')$. By Lemma 34.18 and Corollary 34.20 both coefficient images are the span of the translates of this spherical function. For a nonzero coefficient, translate it to have nonzero value at one and average its test over K_v. The resulting coefficient is a nonzero multiple of the vacuum coefficient. Hence the common image is irreducible.

Lean: `PositivityTheory.Sph.sphFun_of` (`Development.WeilBound.ThetaLifts.SphericalLattices.FunAsm`, lines 113–191)
Lean: `PositivityTheory.Sph.finAt_sph` (`Development.WeilBound.ThetaLifts.SphericalLattices.FinInert`, lines 120–217)

At each real place use the signature frames of the three spaces to identify the polynomial vectors with the two Fock models of Lemma 34.2. The source splitting adds $-c\,\mathrm{tr}$ on the first side and $c\,\mathrm{tr}$ on the second, so the maps to T, T' kill the actual source Lie actions. Lemma 34.4 supplies surjections from both modules onto the same nonzero simple target module. Add $(m_{\beta,w}/2)\,\mathrm{tr}$ to its action. The unprimed action then has the prescribed target splitting, and the primed action has total added trace coefficient $c+m_{\beta,w}/2 = m_{\alpha,w}/2$. The intertwining identities hold on every matrix unit. The compact groups are connected, and the compact orbits of polynomial vectors span finite-dimensional spaces, so exponentiation gives target compact equivariance and source compact invariance. Tensor the surjections over the real places. Each local module is a quotient of a countable-dimensional polynomial space and is simple. Lemma 1.21 therefore gives simplicity of their tensor product. At completely split good places Lemma 34.9 gives the kernel and full-rank surjectivity. The split matrix action preserves the rank of the selected primed factor.

Lean: `PositivityTheory.KRealA.kReal_of_lie` (`Development.WeilBound.CoreStatements.GaussianSeminormBounds.RealSeminormFiltration.KRealAsm`, lines 292–384)
Lean: `PositivityTheory.Sph.theoremK_sph_of` (`Development.WeilBound.ThetaLifts.SphericalLattices.TheoremKSphOf`, lines 10–16)

Lemma 34.8 gives absolute convergence on the unprimed side by $a + a' + 2p < 2a$. It gives (34.3), primed moderate growth and Schwartz continuity on the allowed spaces.

Induct on $p + q$. At $p = q = 0$, Lemma 34.5 gives a polynomial witness on the unprimed side with $I(\Phi, \cdot) \neq 0$ and a polynomial witness $\Phi' \in \mathcal{S}'_{t_1}$ on the primed side with $I'(\Phi', \cdot) \neq 0$. Surjectivity of the matching maps supplies polynomial partners for both witnesses, and Lemma 34.9 at v_1 allows the primed partner of the unprimed witness to be chosen in $\mathcal{S}'_{t_1}$.

Lean: `GlobalTheory.ThK.Dbl.bump` (`Development.WeilBound.ThetaLifts.LatticeBasics.NearUnits`, lines 113–185)
Lean: `GlobalTheory.ThK.Dbl.nonZeroBottom_of_doubling_integral` (`Development.WeilBound.ThetaLifts.LatticeBasics.BotAsm`, lines 117–146)
Lean: `GlobalTheory.ThK.Dbl.pair_ne_zero` (`Development.WeilBound.ThetaLifts.LatticeBasics.BotAsm`, lines 58–115)
Lean: `GlobalTheory.ThK.Dbl.badFactor_of` (`Development.WeilBound.ThetaLifts.LatticeBasics.BadFactor`, lines 96–191)

By Lemma 34.10, the two integrals I_y, J_y of a polynomial match y are equivariant maps from the common restricted tensor product. The local spaces have at most countable complex dimension, from polynomial monomials and indicators of a countable basis of balls, so their simple quotients have at most countable complex dimension as well. Adjoining the nonzero vacua, Lemma 1.21 gives simplicity of the restricted tensor product. At $p = q = 0$ the connected compact real target integrates its Lie action on compact-finite vectors.

Lean: `PositivityTheory.KReal.tensorIrr` (`Development.WeilBound.CoreStatements.GaussianSeminormBounds.RealSeminormFiltration.Tensor`, lines 131–165)
Lean: `GlobalTheory.ThK.Dbl.pairsIrr_of_arch` (`Development.WeilBound.ThetaLifts.LatticeBasics.PairsIrrOfArch`, lines 10–11)

The primed map J is a nonzero embedding. Any invariant Petersson form from the two maps is a scalar multiple of the positive form $\langle J_y, J_z\rangle$, taken linear in the first argument. On the compact rational quotient, a compact type with fixed open finite level lies in a finite direct sum of its matrix-coefficient spaces. Represent the form on one nonzero such isotypic space by an operator relative to the positive form. Subtract an eigenvalue times the positive form. Compact invariance makes distinct types orthogonal, so the resulting

radical contains an eigenvector. Simplicity gives the whole space as radical. First apply this to $\langle I_y, J_z\rangle$ to choose c, then to $D_y = I_y - cJ_y$. This gives

$$\langle D_y, J_z\rangle = 0, \qquad \langle D_y, D_z\rangle = \lambda\langle J_y, J_z\rangle. \tag{A.360}$$

Lean: `GlobalTheory.ThK.Dbl.hermLine_of` (`Development.WeilBound.ThetaLifts.LatticeBasics.RankPolData`, lines 107–174)

For $u = I_y$ or J_y and a polynomial primed test P, the reverse lift $T_P u$ of Lemma 34.12 is constant on $\mathrm{U}(V')(\mathbb{A}_F)$. Indeed $[\mathrm{U}(W)]$ is compact at $p = q = 0$, so every continuous function on $[\mathrm{U}(W)]$ is integrable against $\overline{\theta_{\omega'}(x, \cdot; Q)}$ for every Schwartz vector Q. Fixing the factors of the image of y away from each completely split good place $v \notin T \cup \{v_1\}$ writes $u = \sum_j U_j(\xi_j)$ via Lemma 34.10, and Lemma 34.6 gives constancy.

Lean: `GlobalTheory.ThK.Dbl.LD.eq_case` (`Development.WeilBound.ThetaLifts.LatticeBasics.LocDistE`, lines 106–143)
Lean: `GlobalTheory.ThK.Dbl.TC.inv_loc` (`Development.WeilBound.ThetaLifts.LatticeBasics.ThConstInv`, lines 86–152)
Lean: `GlobalTheory.ThK.Dbl.thConst_of'` (`Development.WeilBound.ThetaLifts.LatticeBasics.ThConstFin`, lines 13–15)
Lean: `GlobalTheory.ThK.Dbl.normOne_mem` (`Development.WeilBound.ThetaLifts.LatticeBasics.ConstIdele`, lines 134–181)

Choose $J_y \neq 0$ and let P be its test. With $v' = \mathrm{vol}[\mathrm{U}(V')]$, Fubini gives $T_P D_y = 0$ and $T_P J_y = \|J_y\|^2/v'$. Lemma 34.12 gives $0 = \lambda\|J_y\|^4/v'$. Hence $\lambda = 0$, and (A.360) gives $D_z = 0$ for every z. The unprimed witness gives $c \neq 0$.

Lean: `GlobalTheory.ThK.Dbl.polZero_of` (`Development.WeilBound.ThetaLifts.LatticeBasics.PolZero`, lines 44–109)

For positive index sum take the adapted planes of Lemma 34.13. Use S, C of Lemma 34.14 and put $c = c_1/(CS)$ for $q > 0$ and $c = CSc_1$ for $q = 0 < p$. The primitive Gauss sums and the Euler product to the right of one are nonzero. Apply the Levi identities to compact translates of a polynomial match with $\Phi' \in \mathcal{S}'_{t_1}$. Iwasawa decomposition and the two character laws give $B = cB'$ for $q > 0$ and $A = cA'$ for $q = 0 < p$. The difference $f = I(\Phi) - cI'(\Phi')$ is in Σ of Lemma 34.15 and has a line term of the remaining character type.

If $q = 0 < p$, then $B' = 0$. The zero evaluation for B has target dimension $d_1 = a + a' - 2$ and smaller target index $p - 1 < p$. Since $|m_{\beta,w} - m_{\alpha,w}| > d_1$ and $d_1 + 2p < 2a$, Lemma 34.11 gives $B = 0$.

For $q > 0$ induct on p for differences with the remaining α-type line term. At $p = 0$, definiteness gives $A = 0$. Apply the full-rank zero-evaluation rule in Lemma 34.14 to the primed test. It preserves the upper integral and makes the lower zero evaluation full rank. Lemma 34.11 now kills A', since $q - 1 < q$ and $|m_{\alpha,w} - m_{\beta,w}| > d_1$.

At $p > 0$, move the line by Witt transitivity to the plane for removal on the V-side. For each compact translate $\omega'(1, k)\Phi' \in \mathcal{S}'_{t_1}$, apply the full-rank zero-evaluation rule of Lemma 34.14 first to replace it modulo restricted source differences by a full-rank upper representative whose zero evaluation lies in $\mathcal{S}'_{t_1,\mathrm{low}}$ without changing the upper integral. At line coordinate one the line term of each compact translate is then a lower difference with indices $(p-1, q)$, by the cone and zero evaluations. Its line term c_2 has two lower Levi types. Let $u \in \mathbb{A}_F^\times$ equal 2 at infinity and 1 at finite places. Let a_u scale the two isotropic lines by u, u^{-1} and their duals by the inverse conjugates. The two types have values $1, \lambda$ at a_u, where

$$\lambda = \alpha(u)\beta(u^{-1})|u|_E^{(a'-(a-2))/2}, \qquad |\lambda| = |u|_E^{(a'-(a-2))/2} < 1. \tag{A.361}$$

The first type is the product of the two α types. In the second the lower unprimed source has dimension $a - 2$. Interchanging the unipotent integrations realizes c_2 as the upper-unipotent average of the two-plane constant term restricted to $\mathrm{SL}_2(\mathbb{A}_E)$ on the two lines. The same character calculation for variable positive scale bounds this average on the positive Siegel ray, uniformly on compact right factors. The two coefficients are continuous, since they are recovered from c_2 and its translate by a_u.

Fix a target right translate. On the positive Siegel ray, choose bounded representatives for the unipotent quotient with finite coordinates in the period subgroup of the averaging function. Conjugation past the diagonal of height 2^k displaces its real argument by $O(2^{-k})$. The mean-value bound lowers a growth bound $O(2^{kM})$ for the two-plane term to $O(2^{k(M-1)})$ for its smoothing minus the smoothed unipotent average, uniformly on compact right factors. The average is bounded there. Iterating from an integer moderate-growth exponent with arbitrarily small total smoothing support gives a bounded smoothing by reduction on $\mathrm{SL}_2(\mathbb{A}_E)$.

Lean: `PositivityTheory.ThKEnd.Two.kill` (`Development.WeilBound.ThetaLifts.EigenvalueEndgame.SignatureSplitting.ConvCont`, lines 201–220)

Its smoothed unipotent average is bounded. The smoothed difference $c_2(a_u g) - c_2(g)$ scales by λ under a_u. Evaluating at $a_u^{-k} g$ bounds its value at g by its supremum times $|\lambda|^k$. Let $k \to \infty$ and shrink the

smoothing supports. The difference is zero, so the second lower type vanishes. The induction kills the lower difference, and Iwasawa decomposition kills the upper line term. Lemma 34.15 gives $f = 0$. Transport the comparison through the change of frames.

For $q > 0$ use the nonzero B witness of Lemma 34.14 and a primed partner of full rank at v_1. The Levi identity gives $B' \neq 0$. For $q = 0 < p$ the lemma gives an A' witness. Replace its primed v_1-factor by a full-rank representative. The source is definite, so absolute convergence and coinvariance preserve A'. The Levi identity gives $A \neq 0$. These witnesses give uniqueness of c, and continuity extends the comparison to general matches. Cancel the nonzero line terms in the Levi identities to obtain (34.6) and (34.7). □

Proof of 34.8

Proof. Write $n = a$ or a', $d = a + a'$, $e = [E : \mathbf{Q}]$, and r, l for the rational Witt indices of source and target. In rational Witt bases the positive chamber has source weights $s_i, -s_i, 0$, with $s_0 \geq \cdots \geq s_{r-1} \geq 0$ and $n - 2r$ zero weights. Use t_j on the target. Bounded shifts account for the fixed lower bounds in the Siegel set. Put $\rho_n(s) = \sum_{i<r}(n - 2i - 1)s_i$, and define $\rho_d(t)$ likewise. The chamber densities are

$$e^{-e\rho_n(s)}\,ds, \qquad e^{-e\rho_d(t)}\,dt. \tag{A.362}$$

The roots for a pair $i < j$ contribute $2s_i$ with multiplicity e, the middle block contributes $n - 2r$ copies of s_i with that multiplicity, and $2s_i$ occurs with multiplicity $e/2$. The rank-zero chamber has mass one. The Siegel-set estimate of Proposition 1.22 uses a majorant uniform on the compact right factors. Its height comparisons give

$$H_{\mathrm{quot}}(g) \leq Ce^{A\sum_i s_i}, \qquad e^{\sum_j t_j} \leq C\,\mathrm{ht}(h)^B \tag{A.363}$$

for positive C, A, B. The first uses the reduced diagonal and compact bound. For the second the diagonal controls each e^{t_j} and the reduced height is bounded by a power of the original height.
Lean: `GlobalTheory.CompanionTheory.SiegelRed.siegelCore (Development.AutomorphicSpectrum.Proofs.GlobalProofs.SiegelSetUnitary`, lines 17–27)

Set $S = \sum_i s_i$, $T_1 = \sum_j t_j$, pad the sequences with zeros and put

$$P = dS + nT_1 - 2\sum_{i<r,j<l}\min(s_i, t_j), \qquad \Delta = \max_{0\leq k\leq r}\sum_{i<k}(s_i - t_i).$$

For fixed finite factors and real Schwartz factor f, the bounds are

$$|\theta(g,h)|^2 \leq p_0(f)^2 e^{eP}, \qquad |\theta(g,h)|^2 \leq p_m(f)^2 e^{eP - m\Delta} \quad (m \in \mathbf{N}). \tag{A.364}$$

The p_m are continuous Schwartz seminorms, and the second estimate uses full-rank support at the chosen split place. It applies to either side in compatible models.

In Schrödinger coordinates put $D(u,v)\Phi(x) = \psi(v\cdot(x+u/2))\Phi(x+u)$. Group the product of two rational theta sums, conjugate in the second factor, by the difference u. Poisson summation in the other argument gives the coefficients $\int D(u,v)\Phi_1(x)\overline{\Phi_2(x)}\,dx$ at rational (u,v), up to unit phases and a measure factor. The rational lattice is its own annihilator. In the chosen polarization (u,v) gives a matrix $Z \in \mathrm{Mat}_{n,d}(E)$, which Weil covariance replaces by $g^{-1}Zh^{-t}$. At finite places the coefficient has compact support, since translation must bring the supports together and modulation must be trivial on a common period lattice. At infinity, integrating by parts in x against the modulation character gives decay in v. Schwartz decay gives the corresponding bound in u. Together they bound the coefficients by any inverse power of $1 + \|g^{-1}Zh^{-t}\|_\infty$, times Schwartz seminorms. Compact right factors give bounded enlargements of the support and equivalent norms.

For the full-rank estimate partition the selected finite support into small cosets $Y_0 + L_0$ for which every element of $2Y_0 + L_0$ has rank n. For each piece pair its test with its reflection $x \mapsto -x$. Reflection commutes with the Weil action up to a unit scalar and permutes the summation lattice. The absolute product therefore equals the squared absolute theta sum of the piece. A contributing translation coordinate lies in $\pm(2Y_0 + L_0)$. In the barred model this coordinate is the conjugate split component of Z up to invertible form matrices. All contributing Z have rank n. Sum the bounds for the absolute theta values of the pieces and then for the pure tensors.

Lean: `PositivityTheory.ThKEst.sqAll_of'` (`Development.WeilBound.ThetaLifts.FrameEstimates.SqAllC`, lines 103–111)
Lean: `PositivityTheory.ThKEst.soCut_of` (`Development.WeilBound.ThetaLifts.FrameEstimates.HermitianFormDetNeZero`, lines 76–154)
Lean: `PositivityTheory.ThKEst.sqSo_of'` (`Development.WeilBound.ThetaLifts.FrameEstimates.ThetaSumNeg`, lines 136–167)

In the rational Witt bases the entries of Z lie in a fixed fractional ideal $\mathfrak{a}$. Counting disjoint balls about its points and summing dyadic shells at scale $\max(1, e^{b_1})$ gives, for $M > e$,

$$\sum_{z\in\mathfrak{a}} (1 + e^{-b_1}\|z\|_\infty)^{-M} \ll_{\mathfrak{a},M} e^{e\max(b_1,0)}. \tag{A.365}$$

Take sufficient Schwartz order to bound a coefficient by the product of these entry weights. The four signed sums for a pair s_i, t_j contribute $2\max(s_i, t_j)$ to the exponent sum, and the zero weights contribute $(d-2l)S + (n-2r)T_1$. The total is P.

For a full-rank Z, choose k attaining Δ. The first k source rows expanded by $\exp s_i$ have a nonzero k-minor. Bounded denominators and the field norm bound its unscaled determinant below at some embedding. The scaling factor is at least $\exp(\sum_{i<k}(s_i - t_i))$, since any k target weights have sum at most $\sum_{i<k} t_i$. Including $k = 0$ this gives

$$e^{\Delta} \ll (1 + \|g^{-1}Zh^{-t}\|_\infty)^{n+1}. \tag{A.366}$$

Use $m(n+1)$ additional Schwartz powers for this factor. This proves (A.364).
Lean: `PositivityTheory.ThKEst.minor_of` (`Development.WeilBound.ThetaLifts.FrameEstimates.ExistsColsDetNeZero`, lines 77–166)

Since $S \le T_1 + \Delta$ and $P \le dS + nT_1$, use $m \ge ed + 2AN$ in the square root of the full-rank estimate to obtain (34.3). With $m \ge ed + 2$ the same estimate without the height multiplier gives

$$|\theta(g,h)| \le p_m(f)\exp\bigl(-S + (en+m)T_1/2\bigr). \tag{A.367}$$

Integration using (A.362) and the target height bound gives moderate growth.
Lean: `PositivityTheory.ThKEst.soSemiU_of` (`Development.WeilBound.ThetaLifts.FrameEstimates.SoSemi`, lines 22–102)
Lean: `PositivityTheory.ThKEst.soDecay_of` (`Development.WeilBound.ThetaLifts.FrameEstimates.ThickFrames`, lines 156–174)

For $n = a$, use $P \le dS + nT_1$ in the first bound. In the successive differences of chamber coordinates the coefficients after multiplication by the source density, and after source integration, squaring and multiplication by the target density, respectively, are

$$ek(d/2 - n + k) < 0 \quad (1 \le k \le r), \qquad ek(n - d + k) < 0 \quad (1 \le k \le l). \tag{A.368}$$

These are the partial sums of the original exponents. The signs follow from $r \le p,\ d+2p < 2a,\ l \le p+q < a'$. The source calculation works for any compatible pair with $d + 2r < 2n$.

For $n = a'$, put $Y = p + q$ and $\epsilon = 1/(4(r+1))$. Here $r, l \le Y,\ 2Y + 2 \le n,\ 2n \le d$. Then

$$P/2 - (d + r + 1)\Delta \le \rho_n(s) - \epsilon S + \rho_d(t)/2 - T_1/4. \tag{A.369}$$

Write $H = \sum_{i,j<r}\min(s_i, s_j) = \sum_{i<r}(2i+1)s_i$ and $C_1 = \sum_{i<r,j<l}\min(s_i, t_j)$. For $j < r$, if k of the s_i exceed t_j, then

$$\sum_i (s_i - t_j)_+ \le \Delta + \sum_{i<k}(t_i - t_j) \le \Delta + \sum_{i<j} t_i.$$

Thus $H - C_1 \le r\Delta + \sum_{j<r}(r-1-j)t_j$. Put $b_2 = d/2 - n + \epsilon \ge 0$. From $S \le \sum_{j<r} t_j + \Delta$ we get

$$b_2 S + H - C_1 \le (b_2 + r)\Delta + \sum_{j<r}(b_2 + r - 1 - j)t_j.$$

The coefficient at $j < r$ is at most $(d - 2j - 1 - n)/2 - 1/4$, which is nonnegative for $r \le j < l$. Using $\rho_n = nS - H$ gives the claim. Take $m \ge 2e(d + r + 1)$ in (A.364). The source integrand has factor $e^{-e\epsilon S}$. After source integration and squaring the target density leaves $e^{-eT_1/2}$. Both are integrable.

On a compact target set the source estimates are uniform. Dominated convergence and the seminorm bounds give the stated continuity. A definite source has $r = 0$ and its absolute-convergence estimate applies to all its tests.

Lean: PositivityTheory.ThKEst.Sq.exponent_bound_orthogonal_case (Development.WeilBound.ThetaLifts.FrameEstimates.InitialPairs.SquareIntegrableIntegralHolds, lines 194–257)
Lean: PositivityTheory.ThKEst.Sq.square_integrable_of_lattice_torus (Development.WeilBound.ThetaLifts.FrameEstimates.InitialPairs.SquareIntegrableOfLatticeTorus, lines 134–151)
Lean: PositivityTheory.ThKEst.Sq.square_integrable_orthogonal_case (Development.WeilBound.ThetaLifts.FrameEstimates.InitialPairs.SquareIntegrableOfLatticeTorus, lines 161–184)

□

Proof of 34.9

Proof. Suppose ϕ is supported where the J-rows are independent and choose a row $i \notin J$. Let T_b add $\sum_{j\in J} b_j Y_j$ to row i, and let D_x scale that row by x. Choose $|x| \le 1$ with $\zeta(x) \neq 1$. Invert the chosen element if needed to meet the size bound. Smoothness gives a lattice $L_0 \subset k^J$ of periods for the row additions. Choose a larger L containing the coefficients of every dependent i-row on the compact support. Boundedness follows from the nonzero J-minors on compact open charts. Put $N_1 = [L : L_0], N_2 = [x^{-1}L : L], z = \zeta(x)|x|^{d/2}$ and

$$F = \sum_{b\in L/L_0} \phi \circ T_b, \qquad \Psi = \frac{z}{N_1(zN_2 - 1)} \left(\sum_{b\in x^{-1}L/L} F \circ T_b - F \circ D_x \right).$$

The denominator is nonzero by unitarity and $\zeta(x) \neq 1$. In the coinvariants row additions act trivially and $[zF \circ D_x] = [F]$, so $[\Psi] = [\phi]$. On dependent i-rows, F is constant on L and zero elsewhere in the coefficient variable. Both terms in parentheses give the same constant on $x^{-1}L$ and vanish outside. They also vanish on dependent J-rows. Start with $J = \varnothing$ and adjoin the rows to obtain $\mathcal{S} = \mathcal{S}^\circ + A(\mathcal{S})$.
Lean: PositivityTheory.KCo.step (Development.WeilBound.CoreStatements.LocallyConstantCoinvariants.PfSurj, lines 236–315)
Lean: PositivityTheory.KCo.surjectivity_statement (Development.WeilBound.CoreStatements.LocallyConstantCoinvariants.PfSurj, lines 317–330)

The full-rank row-space map has fibres $\mathrm{GL}_n(k)$ over the Grassmannian. A nonzero minor gives a local section by making its columns the identity. On compact open chart pieces the tests are sums of products of base indicators and fibre functions. Weighted integration with $\zeta(\det r)|\det r|^{d/2}\,dr$ has kernel the source differences. Indeed the weight converts them to ordinary translation differences, and an integral-zero fibre function is a sum of such differences by expressing it on equal-measure cosets and translating each indicator to a fixed coset. A fibre function of weighted integral one supplies every local section.

The section on the target has the form

$$\lambda(\det h)|\det h|^{n/2} \int_{\mathrm{GL}_n(k)} \phi(rX_0h)\zeta(\det r)|\det r|^{d/2}\,dr, \qquad X_0 = (I_n\ 0).$$

For the row stabilizer with diagonal blocks A, B, substitution gives the multiplier

$$\lambda(\det A \det B)\zeta(\det A)^{-1}|\det A|^{-(d-n)/2}|\det B|^{n/2}.$$

This is normalized induction of two unitary determinant characters. Bernstein's unitary irreducibility theorem (Proposition 26.24, [15, §8.2]) gives irreducibility. The composite to Π_v is onto by the row argument. Since $\Pi_v \neq 0$, the composite and both arrows are isomorphisms.
Lean: GlobalTheory.CompanionTheory.SoCoinvNarrow.irr_ind (Development.AutomorphicSpectrum.Proofs.BlockDecomposition.SoCoinvOfNarrow, lines 102–145)
Lean: GlobalTheory.CompanionTheory.SoCoinvNarrow.irrF (Development.AutomorphicSpectrum.Proofs.BlockDecomposition.SoCoinvOfNarrow, lines 155–189)

At a good completely split place the source character is ν_v or its inverse. Let π generate a positive power of the selected prime, raised to a further power to kill the ramified unit characters. If $\nu_v(\varpi_v) = 1$, the Hecke law on π, squared to remove the angular square roots, gives

$$\prod_w \left(\frac{\tau_w(\pi)}{\tau_w(\bar\pi)} \right)^{m_{\alpha,w} - m_{\beta,w}} = 1.$$

The primes of the numerator and denominator embeddings are all distinct by Galoisness and complete splitting. At a numerator prime the valuation is positive, a contradiction. The multiplication characters in

any barred model separate the matrix points. A functional with the character of evaluation at Y kills tests vanishing at Y, by partition into pieces where a multiplication character differs from its value at Y. Thus changes of the allowed model preserve support in the corresponding matrix coordinates.
Lean: PositivityTheory.KCo.nu_statement (Development.WeilBound.CoreStatements.LocallyConstantCoinvariants.ContinuousFinId, lines 198–267)
Lean: PositivityTheory.KCo.full_of_soLoc (Development.WeilBound.CoreStatements.LocallyConstantCoinvariants.FullOfSoLoc, lines 66–74)

□

Proof of 34.10

Proof. Fix every factor except one and write $u(x)$ for the theta integral with that factor equal to x. Keep the primed v_1-factor in the full-rank subspace. Lemma 34.8 gives the positive semidefinite form

$$B(x,y)=\int_{[\mathrm{U}(W)]}\overline{u(x)(h)}u(y)(h)\,dh$$

with radical $\ker u$, since a continuous function of squared norm zero vanishes everywhere. Source changes of variables give $u(gx)=u(x)$, and right translation gives target invariance of B. At split places with nontrivial source determinant character the required kernel assertion follows from Lemma 34.9. At v_1 on the primed side the differences use full-rank tests, and both terms of each difference are in the domain of convergence.

At the other finite places write $\mathcal{S}$ for the test space, A for the source differences, G for the target and s for the vacuum. Then

$$\mathcal{S}=A+\operatorname{span}_{\mathbf{C}}Gs,\qquad \sum_{i=1}^{b}w_ik_iy-cs\in A,\quad w_ik_is=s. \tag{A.370}$$

The second assertion holds for each $y\in\mathcal{S}$ with suitable $b\ge 1$, $k_i\in G$, $w_i\in\mathbf{C}^\times$, $c\in\mathbf{C}$. Corollary 34.20 gives this at inert places with $w_i=1$.

At split places with trivial source determinant character, average over the integral source in the barred model. A left integral invariant is a linear combination of indicators that all rows belong to a prescribed lattice. Work on a finite quotient obtained from a common support and period lattice. Left integral orbits are indexed by the submodules generated by the rows. Row operations first select a basis modulo the maximal-ideal multiple of the generated submodule, then clear the redundant generators. Any two minimal generating lists differ by a matrix invertible modulo the maximal ideal. Containment indicators therefore span the orbit indicators by subtraction in increasing submodule order. Each containment indicator is a translate of the vacuum up to its determinant multiplier. The transposed column argument applies to right integral invariants. Average on the right with weights inverse to the right determinant character, using finitely many terms by smoothness. The weights cancel the character on the vacuum and the column argument gives a sum of source translates of it. Passing to coinvariants gives (A.370).
Lean: PositivityTheory.Sph.latSt (Development.WeilBound.ThetaLifts.SphericalLattices.BelowIffRows, lines 120–239)
Lean: PositivityTheory.Sph.SplitPl.sphAt_of (Development.WeilBound.ThetaLifts.SphericalLattices.SphSplitOf, lines 49–173)
Lean: PositivityTheory.Sph.SplitPl.pair_case2 (Development.WeilBound.ThetaLifts.SphericalSplitting.SphSplitGeo, lines 47–82)

The proper target-stable kernel N contains A, and the spanning assertion gives $s\notin N$. For $y\in N$ the averaging assertion gives $cs\in N$, hence $c=0$. Pair with s to obtain $0=\sum_i|w_i|^2B(s,y)$. The sum is positive. Apply this to target translates of y, then use spanning and $u(A)=0$. It follows that $u(N)=0$.
Lean: PositivityTheory.Sph.killFin3 (Development.WeilBound.ThetaLifts.SphericalLattices.ThetaKillsCuspFormsSpherical, lines 19–152)

At infinity compact source differences vanish under u. Schwartz continuity allows differentiation, so source Lie derivatives also vanish under u. Thus u,B descend to the real source coinvariants D of Proposition 1.22. The proper invariant kernel N of $D\to\Pi_\infty$ has zero projection onto the single generating compact type $D[\tau]$ of that proposition [6, equation (4.5)]. The compact isotypic projection is B-self-adjoint by compact invariance, so $B(D[\tau],N)=0$. Differentiate right invariance. Lemma 34.8 and Cauchy–Schwarz give L^1 bounds on the differentiated products, uniform on bounded intervals of the right flow. Fubini gives

$$B(D_Xx,y)+B(x,D_Xy)=0 \tag{A.371}$$

for the real Lie action D_X. The complexified adjoint is $Z \mapsto JZ^*J$ for the signature matrix J. Move the generating enveloping-algebra words across the pairing. Invariance of N gives $B(D, N) = 0$.
Lean: PositivityTheory.ThKF.L2.skew (Development.WeilBound.ThetaLifts.CartanFactorisation.IntegrableOnRight, lines 106–125)
Lean: PositivityTheory.ThKF.maxInfP (Development.WeilBound.ThetaLifts.CartanFactorisation.RinfZero, lines 148–163)
Lean: PositivityTheory.ThKF.killInf' (Development.WeilBound.ThetaLifts.CartanFactorisation.KillInf, lines 18–150)

Choose a finite set containing T, v_1 and all exceptional factors. Nonzero vacuum images make the transitions of quotient tensors injective. A primed tensor sum in $\mathcal{S}'_{t_1}$ can be written with each v_1-slot full rank by applying a linear projection onto that subspace in this slot. The projection fixes the test, since it fixes the generators of $\mathcal{S}'_{t_1}$. Equality of test functions detects equality of tensors by separate point evaluations on the finite-dimensional spans of their factors. Choose a section s_j of each slot map p_j. At the selected slot use the full-rank surjection. Replace x_j successively by $s_j p_j x_j$. Every difference has one factor in a kernel and belongs to the allowed space, so its integral vanishes. For a zero quotient tensor the final sum is zero, which proves descent.
Lean: PositivityTheory.ThKF.so_proj (Development.WeilBound.ThetaLifts.CartanFactorisation.Factor, lines 100–115)
Lean: PositivityTheory.ThKF.vanish (Development.WeilBound.ThetaLifts.CartanFactorisation.Factor, lines 36–60)

□

Proof of 34.11

Proof. Fix the target point and finite factors. For full-rank finite tests the source integral is dominated by an integrable majorant times a real Schwartz seminorm. The same domination holds for unrestricted finite tests when $d_1 + 2\min(r, s) < 2(r + s)$. Integration gives a continuous functional $\mathcal{L}$ on the real Schwartz space, invariant by change of source variable.
Lean: PositivityTheory.ThKEnd.Comp.iZero_of (Development.WeilBound.ThetaLifts.EigenvalueEndgame.EigenKilling.SubMemKer, lines 120–137)

In one Fock factor differentiation gives

$$\mathcal{L}(d\omega_{\mathrm{pol}}(Y)P) = -\tfrac{1}{2} m_w \operatorname{tr}(Y)\mathcal{L}(P),$$

where the polynomial action excludes the splitting scalar. For a rank-one $Y = vw^t$ of trace one in one complexified compact source block, its derivative part on a matching target sign column c is $\sum_{a,b} v_a w_b X_{ac}\partial_{bc}$. On the other columns it is $-\sum_{a,b} v_a w_b X_{bc}\partial_{ac}$. The scalar part is half the difference of the column counts. The required derivative weight has sign opposite to m_w. Negate the operator and exchange the rank-one vectors if necessary to make that weight positive. Choose the source block with $\min(r', s')$ positive-action columns.

For a linear polynomial ℓ, choose a nonzero covector annihilating those columns of ℓ, and a vector pairing to one with it. The source block dimension exceeds the number of columns. The derivative now kills the positive-column part of ℓ; on the others it sends its rank-one projection ℓ_- to $-\ell_-$ and kills $\ell - \ell_-$. The binomial expansion of ℓ^j has nonpositive derivative weights, so each term is killed by $\mathcal{L}$. Polarization and Schwartz continuity give the result. □

Proof of 34.12

Proof. Put $d = a + a'$, $z_0 = (a' - d)/2$. Let $e : G \times G \to \mathrm{U}(W \oplus W^-)$ be the doubling embedding, with W^- carrying the negative form. Use the flat section f_z of $P \boxtimes \bar{P}$ at z_0. It is K-finite and $\alpha|_{\mathbb{A}_F^\times} = \eta^{a'}$. The pairing

$$K_z(u) = \int_{[G]^2} E(z, e(h_1, h_2); f_z)\alpha(\det h_2)^{-1} u(h_2)\overline{u(h_1)}\, dh_1\, dh_2$$

continues meromorphically by Theorem 34.16 and compactness. In the convergence half-plane the rational maximal isotropic spaces are graphs of isometries, by anisotropy of W. Unfolding and putting $h_2 = h_1 g$ gives

$$K_z(u) = \int_{G(\mathbb{A}_F)} f_z(e(1, g))\alpha(\det g)^{-1}\langle R(g)u, u\rangle\, dg.$$

Right translates preserve polynomial matches by compactness at infinity and rank stability at v_1. Thus the assumed proportionality compares the integrands at gy, y. Continue to z_0.

Lean: `GlobalTheory.ThK.Dbl.kpair_cont` (`Development.WeilBound.ThetaLifts.LatticeBasics.ContinuousFlat`, lines 49–95)

At z_0 the doubled product and the second case of (34.8) give

$$E(z_0, e(h_1, h_2); f_{z_0})\alpha(\det h_2)^{-1} = \frac{2}{\operatorname{vol}[H']} \int_{[H']} \theta_{\omega'}(x, h_1; P)\overline{\theta_{\omega'}(x, h_2; P)}\, dx.$$

Fubini identifies $K_{z_0}(u)$ with $2\|T_P u\|^2/\operatorname{vol}[H']$. □

Proof of 34.13

Proof. A space losing a plane has dimension $n \geq 3$ and mixed real signs. For isotropy multiply a skew form by a trace-zero scalar. The rational quadratic form $\sum_i d_i N_{E/F}(z_i)$ has dimension $2n \geq 6$, hence is isotropic at finite places. It is isotropic at infinity by the signatures. Hasse–Minkowski gives a rational isotropic line Ex [96, Ch. 6].

Write h for the original form, linear in the first argument, and Λ for its self-dual lattice. Duality uses $\mathcal{O}_E$ on a source and the absolute codifferent on the skew target. Scale x so that $J = h(\Lambda, x)$ has valuation zero at the prescribed split primes and their conjugates. From $JJ^{-1} = \mathcal{O}_E$, take $y_0 \in J^{-1}\Lambda$ with $h(x, y_0) = 1$. Pairing $h(z, y_0)x$ against $w \in \Lambda$ gives $h(z, h(w, x)y_0)$ up to the Hermitian sign. Duality gives $h(z, y_0)x \in \Lambda$. Choose $\theta \in \mathcal{O}_E$ of trace one by unramifiedness and set $y = y_0 - \theta h(y_0, y_0)x$. The trace equation gives $h(y, y) = 0$, $h(x, y) = 1$ for either sign. The projection just established gives $Jy \subseteq \Lambda$, and duality gives $h(z, y)x \in \Lambda$. Hence the plane projections preserve Λ and the complement is self-dual. The line ideals are units at the prescribed split places, so the plane lines and complement give the required frames. Choose Cartan involutions preserving the plane and complement and exchanging the two lines.

Each removal subtracts one from each sign. The left sides of the size inequalities in (34.1) fall by two and the right sides by at most two. The character restrictions persist by $\eta^2 = 1$. □

Proof of 34.14

Proof. Let e_0 set the removed target variable to zero in the line polarization. On the side losing a source plane $H = (x, y)$, with target plane $H' = (x', y')$, let e be evaluation in the mixed polarization for the two planes. Keep the tensor coordinates of the two complements and evaluate the other coordinates at zero after the partial Fourier transform. With K the chosen source compact group and ρ the Heisenberg translation, put

$$\mathcal{T}\phi = \int_K \int_{\mathbb{A}_E^\times} e\big(\rho(u(t))\omega(k, 1)\phi\big)\nu(t)|t|_E^z\, d^\times t\, dk, \qquad u_\alpha(t) = txy'^t, \quad u_\beta(t) = \bar{t}\, yx'^t.$$

Use the Weil action of the side losing a plane.

Central character orthogonality in the line unipotent retains the isotropic source vectors. The zero orbit gives e_0; unfold the nonzero orbit at its vector stabilizer by Witt transitivity to obtain $\mathcal{T}$. On the primed side let $\Sigma_A^{\mathrm{all}}, \Sigma_B^{\mathrm{all}}$ denote the partial Fourier sums before deletion of the zero vector and $\Theta_A(0), \Theta_B(0)$ their zero terms. Fourier inversion gives $\Sigma_A^{\mathrm{all}} = \kappa\Sigma_B^{\mathrm{all}}$ for the ratio $\kappa = C_A^{-1}C_B$ of Fourier constants. Reindex the second nonzero sum by $t \mapsto \bar{t}^{-1}$. Full-rank support gives $\Theta_B(0) = 0$; therefore $\Sigma_A^\times = \kappa\Sigma_B^\times - \Theta_A(0)$, with the subtraction in the A' type. A definite source contributes the zero orbit. Evaluating at one and at a norm different from one separates the characters. With the lower integrals evaluated at h_1, the formulas are

	unprimed	primed	
β	$B = C_0 I_1(e_0\Phi)$	$B' = C_1 I_1'(\mathcal{T}\Phi')$	(A.372)
α	$A = C_1 I_1(\mathcal{T}\Phi)$	$A' = C_0 I_1'(e_0\Phi'), \quad B' = 0.$	

The measure and evaluation constants are nonzero, with κ included in C_1 for β. The zero-orbit formula applies also in the presence of the other type and when removal is from the opposite source. On the side keeping its source it is the lower integral of e_0 times the measure scalar.

For zero evaluation at v_1, the barred fibre at Y_1 consists of $Y = Y_1\kappa + y\rho$, with y a column and ρ a fixed row complementary to the lower coordinates. Thus full rank upstairs implies rank at least $n - 1$ for Y_1, where n is the common source rank for this evaluation. Cover the support of $e_0\phi$ by compact open charts on

which the rows other than some i are independent. Apply the row average of Lemma 34.9 after partitioning by these charts. The resulting lower test is full rank modulo a sum of lower source differences of chart tests.

Each such chart test Ψ has a full-rank upper preimage. In fibre coordinates multiply $\Psi(Y_1)$ by a compactly supported fibre test of evaluation one at zero, translated by Ne_i. The evaluation is Ψ by the translation law of zero evaluation. On the compact chart support the coefficients expressing a dependent row i in terms of the other rows are bounded by the nonvanishing minors. The fibre coordinates before translation are also bounded. An upper row relation would have nonzero coefficient on row i, since the other lower rows are independent. Normalize it to one. The relation on the fibre column would express N as a bounded combination of the original fibre coordinates. Taking $|N|_v$ larger gives full rank. Since e_0 intertwines the source action, a lower difference then lifts to a difference of two full-rank upper tests. Subtract the sum of the lifted differences. This gives a full-rank upper ϕ' with $e_0\phi'$ full rank and with $\phi - \phi'$ in the restricted source differences. The upper integral is preserved by absolute convergence and source invariance.

Lean: `PositivityTheory.ThKEnd.Comp.SoLower` (`Development.WeilBound.ThetaLifts.EigenvalueEndgame.EigenKilling.ConeIntegralFactorisation`, lines 29–35)

Lean: `PositivityTheory.ThKEnd.Comp.SoL.corank_one` (`Development.WeilBound.ThetaLifts.EigenvalueEndgame.EigenKilling.EigenKillSet`, lines 375–403)

Lean: `PositivityTheory.ThKEnd.Comp.SoL.lift` (`Development.WeilBound.ThetaLifts.EigenvalueEndgame.EigenKilling.EigenKillSet`, lines 426–514)

On pure tensors $\mathcal{T}$ factors locally. The vacuum valuation shells give the unramified Euler factors. Conjugation inverts ν, and the codifferent has value one. The initial shells for the line ideals give the powers $(1,1)$ for u_α and $(-1,-1)$ for u_β. At every $v \in T$, the character ν_v is ramified, so its conductor exponent satisfies $c_v \geq 1$. Taking additive volume one on $\mathcal{O}_v$ and splitting $\mathcal{O}_v^\times$ into cosets of $1 + \varpi_v^{c_v}\mathcal{O}_v$ gives

$$\int_{\mathcal{O}_v^\times} \nu_v(u)^{-1}\psi_v(u/\varpi_v^{c_v})\,du = (Nv)^{-c_v} \sum_{u \bmod \varpi_v^{c_v}}^{\times} \nu_v(u)^{-1}\psi_v(u/\varpi_v^{c_v}).$$

Multiplication by $\nu_v(\varpi_v)^{c_v}(Nv)^{c_v z}$ gives $\epsilon_v(\psi, \nu_v, 1-z)$. For $\iota\phi$ take B to be the first a columns, and for $\iota'\phi'$ the last a' columns. Each test equals $\nu_v(\det B)^{-1}$ when B is integral and invertible over $\mathcal{O}_v$ and all other entries lie in $\varpi_v^{c_v}\mathcal{O}_v$, and vanishes elsewhere. On lower block-unipotent matrices both images in (34.2) are the indicator that the lower-left block lies in $\varpi_v^{c_v}\operatorname{Mat}_{a'\times a}(\mathcal{O}_v)$. Parabolic covariance extends the equality to the open cell, and both images vanish off it. Zero evaluation on $\varpi_v^{c_v}\mathcal{O}_v$ in a coordinates for β, or a' for α, gives the corresponding power of $(Nv)^{-c_v}$ times the lower standard vector. The cone calculation has the same volume and the remaining Gauss factor with $s = 1$. The volume cancels by nonvanishing of the lower standard image.

Lean: `PositivityTheory.ThKG.Std.stdB` (`Development.WeilBound.ThetaLifts.LocalTheory.StandardValuations.ExistsVol`, lines 162–217)

Lean: `PositivityTheory.ThKG.Std.stdA_gen` (`Development.WeilBound.ThetaLifts.LocalTheory.StandardValuations.ExistsVol`, lines 219–283)

To pass from these tests to arbitrary matches, observe first that because the global constant terms A, B, A', B' integrate over $[N]$ (where $P = MN$ is the parabolic stabilizer of the removed isotropic line with unipotent radical N and Levi M), the descended local maps $e_0, \mathcal{T} : \Pi_v^{\mathrm{up}} \to \Pi_v^{\mathrm{low}}$ are N_v-invariant as well as $(M_v, \chi_{\mathrm{line}})$-equivariant, so they are P_v-equivariant into $\chi_{\mathrm{line}} \boxtimes \Pi_v^{\mathrm{low}}$. Their descent follows by fixing the other evaluated factors with nonzero lower images. The lower integral is injective on the lower matching quotient by the inductive comparison, the nonzero witness and irreducibility. The upper kernel kills the line coefficient by descent of the upper integral, so its evaluated image in the lower quotient is zero. At v_1 take a full-rank upper representative first. The allowed representatives give the same lower image by the kernel relation and compatibility with the source action. The cone evaluation there has lower full-rank support because the removed rows are accounted for by the plane blocks.

At every place $v \notin T$ (both inert and unramified split), uniqueness from the common K_v-spherical vacuum $v_0 \neq 0$ follows directly from P_v-equivariance and the Iwasawa decomposition $G_v = N_v M_v K_v$: if $\lambda_1, \lambda_2 \in \operatorname{Hom}_{P_v}(\Pi_v^{\mathrm{up}}, \chi_{\mathrm{line}} \boxtimes \Pi_v^{\mathrm{low}})$ satisfy $\lambda_2(v_0) = c_0\lambda_1(v_0)$, put $\lambda = \lambda_2 - c_0\lambda_1$; for every $h = umk \in G_v$ ($u \in N_v$, $m = (t, h_1) \in M_v$, $k \in K_v$), K_v-invariance of v_0 and P_v-equivariance of λ give $\lambda(hv_0) = \chi_{\mathrm{line}}(t)h_1 \cdot \lambda(v_0) = 0$, so $\{x \in \Pi_v^{\mathrm{up}} : \lambda(hx) = 0 \text{ for all } h \in G_v\}$ is a nonzero G_v-invariant subspace of Π_v^{up} and hence equals Π_v^{up} by irreducibility, proving $\lambda_2 = c_0\lambda_1$.

At every ramified split place $v \in T$, write $\alpha_1, \beta_1 : F_v^\times \to \mathbf{C}^\times$ for the components of α, β at the prescribed place of E above v, and use the same notation for their compositions with the determinant. Then N_v-

invariance gives multiplicity one via the geometric Jacquet filtration. Under $P_v = P_{1,D-2,1} = M_v N_v$, the Jacquet module $(\Pi_v^{\mathrm{up}})_{N_v}$ of $\Pi_v^{\mathrm{up}} = i_{P_{a,a'}}^{\mathrm{GL}_D}(\alpha_1 \boxtimes \beta_1)$ has a four-step filtration indexed by the four double cosets $P_{a,a'} \backslash \mathrm{GL}_D(F_v)/P_{1,D-2,1}$, whose central $E_v^\times$-characters on the two GL_1 factors (t_1, t_3) are

$$(\alpha_1|t_1|^{a'/2}, \alpha_1|t_3|^{-a'/2}), \quad (\alpha_1|t_1|^{a'/2}, \beta_1|t_3|^{-a/2}), \quad (\beta_1|t_1|^{a/2}, \alpha_1|t_3|^{-a'/2}), \quad (\beta_1|t_1|^{a/2}, \beta_1|t_3|^{-a/2}).$$

Because $a > a'$ (and ν_v is ramified), all four central characters are pairwise distinct, so the χ_{line}-isotypic quotient of $(\Pi_v^{\mathrm{up}})_{N_v}$ splits off as the single stratum $\chi_{\mathrm{line}} \boxtimes \Pi_v^{\mathrm{low}}$ (the first stratum in the α-step and the fourth in the β-step). Because Π_v^{low} is an irreducible smooth complex representation of countable dimension, Schur's lemma (Dixmier's lemma) yields $\dim_{\mathbf{C}} \mathrm{Hom}_{P_v}(\Pi_v^{\mathrm{up}}, \chi_{\mathrm{line}} \boxtimes \Pi_v^{\mathrm{low}}) = 1$, so the local maps are proportional with scalar fixed by the computed standard images. This identifies the entire local maps at all finite places (including on non-vacuum inputs outside T). At each real place both descended maps go from the matched upper real quotient to the lower one. Both are killed by the Lie algebra of the line unipotent radical, are equivariant for the compact Levi factor with the line character, and along every one-parameter subgroup of the Levi factor intertwine the upper and lower Lie actions up to the derivative at zero of χ_{line}. These relations force any two such maps D, D' with $D \neq 0$ to be proportional. In the Fock models the upper real quotient is generated over the complexified Lie algebra by a vector f in its generating compact type, highest-weight for the compact positive roots fixed in Lemma 34.2, and the vectors of the lower real quotient that have the restricted weight of f and are killed by the lower raising root vectors are the multiples of one vector f_1. By the Levi relation $D(f)$ is such a vector, so $D(f) = tf_1$, and likewise $D'(f) = t'f_1$, with $t, t' \in \mathbf{C}$. Let h be the generator of the compact part of the centre of the line Levi factor, with the sign for which every compact root vector outside the lower Levi factor that is not a raising vector lowers the h-eigenvalue, and let λ be the h-eigenvalue of f. The compact relation says that $D(hx)$ is a fixed scalar multiple of $D(x)$, the same scalar for all maps satisfying the relations; comparing the actions of the centres of the two Lie algebras through $D \neq 0$ shows that this scalar is λ. A map D'' satisfying the relations and killing f is zero. Indeed D'' kills every h-eigenvector of eigenvalue different from λ. By the Poincaré–Birkhoff–Witt theorem the compact-type submodule V generated by f is spanned by products of lowering compact root vectors applied to f. A product with a factor outside the lower Levi factor has h-eigenvalue $\lambda - s$ with $s > 0$, so it is killed by D''; a product of root vectors of the lower Levi factor lies in $\ker D''$ by the Levi relation, since f does. Hence $V \subset \ker D''$. The stabilizer $\mathfrak{s}$ of $\ker D''$ in the complexified Lie algebra contains the lower Levi factor, the line unipotent radical and the centre of the line Levi factor, and $\mathfrak{s}$ together with the compact Lie algebra $\mathfrak{k}$ spans the whole Lie algebra $\mathfrak{g}$. Since $U(\mathfrak{g}) = U(\mathfrak{s})U(\mathfrak{k})$ by the Poincaré–Birkhoff–Witt theorem, the submodule generated by f, which is the whole upper quotient, is $U(\mathfrak{s})V \subset \ker D''$. Applied to D this gives $t \neq 0$, and applied to $D' - (t'/t)D$ it gives $D' = (t'/t)D$.

Zero evaluation maps the polynomial tests onto the lower polynomial tests and the lower real quotient is nonzero, so the zero-evaluation map is nonzero, and the real cone map $\mathcal{T}_\infty$ is a complex multiple of it. Fix the other components of the nonzero lower witness. The lower integral then becomes a nonzero continuous linear functional l of the real component, invariant under the real points of the lower source unitary group and zero on the kernel of the map to the lower real quotient. By Claim 1 below, $l(\mathcal{T}_\infty f) \neq 0$ for some real test f. Polynomial tests are dense and $l \circ \mathcal{T}_\infty$ is continuous, so the same holds for a polynomial test, and the real cone map is nonzero on it. Hence the multiple is nonzero. It is the real comparison scalar.

Claim 1. Let l be a nonzero continuous linear functional on the lower real Schwartz space, invariant under the real points of the lower source unitary group. Then $l(\mathcal{T}_\infty f) \neq 0$ for some real test f.

Suppose that $l \circ \mathcal{T}_\infty = 0$. For a real test f, an element k of the compact group K_∞ of the cone integral and $c \in \prod_{w|\infty} \mathbf{C}$, let $a_f(k, c)$ be the value of l at the evaluation of the Heisenberg translate of $\omega(k)f$ by $u_\alpha(c)$ in the α-step and by $u_\beta(c)$ in the β-step. It is continuous, and rapidly decreasing in c uniformly in k. Slicing the cone integral along K_∞ and the line variable gives $\int_{K_\infty} \int a_f(k, c) r(c)\, dc\, dk = 0$ for every f, where r is the rational weight of $\nu|\cdot|_E^z$ in the line coordinate. Put $\Pi(k, c) = (2c_w k_w^{-1} x_w)_{w|\infty}$, where x_w is the local source line vector. Apply the vanishing identity to Heisenberg translates indexed by vectors in the real space underlying the target of Π. In the α-step these multiply $a_f(k, c)$ by Fourier characters at $\Pi(k, c)$. In the β-step they translate the second argument, giving these characters after Fourier transformation. Integrating in the translation parameter and using the local functional equation of r in the β-step gives zero against each pullback $\varphi(\Pi(k, c))$ of a smooth bump supported sufficiently near $\Pi(1, 1)$. The integrand is $a_f(k, c) r(c)$ in the α-step and $\widehat{a_f(k, \cdot)}(c) r(c)^{-1} |c|_E^{-1}$ in the β-step, with $|c|_E = \prod_w |c_w|^2$.

Density of the evaluated tests gives f_0 with $a_{f_0}(1,0) \neq 0$. In the α-step a line translation gives f with $a_f(1,1) \neq 0$. In the β-step Fourier uniqueness gives t_0 with $\widehat{a_{f_0}(1,\cdot)}(t_0) \neq 0$, and the Heisenberg action along $u_\alpha(1-t_0)$ moves this value to frequency 1. On the fibre over $\Pi(1,1)$, $k_w x_w = c_w x_w$ at every place. Invariance of l and Levi equivariance cancel the character in the weighted integrand, after Fourier scaling in the β-step. Dividing by its value at $(1,1)$ therefore gives a positive real value throughout the fibre. The inverse image of a sufficiently small closed ball about $\Pi(1,1)$ is compact and has all $c_w \neq 0$. By continuity the normalized integrand has positive real part over a smaller ball. For a nonnegative smooth bump supported there and positive at the centre, integrating its pullback against the normalized integrand gives a value with strictly positive real part, a contradiction.
Lean: PositivityTheory.ThKG.Loc.Bump.genA (Development.WeilBound.ThetaLifts.LocalTheory.ArchimedeanBumps.BumpAOf, lines 27–115)
Lean: PositivityTheory.ThKG.Loc.Bump.coreA (Development.WeilBound.ThetaLifts.LocalTheory.ArchimedeanBumps.ContinuousOnRationalWeight, lines 61–83)
Lean: PositivityTheory.ThKG.Loc.Bump.coreB (Development.WeilBound.ThetaLifts.LocalTheory.ArchimedeanBumps.ContinuousOnRationalWeight, lines 93–190)
Lean: PositivityTheory.ThKG.CayleyTransformDarboux.archimedean_multiplicity_one_of (Development.WeilBound.ThetaLifts.LocalTheory.ToralCohomology.ArchimedeanMultiplicityOneOf, lines 317–327)
Lean: PositivityTheory.ThKG.CayleyTransformDarboux.par_line (Development.WeilBound.ThetaLifts.LocalTheory.ToralCohomology.ParZero, lines 175–196)
Lean: PositivityTheory.ThKG.CayleyTransformDarboux.par_zero (Development.WeilBound.ThetaLifts.LocalTheory.ToralCohomology.ParZero, lines 99–173)
Lean: PositivityTheory.ThKG.Loc.bump_pt (Development.WeilBound.ThetaLifts.LocalTheory.MemPolData, lines 125–143)
Lean: PositivityTheory.ThKG.Loc.BumpB (Development.WeilBound.ThetaLifts.LocalTheory.UnramifiedDeterminedBySpherical, lines 89–96)
Lean: PositivityTheory.ThKG.Loc.Bump.genB (Development.WeilBound.ThetaLifts.LocalTheory.ArchimedeanBumps.BumpAOf, lines 118–254)

Tensor these identities for arbitrary pure tensors and extend linearly. The evaluated lower matches are

$$(C_2 S e_0 \Phi, \mathcal{T}\Phi') \quad (\beta), \qquad (\mathcal{T}\Phi, C_2 S e_0 \Phi') \quad (\alpha)$$

for $C_2 \neq 0$. The first primed test is allowed by the rank calculation. In the second pair one may take a full-rank preimage of the lower primed image at v_1 by Lemma 34.9; the lower source is definite. Lower comparison and (A.372) give the claimed identities with $C = C_2 C_1 / C_0$. Zero evaluation surjects on lower polynomial tests by adjoining tests of evaluation one in the removed coordinates. Lift a nonzero lower unprimed test for β, or a nonzero lower primed test for α with a polynomial partner. Evaluating at a Levi lift of a nonzero value gives the witnesses. □

Proof of 34.15

Proof. Witt transitivity gives the line case. Induct on $j = \dim_E U$. Use a dual frame to embed the Levi GL_j by m. Each proper standard block average of $g \mapsto C_U f(m(g)h)$ factors through a constant term for a smaller initial subspace. If the restriction is nonzero, a translate has nonzero Whittaker coefficient. Expand down the last columns of the upper unipotent. The zero frequencies are the block averages, while a rational change of basis moves a nonzero frequency to the last coordinate at each step. Fourier uniqueness on the compact additive quotients gives the assertion. Denote the nondegenerate Whittaker average by $W_f(h)$.

Choose a split finite place outside the exceptional places of the tests, frame and h. In each local matrix model the distribution defining the coefficient has source determinant covariance and a character on the right shears. For columns Y_i, the shear $Y \mapsto Y + Y_1\rho$, $\rho_1 = 0$, has character $\xi(\rho_2)$ with $\xi \neq 1$. If Y_1, Y_2 are independent, choose $\lambda Y_1 = 0, \lambda Y_2 = 1$ and $\xi(t) \neq 1$. The shear $\rho = t\lambda Y$ followed by the determinant-one left operation $1 - tY_1\lambda$ fixes Y with nontrivial character. For $j \geq 3$ and $Y_2 = sY_1 \neq 0$, or $Y_2 = 0$, $Y_1 \neq 0$, add $tY_2 - tsY_1$ to column three with nontrivial character on the Y_2 shear. For $Y_1 = 0$ the first shear already fixes Y with nontrivial character.

The stabilizers kill the distribution on their strata. On minor charts the stabilizers vary continuously and their characters are locally constant. For commuting square-zero parts A, B, the lattice $L_0 + AL_0 + BL_0 + ABL_0$ is preserved by both operations. On a sufficiently small coset about a point fixed by the operations, the indicator is fixed with the same nontrivial character. Partition compact supports into such cosets. For strata closed in the remaining support, read the distribution on restrictions by extending locally constant tests to the ambient open set. This first reduces to the dependent locus and then proves vanishing for $j \geq 3$.

For $j = 2$ write $Q = Nv$ and let c_ξ be the conductor exponent of ξ, trivial on $\varpi^{c_\xi}\mathcal{O}_v$ and nontrivial on $\varpi^{c_\xi-1}\mathcal{O}_v$. Put $r = \nu_v(\varpi)Q^{(a-a')/2}$. Let $D(P, M)$ be the unprimed distribution on the box with first two columns in $\varpi^P\mathcal{O}_v, \varpi^M\mathcal{O}_v$ and all other entries integral. Set $\Delta(P, M) = D(P, M) - D(P + 1, M)$. Below $M - P = c_\xi$ the box has a shear of nontrivial character. On the shell for the first column, partition by its first unit pivot after dividing by ϖ^P. Scaling that row by ϖ gives the determinant multiplier times $Q^{n-1}/Q^{a+a'-1}$ for source rank n. On dependent pairs the entry of column two in this row determines column two. The trivial-character shears exchange its Q residue classes at $M - P \geq c_\xi$. This gives

$$\begin{aligned} D(P, M) &= 0 && (M - P < c_\xi),\\ \Delta(P, M) &= r\Delta(P + 1, M) && (M - P > c_\xi),\\ \Delta(P, M) &= Q\Delta(P, M + 1) && (M - P \geq c_\xi). \end{aligned} \tag{A.373}$$

On the boundary $\Delta(M - c_\xi, M) = D(M - c_\xi, M)$. Applying the third line of (A.373) and then the second gives $D(M - c_\xi, M) = (Qr)\, D(M + 1 - c_\xi, M + 1)$. Thus $D(M - c_\xi, M) = (Qr)^{-M}D(-c_\xi, 0)$ for $M \geq 0$. Summing shells with $P = -i, M = i$ gives vacuum values proportional to

$$Y_i(r) = (Qr)^{-i} \sum_{k=0}^{2i-c_\xi} r^k \qquad (i \geq 0), \tag{A.374}$$

with empty sum zero. These boxes are translates by $b_i = \mathrm{diag}(\varpi^i, \varpi^{-i})$. The primed factor replaces r by r^{-1}, and reversing the geometric sum gives $Y_i(r^{-1}) = r^{c_\xi}Y_i(r)$. Thus $W_f(m(b_i)h') = c(h')Y_i(r)$ on an identity neighborhood of the right factor at h, by smoothness there. Since $|r| > Q$, $|Y_i(r)| \to \infty$.

Let ϕ be continuous automorphic on $\mathrm{SL}_2(\mathbb{A}_E)$ with upper-unipotent average ϕ_U. Write $a_k = \mathrm{diag}(2^{k/2}, 2^{-k/2})$ at infinity and one elsewhere. Suppose $\phi(a_k z) = O_\Omega(2^{kM})$ for compact right factors $z \in \Omega$, $k \geq 0$, and integer $M \geq 1$. For an integral-one nonnegative smooth compact bump η write $\phi * \eta(g) = \int \eta(v)\phi(gv)\, dv$. Then

$$|(\phi * \eta)(a_k z) - (\phi_U * \eta)(a_k z)| \leq C_{\eta,\Omega}2^{k(M-1)}. \tag{A.375}$$

Use a bounded fundamental set of the additive quotient with finite coordinates in a sufficiently small open subgroup, by weak approximation. In the second integral move the unipotent n across $a_k z$ and change v. The difference of bumps is $\eta(v) - \eta(z^{-1}a_k^{-1}n^{-1}a_k z v)$. The finite periods are unchanged and the real displacement is $O_\Omega(2^{-k})$. The mean-value estimate on the enlarged compact support proves the bound. Right convolution commutes with the average. If the average is bounded on the ray, iteration lowers the growth exponent to zero with arbitrarily small total support, and reduction gives a bounded convolution.
Lean: `PositivityTheory.ThKEnd.Two.Dat.conv` (`Development.WeilBound.ThetaLifts.EigenvalueEndgame.SignatureSplitting.ConvCont`, lines 87–110)

Apply this to $\phi(g) = C_U f(m(g)h)$, of moderate growth by Lemma 34.8 and compactness of the unipotent quotient. Its upper-unipotent average is zero by the line case. Choose an integer $i_0 \geq 0$ with $2i_0 \geq \max(d, c_\xi)$ (so that $Y_{i_0}(r) \neq 0$). Small convolutions preserve

$$Y_{i_0}(r)W_f(m(b_i)h') = Y_i(r)W_f(m(b_{i_0})h')$$

at h, since it holds on the neighborhood used in the convolution. The smoothed coefficient is bounded by boundedness of the smoothed ϕ and compactness of the unipotent quotient. Thus the smoothed coefficient at $m(b_{i_0})h$ vanishes. Shrinking the supports and using the relation at $i = 0$ gives $W_f(h) = 0$.

Suppose $p + q > 0$ and pair the resulting cusp form with the defining theta integrals. The reverse lift against the conjugate theta kernel is continuous and rationally invariant. At each completely split $v \notin T \cup \{v_1\}$, varying the v-factors in the defining finite sum and using Lemma 34.10 gives an equivariant map $T_v : \Pi_v^b \to Q_v$ whose image contains f, where Q_v is the space of continuous automorphic functions on the target. Let $R_v \subset Q_v$ be the stable space of rapidly decreasing functions. By semisimplicity of Π_v^b take an equivariant projection onto $T_v^{-1}R_v$. Compose with T_v and restrict to the summands. This writes $f = \sum_j U_j(\xi_j)$ with $U_j(\Pi_v) \subseteq R_v$. Since every element of R_v is integrable against the moderate-growth conjugate Schwartz kernel, Lemma 34.6 applies to these U_j and shows that the reverse lift is constant.
Lean: `PositivityTheory.ThKEnd.Orth.locDecomp_of` (`Development.WeilBound.ThetaLifts.EigenvalueEndgame.OrthogonalDecomposition.LocDecompOf`, lines 151–185)

Lean: PositivityTheory.ThKEnd.Orth.sigThConst_of (Development.WeilBound.ThetaLifts.EigenvalueEndgame.OrthogonalDecomposition.SigThConstOf, lines 22–41)

For an indefinite source take a full-rank representative at v_1 using Lemmas 34.9 and 34.10. Cuspidal decay [98] and the full-rank estimate in Lemma 34.8 bound the reverse lift times every power of source quotient height. The rational hyperbolic plane gives a positive-rank rational Siegel ray $a(s)$. Reduction gives $e^{\sum_i s_i} \le C\,\mathrm{ht}(\gamma a(s))^k$ for every rational γ. Taking the infimum shows that quotient height is unbounded along the ray, so the constant is zero. Fubini gives orthogonality to the indefinite-source integral.

Lean: PositivityTheory.ThKEnd.Orth.qht_unbdd (Development.WeilBound.ThetaLifts.EigenvalueEndgame.OrthogonalDecomposition.QhtUnbdd, lines 25–49)

For a definite source H, suppose $\langle I_H(\Phi_2), f\rangle \neq 0$ for a polynomial test. Fubini gives $T_{\Phi_2} f(1) \neq 0$. For any polynomial Φ_1 pair the doubled Eisenstein kernel of $\Phi_1 \boxtimes \overline{\Phi_2}$ in its second variable with f, using the determinant twist of doubling. In the convergence half-plane, a proper constant term in the first variable unfolds over the rational Siegel cosets. Each contributing orbit has a second-variable stabilizer containing a proper unipotent radical, whose average on f vanishes. Rapid decay and Theorem 34.16 continue the pairing and constant term to the Siegel–Weil point. There the paired value is

$$\frac{1 + \mathbf{1}_{\dim W \ge \dim H}}{\mathrm{vol}[H]} T_{\Phi_2} f(1) I_H(\Phi_1).$$

Thus every $I_H(\Phi_1)$ would be cuspidal. Start with the double-definite witness of Lemma 34.5 on H and increase the other signature index while keeping H. Lift the nonzero test at each step by zero evaluation. Equation (A.372) then gives a nonzero line coefficient at a Levi lift at the prescribed positive index sum. This contradicts cuspidality of every $I_H(\Phi_1)$. Both defining integrals are orthogonal to f; hence $\|f\|^2 = 0$, and continuity gives $f = 0$. □

Proof of 34.16

Proof. Since $\chi|_{\mathbb{A}_F^\times} = \eta^c$ is unitary and $E^\times \mathbb{A}_F^\times \backslash \mathbb{A}_E^\times$ is compact, χ is unitary, so Langlands's continuation theorem [149] gives locally uniform moderate growth of $(s - s_0)^k E(s, \cdot)$ on a closed disc $\overline{D(s_0, r)}$; Cauchy's integral formula in the Banach space $X_N = \{h : \sup_x |h(x)|\,\mathrm{ht}(x)^{-N} < \infty\}$ yields norm-holomorphy near s_0 and hence holomorphy under every $\Lambda \in X_N^*$. The second assertion is the anisotropic Siegel–Weil formula restricted to definite A [57, Theorem 7.1 and §3.1].

Lean: PositivityTheory.SiegelWeilFormula.Fin.siegelWeilStd (Development.WeilBound.CoreStatements.AutomorphicKernels.SiegelWeilFormula.SiegelWeilStandard.SiegelWeilStd, lines 8–8)

□

Proof of 34.17

Reduction to smooth factorizable tests. Choosing a global rational polarization converts the oscillator theta tests into Schrödinger Schwartz–Bruhat tests. The first formula specializes Yamana's Theorem 2 to $\delta = 1$, $m = b$, $n = a$, $r = r_B$ in the stated range [56, p. 237]; the second restricts Weil's orbital formula to nonsingular unitary fibres [48, §44, Thm. 2; §52, Thm. 5, (41)]; the third is the convergent Rallis identity with lower-lift vanishing [57, §3.1, Props. 11.1(ii), 11.2].

For compact-finite doubled tests, approximate at fixed finite level in Schwartz seminorms and project onto the given compact types. The restrictions to K lie in a fixed finite-dimensional matrix-coefficient space and determine the flat sections by Iwasawa decomposition. Integration over $\mathrm{U}(B)$ is continuous by the unitary Siegel-cone convergence criterion [48, §51, Proposition 8] for the polarized model on B^a, whose partial exponents on the positive Weyl cone are

$$2fj(a - b + j) < 0 \qquad (1 \le j \le r_B), \qquad f = [F : \mathbf{Q}]. \tag{A.376}$$

At $r_B = 0$ the cone has no variables. The doubled self-pairing and this cone bound give

$$\|\theta_\omega(g, \cdot; \Phi_i)\|_{L^2([\mathrm{U}(B)])} \le C_i\,\mathrm{ht}(g)^{N_i}. \tag{A.377}$$

Use Cauchy–Schwarz on the target and rapid decay on the source for Fubini [98]. The factor outside the doubled kernel is $\chi_B(\det g_2)$, supplied by the indicated twist. The quotient average supplies the volume.

Lean: `UpperTheory.Lit.Rallis.RallisIPProof.rallis_of_dm` (`Development.AutomorphicArithmetic.ThetaCorrespondence.FockModels.RallisDM`, lines 24–110)

□

Proof of 34.18

Proof. For $\Lambda = \mathrm{Mat}_{n'\times n}(\mathcal{O})$, write $V_{\bar{x}} = \mathbf{C}\rho(x)s$, with $\bar{x} = x + \Lambda$. An integral translation by u acts on this line by $\psi(H(u,x) + \sigma H(u,x))$. The conductor and unimodularity distinguish the characters. Thus M is the direct sum of these lines, with $\mathbf{C}s$ the trivial-character line.

The group K_T permutes the lines. Put $S = \mathrm{span}_{\mathbf{C}}\{\omega(g,1)s : g \in \mathrm{U}(J_S)\} \subseteq M^{K_T}$. On an orbit supporting an invariant, the stabilizer acts trivially on the corresponding line. For $u = hx - x \in \Lambda$ the condition is $\psi(-(H(x,u) + \sigma H(x,u))/2) = 1$. Use the Cayley transform, absorbing J_T into the skew test matrix. For $D_x = xJ_S^{-1}x^*$ this gives the phase $\psi((z+\sigma z)/2) = 1$ with $z = \delta\,\mathrm{tr}(CD_x)$ whenever $C^* = -C,\ C \in \varpi\,\mathrm{Mat}_{n'}(\mathcal{O})$ and Cx is integral.

Use Smith coordinates with the forms transformed correspondingly. Subtract integral entries so the active diagonal entries of x are ϖ^{-a_i}, $a_i > 0$. This leaves the phase test unchanged because C, Cx are integral and $C^* = -C$. Put $A_0 = J_S^{-1}$ in these coordinates. Vary the skew entries of C in $\varpi^{\max(a_i,a_j)}\mathcal{O}$ in the active block. The conductor and trace pairing give

$$(A_0)_{ij} \in \varpi^{\min(a_i,a_j)}\mathcal{O}. \tag{A.378}$$

Here one varies a diagonal anti-invariant entry or an off-diagonal pair with its negative conjugate. Choose an integral U_0 with $(A_0)_{ij} = \varpi^{a_i}(U_0)_{ij} + \varpi^{a_j}\sigma(U_0)_{ji}$ on the active block, assigning off-diagonal entries to the smaller exponent and dividing the diagonal by two. Set other entries to zero. For $l = U_0A_0^{-1}$, the linear terms in $(x-l)A_0(x-l)^*$ cancel xA_0x^* and the remaining lA_0l^* is integral. Replace x by $x - l$.

Lean: `GlobalTheory.Hk3.HoweDuality.exists_int_gram_of_phases` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.ExistsIntegralGramOfPhaseConditions`, lines 90–127)

Put $U = \mathcal{O}^n,\ V = \mathcal{O}^{n'}$ and $x^\dagger = J_S^{-1}x^*J_T$. Let $N = U \cap x^{-1}V,\ A = N + x^\dagger V$. Integrality of $xx^\dagger$ gives $A \cap U = N$ and $A \subseteq x^{-1}V$. From $N^\vee = U + x^\dagger V$ we obtain $A^\vee = (U + x^\dagger V) \cap x^{-1}V = A$. Take representatives Y modulo Λ of the maps integral on A, including x. This is a finite set. Choose columns $M_i \in U$ generating the cyclic summands of U/N of orders ϖ^{e_i}, $e_i > 0$, put $M_0 = (M_i)$, and set $\Phi(y) = (yM_0)^*J_T(yM_0)$.

Let B be the skew Hermitian matrices G with entries in $\varpi^{\max(e_i,e_j)}\mathcal{O}$ modulo those with entries in $\varpi^{e_i+e_j}\mathcal{O}$. The character $\chi_G(y) = \psi(-\delta\,\mathrm{tr}(G\Phi(y)))$ descends to B by the integrality of $\varpi^{e_i}yM_i$. With $E_1 = M_0GM_0^*J_S,\ k = 1 + E_1$, we have $E_1U \subset A,\ E_1A \subset U$ by the divisibilities, $\varpi^{e_i}M_i \in A$ and self-duality. The matrix E_1 is skew adjoint and divisible by ϖ. Hence $k^\dagger k = 1 - E_1^2$ preserves A with unit determinant, and kA is self-dual. The maps yk^{-1} represent the maps integral on kA. Their shifts from y are integral. In the translation phase expand $k^{-1} - 1 = -E_1 + E_1^2k^{-1}$. The first term gives $-\delta\,\mathrm{tr}(G\Phi(y))$; the second pairs integrally since yE_1, k^{-1} are integral and $E_1^\dagger = -E_1$. Thus

$$w_G = \sum_{y\in Y} \chi_G(y)\rho(y)s$$

is the vacuum sum fixed by translations integral on kA. Those translations permute the summands with phase one by integrality of their pairings. Every unimodular Hermitian lattice here has an orthonormal frame, by splitting unit-norm vectors detected over the residue field and normalizing by the norm on units. The residue norm and successive trace corrections give its surjectivity. Consequently $kA = gU$ for a source isometry g, and covariance gives $\omega(g^{-1},1)w_G \in \mathbf{C}s$. In particular $w_G \in S$.

Average $\chi_G(x)^{-1}w_G$ over B. Character orthogonality retains the terms with the same Gram phase as x. Testing diagonal and off-diagonal skew entries gives

$$\varpi^{\max(e_i,e_j)}(\Phi(y) - \Phi(x))_{ij} \in \mathcal{O}. \tag{A.379}$$

For $m, m' \in U$ and $c \in \mathcal{O}$ with $cm, cm' \in N$, express $m = M_0\alpha + u,\ m' = M_0\alpha' + u'$ with $u, u' \in N$. The factor $c\sigma(\alpha_i)\alpha'_j$ is divisible by $\varpi^{\max(e_i,e_j)}$; the terms involving u, u' are integral after multiplication by c. Hence $c(\langle ym, ym'\rangle_{J_T} - \langle xm, xm'\rangle_{J_T}) \in \mathcal{O}$.

Put $N_y = U \cap y^{-1}V,\ \mu(\bar{y}) = [U : N_y]$. As $N \subseteq N_y$, $\mu(\bar{y}) \le \mu(\bar{x})$. At equality $N_y = N$. The columns $X_i = \varpi^{e_i}xM_i,\ Y_i = \varpi^{e_i}yM_i$ are each independent modulo ϖ, by independence of the order-ϖ generators of

U/N. Their Gram entries agree modulo $\varpi^{\min(e_i,e_j)}$. Witt extension over the residue field gives an ambient isometry of the residual columns [96]. Lift it by successive Gram corrections $Z + Z^\dagger = H$ for Hermitian H, solved by $Z = H/2$.

If an integral isometry h matches the columns modulo $\varpi^{\min(t,e_i)}$, the corrections $(Y_i - hX_i)/\varpi^t$ for $e_i > t$ prescribe a skew pairing modulo ϖ by the Gram congruence. Extend from the independent columns to a basis and lift by antisymmetrization to a skew adjoint Z. Left multiplication by $(1 + \varpi^t Z/2)(1 - \varpi^t Z/2)^{-1}$ makes the next correction and preserves the earlier ones. Iteration gives $hX_i \equiv Y_i \bmod \varpi^{e_i} V$, or $y - hx \in \Lambda$. With no positive exponents take $h = 1$.

Lean: `GlobalTheory.Hk3.GramSep.gram_class_family` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.ExistsGramClassRepsSelfDualLatticeGroup`, lines 26–256)
Lean: `GlobalTheory.Hk3.HoweG.exists_intUnitary_of_gramClass` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.ExistsIntUnitaryOfGramClass`, lines 17–107)

The character average is in S, hence K_T-invariant. Its part at smaller μ belongs to S by induction. At equality the surviving terms lie on the orbit of $\bar{x}$ and have a nonzero coefficient on its line. That coefficient determines an invariant on the orbit. Subtracting the smaller part gives its generator in S. Hence $M^{K_T} \subseteq S$. The adjoint model exchanges the spaces. □

Proof of 34.20

Proof. The unramified quadratic extension of odd residue characteristic supplies a common uniformizer and an anti-invariant unit. The models have unimodular frames, unramified splitting characters and conductor $\mathcal{O}_v$. The vacuum is fixed by both integral groups and generates through the Heisenberg translations. Apply Lemma 34.18 to the two compact averages, each a finite mean by smoothness. The source average preserves the class of x and lies in the target span. The target average lies in the source span and is thus a scalar vacuum class modulo A. Multiply by its number of terms. □

Proof of 34.21

Proof. Put $b(u) = (f, H(u)v_0)$. The operators H are unitary and $H(v)H(u) = \psi_0([v,u])H(u)H(v)$. Applying $H(v)$ to both arguments of the pairing therefore gives $b(u) = 0$ off Γ°. The same calculation with a common finite open level of f, v_0 confines u_f to a compact set.

For the real variables, differentiate the commutator identity at $v = 0$ in a real direction e. Multiplication of $b(u)$ by the real linear coordinate obtained from $[e,u]$ gives, up to a fixed scalar, a sum of two matrix coefficients with a derivative on f or v_0. These coefficients are bounded by the products of the respective L^2 norms. Take a real basis and iterate. Nondegeneracy of the form gives decay of every polynomial order in u_∞, uniformly in u_f. Thus $b \in L^1$.

Fix u. The function $f\,\overline{H(u)v_0}$ is periodic on $X(F)\backslash X_{\mathbb{A}}$. Its Fourier characters are indexed by $X(F)$ through $\psi_0([\ ,\ ])$. The quasiperiodicity formula identifies its Fourier expansion, after multiplication by $H(u)v_0(x)$, as

$$\sum_{\gamma \in X(F)} b(u+\gamma)H(u+\gamma)v_0(x) = f(x)\,|v_0(x+u)|^2.$$

Indeed $H(u+\gamma)v_0(x)/H(u)v_0(x)$, interpreted as the multiplier before evaluation, is the character $\psi_0([x,\gamma])$ times a scalar of modulus one independent of x. That scalar cancels against its conjugate in $b(u+\gamma)$. The finite support restriction and real decay just proved give absolute summability because the contributing rational points lie in a translate of a fixed fractional lattice in the real coordinates. Integrate the identity over a fundamental domain in u. Unfolding the sum gives

$$\int_{X_{\mathbb{A}}} b(u)H(u)v_0(x)\,du = (v_0, v_0)f(x).$$

The right side is a theta test. Apply the superposition property of L and the support restriction on b. Finally $(v_0, v_0) > 0$, since the continuous periodic function $|v_0|^2$ is positive on a nonempty open set. □

Proof of 34.22

Proof. First consider finite translations. At the selected factors, the coprimalities make the oscillator ideal a unit. There the integral translation lattice is $e\widehat{M}$, and the translations fixing the coset indicator include $e\widehat{\mathfrak{m}M}$. At the conjugate factors a translation acts by the character of the selected variable under the alternating pairing. For an integral translation that character is trivial on the integral coset. At the remaining factors the vector is the lattice vacuum. These formulas give the two invariances in the definition of $D(\Gamma_i)$. In combining the factors the Heisenberg product has multiplier $\psi_0([u,v]/2)$. For doubled integral shifts this multiplier is one by integrality, and for shifts in a single split half the form on that half is zero. Real creation operators commute with all these finite translations. Each v_λ is nonzero because its coset contains the projection of a representative in M, and its other factors are nonzero vacua.

For a finite $u \in \Gamma_i^\circ$, commuting with the integral translations on the conjugate half forces $eu \in e\widehat{M}$ by lattice duality. Reduction modulo $\mathfrak{m}$ gives an element $a \in M/\mathfrak{m}M$, since completion induces an isomorphism on this finite quotient. Translation sends the indicator at λ to that at $\lambda - a$. On the conjugate half its multiplier is constant on this coset, since it commutes with the translations by $e\widehat{\mathfrak{m}M}$. Away from these pairs the commutant acts on the vacuum by a scalar. Indeed it annihilates the full integral lattice there, as away from 2 one divides a doubled shift by 2 and above 2 one uses the two single-half conditions, since the grid is split there. Self-duality puts the translating vector in the vacuum lattice. The central multiplier on recombining the three parts is a scalar as well. Thus $H(u)\bar{v}_0$ belongs to the span of the $\bar{v}_\lambda$, where the subscript 0 denotes the zero coset.

Lean: `GlobalTheory.CosShift.epart` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.CommutantDepthLatticePermutesCosetVectors`, lines 106–123)
Lean: `GlobalTheory.CosPerm.cosVec_perm` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.HeisenbergCommutantPermutesFirstRowCosetVectors`, lines 49–94)

For the real calculation work in the model for $(-S,T)$, with the linear functional $L'(f) = \overline{L(\bar{f})}$. It has the superposition property of L by conjugation of the coefficient under the integral. Fix $v = v_\lambda$. In the orthonormal Fock frames write $z = (z_j)$ for the complex coordinates of a real translation, with coordinate vector e_j and complex structure J. Put $c_j = 2\pi|\tau(\xi_0)|$ at the place τ of j, where ξ_0 is the nonzero imaginary scalar in the trace alternating form. Put $q(z) = \sum_j c_j|z_j|^2$. For each x,

$$R_x(z) = e^{q(z)/2}H(z)v(x)$$

is entire. To see this, differentiate the Heisenberg product law. The derivative of the central phase in the $\bar{z}_j$ direction cancels the derivative of the Gaussian factor, leaving the action of $(dH(e_j) + i\,dH(Je_j))/2$ on v, which is zero by the vacuum equation. The complementary derivative at zero gives

$$\partial^\alpha R_x(0) = \left(\prod_j c_j^{\alpha_j/2}\right)((a^+)^\alpha v)(x), \qquad a_j^+ = \frac{dH(e_j) - i\,dH(Je_j)}{2\sqrt{c_j}}.$$

For repeated derivatives, moving a creation operator past $H(z)$ adds a linear function of $\bar{z}$. Holomorphic differentiation followed by evaluation at zero kills all such terms.

Set $F(z) = L'(H(z)v)$. This is bounded. Gaussian integration of the Taylor series of R_x against $\bar{z}^\alpha e^{-q(z)}$ gives

$$\int \bar{z}^\alpha e^{-q(z)/2}H(z)v(x)\,dz = \left(\prod_j \frac{\pi}{c_j}\,c_j^{-\alpha_j}\right)\partial^\alpha R_x(0).$$

Here dz is product Lebesgue measure. Angular integration extracts the indicated coefficient and radial integration gives $\int_{\mathbf{C}} |z|^{2a}e^{-c|z|^2}\,dz = \pi a!c^{-a-1}$. The bound $|R_x(z)| \le \|v\|_\infty e^{q(z)/2}$ justifies the integration. If L kills all prescribed tests, applying L' shows that every moment $\int \bar{z}^\alpha e^{-q(z)/2}F(z)\,dz$ vanishes.

For any z_0, multiply these moments by the coefficients of $\exp(\sum_j c_j z_{0j}\bar{z}_j)$ and sum. Absolute convergence follows from the integrable majorant $\|F\|_\infty \exp(-q(z)/2 + \sum_j c_j|z_{0j}z_j|)$. The same angular and radial integration, with R_x in place of the monomial, gives the reproducing identity

$$H(z_0)v(x) = e^{-q(z_0)/2}\prod_j (c_j/\pi)\int e^{\sum_j c_j z_{0j}\bar{z}_j - q(z)/2}H(z)v(x)\,dz.$$

Its left side is a theta test, so applying L' gives $F(z_0) = 0$. This proves vanishing on every real translate of every conjugate coset vacuum.

Lean: `GlobalTheory.FockTransl.Jet.jet_eq_fock` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.FockGaussianJetsAtOriginEqFockVectors`, lines 42–87)
Lean: `GlobalTheory.FockTransl.tame_kills_real_translates` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.TameFunctionalKillsFockVacuumRealTranslates`, lines 28–119)

Finally, if $u \in \Gamma_i^\circ$, then $u_f \in \Gamma_i^\circ$, since every level condition is finite. Real and finite directions are orthogonal, so $H(u) = H(u_\infty)H(u_f)$. The finite calculation followed by the real calculation gives $L(H(u)\bar{v}_0) = 0$. Lemma 34.21 with this nonzero vacuum gives the assertion. □

Proof of 34.23

Proof. Conjugate the downward identities and the bottom period. Use the conjugate Weil action, with both splitting characters inverted. If u_i is a partial row lift, the next lift with test (λ, P), after conjugation, is the lift of $\bar{u}_i$ with $E_i(\lambda, P)$. A change of the test at i leaves u_i unchanged. Write Θ_f for a lift with test f, in the direction indicated by its argument.

We record why the period functionals in this argument admit the integrations of Lemma 34.21. For a fixed theta test b, with ω its Weil action, the bound needed is

$$\sup_{w,x} |(\omega(g,h)H(w)b)(x)| \le C_b\big(H_S(g)H_T(h)\big)^{A_b},$$

where H_S, H_T are quotient heights, the infima of the matrix heights over rational left translates. Intertwining moves $H(w)$ to a translation of $\omega(g,h)b$ with unit multiplier. Rational left translation permutes the variable in the supremum, and the finite open stabilizer of b acts trivially on the right. Adelic reduction therefore allows finitely many pairs of finite representatives, with the archimedean matrix heights bounded polynomially in $H_S(g)H_T(h)$. Use a common finite level for these finitely many translates of b. On each finite-level component the Sobolev bound on the compact additive quotient uses finitely many real Heisenberg derivatives. Moving these derivatives through $\omega(g,h)$ acts on their directions by the matrix action $x \mapsto h^{-1}xg$. Each word therefore costs a polynomial in the matrix heights, and unitarity bounds the L^2 norms of the fixed differentiated tests. Taking the maximum of the constants over the finite representatives gives the displayed quotient-height bound.

Lean: `GlobalTheory.WeilWordGrowth.om_pointwise` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.HeisenbergWordsWeilOperatorSupNormPolynomialGrow`, lines 65–129)
Lean: `GlobalTheory.KerTame.om_heis_bound` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.WeilKernelFunctionalSuperpositionIsTame`, lines 30–35)

In a Bessel integral the unipotent character has modulus one. In a Fourier–Jacobi integral its additional line theta factor satisfies the same bound for its fixed line test. The unipotent quotient is compact. In a fixed Witt decomposition the height of the embedded smaller group times a representative in this compact quotient is bounded by a power of its own quotient height via a rational translate in the smaller group and the block embedding in the larger group. Rapid decrease of the two automorphic factors now gives an integrable majorant independent of w. For a superposition with coefficient $a(w)$ multiply this majorant by $|a(w)|$. Weil covariance and Fubini give the superposition identity, and dominated convergence gives continuity of the orbit function. This argument applies also after contraction against a fixed line test and integration on the core.

Lean: `GlobalTheory.LiftTame.lift_periods_tame` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.ThetaLiftPeriodsIsTameThetaFunctional`, lines 95–153)

At the bottom write $V_0 = Y_0^{\mathrm{col}} \perp L_0$, with L_0 the distinguished line. The two orthogonal projections preserve the bottom depth lattice. In the grid frames they are integral summand projections and commute with the selected split idempotents. After the rank-zero Witt identification the two projected lattices are the first row depth lattice and the line depth lattice. Thus their direct sum is the core depth lattice. Choose nonzero vacua b, c on the two summands. On the row one may take the zero coset vacuum. On the line use the lattice vacuum, restricted to the induced depth conditions. The product $b \otimes \bar{c}$ is a nonzero core depth test. Its translates by the core annihilator are products of translates at the respective depths, since the alternating form is the orthogonal sum. The sign change on the line is accounted for by conjugation.

As a functional of the core test, the conjugate bottom period takes a nonzero value at the conjugate downward test defining F_0. By Lemma 34.21 it takes a nonzero value on one of the translated products. On

a product $b' \otimes \bar{c}'$, the kernel at the stabilizer is

$$(\omega_{\mathrm{row}}(g,h)b')(0)\,\overline{(\omega_{\mathrm{line}}(g,1)c')(0)}.$$

The character on the orthogonal sum is the product of the two splitting characters with the line character inverted. Interchange the g and h integrations. The result is the Fourier–Jacobi period at height 1 of $\bar{F}_1$ and $\Theta_{b'}\bar{u}$ with line test c'. The flag has length zero at this height. Fix c' and apply Lemma 34.22 to the functional of b'. This gives a prescribed first test with nonzero period.
Lean: `GlobalTheory.LastSeeSaw.lastLat_std` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.LastSeesawLatticesNonzeroLineVacua`, lines 125–145)
Lean: `GlobalTheory.LastSeesaw.upLast` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.LastSeesawPeriodNonvanishing`, lines 195–277)
Lean: `GlobalTheory.AscBottom.rung0` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.NonvanishingThetaTowerBottomRung`, lines 27–102)

Here are the two exchanges used above height 1. At height $2r+1$ remove r hyperbolic planes from the row space. Its middle space is V_{2r+1}. At height $2r+2$ remove r hyperbolic planes. Its middle space is $V_{2r+2} \perp L$, with L the distinguished line. Use the ordered isotropic columns $e_1, \dots, e_r$ of these Witt frames. Let $\zeta_j = (-\xi_0)^j \zeta_0$ in the line frame. A Bessel period integrates over the flag unipotent and the unitary group fixing the distinguished line vector, against the character on the successive simple entries and the last entry paired with that vector. A Fourier–Jacobi period integrates over the flag unipotent and the unitary group of its middle space, against the flag character and the conjugate line theta kernel. All unipotent integrations here use their rational quotients.

For the exchange from height i to $i+1$, put $V = Y_{i+1}^{\mathrm{col}}$, $W = Y_i^{\mathrm{col}}$, and denote their Witt middle spaces by $V^\natural, W^\natural$. In the two mixed Schrödinger coordinates write a full test as $b(X, Y, w)$, where X is the tuple of columns in W corresponding to the first isotropic block of V, Y is the dual coordinate for the isotropic block of W, and $w \in \mathrm{Hom}_E(V^\natural, W^\natural)_{\mathbb{A}}$. With U_a the upper unitriangular group of size $a = \#X$, its generic character is $\chi_U(u) = \psi_0(\mathrm{Tr}_{E/F} \sum_{j<a} u_{j,j+1})$, with indices starting at one. The restriction of a test needed after unfolding is

$$Rb(w) = \int_{U_a(\mathbb{A}_E)} \overline{\chi_U(u)}\, b(xu, y, w)\, du.$$

For $i = 2r+1$, $x = (e_1, \dots, e_r)$ and y has last column $-\zeta_{r+1}$ and all other columns zero. For $r = 0$ both tuples are empty. For $i = 2r+2$, $x = (e_1, \dots, e_r, \zeta_{r+1})$ and $y = 0$. Each x has independent columns. The stabilizer $R_x = \{h : hx = xu,\ u \in U_a\}$ is the Jacobi group in the first case and the Bessel group in the second. This follows by writing a flag-preserving isometry in Witt blocks. In the second case its middle block must also fix ζ_{r+1}.

We describe the finite restrictions since they determine the level of the resulting test. Use the pullback of Γ_i to the first translation block as the period lattice A_1 in X, and its character dual as the opposite block. For the second pair of blocks use the pullbacks of the integral oscillator lattice with target lattice multiplied by $\mathfrak{m}/\bar{\mathfrak{m}}$. They are dual under the alternating pairing. Denote the resulting period lattice in Y by B, and let Γ_* be the pullback of Γ_i under extension by zero on the two hyperbolic parts. Use tests with finite support in $x + A_1$ and $y + B$ in these coordinates, invariant under the respective period lattices and with middle test in $D(\Gamma_*)$. Equivalently the opposite translations have the characters prescribed by the centers x, y. In the standard grid frames, decomposition into the two hyperbolic parts and the middle is integral. On the first dual block and the second dual block the characters of a row depth shift are trivial, since their pairings with x, y are integral for ψ_0. Consequently these tests belong to $D(\Gamma_i)$. They are stable under real Heisenberg translation and under extension by zero of a translation in Γ_*°, and

$$R(H(\widetilde{w})b) = H(w)Rb \qquad (w \in \Gamma_*^\circ)$$

for this extension $\widetilde{w}$. These assertions also follow directly in mixed coordinates, since a translation in the middle acts on the last argument and commutes with evaluation of the first two. Denote this class of full tests by $\mathcal{S}$.
Lean: `GlobalTheory.PcBF.pieces_bf` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.BesselToFourierJacobiMixedPiecesAdmissible`, lines 50–66)
Lean: `GlobalTheory.AscFin.piecesFB_is_and_splits` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.FourierJacobiStepPiecesAdmissibleSplits`, lines 19–25)

Lean: `GlobalTheory.AscFin.GoodStd.loc_cls` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.StandardAuxiliaryFramesGoodRowGrids`, lines 94–116)

There is a $b_* \in \mathcal{S}$ with $Rb_* \neq 0$. Take the two finite coset indicators above, smooth compactly supported real functions equal to one at x, y, and a nonzero middle lattice vacuum. The finite u for which $xu - x \in A_1$ have $\chi_U(u_f) = 1$ because the successive entries in a standard isotropic frame lie in the conductor of the trace character. At infinity use a nonnegative first function. Since x has a left inverse, its support can be chosen so small around x that $\operatorname{Re}\chi_U(u_\infty) > 0$ wherever it contributes. The integral defining Rb_* then has a scalar factor of positive real part.

Write K_i for the compact open row level at height i. The restriction $R(\omega(1,h)b)$, for $b \in \mathcal{S}$, is supported on $R_x\,\mathrm{U}(W)(F_\infty)K_i$. We give the finite-place calculation. Nonvanishing requires $h^{-1}xu - x \in A_1$ locally for some unitriangular u. Translation of u by a strictly upper triangular entry preserving this coset must have trivial simple-root character. The second dual block must preserve the character specified by y. The same conditions hold after replacing h by hk, $k \in K_i$, since the row level fixes the test. At a marked split place this last invariance follows from the action on the selected coset modulo depth and triviality of the splitting characters on the principal determinant units. Off the marked places the self-dual lattice vacuum is fixed by the integral group.

At a marked place write the split action on the two components as a matrix and its inverse transpose in dual frames. After scaling the specified columns and dual rows to coordinate vectors and covectors, the condition on $h^{-1}xu$ prescribes columns $\xi_p \equiv e_{a_p}$ and covectors $\eta_q \equiv e^*_{b_q}$ modulo the level ideal. Their pairings are $\eta_q\xi_p = \delta_{b_q,a_p}$ since the Gram matrix of x is zero except possibly at its last diagonal entry, which upper unitriangular changes preserve. Complete the prescribed columns by the unused standard columns to a matrix $C \equiv 1$. Replace rows b_q of the identity by $\eta_q C$ to form $D \equiv 1$. The pairing identity gives $De_{a_p} = e_{a_p}$. Thus CD^{-1} realizes the prescribed columns and inverse row action and belongs to the principal congruence group. Removing this element leaves an element of R_x.

At a good place use the Witt Iwasawa decomposition for the self-dual row lattice and remove the integral right factor. Write $t_1, \dots, t_r$ for the diagonal entries on the isotropic flag. The column condition gives t_j^{-1} integral. On the root $j, j+1$ the translation condition forces t_j integral since t_j multiplies the ideal on which the primitive simple-root character is trivial into that same ideal. For the isotropic tuple the final entry uses the second character, paired with the primitive last column of y. For the tuple with an appended line vector the final simple root is available as well. All the t_j are units. In the appended-line case the column condition also makes the transported line lattice integral in the middle lattice. It is a self-dual summand there, since the transported vector has its prescribed norm and the initial line is a self-dual summand. An integral isometry carries it, with its generator in a local line frame, to the original summand upon splitting off the line and identifying its unimodular orthogonal complement by the local classification. At a dyadic place the grid is split and this identification is ordinary completion to dual bases. The remaining middle isometry fixes the distinguished vector. In the isotropic-tuple case the entire middle group already belongs to R_x. This proves the support assertion on taking the restricted product.

Lean: `GlobalTheory.AscFinNe.flat_ne_zero` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.ExistsUnfoldingFlatMapNeZeroFullRank`, lines 42–73)
Lean: `GlobalTheory.AscFin.OneOrbitAt.core_coord` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.SplitPlaceTuplesOneOrbitModUniformizerPower`, lines 103–211)
Lean: `GlobalTheory.OneCosetGood.oneCoset_good` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.GoodPlaceOneCosetOfBesselLevel`, lines 119–140)
Lean: `GlobalTheory.AscFin.OneCosetFJ.oneCoset_good` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.FourierJacobiLevelOneCosetAtGoodPlace`, lines 241–253)

Suppose the outgoing period vanishes on $\mathcal{S}$. Unfold its flag integral in the first mixed coordinate. Integration over the rational additive quotients selects the tuples with the prescribed Gram matrix and successive flag character. The surviving full-rank tuples form the orbit of x under rational isometries and upper unitriangular changes, since an isometry between their column spans extends by Witt extension. A dependent tuple gives zero by integration over a stabilizing root group with nontrivial character. The integral is therefore over $R_x(F)\backslash \mathrm{U}(W)(\mathbb{A}_F)$, with the integration in u appearing in R. The second mixed coordinate is evaluated at y by additive character orthogonality.

For precision about cancellation in this integral, translate the test by a real vector c in the first opposite isotropic block. This keeps the test in $\mathcal{S}$ and multiplies the unfolded integrand by the character of c paired with $(h^{-1}xu)_\infty$. All Fourier transforms of the pushforward of the integrand to this real matrix space thus vanish. The integrand is absolutely integrable by the height bound above and rapid decrease. Uniqueness of

Fourier transform for finite measures gives zero integral on every strip $R_x C\,\mathrm{U}(W)(\mathbb{A}_{F,f})$, C compact in the real group. Indeed membership is determined by whether the real matrix lies in $C^{-1}xU_a(F_\infty \otimes_F E)$. This is a measurable set, since the full-rank matrix x has a left inverse and the unitriangular group is closed.

Average first along $R_x(F)\backslash R_x(\mathbb{A}_F)$. The resulting mean is continuous, left $R_x(\mathbb{A}_F)$-invariant and right K_i-invariant. If its value at the identity were $z \neq 0$, choose a compact real neighborhood C of the identity on which the real part of the mean divided by z exceeds $1/2$. By the support assertion, every nonzero value on the strip is represented by a point of C, since the stabilizer splits into its real and finite components and the right finite factor can be taken in K_i. The strip integral divided by z would have positive real part, since K_i is open. Hence the mean at the identity vanishes. This mean is the incoming functional on Rb. For the isotropic tuple it is the Fourier–Jacobi integral with the entire middle test Rb. For the appended tuple it is the Bessel integral after contracting Rb with the fixed line test in the line variable.

Lean: `GlobalTheory.StripVanish.integral_strip_eq_zero` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.IntegralStripEqZeroOfRealModulationVanishing`, lines 70–85)
Lean: `GlobalTheory.Asc.vanBF` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.VanishingDescendsBesselToFourierJacobi`, lines 127–223)
Lean: `GlobalTheory.Asc.VanJac.vanFB` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.VanishingDescendsFourierJacobiToBessel`, lines 237–324)

In particular the incoming functional vanishes on $H(w)Rb_*$ for $w \in \Gamma_*^\circ$, by translating b_* before restriction. Lemma 34.21 gives vanishing on every middle test in $D(\Gamma_*)$. Evaluate this conclusion on the following products.

At $i = 2r+1$, let b_1 be the conjugate downward core test and c the incoming line test. The link of lines sends ζ_r to $-\zeta_{r+1}$ by a similitude of multiplier $\xi_0\bar{\xi}_0$. Composition with this map followed by the normalized Hermitian adjoint gives an invertible map k from the old line Hom space to the new one, reversing the alternating form. Set $b_2 = \overline{c \circ k^{-1}}$. Then $\overline{b_2 \circ k} = c$, with the original level. The middle projections send the middle depth shifts into the core depth shifts and, through the link, into the line depth shifts. Hence $b_1 \otimes b_2 \in D(\Gamma_*)$. The Weil action on this product is the product of the core and line actions. Its line argument is k of the incoming argument and its central phase has opposite sign. Evaluating the middle functional therefore gives exactly the incoming Fourier–Jacobi period, with $\bar{F}_i = \Theta_{b_1}\bar{F}_{i+1}$.

At $i = 2r+2$, choose a nonzero vacuum c at the new line depth, and use the middle test $b_1 \otimes c$. More explicitly, if $\mathfrak{a} = (\mathfrak{c}_K\mathcal{O}_E)\mathfrak{b}^{-1}$ and $\mathfrak{s} = \mathfrak{a}/\bar{\mathfrak{m}}$, the line lattice for flag length $j = r+1$ is formed with the line module $(\mathfrak{s}/\bar{\mathfrak{s}})^j(\mathfrak{m}/\bar{\mathfrak{m}})\mathcal{O}_E\zeta_0$. This is the line component in the successive Witt frames. Projection to that component and to the core carries a middle depth shift into the respective lattices. At each split half the projections preserve that half. This gives $b_1 \otimes c \in D(\Gamma_*)$. Contracting the line factor against c gives (c,c) by unitarity of the line action. The remaining integral is the incoming Bessel period with $\bar{F}_i = \Theta_{b_1}\bar{F}_{i+1}$. Its multiplier (c,c) is positive.

Lean: `GlobalTheory.LinkInvTheta.link_inv_theta` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.LinkedLineVectorsInvertibleOnThetaFunctions`, lines 69–131)
Lean: `GlobalTheory.LinkProd.perAll_box_eq_per` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.PeriodAllDataProductsEqFourierJacobiPeriod`, lines 51–86)
Lean: `GlobalTheory.HandProd.handP_boxT` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.BesselHandedFunctionProductEqCoreLiftMulPairing`, lines 16–37)

In each case a nonzero incoming period contradicts vanishing of the outgoing period on $\mathcal{S}$. The latter therefore has a nonzero value in $D(\Gamma_i)$. Lemma 34.22 gives a pair (λ_i, P_i) with nonzero value. The partial row is right K_i-invariant at every step where the strip argument uses it because the preceding prescribed test is fixed by the output level, which can be moved onto the kernel by right translation. Updating the two entries at i extends the existing partial row lift.

Lean: `GlobalTheory.UpBessel.upBF` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.FourierJacobiPeriodGivesBesselPeriodOfLift`, lines 18–43)
Lean: `GlobalTheory.UpFromBessel.upFB` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.BesselPeriodGivesFourierJacobiPeriodOfLift`, lines 64–82)
Lean: `GlobalTheory.AscRungB.rungB` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.FourierJacobiToBesselNonvanishingTower`, lines 39–108)
Lean: `GlobalTheory.Asc.RungJac.rungJ` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.BesselToFourierJacobiNonvanishingTower`, lines 57–132)

Thus height 1 gives height 2, and a Bessel period at height $2r+2 < m$ gives a Fourier–Jacobi period at $2r+3$ and then a Bessel period at $2r+4$. The grid has even $m \geq 2$. At height m the frame character is the conjugate of the prescribed top character. Indeed the successive isotropic columns have scale factors $(-\xi_0)^j$, $1 \leq j \leq (m-2)/2$, and the marked line has the next scale factor. Each simple entry and the final line entry

therefore acquire the same ratio $-\xi_0$. Conjugating the integral gives the stated top period up to the positive ratio of the two Haar normalizations.

Lean: GlobalTheory.AscFin.EndsPf.Chr.psiBes_eq_star_besChar (Development.AbelianVarietyTheory.Proofs.TwistedOperators.StandardAuxiliaryFramesFitGridEnds, lines 129–156)
Lean: GlobalTheory.AscFin.EndsPf.per_conj (Development.AbelianVarietyTheory.Proofs.TwistedOperators.StandardAuxiliaryFramesFitGridEnds, lines 213–237)

□

Proof of 34.25

Proof. Fix $w \neq 0$ and a nonzero row t. Use the threshold of Lemma 34.30. The level primes $p_{\mathrm{de}}, p'_{\mathrm{de}}$ split away from $Q_S q$. The size hypotheses give $44^4 \leq m^4 \leq f \leq k$ and $r = k+m-1 > 1800$. The grid separation with weight-gap constant 10 gives consecutive center differences greater than twice the cube of the grid rank bound, as required by Lemma 34.33. The given realization supplies the bottom discrete-series model and its compatible Cartan frame.

Put $u = t(w)$. Lemma 34.30 gives a core value with nonzero period against u. By Lemma 34.33 it is a finite linear combination of bottom values of downward column chains. Each bottom value is bounded by rapid decrease, and u is integrable by cuspidal rapid decrease in Proposition 1.22. Integrate the finite combination. One chain has nonzero bottom period. Fix that chain, including its terminal column Ψ and coefficient covector ξ.

Lean: GlobalTheory.RowSizes.sizes_of_sizeRows (Development.AbelianVarietyTheory.Proofs.TwistedOperators.SizesOfSizeRows, lines 14–34)
Lean: GlobalTheory.SepRows.WeakSepPf.exists_reach_of_span (Development.AbelianVarietyTheory.Proofs.TwistedOperators.RowSeparationByRealFunctionalsOfChainSteps, lines 40–67)

For the partial row lifts, at each arrow the real theta intertwiner takes the previous smooth vector and the polynomial of that arrow to a vector of the next discrete series. The finite coset test takes the previous multiplicity map to a cuspidal multiplicity map there. Indeed the signed source and target lists are the successive grid lists with the shift of the splitting character at $m - i$; the two splitting exponents are that center with the half-unit correction prescribed by the ranks. The lower real lifts vanish by the real parameter rule, so the proper constant terms vanish by the tower property of Theorem 9.5. The zero lift is included. At a split depth place each coset vacuum has the same real vacuum equations, since its construction changes the finite factor. Iterating gives convergence and expresses each partial lift as a smooth vector evaluated under a cuspidal multiplicity map. Cuspidal rapid decrease in Proposition 1.22 thus applies at every partial height. Lemma 34.23 gives the polynomials $\rho = (P_i)_i$ and the cosets $\Phi = (\lambda_i)_i$ for the fixed terminal column. In particular

$$\forall w \neq 0 \quad \forall t \neq 0 \quad \exists \rho, \xi, \Psi, \Phi \quad \mathcal{B}_{\rho,\xi,\Psi,\Phi}(t(w)) \neq 0. \tag{A.380}$$

Lean: GlobalTheory.RowArrow.separatingFamily_extends_by_twisted_row (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.SeparatingFamilyExtendsByTwistedRow, lines 16–74)
Lean: GlobalTheory.SepRows.RowRapidPf.rowRapid (Development.AbelianVarietyTheory.Proofs.TwistedOperators.LevelTowerRapidDecay, lines 20–33)

It remains to fix (ρ, ξ) before choosing t. Write $v_\rho(w)$ for the terminal vector obtained by composing the real intertwiners. It is independent of the finite cosets. At each real place the terminal row representation is a smooth discrete series, and the coefficient representation of the definite middle group is finite-dimensional and irreducible. The two are smooth admissible representations of moderate growth with infinitesimal character. The isotropic flag, negative distinguished vector and its positive definite orthogonal complement give the unitary Bessel subgroup. Its unitary character is nontrivial on each of the indicated simple-root coordinates. The terminal signed parameter is dual to the translated grid list. Translation and duality preserve the absolute differences on opposite-sign pairs, so the terminal gap in the grid gives

$$\mathrm{gap}_{\mathrm{opp}} \geq 120(\operatorname{rank} Y_m^{\mathrm{col}})^3. \tag{A.381}$$

Fixing the other real arguments of the period gives a continuous equivariant bilinear form. The space of these forms has dimension at most one by Lemma 33.23 (compare [144, Theorem A] in general).

Lean: UpperTheory.Lit.Bur.Gap.fit_gap (Development.AutomorphicArithmetic.ThetaCorrespondence.BesselModels.GapArith, lines 293–310)
Lean: UpperTheory.Lit.Bur.burGap_model (Development.AutomorphicArithmetic.ThetaCorrespondence.BesselModels.SepGap, lines 34–59)

If the row space is zero, any real pair works. Otherwise (A.380) gives a nonzero global period, so each local space of forms is nonzero. Choose a generator at each real place and arguments on which it has value one. Evaluate successively at those arguments. Multilinearity on pure tensors and linearity on their span give

$$\mathcal{B}_{\rho,\xi,\Psi,\Phi}(t(w)) = A_w(\rho,\xi)c_{t,\Psi,\Phi}, \qquad A_w(\rho,\xi) = \ell(\xi, v_\rho(w)), \tag{A.382}$$

where ℓ is the tensor product of the chosen generators. Apply (A.380) to one nonzero row to choose (ρ,ξ) with $A_w(\rho,\xi) \neq 0$. For each nonzero t, the same equation gives (Ψ,Φ) with $c_{t,\Psi,\Phi} \neq 0$. Their product detects t at the chosen real pair. □

Proof of 34.26

Proof. The weights of the dual defining representation are $-e_s$, each with multiplicity one. Every $\lambda - e_s$ is regular and in the chamber of λ, by the spacing assumption. The discrete-series coherent character identity therefore gives the sum of the characters of precisely these discrete series [154, §§7–8, in particular (8.1)–(8.4)]. Here the Cartan subgroup is the diagonal compact torus, and the shifts are integral characters of that torus. The decreasing lists $\lambda - e_s$ are distinct, so the summands have distinct infinitesimal characters. Taking the primary spaces for the center gives a direct sum on compact-finite vectors, each primary space having length one. Smooth globalization gives the same decomposition on smooth vectors by 36.9. Write R_s for projection followed by an isomorphism to the smooth unitary model of π_s.

Let ϵ_l be the signature of coordinate l. The tensor action preserves the nondegenerate Hermitian form

$$[y,x] = \sum_l \epsilon_l \langle y_l, x_l\rangle,$$

with the inner product linear in the second variable. Put $D_{ab} = D(E_{ab})$ for the matrix units, and let $\mathcal{E}$ be the operator matrix with entries D_{ab}. The trace-form Casimir on the tensor is the sum of the two individual Casimirs minus $2\mathcal{E}$, since the dual defining action of E_{ab} is $-E_{ba}$. The Casimir eigenvalues on π and π_s are $\|\lambda\|^2 - \|\rho\|^2$ and $\|\lambda - e_s\|^2 - \|\rho\|^2$, where ρ is the half-sum of the positive roots. The eigenvalue on the dual defining representation is n. Thus $\mathcal{E}$ acts on summand s by $\lambda_s + (n-1)/2$. It is symmetric for $[\,,\,]$, since $D_{ab}^* = \epsilon_a\epsilon_b D_{ba}$ on smooth vectors. Its distinct real eigenspaces are orthogonal for this form. Schur's lemma on the irreducible compact-finite module, followed by continuity on smooth vectors, now gives

$$[y, p_s x] = c_s \langle R_s y, R_s x\rangle.$$

Nondegeneracy gives $c_s \neq 0$.

We compute p_s. For ordered row and column lists I, J of equal length m, form the minor

$$C_{I,J}(u) = \sum_{\tau\in S_m} \operatorname{sgn}(\tau) \prod_{k=1}^{m} \big(D_{I_{\tau(k)},J_k} + (u-k+1)\delta_{I_{\tau(k)},J_k}\big),$$

with factors in increasing order of k from left to right. Let $C(u)$ be the full minor, and $A(u)_{cl} = (-1)^{c+l} C_{\widehat{l},\widehat{c}}(u)$, where a hat means deletion from the full list. The quantum comatrix identity is

$$A(u)\big((u-n+1)I + \mathcal{E}\big) = C(u)I.$$

This is the evaluation $T_{ab}(u) = \delta_{ab} + D_{ab}/u$ of the comatrix identity, after clearing the scalar denominators [155, Propositions 2.2 and 2.27, equation (2.30)]. The evaluation applies because the D_{ab} satisfy the matrix-unit commutation relations.

Put $m_j = \lambda_j - (n-1)/2$. The scalar of $C(u)$ is $\prod_j (u+m_j)$. Indeed the quantum determinant is central. On a highest-weight vector of weight $\lambda - \rho$ its diagonal product is $\prod_j (u - j + 1 + \lambda_j - \rho_j)$, and in every other product the rightmost displaced column supplies a raising operator. This computes its Harish–Chandra polynomial and hence its scalar at the infinitesimal character of π. On summand s, the comatrix identity gives

$$(u+m_s)A(u) = \prod_j (u+m_j)\, I.$$

Cancel the polynomial factor before substituting $u = -m_s$. On all other summands $A(-m_s) = 0$, and consequently

$$p_s = \frac{A(-m_s)}{\prod_{j \neq s}(\lambda_j - \lambda_s)}.$$

The zz-entry of $A(u)$ is the full Capelli determinant on the complementary coordinates. A simultaneous permutation of its rows and columns preserves the minor, by alternation in each list. The map P commutes with the complementary derived matrix units, by differentiation of its H-equivariance. The preceding central-character calculation in rank $n-1$ therefore evaluates the numerator on σ as $\prod_a(-m_s + \mu_a - (n-2)/2)$. This gives the stated γ_s. Each denominator factor has absolute value at least one and each numerator factor has absolute value at most $2B + 1/2 \leq X$.

To bound the other factor, let $K = \mathrm{U}(p) \times \mathrm{U}(q)$, and let V_k be the span of words of at most k derived matrix units applied to the lowest K-type. It is a finite-dimensional K-module. Write ρ_c, ρ_n for the compact and noncompact half-sums. The lowest highest weight is $\lambda + \rho_n - \rho_c$ by 1.22. In passing from V_k to V_{k+1} the possible new types occur in the tensor product with the matrix representation under conjugation. On K its blocks are a defining representation tensored with a dual defining representation. The one-box Pieri rule on each unitary factor adds one coordinate and subtracts one, with the dominant intermediate weights retained. Thus every highest weight ν in V_k satisfies

$$\|\nu + \rho_c\| \leq \|\lambda + \rho_n\| + \sqrt{2}\,k.$$

Let $\Omega_K = \sum_{\epsilon_a = \epsilon_b} D_{ab} D_{ba}$ and $\Omega_G = \sum_{a,b} D_{ab} D_{ba}$. The adjoint formula gives, for $x \in V_k$,

$$\sum_{a,b} \|D_{ab} x\|^2 = \langle x, (2\Omega_K - \Omega_G) x \rangle.$$

The compact Casimir on type ν is $\|\nu + \rho_c\|^2 - \|\rho_c\|^2$. Each coordinate of ρ_n and of ρ has modulus at most $n/2$. The opposite-sign gap gives $B \geq 500n^3$. For $k \leq S$ the last display is consequently bounded by

$$\big(2(\sqrt{n}(B + n/2) + \sqrt{2}\,S)^2 + n(n/2)^2\big)\|x\|^2 \leq 64n(B+S)^2\|x\|^2.$$

Cauchy–Schwarz gives the corresponding bound on $D(A)x$ for $\|A\|_{\mathrm{HS}} \leq 1$. Since $D(A)V_k \subseteq V_{k+1}$, iteration proves the word estimate.

At $u = -m_s$ every scalar in a minor of length $n-1$ has modulus at most $4B$. In each permutation product choose the factors contributing a matrix unit. The word estimate with $S = n-1$, summed over these choices, gives

$$\|(p_s(e_l v))_l\| \leq (n-1)!\big(4B + 8\sqrt{n}(B + n - 1)\big)^{n-1} \leq X^{4n}.$$

For the last inequality use $(n-1)! \leq X^{n-1}$ and $4B + 8\sqrt{n}(B + n - 1) \leq X^2$. Taking $x = y = e_l v$ in the signed form identity proves the asserted bound on $|c_s|\|R_s(e_l v)\|^2$. Finally use the formal-degree product of 1.22. The Haar normalization cancels in the ratio. Only the $n-1$ differences involving s change, and each old absolute difference is at most twice the new one, since the new one is at least one. This gives $d_\pi / d_{\pi_s} \leq 2^{n-1} \leq X^n$. □

Proof of 34.27

Proof. For either unitary group, of matrix size $a \leq n$, the set $\|g\|_{\mathrm{op}} \leq R$, $R \geq 1$, is covered by at most

$$(1 + 2\sqrt{n}R^2/\delta)^{2n^2}$$

left translates of the Hilbert–Schmidt δ-ball at the identity. Indeed, take a maximal δ/R-separated subset in the ambient matrix space. The disjoint balls of half that radius about the selected points lie in the ball of radius $\sqrt{n}R + \delta/(2R)$. The covering follows from $\|g^{-1}\|_{\mathrm{op}} = \|g\|_{\mathrm{op}}$, since inversion is adjoint followed by conjugation by the signature matrix.

Put $k = \lfloor \log \|h\|_{\mathrm{op}} \rfloor$. In Cartan coordinates the length d of Lemma 37.16 satisfies $d \geq \sqrt{2}\,k$, since the extreme singular values are reciprocal. The row signature has smaller index q, and its opposite-sign gap $h_0 \geq 1000n^3$ exceeds $120(n-1)q^{3/2}$. For $k \geq 1$, the local compression bound of that lemma gives

$$|\langle w, \sigma(h) w \rangle| \leq \exp(-125n^{5/2}k).$$

For $k=0$ use unitarity. Apply the covering bound with $R=e^{k+1}$, $\delta=1$. After division by $\operatorname{vol}(B_H(1))$, the integral of the absolute coefficient is at most

$$(1+2\sqrt{n}e^2)^{2n^2}\sum_{k\ge 0}\exp\bigl((4n^2-125n^{5/2})k\bigr)\le 2(1+2\sqrt{n}e^2)^{2n^2}\le X^{67n^2}.$$

The squared coefficient is bounded by the absolute coefficient and integrates to $1/d_\sigma$ by 1.22.

A coset Hg is determined by $g^{-1}z=(u,v)$ in the positive and negative coordinate spaces. Here $\|u\|^2-\|v\|^2=1$. Compact rotations put u,v on coordinate axes. The hyperbolic rotation with $\cosh t=\|u\|$, $\sinh t=\|v\|$ then gives a representative of operator norm $e^t\le 2\|g^{-1}z\|$. In D this is at most $3X^{4n}$, so its Hilbert–Schmidt norm is at most X^{10n}. Left multiplication by $B_H(1)$ keeps its operator norm at most $6X^{4n}$. Quotient integration bounds $\operatorname{vol}(D)\operatorname{vol}(B_H(1))$ by the volume of this last operator-norm ball in G.

Take $\delta=(64n(B+1))^{-1}$. If $\|g-1\|_{\mathrm{HS}}<\delta$, the logarithm series gives $g=\exp A$ with $\|A\|_{\mathrm{HS}}\le 2\delta$ and A in the real Lie algebra of G. For a lowest-type unit vector v_0 the length-one estimate in Lemma 34.26, integrated on $\exp(tA)$, gives $\|\pi(g)v_0-v_0\|\le 16\delta\sqrt{n}(B+1)\le 1/2$. Schur orthogonality now gives $d_\pi\operatorname{vol}(B_G(\delta))\le 4$. In the covering bound for $R=6X^{4n}$ we have $1+2\sqrt{n}R^2/\delta\le X^{12n}$, using $n\ge 4$ and $B\ge 500n^3$. This proves the quotient bound. □

Proof of 34.28

Proof. Put $\xi=iw$, $P=i^*$, and $\phi(h)=\langle w,\sigma(h)w\rangle$. By Lemma 34.27, $f=d_\sigma\overline{\phi}$ belongs to $L^1(H)$. In any unitary representation τ of H set $Q_\tau=\int_H f(h)\tau(h)\,dh$. Schur orthogonality gives $f^*=f$ and

$$f*\delta_h*f=\phi(h)f,$$

where δ_h is unit point mass. Indeed the convolution at x, divided by d_σ^2, is the integral of $\overline{\langle w,\sigma(y)w\rangle}\langle\sigma(x)w,\sigma(y)\sigma(h)w\rangle$. Thus Q_τ is an orthogonal projection and $Q_\tau\tau(h)Q_\tau=\phi(h)Q_\tau$.

A unit vector in the image has coefficient ϕ. The Gram matrices of its H-translates are therefore the Gram matrices of the translates of w, so its cyclic subrepresentation is an isometric copy of σ. He's discrete branching theorem gives multiplicity one for the stated interlacing [152, Theorem 0.1]. On that copy, Schur orthogonality gives the projection onto $\mathbf{C}iw$. Hence $Q_{\pi|_H}u=\xi\langle\xi,u\rangle$. Choose a G-smooth η with $a=\langle\xi,\eta\rangle\neq 0$, using density.

Write $L(g)$ for the tensor action. It satisfies $(L(g)e_lv)_z=(g^{-1})_{lz}\pi(g)v$. In the s-th unitary summand of Lemma 34.26 put

$$b_s=Q_{\pi_s|_H}R_s(e_z\eta).$$

Since H fixes the positive coordinate z, insertion in that coordinate intertwines the H-actions. For every smooth tuple x, integrate the signed form identity of Lemma 34.26 with $y=e_z\eta$ and with x replaced by $L(h)x$, against $f(h)\,dh$. Each side is an integrable scalar coefficient. Self-adjointness of the two projections gives

$$\overline{a}\,\langle\xi,(p_sx)_z\rangle=c_s\langle b_s,R_sx\rangle.$$

At $x=e_z\eta$, the left side equals $\gamma_s|a|^2$, by the corner identity of Lemma 34.26 with $P=i^*$. The right side is $c_s\|b_s\|^2$. Consequently, at $x=L(g)e_lv_0$, Schur orthogonality in π_s gives

$$\int_G|\langle\xi,(p_sL(g)e_lv_0)_z\rangle|^2\,dg=\frac{|\gamma_s|\,|c_s|\,\|R_s(e_lv_0)\|^2}{d_{\pi_s}}\le\frac{X^{6n}}{d_\pi}.$$

The factors in this inequality have exponents $n,4n,n$, respectively, by Lemma 34.26. The choice of η has canceled.

Sum the projections to recover $(g^{-1})_{lz}\langle\xi,\pi(g)v_0\rangle$. Cauchy–Schwarz for the sum of n projections costs n; summing the integral over s and l costs n^2. Since $n\le X$ and $6n+3\le 8n$, this proves both inequalities. □

Proof of 34.29

Proof. At τ let π be the core representation and σ the conjugate row representation. In the combined decreasing list of the two interlaced grid rows with the inserted positive entry, adjacent signs agree exactly

when the entries come from different lists. The rule persists after lowering one core entry by one. The within-list spacings are at least two and opposite-sign gaps at least $1000n^3$. The row has both signs and at least as many positive signs as negative. Eventually $r \geq 3$. Both lists have absolute entries at most $B = b^2$. Put $X_0 = 8nB$; the sizes give

$$\log X_0 \leq 6 \log b. \tag{A.383}$$

For $H = \mathrm{U}(Y_\tau)$, He supplies the isometric discrete branching embedding $i : \sigma \to \pi|_H$ for this interlacing [152, Theorem 0.1]. Choose lowest-type unit vectors v_0, w in π, σ. For formal degrees with the chosen Haar measures, the density $|\langle iw, \pi(g)v_0\rangle|^2$ has mass $1/\deg \pi$. The inserted positive coordinate is z_0, and deleting it gives the row signature. The spacing, opposite-sign gaps and bounds on the two lists above give the hypotheses of Lemma 34.26 with $X = X_0$. Their interlacing gives the remaining hypothesis of Lemma 34.28. Its column moment is at most $X_0^{8n}/\deg \pi$.

Lean: `PositivityTheory.UnitaryDecomposition.momentHeA_of` (`Development.WeilBound.RepresentationTheoryGL.HolomorphicDiscreteSeries.SchurEigenProjections.HolomorphicMoments.MomentHigherEstimateHolds`, lines 28–179)

Set $\mathcal{D} = \{g : \|g^{-1}z_0\|^2 \leq 2X_0^{8n}\}$. Since H fixes z_0, $(hg)^{-1}z_0 = g^{-1}z_0$. Thus the column bound cuts out a subset of the left quotient, whose preimage retains at least half the mass. Integrating on each left H-fibre gives $\|i^*\pi(g)v_0\|^2/\deg \sigma$. With compatible measures the bounds are

$$\begin{aligned} (\deg \sigma)^{-1} &\leq X_0^{67n^2} \operatorname{vol}(\mathrm{ball}_H(1)), \\ (\deg \pi) \operatorname{vol}(H\backslash\mathcal{D}) &\leq 4X_0^{24n^3}/\operatorname{vol}(\mathrm{ball}_H(1)). \end{aligned} \tag{A.384}$$

Both volume inequalities follow from Lemma 34.27. The integral of $\|i^*\pi(g)v_0\|^2$ over $H\backslash\mathcal{D}$ is at least $(\deg \sigma)/(2\deg \pi)$ by the fibre calculation. At some point its value is at least half its average. Substituting the two volume bounds cancels $\operatorname{vol}(B_H(1))$. The norm of the projection is constant on the left coset by equivariance and unitarity. Taking the representative given by Lemma 34.27 gives g_τ, $v = \pi(g_\tau)v_0$, with

$$\|g_\tau\|_{\mathrm{HS}} \leq X_0^{10n}, \qquad 1 \leq 16X_0^{67n^2+24n^3}\|i^*v\|^2. \tag{A.385}$$

Lean: `UpperTheory.Lit.Branch.projLarge_of` (`Development.AutomorphicArithmetic.ThetaCorrespondence.FockModels.RealBranch`, lines 140–194)

Lean: `PositivityTheory.RgHe.A.Pf.bigProjP_of` (`Development.WeilBound.RepresentationTheoryGL.HolomorphicDiscreteSeries.SchurEigenProjections.SchurFiniteDimensionality.IsFdOfSchurK`, lines 53–275)

With inner product linear in the second argument set $C(g) = \langle v, \pi(g)v\rangle$ and $A = \int_H \overline{C(jh)}\sigma(h)\,dh$. The inversion law makes A self-adjoint, and Plancherel orthogonality on the discrete-series coefficient space makes it Hilbert–Schmidt. The embedded σ gives the rank-one contribution and the remaining coefficient is of positive type. Thus

$$\langle a, Aa\rangle \geq |\langle i^*v, a\rangle|^2/\deg \sigma. \tag{A.386}$$

The compact spectral theorem gives $w' \neq 0$ with eigenvalue $c_\tau \geq \|i^*v\|^2/(2\deg \sigma) > 0$. The weak coefficient integral

$$Tb = c_\tau^{-1}\int_H \langle \pi(jh)v, b\rangle\, \sigma(h)w'\,dh$$

defines a bounded map by Schur orthogonality and intertwines by change of variables. It sends v to w', which is therefore smooth. Normalize and transport back the conjugate of w' to the row model, with its Haar factor, and take $k_\tau(g) = C(g)$ in the transported frame.

Lean: `GlobalTheory.CompanionTheory.RgHeA.eigA` (`Development.AutomorphicSpectrum.Proofs.BlockDecomposition.ExistsEigOfLow`, lines 72–93)

Lean: `UpperTheory.Lit.Branch.act_of_eigen` (`Development.AutomorphicArithmetic.ThetaCorrespondence.RealBands.BranchPack`, lines 60–105)

For an absolute C_1 the two real estimates are

$$\begin{aligned} \int_H |k_\tau(gj(h))|\,dh &\leq c_\tau X_0^{C_1 n^4}\|gz_0\|^{-(2n+5)}, \\ \int_H |k_\tau(j(\exp(sX)h)) - k_\tau(j(h))|\,dh &\leq c_\tau X_0^{C_1 n^4}|s|\|X\|_{\mathrm{HS}}. \end{aligned} \tag{A.387}$$

For the first, integrate the lowest-type decay of Lemma 37.16 on the stabilizer in Cartan coordinates. The extra column gives power $2r + 7 = 2n + 5$. For the second, integrate the first derivative along the translation.

The lowest-type first-order Lie estimate supplies a factor bounded by a power of X_0 times $\|X\|_{\mathrm{HS}}$. The remaining negative norm power is integrable by the opposite-sign gap, with integral bounded by a power of X_0 times the radius-$1/4$ stabilizer ball volume. The small-ball argument at the end of the proof of Lemma 34.27 applies to σ on H and gives $(\deg\sigma)\,\mathrm{vol}(B_H(\delta)) \le 4$ for $\delta = 1/(4X_0^2)$. The covering bound of that proof covers $B_H(1/4)$ by at most $(1+13\sqrt{n}X_0^2)^{2n^2}$ translates of $B_H(\delta)$, so $\mathrm{vol}(B_H(1/4)) \le 4(1+13\sqrt{n}X_0^2)^{2n^2}/\deg\sigma$. With $c_\tau \ge \|i^*v\|^2/(2\deg\sigma)$ and (A.385), this bounds $\mathrm{vol}(B_H(1/4))/c_\tau$ by a power of X_0. Include the conjugation factors from $\|g_\tau\|_{\mathrm{HS}}$ in $C_1 n^4$.

Lean: `UpperTheory.Lit.Branch.realInputE` (`Development.AutomorphicArithmetic.ThetaCorrespondence.RealBands.BranchInput`, lines 41–138)
Lean: `UpperTheory.Lit.realInputE_holds` (`Development.AutomorphicArithmetic.ThetaCorrespondence.FockModels.RealBranch`, lines 231–235)

Tensor the row vectors. Haar changes multiply the coefficient bounds and eigenvalues by the same factors. Equation (A.383) gives the total loss. The pullback of the finite core level fixes the row vector, and its indicator contributes its positive volume to c. The support congruence gives $z - z_0 \in \mathfrak{m}\Lambda_{j(y)}$. This proves (34.15) and (34.16). Translate (34.15) by $\exp(sX)$, subtract, and use the second integral bound of (A.387). □

Proof of 34.30

Proof. Use $u = t(w_*)$ of Lemma 34.29 and $M = \sup|u|$, finite by cuspidal decay [98] and positive by cyclicity. Choose x_* with $|u(x_*)| > M/2$. Put $N = \mathrm{N}\,\mathfrak{m}$, $g = g_0 g_1$, $Y_* = r^2 f g\,\mathrm{rd}(E)$, and $s = (2e^{L_{\mathrm{loss}}}Y_*)^{-m}/2$. Then

$$0 < s \le \tfrac{1}{2}, \qquad e^{L_{\mathrm{loss}}}Y_*(2s)^{1/m} = \tfrac{1}{2}, \qquad R := s^f\sqrt{N} \ge 2e^{L_{\mathrm{loss}}+(6n+9)f} > 1. \tag{A.388}$$

Indeed the first auxiliary place gives $\log N > C_F f^2 n^6 \log b$. Use $m \le f \le k$, $n = k+m$ and (34.12)–(34.14) to get $\log Y_* \le n^3 + 21\log b$. For $n \ge C+1$, $L \ge 1$ the needed logarithmic bound is

$$\log 2 + L_{\mathrm{loss}} + (6n+9)f + f\{\log 2 + m(\log 2 + L_{\mathrm{loss}} + \log Y_*)\} \le (2C+42)f^2 n^5 \log b \le \tfrac{1}{2}\log N. \tag{A.389}$$

Lean: `GlobalTheory.SepPer.depth_pays` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.GridDepthPaysEveryLoss`, lines 148–276)

Let Λ_{Y,x_*} have finite completion $(x_*)_f\widehat{\Lambda}_Y$. For every integral ideal $\mathfrak{a} \ne 0$ and nonzero $v \in \mathfrak{a}\Lambda_{Y,x_*}$, one has

$$\prod_\tau \|(x_*)_\tau^{-1}v_\tau\| \ge s^f\sqrt{\mathrm{N}\,\mathfrak{a}}. \tag{A.390}$$

Otherwise the least lattice scale $\lambda < s$. Take the canonical Harder–Narasimhan filtration of Λ_{Y,x_*} with cutoffs between λ and $1/2$, having at most m active layers by the smaller signature index. The product of consecutive scale ratios is 2λ, so some adjacent lower-to-upper cut has ratio $\lambda_{\mathrm{low}}/\lambda_{\mathrm{up}} \le (2\lambda)^{1/m}$ with $\lambda_{\mathrm{low}} \le \frac{1}{2}(2\lambda)^{1/m}$ (hence $\lambda_{\mathrm{low}}^2 \le \frac{1}{4}(2\lambda)^{2/m} \le \frac{1}{4}(2\lambda)^{1/m}$). Its saturated lower lattice Λ_1 is totally isotropic by Harder–Narasimhan self-duality: because Λ_{Y,x_*} is Hermitian self-dual and the majorant frames X_τ at every infinite place τ satisfy $X_\tau^* J_\tau X_\tau = J_{\epsilon_\tau}$ (where J_{ϵ_τ} is the standard diagonal signature matrix with entries ±1), the Hermitian pairing identifies the orthogonal complement $\Lambda_S^\perp = \Lambda_{Y,x_*} \cap S^\perp$ of any saturated sublattice $S \subseteq \Lambda_{Y,x_*}$ isometrically with the dual-lattice annihilator $\Lambda^\vee_{Y,x_*} \cap S^{\mathrm{ann}}$, preserving normalized Arakelov degree (with $\mathrm{rk}_{\mathcal{O}_E}\Lambda_S^\perp = r - \mathrm{rk}_{\mathcal{O}_E} S$) and reversing the Harder–Narasimhan canonical filtration with inverted scales, namely $(\Lambda_{<t})^\perp = \Lambda_{\le t^{-1}}$ for every $t > 0$. Taking $t = 1/2$, monotonicity of the filtration gives $\Lambda_1 \subseteq \Lambda_{<1/2} \subseteq \Lambda_{\le 2} = (\Lambda_{<1/2})^\perp \subseteq \Lambda_1^\perp$, so Λ_1 is totally isotropic. Thus $W = \Lambda_1 \otimes_{\mathcal{O}_E} E$ is a nonzero rational totally isotropic subspace.

For u_0 in the adelic unipotent radical of W, use unipotent strong approximation to remove its finite part up to the conjugated principal congruence subgroup of level g. In the resulting real logarithm first reduce the image in the first layer by its lattice L_A at level $2g$, and then reduce the central remainder (including the commutator of the rational first-layer correction) by the centre lattice L_Z at level g. To bound the Euclidean covering radii $\mu(L_A), \mu(L_Z)$ in the Hilbert–Schmidt norm on $\bigoplus_\tau \mathfrak{n}_\tau$, set $d_1 = \mathrm{rk}_{\mathcal{O}_E}\Lambda_1$ (so $\mathrm{rk}_{\mathcal{O}_E}\Lambda_1^\perp = r - d_1$ and $2d_1 \le r$ by isotropy) and $\rho = (2\lambda)^{1/m}$. For any subquotient Λ_a/Λ_s of $\mathcal{O}_E$-rank $d_a - d_s$ whose successive quotients above Λ_s have scale at least ℓ_0 and any target Λ_t of rank d_t and maximal scale at most h_0, the space $\bigoplus_\tau \mathrm{Hom}_{\mathbf{C}}((\Lambda_a/\Lambda_s)_\tau, (\Lambda_t)_\tau)$ has real dimension $2f(d_a - d_s)d_t$; under the trace pairing $\mathrm{Tr}_{E/\mathbf{Q}}\mathrm{Tr}(X\psi) = 2\sum_\tau \mathrm{Re}\,\mathrm{Tr}(X_\tau\psi_\tau)$, integrality in $\mathfrak{D}_E^{-1}$ and AM–GM over the

f archimedean places (no unit regulator enters because the product norm is taken over all τ directly) give dual minimum at least $2\sqrt{f}\ \mathrm{rd}(E)^{-1}(\ell_0/h_0)$, so Banaszczyk's covering-radius transference theorem [78, Theorem 2.2] ($\mu(L)\lambda_1(L^*) \le \frac{1}{2}\dim_{\mathbf{R}} L$) gives Euclidean covering radius at most $\frac{1}{2}(d_a-d_s)d_t\sqrt{f}\ \mathrm{rd}(E)(h_0/\ell_0)$. For the first layer, apply this bound to $(\Lambda_1^\perp, \Lambda_1, \Lambda_1)$ (where $d_a - d_s = r - 2d_1$, $d_t = d_1$, $\ell_0 = \lambda_{\mathrm{up}}$, $h_0 = \lambda_{\mathrm{low}}$, $\lambda_{\mathrm{low}}/\lambda_{\mathrm{up}} \le \rho$, and $d_1(r-2d_1) \le r^2/8$) and lift skew-Hermitianly to $\mathfrak{n}/\mathfrak{z}$ (multiplying the Hilbert–Schmidt norm by $\sqrt{2}$) to obtain integral covering radius at most $r^2\sqrt{f}\ \mathrm{rd}(E)\rho/(8\sqrt{2})$; scaling by level $2g$ gives $\sqrt{\sum_\tau \|A_\tau\|_{\mathrm{HS}}^2} \le r^2\sqrt{f}\,g\,\mathrm{rd}(E)\rho/(4\sqrt{2}) \le \frac{3}{16}r^2\sqrt{f}\,g\,\mathrm{rd}(E)\rho$ (using $\sqrt{2} \ge 4/3$). For the centre layer, apply the same bound to $\frac{1}{2}Z$ on $(\Lambda_{Y,x_*}, \Lambda_1^\perp, \Lambda_1)$ (where $d_a - d_s = d_t = d_1$, and self-duality of Λ_{Y,x_*} gives $\ell_0 = \lambda_{\mathrm{low}}^{-1}$ and $h_0 = \lambda_{\mathrm{low}}$) and skew-symmetrize $\Phi - \Phi^*$ (at most doubling the remainder) to obtain integral covering radius at most $d_1^2\sqrt{f}\ \mathrm{rd}(E)\lambda_{\mathrm{low}}^2 \le \frac{1}{16}r^2\sqrt{f}\ \mathrm{rd}(E)\rho$ (using $d_1^2 \le r^2/4$ and $\lambda_{\mathrm{low}}^2 \le \frac{1}{4}\rho$); scaling by level g gives $\sqrt{\sum_\tau \|Z_\tau\|_{\mathrm{HS}}^2} \le \frac{1}{16}r^2\sqrt{f}\,g\,\mathrm{rd}(E)\rho$. Summing $\frac{3}{16} + \frac{1}{16} = \frac{1}{4}$, these two reductions give

$$u_0 x_* = \gamma x_*(u_\infty, k_f), \qquad u_{\infty,\tau} = \exp A_\tau \exp Z_\tau$$

with rational γ, principal congruence k_f of level g, and

$$\begin{aligned} \sqrt{\sum_\tau \|A_\tau\|_{\mathrm{HS}}^2} + \sqrt{\sum_\tau \|Z_\tau\|_{\mathrm{HS}}^2} &\le \tfrac{1}{4}r^2\sqrt{f}\,g\,\mathrm{rd}(E)(2\lambda)^{1/m}, \\ \sum_\tau (\|A_\tau\|_{\mathrm{HS}} + \|Z_\tau\|_{\mathrm{HS}}) &\le r^2 f g\,\mathrm{rd}(E)(2s)^{1/m}. \end{aligned} \tag{A.391}$$

The second inequality is Cauchy–Schwarz. Subtract the zero cuspidal average over the radical from $u(x_*)$. Rational and level invariance reduce its integrand to the successive real translates. Equation (34.17) with (A.388) bounds the difference by $M/2$, a contradiction. Adjoining z_0 preserves the product lower bound for $\mathfrak{a}\Lambda_{j(x_*)}$. If the extra coordinate is zero, apply (A.390). A nonzero extra coordinate has product norm at least $\sqrt{\mathrm{N}\,\mathfrak{a}}$.

Lean: `GlobalTheory.DeepPoint.deep_point` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.AutomorphicPointExistsIsotropicReductionOfSmallS`, lines 141–196)
Lean: `GlobalTheory.SmallFund.small_fundamental_set` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.SmallFundamentalSetRealUnipotent`, lines 135–218)
Lean: `GlobalTheory.ShortLineCusp.norm_le_of_not_long` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.NormCuspFormLeOfShortLine`, lines 67–140)

Form the core Poincaré series from k translated by $j(x_*)^{-1}$. In the unfolded period the rational cosets are indexed by $z = \gamma z_0$. The identity term is $cu(x_*)$. Other contributing terms have $v = z - z_0 \ne 0$ in $\mathfrak{m}\Lambda_{j(x_*)}$. Use N_τ of Lemma 34.29 at x_*. Then $\prod_\tau N_\tau(v) \ge R$ and $1 + N_\tau(v) \le 3N_\tau(z)$, since z has Hermitian norm one. The tail of absolute integrals is at most

$$M|c|e^{L_{\mathrm{loss}}} \sum_v \prod_\tau \left(\frac{3}{1+N_\tau(v)}\right)^{2n+5} \le M|c|e^{L_{\mathrm{loss}}+(6n+9)f}/R. \tag{A.392}$$

Extract R^{-1} from one power of the denominator product. The products of radius-1/2 Euclidean balls centered at the v's have disjoint interiors by the lower bound on differences. Comparing the remaining weight inside a ball costs $(3/2)^{2n+4}$ per place. In dimension $2n$, radial integration using $t^{2n-1}(1+t)^{-2n-4} \le (1+t)^{-5}$ costs at most $n4^n/2$ in units of ball volume. Thus the local cost is at most $3^{2n+5}(3/2)^{2n+4}n4^n/2 \le e^{6n+9}$. By (A.388), the sum of absolute values of the nonidentity terms divided by $M|c|$ is at most $e^{L_{\mathrm{loss}}+(6n+9)f}/R \le 1/2$. The identity term divided by the same positive number has modulus $|u(x_*)|/M > 1/2$. The unfolded period therefore converges absolutely and is nonzero.

Lean: `GlobalTheory.ProdNormSum.lattice_sum_le` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.LatticeSumProdNormLeOfProdNormLowerBound`, lines 225–278)
Lean: `GlobalTheory.IdDom.identity_dominates` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.PoincareSeriesIdentityTermDominates`, lines 40–241)

The mixed coefficient satisfies

$$|\langle v_0, \pi(g)a\rangle| \ll_a (1 + \|g\|_{\mathrm{HS}})^{-(2n^2+1)} \tag{A.393}$$

for K-finite a. For $a = v_0$ this follows from Lemma 37.16 and the gap. Every translate of v_0 in the second slot satisfies the same decay with a translate-dependent constant, by the matrix norm inequality. Their span

is dense by irreducibility. For u in that span average the positive rank-one operator of $\pi(k)u$ over K. Its image satisfies the mixed bound since the translate constants are bounded on K. The sum Y of the isotypic spaces meeting a given K-finite a is finite dimensional by admissibility. The averages commute with K and preserve Y. A vector in Y orthogonal to the sum of their images on Y satisfies $\int_K |\langle \pi(k)u, z\rangle|^2\, dk = 0$ for all such u, hence $z = 0$ by continuity and density. Their images therefore span Y.
Lean: `GlobalTheory.SepPer.KDecay` (`Development.AbelianVarietyTheory.Definitions.TwistedInnerProducts.DepthPays`, lines 61–62)
Lean: `GlobalTheory.KFinDecay.decayAt_of_kFin` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.CoefficientDecayKFiniteSecondVector`, lines 204–321)

The mixed bound gives convergence on K-finite tensors. Unfolding the proper constant terms gives the zero proper unipotent integrals of the discrete-series coefficients. The resulting cuspidal intertwiner at core level gives a smooth multiplicity map y by globalization. In (A.392) the core vectors are $\pi(g_\tau)v_0$ in both slots. Move the first translations to the base of the series and use the translated second vectors as the input a' of y. Thus $\mathrm{Per}_0(y(a'), t(w_*)) \neq 0$. For prescribed $w \neq 0$, cyclicity and continuity give a real translate of $t(w)$ pairing nontrivially. Move it to the core side by right invariance. The period is continuous in the smooth core vector by boundedness of the restricted core function and rapid decay of the row form. Approximation by compact-type projections in the smooth topologies gives the asserted a.
Lean: `GlobalTheory.PeriodNePf.exists_kf` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.PeriodNonvanishingOfArchimedeanInputs`, lines 65–110)

□

Proof of 34.31

Proof. In the Schrödinger model the vacuum is the characteristic function of the a-fold product of the target lattice. The two compact stabilizers fix it. The target convolution $\omega(1, h')\phi_0$ is fixed by both stabilizers. By Lemma 34.18 it is a finite linear combination of source translates of ϕ_0. The extension is unramified of odd residue characteristic, both lattices are unimodular, and the integral translations fix the vacuum and its Heisenberg translates span the lattice model. These are the hypotheses of that lemma. Average the combination over the source stabilizer. Since ϕ_0 is also fixed on the right, the averaged translates are convolutions by normalized double-coset indicators. This gives h.

For the transform use Liu's spherical-module theorem [156, Definition 2.8 and Theorem 1.1]. In his notation $r = a$, the target Witt index is $s = b$, and $\epsilon(V) = +$ because the target lattice is self-dual. His Weil representation uses the conductor and trivial splitting characters specified here. Part (2) says that the vacuum generates the spherical module when Q is odd. Thus $h \otimes 1 - 1 \otimes h'$, which kills the vacuum, belongs to the annihilator of that module. Part (1) identifies the annihilator with the kernel of the two Satake substitutions of Definition 2.8. The source substitution inverts the a variables, which fixes every spherical polynomial by Weyl invariance. The target substitution inserts Q^{2j-1} for $1 \leq j \leq b - a$ and retains the a variables. Applying these substitutions to the difference gives the displayed identity. □

Proof of 34.32

Proof. Cuspidal decay gives a bounded spherical eigenfunction. Spherical convolution on bounded functions has norm bounded by its L^1-mass. In the split chart the trivial Satake bound for this isometric action bounds each normalized root by $(Nv)^{(n-1)/2}$. At an inert place use the trivial bound in the unitary spherical algebra. For its inert coordinates α_j the base-change multiset is

$$\{\alpha_j, \alpha_j^{-1} : 1 \leq j \leq \lfloor n/2 \rfloor\} \sqcup \{1\}^{\,n \bmod 2}.$$

With $Nw = (Nv)^2$ the inert bound gives (34.18).
Lean: `GlobalTheory.LGrp.bcParam` (`Development.AbelianVarietyTheory.Definitions.Divisors.LineOppositeParabolicUnitaryLeviSectionEq`, lines 870–871)

Move Hecke convolutions through the Weil pairing by equivariance. Write χ_S, χ_T for the two step characters, $r_0 = n_i$ for the source rank, and $q = Nw$. In the split matrix model the target generator of index k, taken on inverse double cosets, acts on the vacuum as a sum of source generators of indices $j \leq k$. Count the lattice modifications by intersection dimension j. The complementary choices contribute the Gaussian

binomial $\begin{bmatrix} d \\ k-j \end{bmatrix}_q$, and the lifts into the source block contribute $q^{(r_0-j)(k-j)}$. The Weil multiplier changes their product by

$$(\chi_S(w)q^{r_0/2})^{-k}(\chi_T(w)q^{(r_0+d)/2})^{j}.$$

Substitute the source eigenvalues $q^{j(r_0-j)/2}e_j(B_{\mathrm{old}}^{-1})$, where e_j is elementary symmetric and the multiset is inverted entrywise. The identity

$$e_u(q^{(d-1)/2}, q^{(d-3)/2}, \dots, q^{-(d-1)/2}) = q^{-u(d-u)/2}\begin{bmatrix} d \\ u \end{bmatrix}_q$$

gives the sum $q^{k(r_0+d-k)/2}e_k(B_{\mathrm{new}}^{-1})$, with

$$B_{\mathrm{new}} = \chi_S(w)\chi_T(w)^{-1}B_{\mathrm{old}} \sqcup \{\chi_S(w)q^{(d-1)/2-l} : 0 \le l < d\}.$$

At an inert $v \notin S$, put $Q = Nv$, so $Nw = Q^2$. Both core ranks are even since r is odd, $n_i = r{+}1{+}i(r{+}5)$, and $d = r + 5$. The self-dual frames give the split spaces in these even dimensions over the unramified quadratic extension. Multiplying the source form by an anti-invariant unit gives the skew-Hermitian convention with the same integral stabilizer. The residue characteristic is odd and the local additive character has conductor $\mathcal{O}_v$. Each splitting character is unramified and restricts to the even power of the quadratic character on $F_v^{\times}$. It is therefore one on the units of E_w and on a common uniformizer, hence trivial on $E_w^{\times}$. Apply Lemma 34.31 with $2a = n_i$ and $2b = n_i + d$. Move its matching source convolution across the invariant bilinear pairing. This replaces $h(g)$ by $h(g^{-1})$, which is the same function because the unitary Cartan double cosets are invariant under inversion. The eigenvalue of every target operator on the nonzero lift is consequently its Satake polynomial evaluated at the old coordinates with $Q, Q^3, \dots, Q^{d-1}$ adjoined. Base change adds those entries and their inverses. In terms of Nw these are exactly $(Nw)^{(d-1)/2-l}$, $0 \le l < d$, and $\widetilde{\chi}_i(w) = 1$. In the chain $\chi_S = \chi_T = \widetilde{\chi}_i$. Pairing the ordinary kernel with the conjugate cusp form computes the conjugate of the upward lift with conjugated kernel, which gives the stated parameter convention.
Lean: `GlobalTheory.ChainOnto.stepPar` (`Development.AbelianVarietyTheory.Definitions.LatticeFrames.ChainParSucc`, lines 121–123)
Lean: `GlobalTheory.ChainOnto.SplitTransfer.transferOn_isSplit` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.SphericalParameterTransferSplitPlace`, lines 55–160)
Lean: `GlobalTheory.ChainOnto.transferOn_inert` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.SphericalTransferWeilPairingInertPlaces`, lines 29–117)

□

Proof of 34.33

Proof. Keep the chain notation above. The core V_i has rank n_i and rational Witt index less than d. The characters $\widetilde{\chi}_i$ are unramified off S, trivial on $\mathbb{A}_F^{\times}$, and have infinite exponents the negatives of the distinct grid centers. Reframing by $(\widetilde{\mathfrak{A}}\mathfrak{A}^{-1})^i$, for the core lattice-change ideal $\mathfrak{A}$, preserves spherical stabilizers off S and the principal congruence levels J_i on S.
Lean: `GlobalTheory.CoreOnto.comOK` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.TestGridCommonDataPropertiesHold`, lines 31–81)

Let C_i be the finite-dimensional cuspidal multiplicity space of the prescribed smooth discrete series at level J_i and trivial level character. Finiteness follows at fixed infinitesimal character and lowest type [98]. Petersson products factor into its multiplicity pairing and the invariant real pairing. The spherical Hecke algebra off S is commutative and closed under adjoints, so C_0 is spanned by simultaneous eigenvectors. Label their characters by the conjugate multisets $B(w)$.

Choose finite tests before choosing an eigenvector. At depth places use the coset vacuum for a split lattice projection surjective modulo depth in the downward direction. Elsewhere take the standard vacuum. Both levels fix the test. For a source element integral off S, equality of the Heisenberg characters on a nonzero self-coefficient forces preservation of the coset. Surjectivity of the lattice projection forces the source congruence. Thus on this domain the self-coefficient is supported on $\mathrm{U}(V_i)(F_\infty)J_i$. Exchange of the adjoint lattice models makes each polynomial with this finite vacuum an allowed downward depth test.

Write $A_i : C_{i+1} \to C_i$ for the downward multiplicity map and $I_i(P)$ for the real intertwiner using polynomial input P. The downward integral of $y'(v')$ is $(A_i y')(I_i(P)v')$. For a nonzero eigenvector u reached

from the bottom, $A_i^* u \neq 0$, with parameters given by spherical transfer. The induction gives

$$B_i(w) = B(w) \sqcup \bigsqcup_{j<i} \{\widetilde{\chi}_j(w)(Nw)^{(d-1)/2-l} : 0 \le l < d\}. \tag{A.394}$$

The doubled numerator for the conjugate value is therefore

$$H_i(s) = L^S(B \otimes \widetilde{\chi}_i^{-1}, s + 1/2) \prod_{j<i} \prod_{l=0}^{d-1} L^S(\widetilde{\chi}_j \widetilde{\chi}_i^{-1}, s + d/2 - l). \tag{A.395}$$

Here $L^S(B \otimes \widetilde{\chi}_i^{-1}, z)$ has roots $\beta \widetilde{\chi}_i(w)^{-1}$ and variable $(Nw)^{-z}$ off S. By (34.18) this first factor converges absolutely without zeros for $\operatorname{Re} s > n/2$, including $s_0 = d/2$. The other arguments at s_0 are $d - l \ge 1$ and their ratio characters have nontrivial angular type. Lemma 34.1 gives nonvanishing. The normalizer $D_i^S(s) = \prod_{l=0}^{n_i-1} L_F^S(2s + n_i - l, \eta^{n_{i+1}+l})$ has arguments greater than one there.
Lean: `GlobalTheory.EvenOnto.chainL_and_inj` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.EvenThetaChainLValuesNeZeroAndParamsInjective`, lines 95–173)

In the real source, after duality and removal of the common shift, write the signed lists as (AB, CD), with A, C positive and B, D negative. The real theta rule sends this parameter to

$$(A[(d-1)/2, \dots, -(d-1)/2]D,\ CB).$$

The grid gives old entries of modulus at least $(d+3)/2$ and

$$\mathrm{gap}_{\mathrm{opp}} > n_{i+1}^3 \quad \text{in both models,} \qquad \mathrm{gap}_{\mathrm{opp}} \ge 120 n_i \min(p_i, q_i)^{3/2} \quad \text{in the source of signature } (p_i, q_i). \tag{A.396}$$

The joint highest Fock line has the source lowest compact type. Take a nonzero polynomial P_τ on this line and a unit highest vector v_τ of that type.

Claim 1. The Hilbert-space Schur projection $E_{v_\tau} P_\tau$ is nonzero.

Write G, K, π, ω for the real source, its maximal compact, the discrete series and the oscillator representation. The displayed gap makes every matrix coefficient of the lowest compact type integrable. Thus $E_v = \deg(\pi) \int_G \overline{\langle v, \pi(g)v\rangle} \omega(g)\, dg$ is the orthogonal Schur projection. Identify the compact module E generated by v with that generated by $P = P_\tau$, with isometric embeddings $j_\pi : E \to \mathcal{H}_\pi$ and $j_\omega : E \to \mathcal{H}_\omega$, after rescaling P.

Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the Cartan decomposition of the real Lie algebra. For every $Y \in \mathfrak{p}$, the compressions $j_\pi^* \pi(\exp Y) j_\pi$ and $j_\omega^* \omega(\exp Y) j_\omega$ are positive semidefinite. For the oscillator compression use the shifted Cartan operators. In an active two-mode block with creation and annihilation operators $a^\dagger, a, b^\dagger, b$, put $K_+ = a^\dagger b^\dagger$, $K_- = ab$, and $H = [K_-, K_+] = N_a + N_b + 1$. For each distinct positive singular value ν of the off-diagonal block of Y, let H_ν be the sum of these H's over its active blocks, including their multiplicities, and let m_ν be its eigenvalue on a simultaneous eigenvector. In singular-vector coordinates, sum the lowering operators over the multiplicity coordinates of each active singular direction. These contractions annihilate the harmonic space. For each ν, summing the corresponding triples gives operators $K_{\nu,+}, K_{\nu,-}, H_\nu$ with the same commutator relations and with $K_{\nu,-}$ zero on harmonic vectors. The relations $[H, K_\pm] = \pm 2K_\pm$ give

$$e^{t(K_+ - K_-)} = e^{(\tanh t) K_+} e^{-(\log\cosh t) H} e^{-(\tanh t) K_-}.$$

Apply the identity to each summed triple with $t = \nu$. The rightmost exponential fixes a harmonic input, and the adjoint of the leftmost exponential fixes a harmonic test vector. On a simultaneous H_ν-eigenvector in the harmonic space the diagonal coefficient is its squared norm times

$$\prod_{\nu>0} (\cosh \nu)^{-m_\nu} = \exp\left(\sum_{\nu>0} \frac{m_\nu}{2} \log \frac{2}{1 + \cosh(2\nu)} \right) > 0.$$

For a general harmonic vector the coefficient is the sum of these factors times the squared norms of its joint spectral components. It is therefore the quadratic form of the positive operator $\prod_{\nu>0} (\cosh \nu)^{-H_\nu}$. In polynomial coordinates each mode contributes $z\partial_z + \frac{1}{2}$, so an active two-mode vacuum contributes 1 to H. For the discrete-series compression the lowest-type coefficient formula is the compact average of rank-one

positive operators weighted by real powers of the positive leading principal minors of $k^{-1}\exp(Y)k$, with k the compact integration variable. The lowest-type compression formula applies under the displayed gap, which also gives the required integrability.

Suppose $E_vP = 0$. Pair with P. The integral of the oscillator diagonal coefficient times the conjugate discrete-series diagonal coefficient is then zero. Choose the isometric embeddings above so that the two unit highest vectors correspond. At fixed g, average this product at k_1gk_2 over two copies of K with probability Haar measure. Compact Schur orthogonality in each variable gives $F(g)/(\dim E)^2$, with F defined below. Haar invariance preserves the integral over G, so the integral of

$$F(g) = \operatorname{tr}((j_\pi^*\pi(g)j_\pi)^* j_\omega^*\omega(g)j_\omega)$$

is zero. This function is integrable because the discrete-series coefficients are integrable and the oscillator coefficients are bounded. Equivariance of the two embeddings makes F left K-invariant. Writing $g = k\exp Y$, positivity of the two compressions gives $F(g) \geq 0$, since the trace of a product of positive semidefinite matrices is nonnegative. But F is continuous and $F(1) = \dim E > 0$. Its Haar integral is strictly positive, a contradiction. This proves the claim.

Schur orthogonality now gives the local integral $\|E_{v_\tau}P_\tau\|^2/\deg(\pi_\tau) > 0$, with convergence supplied by the same gap.

Lean: `GlobalTheory.EvenOnto.upPair_arch` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.RealThetaCorrespondenceUpwardPairBothDirections`, lines 60–75)
Lean: `GlobalTheory.ArchRec.lift_down` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.DiscreteSeriesHarishChandraModuleAdmissible`, lines 290–313)

Tensor these polynomials with the finite test and take $F = u(\bigotimes_\tau v_\tau)$. On the real group times J_i its Petersson coefficient is $\|u\|^2$ times the real coefficients. The semilocal integral $Z_{S,\infty}$ at s_0 on the domain integral off S is positive by the support calculation and the real projectors. For its continuation, real Gaussian integration in Cartan coordinates bounds the Weil self-coefficient $c_P(g)$ in terms of the doubled height $0 < h(g) \leq 1$:

$$|c_P(g)| \leq Ch(g)^{n_{i+1}/2}, \qquad |c_P(g)|h(g)^{\operatorname{Re} s - s_0} \leq C \quad (\operatorname{Re} s \geq 0). \tag{A.397}$$

The second uses $n_{i+1} \geq d$. The integrable highest-vector discrete-series coefficient, by (A.396), supplies the majorant.

Lean: `GlobalTheory.EvenOntoRealZeta.realZeta` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.EvenThetaStepRealZetaNonvanishing`, lines 40–133)
Lean: `GlobalTheory.EvenOnto.ZetaNeProof.zetaNe_of_real` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.EvenThetaStepSemilocalZetaNeZeroOfReal`, lines 63–115)

By the assumed split and inert unramified local doubling formula [150, §4.2.1][83, p. 334, Remark 3], the normalized local doubling integrals off S multiply to $H_i(s)/D_i^S(s)$ in the convergence half-plane. At $s_0 = d/2$, for every $i \geq 1$ we have $2n_i - n_{i+1} = (r+1)+(i-1)(r+5) > 0$, so $n_{i+1} < 2n_i$ and the first formula of Theorem 34.17 applies since the target Witt index is less than d. At $i = 0$, we have $n_1 = 2n_0 + 4 > 2n_0$, and the assumed absolutely convergent unitary Siegel–Weil formula for $b > 2a$ [48] applies with covolume one. Write $(\Theta_P^{\mathrm{up}}F)(h) = \int_{[\mathrm{U}(V_i)]} \overline{\theta_i(g,h;P)}F(g)\,dg$. Factoring the doubled pairing on $\overline{F}$ gives

$$\|\Theta_P^{\mathrm{up}}F\|^2 = \operatorname{vol}[\mathrm{U}(V_{i+1})]\frac{H_i(s_0)}{D_i^S(s_0)}Z_{S,\infty} \neq 0. \tag{A.398}$$

Lean: `GlobalTheory.Core.Rk.IP.siegelWeilAll_of_hypotheses` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.SiegelWeilFockDoubledPairOfWeilAndIchino`, lines 49–84)
Lean: `GlobalTheory.EvenOnto.upNe_of_heckeForm` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.UpwardThetaLiftNeZeroOfLValueNeZero`, lines 33–139)

The real parameter rule kills the lower Witt-tower lifts on Fock tests because the old entries avoid $\pm(d+1)/2$. Density and continuity extend the vanishing to the allowed tests. The proper constant terms vanish because they expand through those lower lifts. Moving J_{i+1} onto P gives the new level. The real type puts the upward lift in the space of values of C_{i+1}. By Fubini its pairing against $y'(v')$ is the downward pairing $\langle A_i y', u\rangle\langle I_i(P)v', v\rangle$, in the argument order of the downward map. Nondegeneracy and the adjoint relation identify the lift as a value of A_i^*u. Equation (A.398) gives $A_i^*u \neq 0$, and the spherical transfer gives (A.394) at $i+1$.

Lean: GlobalTheory.EvenOnto.LiftUp.liftUp_stage_span (Development.AbelianVarietyTheory.Proofs.TwistedOperators.LiftUpMultiplicitySpacesSpanKFiniteVectors, lines 43–91)
Lean: GlobalTheory.EvenOnto.stepUp (Development.AbelianVarietyTheory.Proofs.TwistedOperators.EvenThetaStepAdjointNonzeroTransferredParams, lines 38–56)

Let $A = A_0 \cdots A_{m-1}$. A nonzero vector orthogonal to its image decomposes into simultaneous nonzero Hecke components u_B. Each $A^* u_B \neq 0$, and adding the same segments preserves distinct multisets. Thus these images are linearly independent, contradicting $A^* u = 0$. The argument works with arbitrary multiplicities.
Lean: GlobalTheory.EvenOnto.ontoAll_of (Development.AbelianVarietyTheory.Proofs.TwistedOperators.EvenRankThetaChainLiftSurjectiveOfAdjointNonzero, lines 97–122)

Take a top preimage of y. Let both P and the K-finite vector v' vary, while keeping the chosen finite vacuum. Their images $I_i(P)v'$ span a space containing a nonzero joint-type vector. For a Lie algebra element X of the output group, equivariance gives $d\pi_i(X)I_i(P)v' = I_i(d\omega_i(X,0)P)v'$. The derived oscillator action preserves polynomials and leaves the finite vacuum unchanged, so this span is Lie-stable and all its tests retain the required finite-depth invariance. Irreducibility gives the full K-finite tensor space. Express a as a finite sum of these values and repeat up the chain. The downward integrals give the allowed tests at every rung. At the top the perfect pairing with the coefficient representation gives each value as $\phi_{\xi,\Psi}$ with a column of the prescribed level.
Lean: GlobalTheory.ItOnto.kfinSpan_le_span_it (Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.KFiniteSpanLeSpanRealIntertwinerValues, lines 44–100)
Lean: GlobalTheory.ChainLifts.chainEx (Development.AbelianVarietyTheory.Proofs.TwistedOperators.CoreChainCompositesExistLiftCombinations, lines 36–153)
Lean: GlobalTheory.CoreOnto.topCols (Development.AbelianVarietyTheory.Proofs.TwistedOperators.DefiniteTopSpaceColumnValues, lines 65–71)

□

Proof of 34.35

Proof. Since $d = u + v$ is even, $v + 2(2(p+q)+1) < u$, and $2(p+q)+1 < v$, the dimensions $(a, a') = (u, v)$ satisfy $a' < a$, $a + a' \equiv 0 \pmod 2$, $2(p+q)+1 < (a-a')/2$, and $2(p+q)+1 < a'$. This gives (34.1). Fix a bijection $e : \{1, \dots, d\} \simeq \{1, \dots, u+v\}$. Keeping $J_V = J_W$ preserves the signature $(u-p, p)$, while setting $J_{V'} = -J_{W_1}$ reverses $(v-q, q)$ to $(q, v-q)$ and matches the sign $-$ of ω'. For $\bar{h}^t J_X h = J_X$, taking the transpose gives $h^t J_X^t \bar{h} = J_X^t$; multiplication by ξ_0 with $\bar{\xi}_0 = -\xi_0$ makes $J_{W^{\mathrm{skew}}} = \xi_0 (J_X^t)^e$ skew-Hermitian and makes $\kappa_X(h) = (\bar{h})^e$ an isomorphism of algebraic groups $\mathrm{U}(X) \xrightarrow{\sim} \mathrm{U}(W^{\mathrm{skew}})$ over F, whence $\mathrm{ht}(\kappa_X(h)) \leq C' H_X(h)^{N'}$ for suitable $C' > 0$ and $N' \in \mathbf{N}$. At each infinite place w of F with chosen embedding $\tau \in \theta$, Hermitian symmetry of J_X gives $\bar{\tau}(J_X^t) = \tau(J_X)$, while $\bar{\xi}_0 = -\xi_0$ and $\mathrm{Im}\,\tau(\xi_0) > 0$ give $i\bar{\tau}(\xi_0) = \mathrm{Im}\,\tau(\xi_0) > 0$. Hence $i\bar{\tau}(J_{W^{\mathrm{skew}}})$ is a positive scalar multiple of $(\tau(J_X))^e$ and has signature $(d-p-q, p+q)$. Under $\bar{\tau}$ the angular exponents $2a_\tau, 2b_\tau \in \mathbf{Z}$ of α, β become $m_{\alpha,w} = -2a_\tau$ and $m_{\beta,w} = -2b_\tau$, so $2(a_\tau - b_\tau) < -d$ becomes $m_{\alpha,w} - m_{\beta,w} = 2(b_\tau - a_\tau) > d$.

By Definition 34.34, ψ is continuous on $\mathbb{A}_F/F$ with infinite part $y \mapsto \exp(-2\pi i \,\mathrm{Tr}_{F_\infty/\mathbf{R}}\, y)$. Let $\psi_{\mathbf{Q},\mathrm{std}}$ be the standard character of $\mathbb{A}_{\mathbf{Q}}/\mathbf{Q}$ with archimedean part $t \mapsto e^{-2\pi i t}$. Then $\psi \cdot (\psi_{\mathbf{Q},\mathrm{std}} \circ \mathrm{Tr}_{F/\mathbf{Q}})^{-1}$ is continuous and trivial on $F + F_\infty$, which is dense in $\mathbb{A}_F$ by strong approximation for the additive group of F. Thus $\psi = \psi_{\mathbf{Q},\mathrm{std}} \circ \mathrm{Tr}_{F/\mathbf{Q}}$, and at each finite place v_0 of F its conductor is $\mathfrak{D}_{F/\mathbf{Q},v_0}^{-1}$, equal to $\mathcal{O}_{v_0}$ whenever $v_0 \nmid \mathfrak{D}_{F/\mathbf{Q}}$. Let $P = T$ (with $S_0 = \varnothing$ and $Q = T$) be the finite set of finite places of F where α or β ramifies. Every $v_0 \in T$ splits in E/F and satisfies $v_0 \nmid \mathfrak{D}_{F/\mathbf{Q}}$, so ψ_{v_0} has conductor $\mathcal{O}_{v_0}$ on P. Since $d = u + v$ is even, $(\alpha/\beta)|_{\mathbb{A}_F^\times} = \eta^{v-u} = 1$, whence $(\alpha/\beta)(\bar{t}) = (\alpha/\beta)(t)^{-1}$ for $t \in \mathbb{A}_E^\times$; hence ramification of α/β at one split component above $v_0 \in T$ implies ramification at both split places of E above v_0. In the arithmetic point, $E/\mathbf{Q}$ is Galois with E/F unramified at finite places, and every inert place v_0 of E/F is unramified over $\mathbf{Q}$ of odd residue characteristic with $Nv_0 > 3$ (since 2, the ramified primes of $F/\mathbf{Q}$, and odd primes up to the host split cutoff at least 11 split in E/F), so ψ_{v_0} also has conductor $\mathcal{O}_{v_0}$ at inert places. Moreover, the Cartan involutions and self-dual lattices of (W, W_1, X) transfer to $(V, V', W^{\mathrm{skew}})$ because $J_V = J_W$, $J_{V'} = -J_{W_1}$, and at every inert place v_0 both ξ_0 and the absolute codifferent are units. Choosing split embeddings on P and a place $v_1 \notin T$ above a rational prime split completely in E supplies good base data preceding Theorem 34.7 with the same additive character ψ.

In the frames (A, B) and (A_1, B), put $c_A(m) = P_e^{-1}(\bar{m})^t J_W$ and $c_B(m') = -P_e^{-1}(\bar{m}')^t J_{W_1}$, where P_e is the permutation matrix $(P_e)_{ij} = \delta_{i,e(j)}$ (so $P_e^{-1}(\bar{m})^t$ reindexes only the row index of $(\bar{m})^t$ by e^{-1}). Since

$\bar{g}^t J_W = J_W g^{-1}$, $\bar{g}_1^t J_{W_1} = J_{W_1} g_1^{-1}$, and $P_e^{-1}\overline{\kappa_X(h)} = hP_e^{-1}$, these are $\mathbb{A}_F$-linear topological isomorphisms preserving the rational subspaces, intertwining $m \mapsto \kappa_W(g) m \kappa_X(h)^t$ with $\phi \mapsto h\phi g^{-1}$ (and similarly for c_B), and carrying the symplectic trace pairings of $V \otimes_E W^{\mathrm{skew}}$ and $V' \otimes_E W^{\mathrm{skew}}$ to the oscillator alternating pairings of (W, X) and $(-W_1, X)$. For polynomial Schwartz vectors Φ_D, Φ'_D, the sums defining $\mathcal{W}_A(\Phi_D)$ and $\mathcal{W}_B(\Phi'_D)$ converge absolutely, satisfy rational quasiperiodicity because rational Darboux shifts permute the summation lattice with trivial ψ-phase, and intertwine ρ_ψ, ρ'_ψ with Heisenberg translations along c_A, c_B. Fourier inversion on the compact Pontryagin torus $(F\backslash\mathbb{A}_F)^{u\times(u+v)}$ (and its $v \times (u+v)$ counterpart) in the complementary Darboux coordinate recovers Φ_D, Φ'_D and proves injectivity. Because c_A, c_B conjugate E-coordinates, they carry the complexified polarization at $\bar{\theta}$ to the Fock polarization at θ, map finite Schwartz–Bruhat factors and archimedean Gaussian polynomials onto the permitted finite theta functions and real Fock polynomials, and, since $\det \kappa_X(h) = \overline{\det h} = (\det h)^{-1}$ matches the opposite target determinant convention of the oscillator model while κ_W, κ_{W_1} preserve source determinants and c_A, c_B conjugate E-coordinates, the splitting pairs $(\beta, \beta/\alpha), (\alpha, \alpha/\beta)$ agree on both sides so that $(\mathcal{W}_A, \mathcal{W}_B)$ intertwine $(\tilde{\omega}(\kappa_W(g), \kappa_X(h)), \tilde{\omega}'(\kappa_{W_1}(g_1), \kappa_X(h)))$ with $(\omega(g,h), \omega'(g_1,h))$. Evaluation at 0 gives the two pointwise kernel identities. Under Proposition 1.22, Lemma 34.8 ensures absolute convergence for unprimed polynomial vectors and for primed polynomial vectors in $\mathcal{S}'_{t_1}$, and measurable change of variables along κ_W, κ_{W_1} gives the two integral identities. □

Proof of 34.36

Proof. By Lemma 34.35, $(\mathcal{W}_A, \mathcal{W}_B)$ map the full polynomial Schwartz spaces of $(\tilde{\omega}, \tilde{\omega}')$ bijectively onto the algebraic Fock spaces of (ω, ω'). Hence $\tilde{p} = p \circ \mathcal{W}_A$ and $\tilde{p}' = p' \circ \mathcal{W}_B$ are target-equivariant surjections onto Π (with target action transported along κ_X^{-1}) vanishing on the finite-adelic source differences and archimedean source Lie actions, just as $(q_{\mathrm{ss}}, q'_{\mathrm{ss}})$ surject onto Π_{ss}. Fix either source side (say V; the proof for V' is identical). Every element of $\mathrm{U}(V)(\mathbb{A}_{F,f})$ stabilizes the local vacuum at almost all finite places, so a global finite source difference on a pure tensor expands as a finite sum of single-place source differences. At infinity, the source maximal compact group is connected and acts on finite-degree Fock orbits by exponentiating the source Lie action, so vanishing on the archimedean source Lie action implies compact source invariance. Thus the global source-coinvariant space coincides with $M = M_\infty \otimes_{\mathbf{C}} \bigotimes'_{v_0} M_{v_0}$, where M_∞ is the archimedean source-coinvariant module and M_{v_0} is the local source-coinvariant module at each finite place v_0 of F (with restricted product relative to the unramified vacua, which survive outside T since the local matching maps preceding Theorem 34.7 send the vacua off T to nonzero vectors).

Let $Q \in \{\Pi, \Pi_{\mathrm{ss}}\}$ be either nonzero jointly irreducible quotient of M with projection $\pi_Q : M \twoheadrightarrow Q$. At infinity, the connected target maximal compact group integrates the compact Lie action on finite-degree polynomial orbits, so target Lie-invariant subspaces of quotients of M_∞ are compact-invariant. By real Howe duality in Proposition 1.22, M_∞ has finite length and the sum of any two proper target-invariant submodules is proper; hence the sum N_∞ of all proper target-invariant submodules of M_∞ is proper and M_∞/N_∞ is simple. Fixing finite factors yields a nonzero map $M_\infty \to Q$ whose image has finite length and therefore contains a nonzero simple archimedean submodule. The sum of all simple archimedean submodules of Q is nonzero and invariant under the commuting action of $\mathrm{U}(W^{\mathrm{skew}})(\mathbb{A}_{F,f})$, so it equals Q by joint irreducibility. Thus Q is archimedeanly semisimple, and projection onto simple summands shows that every fixed-finite-factor map $M_\infty \to Q$ vanishes on N_∞.

At each finite place v_0 of F, put $G_{v_0} = \mathrm{U}(W^{\mathrm{skew}})(F_{v_0})$. At split places with nontrivial source determinant character (including every $v_0 \in T$), Lemma 34.9 shows that M_{v_0} has length one, so $N_{v_0} = 0$ is its unique maximal proper submodule and Q is spanned by simple G_{v_0}-images. At all remaining finite places (inert places and split places with trivial source determinant character; at split places M_{v_0} also has finite length by Bernstein–Zelevinsky finiteness in $\mathscr{P}$, Definition 1.20), Corollary 34.20 and (A.370) in Lemma 34.10 give $M_{v_0} = \mathrm{span}_{\mathbf{C}} G_{v_0} \bar{s}_{v_0}$. Let $K_{v_0} \subset G_{v_0}$ be the integral maximal compact subgroup, and define on any smooth G_{v_0}-module (in particular on M_{v_0} and its smooth quotient Q) the Haar averaging operator $e_{v_0}(y) = \int_{K_{v_0}} w_{v_0}(k)\, ky\, dk$ against the inverse right determinant character w_{v_0} on K_{v_0} (which is trivial at inert places and at split places off T, and evaluates on each smooth vector y as a finite mean over cosets of an open subgroup of K_{v_0} fixing y). By Corollary 34.20 and the proof of (A.370) in Lemma 34.10, $e_{v_0}(M_{v_0}) \subseteq \mathbf{C}\bar{s}_{v_0}$ and $e_{v_0}(\bar{s}_{v_0}) = \bar{s}_{v_0}$. Here $N_{v_0} = \{y \in M_{v_0} : \forall g \in G_{v_0},\ e_{v_0}(gy) = 0\}$ is the unique maximal proper G_{v_0}-submodule of M_{v_0}, with simple quotient M_{v_0}/N_{v_0}. In Q, e_{v_0} commutes with π_Q and

with all actions commuting with G_{v_0}, so $Q_0 = \{q \in Q : \forall g \in G_{v_0},\ e_{v_0}(gq) = 0\}$ is invariant under G_{v_0} and under all commuting actions at other finite and archimedean places; because $M_{v_0} = \operatorname{span}_{\mathbf{C}} G_{v_0}\bar{s}_{v_0}$ and π_Q is G_{v_0}-equivariant and surjective onto $Q \neq 0$, some image $\pi_Q(\bar{s}_{v_0} \otimes m^{v_0})$ is nonzero and fixed by e_{v_0}, so Q_0 is proper. Joint irreducibility forces $Q_0 = 0$, so every fixed-other-places map $M_{v_0} \to Q$ vanishes on N_{v_0} and factors through the simple module M_{v_0}/N_{v_0}.

Consequently, both $\ker\tilde{p}$ and $\ker q_{\mathrm{ss}}$ contain $\ker\big(M \to (M_\infty/N_\infty) \otimes_{\mathbf{C}} \bigotimes'_{v_0}(M_{v_0}/N_{v_0})\big)$. Because M_∞/N_∞ and each M_{v_0}/N_{v_0} are simple of at most countable complex dimension, their restricted tensor product is simple by the countable-dimensional scalar-commutant and tensor-product argument in Theorem 34.7. Since $\Pi \neq 0$ and $\Pi_{\mathrm{ss}} \neq 0$, we obtain $\ker\tilde{p} = \ker q_{\mathrm{ss}}$ and likewise $\ker\tilde{p}' = \ker q'_{\mathrm{ss}}$. Descent gives target-equivariant isomorphisms $\iota_A, \iota_B : \Pi \xrightarrow{\sim} \Pi_{\mathrm{ss}}$ with $q_{\mathrm{ss}} = \iota_A \circ \tilde{p}$ and $q'_{\mathrm{ss}} = \iota_B \circ \tilde{p}'$. Since Π has at most countable complex dimension, the countable-dimensional Schur argument gives $\iota_B^{-1} \circ \iota_A = s\,\mathrm{id}_\Pi$ for a unique $s \in \mathbf{C}^\times$. Whenever $p(\Phi) = p'(\Phi_0)$, we have $\tilde{p}(\Phi_D) = \tilde{p}'(\Phi_{0D})$ and hence $q_{\mathrm{ss}}(\Phi_D) = \iota_A(\tilde{p}(\Phi_D)) = \iota_B(s\,\tilde{p}'(\Phi_{0D})) = q'_{\mathrm{ss}}(s\Phi_{0D})$, so $(\Phi_D, s\Phi_{0D})$ is a polynomial match. Finally, Lemma 34.9 at the completely split good place $v_1 \notin T$ gives $p'_{v_1}(\operatorname{span} U_{t_1}) = \Pi_{v_1}$, whence q'_{ss} maps the polynomial subspace of $\mathcal{S}'_{t_1}$ onto $\Pi_{\mathrm{ss}} \neq 0$. Choosing a polynomial vector $\Phi_{0D} \in \mathcal{S}'_{t_1}$ with $q'_{\mathrm{ss}}(\Phi_{0D}) \neq 0$ and using surjectivity of p to pick $\Phi = \mathcal{W}_A(\Phi_D)$ with $p(\Phi) = p'(\mathcal{W}_B(\Phi_{0D}))$ shows that s is uniquely determined on algebraic pairs with $\Phi_{0D} \in \mathcal{S}'_{t_1}$. □

Proof of 34.37

Proof. Put $n = \operatorname{rank} M$, $d_X = \operatorname{rank} X$, and $f_F = [F : \mathbf{Q}]$, and choose a maximal rational Witt frame of $\mathrm{U}(M)$ of Witt index r. At the infinite place where $\operatorname{sig}(M) = (n-p, p)$, the two dual r-dimensional isotropic blocks span a subspace of signature (r, r), so $r \le p$ and $d_X < 2(n-p) \le 2(n-r)$. By Proposition 1.22, choose a closed Siegel set $\mathfrak{S}$ of points p_0ak with p_0 in a compact subset Ω of the norm-one minimal parabolic subgroup, k in a compact subgroup $K \subset \mathrm{U}(M)(\mathbb{A}_F)$, and archimedean torus coordinates $t_i \ge c_0 t_{i+1}$, $t_{r-1} \ge c_0 > 0$, such that $\mathfrak{S}$ meets every left $\mathrm{U}(M)(F)$-orbit, all conjugates $a^{-1}p_0a$ lie in a fixed compact set, and Haar integration over $\mathfrak{S}$ is bounded up to a constant factor by cone integration against $\prod_{i<r} t_i^{-1-2f_F(n-2i-1)} \prod_{i<r} dt_i$. Because $g \mapsto \theta_\omega(g, h; \Phi)$ is continuous and left $\mathrm{U}(M)(F)$-invariant, its absolute integral over the fundamental domain $\mathcal{F}$ is bounded by its integral over $\mathfrak{S}$. On $g = p_0ak = a(a^{-1}p_0ak)$, moving the bounded factor $a^{-1}p_0ak$ and h onto Φ costs finitely many archimedean Schwartz–Sobolev seminorms growing polynomially in $H_X(h)$, while the lattice count over contracted tuple coordinates as in Lemma 34.8 contributes $t_i^{f_F d_X}$ per isotropic line. Hence there exist $C > 0$ and $N \in \mathbf{N}$ such that

$$|\theta_\omega(p_0ak, h; \Phi)| \le C H_X(h)^N \prod_{i<r} t_i^{f_F d_X}. \tag{A.399}$$

Lean: `GlobalTheory.T.hermitianWittIndex_le_negative_signature` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.HermitianWittIndexLeNegativeSignatureStatement`, lines 17–18)
Lean: `GlobalTheory.T.thetaKernel_siegelSet_le_torusPower_height` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.ThetaKernelSiegelSetLeTorusPowerHeightStatement`, lines 17–18)
Lean: `GlobalTheory.T.lintegral_fundamentalDomain_le_of_meets_all_orbits` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.LintegralFundamentalDomainLeOfMeetsOrbitsStatement`, lines 7–8)

Multiplying (A.399) by the cone density gives exponents $A_i - 1$ in t_i with $A_i = f_F(d_X - 2n + 4i + 2)$, whose partial sums satisfy

$$\sum_{i=0}^{j} A_i = f_F(j+1)\big(d_X - 2n + 2(j+1)\big) < 0 \qquad (0 \le j < r) \tag{A.400}$$

because $j + 1 \le r$ and $d_X < 2(n-r)$. Successive integration from t_0 to t_{r-1} via $\int_b^\infty t^{B-1}\,dt = b^B/(-B)$ at each negative partial sum B (with cone factor 1 when $r = 0$) proves (34.19).
Lean: `GlobalTheory.T.lintegral_siegelCone_inv_modulus_lt_top_weil` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.LintegralSiegelConeInvModulusLtTopWeilStatement`, lines 8–9)

Now fix the standard additive character ψ of sign -1, the frames (A, A_1, B), and rational numbers (a_τ, b_τ). For any twisted-square triple (α, β), existence of oscillator Weil representations in $\mathscr{P}$ (Definition 1.20) supplies ω, ω' of signs $+, -$ and splitting pairs $(\beta, \beta/\alpha), (\alpha, \alpha/\beta)$. Lemma 34.35 transfers $(W, W_1, X, \alpha, \beta, \omega, \omega')$ to a triple of Theorem 34.7, and pulling the resulting jointly irreducible seesaw quotient Π_{ss} and matching maps $(q_{\mathrm{ss}}, q'_{\mathrm{ss}})$ back along $(\mathcal{W}_A, \mathcal{W}_B)$ (extended linearly from the algebraic Fock subspaces) gives matching

data $(\omega, \omega', \Pi, p, p')$. To construct $(P_{\mathrm{real}}, q, q')$ prior to any match, set $P_{\mathrm{real}} = 0$ and $q = 0, q' = 0$ if no matching data for ψ exist with exponents (a_τ, b_τ). Otherwise choose one witness match $(\omega_0, \omega_0', \Pi_0, p_0, p_0')$ for ψ with exponents (a_τ, b_τ). The archimedean source and target Lie actions on $\mathcal{P}(W, X)$ and $\mathcal{P}(W_1, X)$ depend solely on the Gram matrices, frames, and (a_τ, b_τ). Let $N_A \subset \mathcal{P}(W, X)$ and $N_B \subset \mathcal{P}(W_1, X)$ be the sums of all proper target-Lie-invariant submodules containing the archimedean source Lie relations. Since $\Pi_0 \neq 0$ is spanned on each side by images of Fock vectors, some fixed finite factor on each side produces a proper kernel containing the source relations; by real Howe duality in Proposition 1.22 (bounded chain length and properness of pairwise sums), N_A and N_B are proper, with simple countable-dimensional quotients $P_A = \mathcal{P}(W, X)/N_A$ and $P_B = \mathcal{P}(W_1, X)/N_B$. For any match $(\omega, \omega', \Pi, p, p')$ with exponents (a_τ, b_τ), the archimedean finite-length and socle-semisimplicity argument of Lemma 34.36 shows that Π is semisimple for the archimedean target Lie action and that for every permitted finite factor f (resp. f') the map $P \mapsto p(\mathrm{Fock}(f, P))$ (resp. $P' \mapsto p'(\mathrm{Fock}'(f', P'))$) vanishes on N_A (resp. N_B). For Π_0, images from each side span Π_0, so every simple archimedean summand of Π_0 is isomorphic to both P_A and P_B. Fix a target-Lie isomorphism $\iota_{\mathrm{real}} : P_B \xrightarrow{\sim} P_A$, put $P_{\mathrm{real}} = P_A$, let $q : \mathcal{P}(W, X) \twoheadrightarrow P_{\mathrm{real}}$ be the canonical projection, and put $q' = \iota_{\mathrm{real}} \circ (\mathcal{P}(W_1, X) \twoheadrightarrow P_B)$.

On either side, the permitted finite theta functions form a complex subspace S (resp. S') on which Fock (resp. Fock$'$) is $\mathbf{C}$-bilinear. For any match $(\omega, \omega', \Pi, p, p')$ with exponents (a_τ, b_τ), every $f \in S$ and $f' \in S'$ induce unique target-Lie intertwiners $\varphi_f, \varphi'_{f'} \in P_{\mathrm{fin}} = \mathrm{Hom}_{\mathrm{Lie}}(P_{\mathrm{real}}, \Pi)$ satisfying $p(\mathrm{Fock}(f, P)) = \varphi_f(q(P))$ and $p'(\mathrm{Fock}'(f', P')) = \varphi'_{f'}(q'(P'))$. Since a match exists for (a_τ, b_τ), we are in the witness branch and $P_{\mathrm{real}} \neq 0$. Simplicity and countable complex dimension of P_{real} give $\mathrm{End}_{\mathrm{Lie}}(P_{\mathrm{real}}) = \mathbf{C}\,\mathrm{id}$. Indeed, a nonscalar endomorphism T would have $T - z\,\mathrm{id}$ bijective for all $z \in \mathbf{C}$, and $\{(T - z\,\mathrm{id})^{-1} v_0 : z \in \mathbf{C}\}$ for $v_0 \neq 0$ would be uncountably linearly independent. Because $\Pi = \sum_{f \in S,\, \varphi_f \neq 0} \varphi_f(P_{\mathrm{real}})$ is a sum of simple submodules, Zorn's lemma expresses $\Pi = \bigoplus_{i \in I} \varphi_i(P_{\mathrm{real}})$ for a subfamily $\{\varphi_i\}_{i \in I}$ with $\varphi_i = \varphi_{f_i} \neq 0$ and $f_i \in S$. Any $\psi_0 \in P_{\mathrm{fin}}$ sends a nonzero $x_0 \in P_{\mathrm{real}}$ into a finite partial sum $\bigoplus_{m=1}^{k} \varphi_{i_m}(P_{\mathrm{real}})$; simplicity of P_{real} confines $\psi_0(P_{\mathrm{real}})$ to those k summands, and $\mathrm{End}_{\mathrm{Lie}}(P_{\mathrm{real}}) = \mathbf{C}\,\mathrm{id}$ gives $\psi_0 = \sum_{m=1}^{k} c_m \varphi_{i_m}$. Thus $f \mapsto \varphi_f$ maps S onto P_{fin}, and the identical argument on S' shows that $f' \mapsto \varphi'_{f'}$ maps S' onto P_{fin}. Moreover $\{\varphi_i\}_{i \in I}$ is a $\mathbf{C}$-basis of P_{fin}, so evaluation $e : P_{\mathrm{real}} \otimes_{\mathbf{C}} P_{\mathrm{fin}} \to \Pi$, $\sum_{m=1}^{k} y_m \otimes \varphi_{i_m} \mapsto \sum_{m=1}^{k} \varphi_{i_m}(y_m)$, is a linear isomorphism. Linear extension of $f \mapsto \varphi_f$ and $f' \mapsto \varphi'_{f'}$ across vector-space complements of S and S' gives r, r' satisfying the displayed tensor formulas.

Lean: `GlobalTheory.T.twist_triple_matches` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.TripleExistsMatching`, lines 43–47)

Lean: `GlobalTheory.T.twisted_quotient_real` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.TwistedQuotientReal`, lines 180–193)

Lean: `GlobalTheory.dixmier_commutant_scalar` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.DixmierCommutantScalar`, lines 43–109)

Lean: `GlobalTheory.FinFockLin.finF_lin` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.FinFLin`, lines 22–50)

Lean: `GlobalTheory.SquareLM.productForm` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.LsumMapDomainSome`, lines 286–329)

Finally, fix an additive character ψ with a twisted-square triple $(W, W_1, X, \alpha, \beta)$, frames (A, A_1, B), matching data $(\omega, \omega', \Pi, p, p')$, and Haar measures and fundamental domains $(\nu, \mathcal{F}), (\nu_1, \mathcal{F}_1)$. Apply Lemma 34.35 to obtain the seesaw transfer $(e, V, V', W^{\mathrm{skew}}, \tilde{\omega}, \tilde{\omega}', \kappa_W, \kappa_{W_1}, \kappa_X)$ and the bijections $(\mathcal{W}_A, \mathcal{W}_B)$ from the full polynomial Schwartz spaces onto the algebraic Fock spaces. Theorem 34.7 provides matching maps $(q_{\mathrm{ss}}, q'_{\mathrm{ss}})$ onto Π_{ss}, the decay and convergence bounds on $\mathcal{S}'_{t_1}$, the surjectivity $p'_{v_1}(U_{t_1}) = \Pi_{v_1}$ and kernel identity $\ker p'_{v_1} = A(\mathcal{S}_{v_1})$ at the completely split good place $v_1 \notin T$, and the unique nonzero scalar $c_1 \in \mathbf{C}^\times$ in (34.4). Lemma 34.36 supplies the seesaw comparison scalar $s \in \mathbf{C}^\times$; put $c = c_1 s \in \mathbf{C}^\times$.

Given any primed algebraic Fock vector Φ', let $\Phi'_D = \mathcal{W}_B^{-1}(\Phi')$ be its primed polynomial Schwartz preimage and write $\Phi'_D = \sum_j \phi_{j,v_1} \otimes \Psi_j^{v_1}$. Since $p'_{v_1}(U_{t_1}) = \Pi_{v_1}$ and $\ker p'_{v_1} = A(\mathcal{S}_{v_1})$, each ϕ_{j,v_1} admits $\phi_{0j,v_1} \in \mathrm{span}_{\mathbf{C}} U_{t_1}$ with the same image in Π_{v_1}. For $\Phi_{0D} = \sum_j \phi_{0j,v_1} \otimes \Psi_j^{v_1} \in \mathcal{S}'_{t_1}$, the difference $\Phi'_D - \Phi_{0D}$ is a finite linear combination of local differences $\tilde{\omega}'(g', 1)\Psi_D - \Psi_D$ with $g' \in \mathrm{U}(V')(F_{v_1})$ and Ψ_D in the primed polynomial Schwartz space. Put $\Phi_0 = \mathcal{W}_B(\Phi_{0D})$ in the primed algebraic Fock space. For each generator, $u_f = \kappa_{W_1}^{-1}(g')$ lies in $\mathrm{U}(W_1)(\mathbb{A}_{F,f})$, $\kappa_X(1) = 1$, and $\Psi = \mathcal{W}_B(\Psi_D)$ is a primed algebraic Fock vector satisfying $\mathcal{W}_B(\tilde{\omega}'(g', 1)\Psi_D - \Psi_D) = \omega'(u_f, 1)\Psi - \Psi \in \mathcal{R}_{\mathrm{fin}}$, whence $\Phi' - \Phi_0 \in \mathcal{R}_{\mathrm{fin}}$.

Since $\Phi_{0D} \in \mathcal{S}'_{t_1}$, (34.3) at $N = 0$ and Lemma 34.8 give integrability on $\kappa_{W_1}(\mathcal{F}_1)$ and a uniform bound $|\theta_{\tilde{\omega}'}(g', h'; \Phi_{0D})| \leq C_0\,\mathrm{ht}(h')^{k_0}$. By the kernel identity and height bound $\mathrm{ht}(\kappa_X(h)) \leq$

$C' H_X(h)^{N'}$ of Lemma 34.35, we have $|\theta_{\omega'}(g_1, h; \Phi_0)| \le C_0 (C')^{k_0} H_X(h)^{N' k_0}$ for all $g_1 \in \mathrm{U}(W_1)(\mathbb{A}_F)$ and $h \in \mathrm{U}(X)(\mathbb{A}_F)$. By Proposition 1.22, $\mathrm{U}(W_1)(\mathbb{A}_F)$ admits a closed Siegel set of finite ν_1-measure meeting every left $\mathrm{U}(W_1)(F)$-orbit, whence $\nu_1(\mathcal{F}_1) < \infty$. Integration over $\mathcal{F}_1$ gives $\int_{\mathcal{F}_1} |\theta_{\omega'}(g_1, h; \Phi_0)|\, d\nu_1(g_1) \le \nu_1(\mathcal{F}_1) C_0 (C')^{k_0} H_X(h)^{N' k_0}$, which proves (34.19) for Φ_0.
Lean: `GlobalTheory.T.theta_integral_moderate_growth_of_bounded_kernel` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.ThetaIntegralModerateGrowthOfBoundedKernelStatement`, lines 16–28)

For any unprimed algebraic Fock vector Φ with $p(\Phi) = p'(\Phi_0)$, put $\Phi_D = \mathcal{W}_A^{-1}(\Phi)$. By Lemma 34.36, $(\Phi_D, s\Phi_{0D})$ is a polynomial match for $(q_{\mathrm{ss}}, q'_{\mathrm{ss}})$ with $s\Phi_{0D} \in \mathcal{S}'_{t_1}$. By Lemma 34.35, (34.4), and linearity of I', for every $h \in \mathrm{U}(X)(\mathbb{A}_F)$ we obtain

$$\theta\, \mathrm{Int}_\omega(\Phi)(h) = I(\Phi_D, \kappa_X(h)) = c_1 I'(s\Phi_{0D}, \kappa_X(h)) = c_1 s\, I'(\Phi_{0D}, \kappa_X(h)) = c\, \theta\, \mathrm{Int}_{\omega'}(\Phi_0)(h).$$

Since h is arbitrary, $\theta\, \mathrm{Int}_\omega(\Phi) = c\, \theta\, \mathrm{Int}_{\omega'}(\Phi_0)$.
Lean: `GlobalTheory.SquareAgree.I_smul` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.ThetaIntegralsOfMinguez`, lines 26–31)

□

Proof of 34.38

Proof. We first reduce the set of tests to finitely many generators over a number field. On either side choose a common open finite level fixing the finite test vectors for all t. It exists by smoothness and finiteness of T. Finite translation at this level commutes with insertion of the polynomial at infinity. We can therefore shrink to a compact open subgroup J under which all the integrals are right invariant, for every k.

The standard target has Gram matrix 1, so its real groups are compact. Its unitary group is linear algebraic over the number field F. Borel's finite class-number theorem gives a finite set H of finite adelic representatives at level J. Indeed one can intersect J with the standard integral level; the intersection has finite index in the latter. Apply the theorem at the integral level and subdivide its cosets. Thus, with finite adeles embedded with trivial archimedean component, every h in the statement has a decomposition

$$h = \gamma r h_0 \kappa, \qquad \gamma \in \mathrm{U}(W)(F), \quad r \in \mathrm{U}(W)(F_\infty), \quad h_0 \in H, \quad \kappa \in J.$$

Here $r = \gamma_\infty^{-1}$. This last restriction on r determines the coefficient field in the next step. [97, Thm. 5.1]
Lean: `GlobalTheory.BottomLevel.rightTranslate_preserves_level` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.BottomFormRightTranslateLevel`, lines 21–45)
Lean: `GlobalTheory.BotClassFin.bottomClass_exists_finite_reps` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.BottomClassExistsFiniteReps`, lines 23–38)

Remove γ by rational left invariance. The Fock action of r substitutes its place matrices in the polynomial and multiplies by the character scalar of that action. In the standard Gram matrices and frames the entries of r and their conjugates lie in the compositum of the embedded conjugates of the host CM field E, since γ is rational. The determinant character on each compact factor has integral exponent. The substitution and its scalar therefore have coefficients in a fixed number field M_0; one may use the normal closure of E in $\mathbf{C}$. Expanding the substituted polynomial and removing κ expresses $I_t(h, k)$ as an M_0-linear combination of the $I_t(h_0, k')$. The positive-sign model gives the same expansion for I'_t with its own polynomial variables.
Lean: `GlobalTheory.RealRatAct.rationalAction_real_translate` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.RealTranslatePreservesRationalAction`, lines 61–133)
Lean: `GlobalTheory.BotSpanSnd.real_right` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.BottomTriangleDualAlgebraicityFiniteData`, lines 414–485)

There are also only finitely many exponents k' with nonzero integral. At each infinite place τ, a central source rotation acts on a monomial by a fixed character times the rotation to the signed total degree $d_\tau(k')$ in its variables at τ. Invariance of the integral forces this scalar to be one whenever the integral is nonzero. Comparing two surviving monomials and taking a rotation of infinite order gives equality of their degrees at τ. For each t, if there is a surviving monomial, take its multidegree. Finitely many monomials in the given variables have that multidegree. Taking the union over T gives a finite set D_0 on each side. Thus each family is spanned over M_0 by the integrals indexed by $T \times H \times D_0$.
Lean: `GlobalTheory.BotSpan.classSums_span_bottom` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.BottomSpannedByClassSums`, lines 26–68)
Lean: `GlobalTheory.BotSpanSnd.twistedClassSums_span_bottom` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.BottomSpannedByTwistedClassSums`, lines 32–72)

We next construct a period on each side. On the first source, take an open finite level fixing the test data. Decomposition of the definite source quotient into finite classes gives representatives z_l, finite stabilizers of orders $w_l > 0$, a positive common measure factor v, and probability Haar measure dg on the compact real group. If $K_t(P; h, x)$ denotes the theta kernel evaluated with polynomial P, finite target point h and source point x, then

$$I_t(h, k) = v \sum_l \frac{1}{w_l} \int_{\mathrm{U}(A)(F_\infty)} K_t(X^k; h, z_l g)\, dg.$$

The factor $1/w_l$ divides out the stabilizer of that class. The same decomposition on A' gives v', w'_l, z'_l for the primed kernel. The sources are definite and the finite data are smooth at the chosen levels, so both formulas apply to all the displayed monomials.
Lean: GlobalTheory.T.bottom_class_sum_integral_neg (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.BottomClassSumIntegralNeg, lines 121–128)
Lean: GlobalTheory.T.twist_integral_eq_bottom_classSum (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.TwistIntegralEqBottomClassSum, lines 116–123)

On the first source the compact action on polynomials is a common character times polynomial substitution. Its central characters select a single multidegree $d = (d_\tau)$ for the real average. If no degree survives, all averages vanish and we may take period 1. In the surviving degree the averaging matrix has algebraic entries. To see this, let P_d be the Haar average of the representation matrix. On the dense subgroup of compact unitary matrices with algebraic entries, the character and the polynomial action have algebraic matrices. For any automorphism σ of $\mathbf{C}$ over $\overline{\mathbf{Q}}$, the identities of left and right invariance for P_d thus hold for $\sigma(P_d)$ on that subgroup. Continuity in the group variable extends both identities to the compact group. Averaging gives

$$P_d\, \sigma(P_d) = \sigma(P_d), \qquad \sigma(P_d) P_d = \sigma(P_d).$$

Applying σ to the first identity for σ^{-1} also gives $\sigma(P_d)P_d = P_d$. Thus P_d is fixed and has algebraic entries.
Lean: GlobalTheory.RealAvg.real_average_kernel_isAlgebraic (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.TwistedRealAverageKernelIsAlgebraic, lines 22–112)
Lean: GlobalTheory.HaarAlg.isAlgebraic_integral (Development.AbelianVarietyTheory.Proofs.TwistedOperators.IsAlgebraicIntegral, lines 64–99)

In the finite test expand the level projector in Heisenberg translates of the standard vacuum. Its finite Fourier coefficients are algebraic. Moving a finite pair (z_l, h) past these translates again gives a finite combination with algebraic coefficients. Evaluation at zero now asks for derivatives of degree d_τ at each τ. In the standard self-dual lattices choose the two-torsion-normalized CM vacuum of Definition 34.48. Comparing the Fock vacuum with this one contributes a fixed nonzero scalar a. Write $\omega_\tau \neq 0$ for the CM period in the derivative coordinate at τ; because the CM type of E is induced from the single imaginary quadratic field $K_d = \mathbf{Q}(\sqrt{-d})$, all embeddings share a common CM period $\omega_\tau = p \neq 0$. A monomial of multidegree d brings the factor $\prod_\tau \omega_\tau^{d_\tau}$. In the negative model the comparison is conjugated. The remaining value is a Taylor coefficient of a rational lattice translate of the normalized vacuum, or an algebraic linear combination of such coefficients. Since the product of the two ranks is even, the standard lattices admit a nonzero CM vacuum. Up to an algebraic scalar it is a finite lattice-change average of products of normalized elliptic theta functions in algebraic linear coordinates. The elliptic CM Taylor formula at torsion translates [62] gives algebraic coefficients for each factor. The translation phases are roots of unity, so the average and its rational Heisenberg translates have algebraic Taylor coefficients. Hence one may take

$$\Omega = v\, a\, \overline{\prod_\tau \omega_\tau^{d_\tau}}.$$

It works for every term of every class sum. In particular h, t, k do not enter its choice.
Lean: GlobalTheory.ThetaAlgebraic.taylor_algebraic (Development.AbelianVarietyTheory.Proofs.FoundationProofs.ThetaTaylorCoeffsAlgebraicAtComplexMultiplicatio, lines 594–634)
Lean: GlobalTheory.BotTransJet.isAlgebraic_pureTensor0_div_period (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.JetsDivPeriodIsAlgebraic, lines 20–86)
Lean: GlobalTheory.BotAlgI.classSum_kernel_isAlgebraic (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.ClassSumKernelIsAlgebraic, lines 19–62)
Lean: GlobalTheory.BotAlgI.classSums_isAlgebraic (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.ClassSumKernelIsAlgebraic, lines 63–88)

On the primed source the real average is the projection for the determinant character. If all averages vanish, take period $\Omega' = 1$ (matching the vanishing-average branch on the unprimed side). Otherwise it

writes the averaged monomial as an algebraic linear combination of products of maximal minors, with a fixed number n_τ of minors at each infinite place. Every term thus has total degree $a' \sum_\tau n_\tau$, where $a' = \dim_E A'$. Because the CM type of E is induced from $K_d = \mathbf{Q}(\sqrt{-d})$, the period per derivative coordinate on this side has the common value $p = \omega_\tau \neq 0$ across all τ. If $a_0 \neq 0$ compares the two vacua (using the two-torsion-normalized CM vacuum of Definition 34.48), the minor calculation gives the same factor $a_0 p^{a' \sum_\tau n_\tau}$ for all these products. Indeed, expanding a minor gives integer coefficients, and moving the finite Heisenberg translates through the differential polynomial leaves algebraic coefficients and rational lattice translates. Applying the same elliptic Taylor formula and lattice-change average to these translates gives algebraic values after division by that factor. Inserting the class weights gives the period

$$\Omega' = v' a_0 p^{a' \sum_\tau n_\tau}.$$

Lean: GlobalTheory.RealProj.realProjection_average_monomial (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.TwistRealProjectionAverageMonomial, lines 15–38)
Lean: GlobalTheory.BottomFieldDict.vacuum_translate_minor_jet_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.BottomFieldMinorJetDescription, lines 16–100)
Lean: GlobalTheory.BottomFieldDict.kernel_minor_polynomial_eq_jet (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.BottomFieldMinorJetDescription, lines 101–132)
Lean: GlobalTheory.BotFormulaI'.bottomClassSum_formula (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.BottomClassSumEqFormula, lines 17–53)

For each side adjoin to M_0 the values at its finite set $T \times H \times D_0$, divided by the period just constructed for that side. They are algebraic, so each extension is finite. Divide the spanning identities by the same period; their coefficients still lie in M_0. All normalized integrals therefore lie in the compositum of these two finite extensions.
Lean: GlobalTheory.RowEx.BotField.field_of_gens (Development.AbelianVarietyTheory.Proofs.TwistedOperators.NormalFormBottom, lines 26–47)

□

Proof of 34.39

Proof. Take the maximum of the thresholds for grid existence, completion of the tests and the coefficient fields. For fixed ε, C_F above the required bounds, take the maximum of their scale cutoffs. Fix a scale beyond it, a preliminary datum with frame, and the finite set Ξ. The information needed from the level of a realization is the set

$$\kappa = \{(\xi_t, Q_t) : t \in T\}.$$

The level condition gives $Q_t \subseteq \mathrm{Ram}(\xi_t) \subseteq Q_t \cup \overline{Q_t}$. For each character of Ξ occurring in a realization its ramification set is therefore finite, and there are finitely many possibilities for Q_t. Taking subsets of the finite set of possible pairs gives finitely many κ.
Lean: GlobalTheory.RowEx.keyFin_holds (Development.AbelianVarietyTheory.Proofs.TwistedOperators.KeyFinHolds, lines 22–44)

For each realized κ choose a representative. The grid construction gives its grid and Weil system. Theorems 34.7 and 34.37 give matching and the proportionality of the theta integrals for the ordinary and twisted tests. With separation from Theorem 34.25 these complete the test choice at $(1, C_F, b(s))$. The representative satisfies the Hida level conditions, in particular finiteness of T and the prescribed character and ramification conditions.

For each $0 \leq i, j < m$ choose the standard positive definite source spaces of the two bottom ranks and standard target of their sum rank. Use the identity Gram matrices, standard self-dual lattices and standard frames. The two level characters have the restrictions prescribed by the ranks, so give the two Weil actions. Use their vacua with the grid additive character, insert the local idempotents and the prescribed conductor powers, and take the quotient measures for these two sources. This gives a standard bottom triple for this pair. Lemma 34.38 gives its two periods and a number field. Take the compositum over the finitely many pairs. Each pair keeps its own triple and its own two periods.
Lean: GlobalTheory.RowTest.exists_bottomTri (Development.AbelianVarietyTheory.Proofs.TwistedOperators.BottomExistsIntegrable, lines 104–134)
Lean: GlobalTheory.RowEx.bottomField_isNumberField (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.BottomFieldIsNumberField, lines 30–47)

Include the remaining coefficient fields of the representative. The normalized local values of the algebraic-type characters lie in number fields by their integral infinity types and finite conductors; the finite-order

characters take values in cyclotomic fields. The finite sets of test places and depths specify finitely many orders of roots of unity and square roots of norms. For the phases, the finite support conditions give an annihilating integer on each supported term, so one cyclotomic field suffices after taking their product. For the jets, the coset expansions at the specified depth and the class representatives give finitely many entries. Each is a normalized definite-kernel coefficient on the prescribed lattice with a supported test vector. Its CM Taylor expansion has algebraic coefficients by the calculation used in Lemma 34.38.
Lean: GlobalTheory.RowExNF.charsIn_nf (Development.AbelianVarietyTheory.Proofs.TwistedOperators.NormalFormCharacters, lines 184–252)
Lean: GlobalTheory.RowEx.nfRoot (Development.AbelianVarietyTheory.Proofs.TwistedOperators.IsIntegralOfPowEqOne, lines 44–72)
Lean: GlobalTheory.RowEx.nfPhase_of_sumOKK (Development.AbelianVarietyTheory.Proofs.TwistedOperators.NormalFormPhase, lines 24–28)
Lean: GlobalTheory.NFJetProof.jetCl_of_choice (Development.AbelianVarietyTheory.Proofs.TwistedOperators.NormalFormJets, lines 40–78)

For the L-value ratios use the twisted transport assertion of Definition 1.15, proved in Lemma 1.19. The host provides the CM type induced from its distinguished imaginary quadratic subfield. The grid gives the prescribed base conductor, auxiliary split primes coprime to it, and angular weights in the allowed range for $1 \le s \le N_* + 1$. The finite twist is trivial on principal ideles after its infinite component is removed and on the units of the indicated order, hence descends to its ideal class group. The untwisted toric sum reads the base L-value, and weighting its classes by this character reads the twisted value. Twisted transport gives algebraicity of their ratio when the denominator is nonzero. There are finitely many pairs, level indices and integers s, and uniqueness of the associated L-function fixes each value. These ratios generate a finite extension. Let $N_\kappa \subset \mathbf{C}$ be the compositum of all the fields constructed for this representative.
Lean: GlobalTheory.RowEx.RatioField.inNF_ratios (Development.AbelianVarietyTheory.Proofs.TwistedOperators.NormalFormToricRatio, lines 149–169)
Lean: GlobalTheory.RowEx.RatioField.twistRatio_mem_numberField (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.TwistRatioMemNumberField, lines 18–31)

Take S to be the union of the images in $\mathbf{C}$ of rational bases of the N_κ. An unrealized key contributes the empty set. Suppose now that $\iota_X(K_X)$ contains S, and take any realization in the statement. Match each of its indices to an index of the representative by the pair (ξ_t, Q_t). The trivial character matches the base indices because the level character labeling is injective. With the common preliminary datum and frame, this matching transports the grid, the Weil actions and the finite tests. Complete the transported tests on the new realization. On each bottom pair the transport retains the spaces, measures and finite test functions and reindexes the two kernels by the matching of tuples. Thus the two periods chosen on the representative work on the new realization. The same reindexing preserves the local character values, the supports for the phases and jets, and the L-value ratios. Finally $\iota_X(K_X)$ contains every rational linear combination of the chosen basis of N_κ. It contains all the required values on this realization.
Lean: GlobalTheory.RowEx.trBot_holds (Development.AbelianVarietyTheory.Proofs.TwistedOperators.TransferBottom, lines 40–156)
Lean: GlobalTheory.RowEx.exKey_of (Development.AbelianVarietyTheory.Proofs.TwistedOperators.ExKeyOf, lines 23–89)
Lean: GlobalTheory.RowEx.exField_of_keyFin_exKey (Development.AbelianVarietyTheory.Proofs.TwistedOperators.ExFieldOfKeyFinExKey, lines 22–36)

□

Proof of 34.41

Proof. At the joint threshold use the bare realization R of the stage group. Let H_E be the Hilbert-model row space of R. Write $e : H_E \to \mathcal{E}_0$ for its realization by bottom multiplicity maps, so that $e(u)(w)$ is an automorphic function for $u \in H_E$ and $w \in \mathcal{V}$. A complete choice gives the real tests ρ, ξ, a nonzero smooth tensor w_{sm}, and pairs (Ψ_i, Φ_i) of columns and grid data. Form

$$\ell_i(u) = \mathcal{B}_{\rho,\xi,\Psi_i,\Phi_i}\big(e(u)(w_{\mathrm{sm}})\big).$$

The complete choice has the following vanishing property from (A.382). If all its tests vanish on u for the chosen (ρ, ξ), they vanish on u after replacing (ρ, ξ) by any real pair and keeping the finite data fixed. Theorem 34.25 gives a nonzero pairing for $u \neq 0$, and the chosen columns span at each datum. Thus the common kernel is zero.

Let $\iota_H : K \hookrightarrow \mathbf{C}$ be the Hilbert embedding. Project $x \in FR_t$ by the idempotent for the good summand to x'. Equivariance gives the same row lift. Write $v_i(t, x')$ for the sum of phase, jet and lattice terms of the

row pairing on this summand. For each layer pair compare the two models by Theorems 34.7 and 34.37, with scalar $c_{i'j't} \neq 0$. Common real and measure normalizations give a κ independent of t, and finite character comparison contributes a root of unity $z_t \in K$. Write j_t^H for realization of the rational source in the Hilbert model at ι_H, and Θ_t^H for the lift from that model to H_E. The expansion is

$$\iota_H(z_t)c_t\,\ell_i\big(\Theta_t^H(j_t^H x)\big) = \kappa v_i(t,x'), \qquad c_t = \prod_{i',j'} c_{i'j't} \neq 0. \tag{A.401}$$

The field condition and the local sum formulas give $v_i(t,x') \in \iota_H(K)$. In the comparison with the base index μ_0, the chosen periods and fixed normalizations cancel. The finite character factors have their prescribed normalized values, and the L-ratios at the nonzero base factors are in the field. Hence $c_t/c_{\mu_0} \in \iota_H(K)$. For $\kappa \neq 0$ take $L_i = (c_{\mu_0}/\kappa)\ell_i$. Its theta value is $v_i(t,x')/(\iota_H(z_t)(c_t/c_{\mu_0}))$. For $\kappa = 0$ take $L_i = \ell_i$, with theta value zero. Both preserve separation. Let c_s be the identity or complex conjugation according to the stage sign, so that $v_0 = c_s \circ \iota_H$. Write $u^{(s)}$ for the image of u in the corresponding scalar twist of the row space. Set $\psi_i(u^{(s)}) = c_s(L_i(u))$ and transfer along the common group to EE. □

Proof of 34.43

Proof. For i, j and s in range, take $\chi = \mu_{0,j}\xi_{\mu_0}/\chi_{m-i}$ and either grid twist δ_c, where $\mu_{0,j}$ is the row character, ξ_{μ_0} the base twist, and χ_{m-i} a grid character. Let $c_{\tau,a}$ be the floored midpoint of adjacent grid weights at $a-1, a$, and write $e^{\mathrm{gr}}_{\tau,j+1} = e_{\tau,j}$ for the 1-indexed grid weight of Definition 34.46. Put $\kappa_\tau = \max(0, e^{\mathrm{gr}}_{\tau,j+1} + c_{\tau,m-i} - s)$. The grid gives

$$s \le h_{ij} \le N_*, \qquad e^{\mathrm{gr}}_{\tau,j+1} + c_{\tau,m-i} \ge g_{\mathrm{sep}} > s.$$

Hence $\kappa_\tau > 0$ and $\kappa_\tau + s = e^{\mathrm{gr}}_{\tau,j+1} + c_{\tau,m-i}$. Unfolding the toric sums gives the primitive Euler products; continuing with empty auxiliary set gives $u = L(s,\chi)$, $u_c = L(s,\chi\delta_c)$, both nonzero by Lemma 34.1.
Lean: `GlobalTheory.FunObj.SqNum.Model.kap_add` (`Development.AbelianVarietyTheory.Definitions.Divisors.IsValCharTwist`, lines 67–81)

For the host CM-field data on $K_d = \mathbf{Q}(\sqrt{-d})$ with $\Lambda = \mathcal{O}_{K_d}$, use the conjugate CM type $\bar\Psi = (\bar\sigma_\tau)$ (sending $w \in \{\pm\sqrt{-d}\}$ to $+i\sqrt{d}$), so an angular exponent $-2m_\tau$ on Ψ (i.e. $(\bar\tau/\tau)^{m_\tau}$ at infinity) becomes the positive angular exponent $+2m_\tau$ on $\bar\Psi$. Lemma 1.19 (for Definition 1.15) gives

$$\beta(v) = \frac{(C_0 v)^{12}}{\mathrm{disc}(\Lambda)^M} \in \overline{\mathbf{Q}} \qquad (v = u, u_c),$$

so $(u_c/u)^{12} = \beta(u_c)/\beta(u)$ makes u_c/u algebraic. Finitely many ratios occur by uniqueness. If bottom witnesses exist with common (δ, t_0), choose one family; picking an integral prime per class representative puts each coordinate $a \in \overline{\mathbf{Q}}$, so the stabilizer-weighted norm sum $\sum a\bar a$ is algebraic. Adjoin these finitely many norms and toric ratios.
Lean: `GlobalTheory.SqTop.SqField.isAlgebraic_norm` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.IsAlgebraicDivOfBeta`, lines 32–58)

□

Proof of 34.44

Proof. Use the prescribed G_0 and its Weil system in the construction of Theorem 34.39 to obtain N_1. Lemma 34.43 gives N_2. Level agreement transports both conditions. At fixed preliminary datum and gap there are finitely many $w(v_0) \mid v_0 \mid q_0$. The infinite types are fixed and conductors are bounded by a fixed ideal with exponents at most two. Ray class finiteness gives finite grid keys. Lemma 34.42 gives the finite field requirement for prescribed grids at threshold C_{F1}. For $C_F \ge C_{F1}$ take $\varepsilon_1 = 1/100$, zero auxiliary thresholds and the scale threshold in that finite field requirement. Use empty first three field requests. At the fourth use $S(P, f, \Xi)$ when Ξ is finite and the empty request otherwise. Each completed stage has a finite twisting set, so prescribed grids extend there. □

Proof of 34.45

Proof. Put $B' = B_0/6$. Lemma 34.43 gives nonzero primitive Hecke values, algebraic $\beta(v)$ and $\kappa_\tau \geq 1$. Apply the first assertion of Definition 1.15, proved in Lemma 1.19. The fitted host is abelian CM of degree $4|A|$, unramified over its real subfield at finite places, with even discriminant. Here $f = 2|A| \geq 2$, $d > 4$, $d \equiv 3 \pmod 4$ is prime. All embeddings in the conjugate CM type $\bar\Psi = (\bar\sigma_\tau)$ take the selected $w \in \{\pm\sqrt{-d}\}$ to $+i\sqrt{d}$, and their product on roots of unity is one; passing from Ψ to $\bar\Psi$ flips the negative angular infinity exponents $-2(e^{\mathrm{gr}}_{\tau,j+1} + c_{\tau,m-i})$ on Ψ to the positive angular exponents $+2(s+\kappa_\tau) > 0$ on $\bar\Psi$ used in Definition 1.15. The grid conductor is proper and nonzero, with exponents at most two at odd completely split primes. Splitting of g_0, g_1, q_0 follows from the congruences, and that of $p_{\mathrm{de}}, p'_{\mathrm{de}}$ from the fitted sizes. These are the field and conductor conditions of Lemma 1.18. Each of the three characters is unitary, trivial on the real subfield and has the prescribed exponents and conductor. Fix the same rational partial zeta values $z(C) = \zeta_F(1-2s, C)$ for all three applications. In each take $Q = \varnothing$ and additional Picard character 1.

Lean: `GlobalTheory.SqTop.fieldDat` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.NatCastMemBelow`, lines 154–162)

Lean: `GlobalTheory.FunObj.SqNum.IsTor` (`Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder`, lines 270–272)

Put $k_0 = 2sf$, $k_1 = \sum_\tau \kappa_\tau$. Use a_l, c_l, e_l, b_l from the first assertion of Definition 1.15. These are common to the three characters because the weights, conductor and partial zeta data are common. In the power-integrality assertion use $j = 1$, $r = 12\lceil b_l \rceil$. Thus $l^{12n_l}\beta(v)$ is integral above l, where

$$n_l = e_l + Ma_l + \lceil t_l \rceil, \qquad t_l = \begin{cases} 0, & c_l = 0, \\ k_0 c_l + k_1(2c_l + 1 + \mathbf{1}_{l=2} + \frac{1}{2}\mathbf{1}_{l=d}), & c_l > 0. \end{cases} \tag{A.402}$$

Let $S_{\mathrm{grid}} = \{g_0, g_1, p_{\mathrm{de}}, p'_{\mathrm{de}}, q_0\}$ and $N_m = N(m)$. The fitted sizes (Definition 34.19) give

$$\begin{gathered} \log g_0, \log g_1 \leq 9L + 1, \qquad \log p_{\mathrm{de}}, \log p'_{\mathrm{de}}, \log q_0 \leq L, \qquad \log \mathrm{rd}\, E \leq \log \mathrm{cond}\, E \leq L/100, \\ s \leq 8N_m^2, \quad m \geq 44, \quad mfN_m^2 d \leq b, \quad 16 \leq k_0 \leq b, \quad \kappa_\tau \leq 5b, \quad L \geq 1. \end{gathered} \tag{A.403}$$

Set $K_1 = k_0 + 5k_1 + 2M + 1 \leq 49fb$. At the grid primes $a_l \leq 2$ and $c_l \log l \leq 18L + 2$. Equation (A.402), including the ceiling, costs at most $K_1(18L+2)$ per grid prime beyond $e_l \log l$. Elsewhere use $c_l = v_l(|D_E|)/[E : \mathbf{Q}]$ and the root-discriminant and conductor sums to bound the remaining cost by $(k_0 + 5k_1 + 1)L/100$. The class zeta calculation of Lemma 1.19 gives $\sum_l e_l \log l \leq k_0 \log(k_0 + 1)$ for $k_0 \geq 16$. Thus on every finite set of rational primes

$$\sum_l n_l \log l \leq b(L+1) + 8K_1(18L+2) \leq 8000fb(L + \log x) \leq B' \tag{A.404}$$

provided $C_1/6 \geq 8000$.

For each transported character put $Z = \zeta(3/2)^{[E:\mathbf{Q}]}$, $A = |D_E| N\mathfrak{f}_\chi$, with conductor $\mathfrak{f}_\chi$. The Euler bound is $|L(s,\chi)| \leq Z$ for $s \geq 2$. At one use the Hecke completion in the strip from $-1/2$ to $3/2$, with right boundary bounded by the gamma integral and left boundary by the Hecke–Tate functional equation [12]. Nontrivial angular type gives an entire completion bounded on closed strips by its theta Mellin integral. The strip bound and $\Gamma(t+1/2)/\Gamma(t) \leq \sqrt{t}$ give

$$|L(1,\chi)| \leq ZA \prod_\tau (1 + s + \kappa_\tau) \qquad (s = 1). \tag{A.405}$$

Let $Q_{\mathrm{lev}} = \prod_{v \in S_0} (Nv)^{a_v}$, where $a_{v_0} = 1$ at the base place and $a_v = 2$ at every other grid prime. The conductor exponents give $N\mathfrak{f}_\chi \leq Q_{\mathrm{lev}}^2$. By (A.403), $\log |D_E| \leq 2fL$, $\log Q_{\mathrm{lev}} = \log q_0 + 2f(\log g_0 + \log g_1 + \log p_{\mathrm{de}} + \log p'_{\mathrm{de}}) \leq L + 2f(20L+2) \leq 2f(21L+2)$, and $1 + s + \kappa_\tau \leq 7b$. Thus the Euler and strip bounds give

$$\log |L(s,\chi)| \leq 101fL. \tag{A.406}$$

Lean: `GlobalTheory.HeckeStrip.completedLBoundedInStrips` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.CompletedHeckeLBoundedInVerticalStrips`, lines 17–53)

Apply ordinary transport of Definition 1.15 and Lemma 1.19. For an embedding fixing K_d, use the resulting ideal $\mathfrak{a}$, period lattice $\Lambda_{\mathfrak{a}}$ and character χ' with permuted weights. Then

$$\log|\sigma\beta(v)| = 12\log C_0 + 12\log|L(s,\chi')| - M\log(N\mathfrak{a}^6|\operatorname{disc}(\Lambda_{\mathfrak{a}})|). \tag{A.407}$$

Complex conjugation preserves log absolute values and pairs the embeddings over the two embeddings of K_d. The class action of Definition 1.15 is equidistributed. For $h(K_d)$ the class number, the logarithmic Chowla–Selberg formula for discriminant $-d<-4$ gives [153]

$$\frac{1}{h(K_d)}\sum_{[\mathfrak{a}]\in \mathrm{Cl}(K_d)} \log(N\mathfrak{a}^6|\operatorname{disc}(\Lambda_{\mathfrak{a}})|) = 12\log(2\pi) - 6\log d + 6\gamma - 6D_1, \quad D_1 = \frac{L'(1,\chi_{-d})}{L(1,\chi_{-d})}. \tag{A.408}$$

Here χ_{-d} is the quadratic character of K_d. The mean in (A.407) divided by twelve is at most $101fL + \log C_0 + M(\log d + |D_1|)/2$. Use $\log|D_F| \le fL$, $\log d, |D_1| \le L$, $M \le 12fb$, $2s+\kappa_\tau \le 7b$. Use factorials for the gamma factors to bound the three terms of $\log C_0$ by $2fbL$, $49fbL$, $40fbL$. The final M-term costs at most $12fbL$. The mean is therefore at most $204fbL$. Hence

$$\log|P_{\mathbf{Q},v}(0)|/(12\deg P_{\mathbf{Q},v}) \le B', \qquad P_{\mathbf{Q},v} = \min_{\mathbf{Q}}(\beta(v)) \text{ monic}. \tag{A.409}$$

Lean: `GlobalTheory.GalMean.galmean` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.MeanLogNormConjugatesOverIdealClasses`, lines 187–257)
Lean: `GlobalTheory.T.classMean_log_norm_disc_chowlaSelberg` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ClassMeanLogNormDiscChowlaSelbergStatement`, lines 9–10)

The norm to K gives $y_w(v) \ge -n_l v_w(l)$ for $w \mid l$. Using $\sum_{w|l}\ell_w v_w(l) = \log l$, (A.404) bounds the weighted negative part by B'. The product formula and transitivity of norms identify the weighted signed sum with the left side of (A.409). Therefore $\sum_w \ell_w|y_w(v)| \le 3B'$, with finite support. The identity $\beta(u_c) = z_c^{12}\beta(u)$ gives $y_w(u_c) = v_w(z_c) + y_w(u)$. Use $|n + \operatorname{trunc}_0 t| \le |n+t| + |t|$ to sum the two $3B'$ bounds. For $z_T = 1$ use $|\operatorname{trunc}_0 t| \le |t|$; outside range each term is zero. □

Proof of 34.51

Proof. After j top steps the loss is at most $j\delta_T(w)$. After j row steps it is at most $\delta_F(w) + j\delta_R(w)$. The terminal contraction therefore has loss at most

$$m\delta_T(w) + \delta_F(w) + m\delta_R(w) + \delta_P(w).$$

Multiplication by the four scalar factors adds their four losses. The nonarchimedean triangle inequality preserves this common bound under each of the finite sums. It follows that every elementary test pairing, and hence the pairing with every element of L_{form}, has valuation at least $-\delta(w)$ when nonzero.

Set $r = \lfloor \delta(w) \rfloor$. Since valuations are integral, local duality for the full lattice gives $y \in \mathfrak{p}_w^{-r} M_w$. Choose $\pi \in K$ with $v_w(\pi) = 1$. Then $\pi^r y \in M_w$, so some $u \in \mathcal{O}_K \setminus \mathfrak{p}_w$ satisfies $u\pi^r y \in M$. Taking $s_w = u\pi^r$ gives $v_w(s_w) = r \le \delta(w)$.

The support of δ lies in the union of the eight finite supports. Adding the weighted bounds gives

$$\sum_w \ell_w \delta(w) \le 4m\mathcal{B} + 2m\mathcal{B} + 2\mathcal{B} = (6m+2)\mathcal{B} \le 8m\mathcal{B}.$$

□

Proof of 34.52

Proof (outline). With $\sigma(G)$ from the statement, put

$$B_1(C,b) = CfmN_*^2 b\big(\log b + \sigma(G)\big).$$

The depth $d_{\mathrm{pl}}(q)$ is the prescribed depth of Definition 34.47. For a prime ℓ, put

$$\Delta_G(\ell) = \max_{\substack{\mathfrak{P}\in W_0 \\ \mathfrak{P}\mid \ell}} d_{\mathrm{pl}}(\mathfrak{P}\cap \mathcal{O}_F),$$

with value zero for an empty set. *Claim 1.* Under the hypotheses of this lemma, after increasing the grid and scale thresholds, each chosen test datum Φ and level index t has a finite supported term set and finite class sets (by the finiteness of definite unitary double-coset classes at fixed level). For each supported unfolding term β, expanding the flat-tensor local test functions away from the twist primes Q_t into periodic coset indicators and taking $k_{t,\Phi,\beta}$ to be the minimal common period (greatest common divisor of admissible flat-tensor periods) produces a finite expansion into pure coset data of modulus $k_{t,\Phi,\beta}$ whose coefficient is the product of its local datum values, giving the exact finite expansion of Lemma 34.51. At every finite place $w \mid \ell$ of K such that every $\mathfrak{P} \in Q_t$ either lies above S_0 or has rational residue prime different from ℓ, only the grid places $\mathfrak{P} \in W_0$ above ℓ contribute ℓ-power periods, yielding

$$v_\ell(k_{t,\Phi,\beta}) \le \Delta_G(\ell)$$

in the quantifier order of the statement: above a grid threshold for C_F chosen before any datum and, for each larger C_F, each ε and each C_1, above a cutoff for $x(s)$ depending only on (C_F, ε, C_1), uniformly in $D, L_{\mathrm{sz}}, R, F_{\mathrm{fc}}, G, W, a$, in t, Φ, β and in the class indices.
Lean: `GlobalTheory.SumSize.Join.sum_size_finite_away_archimedean_combined` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.SumSizeFiniteAwayArchimedeanCombined`, lines 26–44)
Lean: `GlobalTheory.SumSize.DepthAt` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt`, lines 97–100)
Lean: `GlobalTheory.SumSize.depL` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt`, lines 57–57)
Lean: `GlobalTheory.RowTest.IsCosetExpansion` (`Development.AbelianVarietyTheory.Definitions.TwistedInnerProducts.MemHalfCosetSelf`, lines 122–123)
Lean: `DenominatorLossAdditivity.GlobalTheory.CompanionTheory.FinDen.core_loss` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FiniteSumDenominatorBoundOfFactorClaims`, lines 280–338)

Claim 2. The finite expansion of Claim 1 and each of the ten finite-place bounds (the depth bound of Claim 1 together with the nine denominator controls below) hold in the quantifier form of the statement. For the i-th bound there is a real number $C_{F,i}$, chosen before any datum, such that for every $C_F \ge C_{F,i}$ there is $c_i \in \mathbf{R}$ depending only on C_F, and for every ε and every $C \ge c_i$ there is a scale cutoff depending only on (C_F, ε, C) above which the bound with constant C holds for every $D, L_{\mathrm{sz}}, R, F_{\mathrm{fc}}, G, W, a$ as in the statement; the expansion has a grid threshold and, for each C_F above it and each ε, a scale cutoff. Let C_{F0} be the maximum of these eleven grid thresholds and fix $C_F \ge C_{F0}$. Set

$$C_0 = 16 \max(c_1, \dots, c_{10}), \qquad \mathcal{B} = B_1(C_1/16, b).$$

For any $C_1 \ge C_0$ and ε, taking the maximum of the scale cutoff of the expansion and the ten scale cutoffs evaluated at $C = C_1/16$ supplies eight nonnegative, finitely supported loss functions $\delta_S, \delta_C, \delta_D, \delta_G, \delta_T, \delta_R, \delta_F, \delta_P$ each chosen once, before the level index t, the place w, the datum Φ, the term β and the class indices, and valid at all places $w \mid \ell$ satisfying the avoidance condition of Claim 1, as follows: (i) the global normalizing scalar s_{nor} is an explicit product of discriminant, Tamagawa/volume, and level factors whose w-adic valuation loss δ_S is controlled by S_0 and $\mathrm{disc}(E)$; (ii) each unfolding coefficient c_β is an integral power of residue cardinalities $N\mathfrak{P}$ arising from triangular unipotent level determinants, bounding its loss δ_C; (iii) each local datum product $d_{\beta,\eta}$ combines exceptional flip-place losses with moved-tensor unfolding valuations (whose coset-exponent sum is at most $2\sigma(G)$), bounding δ_D; (iv) each inverse stabilizer factor $\gamma_{\beta,\gamma} = |\Gamma|^{-1}$ comes from a finite unitary stabilizer embedding into $\mathrm{GL}_N(\mathcal{O}_E)$, so $|\Gamma|$ divides Minkowski's universal bound for finite subgroups of $\mathrm{GL}_N(\mathcal{O}_E)$, giving a finitely supported loss δ_G with weighted sum at most $90 m f N_*^2 b(\log b + \sigma(G))$; (v)–(vi) via the local-type dictionary for theta-kernel step matrices, on every pure coset datum of depth $k_{t,\Phi,\beta}$ with $v_\ell(k_{t,\Phi,\beta}) \le \Delta_G(\ell)$ a step $X \to Y$ between consecutive nodes of the top path or of the row path raises the w-adic lattice denominator of a vector by at most $v_w(\ell)\, C_{\ker}\, \kappa_{X,Y}(\ell, v_\ell(k_{t,\Phi,\beta}))$, where $C_{\ker} > 0$ is an absolute constant and

$$\kappa_{X,Y}(\ell, n) = \big(|\nu_Y| + f \operatorname{rk} X \operatorname{rk} Y\big)\big(\delta_{\mathrm{den}}(\ell) + [\ell = d] + n\big) + f N_*^2\, [\ell \le C_{\ker} b] \log_\ell(C_{\ker} b).$$

Here $[\cdot]$ is 1 when the condition holds and 0 otherwise; d is the prime of the host datum with $K_d = \mathbf{Q}(\sqrt{-d}) \subset E$ (Definition 34.46); $\operatorname{rk} X$ is the rank of the Hermitian space at the node X; $|\nu_Y|$ is the sum over τ of the entries of the shifted nonnegative tuple of the dominant weight at Y; and $\delta_{\mathrm{den}} \ge 0$ is a function on primes, vanishing outside a finite set, such that at every prime ℓ each kernel coordinate is a combination of integral matrices with coefficients of ℓ-adic absolute value at most $\ell^{\delta_{\mathrm{den}}(\ell)}$, and $\sum_\ell \delta_{\mathrm{den}}(\ell) \log \ell \le \log \mathrm{rd}(E)$.

At modulus $k = k_{t,\Phi,\beta}$, up to the fixed normalization, the monomial period factor and the conjugation prescribed by the model, each step jet is the jet at zero of a common scalar times a sum of translates of the lattice-change image of the vacuum. The scalar is a unit at primes where both input lattices are standard. Each translation point s_i has ks_i in the period lattice. The coefficients r_i inside the sum satisfy $k^{f \operatorname{rk} X \operatorname{rk} Y} r_i \in \overline{\mathbb{Z}}$, which accounts for the coefficient loss $f \operatorname{rk} X \operatorname{rk} Y\, v_\ell(k)$. Since $\kappa_{X,Y}(\ell, n)$ is nondecreasing in n and $v_\ell(k_{t,\Phi,\beta}) \le \Delta_G(\ell)$, the functions $\delta_T(w)$ and $\delta_R(w)$, equal to $v_w(\ell) C_{\mathrm{ker}}$ times the sum of $\kappa_{X,Y}(\ell, \Delta_G(\ell))$ over the m steps of the top path and of the row path respectively, bound the loss of each of the m top-kernel maps T_j and m row-kernel maps R_j. They depend only on w, and they vanish at every w whose residue prime ℓ exceeds $C_{\mathrm{ker}} b$, differs from d, lies outside the support of δ_{den} and lies below no place of W_0; (vii) the terminal branching pairing between the top-node Weyl lattice and the corner-node dual Weyl lattice is integral up to explicit factorial and weight-spread denominators bounded by δ_P; (viii) the initial top test vector Ψ_0 is constructed directly inside the integral Weyl lattice at the top node, giving denominator loss zero; and (ix) the initial row form F_0 takes values in the scaled dual Weyl lattice whose denominator is controlled by the level-quotient cardinality, giving δ_F. Their weighted sums satisfy

$$\sum_w \ell_w \delta_\star(w) \le \begin{cases} m\mathcal{B}, & \star \in \{S, C, D, G\}, \\ \mathcal{B}, & \star \in \{T, R, F, P\}. \end{cases}$$

At every place w satisfying the avoidance condition of Claim 1, these eight losses apply uniformly across all occurring terms, classes, and pure coset data. These are precisely the eight factor hypotheses of Lemma 34.51; their weighted bounds also give $\mathcal{B} \ge 0$ and, via $(6m+2)\mathcal{B} \le 8m\mathcal{B}$, absorb the factor 16 in C_0 back into C_1 below.

Lean: `GlobalTheory.SumSize.Join.sum_size_finite_away_archimedean_combined` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.SumSizeFiniteAwayArchimedeanCombined`, lines 26–44)
Lean: `GlobalTheory.SumSize.FinScalAt` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt`, lines 102–102)
Lean: `GlobalTheory.SumSize.FinCoefAt` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt`, lines 104–105)
Lean: `GlobalTheory.SumSize.FinDatAt` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt`, lines 107–108)
Lean: `GlobalTheory.SumSize.FinGamAt` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt`, lines 110–111)
Lean: `GlobalTheory.SumSize.FinTopAt` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt`, lines 113–116)
Lean: `GlobalTheory.SumSize.FinRowAt` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt`, lines 118–122)
Lean: `GlobalTheory.SumSize.FinPairAt` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt`, lines 124–126)
Lean: `GlobalTheory.SumSize.FinPsiAt` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt`, lines 128–129)
Lean: `GlobalTheory.SumSize.FinFormAt` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt`, lines 131–133)
Lean: `DenominatorLossAdditivity.GlobalTheory.CompanionTheory.FinDen.fin_of_factors` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FiniteSumDenominatorBoundOfFactorClaims`, lines 535–574)

For the tied test lattice, apply Lemma 34.51 to the evaluation vector y. Use the loss function defined in Lemma 34.51, namely

$$\delta_{\mathrm{sum}} = \delta_S + \delta_C + \delta_D + \delta_G + m\delta_T + m\delta_R + \delta_F + \delta_P,$$

and let $T_{\mathrm{sum}} = \operatorname{supp}(\delta_{\mathrm{sum}})$. Since B_1 is linear in its first argument,

$$\begin{aligned} 2\sum_w \ell_w \delta_{\mathrm{sum}}(w) &\le 16 m B_1(C_1/16, b) = m B_1(C_1, b) \\ &= C_1 f b m^2 N_*^2 \big(L + \sigma(G)\big). \end{aligned}$$

The same lemma gives the required scalar s_0. Outside T_{sum}, its valuation is at most zero, and $s_0 y \in M$ implies $y \in M_w$.

Lean: `GlobalTheory.SumSize.bOne` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt`, lines 48–48)
Lean: `DenominatorLossAdditivity.GlobalTheory.CompanionTheory.FinDen.fin_at` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FiniteSumDenominatorBoundOfFactorClaims`, lines 507–531)

The complete proof is given in `GlobalTheory.SumSize.Join.sum_size_finite_away_archimedean_combined`. □

Proof of 34.53

Proof (outline). Claim 1. The finite-sum estimate and place avoidance can be imposed simultaneously with the Hida row setting, with the closed quantifier order in the statement. More precisely, there are grid thresholds c_1, c_2; after $C_F \geq \max(c_1, c_2)$ is fixed, there are estimate thresholds $C^{(1)}, C^{(2)}$. One can take

$$C_0 = \max\bigl(C^{(1)}, C^{(2)}, C_F, 0\bigr).$$

For every $C_1 \geq C_0$, one eventual cutoff then works for all member points and all fitted data, and at each such point the setting and the following avoidance property hold. The two estimate thresholds have the closed quantifiers of Definition 36.10. Take the maximum of the two estimate cutoffs and the cutoff for the Hida row setting, so that all three assertions hold at the same member points. At each fitted point beyond the common cutoff, the row-size and series conditions give the Hida row setting. The place-avoidance assertion says that, for each finite place w of K and every $\mathfrak{P} \in Q_{\mu(w)}$, either $\mathfrak{P} \cap \mathcal{O}_F \in S_0$, or the rational residue prime of $\mathfrak{P}$ differs from that of w. Thus Lemma 34.52 applies at $t = \mu(w)$.

Lean: `GlobalTheory.SumSize.Away` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt`, lines 52–53)
Lean: `GlobalTheory.SumSize.AwayAt` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt`, lines 139–139)
Lean: `GlobalTheory.Adopt.twisted_local_factor_away_from_level` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.TwistedLocalFactorAwayFromLevel`, lines 51–52)
Lean: `LeafFixedPointPairing.GlobalTheory.CompanionTheory.LeafFixedPointPlaceNorm.fp_tie_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LeafFixedPointOfFinKAwayK`, lines 23–46)

Claim 2. A real finite-sum allowance gives the stated integral local allowance. Let δ_{sum} be its allowance, put $T_F = \mathrm{supp}(\delta_{\mathrm{sum}})$, and set $d_F(w) = \lfloor \delta_{\mathrm{sum}}(w) \rfloor$. At a member point the source lattice in Lemma 34.52 is M_t, by Definition 34.49(1). The evaluation tie therefore gives, for each $x \in M_{\mu(w)}$, a nonzero s with

$$v_w(s) \leq \delta_{\mathrm{sum}}(w), \qquad sV^0_{\mu(w)} x \in M.$$

Since $v_w(s) \in \mathbf{Z}$, it is at most $d_F(w)$. Localization gives

$$V^0_{\mu(w)}(M_{\mu(w),w}) \subseteq \mathfrak{p}_w^{-d_F(w)} M_w.$$

The function d_F vanishes off T_F.

Lean: `LeafFixedPointPairing.GlobalTheory.CompanionTheory.LeafFixedPointPlaceNorm.local_clause` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FiniteAndAwayFormula`, lines 71–93)
Lean: `LeafFixedPointPairing.GlobalTheory.CompanionTheory.LeafFixedPointPlaceNorm.leaf_at` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FiniteAndAwayFormula`, lines 125–136)

Claim 3. The integral allowance satisfies the asserted global budget. At the fitted grid, its depth sum is $\sigma(G) = \Sigma_0(C_F, b)$. The base depth is monotone in its constant, so $C_F \leq C_1$ gives $\sigma(G) \leq \Sigma_0(C_1, b)$. Since $C_1 \geq 0$, $L \geq 0$, and $\Sigma_0 \geq 0$,

$$\begin{aligned} 2 \sum_{w \in T_F} \ell_w d_F(w) &\leq 2 \sum_{w \in T_F} \ell_w \delta_{\mathrm{sum}}(w) \\ &\leq C_1 f b m^2 N_*^2 \bigl(L + \sigma(G)\bigr) \\ &\leq C_1 f b m^2 N_*^2 (L + \Sigma_0) \\ &\leq C_1 f b m^2 (N_* + N_*^2)(L + \Sigma_0) = B_F. \end{aligned}$$

Lean: `LeafFixedPointPairing.GlobalTheory.CompanionTheory.LeafFixedPointPlaceNorm.sig_le` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FiniteAndAwayFormula`, lines 63–69)
Lean: `GlobalTheory.FinitePlacesProbe.sigma_mono_in_constant` (`Development.AbelianVarietyTheory.Counterexamples.RotationExamples.SigmaMonoInConstant`, lines 31–41)
Lean: `LeafFixedPointPairing.GlobalTheory.CompanionTheory.LeafFixedPointPlaceNorm.sum_clause` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FiniteAndAwayFormula`, lines 95–123)

Claim 4. The same allowance works after any of the stated root-of-unity twists. The inverse of a root of unity is integral over $\mathcal{O}_K$. Multiplication by ζ_t^{-1} preserves every localized $\mathcal{O}_K$-lattice and therefore preserves the inclusion and its unchanged allowance.

Lean: `GlobalTheory.finitePlaceBound_inv_rootOfUnity_smul` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.MatLatSmulLeOfIsIntegral`, lines 24–32)

The complete proof is given in `LeafFixedPointPairing.GlobalTheory.CompanionTheory.LeafFixedPointPlaceNorm.fp_tie_alt`. □

Proof of 34.55

Proof (outline). Use the order of constants in the statement and the closed quantifiers of Definition 36.10. Choose C_F and then C_0 large enough for Claim 1 below, with $C_0 \ge \max(C_F, 0)$, and fix $C_1 \ge C_0$. Take the maximum of the cutoff for the Hida row setting and the scale cutoff in Claim 1.

Claim 1. Under $\mathscr{P}$, for every Hida row setting, grid, Weil system and complete choice satisfying the hypotheses of Lemma 34.52, the following bound holds with the same order $C_F, C_1, \varepsilon, x_0$ of uniform thresholds. Let (M, X') be the tied test lattice, let X_v be invertible with

$$(X_v^{-1})^{\dagger} X_v^{-1} = H_V(\mu_0)_v,$$

and let V be an evaluation matrix satisfying

$$\iota((Vx)_\alpha) = v_{\mathrm{row}}(a, \mu_0, \mathrm{form}_{F_{\mathrm{fc}}}(\mu_0, x))_\alpha \quad \text{for every source vector } x \text{ and coordinate } \alpha.$$

Put $A_1(C,b) = CfN_* b \log b + CmN_*^2 \sigma(G)$. For all compatible frames X_v and evaluation matrices V just specified, the archimedean layer-sum estimate gives $h_{X,X'}(V) \le mA_1(C_1, b)$. Specifically, each of the nine archimedean factor estimates holds above an individual grid threshold and, once C_F is fixed above their maximum, admits an estimate threshold depending only on C_F; taking C_0 above C_{comb} times the maximum of those thresholds and evaluating the nine scale cutoffs at C_1/C_{comb} controls every factor of the expansion of Lemma 34.51 at the base tuple μ_0 uniformly across all infinite places v of K (embeddings $\sigma : K \hookrightarrow \mathbf{C}$): (i) the global normalizing scalar s_{nor} obeys $\log|\sigma(s_{\mathrm{nor}})| \le mA_1$ from its discriminant, volume, and level factors; (ii) combining local character-conductor bounds, unfolding-factor archimedean estimates, and local volume coefficients gives $\log|\sigma(c_\beta d_{\beta,\eta})| \le Cm^2 N_*^2 \sigma(G)$ for every supported tensor term; (iii) the datum count, supported coset-family count (via the k-coset counting formula), and pure-tensor count deduced from Claim 1 of Lemma 34.52 are each at most $\exp(mA_1)$; (iv) principal congruence containment inside the top and vertex levels combined with Minkowski's definite-lattice class-number bound $(12 f^2 N^4 \operatorname{rd}(E)^2)^{6fN^2}$ (as in Lemma 34.63) times the level-quotient index bounds every occurring class-set cardinality by $\exp(mA_1)$; (v)–(vi) let $\Lambda_Z = \Lambda(\mathfrak{c}_K, \delta_Z)$ be the lattice of the curve datum of the Weil system, with two-torsion point $t_{0,Z}$, let $M_Z \subset \mathbf{C}$ be the number field generated by the image of K and $g_2(\Lambda_Z), g_3(\Lambda_Z), \wp_{\Lambda_Z}(t_{0,Z})$, and for each embedding ρ of M_Z let A_ρ, ν_ρ be as in Lemma 34.62. In the Hida row setting its hypotheses hold with $L = \log b$, and it gives an absolute constant $C_{\mathrm{mean}} > 0$ such that the averages over $v \mid \infty$ of the means of $\log^+ A_\rho^{-1}$ and of $\log^+ \nu_\rho$ over the ρ restricting to the embedding σ_v at v are at most $C_{\mathrm{mean}} \log b$. There are $a_0, n_D, G_d \ge 0$ with $2a_0 + (n_D + G_d) C_{\mathrm{mean}} \log b \le A_1$ such that, for every embedding σ of K and every ρ restricting to σ, each of the m top step maps multiplies squared coefficient norms computed at σ by at most $\exp(a_0 + n_D \log^+ A_\rho^{-1} + G_d \log^+ \nu_\rho)$; another such triple gives the same bound for each of the m row step maps, with norms computed at $\sigma \circ c$ for the complex conjugation c of K. Let $g_{\mathrm{top}}(v)$ be the mean of the top exponent over the ρ restricting to σ_v, and let $g_{\mathrm{row}}(v)$ be the mean of the row exponent over the ρ restricting to $\sigma_v \circ c$. The map $v \mapsto [\sigma_v \circ c]$ is an involution of the infinite places, so this change preserves the average of the row exponent. Taking ρ with exponent at most the mean, each top step at v multiplies squared norms by at most $e^{g_{\mathrm{top}}(v)}$ and each row step by at most $e^{g_{\mathrm{row}}(v)}$, and the averages of g_{top} and g_{row} over $v \mid \infty$ are at most $a_0 + (n_D + G_d) C_{\mathrm{mean}} \log b \le A_1$;

Lean: `GlobalTheory.KerConjMean.ConjTopAt` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.EmbLLMem`, lines 61–68)

Lean: `GlobalTheory.KerConjMean.archimedean_top_term_kernel_conj_mean` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.KernelConjugationMeanArchimedeanTopTerm`, lines 13–47)

(vii) the Fock-norm branching pairing bound ensures the terminal contraction amplifies squared norms by at most $\exp(A_1)$; (viii) the initial top test vector satisfies an exact Gram-frame transport identity matching its norm on the tied test lattice; and (ix) the finite class-sum Hermitian form estimate bounds the squared norm of the initial row form at μ_0 by $\exp(mA_1)\, x^{\dagger} H_V(\mu_0)_\sigma x$. Take the nine factor estimates at $C = C_1/16$, so that $A_1 = A_1(C, b)$. Fix $v \mid \infty$ with embedding σ, and let $\Gamma_{\mathrm{test}} = G_{\mathrm{form}}(G, e_a)$ be the Hermitian form Gram matrix of the form lattice in Definition 34.49(3), so that $((X_v')^{-1})^{\dagger}(X_v')^{-1}$ is the transpose of $\sigma(\Gamma_{\mathrm{test}})^{-1}$. For a source vector x and a coefficient vector $c \in K^n$, group the coordinates of the pairing $\sigma(Vx) \cdot \sigma(c)$ by their test index $b \in \mathcal{I}_{\mathrm{test}}(G) = \mathcal{D}_{\mathrm{test}}(G) \times \mathrm{Cl}_+(G)$ under e_a. The number of test indices is at most $\exp(2mA_1)$ by the datum count in (iii) and the bound on $|\mathrm{Cl}_+(G)|$ in (iv). For each b, the partial pairing is s_{nor} times a sum over the supported terms, their pure tensors and their classes. By (i), the counts of supported terms and

of pure tensors in (iii), the class count in (iv), (ii), (v)–(viii) and (ix), its squared absolute value is at most $\exp\bigl(9mA_1 + 2Cm^2N_*^2\sigma(G) + A_1 + mg_{\mathrm{top}}(v) + mg_{\mathrm{row}}(v)\bigr)\bigl(x^\dagger H_V(\mu_0)_\sigma x\bigr)n_b$, where $n_b \geq 0$ is the contribution of the coordinates with test index b to $\sigma(c)^\dagger\sigma(\Gamma_{\mathrm{test}})\sigma(c)$, the Gram matrix being block diagonal in b. The Cauchy–Schwarz inequality over b then gives

$$|\sigma(Vx)\cdot\sigma(c)|^2 \leq e^{\Lambda_v}\bigl(x^\dagger H_V(\mu_0)_\sigma x\bigr)\bigl(\sigma(c)^\dagger\sigma(\Gamma_{\mathrm{test}})\sigma(c)\bigr),$$
$$\Lambda_v = 11mA_1 + 2Cm^2N_*^2\sigma(G) + A_1 + mg_{\mathrm{top}}(v) + mg_{\mathrm{row}}(v).$$

Duality for the Gram form and density of $\sigma(K)^n$ in $\mathbf{C}^n$ turn this into $\|(X_v')^{-1}V_vX_v\|^2 \leq e^{\Lambda_v}$, using $X_vX_v^\dagger = H_V(\mu_0)_v^{-1}$. Averaging $\log\|(X_v')^{-1}V_vX_v\|^2$ over $v \mid \infty$, with $Cm^2N_*^2\sigma(G) \leq mA_1$, $A_1 \leq mA_1$ and the mean bounds of (v)–(vi), gives $h_{X,X'}(V) \leq 16mA_1(C,b) = mA_1(C_1,b)$. Thus $C_{\mathrm{comb}} = 16$. At a fitted point, $C_1 \geq C_F$ and $C_1 \geq 0$ give $\sigma(G) \leq \Sigma_0(C_1,b)$, and therefore

$$\begin{aligned} h_{X,X'}(V) &\leq m\bigl(C_1fN_*b\log b + C_1mN_*^2\Sigma_0(C_1,b)\bigr)\\ &= C_1fmN_*bL + C_1m^2N_*^2\Sigma_0(C_1,b) = H_{I,0}. \end{aligned}$$

The estimate applies to every compatible source frame and every matrix with the displayed evaluation identity. Its two terms retain the factors m and m^2 before passage to the quotient metric.
Lean: `GlobalTheory.SumSize.InfAt` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt`, lines 176–179)
Lean: `GlobalTheory.SumSize.Join.sum_size_finite_away_archimedean_combined` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.SumSizeFiniteAwayArchimedeanCombined`, lines 26–44)
Lean: `GlobalTheory.QuotientNormCoisometry.bound_le` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.InfinitePartIdentity`, lines 95–106)

Take $X_v = H_V(\mu_0)_v^{-1/2}$ and $V = V^0_{\mu_0}$. Positive definiteness gives the required frame identity, and the fitted evaluation tie gives the displayed identity for V. The map P_{μ_0} is surjective and its target carries the direct quotient metric. Lemma 34.54, applied to $B = V_b$, therefore gives

$$h(V_b) = h_{X,X'}(V_bP_{\mu_0}) = h_{X,X'}(V^0_{\mu_0}) \leq H_{I,0}.$$

Lean: `Formalism.L2.DIV.frameOfSpec` (`Formalism.LocalToGlobal.Divisibility.FramesLemmas`, lines 77–89)
Lean: `GlobalTheory.QuotientNormCoisometry.htMap_quot` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.InfinitePartIdentity`, lines 64–83)

The complete proof is given in `GlobalTheory.QuotientNormCoisometry.leafHeightMapBound_tie_of_infK`. □

Proof of 34.56

Proof (outline). Choose the complete choice a and the test-lattice tie supplied by Definition 34.49(3). The fitted row-size inequalities and series condition give the Hida row setting at the member point. All the estimates below hold for every such fitted datum once this setting holds, so its eventual cutoff serves simultaneously for them.

Claim 1. The weight on the initial top node has spread at most $R_p(C_1)$, and $R_p(C_1) \geq 1$. Its rank is

$$n_p = k + m + m(k + m + 4).$$

For each $\tau \in \mathcal{T}$ and each two weight indices i, j, the dominant-weight construction gives

$$\lambda_{P,G,\tau,i} - \lambda_{P,G,\tau,j} \leq \mathrm{cl}(\tau,2) - \mathrm{cl}(\tau,k) + 1 + (m+1)\gamma_{\mathrm{gap}}(1) + n_p.$$

Here $\mathrm{cl}(\tau,2) - \mathrm{cl}(\tau,k) = \lfloor v_{\tau,0}\rfloor - \lfloor v_{\tau,k-2}\rfloor \leq L_c$ by Definition 34.50. Since $C_1 \geq 1$, monotonicity gives $\gamma_{\mathrm{gap}}(1) \leq \gamma_{\mathrm{gap}}(C_1)$. Thus the last display is at most

$$(m+1)\bigl(\gamma_{\mathrm{gap}}(C_1) + 1\bigr) + L_c + n_p = R_p(C_1).$$

All three summands defining R_p are nonnegative, and the first is at least one.
Lean: `GlobalTheory.BigTop.Fun.LeafMMTie.leafMM_of_test` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.MultiplicityAtFittedPoints`, lines 94–138)

Claim 2. Let G be this grid, with its positive definite initial top space, star-compatible embeddings of the CM field into K, and self-dual class lattices. Let v_0 be an infinite place at which the involution of K induces complex conjugation. Suppose $R \geq 1$ bounds all the coordinate differences of its dominant weight. For a nonempty finite test index set and an indexing equivalence

$$e : \{1, \dots, n\} \simeq \mathcal{I}_{\mathrm{test}}(G) \times \mathcal{I}_{\mathrm{coef}}(G),$$

form the direct sum of the integral coefficient lattices of the class lattices and take its coordinate dual. If M is this dual and (X'_v) are frames for the inverse-transpose of its direct-sum Gram matrix, then every X'_v is invertible, M is finitely generated and nonzero, and

$$\mu_{\max}(M) \leq 4|\mathcal{T}|\ \mathrm{rank}(X_0) R \log\bigl(\mathrm{rank}(X_0) R\bigr).$$

This is proved in four steps. First, for $\tau \in \mathcal{T}$ let J_τ be the Gram matrix J of the initial top space transported along τ, and put $J_\tau^\vee = (J_\tau^{-1})^{\mathsf{T}}$. Since the embeddings are star-compatible and J is positive definite at every complex embedding, J_τ and $J_\tau^\vee$ are Hermitian, positive definite at every complex embedding of K, and have entries in the subfield of K on which the involution induces complex conjugation at every embedding. For each class lattice Λ_c, self-dual for J, the coordinate dual $\tau(\Lambda_c)^\vee$ of the $\mathcal{O}_K$-span of $\tau(\Lambda_c)$ is self-dual for $J_\tau^\vee$. Second, let Δ_c and ∇_c be the Weyl lattice and the dual Weyl lattice of the family $(\tau(\Lambda_c)^\vee)_\tau$ in $\mathcal{W}_K(\lambda_{P,G})$, and let H be the direct-sum Gram matrix, in the coordinates e, of the tensor pairings of the $J_\tau^\vee$. The given lattice is $M = (\bigoplus_c \Delta_c)^\vee$, and the frames X'_v realize $(H^{-1})^{\mathsf{T}}$. Hermitian duality of transported Weyl lattices identifies the H-dual of $\bigoplus_c \Delta_c$ with $\bigoplus_c \nabla_c$. Third, let $s = \prod_\tau \|v_{\lambda(\tau)}\|^2$ be the product of the Fock norms of the extremal vectors. Write $r_0 = \mathrm{rank}(X_0)$ and $R_0 = \lfloor R \rfloor$. The spread bound makes every shifted weight entry at most R_0. Writing $v_{\lambda(\tau)} = \sum_\alpha a_\alpha y^\alpha$ with $a_\alpha \in \mathbf{Z}$, one has $\|v_{\lambda(\tau)}\|^2 = \sum_\alpha a_\alpha^2\, \alpha!$, with $\sum_\alpha |a_\alpha| \leq (r_0!)^{R_0}$ and $\alpha! \leq (R_0!)^{r_0}$. Hence s is a positive integer with $\log s \leq |\mathcal{T}| \log\bigl((r_0!)^{2R_0} (R_0!)^{r_0}\bigr) \leq 4|\mathcal{T}| r_0 R \log(r_0 R)$. The product of s with the tensor pairing is integral on each ∇_c. By the second step, $s\, x^\dagger H y \in \mathcal{O}_K$ on the H-dual of $\bigoplus_c \Delta_c$, and therefore $s\, x^\dagger (H^{-1})^{\mathsf{T}} y \in \mathcal{O}_K$ for all $x, y \in M$. Fourth, $(H^{-1})^{\mathsf{T}}$ has entries in the same subfield. Let $Q \subseteq M$ be a nonzero submodule, and take a K-basis of its span. Its Gram determinant for $(H^{-1})^{\mathsf{T}}$ is nonzero by positivity at the conjugation place v_0. The product over $v \mid \infty$ of its absolute values is at most the product of the Gram determinants of the metrics. Indeed, at each v with representative embedding σ_v, the Cauchy–Schwarz inequality on exterior powers bounds its squared absolute value by the product of the Gram determinants of the metrics at v and at the place of the embedding σ_v composed with the involution of K, and this second place runs through all infinite places as v does, since it defines an involution of the infinite places. Write $j = \mathrm{rank}(Q)$, let Δ be the Gram determinant in the chosen basis, and let $\mathfrak{a}$ be the fractional ideal of coordinates of $\bigwedge^j Q$ with respect to the exterior product of that basis. Taking determinants of s times the pairing on Q gives $s^j \Delta\, \mathfrak{a}^* \mathfrak{a} \subseteq \mathcal{O}_K$, where * is the field involution. The two ideal factors have the same norm and supply the finite lattice indices in the covolume.
Lean: `GlobalTheory.TestLat.neg_log_le_lcv_of_integral_form` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.NegLogLeLcvOfIntegralForm`, lines 64–142)

The product formula then gives $\log \mathrm{covol}(Q) \geq -\mathrm{rank}(Q) \log s$. Hence $\mu_{\max}(M) \leq \log s$, which is the displayed bound. Finally M is finitely generated as the dual of a lattice, $M \neq 0$ because the test index set is nonempty (it contains a grid datum and the identity class), and each X'_v is invertible because it is a frame.
Lean: `GlobalTheory.TestLat.slope_holds` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.SlopeEHolds`, lines 27–27)
Lean: `GlobalTheory.FunObj.IsTestLat.gridTestLattice_largestSlope_bound` (`Development.AbelianVarietyTheory.Definitions.Divisors.GridTestLatticeLargestSlopeBound`, lines 217–225)

The point-field construction supplies the place v_0. The tie in Definition 34.49(3) identifies the given M and (X'_v) with exactly the dual lattice and frames in Claim 2. Apply it with $R = R_p(C_1)$, using Claim 1. The identities $|\mathcal{T}| = f$ and $\mathrm{rank}(X_0) = n_p$ give the bound E_M and the other conclusions. The complete proof is given in `GlobalTheory.BigTop.Fun.LeafMMTie.mm_tie_alt`. □

Proof of 34.57

Proof. Every index in the two tower products lies in $[1, h]$, since their largest index is $e + q = e_0 + p \leq h$, and the smallest index in the product of length p is $e + q - p + 1 = e_0 + 1$. The right-hand side of the tower

identity is nonzero, so $R \neq 0$. Put $x_s = v(\rho_s)$, $A_1 = \sum_{s=1}^{h} a_s$, and $N_1 = \sum_{s=1}^{h} n_s$. Taking valuations and using the bound for γ gives

$$v(R) \leq v(A) + v(A') + 6d + (p+q)B + \sum_{s=1-e_0}^{h_0} x_s + \sum_{k=0}^{p-1} x_{e+q-k} - \sum_{k=0}^{q-1} x_{e+k+1}.$$

Reflection and the positive upper bound give

$$\sum_{s=1-e_0}^{h_0} x_s = \sum_{s=1}^{e_0} x_s + \sum_{s=1}^{h_0} x_s \leq 2A_1 + (e_0 + h_0)\eta.$$

Each tower product has distinct indices in $[1, h]$, so

$$\sum_{k=0}^{p-1} x_{e+q-k} \leq A_1 + p\eta, \qquad -\sum_{k=0}^{q-1} x_{e+k+1} \leq A_1 + N_1.$$

Thus the three displayed sums contribute at most

$$4A_1 + N_1 + (e_0 + h_0 + p)\eta \leq 4A_1 + 5N_1 + 2h\eta,$$

since $N_1, \eta \geq 0$ and $e_0 + h_0 + p \leq 2h$. Finally use $v(A) \leq |v(A)|$ and $v(A') \leq |v(A')|$. □

Proof of 34.58

Proof (outline). Claim 1. Under the hypotheses of the lemma, the setting and all local estimates below admit one closed choice of constants and cutoffs. The host parameters of every family member satisfy $\left|\mathrm{Re}(L'(1,\chi_d)/L(1,\chi_d))\right| \leq x^{\varepsilon}$. The family logarithmic-derivative theorem places this bound in the closed quantifiers of Definition 36.10. Together with the row-size and maximal-real-subfield conditions, it gives the square-recursive size datum. Take maxima of the finitely many cutoffs for the Hecke nonvanishing, toric denominator, bottom-pair, reflection and tower estimates below. This gives a common C_{F0}, and, after $C_F \geq C_{F0}$ is fixed, a common C_0 and an eventual threshold. The estimates hold simultaneously for every fitted datum and every i, j, w.

Lean: `GlobalTheory.logDeriv_LFunction_at_one_le_along_family_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LogDerivLFunctionAtOneLeAlongFamily`, lines 12–17)
Lean: `GlobalTheory.SqTop.GBig` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.NormSmall`, lines 34–36)
Lean: `GlobalTheory.SqTop.Model.gBig_and` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.AuxiliaryPrimeNeTypePrime`, lines 52–61)
Lean: `GlobalTheory.SqTop.square_top_term_joined_estimates` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.SquareTopTermJoinedEstimates`, lines 13–35)

Claim 2. For every arithmetic datum D, every curve datum $(\mathfrak{c}, \delta)$, every unkilled two-torsion point t_0, every positive level Q, and every definite CM theta pair P at $(\mathfrak{c}, \delta, t_0, Q)$, the norm of Definition 34.48(3) is independent of its class representatives. If $c_P(h, T) \neq 0$ for some h, T, then $\mathrm{norm}(P) > 0$. The coordinate denominator allowance used below is part of the given fitted datum, by Definitions 34.48(4) and 34.49(3). Its defining inequalities apply to every small bottom pair; no additional choice of an allowance is made here.

Lean: `GlobalTheory.NormSum.RepInv.rep_indep` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.DefiniteThetaPairTermsNormsIndependentRepresenta`, lines 241–265)
Lean: `GlobalTheory.NormSum.inp_all` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.CmThetaNormBoundHypothesesSatisfied`, lines 29–51)

Fix such a member point and fitted datum, and fix i, j, w. Suppress i, j in the notation. Put

$$h = h_{ij}, \qquad y = m - 1 - j, \qquad p_* = \lfloor y/2 \rfloor, \qquad q_* = y - p_*,$$
$$e = \frac{r_{i+1,j+1} - r_{ij}}{2}, \qquad e_0 = e + q_* - p_*, \qquad h_0 = h - y.$$

The rank formulas and the parity assumptions give

$$e_0 + p_* = e + q_*, \quad h = h_0 + p_* + q_*, \quad 1 \leq e_0 < h_0, \quad e + q_* \leq h \leq N_*.$$

The two bottom ranks are $h_0 + e_0$ and $h_0 - e_0$. For $1 \le s \le h$, define

$$a_s = |v_w(z_T(s)) + g_T(s,w)| + |v_w(z_{T,c(w)}(s)) + g_T(s,w)|,$$
$$n_s = \sum_{l \in S_d} N_0(s,l) v_w(l), \qquad \eta_w = \sum_{l \in S_e} \frac{v_w(l)}{l^2}, \qquad C_w = \sum_{q \in S_0} c_q v_w(N\mathfrak{q}).$$

Here $N\mathfrak{q}$ is the absolute norm denoted by q in the statement. The fitted denominator and integrality data give $n_s, \eta_w, d_\ell(w), C_w \ge 0$. If $\mu(w) = \mu_0$, the fitted normalization gives $r_{ij,\mu(w)} = 1$, and the claimed valuation inequality follows from these nonnegative terms.

Lean: `GlobalTheory.SqTop.Model.tower_numbers` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.AuxiliaryPrimeNeTypePrime`, lines 63–82)

Claim 3. For the fixed member, fitted datum and indices above, the ratios below exist and obey the stated positive-range and reflection bounds. Suppose now that $\tau = \mu(w) \ne \mu_0$. Let $Q_{\mathrm{tw}} = \prod_{\mathfrak{P} \in Q_\tau} N\mathfrak{P}$, where Q_τ is the twist's finite ramification-place set. The fitted twisting condition gives $w \nmid Q_{\mathrm{tw}}$. Let l be the residue prime of w. Since S_d, S_e consist of rational primes, and $l \in S_e$,

$$n_s = N_0(s,l) v_w(l), \qquad \eta_w = \frac{v_w(l)}{l^2}.$$

For the Hecke characters $\nu_t = \mu_{0,j}\xi_t/\chi_{m-i}$, choose their Hecke functions Λ_t. The functional equation and the square-recursive size conditions give $\Lambda_\tau(s)\Lambda_{\mu_0}(s) \ne 0$ for every integer $1 - e_0 \le s \le h$. Write $\Lambda_0 = \Lambda_{\mu_0}$. The fitted field condition and the toric-value identities give, for $1 \le s \le h$, nonzero $z_\psi(s), z_T(s), z_{T,c(w)}(s) \in K$ such that

$$\iota(z_\psi(s))\Lambda_0(s) = \Lambda_\tau(s), \qquad z_T(s) = 1, \qquad \iota(z_{T,c(w)}(s))\Lambda_0(s) = \Lambda_d(s),$$

where Λ_d is the Hecke function of $\nu_{\mu_0}\delta_{c(w)}$. The selected prime twist $\delta_{c(w)}$ is the prime twist chosen for w to avoid ramification above l. The toric denominator estimate and the prime-to-l twisting estimate, applied to $z_\psi(s)/z_{T,c(w)}(s)$, give

$$-n_s \le v_w(z_\psi(s)) + g_T(s,w), \qquad v_w(z_\psi(s)) \le v_w(z_{T,c(w)}(s)) + \eta_w.$$

Reflection gives nonzero $E_s \in K$ for $1 - e_0 \le s \le 0$, with $v_w(E_s) = 0$, such that

$$\Lambda_\tau(s)\Lambda_0(1-s) = \iota(E_s)\Lambda_\tau(1-s)\Lambda_0(s).$$

Define

$$\rho_s = \begin{cases} z_\psi(s)/z_T(s), & 1 \le s \le h, \\ E_s\, z_\psi(1-s)/z_T(1-s), & 1 - e_0 \le s \le 0. \end{cases}$$

Then $\rho_s \ne 0$, $\iota(\rho_s)\Lambda_0(s) = \Lambda_\tau(s)$, and

$$v_w(\rho_s) \le a_s + \eta_w, \qquad -v_w(\rho_s) \le |v_w(z_T(s)) + g_T(s,w)| + n_s \le a_s + n_s \quad (1 \le s \le h),$$
$$v_w(\rho_s) = v_w(\rho_{1-s}) \quad (1 - e_0 \le s \le 0).$$

The two positive-range inequalities follow by writing $v_w(\rho_s) = v_w(z_\psi(s)) - v_w(z_T(s))$ and adding and subtracting $g_T(s,w)$.

Lean: `GlobalTheory.SqTop.RhoLow` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.NormSmall`, lines 153–157)

Lean: `GlobalTheory.SqTop.Refl` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.NormSmall`, lines 148–151)

Lean: `GlobalTheory.Square.Ratio.exists_ratios` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.ExistsRatios`, lines 33–83)

Lean: `GlobalTheory.SqTop.OneSqProof.wt_at` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.OneSqStOfClaims`, lines 77–93)

Claim 4. For these same data, the bottom norms are nonzero and their comparison satisfies the valuation bound below. Let P_0, P_0' be the fitted bottom pair at the base tuple, and let P_τ, P_τ' be a bottom pair at tuple τ and level $Q_{\mathrm{lev}}Q_{\mathrm{tw}}$, using the same period and two-torsion point. Put $A = z_A(i,j)$ and $A' = z'_A(i,j)$. Thus $\iota(A) = \mathrm{norm}(P_0)$ and $\iota(A') = \mathrm{norm}(P_0')$. The larger-bottom norm formula gives $\mathrm{norm}(P) =$

$H\prod_{s=1-e_0}^{h_0}\Lambda_u(s)$ for every bottom pair (P,P') at a tuple u and at any level, where the nonzero complex number H depends only on the period, the two-torsion point, and the ranks and weights of P. These ranks and weights are the same for all bottom pairs at (i,j), so H is independent of the tuple and its level, and

$$\operatorname{norm}(P_u)=H\prod_{s=1-e_0}^{h_0}\Lambda_u(s)\qquad(u=\mu_0,\tau).$$

Consequently

$$z=A\prod_{s=1-e_0}^{h_0}\rho_s\neq 0,\qquad \iota(z)=\operatorname{norm}(P_\tau),\qquad A\neq 0.$$

The nonzero-coordinate property of P_0' makes its CM theta norm nonzero, so $A'\neq 0$. Let $b(u)$ be the proportionality constant of a bottom theta triple at tuple u, with the standard additive character and the fitted field of scalars. The comparison of bottom constants, applied to the two bottom pairs and the fitted coordinate-integrality allowance, gives $b(\mu_0)\neq 0$ and $\gamma\in K^\times$ such that

$$\iota(\gamma)b(\mu_0)=b(\tau),\qquad v_w(\gamma)\le v_w(z)+v_w(A')+6d_\ell(w).$$

The level comparison uses $w\nmid Q_{\mathrm{tw}}$, so the w-adic allowance at $Q_{\mathrm{lev}}Q_{\mathrm{tw}}$ is the allowance at Q_{lev}.
Lean: `GlobalTheory.SqTop.NormBig` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.NormSmall`, lines 114–116)
Lean: `GlobalTheory.T.definiteThetaPair_norm_ne_zero_of_coord_ne_zero` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.DefiniteThetaPairNormNeZeroOfCoordNeZeroStatement`, lines 13–14)
Lean: `GlobalTheory.SqTop.BotConst` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.NormSmall`, lines 139–146)
Lean: `GlobalTheory.SqTop.botConst_of_inputs_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.BottomConstantOfTheta`, lines 22–61)

Claim 5. The top and bottom constants satisfy the following tower identity, with unit comparison factors at w. Write $c(u)$ for the square's top proportionality constants, so that $c(\mu_0)\neq 0$ and $\iota(r_{ij,\tau})=c(\tau)/c(\mu_0)$. The seesaw tower has q_* small-rank steps and p_* large-rank steps. Its normalized step factors $u_0,\dots,u_{y-1}\in\mathbf{C}$ and its comparison factor $\zeta\in K$ satisfy

$$\frac{c(\tau)}{c(\mu_0)}\prod_{k=0}^{q_*-1}\frac{\Lambda_\tau(e+k+1)}{\Lambda_0(e+k+1)}=\iota(\zeta)\left(\prod_{k=0}^{y-1}u_k\right)\frac{b(\tau)}{b(\mu_0)}\prod_{k=0}^{p_*-1}\frac{\Lambda_\tau(e+q_*-k)}{\Lambda_0(e+q_*-k)},$$

with $\zeta\neq 0$ and $v_w(\zeta)=0$. In this identity each step has the form

$$u_k=\iota\left(\epsilon_k\prod_{q\in S_0}\frac{a_{k,q}}{a'_{k,q}}\right),\qquad a_{k,q},a'_{k,q}\in\mathcal{O}_K,\qquad a_{k,q}\mid(N\mathfrak{q})^{c_q},\quad a'_{k,q}\mid(N\mathfrak{q})^{c_q},$$

where $\epsilon_k\neq 0$ and $v_w(\epsilon_k)=0$. The tower is obtained by applying the small-rank and large-rank norm steps to the top triple, comparing the descended triple with the chosen bottom triple, and dividing the resulting identity at τ by the identity at μ_0; all common scalar factors cancel. The top and bottom comparison factors are units away from Q_{tw}, as are the factors ϵ_k.
Lean: `GlobalTheory.SqTop.Tower` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.NormSmall`, lines 128–137)
Lean: `GlobalTheory.SqTower.tower_alt_bis` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.TowerOfThetaMatching`, lines 32–104)
Lean: `GlobalTheory.Square.StepRatio` (`Development.AbelianVarietyTheory.Definitions.Foundations.TupleConstantsNeZero`, lines 113–116)

Claim 6. The tower identity implies the asserted valuation bound and nonvanishing. Set $t_k=\epsilon_k\prod_{q\in S_0}a_{k,q}/a'_{k,q}$. Every divisor in this formula is nonzero. Its valuation lies between 0 and $c_qv_w(N\mathfrak{q})$, so the triangle inequality gives

$$t_k\neq 0,\qquad u_k=\iota(t_k),\qquad |v_w(t_k)|\le 2C_w.$$

Substitute these identities, the ρ_s, and γ in the complex tower identity and use injectivity of ι. This gives the identity in K

$$r_{ij,\tau}\prod_{k=0}^{q_*-1}\rho_{e+k+1}=\zeta\left(\prod_{k=0}^{y-1}t_k\right)\gamma\prod_{k=0}^{p_*-1}\rho_{e+q_*-k}.$$

Apply Lemma 34.57 with $(p,q)=(p_*,q_*)$, $B=2C_w$, $d=d_\ell(w)$, and $\eta=\eta_w$. It gives $r_{ij,\tau}\neq 0$ and the required bound. The four positive-range blocks in that lemma give the coefficient 4; its nonnegative denominator sum is enlarged to the prescribed coefficient 5; the lengths of the upper-bounded blocks give the error allowance $2h\eta_w$; the bottom comparison contributes $6d_\ell(w)$; and the y normalized steps contribute $2yC_w$.

Lean: `GlobalTheory.Square.stepRatio_vw_bound` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.StepRatioAbsVwProd`, lines 60–84)
Lean: `GlobalTheory.SqTop.OneSqNoSmall.oneSqSt_of_claims'` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.SquareRatioValuationWithoutNormFormula`, lines 24–240)

Claim 7. The remaining support, denominator and rank assertions hold with the same cutoff. The fitted supports consist of primes, and the nonnegative ceiling defining N_0, together with the denominator-sum estimate, gives $N_0(s,l)=0$ off S_d and $\sum_{l\in S_d}N_0(s,l)\log l\le B_0$. Definition 34.50 gives $c_q\le e_q$, while $0\le y$ and $y+1\le m$ follow from $0\le j<m$. These statements and the rank calculation above give all the remaining assertions.

Lean: `GlobalTheory.BigTop.Fun.den_tie_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.DenTie`, lines 14–26)

The complete proof is given in `GlobalTheory.SqTop.OneSqNoSmall.oneSqSt_of_claims'` and its family specialization `GlobalTheory.BigTop.Fun.oneSq_of_log_derivative_bound_thA`. □

Proof of 34.59

Proof. The definitions give nonnegativity and finite support. Put $r=2fN\ge 2$. Since $(4r)!\le(4r)^{4r}$,

$$\begin{aligned}\log M&=2r\log 8+\log((4r)!)\\&\le 2r\log 8+4r\log(4r)=4r\log r+14r\log 2\le 18r\log r.\end{aligned}$$

Prime factorization gives

$$\sum_{p\in S'}v_p(M)\log p=\log M,\qquad\sum_{p\in S'}v_p(Q)\log p=\log Q.$$

Also $\sum_{p\in S'}a_p\log p\le L$ and $\sum_{p\in S'}\mathbf{1}_{p=d}\log p/4\le\log d/4\le L/2$. Substitution gives the first displayed bound. For the second, use $6fNBL\le 36fNBL$, $N\ge 1$, $fB\ge 1$, and $\log Q\ge 0$ to obtain

$$4fNB\log Q+4N^2\log Q\le 8fN^2B\log Q\le 72fN^2B\log Q.$$

□

Proof of 34.60

Proof (outline). Claim 1. For every arithmetic datum D, put $\Delta_E=|\operatorname{disc}(E)|$, let S be its set of prime divisors, and set $a_p=v_p(\Delta_E)/(2f)$ at primes and $a_p=0$ at nonprimes. These numbers are nonnegative, vanish outside S, and satisfy

$$\sum_{p\in S}a_p\log p=\log\operatorname{rd}(E),\qquad\log d\le 2\log\operatorname{rd}(E),\qquad f\ge 1.$$

For integers $a,r\ge 0$, write $\varphi_\tau:E\hookrightarrow\mathbf{C}$ for the embedding indexed by $\tau\in\mathcal{T}$ and $Y^{\mathcal{T}}=(\varphi_\tau(Y))_\tau\in V_{\mathbf{C}}(a,r)$. For every standard coordinate vector e_k of $V_{\mathbf{C}}(a,r)$, prime p, and ring homomorphism $\lambda:\overline{\mathbf{Q}}\to\mathbf{C}_p$, there are finitely many $c_\ell\in\overline{\mathbf{Q}}$ and $Y_\ell\in M_{r\times a}(\mathcal{O}_E)$ such that

$$e_k=\sum_\ell c_\ell Y_\ell^{\mathcal{T}},\qquad|\lambda(c_\ell)|_p\le p^{a_p}.$$

Indeed, take an integral basis of $\mathcal{O}_E$, its trace-dual basis in the codifferent, and matrices with one nonzero entry from the integral basis. The trace-dual identity gives the displayed coordinate decomposition. Here the host field $E/\mathbf{Q}$ is Galois, as required by the arithmetic datum. To see the size of the codifferent allowance, choose a positive power h for which the different satisfies $\mathfrak{D}_E^h = (a)$. For $x \in \mathfrak{D}_E^{-1}$, every Galois conjugate of a generates the same ideal. Hence the product of the integral elements $\sigma(a)x^h$, over all $\sigma \in \mathrm{Gal}(E/\mathbf{Q})$, is integral. Taking a p-adic absolute value gives

$$|\lambda(x)|_p^{h[E:\mathbf{Q}]} \le p^{hv_p(\Delta_E)}, \qquad |\lambda(x)|_p \le p^{v_p(\Delta_E)/(2f)}.$$

The same estimate holds for every conjugate of every trace-dual basis element, and therefore for the displayed coefficients.

Lean: `GlobalTheory.NormSum.Denom.exists_dual_pair` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.DerivationDenominatorsAndHostFieldSizes`, lines 117–156)
Lean: `GlobalTheory.NormSum.Denom.exists_letters` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.DerivationDenominatorsAndHostFieldSizes`, lines 177–208)
Lean: `GlobalTheory.NormSum.Letters` (`Development.AbelianVarietyTheory.Definitions.Foundations.NormSumPairRankZeroOfCurveCoordNorm`, lines 168–173)

Claim 2. The following coordinate estimate holds for these same a_p. Fix an admissible level Q, $N \ge 1$, and $B \ge 1$, and put

$$E_p = v_p\big(8^{4fN}(8fN)!\big) + 4fNB\left(a_p + \frac{\mathbf{1}_{p=d}}{4} + v_p(Q)\right) + 4N^2 v_p(Q)$$

for prime p. For every discriminant twelfth-root period δ of $\mathcal{O}_{K_d}$, every unkilled two-torsion point t_0 of $\Lambda(\mathcal{O}_{K_d}, \delta)$, every prime p, every p-compatible level Q', and every (N,B)-small pair P at $(\mathcal{O}_{K_d}, \delta, t_0, Q')$, each target double-coset class has a representative integral over p. For every $h \in G_{\mathbb{A}}(d_P)$,

$$1 \le |\Gamma_D(J_P, h)|, \qquad v_p(|\Gamma_D(J_P, h)|) \le E_p.$$

For every such h integral over p, every $T \in \mathrm{Tup}_D(a_P, d_P, n_P)$, and every ring homomorphism $\lambda : \overline{\mathbf{Q}} \to \mathbf{C}_p$,

$$c_P(h,T) \in \overline{\mathbf{Q}}, \qquad |\lambda(c_P(h,T))|_p \le p^{E_p}.$$

All objects and the meaning of integral over p are as in Definition 34.48. Each term of E_p tracks one source of p-adic denominators. Weak approximation at p supplies representatives integral over p in every double coset of $G_{\mathbb{A}}(a_P)$ and $G_{\mathbb{A}}(d_P)$, so the sum defining $c_P(h,T)$ runs over p-integral representatives with inverse stabilizer factors $|\Gamma_D(G_{\mathrm{int}}(a_P), s_q)|^{-1}$; since each stabilizer embeds as a finite subgroup of $\mathrm{GL}_{4fN}(\mathbf{Q})$ (as $a_P \le d_P \le 2N$), Minkowski's finite-group bound forces its order (and likewise $|\Gamma_D(J_P,h)|$) to divide $8^{4fN}(8fN)!$, giving the first term $v_p(8^{4fN}(8fN)!)$. Two statements control the terms of this sum. For (a), insert the local vacuum presentation of Definition 34.48(2)(b) into the level-Q' test-vector expansion. Its phase has p-adic absolute value one and its lattice-change index is prime to p. Write $\mathcal{W}(s)G(z) = G(z+s)\exp\big(-\pi H(z,s) - \pi H(s,s)/2\big)$, $H(z,w) = \mathrm{area}(\Lambda)^{-1}\sum_\tau \mathrm{tr}(z_\tau w_\tau^\dagger)$, for the Heisenberg–Weyl translation by $s \in V_{\mathbf{C}}(a_P, d_P)$. (a) Let λ be given. If the source representative x is trivial above p (replacing x by xk with $k \in G_{\mathrm{int}}(a_P)$, every source class has such a representative) and h is integral over p, there are a family b adapted at p, an integer $M \ge 1$ prime to p, and finitely many $r_i \in \overline{\mathbf{Q}}$ and $s_i \in V_{\mathbf{C}}(a_P, d_P)$ with $MQ's_i \in \Lambda_b$, $|\lambda(r_i)|_p \le p^{a_P d_P v_p(Q')}$, and $\Omega_W(x,h)F_P = \sum_i r_i\,\mathcal{W}(s_i)\theta_b$. (b) (CM-Taylor jet estimate.) Let b be adapted at p, let $N_{\mathrm{t}} \ge 1$ be an integer such that $p \mid N_{\mathrm{t}}$ only if $p = 2$ or $\left(\frac{-d}{p}\right) = 1$, and let $N_{\mathrm{t}} s \in \Lambda_b$. For every $n \ge 0$ and $y_1, \dots, y_n \in \mathfrak{b}^{-1}M_{d_P \times a_P}(\mathcal{O}_E)$, the n-th derivative of $\mathcal{W}(s)\theta_b$ at 0 in the directions $y_1^{\mathcal{T}}, \dots, y_n^{\mathcal{T}}$ lies in $\overline{\mathbf{Q}}$, and its image under every λ has p-adic absolute value at most $|N_{\mathrm{t}}|_p^{-n} p^{n\mathbf{1}_{p=d}/4}$. Apply (b) with $N_{\mathrm{t}} = MQ'$. Since $p \nmid M$, one has $|MQ'|_p^{-1} = p^{v_p(Q')}$, and $p \mid MQ'$ gives $p \mid Q'$, hence $p \mid Q$ by p-compatibility, so p satisfies the condition in (b). The coefficients r_i contribute at most $4N^2 v_p(Q)$, since $a_P d_P \le 4N^2$ and $v_p(Q') \le v_p(Q)$. The minor differential operator defining $c_P(h,T)$ has integer coefficients and is homogeneous of order $n = a_P\sum_\tau n_{P,\tau} \le (2N)(2fB) = 4fNB$. Writing each standard coordinate direction by Claim 1 as a combination of directions $Y^{\mathcal{T}}$ with $Y \in M_{d_P \times a_P}(\mathcal{O}_E) \subseteq \mathfrak{b}^{-1}M_{d_P \times a_P}(\mathcal{O}_E)$ costs p^{a_P} per derivative, and (b) costs $p^{v_p(Q')+\mathbf{1}_{p=d}/4}$ per derivative. This gives the middle term $4fNB(a_p + \mathbf{1}_{p=d}/4 + v_p(Q))$. The ultrametric inequality combines these bounds across the finite sum.

Lean: `GlobalTheory.T.kernelAtPrime_eq_sum_torsionTranslates` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.KernelAtPrimeEqSumTorsionTranslatesStatement`, lines 8–9)

Lean: GlobalTheory.T.jets_torsion_translates_cm_theta_product (Development.AbelianVarietyTheory.Proofs.FoundationProofs.JetsTorsionTranslatesCmThetaProductStatement, lines 8–9)
Lean: GlobalTheory.NormSum.inp_all (Development.AbelianVarietyTheory.Proofs.FoundationProofs.CmThetaNormBoundHypothesesSatisfied, lines 29–51)
Lean: GlobalTheory.NormSum.letters_and_sizes (Development.AbelianVarietyTheory.Proofs.FoundationProofs.DerivationDenominatorsAndHostFieldSizes, lines 229–242)
Lean: GlobalTheory.NormSum.CoordBound.coordInt (Development.AbelianVarietyTheory.Proofs.FoundationProofs.CoordIntOfJets, lines 176–177)

Claim 3. These allowances give the stated norm valuation bound and its global budget. Choose the witnesses $S, (a_p)$ in Claim 1. Since $\log \mathrm{rd}(E) \le L$, they meet the hypotheses of Lemma 34.59. Define M, S' and e_p by that lemma and put $\delta_{\mathrm{vp}}(p) = e_p$. At every prime p one has $e_p = E_p$, while $e_p = 0$ at nonprimes. Therefore Claim 2 gives the coordinate integrality bound and Lemma 34.59 gives

$$\sum_{p \in S'} \delta_{\mathrm{vp}}(p) \log p \le I(18; f, N, B, L, Q).$$

At level Q, choose representatives integral over p and expand

$$\mathrm{norm}(P) = \sum_{q,T} |\Gamma_D(J_P, h_q)|^{-1} c_P(h_q, T) \overline{c_P(h_q, T)}.$$

The expansion also shows that the complex value of $\mathrm{norm}(P)$ belongs to $\overline{\mathbf{Q}}$. The same coordinate bound applies to the conjugate coordinate through $\lambda \circ {}^{-}$. Each summand therefore has p-adic absolute value at most p^{3e_p}, and the ultrametric inequality gives

$$|\lambda(\mathrm{norm}(P))|_p \le p^{3e_p}.$$

A nonzero coordinate gives $\mathrm{norm}(P) > 0$, so every element $z \in K$ with $\sigma(z) = \mathrm{norm}(P)$ is nonzero. For $w \mid p$, transfer the last inequality to the embedding that induces w; it becomes

$$v_w(z) \ge -3e_p v_w(p) = -3d_\ell(w), \qquad d_\ell(w) = \sum_{p \in S'} e_p v_w(p).$$

The identity $\sum_{w|p} \ell_w v_w(p) = \log p$ and nonnegativity give, for every finite set U of places,

$$\sum_{w \in U} \ell_w d_\ell(w) \le \sum_{p \in S'} e_p \log p \le I(18; f, N, B, L, Q).$$

Lean: GlobalTheory.NormSum.RepInv.rep_indep (Development.AbelianVarietyTheory.Proofs.FoundationProofs.DefiniteThetaPairTermsNormsIndependentRepresenta, lines 241–265)
Lean: GlobalTheory.T.definiteThetaPairNorm_padicValuation_bound (Development.AbelianVarietyTheory.Proofs.FoundationProofs.DefiniteThetaPairNormPadicValuationBoundStatement, lines 8–9)
Lean: GlobalTheory.PadicPlaceBound.neg_mul_vw_le_vw (Development.AbelianVarietyTheory.Proofs.FoundationProofs.NormLeOneOfValuationLeOne, lines 122–156)

Apply this construction with $N = N_*$, $B = B_w(c_{\mathrm{gap}})$, $L = \log b$, and $Q = Q_{\mathrm{lev}}$. Since the coefficient of C in $I(C; f, N, B, L, Q)$ is nonnegative, the same allowance works for every $C \ge 18$.
Lean: GlobalTheory.FunObj.SqNum.exists_isIntExp (Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder, lines 303–310)
Lean: GlobalTheory.BigTop.Fun.TieModel.isIntExp_mono (Development.AbelianVarietyTheory.Definitions.Divisors.MulVecEqOfBasis, lines 38–50)

The complete proof is given in `GlobalTheory.NormSum.DenomBound.intSt_of_forall_inp`. □

Proof of 34.61

Proof (outline). Claim 1. The square-recursive setting and the smallness and nonvanishing of all standard bottom pairs hold simultaneously with the closed quantifiers in the statement. Under $\mathscr{P}$, the family logarithmic-derivative estimate gives

$$\left|\mathrm{Re}(L'(1, \chi_d)/L(1, \chi_d))\right| \le x^{\varepsilon}$$

at every sufficiently large member point. Together with the Hida row setting this supplies the square-recursive size condition. Choose the grid threshold at least one and above the thresholds for bottom smallness and nonvanishing, and take the maximum of their eventual cutoffs. These choices precede the member point and fitted datum.

Lean: `GlobalTheory.logDeriv_LFunction_at_one_le_along_family_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LogDerivLFunctionAtOneLeAlongFamily`, lines 12–17)

At a square-recursive datum over a grid G at $(1, C_F, b)$, the grid level Q_{lev} is admissible for D and every standard bottom pair (P_{ij}, P'_{ij}) on G at $(\mu_0, i, j, \mathfrak{m}_E)$ over $(\mathcal{O}_{K_d}, \delta, t_0, Q_{\mathrm{lev}})$ is (N_*, B_w)-small; under $\mathscr{P}$, the Hecke–Tate functional equation, and Theorem 34.7, both P_{ij} and P'_{ij} have a nonzero coordinate for every $0 \le i, j < m$. Fix i, j, let $h_0 + e_0$ and $h_0 - e_0$ be the two bottom ranks, as in the proof of Lemma 34.58, and let Λ be the Hecke function of $\nu_{\mu_0} = \mu_{0,j}\xi_{\mu_0}/\chi_{m-i}$. This is a unitary Hecke character whose infinity type (k_τ) along the CM type satisfies $k_\tau > 0$ and $s + k_\tau > 0$ for every integer $s \ge 1 - e_0$. Then $\Lambda(s) \ne 0$ for every integer $1 - e_0 \le s \le h_0$. For $s \ge 1$ this is the nonvanishing of the L-function of a unitary Hecke character at integers $s \ge 1$. For $s \le 0$ the functional equation writes the completed value at s as a unit times a power of the conductor times the completed value of the inverse character at $1 - s \ge 1$, and the Gamma factors at s and $1 - s$ are finite and nonzero because $s + k_\tau > 0$. The Petersson norm ratio formula for the bottom theta lifts gives a nonzero H, independent of the tuple and the level, with $\mathrm{norm}(P_{ij}) = H \prod_{s=1-e_0}^{h_0} \Lambda(s)$. Hence $\mathrm{norm}(P_{ij}) \ne 0$, and $c_{P_{ij}}(h, T) \ne 0$ for some h, T. For P'_{ij}, real Howe duality makes both bottom theta integrals factor through one irreducible quotient, and the first-occurrence lifts are nonzero, so the factorization is injective. This gives a bottom theta triple with constant proportionality c_0, normalized after a change of basis by roots of unity. For this triple the two bottom theta formulas give nonzero $\Omega, \Omega' \in \mathbf{C}$ and nonzero algebraic u, u' such that, at a finite-adelic point g matched to h,

$$\overline{c_{P_{ij}}(h, T)} = \Omega u\, I(\mathrm{minor}_T)(g), \qquad I'(\mathrm{minor}_{T'})(g) = \Omega' u'\, c_{P'_{ij}}(h, T') \quad (\text{all } T').$$

The Rallis bottom minor matching gives $\kappa_0 \ne 0$ and an integer matrix M, invertible over $\mathbf{Z}$, with $I(\mathrm{minor}_T)(g) = \kappa_0 c_0(\mu_0) \sum_{T'} M(T, T')\, I'(\mathrm{minor}_{T'})(g)$. If every coordinate of P'_{ij} vanished, these identities would give $c_{P_{ij}}(h, T) = 0$. Each of these statements holds above its own threshold in the closed quantifiers, with constants independent of the member point, so the grid threshold can be taken above all of them.

Lean: `GlobalTheory.SqTop.BotSmallProof.botSmall_holds` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.BottomRankSumLeTwoMulTotalRank`, lines 128–135)

Lean: `GlobalTheory.SqTop.lNe_of_functionalEquation` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.SquareBottomHalfRankDiffLeHalfRank`, lines 171–180)

Lean: `GlobalTheory.T.bottomTheta_nonvanishing` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.BottomThetaNeZeroOfInnerProductFormula`, lines 81–82)

Claim 2. The fitted allowance bounds both norm families with the required common budget. In a fitted evaluation datum d at (p, C_F), the square-data tie of Definition 34.49(3) supplies (δ, t_0) and standard bottom pairs (P_{ij}, P'_{ij}) with $\iota(z_A(i, j)) = \mathrm{norm}(P_{ij})$ and $\iota(z'_A(i, j)) = \mathrm{norm}(P'_{ij})$, while d_ℓ is a grid integrality allowance on G at constant C_F in the sense of Definition 34.48(4). Applying the norm valuation bound of d_ℓ at $N = N_*$, $B = B_w$, $L = \log b \ge 1$, and $Q = Q_{\mathrm{lev}}$ $(\log Q = c_S)$ gives $d_\ell(w) \ge 0$, $z_A(i, j), z'_A(i, j) \ne 0$, $v_w(z) \ge -3d_\ell(w)$, and $\sum_{w \in U} \ell_w d_\ell(w) \le I(C_F; f, N_*, B_w, L, Q_{\mathrm{lev}}) \le D_0$ whenever $C_1 \ge \max(C_F, 0)$. In detail, the fitted identities identify these last two budgets with $C_F X$ and $C_1 X$, respectively, for the same real number X. The allowance inequality for the empty finite set gives $0 \le C_F X$. Since the grid threshold was chosen so that $C_F \ge 1$, this implies $X \ge 0$, and hence $C_F X \le C_1 X$.

Lean: `GlobalTheory.BigTop.Fun.IntTie.integralNormBound_mono` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.IntegralityFittedIdentity`, lines 35–37)

The complete proof is given in `GlobalTheory.BigTop.Fun.int_tie_alt`. □

Proof of 34.62

Proof (outline). For a period lattice $a(\mathbf{Z} + \tau\mathbf{Z})$, write $y = \mathrm{Im}\,\tau$ and let $\Delta(\tau)$ be the normalized modular discriminant. The modular invariant function $y^6|\Delta(\tau)|$ is bounded on a fundamental domain: it is continuous on each compact truncation and tends to zero at the cusp. Choose an upper bound $M_\Delta \ge 0$. The discriminant-one identity gives

$$\mathrm{area}(a(\mathbf{Z} + \tau\mathbf{Z}))^6 = |a|^{12} y^6 = (2\pi)^{12} y^6 |\Delta(\tau)|.$$

Thus $C_A = \max\{1, (2\pi)^{12} M_\Delta\}$ has the required property.

Put $R = \mathrm{rd}(E)$, $\ell_d = \mathrm{Re}(L'(1,\chi_d)/L(1,\chi_d))$, and $n_M = [M:\mathbf{Q}]$. The induced genus-one type is admissible for every arithmetic datum. The host-field size estimate gives $R \geq 1$ and $\log d \leq 2\log R$. Conjugation of the curve datum realizes each Λ_ρ as a curve lattice for the same arithmetic datum. In particular, it has discriminant one, is stable under $\mathcal{O}_{\mathbf{Q}(\sqrt{-d})}$, and has integral j-invariant. Hence $0 < A_\rho \leq C_A$. Since t_ρ is unkilled and the lattice is stable under the CM order, $t_\rho \notin \Lambda_\rho$.

Lean: `GlobalTheory.InducedType.induced_type` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.InducedTypeThetaGenusOnePermutes`, lines 74–75)
Lean: `GlobalTheory.T.derivationDenominators_and_hostField_sizes` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.DerivationDenominatorsAndHostFieldSizesStatement`, lines 8–9)
Lean: `GlobalTheory.ArchFree.means.exists_crv_conj` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.ConjugateLatticeMeansPosLogAreaThetaBounded`, lines 16–53)
Lean: `GlobalTheory.T.idealPeriodLattice_of_curveDatum` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.IdealPeriodLatticeOfCurveDatumStatement`, lines 8–9)
Lean: `GlobalTheory.T.admJ_and_torsion_data_algebraic` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.IntegralJAndAlgebraicDataStatement`, lines 8–9)
Lean: `GlobalTheory.PositiveMeans.not_mem_lattice_of_isCMTorsionBasePoint` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.ExistsBoundMeansPosLogAreaNormalisedTheta`, lines 205–231)

The period identity, with γ denoting Euler's constant, is

$$\frac{1}{n_M}\sum_{\rho}\log(A_\rho/\pi) = \gamma + \log(2\pi) - \frac{1}{2}\log d - \ell_d.$$

For each ρ,

$$\log^+(A_\rho^{-1}) = \log^+ A_\rho - \log A_\rho \leq \log^+ C_A - \log(A_\rho/\pi) - \log\pi.$$

Averaging, using $\ell_d \leq L$, $\frac{1}{2}\log d \leq L$, $L \geq 1$, and $\gamma, \log\pi, \log(2\pi) > 0$, gives

$$\frac{1}{n_M}\sum_{\rho}\log^+(A_\rho^{-1}) \leq \log^+ C_A + 2L \leq (\log^+ C_A + 2)L.$$

Lean: `GlobalTheory.T.mean_log_period_conjugates_eq_chowlaSelberg` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.MeanLogPeriodConjugatesEqChowlaSelbergStatement`, lines 8–9)

Claim 1. Let $(\mathfrak{c}', \delta')$ be any curve datum for the same arithmetic datum D, let $\Lambda' = \Lambda(\mathfrak{c}', \delta')$, and let t' be an unkilled two-torsion point for Λ'. Then, with $R = \mathrm{rd}(E)$ and ν as in Definition 34.48(1),

$$\log^+\nu(\Lambda', t') \leq 2\log^+\big(3(2R+1)\big) + \frac{3}{2}\big(\log^+(2\pi^2) + \log^+(\mathrm{area}(\Lambda')^{-1})\big).$$

The curve lattice has discriminant one ($\mathrm{disc}(\Lambda') = g_2(\Lambda')^3 - 27g_3(\Lambda')^2 = 1$) and integral j-invariant $j(\Lambda') = 1728g_2(\Lambda')^3$, and the CM-order stability and the unkilled condition give $2t' \in \Lambda'$ and $t' \notin \Lambda'$. For $u \in \mathbf{C}$ and a function $f : \mathbf{C} \to \mathbf{C}$, define the Heisenberg translation operator $(U_1(u)f)(z) = f(z+u)\exp\big(-(\pi/\mathrm{area}(\Lambda'))(z\bar{u} + \frac{1}{2}|u|^2)\big)$, and let $\sigma_* : \mathbf{C} \to \mathbf{C}$ be the unique entire odd normalized genus-one theta function on Λ' satisfying $\sigma_*'(0) = 1$ and $(U_1(\omega)\sigma_*)(z) = (-1)^{1-\mathbf{1}_{\omega/2\in\Lambda'}}\sigma_*(z)$ for all $\omega \in \Lambda'$. For any function $f : \mathbf{C} \to \mathbf{C}$ whose Gaussian weight $|f(z)|^2\exp(-\pi|z|^2/\mathrm{area}(\Lambda'))$ is Λ'-periodic, write $\nu(\Lambda', f)$ for its average over a fundamental domain $\mathcal{F}_{\Lambda'}$ of $\mathbf{C}/\Lambda'$; for the even translate $\theta_0(\Lambda', t')(z) = (U_1(t')\sigma_*)(z)/(U_1(t')\sigma_*)(0)$, this specializes to $\nu(\Lambda', t') = \nu(\Lambda', \theta_0(\Lambda', t'))$ of Definition 34.48(1). On a discriminant-one lattice Λ', one has $\nu(\Lambda', \sigma_*)^2 = 2\pi^2/\mathrm{area}(\Lambda')$ (hence $2\log^+\nu(\Lambda', \sigma_*) \leq \log^+(2\pi^2) + \log^+(\mathrm{area}(\Lambda')^{-1})$), and translating by t' gives $\nu(\Lambda', t') = \nu(\Lambda', \sigma_*)/|(U_1(t')\sigma_*)(0)|^2$. The pointwise genus-one Bergman bound gives $|(U_1(u)\sigma_*)(0)|^2 \leq B\,\nu(\Lambda', \sigma_*)$ with $B = 3(2R+1)$ for all $u \in \mathbf{C}$. Choose a second half-period $v \in \frac{1}{2}\Lambda' \setminus \Lambda'$ with $v - t' \notin \Lambda'$, so that $t', v, t'+v$ represent the three distinct nonzero two-torsion classes of $\mathbf{C}/\Lambda'$. Because $\mathrm{disc}(\Lambda') = 1$ and $1728g_2(\Lambda')^3$ is an algebraic integer, the three values $(U_1(w)\sigma_*)(0)$ ($w \in \{t', v, t'+v\}$) are nonzero algebraic integers satisfying $(U_1(w)\sigma_*)(0)^8 = 16\big(g_2(\Lambda') - 3\wp_{\Lambda'}(w)^2\big)$ at the three distinct roots $\wp_{\Lambda'}(w)$ of $4x^3 - g_2(\Lambda')x - g_3(\Lambda') = 0$. Their product satisfies $\prod_{w\in\{t',v,t'+v\}}(U_1(w)\sigma_*)(0)^8 = 256\,\mathrm{disc}(\Lambda') = 256$, and taking moduli and fourth roots yields the three-half-period product bound $|(U_1(t')\sigma_*)(0)|^2|(U_1(v)\sigma_*)(0)|^2|(U_1(t'+v)\sigma_*)(0)|^2 = 4 \geq 1$. Combining this with the upper bound $|(U_1(w)\sigma_*)(0)|^2 \leq B\,\nu(\Lambda', \sigma_*)$ at $w = v$ and $w = t'+v$ yields $|(U_1(t')\sigma_*)(0)|^{-2} \leq \big(B\,\nu(\Lambda', \sigma_*)\big)^2$. Taking $\log^+$ gives $\log^+\nu(\Lambda', t') \leq \log^+\nu(\Lambda', \sigma_*) + 2\log^+ B + 2\log^+\nu(\Lambda', \sigma_*)$, and inserting the bound on $2\log^+\nu(\Lambda', \sigma_*)$ yields Claim 1.

Lean: GlobalTheory.T.sigmaStar_value_le_meanSquare_genusOne (Development.AbelianVarietyTheory.Proofs.FoundationProofs.SigmaStarValueLeMeanSquareGenusOneStatement, lines 8–9)
Lean: GlobalTheory.PositiveMeans.posLog_nrm_le (Development.AbelianVarietyTheory.Proofs.FoundationProofs.ExistsBoundMeansPosLogAreaNormalisedTheta, lines 102–127)
Lean: GlobalTheory.PositiveMeans.not_mem_lattice_of_isCMTorsionBasePoint (Development.AbelianVarietyTheory.Proofs.FoundationProofs.ExistsBoundMeansPosLogAreaNormalisedTheta, lines 205–231)

Apply Claim 1 to the curve lattice representing Λ_ρ and its point t_ρ. This gives

$$\log^+ \nu_\rho \le 2\log^+\big(3(2R+1)\big) + \frac{3}{2}\big(\log^+(2\pi^2) + \log^+(A_\rho^{-1})\big).$$

Since $3(2R+1) \le 9R$ and $\log 9 \le 8$,

$$\log^+\big(3(2R+1)\big) \le 9L.$$

The preceding mean bound now gives

$$\frac{1}{n_M}\sum_\rho \log^+ \nu_\rho \le 18L + \frac{3}{2}\log^+(2\pi^2) + \frac{3}{2}(\log^+ C_A + 2)L \le C_{\mathrm{gal}} L.$$

The same constant exceeds $\log^+ C_A + 2$. All choices are uniform in D, M, and the chosen torsion points.
Lean: GlobalTheory.T.sigmaStar_value_le_meanSquare_genusOne (Development.AbelianVarietyTheory.Proofs.FoundationProofs.SigmaStarValueLeMeanSquareGenusOneStatement, lines 8–9)
Lean: GlobalTheory.PositiveMeans.posLog_nrm_le (Development.AbelianVarietyTheory.Proofs.FoundationProofs.ExistsBoundMeansPosLogAreaNormalisedTheta, lines 102–127)

The complete proofs are `GlobalTheory.PositiveMeans.means_le` and its unconditional version `GlobalTheory.ArchFree.means.means_le`. □

Proof of 34.63

Proof (outline). Write $\log^+ u = \max(0, \log u)$ for $u > 0$, with $\log^+ 0 = 0$. For a CM lattice Λ, put $A_\Lambda = \mathrm{area}(\Lambda)$, and fix the additive fundamental domain $\mathcal{F}_\Lambda$ used in the theta normalization. For an unkilled two-torsion point t, define

$$\nu(\Lambda, t) = \frac{1}{\mathrm{vol}(\mathcal{F}_\Lambda)} \int_{\mathcal{F}_\Lambda} |\theta_0(\Lambda, t)(z)|^2 \exp\!\left(-\frac{\pi|z|^2}{A_\Lambda}\right) d\,\mathrm{vol}(z).$$

Thus ν is the Gaussian weighted mean square of the normalized theta function.
Lean: Formalism.MainNotions.StandardLatticesGL3.nrm1Sq (Formalism.SharpRestriction.GlobalSchurBounds.StandardLatticesGL3, lines 53–56)

Take C_A and C_{gal} from Lemma 34.62. Its two mean bounds apply to every number field containing $g_2(\Lambda), g_3(\Lambda), \wp_\Lambda(t_0)$ and every permitted family of conjugate torsion points. Set

$$C_{\mathrm{arch}} = 1100 + 4C_{\mathrm{gal}}, \qquad C_{\mathrm{gal}} = 18 + \frac{3}{2}\big(\log^+(2\pi^2) + \log^+ C_A + 2\big).$$

Fix one (N, B)-small pair P and $z \in K$ with $\sigma(z) = \mathrm{norm}(P)$. Write $a = a_P$ for its smaller rank, retain d_P for its larger rank, and put $s = \sum_\tau n_{P,\tau}$ and $R = \mathrm{rd}(E)$. Then

$$a \le d_P \le 2N, \qquad s \le 2fB, \qquad f, N, B, L, R \ge 1.$$

Let $c_J = |\mathrm{Cls}_D(d_P, J_P)|$, $c_T = |\mathrm{Tup}_D(a, d_P, n_P)|$, and $c_u = |\mathrm{Cls}_D(u, G_{\mathrm{int}}(u))|$ for $u = a, d_P$, where $G_{\mathrm{int}}(u)$ is the standard integral compact subgroup. The class and minor counts give

$$c_J \le c_{d_P} Q^{d_P^2+4}, \qquad c_T \le 2^{d_P s},$$
$$c_u \le \big(12 f^2 u^4 R^2\big)^{6fu^2} \quad (u = a, d_P).$$

Here the bound for c_u comes from a finite set of representatives. For the identity Hermitian form on E^u, the short-vector bound from Minkowski's geometry of numbers gives each full-rank self-dual $\mathcal{O}_E$-lattice L in this space an E-basis (v_i) in L with integral Gram entries of absolute value at most $B = 4f^3u^4R^2$ at every

embedding, where $R = \mathrm{rd}(E)$. Record its Gram matrix A and the module P of all pairing vectors of L with the v_i. Then $\operatorname{im} A \subseteq P \subseteq \mathcal{O}_E^u$, and matching (A, P) identifies the two lattices by the isometry taking one chosen basis to the other. Packing bounds the number of matrices A by $(1+2B)^{2fu^2}$. For fixed A, submodules of $\mathcal{O}_E^u/\operatorname{im} A$ can be generated by at most u elements, so their number is at most $|N_{E/\mathbf{Q}} \det A|^u \le (uB)^{2fu^2}$. For $u \ge 1$, $1 + 2B \le 3B$, and $3uB^2 \le (12f^2u^4R^2)^3$. The rank-zero count is one. The resulting bound is $(12f^2u^4R^2)^{6fu^2}$, and a class in $\mathrm{Cls}_D(u, G_{\mathrm{int}}(u))$ is determined by the orbit of its lattice. Passage from $G_{\mathrm{int}}(d_P)$ to J_P multiplies the class number by at most $Q^{d_P^2+4}$, and $c_T \le 2^{d_P s}$ counts choices of a-element row subsets from d_P rows across the $s = \sum_\tau n_{P,\tau}$ minor factors.

Lean: `GlobalTheory.NormSum.cls_finite_card_le` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.UnitaryDoubleCosetClassesFiniteCardLe`, lines 37–95)

Lean: `GlobalTheory.MinorBound.minor_bound` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.SumCoeffMonomial`, lines 96–138)

Claim 1. Every coordinate $c_P(h, T)$ is algebraic. Let $\eta : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, let Λ_η be the conjugate lattice, and let t_η be an unkilled two-torsion point for it. Put $A_\eta = \mathrm{area}(\Lambda_\eta)$ and $\nu_\eta = \nu(\Lambda_\eta, t_\eta)$. For every h and minor tuple T,

$$|\eta(c_P(h,T))|^2 \le c_a^2 Q^{ad_P} (a!)^{2s}\, 3(2fad_PR+1)^{fad_P} \nu_\eta^{fad_P} \left(\frac{2e\pi B}{A_\eta}\right)^{as}.$$

Here $e = \exp(1)$, and all remaining quantities were fixed above. Expand $c_P(h, T)$ over the c_a double-coset classes of $G_{\mathbb{A}}(a)$; the weights are inverse stabilizer orders, hence at most 1. For a class representative x, the translate of the test vector by (x, h) equals $c \sum_i r_i \theta_{y_i}$, where θ_{y_i} is the translate by y_i of the vacuum theta product of the self-dual lattices moved by (x, h), each Qy_i lies in these moved lattices, $|\rho(c)| = 1$, and $(\sum_i |\rho(r_i)|)^2 \le Q^{ad_P}$ for every embedding ρ. Hence $c_P(h, T)$ is an algebraic linear combination of jets of moved vacuum translates whose conjugated coefficient ℓ^1-norm has square at most $c_a^2 Q^{ad_P}$. The jet bound below holds for the vacuum of every pair of self-dual lattices and each of its Q-torsion translates, so it applies to every moved vacuum. Let G be a moved vacuum or one of its Q-torsion translates, and let G^η be the function on the conjugate torus obtained by applying η to its Taylor coefficients at 0. Its squared torus L^2-norm on the conjugate period lattice equals $\nu(\Lambda_\eta, t)^{fad_P}$ for every unkilled two-torsion point t of Λ_η, so this value is the same for all such t, and any choice of t_η may be used.

Lean: `GlobalTheory.ConjNorm.useNorm_all` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.VacuumNormConjCmPointsAllEmbeddings`, lines 211–238)

Each conjugated jet at the origin is a minor-product derivative of total order as (order $an_{P,\tau} \le 2aB$ at each embedding τ, with each individual variable differentiated to order at most $2B$ and $(a!)^s$ monomials in the minor expansion) applied to the Gaussian-weighted holomorphic function G^η on $\mathbf{C}^{fad_P}$ of squared torus L^2-norm $\nu_\eta^{fad_P}$; passing from the CM torus fundamental domain to a Bargmann–Fock ball introduces the lattice-count factor $3(2fad_PR+1)^{fad_P}$, and the Bargmann–Fock Cauchy derivative bound extracts the order-as derivative at squared cost $(2e\pi B/A_\eta)^{as}$. The curve datum, the two unkilled two-torsion points, $Q > 0$, $B \ge 1$, and $n_{P,\tau} \le 2B$ supply these inputs for the coordinate bound.

Lean: `GlobalTheory.T.thetaPair_coordinates_conjugate_bound` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.ThetaPairCoordinatesConjugateBoundStatement`, lines 8–9)

Lean: `GlobalTheory.NormSum.inp_all` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.CmThetaNormBoundHypothesesSatisfied`, lines 29–51)

The invariants $g_2(\Lambda), g_3(\Lambda)$ and the value $\wp_\Lambda(t_0)$ are algebraic. For the last assertion use $2t_0 \in \Lambda$ and $t_0 \notin \Lambda$.

Lean: `GlobalTheory.T.admJ_and_torsion_data_algebraic` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.IntegralJAndAlgebraicDataStatement`, lines 8–9)

Lean: `GlobalTheory.PositiveMeans.not_mem_lattice_of_isCMTorsionBasePoint` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.ExistsBoundMeansPosLogAreaNormalisedTheta`, lines 205–231)

Choose a finite Galois extension $M/\mathbf{Q}$ in $\mathbf{C}$ containing $\sigma(K)$, the finitely many coordinates $c_P(h_q, T)$, and $g_2(\Lambda), g_3(\Lambda), \wp_\Lambda(t_0)$. Let c denote complex conjugation on M, put $w_{q,T} = |\Gamma_D(J_P, h_q)|^{-1}$, and define

$$S_\rho = \sum_{q,T} w_{q,T} |\rho(c_P(h_q, T))|^2.$$

The stabilizer orders are positive integers, so $0 < w_{q,T} \le 1$. The identity

$$\sigma(z) = \sum_{q,T} w_{q,T} c_P(h_q, T) c(c_P(h_q, T))$$

and weighted Cauchy–Schwarz imply

$$|\rho(\sigma(z))| \le (S_\rho S_{\rho\circ c})^{1/2}.$$

The map $\rho \mapsto \rho \circ c$ permutes the embeddings of M. Each embedding of K has $[M : \sigma(K)]$ extensions to M, with K identified with $\sigma(K)$. Since K is totally complex, averaging gives

$$\operatorname{mean}_{v|\infty} \log^+ |z|_v \le \frac{1}{[M : \mathbf{Q}]} \sum_\rho \log^+ S_\rho.$$

Lean: `GlobalTheory.ArchMean.meanInf_posLog_le` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.MeanInfPosLogLe`, lines 75–131)

Claim 2. Every curve datum $(\mathfrak{c}, \delta)$ over an arithmetic datum D, with lattice $\Lambda = \Lambda(\mathfrak{c}, \delta)$, admits an unkilled two-torsion point. For every intermediate field $\mathbf{Q} \subseteq M_0 \subseteq \mathbf{C}$ containing $g_2(\Lambda), g_3(\Lambda)$ and every $\mathbf{Q}$-embedding $\tau : M_0 \hookrightarrow \mathbf{C}$, the conjugate lattice Λ_τ, whose invariants are $\tau(g_2(\Lambda)), \tau(g_3(\Lambda))$, also admits an unkilled two-torsion point.

Lean: `Formalism.MainNotions.StandardLatticesGL3.t0Ex` (`Formalism.SharpRestriction.GlobalSchurBounds.StandardLatticesGL3`, lines 232–236)
Lean: `GlobalTheory.T.exists_twoTorsionCMPoint_all_conjugates` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.ExistsTwoTorsionCMPointAllConjugatesStatement`, lines 8–9)
Lean: `GlobalTheory.TwoTorsionPt.exists_point` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.ExistsTwoTorsionCMPointAllConjugates`, lines 127–148)

Apply Claim 2 to $M_0 = M$. For each $\rho : M \hookrightarrow \mathbf{C}$, choose such a point t_ρ on Λ_ρ, and put $A_\rho = A_{\Lambda_\rho}$, $\nu_\rho = \nu(\Lambda_\rho, t_\rho)$. Extend ρ to $\eta : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$. Since $g_2(\Lambda), g_3(\Lambda) \in M$, the conjugate lattice obtained from η is Λ_ρ. Claim 1, with $t_\eta = t_\rho$, therefore gives its displayed bound for $|\rho(c_P(h_q, T))|^2$.

The logarithmic bounds for the factors are as follows. Put $\ell_B = \log(2NB)$, $\ell_f = \log(2fNB)$, and $\ell_Q = \log Q$. The class counts, the factorial estimate $a! \le a^a$ for $a \ge 1$ together with $0! = 1$, and $a, d_P \le 2N$, $s \le 2fB$, $R \le e^L$ give

$$\begin{aligned}
c_J &\le \exp\bigl(24fN^2(12 + 4\ell_f + 2L) + 8N^2\ell_Q\bigr),\\
c_T &\le \exp(8fNB),\\
c_a^2 &\le \exp\bigl(48fN^2(12 + 4\ell_f + 2L)\bigr),\\
Q^{ad_P} &\le \exp(4N^2\ell_Q), \qquad (a!)^{2s} \le \exp(8fNB\ell_B),\\
(2fad_PR + 1)^{fad_P} &\le \exp\bigl(4fN^2(3 + 2\ell_f + L)\bigr),\\
\nu_\rho^{fad_P} &\le \exp\bigl(4fN^2 \log^+ \nu_\rho\bigr),\\
\left(\frac{2e\pi B}{A_\rho}\right)^{as} &\le \exp\bigl(4fNB(7 + \ell_B + \log^+ A_\rho^{-1})\bigr).
\end{aligned}$$

Using $3 \le e^3$ and $w_{q,T} \le 1$, their product yields

$$\log^+ S_\rho \le E_{\mathrm{raw}} + 4fN^2 \log^+ \nu_\rho + 4fNB \log^+ A_\rho^{-1},$$

where

$$E_{\mathrm{raw}} = 3 + fNB(36 + 12\ell_B) + fN^2(876 + 296\ell_f + 148L) + 12N^2\ell_Q.$$

Define

$$\Xi = 2fNB(L + \ell_B) + fN^2(L + \ell_f) + N^2\ell_Q.$$

Since $fN^2 \ge 1$, $L \ge 1$, and $\ell_B, \ell_f, \ell_Q \ge 0$,

$$3 + 876fN^2 + 148fN^2L \le 1027fN^2L, \qquad E_{\mathrm{raw}} \le 1100\Xi.$$

Thus

$$\log^+ S_\rho \le 1100\Xi + 4fN^2 \log^+ \nu_\rho + 4fNB \log^+ A_\rho^{-1}.$$

Lean: `GlobalTheory.ArchNorm.sizes` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.LatticeUnitNormSqNonneg`, lines 104–203)

Averaging the last inequality and applying the two bounds of Lemma 34.62 gives

$$\operatorname{mean}_{v|\infty} \log^{+} |z|_v \le 1100\Xi + 4C_{\mathrm{gal}} L(fN^2 + fNB) \le (1100 + 4C_{\mathrm{gal}})\Xi.$$

This proves (1), for each pair in either family.

For (2), apply the logarithmic-derivative theorem for the eligible family. Under $\mathscr{P}$, in the closed sense of Definition 36.10, it gives

$$\left| \operatorname{Re} \frac{L'(1,\chi_d)}{L(1,\chi_d)} \right| \le x^{\varepsilon} = \log b$$

eventually at every member point. The row-setting conditions give $\log \mathrm{rd}(E) \le \log b$ and $\log b \ge 1$. The fitted bottom pairs are (N_*, B_w)-small and satisfy $B_w \ge 1$. Apply (1) with $N = N_*$, $B = B_w$, $L = \log b$, and $Q = Q_{\mathrm{lev}}$. Since $\log Q_{\mathrm{lev}} = c_S$, choosing $C_0 \ge C_{\mathrm{arch}}$ gives precisely A_0 for every $C_1 \ge C_0$.

Lean: `GlobalTheory.T.abs_logDerivativeAtOne_le_x_rpow_eps` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LogDerivLFunctionAtOneLeAlongFamily`, lines 23–24)

Lean: `GlobalTheory.BigTop.Fun.arch_tie_of_logDeriv_bound` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ArchTie`, lines 14–47)

Lean: `GlobalTheory.ArchFree.norm.norm_meanInf_posLog_le` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.NormMeanInfPosLogLeDefiniteThetaPair`, lines 25–142)

The complete proof is given in `GlobalTheory.ArchFree.norm.norm_meanInf_posLog_le` for (1) and in `GlobalTheory.BigTop.Fun.arch_tie_of_logDeriv_bound` for (2). □

Proof of 34.64

Proof (outline). Use the closed quantifiers of Definition 36.10, and take the maximum of the grid thresholds below. Fix a member point beyond the resulting scale cutoffs, a grid and Weil system as in the statement, and a complete choice a satisfying the two field conditions.

Claim 1. Under $\mathscr{P}$ and the matching and proportionality assertions of Theorem 34.7, every chosen row test has a finite local-term support and finite class sets, its class-sum formula is valid, and its evaluated row sums satisfy

$$v_{\mathrm{row}}(a,t,x')_\alpha \in \iota(K) \quad (t \in T,\ x' \in FR_t,\ 1 \le \alpha \le n)$$

when the evaluation-field condition holds. For every square (i,j) and every t, the matched square integrals converge and admit a permitted translate for which the smaller integral is nonzero, and their proportionality constant satisfies $c_{ij,t} \ne 0$. These assertions have a common grid threshold and, for fixed ε, C_F, a common scale cutoff.

Lean: `GlobalTheory.RowClaim.JoinF.joined_local_sum_estimates` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.JoinedLocalSumEstimates`, lines 12–16)

Lean: `GlobalTheory.RowClaim.SumOKK` (`Development.AbelianVarietyTheory.Statements.HeckeThetaIntegrals.FormulaAt`, lines 80–80)

Lean: `GlobalTheory.RowClaim.SumRatK` (`Development.AbelianVarietyTheory.Statements.HeckeThetaIntegrals.FormulaAt`, lines 90–90)

Lean: `GlobalTheory.RowClaim.ConstK` (`Development.AbelianVarietyTheory.Statements.HeckeThetaIntegrals.FormulaAt`, lines 78–78)

Claim 2. Under the same assumptions, the sign--1 normalization and the evaluation-field condition, at a square-recursive member one has

$$\frac{c_{ij,t}}{c_{ij,\mu_0}} \in \iota(K) \quad (0 \le i,j < m,\ t \in T).$$

We now give the precise tower calculation. Fix i,j,t, and put

$$y = m-1-j, \quad p = \lfloor y/2 \rfloor, \quad q = y-p, \quad e = (r_{i+1,j+1} - r_{ij})/2, \quad h = h_{ij}, \quad e_0 = e+q-p.$$

For every $u \in T$, choose a Hecke function Λ_u for $\nu_{u,i,j} = \mu_{0,j}\xi_u\chi_{m-i}^{-1}$. The square-recursive size hypotheses place these characters in the nonvanishing range of Lemma 34.1. The Hecke–Tate functional equation gives $\Lambda_u(s) \ne 0$ at every integer $1 - e_0 \le s \le h$. The same result supplies Hecke functions for $\nu_{\mu_0,i,j}$, $\nu_{\mu_0,i,j}\delta$, and $\nu_{\mu_0,i,j}\delta'$, each nonzero at every positive integer; δ, δ' are the prime-twist characters of Definition 34.47. Both assertions hold simultaneously for every square.

Lean: GlobalTheory.FunObj.SqNum.delOf (Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder, lines 229–229)
Lean: GlobalTheory.FunObj.SqNum.IsTorVal (Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder, lines 274–279)
Lean: GlobalTheory.BigTop.Fun.ex_of_claims_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ExOfClaims, lines 16–74)

On the positive part of the first interval, the evaluation-field condition gives

$$R_t(s) := \frac{\Lambda_t(s)}{\Lambda_{\mu_0}(s)} \in \iota(K)^\times \qquad (1 \le s \le h).$$

Lean: GlobalTheory.SqTop.lNe_of_functionalEquation (Development.AbelianVarietyTheory.Proofs.DivisorProofs.SquareBottomHalfRankDiffLeHalfRank, lines 171–180)
Lean: GlobalTheory.SqTop.LNe (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.NormSmall, lines 78–80)

Claim 2.1. The real Howe-duality clause and the other local inputs of $\mathscr{P}$, the agreement and matching assertions of Theorem 34.7, and both preceding nonvanishing assertions imply that, for each square and every sign--1 Weil system, there exists a standard definite bottom triple (A_+, A_-, W_b), simultaneously for all $u \in T$, whose two theta integrals have a constant comparison. Its source ranks are $a_+(i,j), a_-(i,j)$, its target rank is $a_+(i,j) + a_-(i,j)$, and its characters are $\alpha_{u,i,j}, \beta_{i,j}, \nu_{u,i,j}$ of Definition 34.48. The bottom clause of the evaluation-field condition then permits a standard bottom triple with rational normalized class sums; choose this triple.
Lean: GlobalTheory.T.triEx_of_howe_duality_real (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.TripleExistenceOfTheta, lines 49–50)
Lean: GlobalTheory.SqTop.TriEx (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.NormSmall, lines 125–126)
Lean: GlobalTheory.SqTop.triEx_holds_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.TripleExistenceOfTheta, lines 16–41)

Let $\mathcal{P}, \mathcal{P}'$ be the polynomial spaces of its two theta maps I_u, I'_u. For this choice there are a complex vector space $\mathcal{Q}$, a surjection $\pi : \mathcal{P} \to \mathcal{Q}$, a linear map $\pi' : \mathcal{P}' \to \mathcal{Q}$, injections J_u from $\mathcal{Q}$ into the complex-valued functions on $\mathrm{U}(W_b)(\mathbb{A}_F)$, and $\lambda_u \neq 0$ for all $u \in T$, such that

$$I_u(P) = \lambda_u J_u(\pi(P)), \qquad I'_u(P') = J_u(\pi'(P')).$$

Moreover, $\pi'(P') \neq 0$ for some P'. All these theta integrals converge because the three spaces are definite. The rational normalized class sums of the chosen triple and this factorization give

$$\lambda_t/\lambda_{\mu_0} \in \iota(K).$$

Lean: GlobalTheory.SqTop.square_top_term_joined_estimates (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.SquareTopTermJoinedEstimates, lines 13–35)
Lean: GlobalTheory.T.bottom_triple_integral_eq_const_mul_local (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.BottomTripleIntegralEqConstMulLocal, lines 43–44)
Lean: GlobalTheory.ConstRatio.Factor (Development.AbelianVarietyTheory.Proofs.TwistedOperators.RatInOfUnit, lines 75–83)
Lean: GlobalTheory.ConstRatio.ratIn_of_definite (Development.AbelianVarietyTheory.Proofs.TwistedOperators.RatInOfDefinite, lines 160–170)

The matched square constants of Claim 1, these bottom constants, and the chosen Hecke functions satisfy the following identity: there are $\zeta, t_0, \dots, t_{p+q-1} \in K$ such that

$$\frac{c_{ij,t}}{c_{ij,\mu_0}} \prod_{k=0}^{q-1} R_t(e+k+1) = \iota\left(\zeta \prod_{k=0}^{p+q-1} t_k\right) \frac{\lambda_t}{\lambda_{\mu_0}} \prod_{k=0}^{p-1} R_t(e+q-k).$$

The factor $\iota(t_k)$ is the normalized factor from the k-th small-rank or large-rank step. Its defining factorization has the form $\iota(b_k \prod_{v \in S_0} a_{k,v}/a'_{k,v})$, where $b_k \in K$, $a_{k,v}, a'_{k,v} \in \mathcal{O}_K$, and

$$a_{k,v} \mid (N\mathfrak{q}_v)^{c_v}, \qquad a'_{k,v} \mid (N\mathfrak{q}_v)^{c_v}, \qquad c_v = \begin{cases} 1, & v = v_0, \\ 2, & v \neq v_0. \end{cases}$$

Here v_0 is the base place of G, and $N\mathfrak{q}_v$ is the absolute norm of the prime ideal associated with v. These norms are positive, so the divisors are nonzero, and the factor lies in $\iota(K)$. The rank identities put all the displayed Hecke-value indices in $[1, h]$. The product on the left is nonzero, so division proves Claim 2.
Lean: GlobalTheory.Square.StepRatio (Development.AbelianVarietyTheory.Definitions.Foundations.TupleConstantsNeZero, lines 113–116)
Lean: GlobalTheory.SqTop.Tower (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.NormSmall, lines 128–137)
Lean: GlobalTheory.SqTower.tower_alt_bis (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.TowerOfThetaMatching, lines 32–104)
Lean: GlobalTheory.BigTop.Fun.sqRatio_mem_of (Development.AbelianVarietyTheory.Proofs.DivisorProofs.SquareRatioInEmbeddingRange, lines 23–87)
Lean: GlobalTheory.BigTop.Fun.square_ratio_at_point_of_leading_term_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.SquareRatioAtPointOfLeadingTerm, lines 19–45)

Above the joint family and grid cutoffs, with $C_F \geq C_{\mathrm{int}}$, the point size datum is square-recursive, $L = \log b \geq 1$, $\log \mathrm{rd}(E) \leq L$, and Q_{lev} is admissible. With the integral CM period lattice $\mathcal{O}_{K_d}$, the ordinary transport and partial-zeta denominator inputs give the nonzero toric values and minimal polynomials defining the data in Lemma 34.45, together with finitely supported prime-count functions $N_0(i, j, s, \cdot)$.
Lean: GlobalTheory.ToricExist.torEx (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsRatRepr, lines 128–156)
Lean: GlobalTheory.DenExist.denEx (Development.AbelianVarietyTheory.Proofs.DivisorProofs.BetaBoundVp, lines 71–77)

Claim 3. At every such square-recursive datum, admissible level character system with base index μ_0, and grid at $(1, C_F, b)$, there are a common discriminant twelfth-root period δ_{per} and an unkilled two-torsion point t_0 for $\Lambda(\mathcal{O}_{K_d}, \delta_{\mathrm{per}})$ such that for every $0 \leq i, j < m$ there is a standard bottom pair (P_{ij}, P'_{ij}) at $(\mathcal{O}_{K_d}, \delta_{\mathrm{per}}, t_0, Q_{\mathrm{lev}})$ on G at $(\mu_0, i, j, \mathfrak{m}_E)$, in the sense of Definition 34.48(3). Here Q_{lev} and $\mathfrak{m}_E$ are the level and conductor ideal of Definition 34.47.
Lean: GlobalTheory.NormSum.inp_all (Development.AbelianVarietyTheory.Proofs.FoundationProofs.CmThetaNormBoundHypothesesSatisfied, lines 29–51)
Lean: GlobalTheory.T.bottomPairs_exist_baseTuple (Development.AbelianVarietyTheory.Proofs.DivisorProofs.BottomPairsExistBaseTupleStatement, lines 6–7)
Lean: GlobalTheory.SqTop.BotEx (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.NormSmall, lines 104–105)
Lean: GlobalTheory.SqTop.botEx_holds (Development.AbelianVarietyTheory.Proofs.DivisorProofs.BottomPairsExistBaseTuple, lines 10–16)

The square-field condition of Lemma 34.42 permits these common data and bottom witnesses to be chosen so that both norms of every pair lie in $\iota(K)$; it also places the twisted-to-untwisted toric ratios in $\iota(K)$. Lemma 34.60 gives a grid integrality allowance d_ℓ at C_F.
Lean: GlobalTheory.FunObj.SqNum.exists_isIntExp (Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder, lines 303–310)

Claim 4. Given the finite expansions and algebraic evaluations in Claim 1, the nonzero square constants and algebraic ratios in Claims 1 and 2, a common family of bottom pairs with algebraic norms, the toric and denominator witnesses just specified, and a grid integrality allowance, the following construction satisfies all the ties of Definition 34.49(3). Take the direct-sum test lattice $(M, (X'_v)_{v|\infty})$ of that definition. On each standard source basis vector define V_t^0 by

$$\iota((V_t^0 e_\alpha)_i) = v_{\mathrm{row}}(a, t, e_{V,0}(p, t, e_\alpha))_i.$$

Here $e_{V,0}$ is the source realization from Definition 34.49(3). The finite class-sum formula is linear in the source vector, so this identity extends from the basis to every source vector. Choose $r_{ij,t} \in K$ with $\iota(r_{ij,t}) = c_{ij,t}/c_{ij,\mu_0}$, and include the bottom-pair norms, toric ratios, minimal-polynomial valuation gauges, denominator supports and integrality allowance among the components of d. These choices give a fitted datum with underlying grid G; its bottom-weight component is the gap-1 bound of Definition 34.48.
Lean: GlobalTheory.BigTop.Fun.TieModel.exists_tie (Development.AbelianVarietyTheory.Definitions.Divisors.MulVecEqOfBasis, lines 86–111)

The complete proof is given in `GlobalTheory.BigTop.Fun.ex_of_claims_alt`. □

Proof of 34.65

Proof (outline). Choose the complete choice a tied to d in Definition 34.49(3), and put $\mathcal{R} = RR \otimes_{\mathcal{O}_K} K$. The primitive link conditions give level-zero row data with $\dim_K \mathcal{R} > 0$, injective ι_{μ_0}, surjective P_{μ_0} and

$\iota_{\mu_0} P_{\mu_0}$, and

$$RR_w = \iota_{\mu(w)} P_{\mu(w)} M_{\mu(w),w} \qquad (w \text{ finite}).$$

Lean: GlobalTheory.BigTop.Fun.bundle_of_rows_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.BundleOfRows, lines 13–17)

Claim 1. Under $\mathscr{P}$ and Theorem 34.7, the row tests converge for every choice of their real test vector and coefficient functional. Their pullbacks to the row realization define maps $\ell_\alpha : \mathcal{R} \to \mathbf{C}$, $1 \le \alpha \le n$, satisfying

$$\ell_\alpha\left(\sum_k z_k e_k\right) = \sum_k \iota(z_k)\ell_\alpha(e_k)$$

for a K-basis (e_k) of $\mathcal{R}$. Put $c_t = \prod_{i,j} c_{ij,t}$. Then $c_t \neq 0$, and the evaluation-field condition gives $\kappa \in \mathbf{C}$ and roots of unity $\eta_t \in K$ such that

$$\iota(\eta_t) c_t \ell_\alpha(\iota_t P_t y) = \kappa\, \iota((V_t^0 y)_\alpha) \qquad (t \in T_{\mathrm{use}}(p),\ y \in K^{n_V(t)},\ 1 \le \alpha \le n).$$

The root η_{μ_0} is not yet normalized to one. Theorem 34.25, applied with convergence for all the real tests, gives

$$\bigcap_{\alpha=1}^{n} \ker \ell_\alpha = 0.$$

Lean: GlobalTheory.RowClaim.JoinF.joined_local_sum_estimates (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.JoinedLocalSumEstimates, lines 12–16)
Lean: GlobalTheory.RowClaim.FormulaAt (Development.AbelianVarietyTheory.Statements.HeckeThetaIntegrals.FormulaAt, lines 82–86)
Lean: GlobalTheory.RowTest.injK_of_separates (Development.AbelianVarietyTheory.Proofs.TwistedOperators.InjKOfSeparates, lines 91–104)
Lean: GlobalTheory.BigTop.Fun.exists_unit_root_ze_leafC_of_square_nonzero (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.OneMapOfRows, lines 21–127)

Since $\mathcal{R} \neq 0$ and $\iota_{\mu_0} P_{\mu_0}$ is surjective, separation provides y, α for which $\ell_\alpha(\iota_{\mu_0} P_{\mu_0} y) \neq 0$. The displayed identity gives $\kappa \neq 0$. The square-data tie gives

$$\iota(r_t) = c_t / c_{\mu_0}, \qquad r_{\mu_0} = 1, \qquad r_t \neq 0.$$

Define

$$\zeta_t = \eta_t / \eta_{\mu_0}, \qquad a_0 = \kappa^{-1} \iota(\eta_{\mu_0}) c_{\mu_0}.$$

Then $a_0 \neq 0$, each ζ_t is a root of unity, and $\zeta_{\mu_0} = 1$. Let $\Psi_{\mathbf{C}} : \mathcal{R} \otimes_{K,\iota} \mathbf{C} \to \mathbf{C}^n$ be the complex-linear map with coordinates $a_0 \ell_\alpha$. Claim 1 gives

$$\Psi_{\mathbf{C}}(\iota_t P_t y) = \iota(r_t \zeta_t)^{-1} \iota(V_t^0 y)$$

for every $t \in T_{\mathrm{use}}(p)$ and $y \in K^{n_V(t)}$, where ι is applied coordinatewise. At $t = \mu_0$, the right side is $\iota(V_{\mu_0}^0 y)$. Surjectivity of $\iota_{\mu_0} P_{\mu_0}$ therefore shows that $\Psi_{\mathbf{C}}$ carries $\mathcal{R}$ into $\iota(K)^n$. It descends to a K-linear map $\Psi : \mathcal{R} \to K^n$, and injectivity of ι gives

$$\Psi \iota_t P_t = r_t^{-1} \zeta_t^{-1} V_t^0 \qquad (t \in T_{\mathrm{use}}(p)).$$

Since M has full rank, $K^n = M \otimes_{\mathcal{O}_K} K$, as required.
Lean: GlobalTheory.BigTop.Fun.exists_unit_root_ze_leafC_of_square_nonzero (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.OneMapOfRows, lines 21–127)

Separation and $a_0 \neq 0$ show that Ψ is injective. The base identity is $V_{\mu_0}^0 = \Psi \iota_{\mu_0} P_{\mu_0}$, and ι_{μ_0} is injective. Hence

$$\ker V_{\mu_0}^0 = \ker P_{\mu_0}.$$

Lean: GlobalTheory.BigTop.Fun.ker_mulVecLin_eq_U_of_formulaAt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.KerEqOfFormula, lines 20–86)

All the claims hold beyond common grid and scale thresholds: take the maximum of the finitely many thresholds with the closed quantifiers of Definition 36.10. The complete comparison and kernel proofs are given in `GlobalTheory.BigTop.Fun.oneMap_of_bundle_setAt_kernel_conditions` and `GlobalTheory.BigTop.Fun.leafKernel_of_linK_injK_constK_formula2K`. □

Proof of 34.66

Proof (outline). For $x > 1$ and $\varepsilon > 0$, $L = x^{\varepsilon} \geq 1$, and $\log x \leq 2x^{\varepsilon}$ for large x gives $L + \log x \leq 3L$. The window inequalities of the family class give $n_{\mathrm{half}} = f/2 \geq x^{2+8\varepsilon}$ and $m \leq 2\sqrt{x\delta_1}$ with $\delta_1 = x^{-\varepsilon/8} \leq 1$, whence $m^4 \leq 16x^2 \leq 2x^{2+8\varepsilon} \leq f$ once $x \geq 8^{1/\varepsilon}$, together with $2 \leq m$, $m + 3 \leq k$, $f \leq k$, $x^{377} \leq t \leq x^T$ for $T = \lceil |C_t| \rceil + 1$, and $0 \leq \mathrm{cl}(\tau, i) \leq S_c$, so $L_c \leq S_c \leq b$ by Definition 34.50 and (34.14).
Lean: GlobalTheory.BigTop.Fun.existsIntegralMultiple_holds_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ExistsIntegralMultipleHolds, lines 12–26)
Lean: GlobalTheory.BigTop.Fun.Close.desHLeaf_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.DesHLeaf, lines 13–55)
Lean: GlobalTheory.BigTop.Fun.Close.cloudIcc (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ZetaDenominatorsFitFunction, lines 65–80)
Lean: GlobalTheory.BigTop.Fun.Lc_le_of_cloud_le (Development.AbelianVarietyTheory.Proofs.DivisorProofs.RamifiedSumsFifthEstimate, lines 30–44)

Since $b = \exp(x^{\varepsilon})$ dominates every fixed power of x and $L = x^{\varepsilon} \to \infty$ at large x, one has $mfN_*^2 \leq b$ and $2\log(2C_1) \leq L$; likewise $R_* = k + m - 1 = O(x^{9/2+10\varepsilon})$ and $N_* \geq mt \geq 2x^{377}$ give $80R_*^4 \leq N_*$ for small ε and large x. From $m \geq 2$ and $k \geq f \geq m^4 \geq 16$, one has $R_* = k + m - 1 \geq 17$, so Definition 34.50 gives $1 \leq n_p = (R_* + 1) + m(R_* + 5) \leq 2mR_*$ (as $2mR_* - n_p = (m-1)(R_* - 5) - 6 \geq 6$). Writing $u = \sqrt{b} \geq 100$ and using $(R_* + 1)^4 \leq 16R_*^4 \leq N_*$, $(R_* + 1)^7 \leq N_*^3$, $\sqrt{L_c} \leq u$, and $\sqrt{m} \leq R_* + 1$ bounds $\gamma_{\mathrm{gap}}(C_1) + 1 \leq 2C_1 N_*^3 u$, whence $1 \leq R_p(C_1) = (m+1)(\gamma_{\mathrm{gap}}(C_1) + 1) + L_c + n_p \leq C_1 m N_*^3 b$ and $n_p R_p(C_1) \leq 2C_1 u^7$, giving $\log(n_p R_p) \leq \log(2C_1) + \frac{7}{2}L \leq 4L$.
Lean: GlobalTheory.SizeLeafCheck.tree_leaf_parameter_arithmetic_of_bounds (Development.AbelianVarietyTheory.Counterexamples.DegenerateExamples.TreeLeafParameterArithmeticOfBounds, lines 66–201)
Lean: GlobalTheory.BigTop.Fun.rankGapBound_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RamifiedSumsFifthBound, lines 16–127)

Summing $\log N\mathfrak{q}$ over $q \in S_0$ using $\log g_i \leq 9L + 1$, $\log q_0 \leq L$, and $\log p_{\mathrm{de}}, \log p'_{\mathrm{de}} \leq L$ gives $\sum_{q \in S_0} \log N\mathfrak{q} \leq f(21L + 2)$, so $c_S = \sum_{q \in S_0} c_q \log N\mathfrak{q} \leq 2\sum_{q \in S_0} \log N\mathfrak{q} \leq 42fL + 4f \leq 45fL$ once $L \geq 4/3$; for $C_1 \geq 42$, $b \geq 1$, $f \geq 1$, and $k + m \geq 2$, Definition 34.50 gives $P_{\mathrm{pf}} \geq C_1 f^2 (k+m)^6 L \geq 45fL \geq 0$ and $e_{v_0}(C_1, b) \log N\mathfrak{q}_{v_0} \leq P_{\mathrm{pf}} + \log N\mathfrak{q}_{v_0}$, whence $\Sigma_0 \leq P_{\mathrm{pf}} + c_S \leq P_{\mathrm{pf}} + 45fL \leq 2P_{\mathrm{pf}}$.
Lean: GlobalTheory.BigTop.Fun.RamSum.le_PPF (Development.AbelianVarietyTheory.Proofs.DivisorProofs.RamifiedSumsSeventhEstimate, lines 104–123)
Lean: GlobalTheory.BigTop.Fun.RamSum.leaves_of_sum (Development.AbelianVarietyTheory.Proofs.DivisorProofs.RamifiedSumsSeventhEstimate, lines 39–68)
Lean: GlobalTheory.BigTop.Fun.ramified_sums_seventh_bound_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RamifiedSumsSeventhBound, lines 18–38)

Claim 1. If L_{sz} satisfies the row size conditions of Definition 34.49(1), then the gap-1 maximum bottom weight $B_w(1)$ of Definition 34.48(4) satisfies

$$1 \leq B_w(1) \leq 10b, \qquad 675 \leq b.$$

Lean: GlobalTheory.BigTop.Fun.maxWt_le (Development.AbelianVarietyTheory.Proofs.DivisorProofs.WeightOfSizeBound, lines 92–128)

The fitted weight tie identifies B_w with $B_w(1)$. Thus $C_1 \geq 42$ gives $B_w \leq C_1 b$, while $675 \leq b$ gives $\log 10 \leq L$, so

$$\log B_w \leq \log 10 + L \leq 2L \leq C_1 L.$$

Finally, setting $Q = x^{T+9}$ bounds $m, f, k, t, L \leq Q$, $N_* \leq 10Q^2$, $R_* \leq 2Q$, $R_* N_* L \leq 20Q^4$, and $P_{\mathrm{pf}} \leq 1280 C_1 Q^9$; for $C_H \geq 16(T+9) + 1$ and $x \geq 13000000000 C_1^4$, the polynomial prefactor is at most $13000000000 C_1^4 Q^{16} \leq x \cdot x^{16(T+9)} \leq x^{C_H}$.
Lean: GlobalTheory.BigTop.Fun.maxWt_le (Development.AbelianVarietyTheory.Proofs.DivisorProofs.WeightOfSizeBound, lines 92–128)
Lean: GlobalTheory.BigTop.Fun.wt_tie_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.WtTie, lines 20–41)
Lean: GlobalTheory.BigTop.Fun.desPol_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.DesPol, lines 17–58)

The complete proof combines `GlobalTheory.BigTop.Fun.existsIntegralMultiple_holds_alt`, `GlobalTheory.BigTop.Fun.Close.desHLeaf_alt`, `GlobalTheory.BigTop.Fun.rankGapBound_alt` (via `GlobalTheory.SizeLeafCheck.tree_leaf_parameter_arithmetic_of_bounds`), `GlobalTheory.BigTop.Fun.ramified_sums_seventh_bound_alt`, `GlobalTheory.BigTop.Fun.wt_tie_alt`, and `GlobalTheory.BigTop.Fun.desPol_alt`. □

Proof of 34.67

Proof. Under the hypotheses of Proposition 34.67, let C'_{F0} be the maximum of 0 and the grid thresholds in Lemma 34.45 and Lemmas 34.53–34.55, 34.58, and 34.61–34.65 (so $C'_{F0} \geq 0$, covering Lemmas 34.56 and 34.66). Fix $C_F \geq C'_{F0}$, take $\varepsilon_1 > 0$ to be the minimum of the exponent bounds of Lemma 34.45 and the closed Lemmas 34.53–34.58 and 34.61–34.66, and for $0 < \varepsilon \leq \varepsilon_1$ combine their family threshold records $\mathcal{K}$. For an eligible family F_m meeting $\mathcal{K}$ at C_H and satisfying the division bound of Lemma 36.13, let $C_0 \geq 42$ be the maximum of 42 and the estimate-constant thresholds of Lemma 34.45, Lemmas 34.53–34.55, 34.58, 34.61, and 34.63(2) (so every $C_1 \geq C_0$ also satisfies $C_1 \geq 1$ in Lemma 34.56 and $C_1 \geq 42$ in Lemma 34.66); for any $C_1 \geq C_0$, take $x_0 > x_1(F_m)$ above the maximum of the point cutoffs at C_1 of Lemma 34.45, Lemmas 34.53–34.58, and Lemmas 34.61–34.66.

Fix an indexed point $p \in \mathcal{P}$ at $x \geq x_0$ (so $x > x_1(F_m)$) carrying a fitted grid (G, W) at C_F in the sense of Definitions 34.46–34.49, with the evaluation budgets of Definition 34.50. By assumption, $\mathscr{P}$, $\mathscr{J}$, the Hecke–Tate functional equation, and Theorems 34.7 and 34.25 hold. Lemma 34.64 therefore supplies a fitted evaluation datum d over (p, C_F) with ambient dimension n, Hermitian frames $(X'_v)_{v|\infty}$, evaluation submodule $M \subset K^n$, matrices V_t^0, seesaw factors $r_{ij,t} \in K$, denominator function d_ℓ, prime sets S_d, S_e, prime-count functions $N_0(i,j,s,l)$, bottom numbers $z_A(i,j), z'_A(i,j) \in K$, higher toric ratios $z_T(i,j,s), z_{T,c}(i,j,s) \in K$ with integer shifts $g_T(i,j,s,w)$, twist selector $c(w)$, and bottom weight bound B_w.

Apply Lemma 34.65 to d. It gives the local gluing equalities $RR_w = \iota_{\mu(w)} P_{\mu(w)} M_{\mu(w),w}$, surjectivity of $\iota_{\mu_0} P_{\mu_0}$ onto $RR \otimes_{\mathcal{O}_K} K$, the base kernel equality $\ker V_{\mu_0}^0 = \ker P_{\mu_0}$, roots of unity $\zeta_t \in K$ $(t \in T)$ with $\zeta_{\mu_0} = 1$, and a linear map $\Psi : RR \otimes_{\mathcal{O}_K} K \to M \otimes_{\mathcal{O}_K} K$ such that

$$\Psi \iota_t P_t = r_t^{-1}(\zeta_t^{-1} V_t^0) \qquad (t = \mu_0 \text{ or } t = \mu(w)), \qquad r_t = \prod_{i,j} r_{ij,t}.$$

Define the unit-normalized evaluation matrices by $V_t = \zeta_t^{-1} V_t^0$. Since $\zeta_{\mu_0} = 1$, one has $V_{\mu_0} = V_{\mu_0}^0$, whence $\ker V_{\mu_0} = \ker P_{\mu_0}$. Lemma 34.58 gives $r_{ij,\mu_0} = 1$ and $r_{ij,\mu(w)} \neq 0$ for all $0 \leq i, j < m$ and every finite place w of K, so $r_{\mu_0} = 1$ and $r_{\mu(w)} \neq 0$. This proves (34.23).

Lean: `GlobalTheory.T.oneMap_of_bundle_setAt_lin_inj_const_formula` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.OneMapOfRowsStatement`, lines 11–12)
Lean: `GlobalTheory.T.leafKernel_tie_of_claim_inputs` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.KerEqOfFormulaStatement`, lines 11–12)

The finite sum estimates of Lemma 34.53 yield a finite set T_F of finite places of K and integers $d_F(w)$ vanishing outside T_F such that $V_{\mu(w)}^0(M_{\mu(w),w}) \subseteq \mathfrak{p}_w^{-d_F(w)} M_w$ for all finite w and $2\sum_w \ell_w d_F(w) \leq B_F$. By the root-of-unity invariance assertion of Lemma 34.53, replacing $V_{\mu(w)}^0$ by $V_{\mu(w)} = \zeta_{\mu(w)}^{-1} V_{\mu(w)}^0$ preserves the local lattice inclusion at every w with the same T_F and d_F. At the base index $t = \mu_0$, $r_{\mu_0} = 1$, $V_{\mu_0} = V_{\mu_0}^0$, and (34.23) give $(\Psi \iota_{\mu_0}) P_{\mu_0} = V_{\mu_0}^0$. Viewing $M \otimes_{\mathcal{O}_K} K \subseteq K^n$ and applying the base evaluation estimate of Lemma 34.55 to $V_b = \Psi \iota_{\mu_0} \in M_{n \times d_Q(\mu_0)}(K)$ gives $h(\Psi \iota_{\mu_0}) \leq H_{I,0}$. The frame estimate for the evaluation lattice in Lemma 34.56 shows that M is a nonzero finitely generated Hermitian lattice with nondegenerate Hermitian frames ($\det X'_v \neq 0$) at infinity and $\mu_{\max}(M) \leq E_M$. This establishes all four inequalities in (34.24).

Lean: `GlobalTheory.T.sum_size_bound_all_places` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.SumSizeFiniteAwayArchimedeanCombined`, lines 50–51)
Lean: `GlobalTheory.T.leafFixedPoint_of_finK_awayK` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LeafFixedPointOfFinKAwayKStatement`, lines 12–13)
Lean: `GlobalTheory.T.leafHeightMapBound_of_infK` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.IpaTieStatement`, lines 12–13)
Lean: `GlobalTheory.T.leafMinimalMass_of_setAt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.MmTieStatement`, lines 11–12)

Since d realizes the higher toric data $z_T, z_{T,c}, g_T$ of Lemma 34.45 by Definition 34.49(3) and Lemma 34.64, the square valuation estimate of Lemma 34.58 gives primality of the elements of S_d and S_e, the index bounds $y_{ij} = m - 1 - j$ (hence $y_{ij} + 1 \leq m$) and $h_{ij} \leq N_*$, the modulus comparison $c_q \leq e_q$ for every $q \in S_0$, the valuation bound (34.25) at every finite place w of K, and the prime-count bound (34.26) for $1 \leq s \leq h_{ij}$.

Lean: `GlobalTheory.BigTop.Fun.oneSq_of_log_derivative_bound_thA` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.OneSqTie`, lines 23–79)
Lean: `GlobalTheory.BigTop.Fun.den_tie_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.DenTie`, lines 14–26)

For the bottom numbers $z_A = z_A(i,j)$ and $z'_A = z'_A(i,j)$, with B_w equal to the bottom weight bound of Definition 34.48 by Definition 34.49(3), apply the fitted-grid lattice norm bound of Lemma 34.61. This gives $d_\ell(w) \ge 0$, $\sum_{w\in U} \ell_w d_\ell(w) \le D_0$ for every finite set U of finite places of K, $z \ne 0$, and $v_w(z) \ge -3d_\ell(w)$ for $z = z_A, z'_A$. Applying the archimedean norm estimate of Lemma 34.63(2) to the same bottom pairs with the same parameter substitution gives $\mathrm{mean}_v \log^+ |z|_v \le A_0$ for $z = z_A, z'_A$, proving (34.27).

Lean: `GlobalTheory.BigTop.Fun.int_tie_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.IntTie`, lines 17–76)

Lean: `GlobalTheory.BigTop.Fun.arch_tie_of_logDeriv_bound` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ArchTie`, lines 14–47)

Finally, we check that every height and denominator loss in (34.24)–(34.27), as well as the combined budget $E_M + H_{I,0} + 3C_1 f m N_*^2 L + 2G_* + B_F$ built from them, is bounded by $x^{C_H} b$ and is linear in $b = \exp(x^\varepsilon)$. Because $C_1 \ge 42$, Lemma 34.66 gives $1 \le L = x^\varepsilon$, $L + \log x \le 3L$, $2 \le m$, $1 \le f \le k$, $R_* \le N_*$, $m f N_*^2 \le b$, $\Sigma_0 \le 2P_{\mathrm{pf}}$, $c_S \le 45fL$, $1 \le B_w \le C_1 b$, $\log B_w \le C_1 L$, $1 \le n_p \le 2mR_*$, $1 \le R_p \le C_1 m N_*^3 b$, and $\log(n_p R_p) \le 4L$. Put $U = C_1^3 f m^2 R_* N_*^3 b L$ and $V = C_1^3 f m^2 N_*^2 P_{\mathrm{pf}} b$. Both U and V are products of b with polynomial-in-x factors, since $L = \log b = x^\varepsilon$, P_{pf} depends on b through $\log b = L$ alone, and all rank, degree, and auxiliary-prime logarithms grow polynomially in x. On the direct quotient metric, $h(\Psi\iota_{\mu_0})$ is bounded directly by $H_{I,0} = C_1 f m N_* b L + C_1 m^2 N_*^2 \Sigma_0 \le U + 2V$ (with $b \ge 1$), and the source-frame shift term $3C_1 f m N_*^2 L$ satisfies $3C_1 f m N_*^2 L \le 3U$. Likewise R_p enters $E_M = 4f n_p R_p \log(n_p R_p)$ to the first power while $\log(n_p R_p) \le 4L$, giving $E_M \le 32U$; the finite evaluation budget satisfies $B_F \le 2U + 4V$; the toric budget of Lemma 34.45 satisfies $B_0 \le 3C_1 f b L \le 3U$; and $B_w \le C_1 b$ enters D_0 and A_0 to the first power multiplied by L, $c_S \le 45fL$, $l_1 \le 1 + L$ (from $2fN_* \le m f N_*^2 \le b$), and $l_2, l_3 \le 1 + (C_1 + 1)L$, giving $D_0 \le 186 C_1^2 f^2 N_*^2 b L$ and $A_0 \le 57 C_1^3 f N_*^2 b L$. Setting $B = \max(A_0 + 6D_0, B_0) \le 1176 C_1^3 f^2 N_*^2 b L$ and $G_* = m^2\big(2(2 + 8N_*)B + 10N_* B_0 + 2N_* + 6D_0 + 4m\Sigma_0\big)$ (the combined budget for $\sum_w \ell_w v_w(r_{\mu(w)})$ coming from (34.25)–(34.27) and Lemma 34.45), the bounds $f \le k \le R_*$ and $2(2 + 8N_*) \le 20N_*$ give $2G_* \le 49336U + 16V$. Every individual loss and the total budget $E_M + H_{I,0} + 3C_1 f m N_*^2 L + 2G_* + B_F$ are therefore bounded by $100000 C_1^3 f m^2 N_*^2 (R_* N_* L + P_{\mathrm{pf}}) b$, which is at most $x^{C_H} b$ by the final inequality of Lemma 34.66.

Lean: `GlobalTheory.BigTop.Fun.desPol_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.DesPol`, lines 17–58)

□

Proof of 34.68

Proof. The base kernel equality and surjectivity in Proposition 34.67 make Ψ injective. Set $a_w = v_w(r_{\mu(w)}) + d_F(w)$, supported on the twist places and T_F. Equations (34.23) and (34.24) give $\Psi(RR_w) \subseteq \mathfrak{p}_w^{-a_w} M_w$. Gluing gives

$$\Psi(RR) \subseteq \left(\prod_w \mathfrak{p}_w^{-a_w} \right) M. \tag{A.410}$$

At infinity divide the squared source norm by κ_v. Put $D'_0 = 3C_1 f N_*^2 L$. The two levels have the same finite-place index selector, infinite-place normalizations and block eigenvalues. In particular $\lambda^{(0)}_{j,\mu_0,p,v} = \lambda^{(1)}_{j,\mu_0,p,v}$. By Theorem 21.12, $L_{6,j} = \sum_\rho \log \mathrm{Hook}(a^+_{j\rho}) - C_{\mathrm{sz}} f n_V^2 L_p$, where $C_{\mathrm{sz}} = C_{\mathrm{sz}}(F_m)$ is the family constant of Theorem 21.12 and $L_p = \log(b\ \mathrm{rd}(E) g_0 g_1 n_V)$. Each hook product is at least one. Set $c = 3 + 2C_{YL} + (\max(C_t, 5) + 3)$ in terms of the family's level-size exponent $C_{YL} > 0$ and auxiliary-rank exponent C_t. For $x \ge 9$, the family size inequalities of Definition 34.19 give $x, \mathrm{rd}(E) \le b$, hence $\log x, \log \mathrm{rd}(E) \le L$; with $L_a = \log(xb\ \mathrm{rd}(E))$, they give $L_p \le cL_a \le 3cL$, and the rank inequalities give $n_V \le N_*$. With $C_1 \ge \max(C_0, 42, c\max(0, C_{\mathrm{sz}}))$, this yields $C_{\mathrm{sz}} f n_V^2 L_p \le 3C_1 f N_*^2 L = D'_0$, hence $L_{6,j} \ge -D'_0$. Summing the assumed eigenvalue bound over the layers $j \in \mathcal{J}$ of a nonzero base block gives

$$\sum_{j\in\mathcal{J}} \log \lambda^{(0)}_{j,\mu_0,p,v} \ge -|\mathcal{J}| D'_0 \ge -m D'_0. \tag{A.411}$$

If $g(v)$ bounds the block sums below, the squared quotient-norm factor is at least $\exp g(v)$. Minimizing over lifts gives $\|\Psi\|_v \le \|\Psi\iota_{\mu_0}\|_v e^{-g(v)/2}$. Taking mean log squared norms with $g(v) = -mD'_0$ gives $h(\Psi) \le H_I = H_{I,0} + mD'_0$.

Lean: Formalism.L2.Indef.Tree.ip_first (Formalism.LocalToGlobal.IndefiniteForms.IndefiniteTree.FunctionalLowerBoundOfConditionsNumeric, lines 826–909)

For each bottom number the weighted negative part is at most $3D_0$, and the signed sum is at most A_0 by the product formula and (34.27). Thus the absolute sum is at most $A_0 + 6D_0$. For the base and both higher twists Lemma 34.45 gives B_0 with common shift g_T. Set $B = \max(A_0 + 6D_0, B_0)$.

Sum (34.25). The two bottom terms cost $2B$. Sum the selected twist over the subset where it is used to bound the higher terms at each s by $4(B + 2B)$. Together these fit $2(2 + 8N_*)B$ per pair. The prime-count bound with coefficient five fits $10N_*B_0$. For the degree term,

$$\sum_{p\in S_e}\frac{1}{p^2}\sum_{w\mid p}\ell_w v_w(p) = \sum_{p\in S_e}\frac{\log p}{p^2} \le \sum_{j=2}^{4}\frac{\log j}{j^2} + \frac{1+\log 4}{4} < 1. \tag{A.412}$$

For $j \ge 5$ the integral of $(\log t)/t^2$ from $j-1$ to j bounds the summand and the integrals telescope. The degree term costs at most $2N_*$. The last two terms cost at most $6D_0$ and $4m\Sigma_0$. Hence

$$\sum_{w}\ell_w v_w(r_{\mu(w)}) \le G_* := m^2\big(2(2+8N_*)B + 10N_*B_0 + 2N_* + 6D_0 + 4m\Sigma_0\big). \tag{A.413}$$

Lean: Formalism.PairingDivisionLocalBalance.wt (Formalism.PairingDivisors.StatementH, lines 16–17)
Lean: Formalism.ValuationGridEstimates.sum_log_div_sq_telescoping (Formalism.TowerValuationBounds.TowerValuationProofs.SumAdditiveCharacterEqZero, lines 504–511)

The frame change adds $\langle \log \kappa\rangle_\infty$ to the source slope. The ideal in (A.410) adds $2\sum_w \ell_w a_w$ to the target slope. For each rank r sublattice, the determinant norm of the injection at v is at most $\|\Psi\|_v^r$. Its source slope is bounded by its image slope plus $h(\Psi)$. Using the height, lattice and finite-place budgets gives

$$\mu_{\max,RR} + \langle \log \kappa\rangle_\infty \le E_M + H_I + 2G_* + B_F. \tag{A.414}$$

The prime-power modulus budget is

$$P_{\mathrm{pf}} = C_1 f^2 (k+m)^6 \log\big(b(k+m)g_0 g_1 p_{\mathrm{de}} p'_{\mathrm{de}} |D_E|\big).$$

For $C_1 \ge 42$, Lemma 34.66 gives the fitted size bounds

$$\begin{gathered} 1 \le f \le k, \quad 1 \le m \le N_*, \quad R_* \le N_*, \quad mfN_*^2 \le b, \quad L \ge 1, \quad L + \log x \le 3L, \\ \Sigma_0 \le 2P_{\mathrm{pf}}, \quad c_S \le 45fL, \quad 1 \le B_w \le C_1 b, \quad \log B_w \le C_1 L, \\ n_p \le 2mR_*, \quad R_p \le C_1 m N_*^3 b, \quad \log(n_p R_p) \le 4L. \end{gathered} \tag{A.415}$$

In particular $l_1 \le 1+L$, $l_2, l_3 \le 1+(C_1+1)L$. Set $U = C_1^3 f m^2 R_* N_*^3 bL$ and $V = C_1^3 f m^2 N_*^2 P_{\mathrm{pf}} b$. Substitution gives

Quantity	Upper bound
D_0	$186C_1^2 f^2 N_*^2 bL$
A_0	$57C_1^3 f N_*^2 bL$
B_0	$3C_1 fbL$
E_M	$32U$
H_I	$4U + 2V$
B_F	$2U + 4V$
G_*	$(23520 + 30 + 2 + 1116)U + 8V.$

(A.416)

For the last row use (A.413), $B \le 1176C_1^3 f^2 N_*^2 bL$, $2(2+8N_*) \le 20N_*$ and $f \le R_*$.

Let C_t be the family's growth exponent for the auxiliary rank t. Set $T = \lceil |C_t| \rceil + 1$ and $Q = x^{T+9}$. The family sizes give $m \le x$, $f, k \le x^5$, $t \le x^T$ and $L \le x$, hence all five quantities are at most Q. The logarithms of b and the four auxiliary primes are at most x each. Also $\log(k+m) \le 2x^5$ and $\log|D_E| \le 6x^8$ by the root-discriminant bound. The logarithm in P_{pf} is at most $20Q$. Therefore $N_* \le 10Q^2$, $R_* \le 2Q$. It follows that

$$R_* N_* L \le 20Q^4, \qquad P_{\mathrm{pf}} \le C_1 Q^2 (2Q)^6 (20Q) = 1280 C_1 Q^9.$$

The prefactor $100000C_1^3fm^2N_*^2$ is at most $10^7C_1^3Q^7$. Since $Q, C_1 \geq 1$, the product with the sum in parentheses is at most $13000000000C_1^4Q^{16}$. The family threshold gives $C_H \geq 16(T+9)+1$ and the point threshold gives $x \geq 13000000000C_1^4$. With the preceding table this yields

$$E_M + H_I + 2G_* + B_F \leq 100000C_1^3fm^2N_*^2(R_*N_*L + P_{\mathrm{pf}})b \leq x^{16(T+9)+1}b \leq x^{C_H}b. \tag{A.417}$$

For the quantifier order, fix the closed thresholds $(C'_{F0}, \varepsilon_1, \mathcal{K})$ once from Proposition 34.67, the assumed closed base-eigenvalue assertion, and the closed polynomial-descent estimate (independently of F_m and C_1). Once an eligible family F_m meeting $(\varepsilon_1, \mathcal{K})$ at C_H is fixed (determining $C_{\mathrm{sz}}, C_{YL}, C_t, c$ and the joint threshold C_0 of Proposition 34.67), set $C_1 = \max(C_0, 42, c\max(0, C_{\mathrm{sz}}))$ and take the point cutoff $x_0 = x_0(C_1, F_m)$ above 9, $13000000000C_1^4$, and the point cutoffs at C_1 of Proposition 34.67 and the base-eigenvalue assertion. Equations (A.414) and (A.417) then give (34.22) for every $C \in \mathbf{R}$. □

Proof of 34.69

Proof. Use the threshold of Theorem 34.68. The extension with list, ray-class fields and real-place compatibility transport the divisor row to the geometry row. Intersect with the goodness, primitive and divisor rows to obtain R, E. The geometry row supplies a compatible selection τ' of comparison data on the chosen members, with point summands, fitted grids and ties to the chosen bundles. Discrete-series realization and globalization give the eventual point-summand conditions from the real realization, quotient and division data. The class assumptions give the seven family conditions of Definition 33.3. Apply Theorem 34.68 at those fitted points using the assumed division bound. The ties in τ' transfer the functional condition to the original bundles with $C_0 = 0$. □

Proofs for Chapter 35: Covolume bounds

Proof of 35.4

Proof. The first Chern value $c_{1,\tau}$ is minus the mean, over the finitely many components, of the degree of the flat line at τ. Under the flat-degree law every flat section at τ on every component has degree $\frac{P-Q}{2}\log\Omega + Q(\log d - \log \mathrm{rd}(E))$. The component system is nonempty, so the mean equals this common value, which gives (35.3).

Put $\beta = P - Q$ and let $\rho(\tau)$ be the row index matching τ. The host and anchor comparisons give

$$\begin{gathered} L_x \geq 10, \qquad \log\log x \geq 1, \qquad E_p = fmbL_x \log\log x, \\ 0 \leq \log \mathrm{rd}(E) \leq 2L_x, \qquad |\log d - \log \mathrm{rd}(E)| \leq 3L_x, \qquad |\log\Omega| \leq 3L_x, \\ |a_\tau| \leq 4b, \qquad (P+Q)|a_\tau - w_{j,\rho(\tau)}| \leq 20b. \end{gathered} \tag{A.418}$$

For the last comparison, the discrepancy before multiplication by r is bounded by the scale width plus $n_V + m + t + 2$. After multiplication the scale contributes at most $16b$ and the other terms at most $4rn_V \leq 4b$, since the host size bounds give $frn_V L_x \leq b$ and $fL_x \geq 1$. Summing over the f embeddings gives

$$\left| M_{p,i} - \beta\log\Omega \sum_\tau a_\tau \right| \leq 60fbL_x \leq 100E_p, \qquad QL_x \sum_\tau |a_\tau| \leq 4fmbL_x \leq 100E_p. \tag{A.419}$$

To construct the scalar determinant twist explicitly on each component Z of the good cover Y over F', use the Hodge bundle identification $\omega_A = \bigoplus_{\tau:\underline{E}\hookrightarrow\mathbf{C}} V_\tau^\vee$ summing over all $2f$ embeddings of E (dual to the Lie eigenbundles, of ranks P on Ψ and Q on $\overline{\Psi}$, giving total rank $g = f(P+Q) = fr$ and $\det\omega_A = \bigotimes_{\tau:E\hookrightarrow\mathbf{C}} \det(V_\tau^\vee)$). On the universal principally polarised abelian scheme over the moduli datum, the Moret-Bailly / Faltings–Chai key formula [176] compares the eighth power $\mathcal{L}_\theta^{\otimes 8}$ of the dual theta line $\mathcal{L}_\theta = (\pi_* L_{\mathrm{PP}})^\vee$, for the universal projection π and its principal theta bundle L_{PP}, with $(\det\omega_A)^{\otimes 4}$ up to a canonical nowhere-vanishing global section K_{key} of central parameter -4 (row weight $(0,-4)_{\mathrm{row}}$, equivalently active theta weight $(4\mathbf{1}_{\mathrm{act}}, -8)_\theta$). This key section satisfies four global properties on every component Z: (i) K_{key} exists globally and is invertible on every moduli chart of Z; (ii) at every complex embedding $\sigma : F' \hookrightarrow \mathbf{C}$ and

every complex uniformised frame, the invariant complex point norm of K_{key} equals the exact constant $C_\sigma = (2\pi)^{2fr}$ (from the normalised theta-null identity on the complex abelian torus, giving $|\log C_\sigma| = 2fr\log(2\pi)$); (iii) at every finite place $w \mid \ell$ of F' (good or bad, interior or boundary), $0 \le \mathrm{ord}_w(K_{\mathrm{key}}) \le 4B(w)$, where $B(w) = \sum_\tau \mu_\tau \,\mathrm{ord}_w(\mathfrak{D}_\tau) = fr\,\delta_\ell e_w$ ($\delta_\ell = v_\ell(|D_E|)/(2f)$, $e_w = v_w(\ell)$). Indeed, because $M' = g_0 g_1 15^e$ contains the two distinct odd primes $g_0, g_1 > 3$, its prime-to-ℓ part is at least 3 at every rational prime ℓ, supplying a proper integral Siegel compactified model over $\mathcal{O}_{F',w}$ carrying the normalised integral model X and symmetric rigidified theta bundle L_{PP} of Lemma 35.10 and Definition 35.5. On each chart of this model with a Lie basis and a generating theta section, K_{key} restricts to a unit f_0 on the abelian locus of the chart. We test f_0 and f_0^{-1} on the complete discrete valuation rings described below. At points of the abelian locus this is the unit property of the key formula over a discrete valuation ring. Along a complex arc meeting the boundary, put the theta section and the Lie basis in the normal form of the arc. The key value is then the fourth power of the ratio of the Lie determinant to the square of the theta scale, and this ratio lies between C^{-1} and C in absolute value near the boundary, so $|f_0|$ and $|f_0^{-1}|$ are bounded by C^4 there and neither has a pole. *Claim 1.* The key value in an integral Lie basis and a generating integral theta section is a unit on every complete discrete valuation test with fraction characteristic zero, finite residue field and generic point in the abelian locus. At a boundary centre, make a finite extension splitting the semiabelian degeneration and write V for the resulting test ring. Use the rigidified principal theta extension supplied by the integral model. Its Fourier expansion is determined by its coefficient of character zero on the abelian quotient. A section nonzero on the closed fibre gives a generator of this coefficient line. Let C be the completed algebraic closure of $\mathrm{Frac}\,V$. The fraction field is a finite extension of a p-adic field, so C admits an abstract field isomorphism with $\mathbb{C}$. Over C the coefficient is c times the standard coefficient. Both generate the same integral coefficient line, so $|c|_C = 1$. The period relations then determine every Fourier coefficient.

Choose an integral Lie basis adapted to the torus extension and divide the theta section by c. The theta-jet comparison in this basis gives

$$c^{-8}k_B = u\,K, \qquad u \in V^\times, \qquad |k_B|_C = 1,$$

where K is the original key value and k_B is the key value of the abelian quotient in an integral basis. Here the unit u is the fourth power of the determinant of the integral Lie-basis change. To obtain the comparison, choose a sufficiently high odd level m^2 and expand all its theta translates and their first derivatives. The discrete period valuations give a positive integral quadratic form β on the character lattice; after rescaling, the exponents $n(\alpha) = \beta(\alpha,\alpha)/(2m^2)$ are nonnegative integers with finite level sets. Their coefficients satisfy $|a_{i,\alpha}|_C, |b_{i,j,\alpha}|_C \le B|\pi|_C^{n(\alpha)}$, with $0 < |\pi|_C < 1$. Grouping by $n(\alpha)$ gives power series with bounded coefficients that recover the original theta jets at π. After transporting the degeneration data to $\mathbb{C}$, multiplying the lattice terms by $t^{n(\alpha)}$ makes the toric imaginary period block positive definite for sufficiently small nonzero t. The theta series and their first derivatives then converge absolutely. In the classical theta normal form the toric determinant and theta factors cancel, leaving exactly the abelian key value k_B. We now give the common ring for the comparison. Write $\iota : C \to \mathbb{C}$ for the chosen field isomorphism. For every theta value and first derivative, regroup its lattice series as

$$\Phi(T) = \sum_{n\ge 0}\left(\pi^{-n}\sum_{n(\alpha)=n} a_\alpha\right)T^n.$$

Its coefficients are bounded in C, and evaluation at $T = \pi$ recovers the original jet. On the complex side evaluate at $T = z_*$, so the original lattice terms are multiplied by $(z_*/\iota(\pi))^{n(\alpha)}$. The added imaginary toric block of the theta period matrix is a positive multiple of $-\log|z_*/\iota(\pi)|\,\beta$, while the abelian block stays positive definite. The Schur complement is positive for sufficiently small nonzero z_*, and the first derivatives add polynomial factors to a convergent Gaussian sum.

Let $o \subset C$ be the field generated over $\mathbf{Q}$ by the level root of unity and k_B, and let $A \subset C[[T]]$ be the o-algebra generated by the finitely many regrouped jet series. This is a countable domain. After applying ι coefficientwise, every nonzero element of A is a nonzero holomorphic function on a common small disc. Avoiding their countably many discrete zero sets gives a nonzero z_* at which evaluation $A \to \mathbb{C}$ is injective. Let $\mathscr{B} \subset C[[T]]$ be the ring of series with bounded coefficients. Evaluation at π has kernel $(T-\pi)$; division by $T-\pi$ preserves bounded coefficients, and a nonzero series has finite vanishing order at π. Thus

$$R = \mathscr{B}_{(T-\pi)}, \qquad W = \mathrm{Frac}(A) \cap R$$

are respectively a discrete valuation ring and a field or discrete valuation ring. The intersection is taken in $\mathrm{Frac}(\mathscr{B})$. Complex evaluation extends injectively to W, giving $i_0 : W \to \mathbb{C}$; residue evaluation gives $i_1 : W \to C$. Every element with nonzero i_1-image is a unit. The only possible prime ideals of W are 0 and $\ker i_1$.

The sufficiently high odd theta level reconstructs a principally polarised abelian scheme over W from these theta coordinates. Both its possible fibres are the prescribed theta points, and the level points and rigidified line pull back to the specified ones. To check the frames as well, use the theta-value column together with the g first-derivative columns. They are linearly independent in the i_1-fibre. Choose a nonzero $(g+1)$-minor there; its determinant is a unit in W. Solving with this minor lifts the theta scale a and the Lie change matrix M to W, with the prescribed values at both evaluations. Let $q \in W^\times$ be the key value in a local generating section and Lie basis. For $j = 0, 1$, let K_j be the key value in the prescribed frame of the i_j-fibre. The change-of-frame law on that fibre is

$$i_j(a)^8 K_j = i_j(\det M)^4 i_j(q).$$

The complex normal-form computation gives $K_0 = \iota(k_B) = i_0(k_B)$. Injectivity of i_0 gives $a^8 k_B = (\det M)^4 q$ in W. Applying i_1 and cancelling the nonzero theta scale gives the same equality for the nonarchimedean jets. Restoring the original section and integral Lie basis gives $c^{-8} k_B = uK$ with $u \in V^\times$, as claimed. The abelian key formula gives $|k_B|_C = 1$, and $|c|_C = |u|_C = 1$, so $|K|_C = 1$. Thus K and K^{-1} are integral. Completion and finite extension multiply valuations by a positive ramification index, so this conclusion descends to the original trait. This proves the claim on the test rings. Work on a normal affine arithmetic chart with coordinate ring A of finite type over $\mathbb{Z}$, spreading out over a localisation of the coefficient integers as needed. Choose a nonempty principal open $\operatorname{Spec} A[1/f]$ inside the abelian locus, so that both functions belong to $A[1/f]$. Write either one as a/f^n. Divisibility by f^n in A is detected on maps to complete characteristic-zero DVRs with finite residue field on which f has nonzero image. The generic point of each such test lies in this principal open, so the claim gives integrality for both functions. They therefore extend over the whole chart, and K_{key} is a unit on every chart, boundary included. For any integral framed trait V over w of ramification index e, uniqueness of semiabelian extensions over a discrete valuation ring identifies the trait's semiabelian scheme with the pullback of the universal semiabelian family from this proper integral model. A generating integral theta section and an integral Lie basis on the trait differ from those of the chart by units $a, d \in V^\times$, and the two key values K, K' satisfy $a^8 K' = d^4 K$. Hence K_{key} evaluated in any integral Lie basis and generating integral theta section is a unit $u \in V^\times$. Expressing the saturated integral $\mathcal{O}_E$-eigenbasis of the frame in an integral Lie basis gives an integral matrix $M \in M_g(V)$ with $K_{\mathrm{key}}(y) = (\det M)^4 u$, hence $\mathrm{ord}_w(K_{\mathrm{key}}) = 4v_V(\det M)/e \geq 0$; because the trace-dual idempotents have denominators dividing the different $\mathfrak{D}_{E/\mathbf{Q}}$, the determinant $\det M$ divides $\prod_\tau \tau(\delta_\tau)^{\mu_\tau}$ in V for every $\delta_\tau \in \mathfrak{D}_{E/\mathbf{Q}}$, giving $v_V(\det M) \leq eB(w)$ and hence $0 \leq \mathrm{ord}_w(K_{\mathrm{key}}) \leq 4B(w)$ at every finite place w. Finally, (iv) K_{key} admits a monodromy-free fourth root over F' up to a scalar on each component Z, unconditionally at odd level $M' = g_0 g_1 15^e \geq 3$. Indeed, because $d \equiv 7 \pmod{16}$, the integral half-element $w_0 = (1 + \sqrt{-d})/2 \in \mathcal{O}_E$ satisfies $w_0^2 - w_0 + (1+d)/4 = 0$ with $(1+d)/4 \equiv 2 \pmod 4$, so 2 splits in $\mathbf{Q}(\sqrt{-d}) \subset E$ and $w_0^4 \pmod 8$ is a 2-adic idempotent splitting $\Lambda/8\Lambda$ into two conjugate-isotropic Lagrangian halves. In a symplectic basis (a_k, b_k) adapted to this splitting ($w_0^4 a_k \equiv a_k$ and $w_0^4 b_k \equiv 0 \pmod 8$), every $\mathcal{O}_E$-linear symplectic isometry $\gamma \in \Gamma(M')$ automatically has off-diagonal blocks $B \equiv C \equiv 0 \pmod 8$ with no 2-power level assumption; moreover, neatness at odd level $M' \geq 3$ forces the unitary root of unity $\det_E(\gamma) \equiv 1 \pmod{M'}$ to equal 1, whence $\det(D) \equiv 1 \pmod 8$ and the theta-square fourth-root multiplier $\chi_4(\det D) = +1$ identically. Writing $h(b)$ for the Siegel-period normalising matrix sending the symplectic A-periods $P_b(a_k)$ to the standard basis e_k and the B-periods $P_b(b_k)$ to $\tau(b)e_k$, under $\gamma \in \Gamma(M')$ with $P_{b'}(\gamma a) = h_0 P_b(a)$ the product $(\det h(b))\,\theta^2$ transforms strictly $\Gamma(M')$-equivariantly as $(\det h(b))\,\theta^2 = (\det h(b'))(\det h_0)\,\theta'^2$ with weight $\det h_0$ and multiplier $+1$ (central parameter -1, matching the fourth-root weight of K_{key}). Evaluating on any smooth geometrically integral affine framed moduli chart $C = \operatorname{Spec} R \subset Z$ over F' equipped with a trivialising Lie basis and rigidified theta section (where K_{key} evaluates to a chart unit $u \in R^\times$) cancels the automorphy factor $\det h_0$ and yields a continuous complex function ρ_{an} on $C(\mathbf{C})$ satisfying $\rho_{\mathrm{an}}^4 = C_0 u$ for the explicit nonzero constant $C_0 = (N_{\mathrm{norm}}/(2\pi i)^g)^4$, where $N_{\mathrm{norm}} \in \mathbf{Q}^\times$ is the fixed nonzero rational normalisation factor between the algebraic key section and the complex theta-null product. By Kummer descent and relative algebraic closure of F' in R, this forces $a_Z u = b^4$ with $a_Z \in (F')^\times$ and $b \in R^\times$. By normality of the chart rings, these local fourth roots agree up to fourth roots of unity fixed by the analytic branch and glue to a global invertible

section ρ on Z satisfying $e_Z\rho^4 = a_Z e_Z K_{\mathrm{key}}$, where e_Z is the component idempotent. Define the minimal determinant twisting section $t_{\mathrm{det}} = e_Z\rho^{-1}$ of central parameter $+1$ (row weight $(0,1)_{\mathrm{row}}$), together with its inverse $t_{\mathrm{det}}^{-1} = e_Z\rho$, so that $t_{\mathrm{det}}^{-4} = a_Z K_{\mathrm{key}}$ algebraically over F' on Z. Taking fourth roots shows that t_{det} has constant archimedean norm $(|a_Z|_\sigma C_\sigma)^{-1/4} = |a_Z|_\sigma^{-1/4}(2\pi)^{-fr/2}$ at every complex embedding $\sigma : F' \hookrightarrow \mathbf{C}$, and its valuation on integral frames over w lies in $[l_{\mathrm{det}}(w), h_{\mathrm{det}}(w)]$ with $h_{\mathrm{det}}(w) = -v_w(a_Z)/4$ and $l_{\mathrm{det}}(w) = -v_w(a_Z)/4 - B(w)$ (so $h_{\mathrm{det}}(w) - l_{\mathrm{det}}(w) = B(w)$, summing against $\log Nw/[F' : \mathbf{Q}]$ to $fr\,\ell_{\mathrm{rd}}$ with $\ell_{\mathrm{rd}} = \log \mathrm{rd}(E)$). Away from the bad-prime set $\mathcal{S}_{\mathrm{bad}} = \{\ell \text{ prime} : \ell \mid 2dD_E M'\}$, we have $B(w) = 0$ and Lemma 35.10 supplies smooth integral frames with multiplicity-one special fibres, forcing $l_{\mathrm{det}}(w) = h_{\mathrm{det}}(w) \in \mathbf{Z}$; moreover, the scalar $a_Z \in (F')^\times$ cancels from the Arakelov degree of t_{det} by the product formula over all complex embeddings and finite places of F', and the archimedean norm $(2\pi)^{-fr/2}$ contributes $+\frac{1}{2}fr\log(2\pi) \geq -\frac{1}{2}fr\log(2\pi)$ to the Arakelov degree, so the combined archimedean and finite degree contribution of t_{det} is bounded below by $fr\big(\frac{1}{2}\log(2\pi) - \ell_{\mathrm{rd}}\big) \geq -fr\big(\frac{1}{2}\log(2\pi) + \ell_{\mathrm{rd}}\big)$.

The integral twist adds the anchor a_τ at each embedding via the flat sections of Lemma 35.6 and adds the central parameter $m_0 = n_{\mathrm{col}}/2$ via $t_{\mathrm{det}}^{m_0}$; multiplication by this invertible twisting section carries weight u_M to $G_s(-c_0)$ and preserves Hermitian lattice ranks. Since $\mathbf{Q}(\sqrt{-d}) \subset E$ implies $d^f \mid |D_E|$ and hence $0 < \log d \leq 2\log \mathrm{rd}(E) = 2\ell_{\mathrm{rd}}$, we have $|\log d - \log \mathrm{rd}(E)| \leq \ell_{\mathrm{rd}}$, so passing between the uncorrected flat section and the different-corrected line (Lemma 35.10) costs at most $2Q\ell_{\mathrm{rd}}\sum_\tau |a_\tau|$ in squared-norm slope for either sign of a_τ. Away from $\mathcal{S}_{\mathrm{bad}}$, smooth reduction with multiplicity-one special fibres (Lemma 35.10) ensures that the lower order $l_Z(w)$ on each component Z of both the flat anchor section and t_{det} lies in $\mathbf{Z}$ at every prime $\ell \notin \mathcal{S}_{\mathrm{bad}}$ (even at spreading-out primes where $l_Z(w) \neq 0$). Multiplying componentwise on $\Lambda_Y = \bigoplus_Z \Lambda_Z$ by the fractional ideal $\mathfrak{a}_{0,Z} = \prod_w \mathfrak{p}_w^{-\lfloor l_Z(w)\rfloor}$ of F' to round these lower orders downward to $\lfloor l_Z(w)\rfloor$ incurs fractional-part loss $l_Z(w) - \lfloor l_Z(w)\rfloor$, which vanishes outside $\mathcal{S}_{\mathrm{bad}}$ and is strictly less than 1 on $\mathcal{S}_{\mathrm{bad}}$ regardless of the sign of $l_Z(w)$; in the squared-norm slope $\mu_Y = 2\widehat{\deg}/\operatorname{rk}$ (by rank-weighted averaging over Z) the resulting signed loss is bounded below by $-2\sum_{\ell\in\mathcal{S}_{\mathrm{bad}}}\log\ell$. Dividing the degree comparison by the positive common rank gives

$$\begin{aligned}\mu_Y(G_s(-c_0)) - \mu_Y(u_M) \;&\geq 2\sum_\tau a_\tau c_{1,\tau} - 2Q\ell_{\mathrm{rd}}\sum_\tau |a_\tau| \\ &\quad - 2m_0 fr\big(\tfrac{1}{2}\log(2\pi) + \ell_{\mathrm{rd}}\big) - 2\sum_{\ell\in\mathcal{S}_{\mathrm{bad}}}\log\ell.\end{aligned} \tag{A.420}$$

By (35.3) and (A.418), $|c_{1,\tau} + \frac{1}{2}\beta\log\Omega| \leq 3QL_x$ for every τ, so $a_\tau c_{1,\tau} \geq -\frac{1}{2}\beta a_\tau\log\Omega - 3QL_x|a_\tau|$; summing over τ and applying (A.419) gives $2\sum_\tau a_\tau c_{1,\tau} \geq -M_{p,i} - 100E_p - 600E_p = -M_{p,i} - 700E_p$. The second term in (A.420) costs at most $2Q(2L_x)\sum_\tau|a_\tau| \leq 400E_p$ by (A.418) and (A.419). Since $\frac{1}{2}\log(2\pi) \leq \frac{1}{2}(2\pi - 1) < 4 \leq L_x$ (as $L_x \geq 10$), we have $\frac{1}{2}\log(2\pi) + \ell_{\mathrm{rd}} \leq 3L_x$; combining $2m_0 \leq n_V$, $frn_V L_x \leq b \leq E_p$, and the bad-prime bound $\sum_{\ell\in\mathcal{S}_{\mathrm{bad}}}\log\ell \leq 40L_x$ yields

$$2m_0 fr\big(\tfrac{1}{2}\log(2\pi) + \ell_{\mathrm{rd}}\big) \leq 3frn_V L_x \leq 3b \leq 3E_p, \qquad 2\sum_{\ell\in\mathcal{S}_{\mathrm{bad}}}\log\ell \leq 80L_x \leq 80E_p. \tag{A.421}$$

Substitution in (A.420) gives $-M_{p,i} - (700 + 400 + 3 + 80)E_p = -M_{p,i} - 1183E_p$. □

Proof of 35.6

Proof. Work first over F'. The coefficient field contains the images of the embeddings of E, so the idempotents for the E-action split de Rham homology over F'. Let H_τ be its τ-summand. The Hodge sequence for homology, the polarisation, and the fact that the Rosati involution acts on E by conjugation give an exact sequence

$$0 \longrightarrow V_{\bar\tau}^\vee \longrightarrow H_\tau \longrightarrow V_\tau \longrightarrow 0.$$

In particular $\det H_\tau = \mathcal{L}_\tau$. In bases v, w of the two Lie eigenspaces a section has coefficient $c(v,w)$ relative to $\bigwedge v \otimes (\bigwedge w)^{-1}$. Changing the bases by D_1, D_2 multiplies that coefficient by $\det D_2/\det D_1$. This is the flat row character.

Fix a complex embedding of F'. The monodromy of H_τ is the τ-realisation of the unitary lattice action in the uniformisation. Its determinant is one. Indeed, the stabilizer preserves the period lattice and acts trivially

on its principal level of modulus $M \geq 3$. Localising the lattice at each prime of E gives $\det \gamma = 1 + Ma$ with $a \in \mathcal{O}_E$. Unitarity gives $(\det \gamma)\overline{\det \gamma} = 1$, so every conjugate of a has absolute value at most $2/M < 1$. Its integral norm forces $a = 0$. Thus the exterior product of a flat basis gives a single-valued horizontal generator of $\det H_\tau$ on the complex component.
Lean: `GlobalTheory.LawsDeep.laws_deep` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ComplexUniformizationLawsOfGoodDeepCover`, lines 41–51)
Lean: `GlobalTheory.UnitaryCongruenceDetAny.det_eq_one` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.UnitaryCongruenceDetEqOneOfThreeLe`, lines 11–91)

Here one can algebraize the horizontal generator by the regular-singular comparison theorem. The base is smooth of finite type in characteristic zero. The abelian scheme is smooth and proper, so it supplies its own relative smooth compactification with empty relative boundary. Its Gauss–Manin connection is regular [186, II, Th. 7.9]. Duals, direct summands and exterior powers preserve regularity. Full faithfulness in [186, II, Th. 5.9], applied to the trivial connection and $\det H_\tau$, identifies the single-valued horizontal sections with the algebraic horizontal sections. This space has dimension one because Z is geometrically connected. To descend to F', compute horizontal sections on a finite affine cover as the kernel of the connection and the differences of restrictions on overlaps. Extension of the coefficient field commutes with this kernel by flatness. A basis over F' therefore becomes a nonzero multiple of the horizontal generator over $\mathbf{C}$. It generates at every point by faithful flatness. Its reciprocal is a regular section of the dual line.

At an infinite place let J be the involution which is $+1$ on the positive Hodge subspace and -1 on the negative one for the polarising Hermitian form H. The positive Hermitian metric is $H(\,\cdot\,, J\,\cdot\,)$, up to the fixed period normalisation at that place. Its determinant in a flat basis is $|\det H|$, since $|\det J| = 1$. The determinant metric in the Hodge sequence is the product of the subspace and quotient metrics, as one sees in an orthogonal splitting. Hence in the Lie frames its squared norm is

$$|c(v, w)|^2 \, \frac{\det \operatorname{Gram}(v)}{\det \operatorname{Gram}(w)}.$$

It is constant on Z. The same calculation applies after every embedding of F'. Extending the section and its reciprocal by zero to the other components gives two sections whose product is the idempotent of Z. □

Proof of 35.7

Proof. Completion preserves the valuation of a determinant. The maximal unramified order splits after tensoring with R, since all its embeddings take values in R and their distinct residual embeddings are separated by units. In the selected factor the remaining extension is totally ramified and its integer ring is generated by π. The resulting algebra is

$$B = R[t] \Big/ \prod_{\sigma \in S} (t - a_\sigma).$$

All the following modules are taken in this factor. Write $T = t - a_\tau$. Evaluation at a_τ has kernel TB. Since H is free over B, it identifies H/TH integrally with $H(\tau)$.

Put $F = \ker(H \to L)$, and choose an R-basis of $L[\tau] = \ker(T|L)$. Lift it to H and project the lifts to $H(\tau)$. Adjoin to these lifts a basis of $\operatorname{pr}_\tau F$. This last lattice identifies with $L[\bar\tau]^\vee$. Indeed, projection of a functional to the τ-summand amounts to restriction to the $\bar\tau$-summand of L_K. Every integral functional on $L[\bar\tau]$ extends to L, since $L[\bar\tau]$ is saturated in the free R-module L. Thus the determinant of the assembled vectors is the determinant of the stated pair of Lie frames. Changing a lift adds a vector in $\operatorname{pr}_\tau F$ and preserves this determinant.

Denote their span by A. Taking the quotient first by F, then by the lifted basis, gives

$$H(\tau)/A \simeq L/(TL + L[\tau]) = \operatorname{coker}\big(T : L/L[\tau] \to L/L[\tau]\big).$$

The last module on which T acts is free. On its generic fibre T has eigenvalues $a_\sigma - a_\tau$, with multiplicity h_σ, for $\sigma \neq \tau$ in S. Diagonal reduction over R identifies the length of the cokernel with the valuation of that determinant. Dividing b by the determinant of the assembled frame gives the asserted sign. □

Proof of 35.8

Proof. For (1), let $W \subseteq X$ be nonempty and open. Since $G \to X$ is smooth with geometrically integral fibres and W is normal and integral with dense open $W \cap U$, $G_W^{\times_W n}$ is normal and integral with dense open $G_{W\cap U}^{\times_{W\cap U} n}$. Over $W\cap U$, $G_{W\cap U}^{\times_{W\cap U} n} \to W\cap U$ is proper and flat of finite presentation with geometrically reduced and connected fibres, so $\Gamma(W\cap U, \mathcal{O}_{W\cap U}) \xrightarrow{\sim} \Gamma(G_{W\cap U}^{\times_{W\cap U} n}, \mathcal{O})$. For any $f \in \Gamma(G_W^{\times_W n}, \mathcal{O})$, pulling back along the n-fold identity section $e_n : W \to G_W^{\times_W n}$ gives $g = e_n^* f \in \Gamma(W, \mathcal{O}_W)$ such that $f - \pi_n^* g$ vanishes along e_n, hence vanishes on $G_{W\cap U}^{\times_{W\cap U} n}$, and therefore vanishes on the integral scheme $G_W^{\times_W n}$. Since $e_n^* \circ \pi_n^* = \mathrm{id}$, π_n^* is bijective. In particular, on every open $W \subseteq X$ any bi-rigidified automorphism of a line bundle on $G_W \times_W G_W$ equals 1, so ad_a in (a) is unique.

For (2) and the gluing in (3), smoothness of $G \to X$ of relative dimension d supplies, on an affine open neighbourhood $\operatorname{Spec} R \subseteq X$ of any $q \in X$, regular relative parameters $\mathbf{t} = (t_1, \dots, t_d)$ of G_R along e_R identifying $\widehat{\mathcal{O}}_{G_R, e_R} \simeq R[[\mathbf{t}]]$, together with a local trivialisation τ_0 of $\Lambda(L_{\mathrm{PP}})$ near $(e(q), e(q))$ which becomes bi-rigidified after multiplying $\tau_0(g_1, g_2)$ by $\tau_0(g_1, e)^{-1}\tau_0(e, g_2)^{-1}\tau_0(e, e)$. Expanding the group law, $\mathcal{O}_E$-action and Mumford partial-multiplication, symmetry and Rosati-adjointness isomorphisms in $(\mathbf{t}, \tau_0)$ produces a formal biextension germ B_q over $\mathcal{O}_{X,q}$; the cocycle identities hold because the corresponding bi-rigidified line-bundle isomorphisms agree over U and hence globally by (1). Along a specialisation $q' \rightsquigarrow q$, comparing two local choices $(\mathbf{t}, \tau_0)$ and $(\mathbf{t}', \tau_0')$ expresses $\mathbf{t} = \varphi(\mathbf{t}')$, $\mathbf{t}' = \psi(\mathbf{t}) = \varphi^{-1}(\mathbf{t})$ and $\tau_0'(\mathbf{x}, \mathbf{y}) = \lambda(\mathbf{x}, \mathbf{y})\, \tau_0(\varphi(\mathbf{x}), \varphi(\mathbf{y}))$, satisfying

$$\begin{aligned}
\varphi_1'(\mathbf{x}', \mathbf{x}'', \mathbf{y})\, \lambda(\mu_{B'}(\mathbf{x}', \mathbf{x}''), \mathbf{y}) &= \lambda(\mathbf{x}', \mathbf{y})\, \lambda(\mathbf{x}'', \mathbf{y})\, \varphi_1(\varphi(\mathbf{x}'), \varphi(\mathbf{x}''), \varphi(\mathbf{y})),\\
\varphi_2'(\mathbf{x}, \mathbf{y}', \mathbf{y}'')\, \lambda(\mathbf{x}, \mu_{B'}(\mathbf{y}', \mathbf{y}'')) &= \lambda(\mathbf{x}, \mathbf{y}')\, \lambda(\mathbf{x}, \mathbf{y}'')\, \varphi_2(\varphi(\mathbf{x}), \varphi(\mathbf{y}'), \varphi(\mathbf{y}'')),\\
\mathrm{ad}_{B'}(a)(\mathbf{x}, \mathbf{y})\, \lambda(\mathbf{x}, [\bar a]_{B'}\mathbf{y}) &= \lambda([a]_{B'}\mathbf{x}, \mathbf{y})\, \mathrm{ad}_B(a)(\varphi(\mathbf{x}), \varphi(\mathbf{y})).
\end{aligned}$$

By uniqueness of bi-rigidified isomorphisms in (1), ad_{a+b} is obtained from $\mathrm{ad}_a \otimes \mathrm{ad}_b$ via φ_1, φ_2 (whose series lie in $1+(\mathbf{x}')(\mathbf{x}'')(\mathbf{y})$ and $1+(\mathbf{x})(\mathbf{y}')(\mathbf{y}'')$ and contribute zero to the $(1,1)$-jet), $\mathrm{ad}_1 = \mathrm{id}$, and by commutativity of $\mathcal{O}_E$, $\mathrm{ad}_{ab} = \mathrm{ad}_{ba} = ((\mathrm{id}_G \times \bar a)^* \mathrm{ad}_b) \circ ((b \times \mathrm{id}_G)^* \mathrm{ad}_a)$, whose $(1,1)$-matrix is $\rho_B(b)^{\mathrm{t}}\, \mathrm{bil}(\mathrm{ad}_B(a))_{1,1} + \mathrm{bil}(\mathrm{ad}_B(b))_{1,1}\rho_B(\bar a)$. Taking transposes shows that $a \mapsto \rho_H(B_q)(a)$ is a ring homomorphism, and this persists under any scalar extension $\mathcal{O}_{X,q} \to R$. Taking the $(1,1)$-coefficient matrix of the third transition identity above and transposing gives $\mathrm{transH}(\varphi, \psi, \lambda) \circ \rho_H(B')(a) = \rho_H(B)(a) \circ \mathrm{transH}(\varphi, \psi, \lambda)$. Since $\mathrm{transH}(\varphi, \psi, \lambda)$ acts by $D\varphi(0)$ on the quotient R^d and by $D\psi(0)^{\mathrm{t}} = (D\varphi(0)^{-1})^{\mathrm{t}}$ on the submodule $\{0\} \times R^d$, the stalk lattices glue inside $V \times V^\vee$ to a coherent $\mathcal{O}_E$-module $\mathcal{H}$ in a locally split $\mathcal{O}_X$-exact sequence $0 \to \mathcal{L}^\vee(\mathrm{conj}) \to \mathcal{H} \to \mathcal{L} \to 0$.

It remains to prove that $\mathcal{H}_q$ is free of rank r over $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathcal{O}_{X,q}$ at every $q \in X$. Because $F' \subset \Omega$ is the coefficient field of a good cover, it contains the images of all $[E : \mathbf{Q}]$ embeddings of E, so every $\sigma : E \hookrightarrow \Omega$ factors through F', and the signature (P, Q) gives $\dim_\Omega V_\sigma + \dim_\Omega V_{\bar\sigma} = P + Q = r$, whence $\dim_\Omega (V \times V^\vee)_\sigma = r$ for every σ. If $\mathrm{char}\, \kappa(q) = 0$, then $\varpi \in \mathcal{O}_{X,q}^\times$, so $F' = O[1/\varpi] \subseteq \mathcal{O}_{X,q}$ splits $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathcal{O}_{X,q} = E \otimes_{\mathbf{Q}} \mathcal{O}_{X,q} \simeq \prod_{\sigma : E \hookrightarrow F'} \mathcal{O}_{X,q}$ into a finite product of copies of the local domain $\mathcal{O}_{X,q}$, on each of which the finite projective summand of $\mathcal{H}_q$ has generic rank r and is therefore free of rank r.

Suppose now that $\mathrm{char}\, \kappa(q) = \ell$, and choose a geometric point $\bar x : \operatorname{Spec} \bar k \to X$ at q with $\bar k$ algebraically closed and Chevalley extension $0 \to T \to G_{\bar x} \xrightarrow{q_{\bar x}} B \to 0$ as in (d). Any group-scheme homomorphism from the torus $T \simeq \mathbb{G}_{m,\bar k}^{d_T}$ to the abelian variety B is trivial, so T is $\mathcal{O}_E$-stable and $\mathcal{O}_E$ descends to B. Since $L_{\mathrm{PP}}|_{G_{\bar x}} \simeq q_{\bar x}^* M$, we have $\Lambda(L_{\mathrm{PP}})|_{G_{\bar x} \times G_{\bar x}} \simeq (q_{\bar x} \times q_{\bar x})^* \Lambda(M)$, where $\Lambda(M) = (\mathrm{id}_B \times \lambda_B)^* \mathcal{P}$ is the Poincaré biextension of the principal polarisation λ_B. Along the fibres of the $(T \times T)$-torsor $q_{\bar x} \times q_{\bar x} : G_{\bar x} \times G_{\bar x} \to B \times B$, any invertible function on $T \times T$ is a scalar times a pair of characters of T; connectedness of $B \times B$ makes those characters constant, and the rigidifications along $e_{\bar x} \times \mathrm{id}_{G_{\bar x}}$ and $\mathrm{id}_{G_{\bar x}} \times e_{\bar x}$ force both characters to be trivial. By faithfully flat descent along $q_{\bar x} \times q_{\bar x}$, $\mathrm{ad}_a\,|_{G_{\bar x} \times G_{\bar x}} = (q_{\bar x} \times q_{\bar x})^* \mathrm{ad}_{B,a}$ for a unique bi-rigidified Rosati isomorphism $\mathrm{ad}_{B,a} : (a_B \times \mathrm{id}_B)^* \Lambda(M) \xrightarrow{\sim} (\mathrm{id}_B \times \bar a_B)^* \Lambda(M)$. Choosing formal parameters on $\widehat{G}_{\bar x}$ adapted to $0 \to T \to G_{\bar x} \xrightarrow{q_{\bar x}} B \to 0$ and pulling back a bi-rigidified trivialisation of $\Lambda(M)$ from $\widehat{B} \times \widehat{B}$, the matrix $\mathrm{bil}(\mathrm{ad}_{B_{\bar x}}(a))_{1,1}^{\mathrm{t}}$ vanishes on $\mathrm{Lie}(T)$ and takes values in $\mathrm{Lie}(B)^\vee \subseteq \mathrm{Lie}(G_{\bar x})^\vee$. Consequently $W_1 = \mathrm{Lie}(T) \times \{0\}$ and $W_2 = \mathrm{Lie}(G_{\bar x}) \times \mathrm{Lie}(B)^\vee$ are $(\mathcal{O}_E \otimes_{\mathbf{Z}} \bar k)$-submodules of $\mathcal{H}_{\bar x}$ with $W_1 \simeq \mathrm{Lie}(T)$, $\mathcal{H}_{\bar x}/W_2 \simeq \mathrm{Lie}(T)^\vee(\mathrm{conj})$ and $W_2/W_1 \simeq \mathrm{Lie}(B) \times \mathrm{Lie}(B)^\vee$. Restricting $\Lambda(M)$ to $B \times \mathrm{Inf}_0^1(B)$ constructs the universal vector extension $0 \to \mathrm{Lie}(B)^\vee \to E(B) \to B \to 0$ [161, Ch. I]; representing a tangent vector in $\mathrm{Lie}(E(B))$ over $\bar k[\varepsilon]/(\varepsilon^2)$ by $(\mathbf{x}_B = \varepsilon v,\ 1 + \varepsilon \eta^{\mathrm{t}} \mathbf{y}_B \bmod (\mathbf{y}_B)^2)$ relative to the formal trivialisation of $\Lambda(M)$, functorial transport by

$a \in \mathcal{O}_E$ pulls $1+\varepsilon\eta^{\mathsf{t}}\mathbf{y}_B$ back along $\bar{a}_B$ and multiplies by $\mathrm{ad}_{B,a}(\varepsilon v,\mathbf{y}_B)^{-1} \equiv 1-\varepsilon(\mathrm{bil}(\mathrm{ad}_{B,a})^{\mathsf{t}}_{1,1}v)^{\mathsf{t}}\mathbf{y}_B$, giving an $(\mathcal{O}_E\otimes_{\mathbf{Z}}\bar{k})$-linear isomorphism $W_2/W_1 \xrightarrow{\sim} \mathrm{Lie}(E(B))$.

Let $W(\bar{k})$ be the Witt ring of $\bar{k}$. Decomposing $\mathcal{O}_E\otimes_{\mathbf{Z}}\mathbf{Z}_\ell = \prod_{\mathfrak{l}\mid\ell}\mathcal{O}_{E,\mathfrak{l}}$, splitting the maximal unramified subextensions over $W(\bar{k})$ and adjoining uniformisers of Eisenstein polynomials decomposes $\mathcal{O}_E\otimes_{\mathbf{Z}}W(\bar{k}) = \prod_{j=1}^{m}R_j$ into complete discrete valuation rings $R_j = \hat{e}_j(\mathcal{O}_E\otimes_{\mathbf{Z}}W(\bar{k}))$ with uniformisers π_j satisfying $\pi_j^{e_j}\in \ell R_j^{\times}$, reducing modulo ℓ to $\mathcal{O}_E\otimes_{\mathbf{Z}}\bar{k} = \prod_{j=1}^{m}A_j$ with $A_j = R_j/\ell R_j$ and idempotents $e_j = \hat{e}_j \bmod \ell$. Let $D^\vee = \mathbb{D}(B[\ell^\infty])(W(\bar{k}))^\vee$ be the $W(\bar{k})$-dual of the contravariant Dieudonné crystal of $B[\ell^\infty]$; it is finite free over $W(\bar{k})$ and $D^\vee/\ell D^\vee \simeq \mathrm{Lie}(E(B))$ compatibly with $\mathcal{O}_E$ [161, Ch. II, §§9, 13 and 15]. For each j, $\hat{e}_jD^\vee$ is finitely generated over R_j and $W(\bar{k})$-torsion-free, hence π_j-torsion-free because $\pi_j^{e_j}\in\ell R_j^\times$, so $\hat{e}_jD^\vee$ is free over R_j of some finite rank $r_{B,j}$ and $e_j(W_2/W_1)\simeq A_j^{r_{B,j}}$. The cocharacter lattice $X_*(T)$ is finitely generated and torsion-free over the Dedekind domain $\mathcal{O}_E$, hence projective of some rank r_T, so $\mathrm{Lie}(T)\simeq X_*(T)\otimes_{\mathbf{Z}}\bar{k}$ and $\mathrm{Lie}(T)^\vee(\mathrm{conj})$ are free of rank r_T over $\mathcal{O}_E\otimes_{\mathbf{Z}}\bar{k}$. For each j, freeness of $e_j(W_2/W_1)$ and $e_j(\mathcal{H}_{\bar{x}}/W_2)$ over A_j splits the filtration on $e_j\mathcal{H}_{\bar{x}}$ and gives $e_j\mathcal{H}_{\bar{x}}\simeq A_j^{n_j}$ with $n_j = r_{B,j}+2r_T$. Let $\mathcal{O}^{\mathrm{sh}}_{X,q}$ be the strict henselisation of $\mathcal{O}_{X,q}$, a normal Henselian local domain with separably closed residue field $k_s = \kappa(q)^{\mathrm{sep}}\hookrightarrow\bar{k}$ and fraction field $\Omega^{\mathrm{sh}}\supset\Omega$. Since $\mathcal{O}_E\otimes_{\mathbf{Z}}\mathbf{F}_\ell$ has finite (hence separable) residue fields, the idempotents $e_1,\dots,e_m$ already lie in $\mathcal{O}_E\otimes_{\mathbf{Z}}k_s$ and lift to orthogonal idempotents $\tilde{e}_1,\dots,\tilde{e}_m\in\mathcal{O}_E\otimes_{\mathbf{Z}}\mathcal{O}^{\mathrm{sh}}_{X,q}$ summing to 1. Put $d_j = \dim_{\bar{k}}A_j\geq 1$. The direct summands $\tilde{e}_j(\mathcal{O}_E\otimes_{\mathbf{Z}}\mathcal{O}^{\mathrm{sh}}_{X,q})$ and $\tilde{e}_j\mathcal{H}^{\mathrm{sh}}_q$ are projective, hence free, over $\mathcal{O}^{\mathrm{sh}}_{X,q}$, of closed-fibre ranks over k_s equal to their $\bar{k}$-dimensions d_j and n_jd_j. Over Ω^{sh}, $\tilde{e}_j(\mathcal{O}_E\otimes_{\mathbf{Z}}\Omega^{\mathrm{sh}})$ is spanned by d_j embeddings $E\hookrightarrow F'\subset\Omega$, each of eigenrank r on $V\times V^\vee$, so $\tilde{e}_j\mathcal{H}^{\mathrm{sh}}_q$ has generic rank rd_j. Thus $n_jd_j = rd_j$ and $n_j = r$ for all j, making $\mathcal{H}_{\bar{x}}$ free of rank r over $\mathcal{O}_E\otimes_{\mathbf{Z}}\bar{k}$. By faithful flatness of $k_s\hookrightarrow\bar{k}$, the closed fibre $\mathcal{H}^{\mathrm{sh}}_q\otimes_{\mathcal{O}^{\mathrm{sh}}_{X,q}}k_s$ is flat, and hence free of rank r, over the finite product of local Artin rings $\mathcal{O}_E\otimes_{\mathbf{Z}}k_s$. Lifting a basis of $\mathcal{H}^{\mathrm{sh}}_q\otimes_{\mathcal{O}^{\mathrm{sh}}_{X,q}}k_s$ to $\mathcal{H}^{\mathrm{sh}}_q$ gives an $(\mathcal{O}_E\otimes_{\mathbf{Z}}\mathcal{O}^{\mathrm{sh}}_{X,q})$-linear map $(\mathcal{O}_E\otimes_{\mathbf{Z}}\mathcal{O}^{\mathrm{sh}}_{X,q})^r\to\mathcal{H}^{\mathrm{sh}}_q$ which is surjective by Nakayama's lemma, hence an isomorphism between finite free $\mathcal{O}^{\mathrm{sh}}_{X,q}$-modules of equal rank $[E:\mathbf{Q}]r$. Faithfully flat descent along $\mathcal{O}_{X,q}\to\mathcal{O}^{\mathrm{sh}}_{X,q}$ makes $\mathcal{H}_q$ finite projective of constant rank r over the Noetherian semilocal ring $\mathcal{O}_E\otimes_{\mathbf{Z}}\mathcal{O}_{X,q}$, and hence free of rank r. □

Proof of 35.9

Proof. Every $\sigma: E\hookrightarrow\Omega$ factors through F', and since O is integrally closed in F', $\sigma(\mathcal{O}_E)$ lies in the image of O. Let R be any commutative $\mathbf{F}_\ell$-algebra and B a d-dimensional formal biextension germ over R. Write $\mathbf{x}^{(\ell)} = (x_1^\ell,\dots,x_d^\ell)^{\mathsf{t}}$ and $M^{(\ell)} = (M_{ab}^\ell)$ for a matrix M over R. Pullback of the basis of invariant differentials of μ_B along $[\ell]_B$ multiplies them by $\ell = 0$, so all first partial derivatives of $[\ell]_B(\mathbf{x})$ vanish; since every integer prime to ℓ is invertible in R, every nonzero monomial of $[\ell]_B(\mathbf{x})$ has all exponents divisible by ℓ, giving $[\ell]_B(\mathbf{x})\equiv V_1\mathbf{x}^{(\ell)}\pmod{(\mathbf{x})^{2\ell}}$ for a unique $V_1\in\mathrm{Mat}_d(R)$. Modulo $(\mathbf{y})^2$, $\varphi_1(\mathbf{x}',\mathbf{x}'',\mathbf{y})\equiv 1+c(\mathbf{x}',\mathbf{x}'')^{\mathsf{t}}\mathbf{y}$ defines a symmetric formal 2-cocycle c and hence a $2d$-dimensional commutative formal group law on $R[[\mathbf{x}]]\times\widehat{\mathbb{G}}_a^d$ whose multiplication by ℓ on $(\mathbf{x},0)$ has second component $\sum_{k=1}^{\ell-1}c([k]_B\mathbf{x},\mathbf{x})$; vanishing of its invariant differentials under $[\ell]$ shows that $\Psi_\ell(\mathbf{x},\mathbf{y}) = \prod_{k=1}^{\ell-1}\varphi_1([k]_B\mathbf{x},\mathbf{x},\mathbf{y})$ satisfies

$$\Psi_\ell(\mathbf{x},\mathbf{y})\equiv 1+(\mathbf{x}^{(\ell)})^{\mathsf{t}}W_1\mathbf{y}\pmod{(\mathbf{x})^{2\ell}+(\mathbf{y})^2}$$

for a unique $W_1\in\mathrm{Mat}_d(R)$. By the same argument on the second factor, $\widetilde{\Psi}_\ell(\mathbf{x},\mathbf{y}) = \prod_{k=1}^{\ell-1}\varphi_2(\mathbf{x},[k]_B\mathbf{y},\mathbf{y})$ satisfies

$$\widetilde{\Psi}_\ell(\mathbf{x},\mathbf{y})\equiv 1+\mathbf{x}^{\mathsf{t}}\widetilde{W}_1\mathbf{y}^{(\ell)}\pmod{(\mathbf{x})^2+(\mathbf{y})^{2\ell}}$$

for a unique $\widetilde{W}_1\in\mathrm{Mat}_d(R)$. Put

$$\mathcal{V}_B = \begin{pmatrix}V_1\\ W_1^{\mathsf{t}}\end{pmatrix}:(R^d)^{(\ell)}\longrightarrow R^d\times R^d,\qquad \mathcal{F}_B = \begin{pmatrix}-\widetilde{W}_1^{\mathsf{t}} & V_1^{\mathsf{t}}\end{pmatrix}:R^d\times R^d\longrightarrow(R^d)^{(\ell)}.$$

Telescoping the four-variable biextension interchange law first over $\mathbf{y}' = [k]_B\mathbf{y}$, $\mathbf{y}'' = \mathbf{y}$ $(1\leq k<\ell)$ and then over $\mathbf{x}' = [k]_B\mathbf{x}$, $\mathbf{x}'' = \mathbf{x}$ $(1\leq k<\ell)$ yields $\Psi_\ell(\mathbf{x},\mathbf{y})^\ell\,\widetilde{\Psi}_\ell([\ell]_B\mathbf{x},\mathbf{y}) = \widetilde{\Psi}_\ell(\mathbf{x},\mathbf{y})^\ell\,\Psi_\ell(\mathbf{x},[\ell]_B\mathbf{y})$. Since $\varphi_1(\mathbf{x}',\mathbf{x}'',\mathbf{y})-1\in(\mathbf{x}')(\mathbf{x}'')(\mathbf{y})$, for every prime $\ell\geq 2$ one has $(\Psi_\ell(\mathbf{x},\mathbf{y})-1)^\ell\in(\mathbf{x})^{2\ell}(\mathbf{y})^\ell$ and $(\widetilde{\Psi}_\ell(\mathbf{x},\mathbf{y})-1)^\ell\in$

$(\mathbf{x})^{\ell}(\mathbf{y})^{2\ell}$; comparing $(\mathbf{x}^{(\ell)},\mathbf{y}^{(\ell)})$-coefficients modulo $(\mathbf{x})^{2\ell}+(\mathbf{y})^{2\ell}$ gives $V_1^{\mathsf{t}}\widetilde{W_1}=W_1V_1$, hence $\mathcal{F}_B\circ\mathcal{V}_B=0$. Similarly, for any $g(\mathbf{x},\mathbf{y})\in 1+(\mathbf{x})(\mathbf{y})R[[\mathbf{x},\mathbf{y}]]$ and any prime $\ell\geq 2$, $g(\mathbf{x},\mathbf{y})^{\ell}-1\in(\mathbf{x})^{\ell}(\mathbf{y})^{\ell}$ vanishes both modulo $(\mathbf{y})^2$ and modulo $(\mathbf{x})^2$. Thus telescoping the left Rosati cocycle modulo $(\mathbf{x})^{2\ell}+(\mathbf{y})^2$, the right Rosati cocycle modulo $(\mathbf{x})^2+(\mathbf{y})^{2\ell}$, the symmetry cocycle $\widetilde{\Psi}_{\ell}(\mathbf{x},\mathbf{y})\,\mathrm{sy}(\mathbf{x},[\ell]_B\mathbf{y})=\mathrm{sy}(\mathbf{x},\mathbf{y})^{\ell}\,\Psi_{\ell}(\mathbf{y},\mathbf{x})$ modulo $(\mathbf{x})^2+(\mathbf{y})^{2\ell}$, and the φ_1,φ_2 transition cocycles of any formal biextension isomorphism $(\varphi,\psi,\lambda):B\xrightarrow{\sim}B'$ gives, for all $a\in\mathcal{O}_E$:

$$\rho_H(B)(a)\circ\mathcal{V}_B=\mathcal{V}_B\circ\rho_B(a)^{(\ell)},\qquad \mathcal{F}_B\circ\rho_H(B)(a)=(\rho_B(\bar a)^{(\ell)})^{\mathsf{t}}\circ\mathcal{F}_B,$$
$$\mathcal{F}_B=\mathcal{V}_B^{\mathsf{t}}\begin{pmatrix}\mathrm{bil}(\mathrm{sy})_{1,1}^{\mathsf{t}} & I_d\\ -I_d & 0\end{pmatrix},\qquad \mathrm{transH}(\varphi,\psi,\lambda)\circ\mathcal{V}_{B'}=\mathcal{V}_B\circ D\varphi(0)^{(\ell)},$$
$$\mathcal{F}_B\circ\mathrm{transH}(\varphi,\psi,\lambda)=(D\psi(0)^{(\ell)})^{\mathsf{t}}\circ\mathcal{F}_{B'}.$$

On the regular proper integral curve $\widetilde{\Gamma}$ over k', let $\kappa=k'(\widetilde{\Gamma})$ be its function field and $\mathcal{O}_i=\mathcal{O}_{\widetilde{\Gamma},i}$ ($i\in\widetilde{\Gamma}$ closed) its discrete valuation rings with residue fields k_i. Base-changing the stalk germs and specialisation isomorphisms of S from Lemma 35.8(2) along $\operatorname{Spec}\kappa\to X$ and $\operatorname{Spec}\mathcal{O}_i\to X$ yields formal biextension germs B_i over $\mathcal{O}_i$ with reductions B_{k_i} over k_i and compatible transitions to B_κ. Fix a closed point i and an algebraic closure $\bar k_i$ of k_i, with geometric point $\bar x:\operatorname{Spec}\bar k_i\to X$ and extension $0\to T\to G_{\bar x}\xrightarrow{q_{\bar x}}B_{\mathrm{ab}}\to 0$ as in Lemma 35.8(d). In formal parameters $(\mathbf{x}_T,\mathbf{x}_B)$ adapted to $T\simeq\mathbb{G}_{m,\bar k_i}^{d_T}$ and B_{ab} with trivialisation pulled back from $\Lambda(M)$ on $\widehat{B}_{\mathrm{ab}}\times\widehat{B}_{\mathrm{ab}}$, one has $[\ell]_T(\mathbf{x}_T)=\mathbf{x}_T^{(\ell)}$ so $V_{1,T}=I_{d_T}$, while W_1 vanishes on T-variables; hence $\mathrm{rk}_{\bar k_i}\mathcal{V}_{B_{\bar x}}=d_T+\mathrm{rk}_{\bar k_i}\mathcal{V}_{B_{\mathrm{ab}}}$. Under the identification $\mathrm{Lie}(B_{\mathrm{ab}})\times\mathrm{Lie}(B_{\mathrm{ab}})^{\vee}\simeq\mathrm{Lie}(E(B_{\mathrm{ab}}))$ of Lemma 35.8, evaluating $[\ell]_{E(B_{\mathrm{ab}})}$ on $(\mathbf{x}_B=\zeta w,1)$ over $\bar k_i[\zeta]/(\zeta^{2\ell})$ (with $\varepsilon=\zeta^{\ell}$) identifies $\mathcal{V}_{B_{\mathrm{ab}}}$ with the map $\mathrm{Lie}(B_{\mathrm{ab}})^{(\ell)}\to\mathrm{Lie}(E(B_{\mathrm{ab}}))$ induced by Verschiebung on $\mathbb{D}(B_{\mathrm{ab}}[\ell^{\infty}])(W(\bar k_i))^{\vee}/\ell$ [161, Ch. II, §15], which is injective of rank $d_B=\dim B_{\mathrm{ab}}$. Hence $\mathrm{rk}_{k_i}\mathcal{V}_{B_{k_i}}=d_T+d_B=d$, and the matrix relation between $\mathcal{F}_{B_{k_i}}$ and $\mathcal{V}_{B_{k_i}}^{\mathsf{t}}$ gives $\mathrm{rk}_{k_i}\mathcal{F}_{B_{k_i}}=d$. By Nakayama's lemma over each discrete valuation ring $\mathcal{O}_i$ and compatibility with transH, the maps $\mathcal{V}_{B_i}$ and $\mathcal{F}_{B_i}$ glue to an $\mathcal{O}_E$-equivariant exact sequence of vector bundles on $\widetilde{\Gamma}$,

$$0\longrightarrow F_{\widetilde{\Gamma}}^{*}\mathcal{L}_{\widetilde{\Gamma}}\xrightarrow{\mathcal{V}}\mathcal{H}_{\widetilde{\Gamma}}\xrightarrow{\mathcal{F}}F_{\widetilde{\Gamma}}^{*}\big(\mathcal{L}_{\widetilde{\Gamma}}^{\vee}(\mathrm{conj})\big)\longrightarrow 0,$$

where $F_{\widetilde{\Gamma}}:\widetilde{\Gamma}\to\widetilde{\Gamma}$ is the absolute ℓ-power Frobenius morphism and $\mathcal{L}_{\widetilde{\Gamma}}=\nu^{*}(\mathcal{L}|_{\Gamma})$, $\mathcal{H}_{\widetilde{\Gamma}}=\nu^{*}(\mathcal{H}|_{\Gamma})$.

For every $a\in\mathcal{O}_E$, the characteristic polynomial $P_a(T)\in\mathbf{Z}[T]$ of multiplication by a on $\mathcal{O}_E$ splits in $\Omega[T]$ as $\prod_{\sigma':E\hookrightarrow\Omega}(T-\sigma'(a))$; since $\sigma'(a)\in O$ and $O\hookrightarrow\Omega$ is injective, $P_a(T)$ splits in $O[T]$ and hence over the subfield $k_1\subseteq k_0\subseteq k'$ generated over $\mathbf{F}_{\ell}$ by the reductions modulo ϖ of $\{\sigma'(b):\sigma':E\hookrightarrow\Omega,\ b\in\mathcal{O}_E\}$. Because $\mathcal{O}_E$ is finite over $\mathbf{Z}$, k_1 is a finite field. Thus $\mathcal{O}_E\otimes_{\mathbf{Z}}k_1=\prod_{\nu=1}^{s}A_\nu$ decomposes into local Artin k_1-algebras $A_\nu=\epsilon_\nu(\mathcal{O}_E\otimes_{\mathbf{Z}}k_1)$ with residue fields $A_\nu/\mathfrak{m}_\nu=k_1$, and $\mathrm{id}_{\mathcal{O}_E}\otimes F_{k_1}$ permutes the primitive idempotents by $\epsilon_\nu\mapsto\epsilon_{\pi(\nu)}$. For any $\mathcal{O}_E$-bundle $\mathcal{E}$ on $\widetilde{\Gamma}$ we have $\epsilon_{\pi(\nu)}F_{\widetilde{\Gamma}}^{*}\mathcal{E}=F_{\widetilde{\Gamma}}^{*}(\epsilon_\nu\mathcal{E})$, while $\deg_{\widetilde{\Gamma}}(F_{\widetilde{\Gamma}}^{*}\mathcal{E})=\ell\deg_{\widetilde{\Gamma}}(\mathcal{E})$ because $F_{\widetilde{\Gamma}}^{*}$ raises transition determinants to the ℓ-th power at every valuation of κ. Comparing the conjugate sequence above with the pulled-back Hodge sequence $0\to\mathcal{L}_{\widetilde{\Gamma}}^{\vee}(\mathrm{conj})\to\mathcal{H}_{\widetilde{\Gamma}}\to\mathcal{L}_{\widetilde{\Gamma}}\to 0$ of Lemma 35.8(3) gives

$$\deg_{\widetilde{\Gamma}}(\epsilon_{\pi(\nu)}\mathcal{H}_{\widetilde{\Gamma}})=\ell\deg_{\widetilde{\Gamma}}(\epsilon_\nu\mathcal{L}_{\widetilde{\Gamma}})+\ell\deg_{\widetilde{\Gamma}}\big(\epsilon_\nu\mathcal{L}_{\widetilde{\Gamma}}^{\vee}(\mathrm{conj})\big)=\ell\deg_{\widetilde{\Gamma}}(\epsilon_\nu\mathcal{H}_{\widetilde{\Gamma}}).$$

Iterating around the finite π-cycle of ν yields $\deg_{\widetilde{\Gamma}}(\epsilon_\nu\mathcal{H}_{\widetilde{\Gamma}})=0$ for every ν. Let A_{ν_0} be the factor through which the residual homomorphism $\bar\sigma:\mathcal{O}_E\to O\to k_1$ factors, and choose a k_1-composition series $A_{\nu_0}=I_0\supset I_1\supset\cdots\supset I_{e_0}=0$ ($e_0=\dim_{k_1}A_{\nu_0}\geq 1$), so each quotient I_{t-1}/I_t is isomorphic to k_1 with $\mathcal{O}_E$ acting via $\bar\sigma$. By Lemma 35.8(3), $\epsilon_{\nu_0}\mathcal{H}_{\widetilde{\Gamma}}$ is locally free of rank r over $A_{\nu_0}\otimes_{k_1}\mathcal{O}_{\widetilde{\Gamma}}$, hence flat over A_{ν_0}; tensoring the composition series over A_{ν_0} with $\epsilon_{\nu_0}\mathcal{H}_{\widetilde{\Gamma}}$ filters it by e_0 subbundles with successive quotients isomorphic to $\nu^{*}(\mathcal{H}\otimes_{\mathcal{O}_E\otimes_{\mathbf{Z}}\mathcal{O}_X,\sigma\otimes 1}\mathcal{O}_X)$. Hence $e_0\deg_{\widetilde{\Gamma}}(\nu^{*}\mathscr{D}_\sigma)=\deg_{\widetilde{\Gamma}}(\epsilon_{\nu_0}\mathcal{H}_{\widetilde{\Gamma}})=0$ and $\deg_{\widetilde{\Gamma}}(\nu^{*}\mathscr{D}_\sigma)=0$. □

Proof of 35.10

Proof. Let $O=\mathcal{O}_{F',w}\subset F'$ be the valuation ring at w, with uniformiser $\varpi\mid\ell$ and residue field $k_0=O/\varpi O$, and let (P,Q) be the signature of Z with $r=P+Q$. If $r=0$, both Lie eigenspaces are zero, $Q=0$, and $s\in(F')^{\times}$ is a nonzero constant with $\rho_w=v_w(s)$, vanishing outside finitely many places. Assume now that $r\geq 1$, and put $g=\tfrac{1}{2}[E:\mathbf{Q}]r$. By the definition of a good cover, the covering level is $M'=g_0g_1 15^{e}$

where $g_0, g_1 > 3$ are the two distinct odd base-level primes, and F' contains all complex embeddings of E. Choose $n \in \{g_0, g_1\}$ with $n \neq \ell$; then $n \geq 5$ is odd and invertible in O, and $\zeta_n \in F'$ (for a general good cover without $\zeta_n \in F'$, replacing F' by $F'(\zeta_n)$ at a place $w' \mid w$ preserves integrality of Z because F' is relatively algebraically closed in Ω and yields $\rho_w = \rho_{w'}/e(w'/w) \in \mathbf{Q}$). Since O is integrally closed in F', every embedding $\sigma : E \hookrightarrow \Omega$ into the function field Ω of Z factors through F' and maps $\mathcal{O}_E$ into O, and $\zeta_n \in O$.

Base-changing the integral Siegel toroidal compactification of Definition 35.5 at odd full level n [176, Chs. IV–V] along $\mathbf{Z}[\zeta_n, 1/n] \to O$ yields a projective smooth O-scheme $\mathcal{S}$ containing $\mathcal{A}_{g,n,O}$ as a fibrewise dense open subscheme equal to the abelian locus of the universal semiabelian scheme $G_{\mathcal{S}} \to \mathcal{S}$. Because $\mathcal{S}$ is regular and $G_{\mathcal{S}}^{\times sm} \to \mathcal{S}$ $(m \geq 1)$ is smooth with geometrically integral fibres, every prime divisor of $G_{\mathcal{S}}^{\times sm}$ outside $\mathcal{A}_{g,n,O}$ is the preimage of a boundary divisor of $\mathcal{S}$; extending a Weil divisor on the regular scheme $G_{\mathcal{S}} \times_{\mathcal{S}} G_{\mathcal{S}}$ and twisting along $(e_{\mathcal{S}}, e_{\mathcal{S}})$ therefore extends the universal symmetric Poincaré biextension on $\mathcal{A}_{g,n,O}$ uniquely to a symmetric bi-rigidified commutative biextension $\mathcal{P}_{\mathcal{S}}$ on $G_{\mathcal{S}} \times_{\mathcal{S}} G_{\mathcal{S}}$. At any geometric point $\bar{s} : \operatorname{Spec} \bar{k} \to \mathcal{S}$ with Chevalley extension $0 \to T \to G_{\mathcal{S},\bar{s}} \xrightarrow{q_{\bar{s}}} B \to 0$, choose a complete discrete valuation ring V_0 with residue field $\bar{k}$ and characteristic-zero fraction field K_0 mapping into $\mathcal{A}_{g,n,F'}$. By Grothendieck's duality and biextension extension over V_0 [180, §7], [176, Ch. III, §7], the generic principal polarisation extends to an isomorphism $G_{V_0} \xrightarrow{\sim} G_{V_0}^t$ inducing a principal polarisation $\lambda_B : B \xrightarrow{\sim} B^\vee$ and extending the generic Poincaré biextension to a bi-rigidified biextension on $G_{V_0} \times_{V_0} G_{V_0}$ whose closed-fibre restriction is $(q_{\bar{s}} \times q_{\bar{s}})^*(\mathrm{id}_B \times \lambda_B)^*\mathcal{P}$; uniqueness of rigidified extension on the regular scheme $G_{V_0} \times_{V_0} G_{V_0}$ with irreducible closed fibre identifies $\mathcal{P}_{\mathcal{S}}|_{G_{\bar{s}} \times G_{\bar{s}}}$ with $(q_{\bar{s}} \times q_{\bar{s}})^*(\mathrm{id}_B \times \lambda_B)^*\mathcal{P}$.

On the moduli open $U \subset Z$ over F', the fixed Weil pairing and $\zeta_n \in F'$ turn the level-n half-level data into a full symplectic trivialisation of $A_U[n]$. Forgetting the $\mathcal{O}_E$-action and prime-to-n level gives a quasi-finite F'-morphism $f_U : U \to \mathcal{A}_{g,n,F'}$, which is proper (and hence finite) by the valuative criterion: over any characteristic-zero discrete valuation ring R_0 over F' on which A_{K_0} extends to a principally polarised abelian scheme A_{R_0}, the $\mathcal{O}_E$-action extends uniquely by Definition 35.5 [167, Prop. 3.3.1.5] and the K_0-rational half-level sections of the finite étale subgroup scheme of $A_{R_0}[M']$ extend uniquely across R_0 by properness of $A_{R_0} \to \operatorname{Spec} R_0$, remaining disjoint on the closed fibre. Let $\overline{W} \subset \mathcal{S}$ be the schematic closure of $f_U(U)$. Choose the normal integral PEL model with its level and principal theta data, using the prime-to-ℓ part of M' and an auxiliary odd level prime to ℓ. In the integral-model construction, first take the full-level model with theta frames, then form the normalised level change to the prescribed level; choose compatible toroidal fans and a common normal projective refinement over $\overline{W}$, unchanged over U, and call it X [188, Ths. 7.14 and 11.2]. Retain the pulled-back semiabelian scheme, rigidified polarisation and theta extension in this construction. We construct the theta generator before using it. At a point of the normal integral model, pass faithfully flatly to the completed local ring and a finite étale local extension splitting the torus. These rings are Noetherian normal domains. Apply the complete-local degeneration construction to the polarisation and a nonzero generic principal theta section. On its abelian quotient, the direct image of the principal theta line is free of rank one and commutes with base change. Choose its generator b_0 with nonzero reduction. The cubical period relations extend b_0 uniquely to a Fourier family (b_α). Their integrality and period positivity give integral b_α, with b_α divisible by an element of the maximal ideal for $\alpha \neq 0$. Summing on each infinitesimal thickening and using the compatible uniformisation gives an integral section θ; on the closed fibre it is the pullback of b_0, so it is nonzero. The same period relations show that every integral theta section is a unique scalar multiple of θ, and this multiple is nonzero on the closed fibre exactly when its scalar is a unit. To descend existence from the faithfully flat local extension, express θ as a finite linear combination of base sections by flat base change. If all those sections vanished on the closed fibre, their combination would vanish too. One base section therefore has nonzero reduction; after shrinking, smoothness makes its nonvanishing locus meet every fibre. This supplies the local generator on the normal model. Its pullback to the level change or common refinement remains nonzero on every fibre. This section generates the theta direct image locally on the normal model. Indeed, every other section is a rational scalar multiple of it on the generic fibre. A pole of that scalar at a height-one point would give a pole of the multiple where the generator has nonzero reduction. Normality therefore makes the scalar regular. The generator remains nonzero on each fibre after pullback. Prime-to-ℓ torsion sections extend on the abelian locus, and semiabelian uniqueness supplies the good-reduction extension property. These properties, and the formal uniformisations constructed from them below, persist on the common refinement. Then $\pi_X : X \to \operatorname{Spec} O$ is integral, normal, flat and projective, and finiteness of f_U identifies the abelian locus of $G = G_{\mathcal{S}} \times_{\mathcal{S}} X$ in

$X_{F'}$ with the dense open $U \subset X_{F'}$. Pull $\mathcal{P}_{\mathcal{S}}$ back to $\mathcal{P}_X$ on $G \times_X G$. Definition 35.5 [167, Prop. 3.3.1.5] extends the generic $\mathcal{O}_E$-action on A_U uniquely to G. For each $a \in \mathcal{O}_E$, the generic Rosati isomorphism $\mathrm{ad}_{a,U}$ is a rational section of the invertible sheaf $\mathcal{M} = (a \times \mathrm{id}_G)^*\mathcal{P}_X^{-1} \otimes (\mathrm{id}_G \times \bar{a})^*\mathcal{P}_X$ equal to 1 along (e,e). At every codimension-one point η_D of the normal scheme $G \times_X G$ outside $G_U \times_U G_U$, η_D is the generic point of the smooth fibre over a height-one prime $\mathfrak{p}$ of X; since $\mathcal{O}_{X,\mathfrak{p}}$ is a discrete valuation ring, $G_{\mathcal{O}_{X,\mathfrak{p}}} \times_{\mathcal{O}_{X,\mathfrak{p}}} G_{\mathcal{O}_{X,\mathfrak{p}}}$ is regular, and evaluating along (e,e) forces $\mathrm{ord}_{\eta_D}(\mathrm{ad}_{a,U}) = 0$. Hence $\mathrm{ad}_{a,U}$ extends to a bi-rigidified isomorphism $\mathrm{ad}_a : (a \times \mathrm{id}_G)^*\mathcal{P}_X \xrightarrow{\sim} (\mathrm{id}_G \times \bar{a})^*\mathcal{P}_X$ across $G \times_X G$. Since the quadratic host $\mathbf{Q}(\sqrt{-d}) \subseteq E$ has odd discriminant $-d \equiv 1 \pmod 4$, the element $w_0 = (1+\sqrt{-d})/2$ lies in $\mathcal{O}_E$ and satisfies $w_0 + \bar{w}_0 = 1$; define $L_{\mathrm{PP}} = (\mathrm{id}_G, w_0)^*\mathcal{P}_X$ on G, rigidified along e. The biextension partial-group laws give $([-1] \times [-1])^*\mathcal{P}_X \simeq \mathcal{P}_X$, making L_{PP} symmetric, and the partial-group laws, symmetry and ad_{w_0} identify its Mumford biextension as

$$\Lambda(L_{\mathrm{PP}}) \simeq (\mathrm{id}_G \times w_0)^*\mathcal{P}_X \otimes (w_0 \times \mathrm{id}_G)^*\mathcal{P}_X \simeq \big(\mathrm{id}_G \times (w_0 + \bar{w}_0)\big)^*\mathcal{P}_X = \mathcal{P}_X,$$

making L_{PP} cubical with $L_{\mathrm{PP}} \simeq (\mathrm{id}_G, w_0)^*\Lambda(L_{\mathrm{PP}})$. On any geometric special-fibre point $\bar{x} : \operatorname{Spec}\bar{k} \to X$ with $0 \to T \to G_{\bar{x}} \xrightarrow{q_{\bar{x}}} B \to 0$, $M = (\mathrm{id}_B, w_{0,B})^*(\mathrm{id}_B \times \lambda_B)^*\mathcal{P}$ on B satisfies $L_{\mathrm{PP}}|_{G_{\bar{x}}} \simeq q_{\bar{x}}^*M$ and $\Lambda(M) \simeq (\mathrm{id}_B \times \lambda_B)^*\mathcal{P}$, inducing λ_B.

Claim 1. This model satisfies the formal uniformisation hypothesis of Lemma 35.8. Let R be any complete Noetherian normal local domain of characteristic zero mapping to X, with nonempty pulled-back abelian locus and split closed fibre. The abelian locus is open, so it contains the generic point of $\operatorname{Spec} R$. Choose a local theta generator around the image of the closed point and pull its section back to the semiabelian family over R. It is nonzero on every fibre. Apply the complete-local degeneration construction to this polarised semiabelian family. It produces the Raynaud extension, its character and period lattices, and the symmetric cubical period trivialisation. Functoriality and uniqueness extend the $\mathcal{O}_E$-action to this data and identify its adjoint action with the polarisation. The uniformisation identifies the generic abelian scheme with the period quotient. Applying multiplication by each integer to this quotient identifies the induced torsion maps with the torsion of the original family, so they are isomorphisms. At a height-one point of the abelian locus let $m(y)$ be the order of the cubical period section for a lattice vector y, in a local frame. Choose a prime n invertible there. Good reduction makes the n^j-torsion rational over an unramified extension for each j. Under the torsion identification, taking n^j-division points in the period pairing makes $m(y+y') - m(y) - m(y')$ divisible by n^j. The difference is therefore zero. Symmetry gives $m(-y) = m(y)$, so additivity gives $m(y) = 0$. Normality now makes each period section a unit on the abelian locus. This proves the required assertion for every R. The split-fibre property follows from splitting the character lattice after a finite separable residue-field extension. Lemma 35.8 now furnishes the coherent $\mathcal{O}_E$-module $\mathcal{H}$ on X, locally free of rank r over $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathcal{O}_X$, in the locally split $\mathcal{O}_X$-exact sequence $0 \to \mathcal{L}^\vee(\mathrm{conj}) \to \mathcal{H} \to \mathcal{L} \to 0$, where $\mathcal{L} = \mathrm{Lie}(G/X)$. Form the invertible sheaf

$$\mathscr{D}_\tau = \det\big(\mathcal{H} \otimes_{\mathcal{O}_E \otimes_{\mathbf{Z}} \mathcal{O}_X,\, \tau \otimes 1} \mathcal{O}_X\big).$$

Over $X_{F'}$, splitting $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathcal{O}_{X_{F'}}$ identifies $\mathscr{D}_\tau|_{X_{F'}}$ canonically with $\det(\mathcal{L}_{X_{F'},\tau}) \otimes \det(\mathcal{L}_{X_{F'},\bar{\tau}})^{-1}$, extending the line $\mathcal{L}_\tau$ of Lemma 35.6 across $X_{F'}$. Fix a complex embedding $F' \hookrightarrow \mathbf{C}$ and choose an embedded characteristic-zero resolution $\widetilde{X}_{\mathbf{C}} \to X_{\mathbf{C}}$ which is an isomorphism over $U_{\mathbf{C}}$ and has simple normal crossings boundary $\widetilde{X}_{\mathbf{C}} \setminus U_{\mathbf{C}}$ [165]. On $\mathcal{S}_{\mathbf{C}}$, the logarithmic canonical extension $\mathscr{H}_{\mathrm{can}}$ of de Rham homology [176, Ch. VI, Th. 1.1] has unipotent monodromy around the toroidal boundary (as the odd level $n \geq 5$ is neat) and sits in an exact sequence $0 \to \mathrm{Lie}(G_{\mathcal{S},\mathbf{C}}/\mathcal{S}_{\mathbf{C}})^\vee \to \mathscr{H}_{\mathrm{can}} \to \mathrm{Lie}(G_{\mathcal{S},\mathbf{C}}/\mathcal{S}_{\mathbf{C}}) \to 0$; because $\widetilde{X}_{\mathbf{C}} \setminus U_{\mathbf{C}}$ is a simple normal crossings divisor and the monodromy is unipotent, the pullback of $\mathscr{H}_{\mathrm{can}}$ to $\widetilde{X}_{\mathbf{C}}$ coincides with Deligne's canonical extension of $H_1^{\mathrm{dR}}(A_{U_{\mathbf{C}}}/U_{\mathbf{C}})$, so the flat $\mathcal{O}_E$-action on $U_{\mathbf{C}}$ extends uniquely to it compatibly with the pulled-back Hodge sequence, and the determinant of its τ-summand is also canonically isomorphic to $\det(\mathcal{L}_{\widetilde{X}_{\mathbf{C}},\tau}) \otimes \det(\mathcal{L}_{\widetilde{X}_{\mathbf{C}},\bar{\tau}})^{-1}$. Around any boundary point of $\widetilde{X}_{\mathbf{C}}$, the local canonical-extension frame is obtained from a multivalued flat basis by applying the exponential of a linear combination of the commuting nilpotent monodromy logarithms N_i; since the flat $\mathcal{O}_E$-action commutes with N_i, each N_i preserves the τ-summand and has trace zero there, so $\det \exp(\sum_i z_i N_i|_\tau) = 1$. Thus the horizontal generator s of Lemma 35.6 extends to a nowhere-vanishing holomorphic section on $\widetilde{X}_{\mathbf{C}}$, algebraizes by projective GAGA [174] (Definition 35.5), descends along $\widetilde{X}_{\mathbf{C}} \to X_{\mathbf{C}}$ by normality of $X_{\mathbf{C}}$, and by faithful flatness generates $\mathscr{D}_\tau|_{X_{F'}}$ everywhere on $X_{F'}$. Hence $\mathrm{div}_X(s)$ is supported on the special fibre X_{k_0}.

Lean: `GlobalTheory.CompanionTheory.Ar.Flat.A0.flat_zero_of_resolution_gaga` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.FlatZeroFromResolutionAndGaga`, lines 39–56)
Lean: `GlobalTheory.CompanionTheory.Ar.Flat.GenAgree.generic_agree` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.HodgeLatticeGenericAgree`, lines 225–229)

Write $\operatorname{div}_X(s)=\sum_i n_iX_i$ and $\operatorname{div}_X(\varpi)=\sum_i m_iX_i$ $(m_i\ge 1)$ on the irreducible components X_i of X_{k_0}, and put $a/M=\min_i(n_i/m_i)$ with integers $M>0$ and a. By normality of X, s^M/ϖ^a is a global regular section of $\mathscr{D}_\tau^{\otimes M}$ whose effective Cartier divisor omits at least one component X_i. On the projective k_0-variety $X_{i,\mathrm{red}}$, the restricted effective Cartier divisor has degree zero on the normalisation of every complete curve after extending k_0 to $\bar{k}_0$, by Lemma 35.9. If that restriction were nonempty, choose an irreducible component Y of $(X_{i,\mathrm{red}})_{\bar{k}_0}$ meeting the divisor at a closed point p: when $\dim X_i=0$ any nonzero section on a point is invertible; when $\dim X_i=1$ take $\Gamma=Y$; and when $\dim X_i\ge 2$ intersect Y with $\dim X_i-1$ hyperplanes through p meeting the divisor support in dimension 0 (by prime avoidance over the infinite field $\bar{k}_0$) and take a reduced irreducible component Γ through p (which has dimension 1 by Krull's principal ideal theorem). In each positive-dimensional case $\Gamma\subset X_{\bar{k}_0}$ is a complete integral curve meeting the divisor properly at p with positive degree, a contradiction. Every omitted component is therefore disjoint from the divisor support. By Stein factorisation [181, III$_1$, Th. 4.3.1], $\pi_{X,*}\mathcal{O}_X=O$ because F' is relatively algebraically closed in Ω and O is normal, so X_{k_0} is connected; hence $\operatorname{div}_X(s^M/\varpi^a)=0$ and s^M/ϖ^a generates $\mathscr{D}_\tau^{\otimes M}$ everywhere on X.

Let y be any integral frame on Z over a test trait $\operatorname{Spec}V$ of ramification index $e=v_y(\varpi)$ over w, with fraction field K and semiabelian extension $G_V\to\operatorname{Spec}V$. By the valuative criterion of properness, $\operatorname{Spec}K\xrightarrow{y}U\hookrightarrow X$ extends uniquely to an O-morphism $\tilde{y}:\operatorname{Spec}V\to X$, and Definition 35.5 [167, Prop. 3.3.1.5] applied to id_{A_K} identifies $\tilde{y}^*G\xrightarrow{\sim}G_V$ compatibly with $\mathcal{O}_E$ and hence $\tilde{y}^*\mathcal{L}\xrightarrow{\sim}\operatorname{Lie}(G_V/V)$. Injectivity of $\operatorname{Lie}(G_V/V)\hookrightarrow\operatorname{Lie}(A_K/K)$ identifies the integral τ- and $\bar{\tau}$-eigenframes over V with bases of $\operatorname{Lie}(G_V/V)\cap(\operatorname{Lie}(A_K/K))_\tau$ and $\operatorname{Lie}(G_V/V)\cap(\operatorname{Lie}(A_K/K))_{\bar{\tau}}$. Pulling $0\to\mathcal{L}^\vee(\mathrm{conj})\to\mathcal{H}\to\mathcal{L}\to 0$ back along $\tilde{y}$ gives an integral Hodge sequence over V in which $s(y)$ has valuation ea/M relative to a V-generator of $\tilde{y}^*\mathscr{D}_\tau=\det\big((\tilde{y}^*\mathcal{H})(\tau)\big)$. Applying Lemma 35.7 over V yields $v_y(s(y))=e\rho_w$ with

$$\rho_w=\frac{a}{M}-\sum_{\sigma\in S\setminus\{\tau\}}h_\sigma\,v_w(a_\sigma-a_\tau)\in\mathbf{Q},$$

since each a_σ lies in the integer ring of F'_w so $v_y(a_\sigma-a_\tau)=e\,v_w(a_\sigma-a_\tau)$, and $h_\sigma\in\{P,Q\}$ depends only on the generic signature. (Comparing with any other F'-rational horizontal generator $s'=cs$ shifts ρ_w by $+v_w(c)$ for $c\in(F')^\times$.)
Lean: `GlobalTheory.CompanionTheory.Ar.Flat.CPoint.value_at_point` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.EigenLinePAdicFrame`, lines 72–338)

Choose a closed point y_1 of U over a finite extension K_1/F'. Spreading out the abelian variety A_{y_1}, its $\mathcal{O}_E$-action, bases of its two Lie eigenspaces (with their inverse transition matrices) and the nonzero value $s(y_1)\in K_1^\times$ after inverting $\operatorname{disc}E$ and finitely many denominators produces an integral frame over $\mathcal{O}_{K_1,w_1}$ with $v_{w_1}(s(y_1))=0$ at all but finitely many places w_1 of K_1, giving $\rho_w=0$ outside finitely many places w of F'. Moreover, at every good rational prime $\ell\notin\mathcal{S}_{\mathrm{bad}}=\{\ell\text{ prime}:\ell\mid 2dD_EM'\}$, all three hypotheses of Lan's good-reduction integral PEL theorem [167, Th. 6.4.1.1] hold directly: E is unramified at ℓ (as $\ell\nmid D_E$), the Hermitian PEL lattice is self-dual at ℓ (as $\ell\nmid 2dD_E$), and the level M' is prime to ℓ. Passing from the normalised Siegel closure X to the resulting smooth integral PEL toroidal compactification $X'\to\operatorname{Spec}O$ (supplied by Definition 35.5(1)) with reduced split boundary fibres (every irreducible component X'_i of X'_{k_0} has multiplicity $m_i=1$ in $\operatorname{div}_{X'}(\varpi)=\sum_i m_iX'_i$), relaying the horizontal generator s between X and X' over their common generic open U (where the two integral horizontal generators agree up to a scalar in $(F')^\times$), and applying the preceding divisor argument on X' yields $a/M=\min_i(n_i/m_i)=\min_i n_i\in\mathbf{Z}$. Since each $v_w(a_\sigma-a_\tau)$ is also an integer (and vanishes because $\mathcal{O}_E\otimes_{\mathbf{Z}}O$ is étale over O), the formula above gives $\rho_w\in\mathbf{Z}$ for all $w\nmid 2dD_EM'$.

Finally, the row-metric defect is Q times the difference of the orders of the extended different $\tau(\mathfrak{D}_{E/\mathbf{Q}})$ and of d. Since $E/\mathbf{Q}$ is Galois, taking the ideal norm identifies $v_w(\tau(\mathfrak{D}_{E/\mathbf{Q}}))/v_w(\ell)$ at $w\mid\ell$ with $v_\ell(|\operatorname{disc}E|)/[E:\mathbf{Q}]$, while the order of d is $v_w(\ell)\mathbf{1}_{\ell=d}$. Subtracting this defect gives the stated coefficient, whose sum against $\log Nw/[F':\mathbf{Q}]$ contributes $Q(\log d-\log\mathrm{rd}(E))$.
Lean: `GlobalTheory.FlatLine.flDefect_vw_eq` (`Development.AbelianVarietyTheory.Definitions.Foundations.ExistsFlatSecRankZero`, lines 188–210)
Lean: `GlobalTheory.FlatLine.sum_flDefect_vw` (`Development.AbelianVarietyTheory.Definitions.Foundations.ExistsFlatSecRankZero`, lines 219–223)

□

Proof of 35.11

Proof. We first assemble the packages $\mathcal{H}_X$ and $\mathcal{H}_H$ of Definition 35.5 directly from the stated hypotheses of Theorem 35.11. For $\mathcal{H}_X$, the prescribed-fan compactification, semiabelian Lie-bundle seminegativity, and point separation are among the theorem's hypotheses; the cuspidal Green parametrisation follows from Theorem 32.31 and the regular character construction; and the Fubini–Study formula for the Chow height follows from the polarisation, Chow form, and arithmetic intersection calculus in $\mathcal{H}_P$ (Definition 35.5). For $\mathcal{H}_H$, the Siegel toroidal compactification and the arithmetic degree formula are among the theorem's hypotheses, while the prescribed-fan boundary charts and their compatibility with the Siegel charts follow from the stated compactification hypotheses. The section comparison uses the vanishing, extension, algebraisation, and nef intersection statements of $\mathcal{H}_P$, together with its Chow form and the hyperplane-section formula; moreover, toroidal nonvanishing from the compactification models in $\mathcal{H}_H, \mathcal{H}_X$ supplies the nonzero degree-one section w for small-section injectivity below, while Definition 35.5 and the compactification hypotheses furnish the Gram frames, finite generation of integral cusp sections, cusp spanning (deep goodness), and component nonvanishing.

We record the two volume-comparison inputs invoked in Proposition 35.94. First, for the self-intersection comparison, let $p : X \to \operatorname{Spec} \mathcal{O}_K$ be an integral arithmetic projective model of a D-dimensional image of a component Z over a totally complex number field K, equipped with the integral image line $\overline{L}$ generated by the section lattice W over $\mathcal{O}_K$, with the Fubini–Study quotient metric induced by the conjugation-invariant family of Gram norms on the section space, generic image degree $\Delta_Z > 0$, and normalised Fubini–Study Chow height $h_{\mathrm{FS}} = h_{\mathrm{FS}}(F_{\mathrm{Ch}})/((D+1)\Delta_Z)$. For any constant metric shift $t \in \mathbf{R}$, the arithmetic top self-intersection $a = I(\overline{L}_t)$ of the shifted line $\overline{L}_t$ satisfies the exact identity $a = [K : \mathbf{Q}](D+1)\Delta_Z(h_{\mathrm{FS}} + t)$, hence in particular $a \le [K : \mathbf{Q}](D+1)\Delta_Z(h_{\mathrm{FS}} + t)$. Indeed, the Fubini–Study Chow-height formula in $\mathcal{H}_X$ identifies the unshifted self-intersection $I(\overline{L})$ with $[K : \mathbf{Q}](D+1)\Delta_Z h_{\mathrm{FS}}$, and the constant-shift law of the arithmetic intersection calculus in $\mathcal{H}_P$ (item 4 of Definition 35.5) shifts $I(\overline{L})$ by $[K : \mathbf{Q}](D+1)\Delta_Z t$; this matches the generating-frame comparison $h' \ge h_{\mathrm{FS}} - (\log n_1 + 1)/2$ used in the proof of Proposition 35.35.
Lean: `ChowFormHeightComparison₂.chow_form_height_comparison` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.HodgeBundleWeights.ChowFormHeightComparison`, lines 11–12)

Second, for the section-injection comparison on a component Z, write X for the projective image model, $\nu : X^\nu \to X$ for its normalisation, and A for its Hermitian line. Choose an integral degree-one section w of nonzero generic value g_w on Z, and let ψ be the embedding of the component function field in that of X. A small section s of degree $8a$, with generic value g_s, gives the rational representative $\psi(g_s)/\psi(g_w)^{8a}$ relative to the image of w^{8a}. By the trait-valuation divisibility criterion of Lemma 35.13 together with the local carrying-order bound (for any integral section s of degree $8a$ whose valuation on every centred test trait of phase ϕ_{fin} and ramification index e_0 is at least $8a$ times the minimum order of the degree-one generators, attained at some l on that trait, with $\psi(g_w)l_c = \psi(g_l)w_c$ allowing the denominator g_l there), writing w_c for the local coordinate of the image of w in a local trivialisation of ν^*A on each standard affine chart c of the normalisation X^ν, the local section coordinate $(\psi(g_s)/\psi(g_w)^{8a})w_c^{8a}$ of the corresponding section of ν^*A^{8a} has nonnegative valuation along every centred test trait and is therefore regular on that chart, gluing across charts to define a global section in $\widetilde{S}_{8a} = H^0(X^\nu, \nu^*A^{8a})$ (mapping small sections injectively into $\widetilde{S}_{8a}$). Equip ν^*A with the pulled-back metric. The pointwise comparison puts the resulting section in the unit ball of $(\nu^*A)^{8a}$ with metric shifted by the small-section threshold. Equal images give equal rational representatives. Cancel the nonzero denominator and use injectivity of ψ and of generic evaluation on the component space. By Lemma 35.12, the logarithmic small-section counts on X^ν and X, with these metrics, differ by $o(a^{D+1})$ as $a \to \infty$, where $D = \dim X_K$. Thus the injection into $\widetilde{S}_{8a}$ gives the arithmetic-volume comparison used with the Chow bound on self-intersection in 35.94.
Lean: `PositivityTheory.Xj7N.Inj.small_sections_injective_of_self_intersection_bound` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.HodgeBundleWeights.SmallSectionsInjectiveOfSelfIntersectionBound`, lines 20–74)

Fix an eligible family and take its common threshold for the weight and cover conditions. Lemmas 35.20 and 35.21 give the quotient weights and good covers. On every good cover at a nonterminal layer, the weight-gap inequalities give weakly decreasing rows and a wall gap at least one for u_M and for the rows at shifts $-c_0, 0, c_0$ (and automatically at positive Hodge shifts such as $2c_0$ since $c_0 > 0$). For all four defining

weights, the Petersson norms admit Gram frames and the integral cusp sections form a finitely generated lattice spanning the cusp space over the coefficient field. The same conclusions hold on each component for u_M. The components form a finite nonempty set. The weight-gap bound after subtracting the canonical weight gives generation of the cusp bundle on the moduli open by Nadel vanishing. Evaluation there gives a nonzero section in the anchored weight. The flat twist carries it to a nonzero section in shift $-c_0$, so the section lattice there has positive rank. This proves deep goodness as defined in 35.3.

Lean: `GlobalTheory.CovTop.DeepGoodAll.deepGood_alt_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.DeepGoodReductionStatement`, lines 23–44)

Lean: `GlobalTheory.CovTop.LowPos.lowered_rank_pos_alt_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RankPosNegativeWeightBundle`, lines 26–48)

To prove the flat-degree law of 35.3, fix a component and a CM embedding. Use the rational generator on this component constructed in Lemma 35.6, together with its reciprocal. Its norm at each infinite place is constant in the frame. Lemma 35.10 computes the common coefficient-place order ρ_w in integral Lie frames, proves finite support and gives the corrected order after subtraction of the different defect. Write ϕ for the value of the generator on a pair of Lie eigenframes. Under changes of basis A, B in the two frames, its value is multiplied by $\det B/\det A$. On the complex uniformisation, choose an E-basis e from the component lattice. Horizontality gives a constant ρ expressing the generator in the flat determinant trivialisation given by e. Put $a = \det \mathrm{Gram}(e) \in E^\times$ and $\eta = \sqrt{-d} \in E$.

Claim. The degree before the different correction is $\frac{P-Q}{2}\log\Omega$. For the two choices of the CM embedding of $\mathbf{Q}(\eta)$, take the CM elliptic curves in the Chowla–Selberg calculation of Lemma 21.22. Write z_+, z_- for their nonzero complex period factors and $n_+(s), n_-(s)$ for the squared norms of their Lie generators at a complex embedding s. For either sign the negative mean of $\log n_\pm(s)$ over the embeddings of its field of definition is

$$h = \gamma + \log(2\pi/\sqrt{d}) - L'(1,\chi)/L(1,\chi),$$

where χ is the host Kronecker character. At a CM point with pure basis in the two eigenspaces, let T express that basis in e, and let g_- be the Gram matrix of its Q-block. Evaluation in the CM Lie frames gives $r = \phi(y_{\mathrm{CM}})\,\tau(\det T)/\tau(\det g_-)$ in a number field k containing the coefficient field and both curve fields. All three factors are nonzero, and the period comparison gives $rz_+^P = \rho z_-^Q$. Each embedding of either curve field has the same number of extensions to k, so both negative means over k still equal h.

Use τ to view E in k. At $s : k \hookrightarrow \mathbf{C}$, the metric adds half the log determinant of the positive Lie Gram matrix and subtracts half that of the negative one. The CM metric in the two blocks contributes $\log n_\pm(s) + \frac{1}{2}\log d$ per row before taking these half determinants. The pure blocks are orthogonal, so their Gram determinants multiply to $\det T\, a\, \overline{\det T}$. Substituting the CM value of ϕ gives the infinite log norm

$$u_s = \log|sr| + \tfrac{1}{2}\log|sa| + \tfrac{P}{2}\big(\log n_+(s) + \tfrac{1}{2}\log d\big) - \tfrac{Q}{2}\big(\log n_-(s) + \tfrac{1}{2}\log d\big).$$

For each rational prime ℓ, choose a CM point and its pure basis anew as follows. Start with a basis x of the component lattice localised at ℓ whose first P vectors have Gram determinant of the full local norm $\exp(-PD_j)$ at every $j : E \hookrightarrow \mathbb{C}_\ell$. Such a basis is obtained by local diagonalisation after multiplying the inverse-different-valued form by a local generator of the different and simultaneous approximation at the primes over ℓ. Replace its first P vectors by the combinations with coefficient matrix $(I_P\ 0) + \ell Y$, Y integral. Approximation at the complex embeddings permits Y to be chosen so that the Hermitian form with matrix $i\,\tau(\mathrm{Gram})$ is positive definite on this block at every τ in the CM type. The Gram entries change by terms of strictly smaller ℓ-adic norm, so expansion of the determinant preserves its full local norm. The Gram inverse then shows that orthogonal projection onto this block preserves the local lattice. Its kernel is a free direct complement over the semilocal coefficient ring; choose a basis of the complement and clear denominators prime to ℓ. Together the two block bases give a local lattice basis x_ℓ. The blocks are orthogonal, and the signature makes the complementary block negative definite. Choose the period map whose pure positive and negative spaces are these two blocks. An integer annihilating the quotient by their global span is prime to ℓ, so the associated CM construction is an isogeny of degree prime to ℓ and supplies the required integral CM frames. Write $t_\ell = \det T_\ell$ for this basis's determinant in e. The complex period relation for this new CM point has the same nonzero scalar r, since $rz_+^P = \rho z_-^Q$; extend the coefficient field to contain the new point. Normalised embedding means are unchanged by this extension. For an embedding j into $\mathbf{C}_\ell$, abbreviate $\log|jv|_\ell$ to $L_j(v)$, and put $D_j = \log\sup_{v\in\mathfrak{D}_{E/\mathbb{Q}}}|jv|_\ell$, restricting j to E when necessary.

The log Gram determinant of the local basis is $-(P+Q)D_j$. Taking determinants in the change of basis therefore gives, for every $j : E \hookrightarrow \mathbf{C}_\ell$,

$$L_j(t_\ell) + L_j(\bar{t}_\ell) + L_j(a) = -(P+Q)D_j.$$

Where e is already a local basis, t_ℓ is a local unit at every place above ℓ. This holds for all but finitely many ℓ. For $j : k \hookrightarrow \mathbf{C}_\ell$, the Q-block has log Gram determinant $-QD_j$, and the scalings from the two CM Lie frames to integral frames both have log absolute value $D_j - L_j(\eta)$. The CM value and the frame transformation law thus give the ordinary finite log norm

$$u_j = L_j(r) - L_j(t_\ell) - QD_j + (Q-P)\big(D_j - L_j(\eta)\big).$$

Conjugation permutes the embeddings of E into $\mathbf{C}_\ell$, and they extend to k with equal multiplicity. In the mean of u_j we may therefore replace $-L_j(t_\ell)$ by $\big((P+Q)D_j + L_j(a)\big)/2$. The degree is minus the sum of the infinite and finite embedding means. The product formula cancels the terms in r and a. The sum of the finite means of D_j is $-\log \mathrm{rd}(E)$, by the norm of the different, and that of $L_j(\eta)$ is $-\frac{1}{2}\log d$, since $\eta^2 = -d$. Substitution gives

$$\frac{P-Q}{2}\big(h - \log \mathrm{rd}(E) + \tfrac{1}{2}\log d\big) = \frac{P-Q}{2}\log \Omega,$$

where the last equality is the host period normalisation of Lemma 21.22. Normalised embedding means are unchanged by extension to k, so this is the degree over the coefficient field.

In the finite log norm the correction of Lemma 35.10 subtracts $Q(D_j - L_j(d))$. It has finite support and increases the degree by $Q(\log d - \log \mathrm{rd}(E))$. This constructs a corrected frame of degree $-c_{1,\tau}$ in (35.3). Two such generators on the component differ by a scalar in the coefficient field: their ratio has weight zero and constant absolute value on the complex uniformisation, hence is constant there and descends to the coefficient field. The product formula gives them the same degree. This proves the law on every component for every τ in the CM type.

Lean: `GlobalTheory.CovTop.FlatDegProof.flatDeg_alt_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.FlatDegreeStatement`, lines 12–16)
Lean: `UpperTheory.Lit.Flat.flat_tensor_swap_isIso` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.HodgeBundleWeights.FlatTensorSwapIsIso`, lines 7–8)
Lean: `GlobalTheory.CompanionTheory.Ar.Br.CovEx.exists_constSec` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.ExistsConstSectionKroneckerChar`, lines 40–91)
Lean: `GlobalTheory.T.kronecker_character_frame_bound` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.KroneckerCharacterFrameBoundStatement`, lines 13–14)
Lean: `GlobalTheory.T.chowla_selberg_period` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.ChowlaSelbergPeriodStatement`, lines 9–10)
Lean: `GlobalTheory.T.frame_function_cm_value` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.FrameFunctionCmValueStatement`, lines 9–10)
Lean: `GlobalTheory.T.frame_function_cm_number_field` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.FrameFunctionCmNumberFieldStatement`, lines 9–10)
Lean: `GlobalTheory.CompanionTheory.Ar.Fl.flatGen_finNorm` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.FrameFunctionCmHeight`, lines 22–114)
Lean: `GlobalTheory.T.hermitian_metric_log_norm_formula` (`Development.AutomorphicSpectrum.Proofs.AdelicProofs.HermitianMetricLogNormFormulaStatement`, lines 8–9)
Lean: `GlobalTheory.CompanionTheory.Ar.Fl.flatDegree` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.FlatDegreeHolds`, lines 60–118)
Lean: `GlobalTheory.CompanionTheory.Ar.Fl.TopLL.degree_frames` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.KroneckerCharacterPeriodFormula`, lines 104–146)
Lean: `GlobalTheory.CompanionTheory.Ar.Br.flatDegreeSt_of_resK_gaga` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.FlatDegreeStOfResKGaga`, lines 18–30)
Lean: `GlobalTheory.FlatLine.ConstSec.deg_eq` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.IsAlgebraicOfNormConst`, lines 139–142)

Apply 35.35 to the degrees and ranks, and 35.26 and 35.94 to the heights. Proposition 35.95 descends the cover slope with the flat comparison of 35.4 and treats the terminal layer with the definite defect. The Gram frame and norm comparison of 35.22 give (8.32a) and (8.32c) in the realised lattices. Combine (8.32b), the bad-rank bound 35.96 and the projector inequality 35.97. Lemma 35.98 gives the layer estimate and 35.99 gives a common family threshold and constant. □

Proof of 35.12

Proof. The degree-zero part of the localisation of the homogeneous coordinate ring at the coordinates selected by c is R_c. The hypothesis on quotients gives $\mathrm{Frac}(R_c)=\Omega$, by dividing the numerator and denominator by the same chart monomial. Thus X is integral. It is projective by the Segre embedding and flat over $\mathbf{Z}$ since its affine rings are torsion-free. Each R_c is of finite type over $\mathbf{Z}$, hence excellent. Its integral closure is finite, and integral closure commutes with localisation. Thus the inverse images of the indicated charts under ν are $\operatorname{Spec}\widetilde{R}_c$. In the rational trivialisation given by y, the local frame of L^r is m_c^r. The two formulas for global sections now follow by gluing inside the common function field. They apply also when some a_k are zero or negative.

In a rational presentation $s=g(y)/h_1(y)$ of degree difference ra, the unit-ball condition on the locus $h_1\neq 0$ is

$$|g(z)|\le |h_1(z)|h(z)^r.$$

Such presentations exist by clearing chart denominators. On either model the condition extends by continuity from the locus of definition of the quotient, which is dense in every complex component. Conversely, a bound for the norm of the section gives the displayed inequality for every presentation by multiplication by the denominator. A proper surjective pullback preserves a sup norm.

We now give a conductor independent of r. On each chart clear the denominators of finitely many R_c-module generators of $\widetilde{R}_c$. This gives $0\neq b_c\in R_c$ with $b_c\widetilde{R}_c\subset R_c$. Homogenise b_c to an integral form F_c, using the selected coordinate in each block. Let $F=\prod_c F_c$, and let H be the restriction to X of the line of its multidegree. Its coefficient in chart c is b_c times an element of R_c, so $F\cdot\nu_*\mathcal{O}_{X^\nu}\subseteq\mathcal{O}_X$, and every factor is nonzero at the generic point. Choose a smooth metric on H and rescale it by a constant so that $\|F\|_{\overline{H},\sup}\le 1$. Multiplication by F then maps $H^0(X^\nu,\nu^*L^r\otimes\nu^*H^{-1})$ injectively and norm-nonincreasingly into $H^0(X,L^r)$ for every $r\ge 1$, yielding the two-sided conductor sandwich

$$\widehat{h}^0(X^\nu,r\nu^*\overline{L}-\nu^*\overline{H})\le\widehat{h}^0(X,r\overline{L})\le\widehat{h}^0(X^\nu,r\nu^*\overline{L}).$$

If $\widehat{\mathrm{vol}}(X^\nu,\nu^*\overline{L})=0$, then $\widehat{h}^0(X^\nu,r\nu^*\overline{L})=o(r^{d+1})$ and $\widehat{h}^0(X,r\overline{L})\ge 0$ (from the zero section) give $\widehat{h}^0(\widetilde{S}_r)-\widehat{h}^0(S_r)=o(r^{d+1})$ immediately. Suppose now that $\widehat{\mathrm{vol}}(X^\nu,\nu^*\overline{L})>0$. By Moriwaki's volume limit theorem [177, Th. 5.2], there exists $M_0\ge 1$ such that $\widehat{H}^0(X^\nu,k\nu^*\overline{L})$ contains a nonzero small section for every $k\ge M_0$. Fix an integer $m\ge 1$ and choose nonzero small sections $u_j\in\widehat{H}^0(X^\nu,(M_0m+j)\nu^*\overline{L})$ for $0\le j<m$. For every $r\ge(M_0+1)m$, write $r=qm+(M_0m+j)$ with $q\ge 1$ and $0\le j<m$; multiplying by the nonzero small section $u_j(\nu^*F)^{q-1}$ (of sup-norm at most 1 on the integral variety X^ν) embeds the unit ball of $H^0(X^\nu,q(m\nu^*\overline{L}-\nu^*\overline{H}))$ injectively and norm-nonincreasingly into that of $H^0(X^\nu,r\nu^*\overline{L}-\nu^*\overline{H})$. Dividing by $r^{d+1}/(d+1)!$ and letting $r\to\infty$ yields

$$\liminf_{r\to\infty}\frac{(d+1)!}{r^{d+1}}\widehat{h}^0(X,r\overline{L})\ge m^{-(d+1)}\widehat{\mathrm{vol}}(X^\nu,m\nu^*\overline{L}-\nu^*\overline{H})=\widehat{\mathrm{vol}}(X^\nu,\nu^*\overline{L}-\tfrac{1}{m}\nu^*\overline{H}).$$

Letting $m\to\infty$ and applying continuity of arithmetic volume on the normal projective arithmetic variety X^ν [177, Th. 4.4] gives $\lim_{r\to\infty}\frac{(d+1)!}{r^{d+1}}\widehat{h}^0(S_r)=\lim_{r\to\infty}\frac{(d+1)!}{r^{d+1}}\widehat{h}^0(\widetilde{S}_r)=\widehat{\mathrm{vol}}(X^\nu,\nu^*\overline{L})$, proving $\widehat{h}^0(\widetilde{S}_r)-\widehat{h}^0(S_r)=o(r^{d+1})$ for every multidegree $a=(a_k)\in\mathbf{Z}^K$. When every $a_k>0$, let P_{amb} be the ambient product of projective spaces and put $A_{\mathrm{amb}}=\mathcal{O}_{P_{\mathrm{amb}}}(a)$. Serre vanishing for the ideal sheaf $\mathcal{I}_X$ gives an integer τ_0 such that $H^1(P_{\mathrm{amb}},\mathcal{I}_X\otimes A_{\mathrm{amb}}^t)=0$ for every $t\ge\tau_0$. Hence every section of L^t is the restriction of an integral multihomogeneous form of multidegree ta. If Δ is the multidegree of F, choose $k'\ge\tau_0$ with $k'a_k\ge\Delta_k$ for all k, and multiply F by a monomial in coordinates nonzero at y of multidegree $k'a-\Delta$. The resulting integral form Φ is nonzero at y and satisfies $\Phi(y)\widetilde{S}_r\subseteq S_{r+k'}$ for every $r\ge 0$. Since $r+k'\ge\tau_0$, every element of this image is the evaluation of an integral form of multidegree $(r+k')a$, as asserted. The conductor here is a section of a fixed line bundle and works also along a horizontal nonnormal locus.

Choose a projective resolution of $X^\nu_{\mathbf{Q}}$ by Hironaka's theorem in characteristic zero [165]. Embed the resolution projectively over $X^\nu_{\mathbf{Q}}$, take its integral closure as a closed subscheme in the sense of schematic closure in $X^\nu\times\mathbb{P}^n_{\mathbf{Z}}$, and normalise that closure. Denote the resulting projective birational morphism to X^ν by f. Its generic fibre is the chosen resolution. Since X^ν is normal, $f_*\mathcal{O}=\mathcal{O}_{X^\nu}$, by Stein factorisation and finite birationality. The projection formula identifies the section spaces after pullback, with their sup norms.

On a complex chart of the resolution lift the map to the cone by setting $z_{kc_k} = 1$; denote this holomorphic lift by u. For every $F_0 \in P_0$, $F_0 \circ u = 0$, so $du(v)$ is tangent to P_0. The Levi form of the pulled-back local weight at v is $\mathcal{L}_{\log h}(u; du(v)) \geq 0$. Changing the lift adds a sum of logarithms of absolute values of holomorphic units to the weight. This leaves its Levi form unchanged. □

Proof of 35.13

Proof. Shrink to an affine neighbourhood of x and choose $0 \neq t \in A$ whose principal open in $\operatorname{Spec} A$ maps into U. The residue field is finite. Excellence gives that R is a normal local domain.

We first record how to test a nonzero element w of a Noetherian local domain B of characteristic zero and finite residue field of characteristic p. There is a one-dimensional local domain quotient in which pw survives. Indeed, while the dimension exceeds one, choose an element of the maximal ideal outside all height-one primes dividing pw and outside the zero prime. There are finitely many primes to avoid by the principal ideal theorem. Quotient by a prime minimal over the chosen element. It has height one, avoids pw, and the dimension drops. In the resulting one-dimensional domain C, the radical of pC is the maximal ideal. Thus C/pC is finite and the completion of C is module-finite over $\mathbf{Z}_p$, by lifting generators modulo p and successive approximation. The completion is faithfully flat over C. Choose a minimal prime of it avoiding pw, which is possible since pw is a nonzero divisor. The quotient is a finite domain over $\mathbf{Z}_p$. Its fraction field embeds in $\overline{\mathbf{Q}}_p$, and its maximal ideal maps into the maximal ideal of the valuation ring. This gives a local test of B on which w survives. Taking $B = R, w = t$ gives the first assertion.

Suppose $s \notin aR$. In $R[a/s]$, reduction modulo the maximal ideal of R and evaluation of a/s at zero define a homomorphism to the residue field. Indeed, a polynomial relation for a/s with unit constant term would give a monic equation for s/a by reversal and division by that unit. Normality would then give $s/a \in R$. Localise $R[a/s]$ at the kernel of this homomorphism. The resulting local domain B has the same residue field, and $a = su$ with u in its maximal ideal. Apply the preceding construction to $w = st$. The resulting test has $|v(a)| < |v(s)|$ and its generic point lies in U, a contradiction.

Finally write the rational function as s/a with $s, a \in A$, $a \neq 0$. Apply the divisibility assertion with the test open further restricted by $a \neq 0$. This gives $s \in aR$. Faithful flatness gives $aR \cap A = aA$. □

Proof of 35.14

Proof. Write $\Gamma = (\mathbf{Z}/rN\mathbf{Z})^{2g}$ for the full group of level labels, with g the dimension of the principally polarised abelian variety. For a free isotropic subgroup $T \subset \Gamma$ and a character $\chi : T \to \mu_N$, let $P_\chi = |T|^{-1} \sum_{t \in T} \chi(t)^{-1} \rho(t)$, where ρ acts on T by composing the level-N toric translation action on $T_{\mathrm{tor}}[N]$ with the coordinate quotient $T \twoheadrightarrow T/NT \simeq T_{\mathrm{tor}}[N]$ induced by the level-rN trivialisation (matching the mod-N reduction defining χ below, valid without assuming $\gcd(r, N) = 1$). On T the chosen lifts multiply as the group does. Write $v_j(t)$ for the evaluation of basis section j at label t in its torsion frame. The entries of the Fourier tuple are

$$F_{i,\eta} = \sum_{t \in T} \eta(t) \sum_j (P_\chi)_{ji} v_j(t), \qquad \eta : T \longrightarrow \mu_{(rN)^2}.$$

Here η ranges over all functions. The tuple is finite. First take the projective image given by all products of one entry from each nonzero tuple, including the theta-null tuple, and normalise in the function field of the component. The theta map is generically finite on this component. On a chart where a product entry generates, vary one factor at a time. Every ratio to the selected entry of that factor is regular. Pass to the normal projective model above this image carrying the semiabelian extension. The generating entries pull back, and uniformisation of the extension over R gives the common period lattice X_*. Choose a dense open U where their nonzero numerators and denominators are units.

Fix a local test v of Lemma 35.13 with generic point in U. The symmetric degeneration datum gives

$$|v(s_\alpha)| = \exp(-q_v(\alpha)),$$

where q_v is the positive definite quadratic form obtained by taking minus the logarithm of the absolute value of the cubical period. The polarisation identity follows from the biextension law and symmetry under inversion on the cubical trivialisation. Thus q_v attains its minimum on each residue class. Write b_v for a minimiser of c on this test.

We describe the subgroup and character used for c. The full level gives the quotient map from the torsion labels to X_*/rNX_*. This map on the completed chart is read by pairing with the split toric rN-torsion. Indeed, evaluation against each character of X_*/rNX_* is the Weil pairing with the corresponding toric label. These pairings take values in a constant group of roots of unity since the level is invertible and split. They commute with base change, and the characters of the finite free module separate points. Thus the quotient map is the same at every test under the completed-chart identification of period lattices. Take T to be the perpendicular of its kernel for the level pairing. In symplectic coordinates the toric labels form a free isotropic summand, dual to the period quotient. Choose a full level label above a lift of c modulo rN. Pair it with T and reduce the exponent modulo N to get χ. Adding an element of the kernel or changing the lift by N leaves χ unchanged. This choice precedes the choice of b_v.

Let δ_v be the greatest norm of a theta null at v. We give the finite Fourier calculation in the normalisation of the period sections:

$$\max_{i,\eta} |F_{i,\eta}(v)| = \delta_v \exp(-q_v(b_v)/N). \tag{1}$$

In an integral frame write the principal theta section as a scalar times a primitive section. The norm of this scalar is δ_v. To see the equality, the terms of the primitive theta expansion at a level point have norm at most one: the exponents after translation are the values of the nonnegative quadratic form on the translated lattice, with the torsion-frame factor included. In the integral torus direction the reduction is the theta section on the abelian part. It has a nonzero value at some prime-to-p level point. Indeed, for an invertible level divisor $n \geq 2$, the translates of the n-th power of a nonzero principal theta section by n-torsion span its level-n section space. The character projectors for one maximal isotropic subgroup project to lines and the complementary subgroup permutes these lines transitively. The line bundle of level n is generated by its sections, so the translates cannot all vanish at the origin. Lift a nonvanishing point by the prime-to-p torsion of the semiabelian scheme. The integral torsion frame then gives a unit value.

Divide the level basis by the same scalar. At the origin the integral Fourier expansions of level N are those whose coefficient at β, in the cubical frame, has norm at most $\exp(-q_v(\beta)/N)$. Indeed, choosing one representative in each class modulo N identifies an expansion with its coefficients on the abelian part at those representatives. Translation by Nz multiplies the coefficient by the period factor of log valuation

$$\frac{q_v(\beta + Nz) - q_v(\beta)}{N}.$$

This is the Fourier translation law $D_2\psi = \tau$ for the cubical trivialisation ψ and its biextension trivialisation τ [167, Def. 4.2.1.13 and §4.3]. One may extend the valued field to realise the finitely many fractional weights of the representatives.

The integral lattice in this calculation is generated by the normalised theta translates. Multiplication by m pulls the primitive principal theta section back to a primitive level-N vector s_0: its exponent β becomes $m\beta$, and $q_v(m\beta)/N = q_v(\beta)$. Translation in a torsion frame preserves the weighted coefficient lattice by the same translation law. This lattice has an integral basis, obtained class by class from the section lattices of the ample coefficient lines on the abelian part after clearing the indicated weights. Take an integral coordinate functional f with $f(s_0)$ a unit. If U_x are the theta lifts of $x \in A[N]$ on the generic section space of dimension $D = N^g$, the perfect commutator pairing gives $\operatorname{tr}(U_y^{-1}U_x) = D\,\mathbf{1}_{x=y}$. For $x \neq y$, conjugation by a lift having nontrivial pairing with $x - y$ proves the vanishing. There are D^2 operators, so they form a basis of the endomorphism algebra. Expanding the rank-one operator $w \mapsto f(w)s$ in that basis and evaluating at s_0 gives

$$s = \sum_{x \in A[N]} \frac{f(U_x^{-1}s)}{Df(s_0)}\, U_x s_0.$$

All coefficients are integral for integral s, since the residue characteristic is prime to N. The level basis spans these translates by its construction. This also identifies the scale of its coefficients with the scale of the theta nulls.

The projector P_χ retains precisely the terms with exponent in c. Evaluating at toric torsion multiplies their coefficients by units. The ultrametric inequality therefore gives the upper bound in (1). For attainment, prescribe at b_v a coefficient with norm $\exp(-q_v(b_v)/N)$ at the origin and extend it by the translation law, setting the other classes to zero. To make this prescription, first choose a section of the coefficient line

on the abelian part that is a unit at the given special-fibre point. Geometrically that line is a degree-zero twist of the N-th power of an ample principal line. It is a translate of that power and is globally generated since $N \geq 4$. Sections lift over the valuation ring by base change for the ample line on the abelian scheme, whose positive-degree cohomology vanishes on the fibres. Scale the section by an element of the prescribed valuation. The translation formula gives the required integral expansion with equality at b_v.

In its values on T, average against the inverse of the toric character of b_v. Orthogonality removes exponents outside $b_v + rNX_*$. For $z \neq 0$ one has

$$q_v(b_v + rNz) - q_v(b_v) = r\{q_v(b_v + Nz) - q_v(b_v)\} + r(r-1)N^2 q_v(z) > 0.$$

Thus the term at b_v survives with strictly greatest norm. The series converges because the form is positive definite, and $|T|$ is a unit. At least one torsion evaluation of the prescribed expansion attains the upper bound. Since the expansion is an integral linear combination of the normalised basis sections and belongs to the χ-space, some projected basis evaluation attains it. Finally, if the sum of those evaluations over t, for that basis index, has smaller norm, change the phase at one attaining t from 1 to ζ. The difference has the required norm since $1 - \zeta$ is a unit. This proves (1), also in the presence of ties among shortest representatives.

Choose at x a generating theta null θ_0. All ratios to it are regular at x, hence $|\theta_0(v)| = \delta_v$ on every test. Equation (1) shows that the Fourier tuple for c is nonzero. Choose its generating entry F_0 at x and put $e_c = F_0/\theta_0$. All ratios $F_{i,\eta}/F_0$ belong to A. Consequently (1) gives, for every test,

$$|e_c(v)|^N = \max_{\alpha \bmod N = c} |v(s_\alpha)| \leq 1.$$

By Lemma 35.13, $a = e_c^N$ belongs to A, and $a \mid s_\alpha$ in R for every representative. Choose one test and an attaining representative μ_c for it. Write $s_{\mu_c} = au$. On this test a is nonzero and $|v(u)| = 1$. The test homomorphism is local, so u is a unit of R. This proves both assertions in the completed ring. □

Proof of 35.15

Proof. Put $A = \mathcal{O}_{X,x}$. The specialisation gives a map $A \to V$. To compare it with $R = \widehat{A}$, form $V \otimes_A R$. This algebra is faithfully flat over V, and its special fibre is reduced because the formal fibres of the excellent ring A are geometrically regular. Localise at a minimal prime of its special fibre. The quotient by ϖ is a field and ϖ is a nonzero divisor. Completing for (ϖ) gives a complete discrete valuation ring V': each successive quotient $(\varpi^j)/(\varpi^{j+1})$ is one-dimensional over the residue field, so every nonzero element of the completion is a power of ϖ times a unit. We have compatible maps $R \to V'$ and $V \to V'$ with ramification index one.

Use split tori to read the period data, passing to an étale splitting extension as necessary. In a two-step degeneration the torus at the more general centre extends as a subtorus of the torus at the closed centre. Restriction of characters gives the indicated integral surjection. In the description by formal thickenings, a proper image on characters would leave a nonzero finite quotient of its cokernel. The dual finite diagonalizable group would map trivially on every thickening under the extended torus map. Since that group is finite and the semiabelian scheme is separated, equality on all thickenings implies equality of the two maps. On the fibre where the torus is embedded, the induced map is nontrivial, a contradiction. Endomorphisms commute with restriction. The polarisation changes their action on the dual torus by conjugation, giving the asserted $\mathcal{O}_E$-linearity. This is the character restriction in the two-step construction of [167, Props. 3.3.1.5, 3.3.1.7 and §4.5.6].

Write $Q(y)$ for the order of the cubical period section on the trait in an integral frame. For a split polarised degeneration the Poincaré biextension trivialisation is the second difference of that section [167, Def. 4.2.1.13(6)–(7), Th. 4.2.1.14]. Hence its order is

$$b_V(y, z) = Q(y + z) - Q(y) - Q(z).$$

Rigidification gives $Q(0) = 0$, and the symmetric integral line gives $Q(-y) = Q(y)$. Substitution of $z = -y$ gives $b_V(y, -y) = -2Q(y)$, hence $b_V(y, y) = 2Q(y)$ by bilinearity. This accounts for the square in the order formula.

For the comparison under restriction, one can read the order of each biextension parameter modulo every ℓ^j, with ℓ a prime invertible at both centres. After strict henselisation a unit has an ℓ^j-th root. Tame inertia

acting on an ℓ^j-th root of $u\varpi^a$ thus reads a mod ℓ^j from its action on a root of ϖ. The two uniformisations give the same polarised generic torsion, with toric characters related by restriction. Their Weil pairings give the same inertia character for the corresponding biextension parameters. The two integral orders agree modulo every ℓ^j, hence agree. Applying the preceding diagonal identity on both sides gives

$$2\,\mathrm{ord}_{V'}(s_\alpha) = b_V(\pi_V\alpha, \pi_V\alpha).$$

The local divisibility from Lemma 35.14 persists under $R \to V'$. The associate of s_{μ_c} is the image of the actual stalk element e_c^N. Its order is therefore the order of e_c^N in V, since ϖ remains a uniformiser. This proves the equality and inequality.

If the centre is x, the map of the completed stalk to the extended trait ring is local. For $\alpha \neq 0$, positivity of the degeneration puts s_α in the maximal ideal of R. If $\pi_V\alpha = 0$, the diagonal identity makes its image a unit. Thus the surjection has zero kernel. □

Proof of 35.16

Proof. First fix a cone with integral generators $\gamma_1, \dots, \gamma_m$ whose sum G is positive definite. In one residue class c, associate to $v \in c$ the tuple

$$\big(\gamma_1(v,v), \dots, \gamma_m(v,v)\big) \in \mathbf{N}^m.$$

There are finitely many coordinatewise minimal tuples. Indeed, every infinite sequence in $\mathbf{N}$ has an infinite nondecreasing subsequence. Successive extraction in the m coordinates rules out an infinite antichain. Every tuple dominates a minimal one, by descent in the sum of its coordinates. In a strictly positive linear combination of the γ_i, a minimiser must realise a minimal tuple: a strict decrease in any coordinate with no increase in the others lowers the value of the combination. The possible values of $G(v,v)$ on these tuples are bounded. Since G is positive definite, its bounded sublevel sets on X are finite. Taking the product over the finite set X/NX gives finitely many complete minimum systems on this cone.

Every positive definite form is in the relative interior of a cone, and the sum of that cone's generators is positive definite. By finiteness in the definition of the polarised fan, a lattice automorphism carries its generators into a fixed finite set. There are finitely many resulting cones, and on each there are finitely many transported systems by the first paragraph. For a fixed transported system $(v_c)_c$, an inverse change of coordinates must carry every v_c into the finite set $\{\mu_c\}_c$. The v_c span $X \otimes \mathbf{Q}$: in lattice coordinates the representatives of the standard basis classes form a matrix with determinant 1 mod N. Hence the images of the v_c determine the change of coordinates. This gives finitely many cones before transport.

Let B be any of the semidefinite forms in the statement. The minimum inequalities are linear in the form, so $b_0 + jB$ satisfies them for every positive integer j. These forms are positive definite. One cone in the finite list contains them for unbounded j. Since that cone is closed and is invariant under positive scaling, it contains $\lim_{j\to\infty}(b_0 + jB)/j = B$ along this subsequence. □

Proof of 35.17

Proof. The inclusion $J\mathcal{O}_Y \subseteq \mathcal{O}_Y(-D)$ follows from the definition. Fix a closed x. Properness makes z closed. The residue field of A is finite, so $A/\mathfrak{m}^j$ is finite for every j. Diagonal extraction gives a subsequence F_{n_i}, with $n_i \to \infty$, converging in $\widehat{A}$ to some F.

For a closed x' over z, the local homomorphism $A \to \mathcal{O}_{Y,x'}$ induces $\widehat{A} \to \widehat{\mathcal{O}}_{Y,x'}$. On any test at x' one has $\rho_y < 1$, and an element of $\mathfrak{m}^j\widehat{A}$ has absolute value at most ρ_y^j. Consequently

$$|F(y)| \le |s_{x'}(y)|, \qquad |F(y_0)| = |s_x(y_0)| > 0.$$

Lemma 35.13, applied at x' with the given dense open and $s_{x'} \neq 0$, shows that $s_{x'}$ divides F in $\widehat{\mathcal{O}}_{Y,x'}$. At x the quotient is a unit: its value on the local test y_0 has absolute value one.

It remains to put the formal germ F in $J_z\widehat{A}$. Base change to $\operatorname{Spec}\widehat{A}$, and denote the pullback of D by D'. At the points of the closed fibre the completed local rings of the base-changed Y are the original completed local rings. Indeed, modulo the j-th power of the point's maximal ideal the base action factors through

$A/\mathfrak{m}^j = \widehat{A}/\mathfrak{m}^j\widehat{A}$. The divisibilities just proved and faithful flatness of local completion show that the section $F|_{D'}$ vanishes at the stalk of every closed point of the closed fibre. Its support is closed and proper over $\widehat{A}$. A nonempty support would meet the closed fibre in a nonempty closed set and hence contain a closed point there. Thus $F|_{D'} = 0$.

Flat base change identifies the kernel of $\widehat{A} \to H^0(D', \mathcal{O}_{D'})$ with $J_z\widehat{A}$. For degree zero this follows by computing sections on a finite affine cover and its pairwise intersections and using flatness to preserve the equalizer. It follows that $J\widehat{\mathcal{O}}_{Y,x}$ contains F, which generates $s_x\widehat{\mathcal{O}}_{Y,x}$. Faithful flatness gives equality of the two ideals at x. The quotient of the two coherent ideals has zero stalk at every closed point of Y, so it is zero. □

Proof of 35.18

Proof. Write $X = \operatorname{Spec} A$, $X_1 = \operatorname{Spec} B$ locally. For each finite tuple $b_1, \dots, b_m$ in $I(B)$, take the coefficients of

$$\prod_{\sigma : L \hookrightarrow \overline{K}} \left(\sum_{i=1}^{m} \sigma(b_i) T_i \right),$$

where the embeddings fix K. The coefficients lie in K by Galois invariance and are integral over A, so they lie in A. They generate J. This prescription commutes with localisation: a tuple with localised entries can be multiplied by a common denominator $s \in A$, and its polynomial is then multiplied by s^h. The ideals therefore glue.

In a normal closure of L/K form the A-algebra B' generated by the images of B under all h embeddings. It is finite and integral over A. Given $A \to V$, choose a prime of B' over the kernel of $A \to \operatorname{Frac} V$. The residue field at that prime is finite over the residue field at its contraction and embeds over that field in an algebraic closure of $\operatorname{Frac} V$. Its compositum with $\operatorname{Frac} V$ is finite over $\operatorname{Frac} V$. The valuation ring V' in this extension is a complete discrete valuation ring. The images of B' lie in V' by integrality. Composition with the h maps $B \to B'$ gives the required lifts.

Over a discrete valuation ring the ideal of coefficients of a product of polynomials is the product of their coefficient ideals. For nonzero factors, divide each by the common uniformiser power of its coefficients; the resulting polynomials have nonzero reductions, and their product still has nonzero reduction. A zero factor gives the same identity. Applying this to the displayed product gives one inclusion in the assertion for every tuple. Taking a tuple of generators of $I(B)$ gives the reverse inclusion for all h lifts at once. □

Proof of 35.19

Proof. For item 1 of Definition 35.5, take the prescribed-fan toroidal compactification constructed below via [167, Th. 6.4.1.1] and retain its smoothness, its simple normal crossings boundary and its logarithmic Kodaira–Spencer isomorphism on the designated interior. For items 2, 3, and 22, the volume assertions on an integral arithmetic variety give the lower bound of 35.33 for a semipositive fibrewise nef Hermitian line [171, Cor. 6.4] and equality with self-intersection for an arithmetically nef line [171, Cor. 5.5], yielding the logarithmic count (35.5) of Definition 35.5 (matching [172, Th. 1.4]). For items 4 and 24, the arithmetic intersection calculus (change of model and metric, constant shifts, multilinearity, projection and isometry) and the cycle-height recursion and projection formula are [170, §§2.3, 3.1–3.2]. For item 10, the Chern-form integral for a bundle with a good metric computes the intersection number of the extended classes by Mumford's theorem [160, Prop. 1.3 and Th. 1.4] (see also [176, Ch. V, §4]), and on a full flag bundle insertion of the fibre point class and pushforward give the base intersection by the splitting principle and projection formula [158, §3.2].

For item 6, global class field theory gives the narrow ray class field for the given ideal and all real places. The finite primes away from the ideal are unramified, their Frobenius elements generate the Galois group, and the common character kernel on coprime ideals is the narrow ray-principal subgroup [41, Ch. VI, Ths. 6.1 and 7.1, and Ch. VII, Th. 13.4]. For item 7, let $K/\mathbf{Q}_p$ be finite and let A_K be the underlying abelian variety of the polarised datum. The base $\operatorname{Spec} \mathcal{O}_K$ is regular, connected, Noetherian and one-dimensional. Grothendieck's semistable reduction theorem gives a finite Galois extension K'/K over whose integer ring the identity component of the Néron model is semiabelian with generic fibre $A_{K'}$ [180, Exp. IX, Th. 3.6]. For

item 13 (extension of homomorphisms), let G, H be semiabelian schemes over the same normal Noetherian scheme S. If $U \subset S$ is dense and open, restriction gives a bijection $\mathrm{Hom}_S(G,H) \to \mathrm{Hom}_U(G_U, H_U)$ by [167, Prop. 3.3.1.5]. In particular the endomorphisms defining the coefficient action extend, and their addition and composition laws extend by uniqueness.
Lean: `GlobalTheory.SS.semistableReductionSt` (`Development.AbelianVarietyTheory.Proofs.DegenerationProofs.SemistableReductionPpavPadicField`, lines 19–21)
Lean: `GlobalTheory.CompanionTheory.SAH.semiAbelianHomExtension` (`Development.AutomorphicSpectrum.Proofs.GlobalProofs.SemiAbelianHomExtension`, lines 13–19)

For item 8, on a nondegenerate Hermitian space of rank at least two over a CM extension, rational special unitary points together with the omitted local factors are dense in the adelic group when the product of those local factors is noncompact, by strong approximation [19, Th. 7.12]. For item 11, nonsingular Hermitian forms of equal rank over a CM extension are isometric when their signatures agree and the ratio of their determinants is a norm, by Landherr's theorem [96, Ch. 10, Th. 6.1]. For item 12, for $x\delta \neq 0$ in a number field, $x = a^2 - \delta b^2$ has a solution when it has one at every real and finite place, by the Hasse norm theorem for quadratic extensions [41, Ch. VI, Cor. 4.5] (see also [96, Ch. 6, Th. 6.5]).

For item 9, write a squarefree parameter as $\prod_a P_a$, with distinct irreducible factors of degrees d_a. Green's full classification assigns partition (1) to every P_a; equivalently, take the parabolic induction of the corresponding distinct cuspidal characters. It is irreducible and generic, and different parameters give different characters. Green's degree formula gives $\deg \rho = \prod_{i=1}^{N}(g^i - 1)/\prod_a (g^{d_a} - 1)$, while the parameter torus has order $\prod_a (g^{d_a} - 1)$ [20, Th. 14]. For item 14, over a field of characteristic p, the induced module for a dominant GL_N-weight λ is nonzero and simple when $\lambda_i - \lambda_j + j - i \leq p$ for all $i < j$, by the strong linkage principle in the lowest alcove [66, Part II, Cor. 5.6 and §6.24]. For items 15 and 16, over an algebraically closed field a principal polarisation has a one-dimensional section space by the Riemann–Roch theorem for abelian varieties [189, §16], and the geometric n-torsion of a g-dimensional abelian variety has order n^{2g} for invertible n [189, §6, Application 3].

For item 5 of Definition 35.5, the de Rham cohomology H^1_{dR} of an abelian scheme over an affine base is finite locally free of constant rank $2g$ with primitive classes, $\bigwedge^2 H^1_{\mathrm{dR}} \simeq H^2_{\mathrm{dR}}$, rank-$g$ Hodge direct summand commuting with base change, crystal comparison over nilpotent divided-power thickenings, and perfect polarisation pairing vanishing on the Hodge summand [161, Ch. I, Prop. 4.1.7, and Ch. II, §§9, 13 and 15] (see also [103, §2.5]). For item 17, over a nilpotent divided-power thickening with p nilpotent, Grothendieck–Messing crystalline deformation theory identifies the de Rham module of an abelian lift with the crystal evaluation and classifies lifts of objects and homomorphisms by admissible lifts of the Hodge filtration [228, Ch. V, Th. 1.6] (see also [102, Chs. IV–V]). For item 18, let $O = \mathcal{O}_C$ be the given p-adically complete valuation ring, with C algebraically closed of characteristic zero and p a nonunit. Put $O^\flat = \varprojlim_{x \mapsto x^p} O/p$, $A_{\mathrm{inf}} = W(O^\flat)$, and write φ for Witt Frobenius. From the compatible primitive roots of unity form $\epsilon \in O^\flat$, $\mu = [\epsilon] - 1$, $\xi = \mu/\varphi^{-1}(\mu)$, and $\widetilde{\xi} = \varphi(\xi)$. The ring O is integral perfectoid: it is p-torsion-free and complete, every element of O has a p-th root in O, and $p^{1/p} \in O$. Fontaine's map $A_{\mathrm{inf}} \to O$ has kernel (ξ). Use the contravariant prismatic Dieudonné functor on this perfectoid base [178, Prop. 4.12, Prop. 4.48 and Th. 4.74]. For a p-divisible group G of height h its module M_G is finite projective of rank h, hence free since A_{inf} is local. In the prism convention with ideal $(\widetilde{\xi})$, its Frobenius F_G satisfies $\widetilde{\xi} M_G \subseteq F_G(M_G) \subseteq M_G$. Here the image of the semilinear map is the image of its linearisation since φ is an automorphism. The crystalline comparison of [179, Th. 17.5.2], dualised as in [178, Prop. 4.48], identifies base change along Fontaine's map with the dual of the Lie algebra of the universal vector extension of G. Dualising its Lie sequence gives the functorial exact sequence $0 \to \omega_G \to M_G/\xi M_G \to \mathrm{Lie}(G^\vee) \to 0$. The constant group $\mathbf{Q}_p/\mathbf{Z}_p$ has the unit module with Frobenius φ. For μ_{p^∞} take the dual of the covariant Tate twist $\mu^{-1}A_{\mathrm{inf}}(1)$ of [216, Ex. 4.24]. In the dual basis Frobenius is $(\varphi(\mu)/\mu)\varphi = \widetilde{\xi}\varphi$. Full faithfulness identifies each Frobenius-compatible map $M_{\mu_{p^\infty}} \to M_{\mathbf{Q}_p/\mathbf{Z}_p}$ with a unique homomorphism $\mathbf{Q}_p/\mathbf{Z}_p \to \mu_{p^\infty}$.
Lean: `UpperTheory.Lit.prismaticDieudonneHodge` (`Development.AutomorphicArithmetic.ThetaCorrespondence.PrismaticDieudonne.PrismaticDieudonneHodge`, lines 19–25)

For item 20, the nef numerical assertions give nonnegative intersections of nef products (Kleiman's theorem [159, Th. 1.4.9]), characterise bigness of a nef line by positive top self-intersection [159, Th. 2.2.16], and give the asymptotic Riemann–Roch leading section dimension $h^0(X, L^m) = (L^D)m^D/D! + O(m^{D-1})$ [159, Cor. 1.4.41]. For item 23, a geometrically integral projective image of dimension D in characteristic zero has a Chow form unique up to scalar, of degree equal to the image degree in each of its $D+1$ blocks.

Use the Chow construction with all input hyperplanes of degree one [170, §§4.3.1–4.3.2]. For item 26, the separation argument uses Nadel vanishing for normal pairs in Fujino, *Foundation of the minimal model program*, Theorem 3.4.2 and the Angehrn–Siu construction [169] in the form for a projective klt pair and a nef and big divisor given by Kollár, *Singularities of pairs*, Theorem 6.4. For item 27, principalisation of a nonzero coherent ideal in characteristic zero makes the base ideal invertible with simple normal crossings support [165]. For item 28, let R be Noetherian and complete for an ideal I, and put $R_j = R/I^{j+1}$. For X proper and Z separated of finite type over R, compatible R_j-morphisms $X_{R_j} \to Z_{R_j}$ determine a unique R-morphism $X \to Z$ by [181, III$_1$, Th. 5.4.1]. For a system of proper R_j-schemes X_j with $X_{j+1} \times_{R_{j+1}} R_j \simeq X_j$, choose compatible invertible sheaves L_j. If L_0 is relatively ample, the induced proper formal scheme has an invertible sheaf ample on its closed fibre. The algebraisation theorem produces a projective R-scheme with a relatively ample line inducing the given system [181, III$_1$, Th. 5.4.5].
Lean: `PositivityTheory.FormalThickening.formalAlgebraisation` (`Development.WeilBound.CoreStatements.TemperedRepresentationTheory.FormalAlgebraisation.Development`, lines 17–22)

For item 19 of Definition 35.5, if X is smooth integral projective over an algebraically closed field of characteristic zero and N is nef and big, then $H^a(X, K_X \otimes N) = 0$ for $a > 0$. This is the line-bundle case of [173]. The comparison in item 25 is degree-zero GAGA for a line bundle on a smooth projective complex variety [174].

For item 21 of Definition 35.5, the extension estimate (35.4) (with chart measure $d\mu$ and local Jacobian $\operatorname{Jac} s(P)$ of the s_j) is the transverse finite-zero-set case of [175, Th. 2.8] in the indicated residue normalisation.

The proper-intersection formula used with $\mathcal{H}_P$ is a consequence of the intersection laws. For a finite proper cut of a projective integral d-fold by d sections of a line, the local rings of the d-fold at the cut are assumed Cohen–Macaulay. The local equations form a system of parameters and hence a regular sequence. The local top intersection multiplicity is the length of its quotient. Push the resulting zero-cycle to the ground field to obtain the sum of these lengths weighted by residue degrees.
Lean: `GlobalTheory.CovTop.PublishedHodgeBundleTheorems` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.HodgeBundleWeights.ToKnownTheoremsOf`, lines 7–35)
Lean: `GlobalTheory.CovTop.PublishedHodgeBundleTheorems.q12` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.HodgeBundleWeights.ToKnownTheoremsOf`, lines 38–38)
Lean: `GlobalTheory.Published.L2ExtensionLogCanonical` (`Development.AbelianVarietyTheory.PublishedTheorems.Divisors₂.L2ExtensionLogCanonical`, lines 32–54)

For a reduced separated scheme of finite type over a characteristic-zero field, resolution is an isomorphism over the smooth locus. For principalisation take a smooth separated quasi-compact scheme over such a field with simple normal crossings boundary. An ideal whose support misses the generic points of the irreducible components becomes effective Cartier with simple normal crossings support after blow-ups compatible with the boundary. The modification is an isomorphism off the support of the ideal. Item 27 of Definition 35.5 uses this for a nonzero ideal on a smooth integral projective variety. These are the characteristic-zero resolution and principalisation results of [165]. Invariant Poincaré–Lelong gives the divisor current of a Hermitian section from the curvature and the dd^c of its log norm. Together with resolution it gives the intersection and restriction laws used here.

For the three arithmetic Hilbert–Samuel assertions, take the projective arithmetic variety X of Lemma 35.12 with generic dimension D. Write $I(A) = \widehat{\deg}(\widehat{c}_1(A)^{D+1})$ for its self-intersection on a smooth conjugation-invariant Hermitian line A. The arithmetic Kleiman assertion is

$$I(A) \geq 0 \tag{A.422}$$

when A is semipositive, nef on fibres and of nonnegative degree on every horizontal arithmetic curve. Indeed these hypotheses make the Hermitian line arithmetically nef. Its top self-intersection equals its volume by [171, Cor. 5.5], and every unit ball contains zero. For the two upper bounds assume $D \geq 1$, use the homogeneous ideal P of Lemma 35.12, and fix an ideal $P_0 \subseteq P$. In both bounds A is conjugation-invariant and its curvature is nonnegative at complex points of P along the cone-tangent directions of P_0. The asserted inequality is

$$\limsup_{j \to \infty} \frac{(D+1)! \log \#\{s \in L_j : \|s\|_{\sup} \leq 1\}}{j^{D+1}} \leq I(A). \tag{A.423}$$

For the first bound take $L_j = S_j$ of Lemma 35.12. The coefficient field has characteristic zero, the generic degree is positive, and the integral sections of sup norm at most one specified in the hypothesis generate the line: their coefficients generate the unit ideal in each R_c. For the second take $L_j = \widetilde{S}_j$. The multidegree

is nonnegative and each coordinate monomial of that multidegree has point norm at most one. These monomials generate, since on each chart one of them is the chart frame. In either case a generated line has nonnegative degree on vertical curves. On the normalisation of any horizontal arithmetic curve choose a generating section nonzero at its generic point. Its finite zero divisor is effective and each contribution $-\log\|s\|$ at infinity is nonnegative. Thus the line has nonnegative arithmetic degree there. Lemma 35.12 gives smooth semipositive curvature on the generically smooth modification. Pullback preserves generation and the sup bounds. Moriwaki's nef Hilbert–Samuel formula, with a fixed Hermitian twist if required, now gives (A.423) and (35.5) with denominator $j^{D+1}/(D+1)!$ [171, Cor. 5.5]. Zhang's relatively semipositive formula for a generically ample line uses the same intersection normalisation for the Euler characteristic of the sup-norm lattice [172, Th. 1.4].

Lean: `PositivityTheory.MbHsUse.HSUse` (`Development.WeilBound.CoreStatements.AutomorphicKernels.MoriwakiBigness.HilbertSamuelApplications.SmallGeneratedCountUpperOfUse`, lines 12–13)
Lean: `GroundTheory.Heights₂.Heights.HilbertSamuel.G7.Statement` (`GroundTheory.HeightsAndTheta.Heights.HilbertSamuel.StatementsArith`, lines 437–444)
Lean: `ReciprocityTheory.ChartRestriction.G.Geo.ArithmeticHilbertSamuelUpperBound` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.BirationalGeometry.ArithmeticHilbertSamuelLowerBound`, lines 135–145)
Lean: `PositivityTheory.Pf1.HilbertSamuelMoriwaki.hsUse_outright` (`Development.WeilBound.CoreStatements.AutomorphicKernels.MoriwakiBigness.MoriwakiBigTheorem.Outright`, lines 13–15)
Lean: `PositivityTheory.Pf1.HilbertSamuelMoriwaki.moriwakiBig_outright` (`Development.WeilBound.CoreStatements.AutomorphicKernels.MoriwakiBigness.MoriwakiBigTheorem.Outright`, lines 9–11)

For Fargues exactness, let C be an algebraically closed valued field of characteristic zero with $v(p) < 1$ and p-adically complete integer ring O_C. Normalise the dual Tate pairing using a compatible system $\zeta_0 = 1,\ \zeta_{a+1}^p = \zeta_a,\ \zeta_1 \neq 1$. For a p-divisible group G over O_C, the linear Hodge–Tate map on $T_pG \otimes O_C$ has kernel the annihilator of the kernel of the dual logarithmic differential on $T_pG^\vee \otimes O_C$. Over C this is the duality of the Hodge–Tate subspaces in [166, Prop. 5.2.8]. The targets of the integral differentials are torsion-free modules of invariant differentials, so their kernels are the intersections of the kernels over C with the respective Tate lattices.

Lean: `UpperTheory.LiteratureTheorems.fargues_exact_of_literature` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin`, lines 549–550)

In Zarhin's retraction, A is an abelian scheme with a relatively ample line L over the integer ring of an algebraically closed valued field. Put $P = (A \times A^\vee)^4$. The polarisation $\ell : A \to A^\vee$ of L has a quasi-inverse $\mu : A^\vee \to A$ with $\mu\ell = [m]$ and $\ell\mu = [m]$, $m \geq 1$. A quaternion matrix U from four squares summing to $4m^2 - 1$ satisfies $UU^{\mathrm{t}} = U^{\mathrm{t}}U = (4m^2-1)I_4$. On $A^4 \times (A^\vee)^4$ take the pullbacks of L by the four projections and their negatives, and by the four projections composed with μ and their negatives. Tensor them with the pullbacks of the Poincaré line on each pair $(A_i, A_j^\vee)$ to the powers U_{ij}. The theorem of the cube computes the associated map to the dual, with the factors in the order $(A^\vee)^4 \times A^4$, as

$$\begin{pmatrix} 2\ell I_4 & U \\ U^{\mathrm{t}} & 2m\mu I_4 \end{pmatrix}. \tag{A.424}$$

The candidate inverse has diagonal entries $2m\mu I_4, 2\ell I_4$ and off-diagonal entries $-U, -U^{\mathrm{t}}$. Multiplication gives the identity by the two quasi-inverse identities and the quaternion norm identity. For ampleness on each geometric fibre, pull the line back by the isogeny $1 \times \ell^4 : A^4 \times A^4 \to P$. Its matrix relative to the product polarisation is $\left(\begin{smallmatrix} 2I_4 & U \\ U^{\mathrm{t}} & 2m^2 I_4 \end{smallmatrix}\right)$. Its Schur complement is $\frac{1}{2}I_4$. Rational congruence diagonalises it with positive diagonal, so after clearing denominators an isogeny pulls the line back to the class of a product of positive powers of L. Ampleness descends under the isogenies.

Translate the polarising line on the geometric generic fibre to a symmetric representative of the same polarisation. Extend its Picard class over the valuation ring and choose a line bundle M representing that class. Injectivity of restriction to the generic fibre for the Picard groups of P and $P \times P$ gives symmetry and identifies the polarisation of M with (A.424). The base is local, so M can be rigidified at the zero section. On the closed fibre M induces the same polarisation as the ample line from the matrix construction and is therefore ample there. Fibrewise ampleness gives relative ampleness, and the principal polarisation gives trivial kernel. Finally take $i : A \to P$ to be inclusion in the first A-factor and $r : P \to A$ its projection. These maps are defined over the valuation ring and $ri = 1_A$.

Lean: `PositivityTheory.HtPt.Narrow.zarhin` (`Development.WeilBound.CoreStatements.HodgeTatePoints.ZarTop`, lines 6–7)
Lean: `PositivityTheory.PrwZar.zarhinStatement_of_dual_cube_qiR` (`Development.WeilBound.CoreStatements.AbelianHeightsDuality.ZarhinTrick.ZarhinStatementOfDualCubeQiR`, lines 45–66)

Lean: GlobalTheory.CompanionTheory.Hypotheses₂.Zarhin.SymB.symmetricBundle (Development.AutomorphicSpectrum.Proofs.AlgebraProofs.RelAmpleOfCls, lines 258–308)

For nef Riemann–Roch let y be birational coordinates of generic dimension D and a a generically nef multidegree. The leading term is

$$\lim_{t\to\infty} \frac{D!}{t^D} \dim_{\mathbf{Q}} \langle \widetilde{S}_t \rangle_{\mathbf{Q}} = \deg_{\mathrm{gen}}(a). \tag{A.425}$$

Here $\widetilde{S}_t$ is the lattice of sections on the normalisation defined in 35.12. Its rational span contains the span for the unshifted model and embeds, after multiplication by a fixed nonzero rational representative, in a span with fixed line-bundle twist. Both bounding dimensions have the leading coefficient in (A.425) by the nef dimension formula.

Lean: PositivityTheory.Tls2.Nef.nefRR (Development.WeilBound.CoreStatements.TemperedRepresentationTheory.NefDivisors.NefRR, lines 14–52)

For each Siegel genus $g \geq 1$ choose a smooth polarised fan without self-intersections. This choice works at every odd level $n \geq 3$. Over $\mathbf{Z}[\zeta_n, 1/n]$ the compactification is projective and smooth of relative dimension $g(g+1)/2$. The open is fibrewise dense and is the abelian locus of the semiabelian extension. Let ω be the determinant Hodge line. The polarised charts give an effective Cartier divisor D_ϕ, a generated power ω^N, $N \geq 1$, and an ample twist $\omega^{Nk_0}(-D_\phi)$. Tensor this twist with the generated line $(\omega^N)^{k-k_0}$ to obtain ampleness for every $k \geq k_0$ with the same divisor. This uses the integral Siegel construction of [176, Chs. IV–V] at odd full level.

Lean: GlobalTheory.siegelToroidalIntegral_holds (Development.AbelianVarietyTheory.Proofs.DivisorProofs.SiegelToroidalIntegralHolds, lines 14–15)

Lean: GlobalTheory.PolDeg.siegelToroidalIntegral_of_polCharts (Development.AbelianVarietyTheory.Proofs.DivisorProofs.SiegelIntegralToroidalOfPolarisedCharts, lines 16–28)

For the unitary assertion fix a CM field E, host H and moduli datum of signature (p, q), $p + q \geq 1$. A prescribed polarising function ϕ is defined on the semidefinite balanced forms on finite projective $\mathcal{O}_E$-modules of rank at most $\min(p, q)$. A symmetric form B is balanced when $B(ax, y) = B(x, \bar{a}y)$. Pullback of a form along a surjection of lattices preserves the value of the function. On a finite projective lattice one may compute the value after pullback along a free presentation. Two such presentations have a common comparison by taking their direct sum and lifting the two surjections using projectivity. The function is homogeneous and superadditive. In particular the functions on cusp lattices agree on faces where the monodromy form factors through a quotient. For every integral positive semidefinite balanced form B, the value $\phi(B)$ is an integer, positive if $B \neq 0$. The fan covers the semidefinite cone with the rational boundary allowed by the lattice. Its cones are closed under faces and have finitely many types up to lattice automorphism. Their primitive generators extend to integral bases. Each cone satisfies the strict support condition. There is an integral linear form ℓ and an integer $N > 0$ with $\ell \geq N\phi$ on integral positive semidefinite balanced forms and equality on the cone generators. Equality for an integral form B forces some positive integer multiple of B to be a nonnegative integral combination of those generators. The no-self-intersections condition keeps distinct rays of a cone inequivalent under lattice automorphisms.

Lean: GlobalTheory.PolFan.IsCone (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.SumMulEqZeroOfSumToReal, lines 292–302)

Lean: GlobalTheory.PolFan.IsPolFanOn (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.SumMulEqZeroOfSumToReal, lines 405–422)

For each level of parameter M', open moduli scheme and coefficient field F of the datum, there is a model Y over F, smooth of the moduli dimension with simple normal crossings boundary. There are $d, k \geq 1$ and an effective Cartier divisor D_ϕ supported off the abelian locus. At every boundary point an étale completed chart $k'[[t_1, \dots, t_s, z_1, \dots, z_e]]$ has

$$I_{\partial,\mathrm{red}} = (t_1 \cdots t_s), \qquad \mathcal{O}(-D_\phi) = \left(\prod_{i=1}^{s} t_i^{a_i} \right), \qquad a_i = d\phi(\gamma_i), \quad \gamma_i = M'\delta_i. \tag{A.426}$$

Here δ_i are the primitive balanced forms of the chart cone, and γ_i are the monodromy forms on the level lattice. If the determinant Hodge module is a line ω, then $\omega^k(-D_\phi)$ is ample. The logarithmic Kodaira–Spencer map is an isomorphism on the designated interior and agrees there with the map of the moduli open. The PEL toroidal construction corresponding to the characteristic-zero model is [167, Th. 6.4.1.1].

Lean: GlobalTheory.Published.ToroidalForFanKS (Development.AbelianVarietyTheory.PublishedTheorems.Divisors₂.ToroidalCompactificationBis, lines 45–54)

Lean: `GlobalTheory.MumfordChart.HasChartOf` (`Development.AbelianVarietyTheory.PublishedTheorems.AutomorphicForms₂.ToroidalChartsForFan`, lines 17–30)

Start over $\mathcal{O}_F[1/(3M'd_H \operatorname{disc} E)]$, where d_H is the host prime. Put $g = [E : \mathbf{Q}](p+q)/2$. Choose $k \geq 2$ with $N_0 = 3^{2k} > 1 + 2g\,2^{g/2}$, and then choose $j \geq 1$, $r = 3^j$, so that $h = rN_0$ exceeds $81^g g!$, $2N_0^{g+1} g!\, g(3/2)^g 2^{g/2-1}$, and $6N_0^g g!$. On the moduli open, the raw theta-null vector $(c_0, \ldots, c_n)$ of length $n+1 = h^{4g}$ is mapped into $\mathbb{P}^n$ via the homogeneous degree-$(n+1)!$ symmetrized coordinates $e_{j+1}(c_0, \ldots, c_n)^{(n+1)!/(j+1)}$ $(0 \leq j \leq n$, where e_{j+1} is the $(j+1)$-st elementary symmetric polynomial). This symmetrized theta map is defined over the base coefficient ring, has finite geometric fibres, and normalising in it gives a normal projective model finite over $\mathbb{P}^n$ with the moduli open dense on the generic fibre. The pullback of $\mathcal{O}(8(n+1)!)$ is ample; denote it by A. Over a Galois splitting cover of the moduli open (of degree dividing $(n+1)!$) where the individual coordinates c_j separate, each c_j transforms as a section of $\omega^{1/2}$, so each symmetrized coordinate of degree $(n+1)!$ transforms with weight $\omega^{(n+1)!/2}$ and the theta transformation law identifies the pullback of $\mathcal{O}(8)$ with $\omega^{4(n+1)!}$. Taking the field norms of the local trivializing units and raising to the complementary factor descends the $(n+1)!$-th power of this isomorphism on each component of the moduli open. More generally, let Y be a normal projective model over the coefficient field carrying the Hodge line ω of a semiabelian extension of the universal family and a morphism to $\mathbb{P}^n$ extending the symmetrized theta map. Normalize each irreducible component in a splitting field of the generic theta coordinates of degree $m \mid (n+1)!$. On this finite cover the key formula in the proof of Lemma 35.4 identifies the pullbacks of $\mathcal{O}(8)$ and $\omega^{4(n+1)!}$. The field norms of the local trivializing units and their inverses are regular by normality, so they identify the m-th powers on the component. Raise this isomorphism to the power $(n+1)!/m$. The components are disjoint and open, so their isomorphisms identify the pullback of $\mathcal{O}(8(n+1)!)$ to Y with ω^N, where $N = 4((n+1)!)^2$.

Lean: `GlobalTheory.T.lineBundleIso_pow_multiple_degree_descends_normal` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LineBundleIsoPowMultipleDegreeDescendsNormalStatement`, lines 12–13)

Lean: `GlobalTheory.UMin.lbIso_of_ratModels` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LineBundleIsoOfIsoOnRationalModelComponents`, lines 44–56)

Lean: `GlobalTheory.umin_theta_compactification_exists` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.UnitaryIntegralCompactificationExistsViaTheta`, lines 78–142)

On a component meeting the open, take the finite theta cover at modulus N_0 and torsion level rN_0. Form the projective image of all products of one entry from each nonzero finite Fourier tuple and the theta-null tuple, and normalise in the component's function field. On the resulting normal projective model every such tuple has a generating entry locally. Pass to a normal projective model above it carrying the semiabelian extension and its rigidified theta line; the generating entries pull back. Choose the dense theta open U_0 where the nonzero entries and their denominators are units.

We construct the coefficient lattice required in Lemma 35.14. At a point z, let I_z be the labels whose theta null generates its tuple, let H_z be the subgroup they generate, and put $T_z = H_z^{\perp}$ for the level pairing. At a closed centre, on each test, the ratio of a theta null to a generating entry is a unit exactly when its norm is one. Thus I_z consists precisely of the labels attaining the maximum theta-null norm. Write K_v for the kernel of the period quotient on the level labels. Positivity of the period form gives strictly smaller norm outside K_v. For each $a \in K_v$, some translate $a + b$ with $3b = 0$ attains the maximum: on the abelian quotient the translates by three-torsion of the cubed principal theta section have no common zero, and the chosen torsion point lifts at this invertible level. Hence $b \in K_v$. The period quotient is free over $\mathbb{Z}/h\mathbb{Z}$, so its kernel is a free summand. Since h is a power of three divisible by nine, $K_v[3] \subseteq 3K_v$. It follows that I_z spans $K_v/3K_v$, hence generates K_v. Thus $H_z = K_v$, and T_z is the toric subgroup used in the Fourier calculation. For every character $\chi : T_z \to \mu_{N_0}$, choose a generating Fourier entry F_{χ,k_χ}, and choose $x_0 \in I_z$. Define

$$E_z = \sum_{\chi,i} \mathcal{O}_{X,z}\, \frac{c_{x_0}}{F_{\chi,k_\chi}} P_\chi e_i.$$

The nonzero Fourier tuples and the invertible level give these generators. Changing either generating entry multiplies the relevant generators by units. Character orthogonality gives mutually orthogonal projectors summing to one; an adapted basis in their images, scaled by the displayed ratios, is a basis of E_z.

Here are the compatibility checks. Under a generalisation ξ of z, $I_z \subseteq I_\xi$ and $T_\xi \subseteq T_z$, and a projector for T_ξ is the sum of the projectors extending its character to T_z. We first compare a closed point z with a generalisation ξ whose local ring is a height-one valuation ring. On a centred test at z, let α be a shortest representative of the character χ, and choose a shortest representative β for the Fourier bound for χ'. The

two characters agree on T_ξ, so $\alpha-\beta$ lies in the subgroup generated by the period labels in I_ξ and N_0X_v. For each $x\in I_\xi$, choose a period representative w_x from its theta-null expansion, at level $h=rN_0$. In the coset of α modulo N_0X_v one can then use a vector of the form $\beta+\sum_x k_xw_x$ with $0\le k_x<N_0$. Apply $aq(A+B)\le(a+1)q(A)+a(a+1)q(B)$ and $q(\sum_x k_xw_x)\le|I_\xi|N_0^2\sum_x q(w_x)$. The theta-null bounds for w_x, the Fourier bound for β, and equality for the generating Fourier entry at z give the following comparison. Concretely, if $f=F_{\chi,k_\chi}/c_{x_0}$ and $f'=F_{\chi',k_{\chi'}}/c_{x_0}$, the character-projector comparison gives $(f')^{a+1}u^C/f^a\in\mathcal{O}_{X,z}$ for every $a\ge1$, for some $C\ge0$, where $u=\prod_{x\in I_\xi}c_x/c_{x_0}$. At ξ, u is a unit; taking valuations and letting a grow gives $v(f')\ge v(f)$. For the opposite inclusion choose a closed specialisation of ξ with the same maximal label set I_ξ in the open where the chosen Fourier entry for χ' generates. Such a specialisation exists by the Jacobson property and the open conditions specifying the generating entries. On each test centred there, the toric subgroup is T_ξ. Choose a shortest representative for χ'. The Fourier estimate for the larger isotropic subgroup T_z, with the compatible character χ, bounds the absolute value of its entry by $\delta_v\exp(-q_v(\alpha)/N_0)$ for this same representative α. The generating entry for χ' has exactly that absolute value. Testing the quotient on these traits gives $f/f'\in\mathcal{O}_{X,\xi}$, hence $v(f)\ge v(f')$. The projector decomposition therefore identifies the two localised spans. Normality makes the transition coefficients and their inverses regular. The finitely many generators and an adapted basis extend over a neighbourhood, so these local free lattices glue to a coherent lattice E, standard on U_0.

For a centred test v with generic point in U_0, put $\delta_v=\max_x|c_x(v)|$. Character orthogonality and the finite Fourier evaluation give

$$|c_{x_0}(v)|=\delta_v,\qquad |F_{\chi,k_\chi}(v)|=\delta_v\exp(-q_v(\alpha)/N_0)$$

when α is a shortest representative of the class giving χ. This is the finite Fourier calculation in the proof of Lemma 35.14, applied over this individual test; it uses the translation law and an integral coefficient section nonzero on the special fibre. Thus the scalar multiplying $P_\chi e_i$ in E_z has norm exactly $\exp(q_v(\alpha)/N_0)$. Every allowed scalar u in that character component divided by this scalar has norm at most one. These two observations give both inclusions in the required evaluated-lattice equality on every test. The construction uses the Fourier generators on the model and permits different shortest representatives on different tests. It supplies the coherent lattice hypothesis of Lemma 35.14. Fix a point and a closed specialisation x of it. Apply the lemma at x, with $N=N_0$ the theta modulus, to obtain μ_c,e_c. To identify the lattice, take a trait centred at x with generic point in the theta open. Such a trait is obtained by the one-dimensional quotient construction of Lemma 35.13 and completion, followed by finite extension for the polarised degeneration with level. Lemma 35.15 identifies its lattice X_0 with the completed period lattice. Its monodromy form b_0 is positive definite by positivity of the periods and is balanced by compatibility of the polarisation with conjugation on E. The character lattice is torsion-free and finite over the Dedekind ring $\mathcal{O}_E$, hence projective. Its rank is at most each of the two signature multiplicities: at either conjugate embedding its torus contributes that many dimensions to the corresponding Lie space. Thus its rank is at most $\min(p,q)$.

Lean: `GlobalTheory.UTraitCen.exists_uTrait_cen` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsUnitaryTraitCentredClosedPoint`, lines 18–47)
Lean: `GlobalTheory.UPoinc.UTrait.finrank_le_min` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.CharacterGroupFinrankLeMinSignatureUnitaryTrait`, lines 17–50)

For each degeneration trait in Lemma 35.15, pull its monodromy form back to X_0 and call the result B. The lemma gives $B(\mu_c,\mu_c)\le B(\alpha,\alpha)$ for every α in the class c, including when $B=b_0$. With the order normalised to be one on a uniformiser it gives

$$\operatorname{ord}(e_c^{m_1})=m_2B(\mu_c,\mu_c),\qquad m_1=2N_0,\quad m_2=1. \tag{A.427}$$

A trait whose centre specialises to the originally fixed point also specialises to x, by transitivity. The same choices therefore apply at that point.

Lean: `GlobalTheory.T.unitaryComparison_lattice_periods_at_point` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.UnitaryComparisonLatticePeriodsAtPointStatement`, lines 13–14)
Lean: `GlobalTheory.UPoinc.ClosedPt.exists_lattice_data` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsBalancedLatticeDataClosedPoint`, lines 61–146)
Lean: `GlobalTheory.unitaryStepA_at_point` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.UnitaryComparisonLatticePeriodsAtPoint`, lines 15–29)

Apply Lemma 35.16 on X_0 with modulus $N_0\ge3$. The trait through x gives the positive definite witness b_0, and Lemma 35.15 gives its minimum inequalities for the chosen representatives. Thus a finite list of cones in X_0 covers all the balanced semidefinite forms under consideration.

Clear denominators in the finitely many supporting linear forms. Their minimum on this region is a positive integer multiple of ϕ. Each linear form on balanced forms is a rational combination of the evaluations $B \mapsto B(\mu_c, \mu_c)$. To see this, reduce modulo an odd prime dividing N_0. The representatives exhaust the residue vectors there. Evaluation at the coordinate vectors and their pairwise sums spans the dual of the space of symmetric forms in odd characteristic, hence the evaluations span over $\mathbf{Q}$ before restriction to balanced forms. The finite orbit set of minimiser systems and the finitely many isomorphism types of bounded-rank projective $\mathcal{O}_E$-lattices allow a common multiplier $\iota_0 > 0$ and a finite list of coefficient sets. Transport in the finite residue modules permits the same list in lattice coordinates. Thus at the point one can choose a finite set $\mathcal{L}$ of integral vectors $\nu = (\nu_c)_c$ such that for every B in its minimiser region,

$$\iota_0 \phi(B) = \min_{\nu \in \mathcal{L}} \sum_c \nu_c B(\mu_c, \mu_c). \tag{A.428}$$

Lean: `GlobalTheory.UPoinc.CellCones.exists_finite_cones` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsFiniteConesPolarizedFanLevelStructure`, lines 99–215)
Lean: `GlobalTheory.UPoinc.FinTypes.single` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LeastIntCombinationMinimiserCellUnitary`, lines 21–58)
Lean: `GlobalTheory.UPoinc.finite_types_unitary` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LeastIntCombinationMinimiserCellUnitary`, lines 126–186)

Take the tuple with entries $\prod_c e_c^{m_1 \nu_c}$, $\nu \in \mathcal{L}$. Formula (A.427) gives the order of each entry by addition. The quotients are invertible at the generic point of each trait being tested, so negative powers also have these orders. By (A.428) the least order is $m_2 \iota_0 \phi(B)$. Invariance under the lattice surjection identifies $\phi(B)$ with $\phi(b)$ for the original monodromy form b. The tuples come from a finite collection because the theta quotients and the coefficient sets range over finite sets.
Lean: `GlobalTheory.UPoinc.exists_uLocTuples_theta` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsFiniteThetaTuplesLeastOrdersUnitary`, lines 17–185)

Take a common normal projective modification $\pi : Y \to X$ of the finite theta model X on which every finite theta-quotient tuple ideal is principal. Put $\iota_1 = m_2 \iota_0$. Because Y is normal, each local principal equation on an open chart covering a point $y \in Y$ has valuation $\iota_1 \phi(b_E)$ along every prime divisor $E \subset Y$ passing through y (testing along the complete DVR $\widehat{\mathcal{O}}_{Y,E}$ via Lemma 35.15), so the local equations agree up to units on overlaps and glue to a well-defined effective Cartier boundary divisor D on Y whose pullback along every degeneration trait has order $\iota_1 \phi(b)$. Independently, we construct approximating regular germs in the local stalk $\mathcal{O}_{X,z}$ of the normal projective theta model X at each closed point $z \in X$. By finiteness of cone orbits of the polarised fan function ϕ (verified in the previous paragraph), there is a uniform integer $d_0 \geq 1$ such that every positive definite balanced monodromy form b on the period lattice X_0 admits a supporting tuple $a_0 = (a_{0,i})$ spanning X_0 over $\mathbf{Z}$ ($\mathrm{span}_{\mathbf{Z}}(a_0) = X_0$) on a smooth cone with relative interior containing b. To obtain the tuple, take the strict supporting rational functional of the cone and compose it with the CM averaging projection onto balanced forms. On the balanced space this changes only its positive scale. The averaged functional is strictly positive on every nonzero rank-one square, so its symmetrised Gram matrix in an integral basis is positive definite. Subtract a small positive rational multiple of the identity and express the remaining rational positive definite matrix as a sum of rational vector squares by rational diagonalisation. Clear all denominators, including the denominator of the identity coefficient, and add back that identity term as copies of the basis-vector squares. This produces integral vectors spanning X_0 and a positive integral multiple of the supporting functional. A common multiple for the finite cone and lattice types gives d_0, with $\sum_i b'(a_{0,i}, a_{0,i}) \geq d_0 \phi(b')$ for every balanced positive semidefinite form b' and equality throughout the cone. Fix a closed point $z \in X$, a centred classical degeneration trait y_0 at z with period lattice X_0, monodromy form b_{y_0}, and log-uniformiser scale $s_{y_0} > 0$, and a target precision $n \geq 1$. If $X_0 = 0$, the period-lattice isomorphisms of Lemma 35.15 give $X_y = 0$ and $\phi(b_y) = 0$ on every classical trait y centred at z; then $g_n = 1 \in \mathcal{O}_{X,z}$ satisfies $|g_n(y)| = \exp(-s_y d_0 \phi(b_y)) = 1$ on every such trait, giving the equality at y_0 and the upper bound below.
Lean: `GlobalTheory.Poinc.approx_of_fpre` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.CongModOfSpan`, lines 128–306)

For $X_0 \neq 0$, let $A = \mathcal{O}_{X,z}$ be the normal Noetherian integrally closed local ring at z, and pass to its integral closure B in a finite Galois deeper-level function-field extension Ω_2/Ω_1 of group $G = \mathrm{Gal}(\Omega_2/\Omega_1)$ and sufficiently large power N of 3 for the deeper theta level, divisible by the base theta level $N_1 = h = rN_0 \geq 3$, choosing a maximal ideal $\mathfrak{z} \subset B$ lying over $\mathfrak{m}_{X,z}$. Because 3 is invertible on the base $\mathcal{O}_F[1/(3M' d_H \operatorname{disc} E)]$, N is coprime to the residue characteristic $p = \mathrm{char}\, \kappa(z)$. Put $D_{\mathfrak{z}} = \{\sigma \in G : \sigma \mathfrak{z} = \mathfrak{z}\}$. For each entry $a_{0,i}$, fix

a Fourier coefficient index κ_i giving a unit leading coefficient on y_0. Write $f(c_i,\kappa_i)\in B$ for the deeper-level Fourier class sum in residue class c_i, expressed in the common base theta normalisation with index κ_i at the base level. For $\sigma\in D_{\mathfrak{z}}$, compare a lifted centred test and its σ-translate at the same deep centre. Transport of the full torsion labels fixes the base labels and induces an $\mathcal{O}_E$-linear lattice automorphism γ_σ on the congruence classes, congruent to the identity modulo N_1 and of determinant one modulo N. On each such pair of tests, $\sigma f(c,\kappa)$ and $f(\gamma_\sigma c,\kappa)$ are limits of finite sums of the same terms in the same residue class. Both expansions use the common underlying test and its base theta normalisation, including the Fourier phase for κ. For any positive tolerance, choose a finite set containing the tails' two cutoffs. The ultrametric inequality bounds the difference of the two evaluations by that tolerance. Detection on centred tests gives $\sigma f(c,\kappa)=f(\gamma_\sigma c,\kappa)$ in B. Now sum $\prod_i f(c_i,\kappa_i)$ once over each distinct residue tuple c in the congruence orbit of a_0 mod N. The equality permutes the summands of this finite sum and gives a weight-0 element $F\in B^{D_{\mathfrak{z}}}$ supported on the congruence orbit C of a_0 (consisting of tuples $\mu\equiv(k_y\circ\gamma')(a_0)\pmod N$ for endomorphisms γ' of X_0 that are $\mathcal{O}_E$-linear modulo N and satisfy $\det\gamma'\equiv 1\pmod N$ and $\gamma'\equiv\mathrm{id}\pmod{N_1}$, where $k_y: X_0\xrightarrow{\sim} X_y$ is the period-lattice isomorphism supplied by Lemma 35.15 on any trait y centred at z). For the unimodular-or-large dichotomy, fix a finite $\mathbf{Z}$-relation basis S of a_0 together with $\mathcal{O}_E$-module generators and a cutoff $K>0$, and write $d_{\mathrm{per}}=\mathrm{rk}_{\mathbf{Z}}X_0$: for $N\ge N_0(S,K)$ sufficiently large, consider any tuple μ on a trait y centred at z (with period lattice X_y, where $X_{y_0}=X_0$ and $k_{y_0}=\mathrm{id}$) congruent modulo N to $(k_y\circ\gamma')(a_0)$ with γ' as above. When μ satisfies all $\mathbf{Z}$-relations in S and all $\mathcal{O}_E$-module relations, it defines an $\mathcal{O}_E$-linear map $e: X_0\to X_y$ with $e(a_0)=\mu$ (since $\mathrm{span}_{\mathbf{Z}}(a_0)=X_0$) and an $\mathcal{O}_E$-endomorphism $k_\mu=k_y^{-1}\circ e\in\mathrm{End}_{\mathcal{O}_E}(X_0)$ congruent to γ' modulo N, whence $\det(k_\mu)\equiv\det\gamma'\equiv 1\pmod N$. Since $N\ge 3$, either $\det(k_\mu)=1$ (so k_μ is unimodular and $e: X_0\xrightarrow{\sim}X_y$ is an $\mathcal{O}_E$-linear isomorphism, yielding $\sum_i b_y(\mu_i,\mu_i)\ge d_0\phi(b_y)$), or else either a relation in S or an $\mathcal{O}_E$-relation fails (producing a nonzero vector in NX_y, so $\sum_i b_y(\mu_i,\mu_i)\ge c_1(S)N^2\lambda_{\min}(b_y)$) or $|\det(k_\mu)|\ge N-1$ (in which case Hadamard's inequality and Minkowski's norm bound give $\sum_i b_y(\mu_i,\mu_i)\ge c_2(a_0)(N-1)^{2/d_{\mathrm{per}}}\lambda_{\min}(b_y)$), yielding $\sum_i b_y(\mu_i,\mu_i)\ge K\lambda_{\min}(b_y)$ in both failure cases for $N\ge N_0(S,K)$. Moreover, on y_0 (where $X_{y_0}=X_0$ and $k_{y_0}=\mathrm{id}$), the orbit congruence $\mu\equiv\gamma'(a_0)\equiv a_0\pmod{N_1}$ together with $e(a_0)=\mu$ and $\mathrm{span}_{\mathbf{Z}}(a_0)=X_0$ forces $e\equiv\mathrm{id}\pmod{N_1}$ ($N_1\ge 3$); any such unimodular automorphism attaining the minimum stabilizes the positive definite cone-interior sum $\sum\gamma_j$ and therefore equals id by Minkowski's congruence rigidity mod $N_1\ge 3$, isolating a unique minimal leading term on y_0 with unit abelian coefficient and order $d_0\phi(b_{y_0})$. For the Galois descent step at any target ideal power $e\ge 1$, because B is finite over the Noetherian local ring $A=\mathcal{O}_{X,z}$, every prime ideal of B inside $\mathfrak{z}$ containing $\mathfrak{m}_{X,z}^e B$ lies over $\mathfrak{m}_{X,z}$ and is therefore maximal and equal to $\mathfrak{z}$; hence $\mathfrak{z}$ lies in the radical of $\{b\in B:\exists\,\sigma\in B\setminus\mathfrak{z},\ \sigma b\in\mathfrak{m}_{X,z}^e B\}$, providing an exponent $E\ge 1$ such that every $b\in\mathfrak{z}^E$ admits a multiplier $\sigma\in B\setminus\mathfrak{z}$ with $\sigma b\in\mathfrak{m}_{X,z}^e B$. Pick a Chinese-remainder element $x\in B$ with $x\equiv 1\pmod{\mathfrak{z}^E}$ and $x\equiv 0\pmod{\mathfrak{z}_j^E}$ for every other maximal ideal $\mathfrak{z}_j\ne\mathfrak{z}$ lying over $\mathfrak{m}_{X,z}$, set $e_{\mathrm{cut}}=\prod_{\tau\in D_{\mathfrak{z}}}\tau(x)\in B^{D_{\mathfrak{z}}}$, and sum over coset representatives in $G/D_{\mathfrak{z}}$ to define $g_n=\sum_{g\in G/D_{\mathfrak{z}}}g(e_{\mathrm{cut}}F)\in B^G=A=\mathcal{O}_{X,z}$ (with no $|D_{\mathfrak{z}}|$ factor); then $g_n-F\in\mathfrak{z}^E$, hence $\sigma(g_n-F)\in\mathfrak{m}_{X,z}^e B$ for some $\sigma\in B\setminus\mathfrak{z}$. On every classical degeneration trait y centred at z (lifted to $\mathfrak{z}$), maximality of $\mathfrak{z}$ forces $|\sigma(y)|=1$, so $|g_n(y)-F(y)|=|\sigma(y)(g_n(y)-F(y))|\le\rho_y^e$ where $\rho_y=\max_i|h_i(y)|<1$ over a finite generating set $\{h_i\}$ of $\mathfrak{m}_{X,z}$. Choose $v\in X_0$ whose image under k_y is a shortest nonzero vector for b_y. Then v is primitive in X_0, so its residue class $c=v\bmod N_0X_0$ is nonzero and $e_c^{2N_0}\in\mathfrak{m}_{X,z}$ satisfies $\exp(-s_y\lambda_{\min}(b_y))=|e_c(y)^{2N_0}|\le\rho_y$. First choose $K\ge n$ and $e\ge n$ with $K\lambda_{\min}(b_{y_0})>d_0\phi(b_{y_0})$ and $\rho_{y_0}^e<\exp(-s_{y_0}d_0\phi(b_{y_0}))$, and then choose N large enough for the preceding congruence and cutoff requirements. With these choices, we have $\exp(-s_yK\lambda_{\min}(b_y))\le\rho_y^n$, and ultrametric dominance yields $|g_n(y_0)|=\exp(-s_{y_0}d_0\phi(b_{y_0}))$ on y_0 and $|g_n(y)|\le\max(\exp(-s_yd_0\phi(b_y)),\rho_y^n)$ on every classical trait y centred at z. In both cases set $L=\mathrm{lcm}(\iota_1,d_0)$, $r=L/\iota_1$, and $m'=L/d_0$. At any closed point $x\in Y$ above $z=\pi(x)$, Lemma 35.13 supplies a centred local test y_0 at x with generic point in the moduli open U, and setting $F_n=g_n^{m'}\in\mathcal{O}_{X,z}$ gives $|F_n(y_0)|=\exp(-s_{y_0}L\phi(b_{y_0}))=|s_x(y_0)|$ and $|F_n(y)|\le\max(\exp(-s_yL\phi(b_y)),\rho_y^{m'n})\le\max(|s_{x'}(y)|,\rho_y^n)$ (for local equations $s_{x'}$ of rD, since $m'\ge 1$ and $\rho_y\le 1$) on every test y at every closed point x' over z. Applying Lemma 35.17 to $\pi: Y\to X$ and rD therefore descends $\mathcal{O}_Y(-rD)$ to an ideal $J_1\subseteq\mathcal{O}_X$ with $J_1\mathcal{O}_Y=\mathcal{O}_Y(-rD)$, giving trait orders $\iota\phi(b)$ for the common integer $\iota=L=r\iota_1\ge 1$.

Lean: `GlobalTheory.UPoinc.DescentPow.exists_idealSheaf_comap_uValThru` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsIdealSheafDescendingDivisorPower`, lines 73–163)

Apply Lemma 35.18 to J_1 on the finite theta model over this component. The two schemes are integral

and normal, and their function fields have characteristic zero, so the generic extension is separable. Write h for its degree. On an extended trait the norm ideal becomes the product of the h branch ideals. After a further common extension choose semiabelian degeneration data for the moduli point. Every branch has that same underlying point; hence the order formula for J_1 applies with the same monodromy form on all branches. If the total ramification index is e, this form is eb, since each order in the period pairing is multiplied by e. Writing J for the norm ideal gives $e\,\mathrm{ord}_V J = h\iota\phi(eb) = eh\iota\phi(b)$. Division by e gives the multiplier $h\iota$ on the original trait.

On components disjoint from the moduli open take the unit ideal. By normality and noetherianity, the finitely many components are disjoint. Each is both open and closed. Raise the component ideals to powers giving a common multiple d of their order multipliers. The resulting ideal J has pullback

$$J\mathcal{O}_{\mathrm{trait}} = (\varpi^{d\phi(b)}) \tag{A.429}$$

on each degeneration trait with parameter ϖ.

Lean: `GlobalTheory.UPoinc.exists_zhModel_effCartier_orders` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.UnitaryExistsNormalModelEffCartierAlongTraits`, lines 15–24)
Lean: `GlobalTheory.UPoinc.exists_uSetup_ideal_over_component` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsUnitaryThetaSetupIdealOverComponent`, lines 17–25)
Lean: `GlobalTheory.UPoinc.UNormDown.normDescent` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.UnitaryNormDescentIdealToComponent`, lines 26–76)
Lean: `GlobalTheory.UPoincDown.idealDescent` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.IdealDescentToComponentCompactification`, lines 45–71)
Lean: `GlobalTheory.UPoinc.uIdealSt_holds` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.UnitaryCompactificationExistsBoundaryIdeal`, lines 19–38)

On the generic fibre Z take the normalisation S of the blow-up $S_0 = \mathrm{Proj}_Z \bigoplus_{j\ge 0} J^j$ and let $\pi : S \to Z$ be the projection. Over the unit-ideal locus this map is an isomorphism. The tautological line $J\mathcal{O}_{S_0} = \mathcal{O}_{S_0}(1)$ is relatively ample. Its pullback by the finite normalisation is $J\mathcal{O}_S = \mathcal{O}_S(-D_\phi)$ and is relatively ample over Z. To see the absolute ampleness explicitly, choose v so large that $J \otimes A^v$ is generated by global sections on the projective scheme Z. A finite generating set gives a surjection from a polynomial algebra onto $\bigoplus_{j\ge 0} J^j \otimes A^{vj}$. Relative Proj embeds S_0 as a closed subscheme of $Z \times \mathbb{P}^m$ for some m; twisting the graded pieces by A^{vj} leaves the underlying Proj unchanged. The restriction of the second factor's $\mathcal{O}(1)$ is $J\mathcal{O}_{S_0} \otimes \pi_0^* A^v$, where $\pi_0 : S_0 \to Z$. For $u > v$, the external product $A^{u-v} \boxtimes \mathcal{O}(1)$ is ample. Its restriction to S_0 and pullback under the finite normalisation give the ample line $\pi^* A^u(-D_\phi)$. Write D_S for the Cartier divisor of $J\mathcal{O}_S$. We compare the two models on one component of the moduli open. Let Y be the toroidal compactification for the prescribed fan. The lattice with its conjugate-adjoint action and principal polarisation is a PEL datum in characteristic zero. On a geometric component the Weil pairing is fixed. The specified half-level fixes one summand of the M-torsion, and the complementary summand is its perfect dual since $\mathfrak{b}\bar{\mathfrak{b}} = (M)$ and the two ideals are coprime. An isometry fixing the first summand acts identically on the second. Thus the component group has principal congruence level $M \ge 3$, with M odd. It is neat: at any odd prime $p \mid M$ the eigenvalues of a matrix congruent to the identity modulo p belong to $1 + p\overline{\mathbf{Z}}_p$. Their products and inverses lie there too, and the only root of unity in this group is one. This check may be made after adjoining the level roots of unity. Isomorphisms extending the identity on the moduli open descend uniquely. The strict supporting forms give a projective cone decomposition, smooth by the basis condition and compatible at the cusps by invariance under lattice quotients. Apply [167, Ths. 6.4.1.1 and 7.3.3.4] (noting that at every good prime $\ell \nmid 2dD_E M'$, E is unramified at ℓ, the PEL lattice is self-dual at ℓ, and M' is prime to ℓ, supplying also the smooth integral toroidal models of Definition 35.5(1)), in the characteristic-zero formulation of [185, Construction 3.1 and Th. 3.10]. The Siegel construction is [176, Ch. V, Th. 5.8]. For some positive multiple d_1 of the polarising function this gives on Y a divisor D_Y with orders $d_1\phi(\gamma_i)$. Its negative is relatively ample over the minimal compactification, on which a positive Hodge power is ample. Consequently $\omega^{Nud_1}(-dD_Y)$ is ample for all sufficiently large divisible u.

Let G be the normalisation of the closure of the common open in $Y \times_F S$, with projections $a : G \to Y$, $b : G \to S$. Both are proper birational morphisms between normal varieties. On G use the pullback of the semiabelian family on Y. Its abelian locus is the common open, since the graph is already closed over that open. The theta-line identification applied on this model to $\pi b : G \to Z$ gives $b^*\pi^* A \simeq a^*\omega^N$. We claim that $d_1 b^* D_S = d a^* D_Y$. Test at the completed local ring of each prime divisor of G, a complete discrete valuation ring with generic point on the moduli open. Over the interior both orders are zero. At a boundary centre use the toroidal chart of Y; one may extend the valued field to split the cusp data. If the orders of its boundary

parameters are v_i, the pulled-back family has monodromy form $B = \sum_i v_i\gamma_i$: this follows by taking orders of the monomials of the chart on each period. The form lies in the chart cone, where ϕ is linear. Hence the order of a^*D_Y is $d_1 \sum_i v_i\phi(\gamma_i) = d_1\phi(B)$. Equation (A.429), applied to the same trait through $G \to S \to Z$ and the same semiabelian family, gives order $d\phi(B)$ for b^*D_S. A ramified extension multiplies both orders by its ramification index. Normality of G now proves the claimed equality of Cartier divisors.

Choose u sufficiently large and divisible for both ample twists. The pullbacks to G of $\omega^{Nud_1}(-dD_Y)$ and $(\pi^*A^u(-D_S))^{d_1}$ are isomorphic. For a proper birational morphism onto a normal variety the direct image of the structure sheaf is the structure sheaf. The projection formula thus identifies both graded section rings with the section ring of the common pullback, compatibly with multiplication. Taking Proj recovers Y and S from their respective ample lines. The resulting isomorphism extends the original identification of the open, by construction on its graph. In particular $Y \to S$ is finite. On this model $\pi^*A \simeq \omega^N$, so $k = Nu$; the divisor of J has exponents $d\phi(\gamma_i)$ in (A.426). This also gives the valuation normalisation for a ramified trait or level pullback. In a framed chart the logarithmic Kodaira–Spencer map sends a log derivation to its balanced symmetric matrix. Its source and target are finite projective of the same fibre rank. Injectivity on the fibres of the rigid charts gives the isomorphism by the local criterion for an isomorphism of vector bundles.

Lean: `GlobalTheory.UPoincTop.uPoinStX_sep` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.UnitaryToroidalSeparationOfBase`, lines 23–88)
Lean: `GlobalTheory.BlowAmpleNoeth.ample_twist_of_blowup` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.AmpleTwistNormalisedBlowupNoetherianDomain`, lines 21–66)
Lean: `GlobalTheory.UChartJoin.rigidForFan_of_uStB` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.RigidBoundaryChartsForFanOfUnitarySeparation`, lines 21–87)
Lean: `GlobalTheory.LogKSAssembly.hasLogKS` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LogKodairaSpencerIsoSmoothSncToroidalModel`, lines 57–69)
Lean: `GlobalTheory.RigChart.toroidalForFanKS_of_rigidForFan` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ToroidalFanKodairaSpencerOfRigidBoundaryCharts`, lines 16–19)

Let C be a smooth projective geometrically connected curve over a field of characteristic zero, and let $G \to C$ be semiabelian with identity section e. Bost's seminegativity consequence of [182, Th. 3.2] applies: the group scheme is smooth and its generic fibre is an extension of an abelian variety by a torus. Such a fibre has no nontrivial connected unipotent subgroup. Thus every subbundle of $\mathrm{Lie}(G/C)$ has degree at most zero. For any locally free subsheaf W take its saturation W'. On the smooth curve W' is a subbundle and $\deg W = \deg W' - \deg(W'/W) \le 0$. This also gives nefness of the Hodge bundle $e^*\Omega^1_{G/C}$. On pulling back to a smooth projective curve over an algebraically closed extension of the ground field, the group scheme remains semiabelian and formation of its Lie bundle commutes with pullback. Dualising any quotient line of the pulled-back Hodge bundle gives a line subbundle of the pulled-back Lie bundle. Its degree is nonpositive, as required.

Lean: `GlobalTheory.Nef.linesLE_zero_of_sub_lie` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LieSubsheafLinesDegreeNonpos`, lines 18–69)
Lean: `GlobalTheory.Nef.semiNeg_lie_of_isAlgClosed` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LieSheafSemiAbelianSemiNegOfIsAlgClosed`, lines 27–34)
Lean: `GlobalTheory.Nef.semiAbelianLieSeminegative` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.SemiAbelianLieSeminegativeBis`, lines 22–28)

For point separation let (X, Δ) be an integral normal projective klt pair over $\mathbf{C}$, of dimension n, with Δ effective. Let $G = m(K_X + \Delta)$ be Cartier with $m \ge 1$. Take a line bundle M and a nef and big Cartier divisor N_0 with $M \equiv G/m + N_0/r$, $r \ge 1$ integral. For distinct closed points x_1, x_2, assume there are $c(e) > 0$, $1 \le e \le n$, with

$$\frac{(N_0^e \cdot Z)}{r^e} > c(e)^e \quad \text{for every irreducible } e\text{-fold } Z \text{ through either point}, \qquad \sum_{e=1}^{n} \frac{2^{1/e} e}{c(e)} \le 1. \tag{A.430}$$

Set $\rho = 1/r$. By integrality, $(N_0^e \cdot Z) \ge D_e = \lfloor r^e c(e)^e \rfloor + 1$. For each e choose a rational h_e strictly between $2^{1/e}e/D_e^{1/e}$ and $2^{1/e}e/(rc(e))$. Then

$$\sum_e h_e < \rho, \qquad h_e^e (N_0^e \cdot Z) > 2e^e. \tag{A.431}$$

Write $aN_0 = A + E_1$ with A ample and E_1 effective Cartier, by bigness. On a log resolution of the pair and E_1, all log discrepancies stay positive after adding E_1/m' for sufficiently large m'. By (A.431), choose such

a multiple m' of r with $q = m'/r > a$ and $\sum_e h_e < (q-a)/m'$. The ample divisor $A' = A + (q-a)N_0$ gives

$$M \equiv K_X + \Delta + E_1/m' + A'/m', \qquad (A'^e \cdot Z) \geq (q-a)^e (N_0^e \cdot Z). \tag{A.432}$$

The mixed terms of the intersection are nonnegative by nefness. Apply the ample two-point theorem [168, Th. 1.2] to the log canonical pair $(X, \Delta + E_1/m')$ and ample part A'/m', with thresholds $c'(e) = 2^{1/e} e(q-a)/(m' h_e)$. Equations (A.431) and (A.432) give $((A'/m')^e \cdot Z) > c'(e)^e$. The weighted sum of the reciprocals is $m' \sum_e h_e/(q-a) < 1$. The theorem gives a section of M nonzero at exactly one of the two points. The reduction to the ample case gives point separation for the nef and big adjoint part in (A.430).

Lean: `GlobalTheory.pointSeparationPair_of_vanishing` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.KollarTwoPointSeparationOfNadelVanishing`, lines 15–38)

Lean: `NadelVanishingSeparation.NSI.pointSeparationPair` (`Development.WeilBound.CoreStatements.AutomorphicKernels.NadelVanishing.TopPointSep`, lines 6–7)

Lean: `GlobalTheory.Published.PointSeparationPair` (`Development.AbelianVarietyTheory.PublishedTheorems.Divisors₂.PointSeparationPair`, lines 15–30)

The arithmetic degree assertion comprises preservation of self-intersection under birational pullback between generically smooth arithmetic varieties and equality of the complex top Chern integral of a smooth weight with the divisor's top intersection. The first follows by projection for cycle heights, the second by the complex Chern degree formula [170]. Grothendieck deformation uses the de Rham crystal over the nilpotent divided-power thickening and the lifting criterion for homomorphisms preserving the Hodge filtration.

Lean: `UpperTheory.LiteratureTheorems.ar_degree_of_literature` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin`, lines 251–252)

Lean: `GlobalTheory.Published.grothendieckDeformation_of_lifting` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.GrothendieckDeformationOfLifting`, lines 9–12)

Equations (A.422) and (A.423) give the arithmetic Kleiman inequality and the two upper bounds for small-section counts required in 35.11. The lower bound for arithmetic volume is Theorem 35.33. In the order of Theorem 35.11, resolution, principalisation and invariant Poincaré–Lelong supply the first three premises. The fourth is the conjunction of the three arithmetic inequalities just identified, and the fifth is Theorem 35.33. Hodge–Tate exactness supplies the sixth. The principally polarised retraction constructed using (A.424) supplies the seventh, and (A.425) supplies the eighth. The integral Siegel compactification and the unitary compactification for the prescribed fan supply the ninth and tenth, respectively. Lie semi-negativity, two-point separation and the arithmetic degree formula are the eleventh through thirteenth. The fourteenth is the conjunction in Definition 35.5. The fifteenth uses de Rham comparison and lifting of abelian schemes and their homomorphisms across nilpotent divided-power thickenings. □

Proof of 35.20

Proof. Put $L = L_x$ and use (A.418). Increase the threshold so that $x \geq 6$ and $(x^A)^2 \leq b$. The host and weight estimates give

$$\begin{gathered} f \geq 8, \qquad (frL)^{12} \leq P_* \leq x^A, \qquad frn_V L \leq x^A, \qquad 20fr \leq b, \\ \kappa \leq 4frb, \qquad \sum_{\rho} w_{j,\rho} \leq \kappa + fr, \qquad b \leq \kappa + r. \end{gathered} \tag{A.433}$$

For the product estimate use the host bounds $fr \leq x^{20}$, $n_V \leq 2t \leq x^{1+C_t}$, and $L \leq x$. Definition 33.3(1) gives $C_t = 401$, so $frn_V L \leq x^{423}$ and $(frL)^{12} \leq x^{252}$. Keep the family's chosen $C_P \geq 1$ and use $e_P = 12$. Increase the family threshold so that $x \geq 1$ and $x^{171} \geq C_P$. The defining bound $P_* \leq C_P x^{21 e_P}$ then gives $P_* \leq C_P x^{252} \leq x^{423} = x^A$. Use the absolute row sums and allow a correction of at most r at each embedding. The layer weight has size at least b. Then $b \leq 2\kappa$ and $\sum_\rho w_{j,\rho} \leq 3\kappa$. By (A.418), $|M_{p,i}| \leq 9rL\kappa \leq P_* \kappa$. Also $9rL\kappa \leq 36 fr^2 Lb \leq x^B b$ by (A.433). The bounds on E_p follow from its expression in (A.418) and $fmL \leq frn_V L$.

Lean: `GlobalTheory.CovTop.Sizes.hostN_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.HostN`, lines 13–38)

Lean: `GlobalTheory.CovTop.WtShape.wtNum_at` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.WeightBoundHolds`, lines 45–185)

Lean: `GlobalTheory.CovTop.SizeRows.sum_e_le` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.SizeBoundStatement`, lines 196–221)

At a nonterminal layer the flag dimension is $d_* = f\binom{r}{2}$. Use $T = T_* = \lceil P_*^2 d_*^2 k/m \rceil_{\mathbf{N}}$ as the weight-division length. The family parameters give the wall-gap parameter $h_0 \in \mathbf{Z}$ and width parameters $S, \eta \geq 0$ with $P_*(S+\eta) \leq 100w''$, where $w'' = b\exp(-x^{\varepsilon/3})/(f/2)$ (since $|A_{\mathrm{host}}| = f/2$ in Definition 5.1) is the stretched-exponential scale bound of the window (15.9, 36.2); because $h_0 = \eta/m = H_0$ is the height parameter of the stage-three window, and the stage-three parameter conditions (with $H_0 \geq (NfB_{\mathrm{win}})^{C_{\mathrm{count}}}$ and $C_{\mathrm{count}} \geq 4e_P + 11$, Definition 14.11) bound $4T^2$ and $P_* T r f^2 m^2 k d_*$ by H_0 beyond the threshold, we obtain the spacing bounds

$$T \geq 4d_* + 4, \quad T \geq P_*^2 d_*^2, \quad h_0 \geq 4T^2, \quad h_0 \geq P_* T r f^2 m^2 k d_*, \quad P_* \geq 6400 r^2. \tag{A.434}$$

Sum the floored adjacent differences in (35.6) from the zero anchor at $k-2$ to obtain u_1. The weight-gap conditions give strict decrease in each block and an unshifted wall gap at least $h_0/(2T)$ (moreover, writing $N = N_m$, $B_{\mathrm{win}} = x^{\varepsilon}$, Δ, $H_0 = \eta/m$, and C_{count} for the stage-three window parameters of Definitions 14.6 and 14.11, with $P_* = C_P Z_0^{e_P}$, whenever $C_{\mathrm{count}} \geq 4e_P + 11$ we have $T(T + d_* + 1) \leq 2C_P^4 (NfB_{\mathrm{win}})^{4e_P+10} \leq \min(4\Delta, H_0)$ beyond the threshold. Since $h_0 = H_0$, $d_* + 2 + 2r \leq T$ eventually and $\min(\Delta, H_0) \geq (NfB_{\mathrm{win}})^{4e_P+11}$, increasing the threshold gives $\min(4\Delta, H_0, h_0) \geq (T + d_* + 2)(T + d_* + 2 + 2r) + 1$, floored within-block adjacent gaps $u_{1,\tau,s} - u_{1,\tau,s+1} \geq T + d_* + 2r \geq T$, and cross-wall gap at least $h_0/(2T) \geq 2T \geq T$). Subtracting $3r\eta_H$ from u_1 decreases each cross-wall difference by $3r$ and leaves it positive. Its relative loss is at most $6rT/h_0$. Within-block differences stay fixed. With V the product of differences over pairs of columns and embeddings, let $\rho = 1 - V(u')/V(u_1)$. There are at most fr^2 cross-wall factors, so the bound on the loss of a product gives

$$0 \leq \rho < 1, \qquad \rho h_0 \leq 6fr^3 T. \tag{A.435}$$

Together (A.434) and (A.435) give $d_*\rho \leq 6/6400$, $d_*^2/T \leq 1/100$ and $T \geq 3d_* + 5$. Thus the parameter $z = 7(d_*^2/T + d_*\rho)$ of (8.28a) is at most $7(1/100 + 6/6400) < 1/2$. □

Proof of 35.21

Proof. In the quadratic host $K = \mathbf{Q}(\sqrt{-d})$ of discriminant $-d$, the splitting tests at $3, 5$ give $s^2 \equiv -d \bmod 15$ with $(2s, 15) = 1$. Thus the ideal $\mathfrak{J} = (15, s + \sqrt{-d})$ satisfies $\mathfrak{J} + \bar{\mathfrak{J}} = (1)$. The generators of the product $\mathfrak{J}\bar{\mathfrak{J}}$ are divisible by 15, and generate 15^2 and $30s$, so $\mathfrak{J}\bar{\mathfrak{J}} = (15)$. In particular $\mathfrak{J}^e$ and $\bar{\mathfrak{J}}^e$ are complementary halves of (15^e), with generators in $\mathbf{Z}[w_0]$ for $w_0 = (1 + \sqrt{-d})/2$. They define the cover at $3, 5$. Retain the base level $g_0 g_1$ away from $3, 5$. The auxiliary level parameter is one, so the full covering level is M'.

Let $N = M'$ and R be the ray class field of K modulo $N\mathcal{O}_K$. It contains the needed roots of unity and Weber values. Conjugation preserves the modulus. The ray class sequence bounds $[R : K]$ by $h_K \#(\mathcal{O}_K/N\mathcal{O}_K)^{\times}$, where h_K is the class number. The quadratic class-number bound $h_K \leq d$ and the conductor-discriminant formula give

$$[R : K] \leq h_K N^2 \leq dN^2, \qquad \log \mathrm{rd}(R) \leq \tfrac{1}{2}\log d + \log N. \tag{A.436}$$

Indeed every character conductor divides $N\mathcal{O}_K$ of norm N^2, and the discriminant formula divides its logarithmic contribution by $2[R : K]$. Take $F' = FR$ for the original coefficient field $F \subset \mathbf{C}$ (with distinguished embedding $\sigma_0 : F \hookrightarrow \mathbf{C}$ containing the small-field generators). Because the ray class field $R/\mathbf{Q}$ of K modulo the conjugation-invariant modulus $N\mathcal{O}_K$ (from Definition 35.5(6)) is Galois over $\mathbf{Q}$ and contains ζ_N and the Weber values, the compositum $F' = FR \subset \mathbf{C}$ is finite Galois over F and every embedding $\sigma' : F' \hookrightarrow \mathbf{C}$ extending $\sigma_0 : F \hookrightarrow \mathbf{C}$ maps R onto R and therefore satisfies $\sigma'(F') = \sigma_0(F)\sigma'(R) = FR = F'$, preserving both the small-field generators and the ray class field. For every $\gamma \in \mathrm{Aut}(\mathbb{C}/F')$, the Weber comparison supplies a CM elliptic curve and an isomorphism from its γ-conjugate which carries every labelled N-torsion point to the point with the same label. We use this simultaneous isomorphism in transporting the level structures. *Claim 1.* The field F' is relatively algebraically closed in each component function field. The host supplies a positive rank-one datum, and the layer has rank at least three. For each component lattice, form a split reference datum of the same signature from positive and negative rank-one data. In the indefinite case, let L be the component lattice and $L_0 = L_+ \perp L_-$ the reference lattice. Their perfect trace pairings and signatures put them in the same self-dual local genus and identify their rational Hermitian spaces. Let π be the self-adjoint projector onto L_+. Choose $b > 0$ with $bL \subset L_0$ and $bL_0 \subset L$. Weak approximation at the primes dividing b, applied to the nonempty local isometry sets, gives a rational unitary map γ_0 and k coprime to b with

$$k\gamma_0 L \subset L_0, \qquad k\gamma_0^{-1} L_0 \subset L.$$

The second inclusion also follows from the first by trace self-duality. Thus $\gamma_0^{-1}\pi\gamma_0$ preserves $L_{(b)}$, where integers coprime to b are inverted. The indefinite space of rank at least three is isotropic. Choose a nonisotropic vector in $\operatorname{im}\pi$. A quasi-reflection on that vector removes $\det\gamma_0$, commutes with π, and leaves a product m of unitary transvections with $m^{-1}\pi m = \gamma_0^{-1}\pi\gamma_0$. For each transvection $1+D$ in the product, choose an integer $d>0$ with $dD(L)\subseteq L$. Choose an integer c divisible by the part of d prime to b and congruent to one to arbitrarily high precision at every prime dividing b. Multiplying the transvection parameter by c gives $1+cD$, with inverse $1-cD$ since $D^2=0$. Both preserve $L[1/b]$. The finite product can thus approximate m at the primes dividing b closely enough that its conjugate projector still preserves $L_{(b)}$. This gives a product q for which $q^{-1}\pi q$ preserves $L_{(b)}$, and both q and q^{-1} preserve $L[1/b]$. Since π preserves $L[1/b]=L_0[1/b]$, its conjugate preserves $L_{(b)}\cap L[1/b]=L$. Consequently $L=(q^{-1}\pi q)L\perp(1-q^{-1}\pi q)L$, with the required positive and negative signatures. In the definite cases the single pure summand suffices. Choosing the period map on these pure summands gives a split CM point in each period component with its existing level marking. Let $\gamma\in\mathrm{Aut}(\mathbb{C}/F')$. Choose the CM isomorphism supplied by the Weber comparison for this γ. Transporting the split period point through it gives an $\mathcal{O}_E$-linear isometry $\Phi: L\to L^\gamma$ preserving the two pure summands. The marking comparison gives

$$t^\gamma(v)-\Phi t(v)\in AL^\gamma \quad (v\in L_{\mathrm{ref}}), \qquad t^\gamma(v)-\Phi t(v)\in ML^\gamma \quad (v\in \overline{\mathfrak{b}}L_{\mathrm{ref}}),$$

where A is the auxiliary level, M the main level, and $\mathfrak{b}$ its half-level ideal; both divide the torsion modulus $N=M'$. In this cover $A=1$. These are exactly the two marking congruences in the moduli equivalence relation. Preservation of the pure summands makes the real-linear extension of Φ complex linear for the chosen split periods. It therefore identifies the conjugate marked moduli point with the original point. Conjugation permutes the disjoint period components; the fixed split point in each component forces this permutation to be the identity.

Let a in a component function field be algebraic over F'. Normality puts a in every local ring of that component, since its monic minimal polynomial has coefficients in F'. Its value on the connected period domain belongs to the finite set of roots of that polynomial and is therefore constant. Since every $\gamma\in\mathrm{Aut}(\mathbb{C}/F')$ preserves the period component, this constant is fixed by every such γ. It consequently belongs to F'. This proves the claim with $F'=FR$. Fix a lattice embedding for uniformisation. At each good embedding of F' take an orthogonal frame compatible with that same lattice embedding. These frames give the unit minors and models required for goodness.

Lean: `GlobalTheory.CovTop.Inhab.exists_half` (`Development.AbelianVarietyTheory.Definitions.Divisors.NormNegCmConstantEqAtAllEmbeddings`, lines 324–384)
Lean: `GlobalTheory.CoverHalf.exists_coverLevel` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.ExistsHalfOfBasePins`, lines 144–172)
Lean: `GlobalTheory.CovTop.SizedCover.exists_ray` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsGoodCoverOverSizedField`, lines 28–107)
Lean: `GlobalTheory.CovTop.SizedCover.exists_good_sized` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsGoodCoverOverSizedField`, lines 111–160)

For a number field F_0 write $\mathrm{sz}(F_0)=\log[F_0:\mathbf{Q}]+\log\mathrm{rd}(F_0)$. The tied field and host estimates give

$$\mathrm{sz}(F)\le 217(frL_x)^2, \qquad \log g_a\le 9L_x+1 \quad (a=0,1), \qquad \log d\le L_x. \tag{A.437}$$

The degree of a compositum is at most the product of degrees and its root discriminant is at most the product of root discriminants, by the relative different bound under base change. Thus (A.436) bounds $\mathrm{sz}(F')-\mathrm{sz}(F)$ by $20(\log N+\log d+1)$. The definition of e gives $e\log 15\le 9L_x$, so by (A.437) one has $\log M'\le 27L_x+2\le 30L_x$, and the increase is at most $20(28L_x+3)\le 174(frL_x)^2$, since $f\ge 8$ and $L_x\ge 10$. Moreover, because $E/\mathbf{Q}$ is Galois, every prime $\ell\mid D_E$ has ramification index $e_\ell\ge 2$ at all places above ℓ and hence $v_\ell(|D_E|)/[E:\mathbf{Q}]\ge(e_\ell-1)/e_\ell\ge 1/2$; summing over $\ell\mid D_E$ gives $\sum_{\ell\mid D_E}\log\ell\le 2\log\mathrm{rd}(E)\le 4L_x$, whence $\sum_{\ell\mid 2dD_EM'}\log\ell\le\log 2+\log d+2\log\mathrm{rd}(E)+\log M'\le 37L_x\le 40L_x$.

Lean: `GlobalTheory.CovTop.RecTied.core_size_le` (`Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol`, lines 192–193)
Lean: `GlobalTheory.CovTop.DeepExist.size_le` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.DeepExistenceStatement`, lines 29–56)

□

Proof of 35.22

Proof. Primitive uniformisation and the core lattice regularity give the core Gram frame and full cusp lattice. Pull back the Hilbert inner product by the row realisation. The ambient realised lattice is the integral span of the semilinear image of the core lattice. The primitive row laws give the good lattice of dimension $d_W(i) = D_{\mathrm{norm}} > 0$.

Lean: `GlobalTheory.CovTop.GramPt.hasGram_of_unif` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.RowNormHasGram`, lines 25–36)

Lean: `GlobalTheory.CovTop.latAll_fg_at_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LatAllFgAt`, lines 15–30)

Lean: `GlobalTheory.CovTop.good_lattice_ne_bot_at_point` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LatGoodNeBotAt`, lines 12–24)

The squared-norm ratio is $c^2 = V \prod_\tau d_\tau$, where V is the row fundamental-domain volume and $d_\tau \geq 1$ is the dimension of the lowest compact type at τ. Put $L' = 2L_x$. Let $\mathrm{mass}_I = \sum_{\Lambda \in \mathrm{gen}(I_r)} |\operatorname{Aut}(\Lambda)|^{-1}$ be the definite unitary genus mass of the standard rank-r self-dual Hermitian lattice I_r over $\mathcal{O}_E$. For each definite self-dual Hermitian $\mathcal{O}_E$-lattice $\Lambda \in \mathrm{gen}(I_r)$, equip Λ with the trace Euclidean metric $\mathrm{Tr}_{E/\mathbf{Q}} H$; because Λ is Hermitian self-dual, its $\mathbf{Z}$-dual is $\Lambda^* = \mathfrak{D}_E^{-1}\Lambda$, where AM–GM on nonzero elements gives $\lambda_1(\Lambda^*)^2 \geq 2f/\operatorname{rd}(E)^2$, and Banaszczyk's transference theorem $\lambda_{2fr}(\Lambda)\lambda_1(\Lambda^*) \leq 2fr$ (cf. Lemma 33.11) extracts an E-basis $v_1, \ldots, v_r \in \Lambda$ whose Hermitian Gram matrix $A = (H(v_i, v_k))_{i,k} \in M_r(\mathcal{O}_E)$ satisfies $|\sigma(H(v_i, v_k))| \leq B = 4f^3 r^4 \operatorname{rd}(E)^2$ at every embedding $\sigma : E \hookrightarrow \mathbf{C}$ (without assuming Λ is $\mathcal{O}_E$-free). Counting the r^2 entries $H(v_i, v_k) \in \mathcal{O}_E$ inside the polydisc of radius B gives at most $(1+2B)^{2fr^2}$ Gram matrices A, and for each sublattice $L_0 = \sum_{i=1}^r \mathcal{O}_E v_i$ of Gram matrix A, counting $\mathcal{O}_E$-overlattices $\Lambda/L_0 \subseteq L_0^\vee/L_0$ via pairings in $\mathcal{O}_E^r/A\mathcal{O}_E^r$ gives at most $|N_{E/\mathbf{Q}}(\det A)|^r \leq (rB)^{2fr^2}$ choices, yielding $\mathrm{mass}_I \leq (1+2B)^{2fr^2}(rB)^{2fr^2} \leq (3(2f)^2 r^4 \operatorname{rd}(E)^2)^{6fr^2}$. For the lower mass bound, retain the single standard-lattice term $|\operatorname{Aut}(I_r)|^{-1}$: every unitary automorphism of I_r has entries in $\mathcal{O}_E$ of complex absolute value at most 1 at every embedding, and in the $2f$-dimensional Minkowski embedding of $\mathcal{O}_E$ at most 3^{2f} algebraic integers lie in the closed unit polydisc (since distinct algebraic integers have sup-norm separation at least 1, so open polydiscs of radius $1/2$ around them are disjoint inside the polydisc of radius $3/2$), giving $|\operatorname{Aut}(I_r)| \leq 3^{2fr^2}$ and hence $\mathrm{mass}_I \geq 3^{-2fr^2}$. Likewise, the finite-level subgroup $K_{\mathrm{row}} \subseteq K_0$ contains the principal congruence subgroup modulo M', so its index $n_\Gamma = [K_0 : K_{\mathrm{row}}]$ is bounded by $|\mathrm{GL}_r(\mathcal{O}_E/M'\mathcal{O}_E)| \leq (M')^{2fr^2}$, which together with $\log M' \leq 30L_x = 15L'$ (and the auxiliary factor from the row level) yields $\log n_\Gamma \leq 38fr^2 L'$. Let μ be adelic Haar measure giving the row fundamental domain volume V, m the product of the archimedean Haar measures whose densities at the identity in exponential coordinates are $2^{-r(r-1)/2}$ relative to the Lie algebra Haar measure whose product with itself pushes forward under $(X, Y) \mapsto X + iY$ to standard Lebesgue measure on $M_r(\mathbf{C})$, rp the archimedean projection and Stab the finite-lattice stabiliser. *Claim 1.* With these normalisations, $V\, m(B) = \mu(\mathrm{Stab} \cap \mathrm{rp}^{-1}(B)) \operatorname{vol}(U(r))^f \mathrm{mass}_I$ for every measurable B. To prove this, let $\nu_{P,Q}$ denote the compact-dual normalised archimedean Haar measure, with $\nu_{r,0}(U(r)) = 1$. Normalise adelic measure μ_0 by giving the full finite lattice stabiliser volume one and using these archimedean factors. The cylinder formula for the restricted product then gives

$$\mathrm{rp}_*(\mu_0|_{\mathrm{Stab}}) = \prod_\tau \nu_{P,Q}.$$

Its adelic quotient volume is mass_I. Indeed, for a definite space, the double-coset decomposition gives precisely $\sum_\Lambda |\operatorname{Aut}\Lambda|^{-1}$. For an indefinite space, unfolding the rank-one Siegel mean with the same self-dual local measures gives the quotient-volume ratio for adjoining a positive or negative line. The finite local factors are those of the corresponding definite mass ratio; the coordinate Jacobians cancel by the product formula. With compact-dual probability measures the archimedean ratio is the same. Thus the quotient volume divided by the definite mass of its rank is unchanged by either line step. The positive-line and negative-line steps through a space with at least two positive directions connect the prescribed signature to a definite one. The signatures with zero or one positive direction follow by adjoining positive lines and cancelling the positive mass ratio. This gives $\mu_0(\mathcal{F}) = \mathrm{mass}_I$ for an adelic fundamental domain $\mathcal{F}$.

The density of $\nu_{P,Q}$ at the identity in the stated coordinates is $(2^{r(r-1)/2} \operatorname{vol}(U(r)))^{-1}$. Comparing with the density $2^{-r(r-1)/2}$ of m_τ, uniqueness of Haar measure gives $m_\tau = \operatorname{vol}(U(r))\nu_{P,Q}$. Finally write $\mu = a\mu_0$,

$a > 0$. Then $V = a\,\mathrm{mass}_I$, while

$$\mu(\mathrm{Stab} \cap \mathrm{rp}^{-1}(B)) = a\,(\prod_{\tau} \nu_{P,Q})(B) = a\,m(B)/\,\mathrm{vol}(U(r))^f.$$

This proves the claim. Choose a measurable box B with $0 < m(B) < \infty$. For the upper volume bound, every element of the lattice stabiliser whose archimedean component lies in B maps into the full-level subgroup K_0 at finite places, and K_0 is a union of $n_\Gamma = [K_0 : K_{\mathrm{row}}]$ cosets of the row level K_{row}; thus the measure of the stabiliser over B is at most $n_\Gamma m(B)$ in the row normalisation, and cancelling $m(B)$ in the Tamagawa measure identity $V\,m(B) = \mu(\mathrm{Stab} \cap \mathrm{rp}^{-1}(B))\,\mathrm{vol}(U(r))^f\,\mathrm{mass}_I$ gives $V \le n_\Gamma\,\mathrm{vol}(U(r))^f\,\mathrm{mass}_I$. Here $\mathrm{vol}(U(r)) \le (r+3)^{r^2}$; with $fr^2 \le e^{L'}$ and $\log \mathrm{rd}(E) \le L'$ the mass and index logarithms sum to at most $(38+81)fr^2L' = 119fr^2L'$, giving the first bound in (A.438).

Lean: `MainStatements.Tam.Plug.tamagawa_pairs_compatible` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.HodgeBundleWeights.PlugPairs`, lines 12–12)

Lean: `MainStatements.Tam.A.up_core` (`Development.WeilBound.RepresentationTheoryGL.TubeMeasures.DisjointTubes.ExponentialThreeBound.LeExpThree`, lines 91–179)

For each τ, the Weyl dimension d_τ is the product over pairs $a < b$ within the two compact blocks. If κ_a, κ_b are the corresponding entries of the lowest compact weight, each factor has absolute value at most $1 + |\kappa_a - \kappa_b|$, since $b - a \ge 1$. The row size estimate bounds the difference by $e^{2L'}$. Since $e^{L'} \ge 2$, each factor contributes at most $\log(1 + e^{2L'}) \le 3L'$ to the logarithm. The two blocks supply at most $r^2/2$ factors, so $\log d_\tau \le 3r^2L'/2$. Sum over the f embeddings to obtain the second bound:

$$\log V \le fr^2 \log(r+3) + 119fr^2L', \qquad \sum_{\tau} \log d_\tau \le \tfrac{3}{2} fr^2 L'. \tag{A.438}$$

Lean: `GlobalTheory.OneMeasDim.log_dimAt_le` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LogLowestKTypeWeylDimensionLeOfSizeBounds`, lines 77–143)

At the family threshold $\log(r+3) \le L'$, this gives $\log c^2 \le 122fr^2L'$. For the lower bound, the row level subgroup K_{row} over B is contained in the stabiliser over B and has measure exactly $m(B)$; cancelling $m(B)$ in the Tamagawa identity gives $V \ge \mathrm{vol}(U(r))^f\,\mathrm{mass}_I$. Using $\mathrm{vol}(U(r)) \ge r^{-r^2}$ and the definite standard-lattice mass lower bound $\mathrm{mass}_I \ge 3^{-2fr^2}$ (and $d_\tau \ge 1$ for every τ, so $\log d_\tau \ge 0$) yields an absolute $C \ge 0$ (indeed $C = 3$ suffices) with

$$\log V \ge -C\{fr^2(\log(fr+3)+1) + \log|\operatorname{disc} E|\}, \qquad \sum_{\tau} \log d_\tau \ge 0. \tag{A.439}$$

By (A.433) and the threshold in 35.20, $r^2 \le b$ and eventually $\log(fr+3) \le 2L_x$. Equation (A.418) gives $\log|\operatorname{disc} E| \le 4fL_x$. Substitute in (A.438) and (A.439) and use the formula for E_p in (A.418). This gives $|\log c| \le C_{\mathrm{meas}} E_p$, for example with $C_{\mathrm{meas}} = (244 + 6C + 3)/2$.

Lean: `GlobalTheory.OneMeasureRatio.log_covolume_ratio_le` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.HodgeBundleWeights.LogMeasureRatioBounded`, lines 16–37)

Lean: `MainStatements.Tam.low` (`Development.WeilBound.RepresentationTheoryGL.TubeMeasures.DisjointTubes.BoxExistence.ExistsBox`, lines 64–126)

Lean: `GlobalTheory.CovTop.MeasUp.ratAt_pos_le` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LogRatioUpperBound`, lines 166–215)

Lean: `GlobalTheory.CovTop.MeasLow.nef_share_lower_bound_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CovolumeLowerBoundOfCover`, lines 21–54)

In the Petersson estimate let s be the sum of the entries of the two algebraic highest-weight partitions over all embeddings, $t = frn_V$ the column weight and $q = Q$ the negative signature index. Write $w = s + t + 1$. First, we establish the thick-region collar bound $\varepsilon_{\mathrm{lev}}(E, d, M', w) = \min\bigl(1, \pi/\bigl(2M'\sqrt{d}\,(fw+2)\log 2\bigr)\bigr) \in (0,1]$ (so $\varepsilon_{\mathrm{lev}}^{-1} \le B := 3fwM'\sqrt{d}$, since $2(fw+2)\log 2 \le 3\pi fw$ for $fw \ge 1$) for every nonzero holomorphic cusp form φ of weight parameter w: at any domain point y_0 attaining at least $\frac{1}{2}\|\varphi\|_{\sup}$, every nonzero integral vector $v \in \Lambda_{\mathrm{lev}}$ of the level-M' integral lattice satisfies $\max_\tau \|v_\tau\|_{y_0}^2 \ge \varepsilon_{\mathrm{lev}}$, and every nonzero vector $v \in \Lambda$ of the self-dual lattice Λ satisfies $\lambda_1(y_0)^2 = \min_{0 \ne v \in \Lambda} \sum_\tau 2\xi_{\mathbf{R}} \|v_\tau\|_{y_0}^2 \ge 2\varepsilon_{\mathrm{lev}}/\,\mathrm{rd}(E)^2$ (matching Lemma 35.44). Indeed, if $v \in \Lambda_{\mathrm{lev}}$ is non-isotropic, its Hermitian pairing $H(v,v) \in \mathcal{O}_{E^+}$ is a nonzero algebraic integer in the totally real subfield E^+, so $1 \le |N_{E^+/\mathbf{Q}}(H(v,v))| \le \prod_\tau \|v_\tau\|_{y_0}^2 \le (\max_\tau \|v_\tau\|_{y_0}^2)^f$, forcing $\max_\tau \|v_\tau\|_{y_0}^2 \ge 1 \ge \varepsilon_{\mathrm{lev}}$. If $v \in \Lambda_{\mathrm{lev}}$ is isotropic, restrict φ to the one-complex-variable diagonal transvection flow $\zeta \mapsto \mathrm{flow}(v, (\zeta, \dots, \zeta), z_0)$ along the rational one-parameter subgroup $M'\mathbf{Z} \subset \mathcal{O}_{E^+}$; this yields a one-variable M'-periodic holomorphic function $G(\zeta)$ vanishing as $\mathrm{Im}\,\zeta \to +\infty$ by cuspidality (with no $\mathrm{rd}(E^+)^{f/2}$

factor), and applying the one-variable periodic Schwarz lemma on the strip $\operatorname{Im}\zeta \geq -t$ with $t = (4\sqrt{d}\,\varepsilon_{\mathrm{lev}})^{-1}$ (so $2\pi t/M' = \pi/(2M'\sqrt{d}\,\varepsilon_{\mathrm{lev}}) \geq (fw+2)\log 2$, giving $e^{-2\pi t/M'}\cdot 2\cdot 2^{w/2} < 1$) shows that $\max_\tau \|v_\tau\|_{y_0}^2 < \varepsilon_{\mathrm{lev}}$ would force $\|\varphi(y_0)\| < \frac{1}{2}\|\varphi\|_{\sup}$, a contradiction. Choosing $b \in E^\times$ by Minkowski's theorem with $b\Lambda \subseteq \Lambda_{\mathrm{lev}}$ and $|\tau(b)|^2 \leq \xi_{\mathbf{R}} \operatorname{rd}(E)^2$ transfers $\max_\tau \|(bv)_\tau\|_{y_0}^2 \geq \varepsilon_{\mathrm{lev}}$ to Λ and yields $\sum_\tau 2\xi_{\mathbf{R}}\|v_\tau\|_{y_0}^2 \geq 2\varepsilon_{\mathrm{lev}}/\operatorname{rd}(E)^2$ for every nonzero $v \in \Lambda$. On this thick region, define four explicit comparison parameters. (i) The invariant complex flag-chart ball radius $r_{\mathrm{ch}} = \big(200r^3(w+f)\big)^{-1} > 0$ (independent of $\varepsilon_{\mathrm{lev}}$) on the flag chart of complex dimension $d_{\mathrm{ch}} = d_{\mathrm{fl}} = f\binom{r}{2} \leq fr^2/2$, on which the holomorphic weight potential varies by at most $\log 2$. (ii) The finite chart-overlap multiplicity bound $M_{\mathrm{ov}} = \big(1+2(2fr)^2|\operatorname{disc} E|\operatorname{rd}(E)^2/(2\varepsilon_{\mathrm{lev}})\big)^{2fr^2} \geq 1$ counting Γ-translates of $B(y_0, r_{\mathrm{ch}})$ meeting a fixed ball. To derive this bound, orthonormalise y_0 to a frame x with $\lambda_1(x)^2 \geq 2\varepsilon_{\mathrm{lev}}/\operatorname{rd}(E)^2$ on Λ; by Banaszczyk transference for the positive majorant at x, with dual lattice $\xi_{\mathbf{R}}^{-1} J_x \mathfrak{D}_E^{-1}\Lambda$ for the isometric sign involution J_x and $0 < \xi_{\mathbf{R}} \leq 1$, there exists an E-basis $w_1, \dots, w_r \in \Lambda$ with squared majorant norm at most $B_d = (2fr)^4|\operatorname{disc} E|^2\operatorname{rd}(E)^2/(8\varepsilon_{\mathrm{lev}})$. Because $r^2 r_{\mathrm{ch}} \leq 1/100$, any $\gamma \in \Gamma$ with $\gamma B(y_0, r_{\mathrm{ch}}) \cap B(y_0, r_{\mathrm{ch}}) \neq \varnothing$ modulo the Borel action multiplies squared majorant norms by at most 4, confining each $\gamma w_i \in \Lambda$ to the Euclidean ball of radius $\sqrt{4B_d}$ inside $\mathbf{R}^{2fr}$ with minimum lattice spacing $\sqrt{2\varepsilon_{\mathrm{lev}}}/\operatorname{rd}(E)$. Packing disjoint balls of radius $\frac{1}{2}\sqrt{2\varepsilon_{\mathrm{lev}}}/\operatorname{rd}(E)$ bounds the choices for each γw_i by $\big(1+2\sqrt{4B_d\operatorname{rd}(E)^2/(2\varepsilon_{\mathrm{lev}})}\big)^{2fr} = \big(1+2(2fr)^2|\operatorname{disc} E|\operatorname{rd}(E)^2/(2\varepsilon_{\mathrm{lev}})\big)^{2fr}$; since γ is E-linear and uniquely determined by $(\gamma w_1, \dots, \gamma w_r)$, raising to the r-th power yields the overlap bound M_{ov}. (iii) The level-M' Tamagawa volume and mass factor $M_{\mathrm{lev}} = n_\Gamma \operatorname{vol}(U(r))^f \operatorname{mass}_I \geq 1$ converting the probability-normalised Petersson integral to local chart Lebesgue measure, bounded as in (A.438) by $\log M_{\mathrm{lev}} \leq 130fr^2L'$. (iv) The nonvanishing frame radius $r_{\mathrm{fr}} = (10\sqrt{wr})^{-1}$, the explicit thick-region covering radius $\mu_{\mathrm{th}} = 4(fr)^3|\operatorname{disc} E|/\lambda_*$ with $\lambda_* = \sqrt{2\varepsilon_{\mathrm{lev}}/(f\operatorname{rd}(E)^2)}$ (since $\lambda_1(y_0)^2 \geq 2\varepsilon_{\mathrm{lev}}/\operatorname{rd}(E)^2$ gives $\max_\tau 2\xi_{\mathbf{R}}\|v_\tau\|_{y_0}^2 \geq \lambda_*^2$, and $2\xi_{\mathbf{R}} \leq 2/\sqrt{d} < 1$), and the Diophantine rational CM approximation radius $R_{\mathrm{ap}} = 4\sqrt{r}\,\mu_{\mathrm{th}}/(\frac{3}{5}r_{\mathrm{fr}}) \geq 1$ (where the extra $\log\operatorname{rd}(E) \leq L'$ factor stays absorbed inside the $2fL'$ slack of $\log|\operatorname{disc} E| \leq 2fL'$, preserving $\log(2R_{\mathrm{ap}}) \leq (4f+32)L'$). Set the CM-box length factor $L_{\mathrm{cm}} = (2R_{\mathrm{ap}})^{2q+1}$, the mean logarithmic Petersson loss

$$\lambda_0 = \big(\tfrac{1}{2}(1+|\log\Omega|) + \tfrac{1}{2}\log L_{\mathrm{cm}}\big)(s+t) + s\log(rL_{\mathrm{cm}}),$$

and the uniform supremum-to-Petersson comparison constant $C_{\sup} = (8d_{\mathrm{ch}}!/(\pi r_{\mathrm{ch}}^2)^{d_{\mathrm{ch}}})\cdot M_{\mathrm{ov}}\cdot M_{\mathrm{lev}} \geq 1$. Because the holomorphic mean-value property on $B(y_0, r_{\mathrm{ch}})$ bounds $\|\varphi(y_0)\|^2$ by $(2d_{\mathrm{ch}}!/(\pi r_{\mathrm{ch}}^2)^{d_{\mathrm{ch}}})$ times the local L^2-integral (losing a factor 4 from potential oscillation and $\frac{1}{2}\|\varphi\|_{\sup} \leq \|\varphi(y_0)\|$), unfolding over the M_{ov}-overlapping translates and scaling by M_{lev} gives $\|\varphi\|_{\sup}^2 \leq C_{\sup}\|\varphi\|_{\mathrm{Pet}}^2$ for every holomorphic cusp form φ of weight w; put $s_0 = \log C_{\sup}$. Use the weight comparison $s+t+1 \leq 10frb$ and (A.433), so $\log w \leq 4L_x = 2L'$ and $\log r \leq \log(fr) \leq L'$ at the family threshold. The approximation-radius bound gives $Y = \log L_{\mathrm{cm}} = (2q+1)\log(2R_{\mathrm{ap}}) \leq (2q+1)(4f+32)L'$. Bounding $s(\log r + Y) \leq (s+t)(L'+Y)$ and $\frac{3}{2}Y \leq (2q+1)(6f+48)L'$ yields the first line of (A.440). For s_0, with $u = r_{\mathrm{ch}}^{-1} \leq 400r^3fw$ we have $d_{\mathrm{ch}}u^2/\pi \leq (80000/\pi)f^3r^8w^2 \leq 2^{16}(fr)^8w^2$, giving $\log(8d_{\mathrm{ch}}!/(\pi r_{\mathrm{ch}}^2)^{d_{\mathrm{ch}}}) \leq 3\log 2 + \frac{1}{2}fr^2(12L'+16\log 2) \leq 3\log 2 + 14fr^2L'$; for M_{ov}, $\log B \leq 2\log 2 + 35L'$ and $\log(\operatorname{rd}(E)^2) \leq 2L'$ give $\log M_{\mathrm{ov}} \leq 2fr^2(6\log 2 + 39L' + 4fL') \leq (90fr^2 + 8f^2r^2)L'$. Adding $\log M_{\mathrm{lev}} \leq 130fr^2L'$ yields $s_0 = \log C_{\sup} \leq 3\log 2 + 234fr^2L' + 8f^2r^2L' \leq 250f^2r^2L' = 500f^2r^2L_x$:

$$\begin{aligned} &\lambda_0 \leq \big((1+|\log\Omega|)/2 + 2L_x + (2q+1)(6f+48)2L_x\big)(s+t),\\ &s_0 \leq 500f^2r^2L_x. \end{aligned} \tag{A.440}$$

From $\|\varphi\|_{\sup}^2 \leq C_{\sup}\|\varphi\|_{\mathrm{Pet}}^2$ we deduce both the rank bound and the maximal sublattice slope bound for the core cusp lattice Λ_{core} over F, as well as for each individual algebraic component Z. First, let $e_1, \dots, e_d$ be a Petersson-orthonormal basis of the complex cusp space of dimension $d = \operatorname{rk}_F(\Lambda_{\mathrm{core}})$. At any domain point x, choosing the unit vector $\varphi = \sum_{i=1}^d \overline{e_i(x)}e_i/(\sum_i |e_i(x)|^2)^{1/2}$ gives the Bergman kernel bound $\sum_{i=1}^d |e_i(x)|^2 \leq C_{\sup}$, and integrating over the probability-normalised adelic quotient yields $d = \sum_{i=1}^d \|e_i\|_{\mathrm{Pet}}^2 \leq C_{\sup}$, hence $\log\operatorname{rk}_F(\Lambda_{\mathrm{core}}) \leq s_0$. Furthermore, let $p_Z \in (0,1]$ be the mass of Z in the global probability measure, and use conditional probability for $\|\cdot\|_{2,Z}$. Zero extension $\widetilde{\varphi}$ satisfies $\|\widetilde{\varphi}\|_{\sup} = \|\varphi\|_{\sup}$ and $\|\widetilde{\varphi}\|_{\mathrm{Pet}}^2 = p_Z\|\varphi\|_{2,Z}^2$. Since $p_Z \leq 1$, the global comparison exports both $\|\varphi\|_{\sup}^2 \leq C_{\sup}\|\varphi\|_{2,Z}^2$ and $\log\operatorname{rk}_F(\Lambda_{u_1,Z}) \leq \log C_{\sup}$ to every individual component Z (in particular, at an enlarged weight scale w with $\log w \leq 25L_b$ and $L_b = L_x$, taking $L' = \frac{25}{2}L_b$ gives $\log C_{\sup} \leq 250f^2r^2(\frac{25}{2}L_b) = 3125f^2r^2L_b \leq 4000(frL_b)^3$, for use in the component degree and height estimates below). Second, for any nonzero integral cusp section $\varphi \in \Lambda_{\mathrm{core}}$, pick a point y_0 in the

thick region with $\|\varphi(y_0)\| \geq \frac{1}{2}\|\varphi\|_{\sup}$. On the recentered flag polydisc $B(y_0, r_{\mathrm{fr}})$ of radius $r_{\mathrm{fr}} = (10\sqrt{wr})^{-1}$, the normalised holomorphic function $g(z) = \varphi(y_0 n(z))/\varphi(y_0)$ has $g(0) = 1$ and satisfies $|g(z)| \leq 2e^{wr\cdot r_{\mathrm{fr}}^2} \leq 2e^{1/100} < 2.1$ by quadratic control of the weight potential. By Cauchy's estimate on $B(y_0, r_{\mathrm{fr}})$, every point z in the concentric ball $B(y_0, r_{\mathrm{fr}}/5)$ satisfies $|g(z) - 1| \leq \frac{1/5}{1-1/5} \cdot 2.1 = 0.525 < 1$, so φ vanishes nowhere on $B(y_0, r_{\mathrm{fr}}/5)$. Because the self-dual lattice Λ has minimum $\max_\tau \|v_\tau\|_{y_0} \geq \lambda_*$ and covering radius bounded by $\mu_{\mathrm{th}} = 4(fr)^3|\operatorname{disc} E|/\lambda_*$, scaling target column vectors by $R_{\mathrm{ap}} = 4\sqrt{r}\,\mu_{\mathrm{th}}/(\frac{3}{5}r_{\mathrm{fr}})$ and rounding to lattice vectors within distance μ_{th} approximates each of the r frame columns to within $\frac{3}{5}r_{\mathrm{fr}}/(4\sqrt{r})$. Put $R = R_{\mathrm{ap}}$, $\mu = \mu_{\mathrm{th}}$, and $\rho = 3r_{\mathrm{fr}}/5$, and let $u_1, \ldots, u_q$ be the rounded negative columns. At each embedding write their matrix as $\binom{A}{B}$ in the orthonormal signature coordinates. Then $\|A\|_{\mathrm{op}}, \|B - RI\|_{\mathrm{op}} \leq \sqrt{r}\,\mu = R\rho/4$, so B is invertible and $\|AB^{-1}\|_{\mathrm{op}} \leq \rho/(4-\rho) \leq \rho/3$. Their span is negative definite. Put $C = (-H(u_i, u_j))_{i,j}$ and $\Delta = \det C$. The entries of C are integral, $\Delta \neq 0$, and Hadamard's inequality gives $|\tau(\Delta)| \leq (2R)^{2q}$ at every embedding. For $v \in \Lambda$ define

$$T(v) = \Delta v + \sum_i u_i\big(\operatorname{adj}(C)(H(u_j, v))_j\big)_i.$$

The identity $C\operatorname{adj}(C) = \Delta I$ gives $H(u_j, T(v)) = 0$; all coefficients in this formula are integral, so $T(\Lambda) \subseteq \Lambda$. The short spanning family supplied by transference has majorant norm at most μ. Choose $r - q$ independent images of its members under T. For a graph of operator norm at most $\rho/3 \leq 1/3$, the squared majorant norm of its Hermitian orthogonal projection is at most twice the original squared norm. Thus each chosen image has squared majorant norm at most $2\mu^2(2R)^{4q} \leq (2R)^{4q+2}$. Together with the u_j, these images form an integral E-basis adapted to the two signature subspaces, with $|\tau H(v_j, v_j)| \leq L_{\mathrm{cm}}^2$. The graph estimate places the corresponding signature subspace, with the recentered within-block flag coordinates, inside $B(y_0, r_{\mathrm{fr}}/5)$; the integral basis is used as a coordinate basis for the frame polynomial. In particular $\varphi(y_{\mathrm{CM}}) \neq 0$. The only denominator in the orthogonal projection was Δ, and the displayed formula has cleared it at every finite place. For the integral sublattice L_0 generated by this basis, self-duality gives $[\Lambda : L_0]^2 = |N_{E/\mathbf{Q}} \det(H(v_i, v_j))|$; applying Hadamard separately to the definite blocks bounds this by $L_{\mathrm{cm}}^{2r[E:\mathbf{Q}]}$. The local determinant-line comparison below therefore uses the same integral basis and the same box bound. At this split CM abelian variety $A_{y_{\mathrm{CM}}}$ (isogenous over the CM basis of box length L_{cm} to a product of rank-one $\mathcal{O}_E$-CM factors, each of complex dimension f), the algebraic frame polynomial G of φ is nonzero over a finite extension K_1/F. We now give the local coefficient calculation. At a finite place w of K_1, let t_w generate the dual CM theta-line ideal and let $c_{w,j}$ generate the CM Lie-line ideals. Write U_w for the change from the CM vectors to an integral Lie eigenbasis, with the transpose convention for polynomial substitution. The integral CM model gives $c_{w,j}(U_w)_{jk} \in \mathcal{O}_w$. Let U be the matrix of the CM vectors in the fixed global coordinates, and let d_w generate the local determinant-line ideal in those coordinates. Taking top exterior powers gives

$$v_w\left((\prod_j c_{w,j}) \det U\right) \geq v_w(d_w).$$

These inclusions follow by extending the CM factor maps over the local model and taking their differentials and top exterior powers. The principal theta generator differs from its local ideal generator by a unit. Thus integrality of φ on integral frames, after every finite extension of the complete local field, implies that $t_w^{-N}G$ evaluated in any such integral eigenframe is integral, where N is the theta-column exponent. Consequently each coefficient satisfies

$$v_w(G_\mu) + \sum_j \mu_j v_w(c_{w,j}) - N v_w(t_w) \geq 0;$$

μ_j denotes the total exponent using the j-th Lie vector. In block coordinates put $P(X) = t_w^{-N}G(U_w^{-\mathsf{T}}X)$. The frame bound makes $P(X)$ integral on products of invertible integral matrices over every finite extension. If a coefficient had negative valuation, scaling to least valuation zero and reducing would give a nonzero polynomial vanishing on all invertible residue matrices over the algebraic closure. Thus P has integral coefficients. Put $C_w = \operatorname{diag}(c_{w,j})$ in the corresponding blocks. The matrix $U_w^{\mathsf{T}}C_w$ is integral, so $P(U_w^{\mathsf{T}}C_w X) = t_w^{-N}G(C_w X)$ has integral coefficients. Reading its coefficient of index μ gives the displayed inequality. This proves the inequality at every finite place. Taking its weighted sum and applying the product formula to one nonzero G_μ gives the required coefficient contribution; the determinant inequality accounts for the change of

Lie frame without an additional denominator term. Combining these local bounds for a nonzero coefficient of G with the product formula and the rank-one $\mathcal{O}_E$-CM Chowla–Selberg period evaluation of 21.22 on each CM factor (contributing $\frac{1}{2}(1+|\log\Omega|)+\frac{1}{2}\log L_{\mathrm{cm}}$ per weight unit from the CM period metric and box rescaling, plus $s\log(rL_{\mathrm{cm}})$ from the algebraic representation matrix of weight sum s) bounds the mean logarithmic sup norm of φ over embeddings $\sigma : F \hookrightarrow \mathbf{C}$ below by $-\lambda_0$, so $\|\varphi\|_{\mathrm{sup}}^2 \le C_{\mathrm{sup}}\|\varphi\|_{\mathrm{Pet}}^2$ bounds the mean logarithmic Petersson norm of every nonzero $\varphi \in \Lambda_{\mathrm{core}}$ below by $-\lambda_0 - \frac{1}{2}s_0$. Applying Minkowski's first theorem to any nonzero sublattice $M \subseteq \Lambda_{\mathrm{core}}$ of rank $d_M \le d$ yields a nonzero vector $\varphi \in M$ whose mean logarithmic Petersson norm is at most $-\frac{1}{2}\mu(M)+\frac{1}{2}\log d_M+\frac{1}{2}\log \mathrm{rd}(F)$; comparing the two bounds and multiplying by 2 (in the squared-norm slope convention, using $\log d_M \le \log d \le s_0$) gives

$$\mu_{\max}(\Lambda_{\mathrm{core}}) \le 2\lambda_0 + 2s_0 + \log \mathrm{rd}(F) \le x^{2A+4}b. \tag{A.441}$$

In the last inequality, $q \le r$ and $f \ge 8$ bound $(2q+1)(6f+48)$ by $36fr$. With (A.418) and (A.433), the first line of (A.440) is at most $760x^{2A}b$. The second line and (A.437) bound the remaining terms in (A.441) by $1217x^{2A}b$. Use $x \ge L_x \ge 10$.

Choose a sublattice of Λ_{all} attaining its maximal slope and extend it to the normal closure of the realisation field over F. With the extended realised norms, its normalised slope is preserved. Rescale those norms at each embedding by c^{-1}, with c taken at the corresponding embedding of the realisation field. The rescaled norms on the extended core space are the scalar extensions of the Petersson norms. For a rank-a sublattice the rescaling changes the normalised logarithmic squared covolume by at most $2aC_{\mathrm{meas}}E_p$. Its slope therefore changes by at most $2C_{\mathrm{meas}}E_p$. The extended sublattice lies in the integral span of the extended core lattice. For the rescaled norms, Galois descent of the maximal slope bounds its slope by $\mu_{\max}(\Lambda_{\mathrm{core}})$. Applying this to the chosen maximal-slope sublattice and using (35.7) and (A.441) gives

$$\mu_{\max}(\Lambda_{\mathrm{all}}) \le x^{2A+4}b + 2C_{\mathrm{meas}}x^{B}b. \tag{A.442}$$

Increase the family exponent and threshold to absorb the sum.

Lean: `UpperTheory.Lit.Tam.slopeMax_sq_latW_le_pow_mul` (`Development.AutomorphicArithmetic.ThetaCorrespondence.FockModels.TamagawaPeterssonLambdaSqNonneg`, lines 596–670)
Lean: `GlobalTheory.CovTop.LamCore.collect` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.SlopeMaxSquaredBound`, lines 49–70)
Lean: `GlobalTheory.SlopeMaxExt.slopeMax_sq_span_le` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.MaxSlopeSqFieldExtensionLeComparableNorms`, lines 142–205)
Lean: `GlobalTheory.CovTop.LamAll.slopeMax_all_le` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LambdaBoundFromMeasure`, lines 42–65)

□

Proof of 35.23

Proof. These two inequalities adapt Zhang's successive-minima intersection bounds [172, Th. 5.2] to birational multihomogeneous models with cone metrics. For (1), subtract a from the logarithmic metric weight of A. The curvature form and the geometric fibre degrees are unchanged, while on each horizontal curve W the arithmetic degree decreases by $a\deg_{\mathbf{C}} W$ and remains nonnegative. By multilinearity and the constant-shift recursion, the top self-intersection decreases by $(D_A+1)ad_A$. Passing to the projective birational modification of Lemma 35.12 via characteristic-zero resolution produces a projective flat integral model with smooth generic fibre on which the pulled-back shifted metric is smooth and semipositive while fibrewise nefness, nonnegativity on horizontal curves and top self-intersection are preserved; arithmetic Kleiman (A.422) gives nonnegativity of the shifted self-intersection, proving (35.9).

For (2), when $D_A = 0$ the generic fibre $X_{\mathbf{Q}}$ is a single horizontal orbit, so Zariski density puts X in the collection and $I = h_A(X)d_A \le ad_A$. When $D_A \ge 1$, nefness on geometric fibres gives $d_A \ge 0$. Suppose $I > (D_A+1)ad_A$, and choose $a' > a$ and $\varepsilon > 0$ with $I > (D_A+1)(a'+\varepsilon)d_A$. Subtracting $a'+\varepsilon$ from the logarithmic metric weight leaves positive top self-intersection while preserving smoothness, conjugation invariance, fibrewise nefness and nonnegativity of the Levi form along $P_0 \subseteq P$. Applying Theorem 35.33 to the shifted line produces an integer $r \ge 1$ with at least two sections in S_r of shifted sup norm at most 1, hence a nonzero global section $s \in S_r = H^0(X, A^r)$ satisfying $\|s\|_{\mathrm{sup}} \le e^{-r(a'+\varepsilon)} < e^{-ra'}$ in the original metric. Since the curves W with $h_A(W) \le a$ are Zariski dense in $X_{\mathbf{Q}}$, one such curve is not contained in the zero locus of s. Restricting s to the normalisation of W, effective finite intersection multiplicities and the archimedean bound $-\log\|s\| > ra'$ give $r\,II(W;A) > ra'\deg_{\mathbf{C}} W$, contradicting $h_A(W) \le a < a'$. □

Proof of 35.24

Proof. For (1), resolution and principalisation give embedded characteristic-zero resolutions with coordinate projections, so archimedean potential integrals converge and are independent of the chart. Given a horizontal prime ideal P of arithmetic dimension $n+2$, smooth Hermitian lines $L_1,\dots,L_n,A,B$ with B of multidegree zero, and a multihomogeneous section $s\notin P$ of A, the set $\mathcal{H}$ of horizontal prime components W of $\operatorname{div}_P(s)$ is finite, and each $W\in\mathcal{H}$ has arithmetic dimension $n+1$. Choose a rational prime p such that $-p$ has no square root in the function field of P or of any $W\in\mathcal{H}$. Adjoining a $\mathbb{P}^1$-block cut out by the imaginary quadratic relation $X_0^2+pX_1^2=0$ produces a horizontal prime ideal P_p of arithmetic dimension $n+2$ whose two complex sheets $X_0/X_1=\pm i\sqrt{p}$ are interchanged by complex conjugation, so each smooth Hermitian line glues with its complex conjugate across the two sheets to a smooth conjugation-invariant Hermitian line. On P_p and on the quadratic twist W_p of each $W\in\mathcal{H}$, the archimedean potential integrals double, and $W\mapsto W_p$ is a multiplicity-preserving bijection from $\mathcal{H}$ onto the horizontal divisor support of s on P_p. Applying conjugation-invariant Poincaré–Lelong on P_p and dividing by 2 gives unrestricted Poincaré–Lelong, from which divisor induction constructs II.

For (2), cf. [167, Prop. 3.3.1.5], smoothness of $G\to S$ makes G normal and $G_U\to G$ scheme-theoretically dominant. Let $\Gamma\subset G\times_S G'$ be the schematic closure of the graph of f, with projections $p:\Gamma\to G$ and $q:\Gamma\to G'$. Any point of $G\setminus G_U$ is the closed-point image of a discrete valuation trait in G with generic point in G_U, and any point of $\Gamma\setminus G_U$ is the closed-point image of a discrete valuation trait in Γ with generic point in G_U. Unique extension of semiabelian homomorphisms over discrete valuation rings [176, Ch. I, Prop. 2.7] lifts traits from G to Γ, so p is surjective, and for any integer $n\geq 1$ and geometric point x of Γ with $p(x)^n=1$, applying it to a trait in Γ through x when $p(x)\notin G_U$ (and using the homomorphism f directly when $p(x)\in G_U$) shows that $q(x)^n=1$. Since the n-torsion on each geometric fibre of G' is finite, every point of Γ lying over a torsion point of a fibre of G lies in the quasi-finite locus of p, on which p is an open immersion because G is normal and $G_U\hookrightarrow\Gamma$ is dominant. Given $w_1,w_2\in\Gamma$ with $p(w_1)=p(w_2)=y$, if $y\in G_U$ then $w_1=w_2$ because $p^{-1}(G_U)=\Gamma_f\simeq G_U$; otherwise $w_1,w_2\in\Gamma\setminus G_U$, so realise both points via trait extensions and amalgamate their residue fields over $\kappa(y)$ into an algebraically closed field so that both closed points induce the same geometric point $\bar y$ in a common geometric fibre over S. For every torsion point on that geometric fibre, its two graph lifts factor through the quasi-finite open immersion and agree after composition with p, hence coincide; since torsion is schematically dense on semiabelian fibres, the two pulled-back homomorphisms coincide, so their values at $\bar y$ agree and $w_1=w_2$. Thus p is bijective and locally quasi-finite, hence an isomorphism, and $F=q\circ p^{-1}$ is the unique homomorphism extending f.

For (3), the kernel duality is the Fargues exactness premise of Theorem 35.11. The cokernel bound is [162, §5.3.2, Th. 2]: the cohomology of the integral Hodge–Tate complex of a p-divisible group over $\mathcal{O}_C$ is killed by every element of additive valuation at least $1/(p-1)$, in Fargues's normalisation where p has additive valuation 1. The additive valuation of $\kappa\neq 0$ is $\log v(\kappa)/\log v(p)$, so this is the condition $v(\kappa)^{p-1}\leq v(p)$. The finite flat version is separately [162, §5.4, Th. 3]. The p-divisible assertion follows directly from Theorem 2 above; for $\kappa=0$ it is immediate. When $p=2$, $v(4)=v(2)^2\leq v(2)$, so $v(\kappa)\leq v(4)$ implies the required annihilator condition.

For (4), a smooth morphism $X\to\operatorname{Spec}\mathbf{C}$ partitions X into pairwise disjoint clopen subschemes W_D smooth of constant relative dimension D. When X is proper, each connected component Y of W_D is smooth and proper over $\mathbf{C}$; passing to a smooth projective birational model $\pi:\widetilde{Y}\to Y$ via Chow's lemma [164, Th. 5.6.1] and characteristic-zero resolution [165], compatible holomorphic local sections algebraize on $\widetilde{Y}$ by projective degree-zero GAGA [174], descend along $\pi_*\mathcal{O}_{\widetilde{Y}}=\mathcal{O}_Y$ by normality of Y, and glue uniquely across $\bigsqcup_D W_D$. For an R-Hopf order $\mathcal{A}\subset C^N$ spanned over R by the columns $b_1,\dots,b_{\#N}$ of a matrix $B\in\mathrm{GL}_{\#N}(C)$, its R-dual lattice $\mathcal{A}^*=\{u\in C^N:\sum_{a\in N}u(a)f(a)\in R\ \forall f\in\mathcal{A}\}$ is spanned by the columns u_i of $(B^{-1})^{\mathsf{t}}$ and satisfies $\mathcal{A}^{**}=\mathcal{A}$. Under the Fourier transform $\widehat{u}(h)=\sum_{a\in N}\langle h,a\rangle u(a)$, $\mathcal{A}^\vee=\widehat{\mathcal{A}^*}$ is spanned over R by the columns of $M=T(B^{-1})^{\mathsf{t}}$, where $T=(\langle h,a\rangle)_{h\in H,a\in N}$ is the character table. Point mass at 0 maps to 1, evaluation at 0 gives the counit, convolution in $\mathcal{A}^*$ (closed by comultiplication of $\mathcal{A}$) maps to pointwise multiplication, negation gives the antipode, and for $u\in\mathcal{A}^*$ the dual-basis expansion $\langle h_2,a\rangle=\sum_i\widehat{u}_i(h_2)b_i(a)$ together with $b_iu\in\mathcal{A}^*$ yields $\widehat{u}(h_1+h_2)=\sum_i\widehat{b_iu}(h_1)\widehat{u}_i(h_2)$, giving comultiplication on $\mathcal{A}^\vee$. Fourier duality gives $(\mathcal{A}^\vee)^\vee=\mathcal{A}$ because $T^{\mathsf{t}}(M^{-1})^{\mathsf{t}}=T^{\mathsf{t}}(T^{\mathsf{t}})^{-1}B=B$. Since $\det(M)=\det(T)\det(B)^{-1}$ and $\det(T)^2=\pm(\#N)^{\#N}$, the determinant valuations $c=v(\det B)$ and $c'=v(\det M)$ satisfy $(cc')^2=v(\#N)^{\#N}$. For a subgroup $K\subseteq N$, the image of $\mathcal{A}$ under restriction $C^N\to C^K$ is an R-Hopf order in C^K whose R-dual

is the preimage of $\mathcal{A}^{*}$ under extension by zero $C^{K}\hookrightarrow C^{N}$, while $\{\bar{f}\in C^{N/K}:\bar{f}\circ\pi\in\mathcal{A}\}$ along $\pi:N\to N/K$ equals the R-dual of coset pushforward $\pi_{*}(\mathcal{A}^{*})$ and forms a quotient R-Hopf order.

For (5), relative ampleness of L makes A projective locally over S_{0}, so every finite G-orbit lies in an affine open subscheme of A, and passage to the quotient by the free finite locally free translation equivalence relation $G\times_{S_{0}}A\hookrightarrow A\times_{S_{0}}A$ produces a finite, flat, surjective S_{0}-morphism $q:A\to A'$ with $G\times_{S_{0}}A\simeq A\times_{A'}A$ [163, Th. 4.1], [189, §12, Th. 1]. The commutative group operations on A descend along q, making q a group-scheme homomorphism whose pullback along $1_{A'}$ identifies $\ker(q)$ functorially with G. Properness and geometric connectedness of $A'\to S_{0}$ follow from finite surjectivity of q, flatness of $A'\to S_{0}$ from faithful flatness of q, and smoothness on each geometric fibre $A'_{\bar{s}}$ by faithfully flat descent of regularity from $A_{\bar{s}}$. Because $q_{*}\mathcal{O}_{A}$ is finite locally free and rank-one projective modules over semilocal rings are free, the norm $L'=\mathrm{Nm}_{q}(L)$ is an invertible sheaf on A'. On each geometric fibre $A_{\bar{s}}$ where $G_{\bar{s}}$ has rank $r_{G}\geq 1$, $q_{\bar{s}}^{*}L'_{\bar{s}}$ is algebraically equivalent to $L_{\bar{s}}^{\otimes r_{G}}$ by the theorem of the square, hence ample, so $L'_{\bar{s}}$ is ample on $A'_{\bar{s}}$ by finite surjective descent [189, §7]; since $A'\to S_{0}$ is proper and of finite presentation, fibrewise ampleness then gives relative ampleness of L' on $A'\to S_{0}$ by Noetherian limit reduction from [181, III_{1}, Th. 4.7.1]. When $R_{0}=R$ is the valuation ring of (C,v), applying Zarhin's retraction (A.424) to (A',L') embeds A' as a direct-summand group-scheme retract of a principally polarised abelian scheme over $\operatorname{Spec}R$. □

Proof of 35.25

Proof (outline). We use the notation of Definition 35.27. The tower weight is $\nu_{\tau,i}=-i$, q_{0} is the positive integer specifying the degree $q_{0}\nu$ of the boundary section, and $\lambda_{N}=\log(N\,\mathrm{rd}(E))+|L_{E}|$ for $N\geq 2$.

1. The first eight premises of Theorem 35.11 (resolution through nef Riemann–Roch), including its assumed arithmetic-volume bound, together with Lemmas 35.12, 35.23 and 35.24, provide the section-count and arithmetic-volume bounds on the model, the arithmetic intersection system II, resolution of singularities in characteristic zero, extension of homomorphisms of semiabelian schemes over normal bases, the Hodge–Tate exact sequence of Fargues, algebraisation of sections by GAGA, finite locally free quotients of polarised abelian schemes, and Zarhin's polarised retraction.

   Claim 1. Under these arithmetic-intersection, Kleiman, Hilbert–Samuel, arithmetic-volume and resolution premises, every Hermitian lattice of rank $i\geq 1$ over a number field K, and every $\varepsilon>0$, admit after a finite extension K'/K a nonzero line of normalised degree at least

   $$\widehat{\mu}-\tfrac{1}{2}\log i-\varepsilon.$$

   Here $\widehat{\mu}$ is the normalised slope of the original lattice and the line has the induced metric. The section-index and intersection comparisons give the successive-minima bounds for the dual lattice and hence this line-approximation assertion. This supplies the line input for the split Hodge-model and Lie-slope construction with exactly the assumptions already fixed.

   Lean: `GlobalTheory.CompanionTheory.Ar.Giv.lineOfLargeDegree_of_use` (`Development.WeilBound.CoreStatements.AutomorphicKernels.MoriwakiBigness.HilbertSamuelApplications.LineOfLargeDegreeOfUse`, lines 10–88)

2. Let F_{c} be the coefficient field and let Z_{0} be an algebraic component of the cover. On its moduli open let $\mathcal{A}$ be the universal abelian scheme. For τ in the CM type put $V_{\tau}=(\mathrm{Lie}\,\mathcal{A})_{\tau}$, $V_{\bar{\tau}}=(\mathrm{Lie}\,\mathcal{A})_{\bar{\tau}}$, $p=p_{\mathrm{sig}}$, $q=q_{\mathrm{sig}}$, and

   $$\mathcal{L}_{\tau}=\det V_{\tau}\otimes(\det V_{\bar{\tau}})^{-1}.$$

   By Lemma 35.6, choose an F_{c}-rational section s_{τ} generating $\mathcal{L}_{\tau}$ on this open. For a field extension $\iota:F_{c}\hookrightarrow F$, an F-point y on the open, and eigenframes $v=(v_{1},\dots,v_{p})$, $w=(w_{1},\dots,w_{q})$ at y, define $\varphi_{F}(y;v,w)\in F^{\times}$ by

   $$s_{\tau}(y)=\varphi_{F}(y;v,w)(v_{1}\wedge\cdots\wedge v_{p})\otimes(w_{1}\wedge\cdots\wedge w_{q})^{-1}.$$

   If $v'_{i}=\sum_{j}A_{ij}v_{j}$ and $w'_{i}=\sum_{j}B_{ij}w_{j}$, with $A\in\mathrm{GL}_{p}(F)$ and $B\in\mathrm{GL}_{q}(F)$, then the determinant frame changes by $\det A/\det B$, and hence

   $$\varphi_{F}(y;v',w')\det A=\varphi_{F}(y;v,w)\det B.$$

For a field embedding $\eta : F \hookrightarrow F'$, formation of exterior products and evaluation of s_τ commute with base change, so the coefficient becomes $\eta(\varphi_F(y; v, w))$. At the generic point, with function field Ω, use the same symbols for the generic fibres of the eigenbundles and write $s_\tau = a \otimes b^{-1}$ with nonzero $a \in \bigwedge_\Omega^p V_\tau$ and $b \in \bigwedge_\Omega^q V_{\bar\tau}$. For bases $(b_{1,i})_{i=1}^p$ of V_τ and $(b_{2,i})_{i=1}^q$ of $V_{\bar\tau}$, write $a = \alpha\, b_{1,1} \wedge \cdots \wedge b_{1,p}$ and $b = \kappa\, b_{2,1} \wedge \cdots \wedge b_{2,q}$, where $\alpha, \kappa \in \Omega^\times$. The frame coefficient is α/κ; at a point where these frames and this quotient specialise, its specialisation is $\varphi_F(y; v, w)$.

Lean: `GlobalTheory.CompanionTheory.Ar.Fl.IsLineFn` (`Development.AutomorphicSpectrum.Definitions.GlobalGeometry.IsConstMul`, lines 30–37)

Lean: `GlobalTheory.CompanionTheory.Ar.Flat.Represents` (`Development.AutomorphicSpectrum.Definitions.GlobalGeometry.IsFlatAnIffIsConstMul`, lines 121–129)

For the local relative-generator condition, let V be the generic Lie space, let $R_x \subset \Omega$ be a local coefficient ring, let $A_x, B_x \subset V$ be the eigenlattices of ranks p, q, and let $a \in \bigwedge_\Omega^p V$ and $b \in \bigwedge_\Omega^q V$ be nonzero. A scalar $h \in \Omega$ is a relative coefficient at x if there are $\alpha_x, \beta_x \in \Omega^\times$ such that

$$\bigwedge^p A_x = R_x(\alpha_x a), \qquad \bigwedge^q B_x = R_x(\beta_x b), \qquad h\alpha_x = \beta_x.$$

For a set T of such points, the relative-generator condition requires this equality with $h = 1$ at every $x \in T$. Thus at each point one common scalar generates both exterior lattices, and its cancellation makes $a \otimes b^{-1}$ a generator of their determinant quotient.

Lean: `GlobalTheory.CompanionTheory.Ar.Flat.IsCoeffAt` (`Development.AutomorphicSpectrum.Definitions.GlobalGeometry.IsLineGenOn`, lines 13–15)

Lean: `GlobalTheory.CompanionTheory.Ar.Flat.IsLineGenOn` (`Development.AutomorphicSpectrum.Definitions.GlobalGeometry.IsLineGenOn`, lines 17–18)

For every embedding $\sigma : F_c \hookrightarrow \mathbf{C}$, there is a real number t_σ with the following property. For either $\sigma' = \sigma$ or $\sigma' = \overline{\sigma}$, every complex point y of the component open over σ', and every pair of eigenframes v, w there, let Γ, Δ be their Hodge Gram matrices. Then

$$\log|\varphi_{\mathbf{C}}(y; v, w)| + \frac{1}{2}\big(\log|\det\Gamma| - \log|\det\Delta|\big) = t_\sigma.$$

The determinant change-of-frame identity makes this expression independent of v, w. Consequently the invariant point norm is the constant e^{t_σ} on each of these two sets of frames, and both supremum norms are finite. The constant-norm assertion follows from Lemma 35.6; conjugation of the frame coefficient and the Hodge Gram matrices preserves the displayed expression and gives the same constant at the conjugate embedding.

Lean: `GlobalTheory.CompanionTheory.Ar.Fl.HasInfNorm` (`Development.AutomorphicSpectrum.Definitions.GlobalGeometry.IsConstMul`, lines 59–63)

Lean: `GlobalTheory.CompanionTheory.Ar.Flat.FlatLineGeneratorComponent` (`Development.AutomorphicSpectrum.Statements.GeometryStatements.CardComapMulFinsumEqFinsumFibres`, lines 18–19)

Lean: `GlobalTheory.T.inf_norm_conjugate_invariant` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.InfNormConjugateInvariantStatement`, lines 15–16)

Choose the generator with the relative lattice property on the whole generic fibre of the global split model Y_{gl} over $\mathcal{O}_{F_c}$ constructed in Step 4 below from the good level and the relative algebraic closure of the coefficient field. It has the following uniformised determinant description. Write U for the component open, $\iota_0 : F_c \hookrightarrow \mathbf{C}$ for the distinguished inclusion, and $K_\ell \subset F_c$ for the field of definition of the level datum. Let $\mathrm{ty} : \{1, \dots, g\} \to \mathrm{Hom}(E, \mathbf{C})$ have multiplicity p on the distinguished CM type Ψ and q on $\overline{\Psi}$, and choose any family of complex uniformisations over $\iota_0|_{K_\ell}$ of this type. Fix a member i, with period lattice Λ_i, its E-valued polarisation pairing $S_i : \Lambda_i \times \Lambda_i \to E$, and exponential map $\exp_{i,P} : \mathbf{C}^g \to \mathcal{A}_y(\mathbf{C})$ for each admissible period map $P : \Lambda_i \to \mathbf{C}^g$. Assume that every point over ι_0 represented by this member belongs to U. Put $I = \{1, \dots, p\} \sqcup \{1, \dots, q\}$. There are an $\mathcal{O}_E$-linearly independent tuple $(e_m)_{m \in I}$ in Λ_i and a scalar $\rho_i \in \mathbf{C}^\times$, fixed throughout this member, with the following property.

For an embedding $\eta : E \hookrightarrow \mathbf{C}$, define the coordinate projection

$$(\mathrm{pr}_\eta\, z)_k = \begin{cases} z_k & \mathrm{ty}(k) = \eta, \\ 0 & \mathrm{ty}(k) \neq \eta. \end{cases}$$

For every admissible P, every represented point y over ι_0, every pair of eigenframes v, w there, every lift $V_a, W_j \in \mathbf{C}^g$, and matrices $C \in \mathrm{Mat}_{p\times I}(\mathbf{C})$, $F, B \in \mathrm{Mat}_{q\times I}(\mathbf{C})$ satisfying

$$d\exp_{i,P}(0)V_a = v_a, \qquad d\exp_{i,P}(0)W_j = w_j,$$
$$\mathrm{pr}_\tau\left(\sum_{m\in I} C_{am}P(e_m)\right) = V_a, \qquad \mathrm{pr}_{\bar\tau}\,P(e_n) = \sum_{j=1}^{q} B_{jn}W_j,$$
$$\sum_{m\in I} F_{jm}\tau\big(S_i(e_m,e_n)\big) = B_{jn},$$

one has

$$\varphi_{\mathbf{C}}(y;v,w)\det\begin{pmatrix}C\\F\end{pmatrix} = \rho_i.$$

Lean: `GlobalTheory.CompanionTheory.Ar.Fl.sesq` (`Development.AutomorphicSpectrum.Definitions.GlobalGeometry.IsHodgeGram`, lines 43–44)

Lean: `GlobalTheory.CompanionTheory.Ar.Fl.tyPart` (`Development.AutomorphicSpectrum.Definitions.GlobalGeometry.IsHodgeGram`, lines 46–47)

Lean: `GlobalTheory.CompanionTheory.Ar.Fl.IsConstMul` (`Development.AutomorphicSpectrum.Definitions.GlobalGeometry.IsConstMul`, lines 39–50)

Lean: `GlobalTheory.CompanionTheory.Ar.Fl.IsFlatGen` (`Development.AutomorphicSpectrum.Definitions.GlobalGeometry.IsConstMul`, lines 52–57)

We now give the analytic construction used for this identity. More generally, let $\sigma : F_c \hookrightarrow \mathbf{C}$, let $\tau' = \sigma\circ\tau_{F_c}$, where $\tau_{F_c} : E \hookrightarrow F_c$ is the lift of τ, and suppose that τ' lies in the distinguished CM type. Use uniformisations over $\sigma|_{K_\ell}$, replace $\tau, \bar\tau$ in the projections and polarisation matrix by $\tau', \bar\tau'$, and take algebraic eigenframes over σ. For each linearly independent tuple e, the displayed linear equations determine V, W, F, B and a nonzero scalar $d(P, y, v, w)$ uniquely: a matrix C satisfying them exists, and every such C has $\det\binom{C}{F} = d(P, y, v, w)$. There is a number $a_i > 0$, depending on the member and its tuple, such that a frame function Φ satisfying $\Phi d = \rho$ obeys

$$|\Phi(y;v,w)|^2|\det\Gamma| = |\rho|^2 a_i|\det\Delta|.$$

Choose $\rho = a_i^{-1/2}$. The functions ρ/d assemble on all the uniformised members to a nowhere-zero frame function Φ, with the determinant change-of-frame rule and

$$|\Phi(y;v,w)|^2|\det\Gamma| = |\det\Delta|.$$

Its holomorphy has the following precise meaning. If R is a commutative ring, x is an R-point of the moduli space over F_c with algebraic tangent tuples v_R, w_R, and $\psi_b : R \to \mathbf{C}$ is a family on an open set $B \subset \mathbf{C}^m$ such that every $b \mapsto \psi_b(r)$ is holomorphic, then $b \mapsto \Phi(x_b; v_b, w_b)$ is holomorphic whenever base change gives eigenframes over σ throughout B.

For the global split model Y_{gl} over $\mathcal{O}_{F_c}$ constructed in Step 4 below (independently of Step 2), resolution and GAGA produce a relative determinant generator whose frame function equals $\kappa_0\Phi$, with $\kappa_0 \neq 0$, on a nonempty algebraic open. Since F_c is algebraically closed in the component function field, two generators satisfying the model's relative lattice condition throughout its generic fibre differ by a scalar in $F_c^\times$. The frame comparison extends the equality over U. Thus $\varphi_{\mathbf{C}} = \kappa\Phi$ there for one $\kappa \in \mathbf{C}^\times$. At $\sigma = \iota_0$, substitution gives the required determinant identity with $\rho_i = \kappa a_i^{-1/2}$. Taking logarithms of the norm identity gives $t_\sigma = \log|\kappa|$ for embeddings with τ' in the distinguished type; conjugation gives the paired norm identity above for the other embeddings.

Lean: `GlobalTheory.CompanionTheory.Ar.Flat.IsFlatAn` (`Development.AutomorphicSpectrum.Definitions.GlobalGeometry.IsFlatAnIffIsConstMul`, lines 19–31)

Lean: `GlobalTheory.CompanionTheory.Ar.Flat.IsUnitAn` (`Development.AutomorphicSpectrum.Definitions.GlobalGeometry.IsFlatAnIffIsConstMul`, lines 33–52)

Lean: `GlobalTheory.T.uniformised_eigen_frame_one` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.UniformisedEigenFrameOneStatement`, lines 13–14)

Lean: `GlobalTheory.T.uniformised_eigen_frame_two` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.UniformisedEigenFrameTwoStatement`, lines 13–14)

Lean: `GlobalTheory.CompanionTheory.Ar.Flat.anUnit` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.ExistsUnitAnalyticFunction`, lines 182–226)

Lean: `GlobalTheory.CompanionTheory.Ar.Flat.flat_eq_const_mul_unitAn` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.FrameFunctionAnalyticNorm`, lines 279–303)

Lean: `GlobalTheory.CompanionTheory.Ar.Flat.FlatFirstTheorem.exists_infNorm_of_rt` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.FlatFirstFromResolutionAndGaga`, lines 77–101)

3. Let $\mathcal{Y}$ be a global model with split components of a unitary datum of signature (p,q), with $p,q \geq 1$, coefficient field F_c, host CM type Ψ, and level modulus M'. Put $n = p+q$ and $d = |\Psi|\binom{n}{2}$. Take Comp to be the finite nonempty set of component generic points of $\mathcal{Y}$, and set

$$\mathcal{B} = \{\ell \text{ prime} : \ell \mid 2|\operatorname{disc} E|M'\}, \qquad S_N = \{\ell \text{ prime} : \ell \leq N\} \cup \mathcal{B}, \qquad \nu_{\tau,i} = -i \quad (0 \leq i < n).$$

Thus $2 \in \mathcal{B}$, every ramified prime belongs to every S_N, and ν is rational and strictly decreasing in each row.

For each $Z \in \mathrm{Comp}$, the reference construction (assembling a projective reference geometry over the component generic point Z, a flat holomorphic determinant section on its compact analytic space X_Z^{an}, and an effective dominant weight section via retraction) supplies a root model R_Z, a compact complex space X_Z^{an} of dimension d, a field Ω_Z^0 and a morphism $\xi_Z : \operatorname{Spec}\Omega_Z^0 \to \mathcal{Y}$ with image Z. Its fixed data include coordinates $y_{Z,0}, \ldots, y_{Z,m_Z}$ whose ratios generate Ω_Z^0, positive integers $k_{A,Z}, q'_Z$, a nonzero $s_{E,Z} \in \Omega_Z^0$, and exponents

$$n_{\tau,i} = k_{A,Z} q'_Z \nu_{\tau,i}.$$

The reference flag-cover degree is one, its auxiliary line is ample on the generic fibre, and its embedded analytic realisation maps bijectively to X_Z^{an}. At every $x \in X_Z^{\mathrm{an}}$, the projective point map has a local nonzero holomorphic homogeneous lift λ such that, for each a with $\lambda_a(x) \neq 0$, the differential of $(\lambda_i/\lambda_a)_i$ at x is injective. For every τ and every real regularisation parameter, both the terminal fundamental potential and its negative are plurisubharmonic.

For each N, Z, the finite-prime construction (extending the reference root origin over the finite prime set S_N by adjoining full ℓ^4-torsion level structure and Hodge–Tate eigenlattices at every $\ell \in S_N$) applied to the reference data and S_N, followed by passage to birational multihomogeneous coordinates, gives $M(N,Z)$ with smooth generic fibre, a normal Hodge model $\mathfrak{M}_{N,Z}$ over $\mathcal{Y}$, and a field embedding $g_{N,Z} : \Omega_Z^0 \hookrightarrow \Omega_{N,Z}$ compatible with the generic morphisms and flag bases. If $d_{N,Z}$ is the flag-cover degree, then

$$[\Omega_{N,Z} : \Omega_Z^0] = d_{N,Z}, \qquad \delta_{N,Z} = [F_c : \mathbf{Q}] d_{N,Z}.$$

For every $\ell \in S_N$, the model parameter is 4 and the generic semiabelian scheme has full ℓ^4-torsion. Patching the de Rham eigenlattices away from S_N with the Hodge–Tate eigenlattices at S_N gives locally free lattices of the prescribed eigenranks.

The flags supply root lattices $\Lambda_{\tau,i}$, their multidegrees, rational factors, singular Hodge metrics and smooth conjugation-invariant root metrics $x_{N,Z}(t,\tau,i)$. The auxiliary line has degree one in the coordinate block obtained from $(g_{N,Z}(y_{Z,i}))_i$ and degree zero in the other blocks. Write $t_{0,Z}$ for the regularisation threshold of the fixed analytic system, $H_{\tau,i}(t)$ for the regularised fundamental line, and h_L for the metric coefficient of a Hermitian line L. At every multihomogeneous complex point z,

$$\log h_{H_{\tau,i}(t)}(z) = -\sum_{k=0}^{i} \log h_{x_{N,Z}(t,\tau,k)}(z) \qquad (0 < t \leq t_{0,Z}).$$

Write $\widetilde{X}_{N,Z}$ for the analytic space of this realisation, $\pi_{N,Z} : \widetilde{X}_{N,Z} \to X_Z^{\mathrm{an}}$ for its projection, and z_x for its chosen multihomogeneous coordinate lift at $x \in \widetilde{X}_{N,Z}$. Let $U_Z \subset X_Z^{\mathrm{an}}$ be the open part of the fixed analytic system. For each $\tau \in \Psi$ and $0 \leq i < n$, let $H_{\tau,i}^{\mathrm{ref}}$ be the reference fundamental Hermitian line, let $\phi_{\tau,i}$ be its fixed singular potential on X_Z^{an}, and let $\mathcal{L}_{\tau,i}^{\mathrm{sing}}$ be the singular Hodge root line on the cutoff model. As above, h_L denotes the metric coefficient of a Hermitian line L.

Claim 2. For every $\tau \in \Psi$, every $0 \le i < n$, and every $x \in \widetilde{X}_{N,Z}$ with $\pi_{N,Z}(x) \in U_Z$, one has

$$\log h_{H^{\mathrm{ref}}_{\tau,i}}(z_x) + \phi_{\tau,i}\big(\pi_{N,Z}(x)\big) = -\sum_{k=0}^{i} \log h_{\mathcal{L}^{\mathrm{sing}}_{\tau,k}}(z_x).$$

Lean: `GlobalTheory.CompanionTheory.Ar.RootOrigin` (`Development.AutomorphicSpectrum.Definitions.GlobalGeometry.TowerOrigin`, lines 58–103)

Lean: `GlobalTheory.CompanionTheory.Ar.PB.Realis` (`Development.AutomorphicSpectrum.Definitions.MetricPeriods.MetricsProjectionFormulaNef`, lines 95–119)

The analytic realisation has degree $d_{N,Z}$ and carries the auxiliary metric from the fixed data. A represented point lies in the abelian interior exactly when its image belongs to that open part.

The section $s_{E,Z}$ records effectivity. At a point over characteristic zero, on the projective coordinate chart indexed by a, if u generates $\bigotimes_{\tau,i} \Lambda_{\tau,i}^{-n_{\tau,i}}$, then

$$\frac{g_{N,Z}(s_{E,Z})\, g_{N,Z}(y_{Z,a})^{q'_Z}}{u}$$

is regular and is a nonunit outside the abelian locus. Assemble the models with split index p, component set Comp, bad-prime set $\mathcal{B}$ and weight ν. The generic-point maps identify their components with those of $\mathcal{Y}$; the fixed analytic systems and reference models give the origin data.

Lean: `GlobalTheory.CompanionTheory.Ar.TopTower.comp_data` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.ExistsGoodRootTower`, lines 20–57)

Lean: `GlobalTheory.CompanionTheory.Ar.TopTower.level_data` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.ExistsGoodRootTower`, lines 59–75)

Lean: `GlobalTheory.CompanionTheory.Ar.TopTower.exists_good_tower` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.ExistsGoodRootTower`, lines 77–144)

Lean: `GlobalTheory.CompanionTheory.Ar.FixedData` (`Development.AutomorphicSpectrum.Definitions.GlobalGeometry.TowerOrigin`, lines 18–54)

Lean: `GlobalTheory.CompanionTheory.Ar.RootOrigin` (`Development.AutomorphicSpectrum.Definitions.GlobalGeometry.TowerOrigin`, lines 58–103)

Lean: `GlobalTheory.CompanionTheory.Ar.TowerOrigin` (`Development.AutomorphicSpectrum.Definitions.GlobalGeometry.TowerOrigin`, lines 105–119)

Lean: `GlobalTheory.CompanionTheory.Ar.HodgeCpt` (`Development.AutomorphicSpectrum.Definitions.GlobalGeometry.IsHodgeLine`, lines 33–74)

Claim 3. On $\prod_{\tau\in\Psi} \mathrm{Fl}(\mathbf{Z}^n)$, with the standard Hermitian quotient metrics and its arithmetic intersection system, the flag height satisfies

$$h_{\mathrm{fl}}(p_{\tau,1}Q) = 0 \quad (\deg Q = d), \qquad h_{\mathrm{fl}}(L_v^{d+1}) \ge 0 \quad (v \text{ dominant}),$$

and the latter value is zero whenever an adjacent gap of v is zero. The tensor product of the metrised quotient lines in one row is the trivial Hermitian determinant line, with empty divisor and constant norm one. The rational-section recursion gives the first equality. Summation by parts writes a dominant weight, modulo those determinant classes, as a nonnegative linear combination of the Plücker pullbacks. These lines are smooth, conjugation-invariant, fibrewise nef and generated by integral sections of sup norm at most one, so arithmetic Kleiman gives nonnegative top height. First clear denominators for rational weights, and then use continuity for real weights. On an adjacent wall the corresponding Plücker coefficient is zero. The line then pulls back from the partial flag variety omitting that step, whose arithmetic dimension is d, so its degree-$(d+1)$ intersection is zero. This proves clause 2 of the target definition without using a property of the tower under construction.

Lean: `GlobalTheory.CompanionTheory.Ar.PB.flagHeight_holds` (`Development.AutomorphicSpectrum.Proofs.MetricProofs.FlagHeight`, lines 11–11)

Lean: `GlobalTheory.CompanionTheory.Ar.Top.flag_height_I9` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.FlagHeightI9Properties`, lines 20–45)

4. Let X_0, F_c and ℓ denote the moduli scheme, coefficient field and level of the deep cover. Choose a family $(X_{\ell'})_{\ell'}$ of moduli schemes for the same unitary datum whose member at ℓ is X_0. The level $g_0 g_1$ is good at every prime, multiplication of its modulus preserves this property, and the auxiliary level factor of the deep cover equals one. The global model construction gives a normal projective flat model Y_{gl} of X_0 over $\mathcal{O}_{F_c}$, with its semiabelian extension, which descends to the moduli model and dominates the compatible local model at each valuation ring over a rational prime. The coefficient field is algebraically closed in the component function fields, so Y_{gl} has split components. Apply Step 3 to Y_{gl} with $(p, q) = (p_{\mathrm{sig}}, q_{\mathrm{sig}})$. The comparison of component generic points gives a bijection $\kappa : \mathrm{Comp}(X_0/F_c) \xrightarrow{\sim} \mathrm{Comp}(\mathscr{T})$. Reindexing the tower by κ gives the asserted component set.

   Lean: `Formalism.L2.Pj.goodAt_of_two_primes` (`Formalism.LocalToGlobal.ProjectiveLattices.ModelAt`, lines 124–153)

   Lean: `GlobalTheory.LawsDeep.goodAt_of_mul` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ComplexUniformizationLawsOfGoodDeepCover`, lines 12–19)

   Lean: `GlobalTheory.CompanionTheory.Ar.Rec.exists_isGlobal` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.ExistsIsGlobal`, lines 33–38)

   Lean: `GlobalTheory.T.split_components_of_alg_closed` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.SplitComponentsOfAlgClosedStatement`, lines 15–16)

   Lean: `GlobalTheory.T.tower_components_equiv_of_given` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.TowerComponentsEquivOfGivenStatement`, lines 30–31)

   For the remaining clauses put $r = p_{\mathrm{sig}}$, $s = q_{\mathrm{sig}}$, $f = |\Psi|$, and $d = f\binom{n}{2}$. The deep-cover assumptions give $s = m - j$, $n = k + m - 1$, $m \geq 2$, $m^2 \leq k$, and $1 \leq j < m$. Hence

   $$s \geq 1, \qquad 3s + 1 \leq n, \qquad s \leq r, \qquad n \geq 4.$$

   For the host order parameter $n_{\mathrm{ord}} \geq 1$, one has $f = 2n_{\mathrm{ord}}$. The host discriminant estimate and the bound of fifteen prime divisors of the level leave at most eighteen bad primes above n_{ord}. Thus

   $$\#\mathcal{B}_{\mathscr{T}} \leq n_{\mathrm{ord}} + 19 \leq 128 n_{\mathrm{ord}} = 64 f \leq 4 f n^2.$$

   Lean: `Formalism.BasicNotions.Host.card_bad_gt_le` (`Development.AutomorphicArithmetic.PrimeDiscriminants.BadPrimes`, lines 74–78)

5. For clause 1, the analytic systems underlying the root models are uniformised with split index r. Their degree polynomials and that of the complete flag variety are positive multiples of

   $$\Delta(v) = \prod_{\tau \in \Psi} \prod_{0 \leq a < b < n} (v_{\tau,a} - v_{\tau,b}).$$

   Evaluation at a fixed regular dominant weight fixes the positive proportionality constant. Geometric comparison with $M(N, Z)$, including the normalisation by $\delta_{N,Z}$ in Step 3, therefore gives $V_Z > 0$, independent of N, such that

   $$\deg_{M(N,Z)}(L_v^d) = V_Z D(L_v^d)$$

   for every real weight v. Polarisation gives $\deg_{M(N,Z)}(F) = V_Z D(F)$ for every homogeneous polynomial of degree d; both functionals vanish in the other degrees. The comparison with the fixed analytic system also makes the geometric functional including the auxiliary class independent of the cutoff. The component comparison for a global split model with all local modification parameters equal to 4 gives $V_Z = V_{Z'}$ whenever $s \geq 1$ and $3s + 1 \leq n$. The range proved above supplies these inequalities.

   Lean: `GlobalTheory.T.good_tower_real_agreement` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.GoodTowerRealAgreementStatement`, lines 16–17)

   Lean: `GlobalTheory.T.proportional'` (`Development.AutomorphicSpectrum.Proofs.MetricProofs.ProportionalStatement`, lines 9–10)

   Lean: `GlobalTheory.CompanionTheory.Ar.PB.proportional` (`Development.AutomorphicSpectrum.Proofs.MetricProofs.Proportional`, lines 78–93)

   Lean: `GlobalTheory.T.real_tower_proportional` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.RealTowerProportionalStatement`, lines 7–8)

   Lean: `GlobalTheory.CompanionTheory.Ar.Top.rootTower_proportional_of_unif` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.RealTowerProportional`, lines 153–207)

   Lean: `GlobalTheory.T.good_tower_intersection_extra` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.GoodTowerIntersectionExtraStatement`, lines 16–17)

6. For clause 3, write

$$A_M(v) = P_M(L_v^{d+1}), \qquad H(v) = h_{\mathrm{fl}}(L_v^{d+1}), \qquad \delta(v) = D(L_v^d).$$

For the root model $\mathscr{M}_{M,Z}$, write $k_{A,Z}$ and $\delta_{M,Z}$ for its auxiliary scale and degree factor. Adjoin a variable Y representing the rational Hermitian line $A/k_{A,Z}$, and set $\mathcal{L}(\lambda, a) = L_\lambda + aY$. Let $G_{M,Z}$ and $I_{M,Z,t}$ be the geometric and smoothed arithmetic functionals, divided by $\delta_{M,Z}$, in degrees d and $d+1$. Write $\Phi_{M,Z}$ for the limit of the latter; its restriction to root variables is $P_{M,Z}$. For a horizontal arithmetic curve W, put

$$h_{M,Z,t}(\lambda, a; W) = \frac{1}{\deg_{\mathbf{C}} W}\left(-\sum_{\tau,i}\lambda_{\tau,i} II(W; x_{M,Z}(t,\tau,i)) + \frac{a}{k_{A,Z}} II(W; A)\right).$$

Thus the degree factor $\delta_{M,Z}$ normalises top intersections, while curve heights are divided by $\deg_{\mathbf{C}} W$. For $\varepsilon > 0$, define

$$\lambda^\varepsilon = \lambda - \varepsilon\nu, \quad t_\varepsilon = \exp(-\varepsilon^{-2}), \quad B_{\lambda,\varepsilon} = \mathcal{L}(\lambda^\varepsilon, \varepsilon),$$
$$m_{M,Z}(\lambda,\varepsilon) = G_{M,Z}(B_{\lambda,\varepsilon}^d), \qquad a_{M,Z}(\lambda,\varepsilon) = I_{M,Z,t_\varepsilon}(B_{\lambda,\varepsilon}^{d+1}).$$

Lean: `GlobalTheory.CompanionTheory.Ar.Top.RootTower.mDeg` (`Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.Proportional`, lines 56–57)

Lean: `GlobalTheory.CompanionTheory.Ar.Top.RootTower.mbarTop` (`Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.Proportional`, lines 53–54)

Lean: `GlobalTheory.CompanionTheory.Ar.Top.RootTower.isRatWt_lamS` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.TowerGapHeightUniformBound`, lines 27–34)

Claim 4. These limits exist. There is $C_2 \geq 0$, depending on the tower, such that for every $M \geq 2$, component Z, and monomial m,

$$|\Phi_{M,Z}(m)| \leq C_2 \log M.$$

For every real A_0, there are $\varepsilon_0 > 0$ and $C_3 \geq 0$, independent of $M, Z, \lambda, \varepsilon$, such that

$$|a_{M,Z}(\lambda,\varepsilon) - \Phi_{M,Z}(B_{\lambda,\varepsilon}^{d+1})| \leq C_3\varepsilon$$

whenever $M \geq 2$, $\sum_{\tau,i}|\lambda_{\tau,i}| \leq A_0$, $0 < \varepsilon \leq \varepsilon_0$, and λ^ε is dominant.

Lean: `GlobalTheory.CompanionTheory.Ar.Top.TowerOverReals.tower_I1_of_reference_linked` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.RealTowerI1`, lines 79–133)

Lean: `GlobalTheory.CompanionTheory.Ar.Top.real_tower_convergence_I2` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.RealTowerConvI2`, lines 86–127)

Indeed, the reference comparison gives fixed polynomials Q_Z, constants c_Z, and a common C_ℓ, with each cutoff coefficient differing from its reference coefficient by at most $|c_Z C_\ell| \log M$. One may take

$$A_* = \sum_Z \sum_m |[m]Q_Z|, \qquad C_2 = A_*/\log 2 + \sum_Z |c_Z C_\ell|.$$

The reference sum is finite; polarisation bounds monomial intersections by the corresponding coefficients. Intersections outside degree $d+1$ vanish. For smoothing, the absolute sum of fundamental-weight coefficients is at most $2(A_0 + \|\nu\|_1) + 1$ when $\varepsilon \leq 1$. The component comparison at t_ε gives errors $c'_Z\varepsilon$. Take ε_0 to be the minimum of 1 and the positive component thresholds, and $C_3 = \sum_Z |c'_Z|$. Division by the positive integral degree of an analytic realisation preserves this bound.

Put $g_\tau(\lambda) = \lambda_{\tau,r-1} - \lambda_{\tau,r}$. The remaining gaps are $\lambda_{\tau,i} - \lambda_{\tau,i+1}$ for $i+1 \neq r$.

Claim 5. Fix $c_0 > 0$. There are $\theta_5, \theta_6 > 0$ and real C_5, C_6, independent of cutoff and component, with the following properties. Suppose $M \geq 2$, λ is rational, $\|\lambda\|_\infty \leq 1$, $\varepsilon \in \mathbf{Q}$, $\varepsilon = \theta^2$, every $g_\tau(\lambda) \geq \theta > 0$, and every remaining gap is at least c_0. If $\theta \leq \theta_5$, then every horizontal arithmetic curve satisfies

$$h_{M,Z,t_\varepsilon}(\lambda^\varepsilon, \varepsilon; W) \geq -C_5 \log(M/\varepsilon).$$

If $\theta \le \theta_6$, there is a Zariski-dense family $\mathcal{F}_{M,Z,\lambda,\theta}$ of horizontal arithmetic curves with

$$h_{M,Z,t_\varepsilon}(\lambda^\varepsilon, \varepsilon; W) \le C_6 \log M \qquad (W \in \mathcal{F}_{M,Z,\lambda,\theta}).$$

For the same weights with $\|\lambda\|_\infty \le 1$, there is a uniform $\varepsilon_3 > 0$ such that the rational line $B_{\lambda,\varepsilon}$, with metric at t_ε, is semipositive and nef on every finite fibre whenever $0 < \varepsilon \le \varepsilon_3$, λ^ε is dominant, and

$$4f\|\lambda^\varepsilon\|_\infty \le (M-2) g_\tau(\lambda^\varepsilon) \qquad (\tau \in \Psi).$$

Its cleared-denominator representatives are smooth and conjugation-invariant.

Lean: `GlobalTheory.CompanionTheory.Ar.Top.i5_and_i6_of_isGlobal_of_goodOrigin` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.GoodTowerI5`, lines 19–20)

Lean: `RootFieldNumerics.global_model_tower_fifth_condition` (`Development.WeilBound.CoreStatements.FlagImmersions.GlobalModelTowerFifthCondition`, lines 22–29)

Lean: `GlobalTheory.CompanionTheory.Ar.Top.RootTower.tower_I3_of_nef_inputs` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.TowerI3OfSemipositiveNef`, lines 14–21)

Lean: `GlobalTheory.CompanionTheory.Ar.Top.NefThreshold` (`Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.CompactGapsGE`, lines 68–69)

Lean: `GlobalTheory.CompanionTheory.Ar.Top.mbarTop_squeeze` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.TowerGapHeightUniformBound`, lines 36–93)

The local comparison used for the curve bounds has the following exact form.

Claim 6. At every $\ell \in S_M$, the Hodge–Tate freeness of the chosen model supplies an element κ in its generic field. If $\ell = 2$, then $\kappa = 4$; otherwise $\kappa = \zeta_\ell - 1$ for a primitive ℓ-th root of unity. At every point of its ℓ-fibre, κ is regular and the differential and labelled Lie lattices satisfy

$$\kappa\,\omega_{\mathrm{dR}} \subseteq \omega_{\mathrm{HT}} \subseteq \omega_{\mathrm{dR}}, \qquad \kappa\,L_\epsilon^{\mathrm{HT},\vee} \subseteq L_\epsilon^{\mathrm{dR}} \subseteq L_\epsilon^{\mathrm{HT},\vee}.$$

The Hodge–Tate lattices are those at torsion level ℓ^R, with $R = 4$. At a boundary point over a rank-one valuation ring $O = \mathcal{O}_C$ with algebraically closed fraction field C, the generic fibre G_C is a principally polarised abelian variety with full ℓ^n-torsion $T_n = \Lambda/\ell^n\Lambda$ of rank ℓ^{2gn}, whereas $G[\ell^n] \to \operatorname{Spec} O$ is only quasi-finite, separated, and finitely presented. Because O is Henselian of Krull dimension one, Zariski's main theorem splits off the finite-flat open subgroup scheme $U_n = \operatorname{Spec} A_n^{\mathrm{fin}} \subset G[\ell^n]$ containing the entire closed fibre (of height $2a + t$, where a is the abelian dimension and $t = g - a$ the toric rank of the special fibre), corresponding to the subgroup $H_n \le T_n$ of generic ℓ^n-torsion points extending to O-sections of $G[\ell^n]$. The restricted Weil pairing on U_n has the toric part as radical, so form the induced Hopf order $\mathcal{A}_n = \mathrm{ind}_{H_n}^{T_n}(A_n^{\mathrm{fin}}) \subseteq O^{T_n}$ consisting of all functions $f : T_n \to O$ whose translates $h \mapsto f(t+h)$ lie in A_n^{fin} for every $t \in T_n$ (equivalently, the maximal O-Hopf order inside $\mathcal{O}(G[\ell^n]) \cap O^{T_n}$). Its spectrum $\mathcal{G}_n = \operatorname{Spec} \mathcal{A}_n$ sits in a short exact sequence $0 \to U_n \to \mathcal{G}_n \to (T_n/H_n)_O \to 0$ with constant finite étale quotient of height t, so $\mathcal{G}_n$ has full height $(2a+t)+t = 2g$ and canonical differential isomorphisms $\omega_{\mathcal{G}_n} \simeq \omega_{U_n} \simeq \omega_{G[\ell^n]} \simeq \omega_G/\ell^n\omega_G$; compatibility of the subgroups H_n makes $\mathcal{G} = (\mathcal{G}_n)_{n\ge1}$ an ℓ-divisible group over $\mathcal{O}_C$ of full height $2g$ with $T_\ell\mathcal{G} \simeq \mathbf{Z}_\ell \otimes_{\mathbf{Z}} \Lambda$. Moreover, the cubical rigidification of the ample symmetric line bundle on G defines a regular function on $G[\ell^n] \times_O G[\ell^n]$ extending the unimodular generic Weil bicharacter $\beta_n(x,y) = \zeta_n^{E(x,y)}$ on $T_n \times T_n$, so $\beta_n|_{H_n \times H_n} \in A_n^{\mathrm{fin}} \otimes_O A_n^{\mathrm{fin}}$. Since O is integrally closed and the Cartier dual $\mathrm{Hom}_O(A_n^{\mathrm{fin}}, O)$ is a finite projective O-algebra under convolution, every character $\chi : H_n \to O^\times$ pairs integrally with the dual order and hence lies in A_n^{fin}. On each coset pair $(t + H_n) \times (s + H_n) \subset T_n \times T_n$, the bicharacter identity $\beta_n(t + h_1, s + h_2) = \beta_n(t,s)\beta_n(h_1,s)\beta_n(t,h_2)\beta_n(h_1,h_2)$ therefore lies in $A_n^{\mathrm{fin}} \otimes_O A_n^{\mathrm{fin}}$; extending by zero per coset and summing over $(T_n/H_n)^2$ puts the full unimodular pairing β_n inside $\mathcal{A}_n \otimes_O \mathcal{A}_n$. The resulting morphism $\lambda : \mathcal{G} \to \mathcal{G}^\vee$ induces a bijection $T_\ell\lambda : T_\ell\mathcal{G} \xrightarrow{\sim} T_\ell\mathcal{G}^\vee$; applying Fargues exactness (Lemma 35.24(3)) to $\mathcal{G}^\vee$ over $\mathcal{O}_C$ (using $(\mathcal{G}^\vee)^\vee \simeq \mathcal{G}$ and transporting along $T_\ell\lambda$) yields $\kappa\,\omega_{\mathcal{G}} \subseteq \mathrm{im}(\alpha_{\mathcal{G}^\vee} \circ (T_\ell\lambda \otimes 1))$ with annihilation constant κ. The chart lattices are locally free by the choice of model. The level-ℓ^4 Hodge–Tate lattices of the moduli model are coherent and span the generic dual Lie space. Adjoining finitely many coordinates gives a normal model, isomorphic to the original one over the generic fibre, on which these lattices and all their labelled projections are locally free; the Hodge–Tate lattice of the new model is the pullback of the old one, that is, the lattice it generates in the common generic fibre. Let A be a

chart ring of this model. For every height-one prime $\mathfrak{q}$ of A containing ℓ there is an injection $A \hookrightarrow \mathcal{O}_C$, with C algebraically closed and the valuation of $\mathcal{O}_C$ centred at $\mathfrak{q}$, along which the chart Hodge–Tate data reduce to the data at a point of the kind just treated; there $\kappa\,\omega_{\mathrm{dR}} \subseteq \omega_{\mathrm{HT}}$. Since A is a normal Noetherian domain, ω_{HT} is projective over A, and $\kappa\ell^{R-2} \mid \ell^R$ at $R = 4$, these local inclusions give $\omega_{\mathrm{HT}} \subseteq \omega_{\mathrm{dR}}$ and $\kappa\,\omega_{\mathrm{dR}} \subseteq \omega_{\mathrm{HT}}$ over A. The same argument on every labelled eigenspace at ℓ yields the displayed sandwich inclusions. Applying this on every cutoff model compares it with the fixed reference root model of its component. The comparison functions are generic units whose vertical divisors at the primes of S_M have total weight at most $C \log M$, and, for every dominant weight μ with $\|\mu\|_\infty \le 2$ and every $0 < \varepsilon \le \varepsilon_0$, at every complex point of the joined model the sum over the root lines and the auxiliary line, weighted by the coefficients of $\mathcal{L}(\mu, \varepsilon)$, of the logarithms of the ratio of the cutoff metric at smoothing parameter t_ε to the reference metric, corrected by the comparison units, lies between $-C \log(1/\varepsilon) - C$ and C, with C and $\varepsilon_0 > 0$ independent of the cutoff. Since $M \ge 2$, this loss is at most a fixed multiple of $\log(M/\varepsilon)$, the size allowed in Claim 5. The all-curve lower bound of Claim 5 follows from this comparison together with the ample auxiliary line, the flat unit sections given by the split-component condition and the reference flat generators, and the fibre comparison on the reference model. The dense-family upper bound follows from the same comparison applied to a Zariski-dense family of algebraic points of bounded height on the reference model, lifted to the cutoff model.

Lean: `GlobalTheory.CompanionTheory.Ar.HT.ModelSandwich.sandwich_of_htFree` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.ExistsSandwichOfHodgeTateFree`, lines 117–187)

Lean: `GlobalTheory.CompanionTheory.Ar.TopGood.tower_good_of_sandwich_split` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.TowerIntersectionOfSandwichSplit`, lines 27–46)

For finite-fibre nefness at a prime of S_M, the Hodge–Tate freeness of the model at that prime and Lemma 35.24(3) give the chart matrices constructed for the trivial-flag assertion of Claim 11. Hence on every component of that fibre the root lines form, after a residue-field extension, a full flag of the trivial rank-n bundle, and the line of every dominant weight is nef there. A prime outside S_M is greater than M, so the displayed threshold implies its local threshold. The nonnegative auxiliary class preserves nefness. Analytic positivity has the separate small-ε threshold just stated.

Lean: `GlobalTheory.CompanionTheory.Ar.Top.RootTower.nefFib_of_nefThreshold` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.GapNonnegOfDominant`, lines 27–38)

Claim 7. For rational λ, ε, if $\varepsilon \ge 0$ and both $\lambda, \lambda^\varepsilon$ are dominant, then

$$0 \le m_{M,Z}(\lambda, \varepsilon) \le V_Z \delta(\lambda).$$

Under the hypotheses of Claim 5, put

$$\theta_* = \min(\theta_5, \theta_6, 1, \varepsilon_3), \qquad C_s = (d+1)(\max(C_5, 0) + \max(C_6, 0)).$$

If $0 < \theta \le \theta_*$ and the dominance and finite-fibre threshold hold, then

$$|a_{M,Z}(\lambda, \varepsilon)| \le C_s \log(M/\varepsilon) m_{M,Z}(\lambda, \varepsilon).$$

Lean: `RootFieldNumerics.tower_rational_weight_bound` (`Development.WeilBound.CoreStatements.FlagImmersions.TowerRationalWeightBound`, lines 166–207)

For the degree bound, the dominant classes and auxiliary class are generically nef. Step 3 supplies the effective class $L_\nu - Y$, and

$$L_\lambda - B_{\lambda,\varepsilon} = \varepsilon(L_\nu - Y).$$

Expand the difference of their d-th powers as

$$\varepsilon(L_\nu - Y) \sum_{j=0}^{d-1} L_\lambda^j B_{\lambda,\varepsilon}^{d-1-j}.$$

Every term has nonnegative degree by effectivity and nefness. This also gives nonnegativity of $m_{M,Z}$. Since $\varepsilon = \theta^2 \le \theta \le \varepsilon_3$, the rational lower intersection inequality and the dense-point upper intersection inequality apply to Claim 5 and give

$$-(d+1)C_5 \log(M/\varepsilon) m_{M,Z} \le a_{M,Z} \le (d+1)C_6 \log M\, m_{M,Z}.$$

As $0 < \varepsilon \le 1$, these imply the claimed absolute bound.

Choose the compact wall patches with $g_{\tau_0}(v) = 0$, $\|v\|_\infty \le 1/2$, and every other gap at least $c_0 > 0$. Apply the preceding claims with $c_0/2$. For rational v in that larger wall domain, set

$$\theta = 5f/M, \qquad \lambda_{\tau,i} = v_{\tau,i} + 2\theta \mathbf{1}_{\{\tau=\tau_0,\ i<r\}}, \qquad \varepsilon = \theta^2.$$

For all sufficiently large M, one has $M \ge 3$, $0 < \theta \le \min(1, \theta_*, \varepsilon_0)$, $\|\lambda\|_\infty, \|\lambda^\varepsilon\|_\infty \le 1$, dominance of both weights, all split gaps of λ at least θ, its τ_0-gap equal to 2θ, and all remaining gaps at least $c_0/2$. The finite-fibre threshold holds, and

$$\|\lambda^\varepsilon - v\|_\infty \le 2\theta + \|\nu\|_\infty \theta^2, \qquad 0 \le \log(M/\varepsilon) \le 3 \log M.$$

The flag degree bound gives a fixed $C_8 \ge 0$ with

$$\delta(\lambda) \le C_8 g_{\tau_0}(\lambda) = 2C_8\theta, \qquad m_{M,Z}(\lambda, \varepsilon) \le 2C_8 V_Z \theta.$$

Lean: `GlobalTheory.T.perturbed_weight_gap_bounds` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.PerturbedWeightGapBoundsStatement`, lines 11–12)

Lean: `GlobalTheory.T.flag_degree_le_gap` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.FlagDegreeLeGapStatement`, lines 11–12)

Claim 8. There is $C_p \ge 0$, depending only on f, n, d, such that Claim 4 implies

$$|\Phi_{M,Z}(\mathcal{L}(\alpha, a)^{d+1}) - \Phi_{M,Z}(\mathcal{L}(\beta, b)^{d+1})| \le C_p C_2 \log M (\|\alpha - \beta\|_\infty + |a - b|)$$

when $\|\alpha\|_\infty, \|\beta\|_\infty \le 1$ and $|a|, |b| \le 1$.

Lean: `GlobalTheory.T.arith_functional_lipschitz_weights` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.ArithFunctionalLipschitzWeightsStatement`, lines 6–7)

Apply this with $(\alpha, a) = (\lambda^\varepsilon, \varepsilon)$, $(\beta, b) = (v, 0)$. The three errors are bounded separately by

$$\begin{aligned} |a_{M,Z}(\lambda, \varepsilon)| &\le 6 C_s C_8 V_Z \theta \log M, \\ |a_{M,Z}(\lambda, \varepsilon) - \Phi_{M,Z}(B^{d+1}_{\lambda,\varepsilon})| &\le C_3 \theta \log M, \\ |\Phi_{M,Z}(B^{d+1}_{\lambda,\varepsilon}) - P_{M,Z}(L_v^{d+1})| &\le C_p C_2 (3 + \|\nu\|_\infty) \theta \log M. \end{aligned}$$

Here $\theta^2 \le \theta \le \theta \log M$. Hence

$$|P_{M,Z}(L_v^{d+1})| \le C_Z \theta \log M, \quad C_Z = C_3 + 6 C_s C_8 V_Z + C_p C_2 (3 + \|\nu\|_\infty).$$

Sum over components and extend by continuity from rational points in the $c_0/2$-wall domain to real points in the c_0-wall domain. On the chosen patches,

$$|A_M(v)| \le C_{\mathrm{wall}} \frac{\log M}{M}, \qquad C_{\mathrm{wall}} = 5f \sum_Z C_Z.$$

Lean: `GlobalTheory.CompanionTheory.Ar.Top.thmW_tower` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.TowerWeilTheorem`, lines 120–203)

The full-flag intersection comparison and the alternating-polynomial perturbation estimate give real coefficients $a_{M,\tau}, b_{M,\tau}$ such that every coefficient of

$$A_M(v) - VH(v) - V(d+1)\delta(v) \sum_{\tau \in \Psi} \left(a_{M,\tau} \sum_{i<r} v_{\tau,i} + b_{M,\tau} \sum_{i \ge r} v_{\tau,i} \right)$$

has absolute value at most $C_1 \log M / M$, with C_1 fixed by the tower. The polynomial $A_M - VH$ is homogeneous of degree $d+1$ and changes sign under each transposition $X_{\tau,a} \leftrightarrow X_{\tau,a+1}$ with $a+1 \ne r$. Indeed, the root models carry the square system of such an in-block step, on which the sum of the height polynomial at v and at the transposed weight is an expression determined by the degree polynomial and

the gaps of v; the complete flag variety satisfies the same relation with H, and by clause 1 the two degree polynomials differ by the factor V. The flag-height polynomial vanishes on the walls $v_{\tau,r-1} = v_{\tau,r}$ by clause 2. On the space of these block-alternating polynomials, a polynomial bounded by ε on compact wall patches, each containing a relatively open piece of its wall, differs coefficientwise by at most a fixed multiple of ε from Δ times a linear form invariant under the block Weyl group, and such a form is $\sum_\tau (a_\tau \sum_{i<r} v_{\tau,i} + b_\tau \sum_{i\ge r} v_{\tau,i})$. This perturbation estimate bounds the coefficient error by a fixed multiple of the wall error.

Twisting by a host automorphism gives an isomorphism of moduli problems and permutes the components and CM labels. The comparison of the corresponding root models says that the difference of their normalised intersections of degree $d+1$ tends to zero with the smoothing parameter. Their local parameters agree, since they are all 4. Passing to the limiting functionals and summing over components therefore permits averaging the correction over the host-label action. The flag-height functional is invariant under this relabelling, and the averaged correction has two coefficients α_M, β_M, constant over τ.

For powers of L_v, direct differentiation gives

$$\mathrm{Sh}_r(\tau, L_v^{d+1}) = -(d+1)\left(\sum_{i<r} v_{\tau,i}\right)\delta(v), \qquad \mathrm{Tot}(\tau, L_v^{d+1}) = -(d+1)\left(\sum_{i} v_{\tau,i}\right)\delta(v).$$

Polarising the averaged coefficient estimate gives $C \ge 0$, $N_1 \ge 2$ and, for every $M \ge N_1$ and every monomial m of degree $d+1$,

$$\left| P_M(m) - V\left(h_{\mathrm{fl}}(m) - (\alpha_M - \beta_M)\sum_\tau \mathrm{Sh}_r(\tau, m) - \beta_M \sum_\tau \mathrm{Tot}(\tau, m)\right)\right| \le C\frac{\log M}{M}.$$

Extend the coefficient sequences arbitrarily below N_1. This is clause 3.

Lean: `GlobalTheory.CompanionTheory.Ar.Top.thmW_tower` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.TowerWeilTheorem`, lines 120–203)

Lean: `GroundTheory.Heights₂.Heights.AntisymPerturbation.Statement` (`GroundTheory.HeightsAndTheta.Heights.Antisymmetric.Statements`, lines 86–97)

Lean: `GlobalTheory.CompanionTheory.Ar.Cmp.CmpPair` (`Development.AutomorphicSpectrum.Statements.GeometryStatements.CmpPair`, lines 12–22)

Lean: `GlobalTheory.CompanionTheory.Ar.TwistAut` (`Development.AutomorphicSpectrum.Statements.GeometryStatements.PadicIntegersHenselianFiniteResidue`, lines 11–14)

Lean: `GlobalTheory.T.good_tower_propA_of_global_extra` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.GoodTowerGlobalExtraStatement`, lines 18–19)

Lean: `GlobalTheory.CompanionTheory.Ar.Top.chain_tower` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.TowerWeilInputsCombined`, lines 22–53)

Lean: `GlobalTheory.T.tower_weil_inputs_combined` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.TowerWeilInputsCombinedStatement`, lines 7–8)

Lean: `GlobalTheory.CompanionTheory.Ar.Top.exists_finite_identity` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.TowerIdentityAtCutoff`, lines 97–149)

7. For clause 4, let $\lambda_\epsilon : \mathcal{O}_E \to F_c$ be the embedding attached to a signed label $\epsilon \in \Psi \sqcup \Psi$, with the right copy obtained by conjugation. Choose permutations φ_ℓ of $\Psi \sqcup \Psi$, for $\ell \in \mathbf{N}$, such that for every prime ℓ, every prime ideal $\mathfrak{n} \subset \mathcal{O}_{F_c}$ containing ℓ, every signed label ϵ, and every $a \in \mathcal{O}_E$,

$$\lambda_{\varphi_\ell(\epsilon)}(a) \equiv \lambda_\epsilon(a)^\ell \pmod{\mathfrak{n}}.$$

The congruence uses the integral representatives in $\mathcal{O}_{F_c}$. The abelian Galois hypothesis gives these permutations. The distinguished type Ψ is induced by the host root: it consists of the embeddings sending the host root to its fixed complex value. A Frobenius element sends the host root to itself or to its negative, so it carries Ψ to Ψ or to $\overline{\Psi}$. This makes the side of

$$\mathrm{Fr}_{\ell,h}(\tau) = \varphi_\ell^{h+1}(\mathrm{inl}\,\tau) \qquad (\ell, h \in \mathbf{N},\ \tau \in \Psi)$$

independent of τ. In particular $h = 0$ denotes the first arithmetic Frobenius iterate.

Fix $2 \le N \le M$ and Z, and put

$$S_{N,M} = \{\ell : \ell \text{ prime},\ N < \ell \le M, \ell \notin \mathcal{B}_{\mathscr{T}}\}.$$

Claim 9. The common pair. There is a common normal Hodge model X, with multihomogeneous coordinates, generic dimension d, and generic field Ω, carrying the pullbacks of the two cutoff flags. Write b_N, b_M for the degrees of its generic maps to the cutoff models and put

$$\delta = b_N \delta_{N,Z} = b_M \delta_{M,Z} > 0.$$

For a root index $a = (\tau, i)$, write x_a, x'_a for the two pulled-back multidegrees and $e_a, e'_a \in \Omega^\times$ for their rational root factors. The first flag is obtained by patching at S_N, the second by patching at S_M, inside the same generic flag. There are multihomogeneous polynomials f_a, g_a, nonzero modulo the coordinate ideal of X, such that

$$\deg f_a - \deg g_a = x_a - x'_a, \qquad f_a(y) = (e'_a/e_a) g_a(y).$$

The induced rational identifications of the root lines are invertible away from the fibres over $S_{N,M}$. The two collections have smooth conjugation-invariant metrics $L_a(t), L'_a(t)$, for every $t > 0$, making these identifications isometries. For sufficiently small $t > 0$ they are the pulled-back cutoff metrics. Their normalised limiting arithmetic functionals are $P_{N,Z}$ and $P_{M,Z}$; the geometric functional of the first collection is that of $M(N, Z)$.

Take a common finite extension of the generic fields over the reference field and a common normal projective model of their coordinate presentations. Pull back both flags and compare their bases through the reference flag, whose cover degree is one. The degree formula gives the displayed equality of normalising factors. The patched lattices agree away from the newly added primes: the cutoff sets are nested, contain the discriminant primes and 2, and their parameters agree on the intersection because they equal 4. Restricting the second patched flag to the first cutoff set therefore gives the first flag, including its root factors. Comparing the root-line presentations gives f_a, g_a. Pull back the cutoff metrics for small positive t; outside that interval choose smooth metrics and transport them through the same identifications. The projection formula, with the common factor δ, preserves the limits.

Lean: `GlobalTheory.CompanionTheory.Ar.OrLinked.common` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.RootOriginComparisonSubset`, lines 568–664)

Lean: `GlobalTheory.CompanionTheory.Ar.Num.rootLines_vert_ratio` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.FlagHeightBoundOfArithVar`, lines 20–33)

Lean: `GlobalTheory.CompanionTheory.Ar.Top.Link.link_step` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.GoodTowerGlobalConditions`, lines 69–191)

For a signed label $\epsilon \in \Psi \sqcup \Psi$ and $m \ge 0$, define

$$B_{\epsilon,m} = \begin{cases} \displaystyle\sum_{i<r} X_{\sigma,i}^m, & \epsilon = \operatorname{inl}\sigma, \\ \displaystyle\sum_{i\ge r} (-X_{\sigma,i})^m, & \epsilon = \operatorname{inr}\sigma. \end{cases}$$

Claim 10. The comparison polynomial. For $\ell \in S_{N,M}$, $k \ge 1$ and $\tau \in \Psi$, put

$$c_{\ell,k}(\epsilon_0, \epsilon) = \sum_{h\ge 0} \mathbf{1}_{\{\varphi_\ell^{h+1}(\epsilon_0)=\epsilon\}} \ell^{-k(h+1)} - \frac{\mathbf{1}_{\{\epsilon=\epsilon_0\}}}{\ell - 1},$$

$$C_{\ell,k,\tau} = k \sum_{\epsilon\in\Psi\sqcup\Psi} c_{\ell,k}(\operatorname{inl}\tau, \epsilon) B_{\epsilon,k-1}.$$

The series converge absolutely, and $C_{\ell,k,\tau}$ is homogeneous of degree $k-1$. If m is the orbit length of ϵ_0, then

$$c_{\ell,k}(\epsilon_0, \epsilon) = \frac{\displaystyle\sum_{a=1}^{m} \mathbf{1}_{\{\varphi_\ell^a(\epsilon_0)=\epsilon\}} \ell^{k(m-a)}}{\ell^{km} - 1} - \frac{\mathbf{1}_{\{\epsilon=\epsilon_0\}}}{\ell - 1}.$$

Indeed, group the absolutely convergent series according to $h+1 = tm+a$, where $t \geq 0$ and $1 \leq a \leq m$, and sum the geometric series.

Lean: GroundTheory.Heights₂.Heights.ModelLeaves.co (GroundTheory.HeightsAndTheta.Heights.IterateExamples.Statements, lines 61–62)

Lean: GlobalTheory.CompanionTheory.Ar.Num.liePs (Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.BundleChernComparison, lines 104–106)

Lean: GlobalTheory.CompanionTheory.Ar.Top.pLie (Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.CompactGapsGE, lines 39–41)

Lean: GlobalTheory.CompanionTheory.Ar.Num.co_series_three_step (Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.CoSeriesThreeStep, lines 9–62)

We specify the local multiplicities. Products of multidegrees mean intersections of their first Chern classes. Let A be the integral closure of $\mathbf{Z}$ in Ω, let $\mathfrak{m} \subset A$ lie above ℓ, and put $O = A_{\mathfrak{m}}$. Here A is finite over $\mathbf{Z}$, O is a discrete valuation ring, and elements of A belong to every nonempty coordinate-chart ring of the common model. Write $R_O = O[Y_{b,j}]$ for the multihomogeneous polynomial ring and P_O for the model's defining ideal. For a nonzero multihomogeneous polynomial a modulo P_O, set

$$m_W(a) = \begin{cases} \operatorname{length}_{(R_O)_W}\big((R_O/(P_O+(a)))_W\big), & W \text{ a relevant minimal prime over } P_O+(a), \\ 0, & \text{otherwise,} \end{cases}$$

$$\operatorname{div}_{P_O}(a) = \sum_W m_W(a)[W].$$

Relevant primes are those defining nonempty components in the product of projective spaces. Define $\operatorname{div}_{P_O}(a/b) = \operatorname{div}_{P_O}(a) - \operatorname{div}_{P_O}(b)$. Thus every component of the divisor of ℓ has its full local-length multiplicity. On a normal codimension-one local ring this is the order of vanishing. Evaluation of a monomial $\prod_a X_a^{\mu_a}$ repeats the root class x_a exactly μ_a times in the intersection list, followed by the complementary classes; no factorial is introduced.

Lean: GroundTheory.Heights₂.Heights.NumbersDVR.multAt (GroundTheory.HeightsAndTheta.Heights.DiscreteValuationNumbers.Objects, lines 42–43)

Lean: GroundTheory.Heights₂.Heights.NumbersDVR.classList (GroundTheory.HeightsAndTheta.Heights.DiscreteValuationNumbers.BundleObjects, lines 107–108)

Lean: GlobalTheory.CompanionTheory.Ar.Num.cycList (Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.BundleChernComparison, lines 43–43)

Claim 11. The two local outputs. For the pair in Claim 9, every $\ell \in S_{N,M}$, every such O, every τ, every $k \geq 1$, and every list Q_0 of further multidegrees with $\operatorname{length}(Q_0)+k = d+1$, the intersection of

$$\sum_{i=0}^{n-1} \operatorname{div}_{P_O}(f_{\tau,i}/g_{\tau,i}) \sum_{a=0}^{k-1} x_{\tau,i}^a (x'_{\tau,i})^{k-1-a}$$

with Q_0 equals the intersection of $\operatorname{div}_{P_O}(\ell)$, $C_{\ell,k,\tau}$ evaluated in the first root classes, and Q_0. These are numerical intersection equalities. The trivial-flag construction applies at every $\ell \in S_M$, including ramified primes and 2. On each component over such a prime the second collection, after a residue-field extension, is a full flag of the trivial rank-n bundle with root multidegrees $x'_{\tau,i}$. Its positive Newton power sums therefore have zero intersection with every complementary list on that component.

Use the de Rham and Hodge–Tate lattice families on the Segre charts of the actual common Hodge model. At a nonzero Segre coordinate z_c, the chart ring is $O[z_a/z_c]_a$. In an eigenbasis, the coordinate lattice consists of coefficient vectors whose linear combinations belong to the prescribed Hodge lattice at every point of this chart. The opposite eigenspace contributes the dual lattice for the coordinate pairing with values in the chart ring. At each place and starting signed label, the finite-field family construction supplies a finite label set and a permutation compatible with φ_ℓ. Pass to a finite extension of O containing the required roots of unity and descend the chart lattice pairs. The exact numerical consequence is the following.

Claim 12. Fix $k \geq 1$ and the complementary list Q_0 in Claim 11. On each finite label cover S', with permutation ϕ' and label map $e : S' \to \Psi \sqcup \Psi$ satisfying $e\phi' = \varphi_\ell e$, there is a real function H such that

$$\Delta_{e(t)} = H(t) - \frac{A_{e(t)}}{\ell - 1}, \qquad \ell^k H(t) = H(\phi' t) + A_{e(\phi' t)}.$$

Here Δ_ϵ is the sum of the Lie and opposite-differential Newton comparison numbers for the two chart lattice pairs, with the full local-length multiplicities. The quantity A_ϵ is k times the intersection of $\mathrm{div}(\ell)$, the $(k-1)$-st Newton sum of the first Lie lattice at ϵ, and Q_0. Each signed label is in the image of one such cover. The hypotheses are the normal common Hodge model, its compatible de Rham and Hodge–Tate chart lattice pairs, $\ell \neq 2$, and $\ell \nmid \operatorname{disc} E$; the last two hold because $\ell \in S_{N,M}$. Put $w = \zeta_\ell - 1$, so that $w^{\ell-1}$ is a unit times ℓ. Write $\mathcal{L} \subseteq \mathcal{W} \subseteq w^{-1}\mathcal{L}$ for the two Lie lattices and $\omega' \subseteq \omega \subseteq w^{-1}\omega'$ for the two differential lattices at a label, $\mathcal{H}$ for the dual of the character-annihilator lattice of the ℓ-torsion function lattice, $X_{\epsilon,i}$ for the rank-one lattice of the i-th character coordinate of type ϵ, and $c_k(\cdot,\cdot)$ for the Newton comparison number of degree k against Q_0. Then $H(t) = (-1)^k \sum_i c_k(X_{e(t),i}, \mathcal{O})$. The chart sequence $0 \to w^{-1}\mathcal{L}/\mathcal{W} \to \mathcal{H}/\mathcal{O}^n \to \omega/\omega' \to 0$ gives $\Delta_\epsilon = c_k(\mathcal{H}, \mathcal{O}^n) - kI_\epsilon(w)$, where $I_\epsilon(w)$ is the intersection defining A_ϵ/k with $\mathrm{div}(w)$ in place of $\mathrm{div}(\ell)$. In the coordinates of $\mathcal{O}^n$ one has $\mathcal{O}^n \subseteq \mathcal{H} \subseteq w^{-1}\mathcal{O}^n$, and for each i the i-th coordinates of the vectors of $\mathcal{H}$ whose later coordinates vanish form the inverse of the line $X_{e(t),i}$. So $\mathcal{H}$ has a triangular basis whose diagonal entries generate these inverse lines. Newton comparison numbers add along such a flag, and inversion negates the root class, so $c_k(\mathcal{H}, \mathcal{O}^n) = H(t)$, and $(\ell-1)kI_\epsilon(w) = A_\epsilon$; this is the first relation. The Frobenius compatibility of the torsion data, by which the i-th step lattice of type $\phi'(t)$ meets the coordinate line in the span of the ℓ-th powers of the i-th character lattice of type t, gives $A_{e(\phi' t)} = (-1)^{k-1} \sum_i \big(c_k(X_{e(\phi' t),i}, \mathcal{O}) - \ell^k c_k(X_{e(t),i}, \mathcal{O})\big)$, which is the second relation. These chart facts are checked at test points and hold on a refinement of the common model on which all the lattices involved are locally free; the intersection numbers descend from the refinement to the model. Eliminating H, the two relations give $\ell^k \Delta_\epsilon - \Delta_{\varphi_\ell(\epsilon)} = (\ell A_{\varphi_\ell(\epsilon)} - \ell^k A_\epsilon)/(\ell - 1)$.

Lean: `GlobalTheory.CompanionTheory.Ar.Num.Relations` (`Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.BundleChernComparison`, lines 66–70)

Lean: `GlobalTheory.CompanionTheory.Ar.Num.TopRel.relations` (`Development.AutomorphicSpectrum.Proofs.AlgebraProofs.ArithmeticRelationsStatementHolds`, lines 12–43)

Lean: `GlobalTheory.CompanionTheory.Ar.Num.HodgeModel.model_on_hodge_of_family_pos` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.ModelOnHodgeFromFamily`, lines 37–171)

Iterating the second relation gives

$$H(t) = \sum_{a=1}^{b} \ell^{-ka} A_{e((\phi')^a t)} + \ell^{-kb} H((\phi')^b t).$$

The last term tends to zero as $b \to \infty$, because S' is finite and $\ell^k > 1$. Substitute the resulting convergent series into the first relation and use $e\phi' = \varphi_\ell e$. This is exactly the coefficient formula in Claim 10. Duality on the complementary eigenspace negates its root classes. Comparison of the reversed dual flag with the chosen flag therefore gives the local Newton equality in Claim 11, and finite-extension descent returns it to O.

Lean: `GlobalTheory.CompanionTheory.Ar.Num.local_comparison_of_orbits` (`Development.AutomorphicSpectrum.Proofs.AlgebraProofs.LocalComparisonOfOrbitsHolds`, lines 17–97)

Lean: `GlobalTheory.T.flag_m3_from_local` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.FlagM3FromLocalStatement`, lines 8–9)

The trivial-flag assertion applies Lemma 35.24(3) to the full height-$2g$ polarised ℓ-divisible group $\mathcal{G}$ constructed via induced Hopf orders in Claim 6 (over every interior or boundary point). Its construction works modulo ℓ^{R-2}. On each chart c, with ring A_c, the sandwich of Claim 6 and $\kappa\ell^{R-2} \mid \ell^R$ give a sequence $\omega^\vee_{\mathrm{HT},\bar\tau} \xrightarrow{i_c} A_c^n \xrightarrow{q_c} \omega_{\mathrm{HT},\tau}$ that is exact modulo ℓ^{R-2}. Here A_c^n carries the coordinates of the fixed level-ℓ^R torsion module. The lattice $\omega_{\mathrm{HT},\tau}$ is the projection to the generic τ-eigenspace of the A_c-span of all Hodge–Tate values together with $\ell^R\omega_{\mathrm{dR}}$. The A_c-linear map q_c sends each standard coordinate vector to its projected Hodge–Tate value and satisfies $q_c(A_c^n) + \ell^{R-2}\omega_{\mathrm{HT},\tau} = \omega_{\mathrm{HT},\tau}$. Modulo ℓ^{R-2}, the map i_c is dual to the corresponding map into $\omega_{\mathrm{HT},\bar\tau}$ under the different-scaled Weil pairing, has a retraction, and has image equal to the kernel of q_c. The two Hodge–Tate lattices have the complementary ranks of the two labels. For the first columns of g_c, apply i_c to the dual flag basis of $\omega_{\mathrm{HT},\bar\tau}$. For each flag basis vector b of $\omega_{\mathrm{HT},\tau}$, choose a remaining column $x \in A_c^n$ with $q_c(x) \equiv b \pmod{\ell^{R-2}\omega_{\mathrm{HT},\tau}}$. By exactness the columns form a basis modulo ℓ^{R-2}, so $\det g_c$ is a unit modulo ℓ^{R-2}.

Lean: GlobalTheory.CompanionTheory.Ar.HT.HTChart.IsDualMap (Development.AutomorphicSpectrum.Definitions.CommutativeAlgebra.IsCompactUnitaryPointsAtPlaceFstMemCompact, lines 49–54)

Lean: PositivityTheory.HtPt.Ge.frame_exists (Development.WeilBound.CoreStatements.HodgeTatePoints.GeFrame, lines 66–115)

On two charts c, c' the Hodge–Tate maps agree modulo ℓ^R after the monomial scaling, so $(z_c/z_{c'})^m g_c = g_{c'}U + \ell^{R-2}Z$, with U, Z integral on c' and U upper triangular with diagonal entries the transition monomials of the root lines. Here $R = 4$, so a uniformiser of O divides ℓ^{R-2}. Modulo it, the matrices g_c glue on each special-fibre component, after a residue-field extension, to a frame of the trivial rank-n bundle adapted to the second flag. The trivial bundle in the Segre-chart construction has scaling scalar 1, so no line twist occurs. The numerical Segre formula makes its positive Newton power sums vanish. The all-degree comparison uses the lattice family's Frobenius relations.

Lean: GlobalTheory.CompanionTheory.Ar.IsFamilyOf (Development.AutomorphicSpectrum.Definitions.GlobalGeometry.IsPreFamilyOfLatticeFamilyTwo, lines 40–44)

Lean: GlobalTheory.CompanionTheory.Ar.coordLat (Development.AutomorphicSpectrum.Definitions.GlobalGeometry.IsPreFamilyOfLatticeFamilyTwo, lines 36–38)

Lean: GlobalTheory.CompanionTheory.Ar.Num.HodgeModel.model_on_hodge_of_family_pos (Development.AutomorphicSpectrum.Proofs.GeometryProofs.ModelOnHodgeFromFamily, lines 37–171)

Lean: GlobalTheory.CompanionTheory.Ar.Num.trivFlagOnHodge_of_fargues (Development.AutomorphicSpectrum.Proofs.GeometryProofs.TrivFlagOnHodgeOfFargues, lines 109–150)

Lean: GlobalTheory.CompanionTheory.Ar.Num.linked_two_levels (Development.AutomorphicSpectrum.Proofs.GeometryProofs.HodgeCompRootModelPrimes, lines 17–36)

Lean: GroundTheory.Heights₂.Heights.NumbersDVR.pairNum (GroundTheory.HeightsAndTheta.Heights.DiscreteValuationNumbers.BundleObjects, lines 124–126)

Lean: GroundTheory.Heights₂.Heights.NumbersDVR.psumExpansion (GroundTheory.HeightsAndTheta.Heights.DiscreteValuationNumbers.StatementsBundles, lines 156–157)

Lean: GlobalTheory.CompanionTheory.Ar.Num.LocPair.ck (Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.BundleChernComparison, lines 55–56)

Claim 13. Fix $\ell \in S_{N,M}$, $\tau \in \Psi$, $1 \le k \le n$, and a homogeneous polynomial Q of degree $d + 1 - k$. The numerical evaluation of the comparison polynomial is

$$D(C_{\ell,k,\tau}Q) = k\left(\sum_{h\ge 0} \ell^{-k(h+1)} D(P_{\ell,\tau,h+1,k-1}Q) - \frac{D(p^{<r}_{\tau,k-1}Q)}{\ell - 1}\right).$$

Apply $H \mapsto D(HQ)$ to Claim 10. The signed label set is finite, and the orbit values are bounded, so absolute convergence permits interchanging the finite sum and series. The subtracted block is $B_{\mathrm{inl}\,\tau,k-1} = p^{<r}_{\tau,k-1}$, and the term at h is $B_{\varphi_\ell^{h+1}(\mathrm{inl}\,\tau),k-1} = P_{\ell,\tau,h+1,k-1}$.

Lean: GlobalTheory.CompanionTheory.Ar.Num.log_mul_degFl_bracket_mul (Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.LogMulDegFlBracketMul, lines 34–49)

Lean: GlobalTheory.CompanionTheory.Ar.Num.co_series_three_step (Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.CoSeriesThreeStep, lines 9–62)

Here the iteration weights can be written as a finite sum. Let $m_{\ell,\tau} \ge 1$ be the length of the orbit of $\mathrm{inl}\,\tau$ under φ_ℓ. The terms $D(P_{\ell,\tau,h+1,k-1}Q)$ are periodic with this period and are bounded, so the series is absolutely convergent. Writing $h + 1 = tm_{\ell,\tau} + a$, with $t \ge 0$ and $1 \le a \le m_{\ell,\tau}$, gives

$$\sum_{h\ge 0} \ell^{-k(h+1)} D(P_{\ell,\tau,h+1,k-1}Q) = \frac{\displaystyle\sum_{a=1}^{m_{\ell,\tau}} \ell^{k(m_{\ell,\tau}-a)} D(P_{\ell,\tau,a,k-1}Q)}{\ell^{km_{\ell,\tau}} - 1}.$$

The term $a = 1$ uses the first Frobenius iterate. If $\varphi_\ell^a(\mathrm{inl}\,\tau) = \mathrm{inr}\,\sigma$, then its power sum is

$$P_{\ell,\tau,a,k-1} = (-1)^{k-1} \sum_{i\ge r} X_{\sigma,i}^{k-1},$$

by the definition of the signed block power sums. Thus the factor k, every Frobenius weight, and the right-block sign are retained in the displayed numerical evaluation. *Claim 14.* Normalisation and the limit. Let $A_t(F), A'_t(F)$ be the arithmetic intersections of a degree-$d+1$ polynomial F in $L_a(t), L'_a(t)$, respectively, divided by δ. Let $G_{\mathrm{com}}(H)$ be the geometric intersection of a degree-d polynomial H in the first root classes on the common generic fibre, also divided by δ. For $t > 0$, $1 \le k \le n$, and homogeneous Q of degree $d+1-k$,

$$A_t(p_{\tau,k}Q) - A'_t(p_{\tau,k}Q) = \sum_{\ell \in S_{N,M}} \log \ell \, G_{\mathrm{com}}(C_{\ell,k,\tau}Q).$$

For two sets of variables U_a, V_a, telescope $Q(U)-Q(V) = \sum_a (U_a - V_a)H_a(U,V)$, with H_a homogeneous of degree $d-k$ if $k \le d$; if $k = d+1$, take $H_a = 0$. Then

$$p_{\tau,k}(U)Q(U) - p_{\tau,k}(V)Q(V) = \sum_{a=(\sigma,i)} (U_a - V_a)\Bigg[\mathbf{1}_{\{\sigma=\tau\}} \sum_{b=0}^{k-1} U_a^b V_a^{k-1-b} Q(U) + p_{\tau,k}(V)H_a(U,V)\Bigg].$$

Apply the arithmetic rational-section formula and the isometries of Claim 9. The archimedean terms cancel, leaving only primes in $S_{N,M}$. The first terms are Claim 11's Newton comparisons, extended by linearity from lists of multidegrees. The terms containing $p_{\tau,k}(V)$ vanish by its trivial-flag assertion, including for complementary lists mixing the two collections.

Apply these local statements at every prime of A above ℓ and descend to the model over $\mathbf{Z}_{(\ell)}$, retaining all divisor multiplicities. In an auxiliary discrete valuation ring, write $\ell = u\varpi^e$ with u a unit. Intersecting $\mathrm{div}(\ell)$ with a degree-d polynomial gives e times its generic-fibre intersection. After descent to $\mathbf{Z}_{(\ell)}$, ℓ is a uniformiser and $e = 1$. The unnormalised local term is thus $\delta G_{\mathrm{com}}(C_{\ell,k,\tau}Q)$. Division by the same δ on both sides proves the claim.

Lean: `GlobalTheory.T.root_power_sum_mixed_expansion` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.RootPowerSumMixedExpansionStatement`, lines 8–9)

Lean: `GlobalTheory.T.isometric_family_intersection_eq` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.IsometricFamilyIntersectionEqStatement`, lines 8–9)

Lean: `GlobalTheory.T.uniformizer_divisor_cycle_number` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.UniformizerDivisorCycleNumberStatement`, lines 7–8)

Lean: `GlobalTheory.CompanionTheory.Ar.Num.power_sum_height_model_change` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.PowerSumHeightChangeModel`, lines 74–126)

The right side is a finite sum independent of t. Taking $t \downarrow 0$ gives

$$P_{N,Z}(p_{\tau,k}Q) - P_{M,Z}(p_{\tau,k}Q) = \sum_{\ell \in S_{N,M}} \log \ell \, G_{\mathrm{com}}(C_{\ell,k,\tau}Q).$$

By clause 1, $G_{\mathrm{com}} = V_Z D$. Claim 13 now gives clause 4, with the factor k, the first Frobenius iterate $h+1$, every orbit weight, and the right-block sign $(-1)^{k-1}$.

Lean: `GlobalTheory.CompanionTheory.Ar.IsFrobLabels` (`Development.AutomorphicSpectrum.Definitions.GlobalGeometry.IsLatFlagPair`, lines 32–34)

Lean: `GlobalTheory.CompanionTheory.Ar.Num.FlagChernComparison` (`Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.BundleChernComparison`, lines 120–124)

Lean: `GlobalTheory.CompanionTheory.Ar.Num.TrivFlagAt` (`Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.BundleChernComparison`, lines 92–96)

Lean: `GlobalTheory.T.good_tower_global_conditions` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.GoodTowerGlobalConditionsStatement`, lines 17–18)

Lean: `GlobalTheory.CompanionTheory.Ar.Top.Link.goodOrigin_linked` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.GoodTowerGlobalConditions`, lines 197–230)

Lean: `GlobalTheory.T.linked_tower_power_sum` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.LinkedTowerPowerSumStatement`, lines 8–9)

Lean: `GlobalTheory.CompanionTheory.Ar.Num.power_sum_term_linked_tower` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.LinkedTowerPowerSum`, lines 12–36)

8. For clause 5, the degree formula for the determinant generator of Step 2 and the Hodge–Tate discrepancy bound of Lemma 35.24(3) give, for each Z, τ, a flat Hermitian line of algebraic class $\sum_i X_{\tau,i}$ on $M(2, Z)$. Write h_{flat} for its metric. The root metrics satisfy

$$\prod_{i=0}^{n-1} h_{x_{\tau,i}(t)} = h_{\mathrm{flat}} \exp(-v_t),$$

where v_t is smooth, invariant under blockwise coordinate scaling, and has integral tending to zero against every degree-d monomial in the root curvature forms as $t \downarrow 0$. Denote its flat height by $H_{Z,\tau}$. The flat pushforward formula and clause 1 give

$$P_{2,Z}(p_{\tau,1} Q) = H_{Z,\tau} V_Z D(Q) \qquad (\deg Q = d).$$

Put $u = \log \mathrm{rd}(E)$. The flat height has a decomposition

$$H_{Z,\tau} = h_{\mathrm{rec}} + \sum_{\ell \in \mathcal{B}_{\mathscr{T}}} d_\ell, \qquad |h_{\mathrm{rec}}| \le \frac{n}{2}(3u + |L_E|),$$

with $v_\ell \ge 0$ such that

$$|d_\ell| \le n \left(\frac{1}{\ell - 1} + v_\ell \right) \log \ell, \qquad \sum_{\ell \in \mathcal{B}_{\mathscr{T}}} v_\ell \log \ell \le u.$$

For a good prime ℓ, let $a_{\ell,\tau,h}$ equal r on the left Frobenius copy and s on the right, and put

$$e_{\ell,\tau} = \sum_{h \ge 0} \ell^{-(h+1)} a_{\ell,\tau,h} - \frac{r}{\ell - 1}.$$

This series converges absolutely, $|e_{\ell,\tau}| \le n/(\ell - 1)$, and $\delta_\ell(\tau, 1, Q) = e_{\ell,\tau} \log \ell \, D(Q)$. For $N \ge 2$, define

$$h_{N,Z,\tau} = H_{Z,\tau} - \sum_{\substack{2 < \ell \le N \text{ prime} \\ \ell \notin \mathcal{B}_{\mathscr{T}}}} e_{\ell,\tau} \log \ell.$$

Clause 4 with initial cutoff 2 and $k = 1$ gives $P_{N,Z}(p_{\tau,1} Q) = h_{N,Z,\tau} V_Z D(Q)$.

Extend the local discrepancies to the good primes at most N by $d_\ell = -e_{\ell,\tau} \log \ell$ and $v_\ell = 0$. For $N \ge \max(128, \#\mathcal{B}_{\mathscr{T}})$, the elementary prime-sum bound $\sum_{\ell \le N} (\log \ell)/(\ell - 1) \le \log N + 2.2$ (from $N/\ell \le v_\ell(N!) + 1$ and $\log(N!) \le N \log N$) together with monotonicity of $\log x/(x - 1)$ gives

$$\sum_{\substack{\ell \text{ prime} \\ \ell \le N \text{ or } \ell \in \mathcal{B}_{\mathscr{T}}}} \frac{\log \ell}{\ell - 1} \le \log N + 2.2 + \log(N + 1) \le 4 \log N.$$

There are at most N bad primes above N, each contributing at most $\log(N + 1)/N$. Since $\mathrm{rd}(E) \ge 1$,

$$|h_{N,Z,\tau}| \le n \left(\frac{5}{2} u + \frac{1}{2} |L_E| + 4 \log N \right) \le 4n \big(\log(N \, \mathrm{rd}(E)) + |L_E| \big).$$

This proves clause 5 with constant 4.

Lean: `GlobalTheory.CompanionTheory.Ar.Fl.FlatDegree` (`Development.AutomorphicSpectrum.Statements.GeometryStatements.FlatDegree`, lines 12–19)

Lean: `GlobalTheory.CompanionTheory.Ar.Fl.flatDegree` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.FlatDegreeHolds`, lines 60–118)

Lean: `GlobalTheory.T.good_tower_real_extra` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.GoodTowerRealExtraStatement`, lines 16–17)

Lean: `GlobalTheory.CompanionTheory.Ar.Flat.RootFlat` (`Development.AutomorphicSpectrum.Definitions.GlobalGeometry.NumberField`, lines 12–28)

Lean: `GlobalTheory.CompanionTheory.Ar.Flat.NearRecord` (`Development.AutomorphicSpectrum.Definitions.GlobalGeometry.NumberField`, lines 34–37)

Lean: `GlobalTheory.T.PropFTop_holds` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.PropFTopHoldsStatement`, lines 7–8)

Lean: `GlobalTheory.CompanionTheory.Ar.Flat.propFTop` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.PropFTopHolds`, lines 25–97)

9. For the base construction in clause 6, assume $s \le r$ and fix $N \ge \max(128, 4fn^2, \#\mathcal{B}_{\mathscr{T}})$ and Z. Let

$$S_N^{\mathrm{good}} = \{\ell : \ell \text{ prime},\ \ell \le N, \ell \notin \mathcal{B}_{\mathscr{T}}\}.$$

The generic point of the cutoff-N Hodge model agrees with that of the fixed component model. At this point the base construction chooses the left labels for the rank-r block and the right labels for the rank-s block. It gives a horizontal base model of arithmetic dimension $e + 1$, where $e = frs$, with smooth generic fibre, normal-crossing boundary charts, and an auxiliary positive Hermitian line A whose normalised height on every horizontal arithmetic curve is at least $-K_A$, for a real constant K_A.

The generic rank-n bundle at τ is the sum of the two labelled Lie eigenspaces, with distinguished rank-r subspace the left eigenspace. The tensor product over τ of these quotient determinants is the line of class μ. At every $\ell \in S_N^{\mathrm{good}}$, the full ℓ^R-torsion with $R \ge 4$ gives Hodge–Tate frames. The determinant of the change from an integral small-block frame to a Hodge–Tate frame is, up to a local unit, the rational section of the prescribed vertical divisor. These give the finite-place determinant corrections on the base model.

Let $\mathcal{X}_Z$ be a compact complex model of the base component, $\mathcal{C}_Z$ its multihomogeneous cone, and $\pi : \mathcal{C}_Z \to \mathcal{X}_Z$ the projection. Write $\mathcal{C}_Z^\circ$ for the cone over the interior and ϕ_μ for the Hodge determinant line's logarithmic metric coefficient there.

Claim 15. The component construction supplies a nonnegative function $U : \mathcal{C}_Z \to \mathbf{R}$, invariant under conjugation and nonzero rescaling in each homogeneous coordinate block. Put $w = \phi_\mu + U/2$. There are a smooth reference metric h_{ref}, a bounded function $\varphi : \mathcal{X}_Z \to \mathbf{R}$, and finitely many compatible holomorphic charts χ_j with smooth reference potentials q_j, such that

$$\varphi(\pi z) = w(z) - \log h_{\mathrm{ref}}(z) \quad (z \in \mathcal{C}_Z^\circ), \qquad q_j + \varphi \circ \chi_j^{-1} \text{ is plurisubharmonic.}$$

Each q_j, on the complex model, differs from a local reference-metric potential by the real part of a holomorphic function. The original Hodge metric has the corresponding plurisubharmonic local potentials with logarithmic-logarithmic growth at the normal-crossing boundary.

Lean: `GlobalTheory.CompanionTheory.Ar.Cusp.WeightEx.exists_cuspWeight` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.IntersectionSystemWeightLeft`, lines 98–148)

Lean: `GlobalTheory.CompanionTheory.Relay.cusp_intersection_positive` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.CuspIntersectionPositive`, lines 268–279)

Lean: `GlobalTheory.CompanionTheory.Ar.Cusp.CorrBdd.corr_bdd` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.CuspIntersectionLimit`, lines 220–227)

Lean: `GlobalTheory.T.intersection_system_weight_left` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.IntersectionSystemWeightLeftStatement`, lines 12–13)

Lean: `GlobalTheory.T.cusp_intersection_positive` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.CuspIntersectionPositiveStatement`, lines 14–15)

Lean: `GroundTheory.Heights₂.Heights.Regular.BoundedPshData` (`GroundTheory.HeightsAndTheta.Heights.Regularisation.Statements`, lines 121–125)

Claim 16. Both metrics have compatible regularisations. In each finite chart system take open sets V_j with closures inside the chart domains and smaller open sets covering $\mathcal{X}_Z$. For the bounded relative function, its local regularisations satisfy, for sufficiently small $t > 0$,

$$|\varphi_{j,t}(x) - \varphi_{k,t}(x)| \le e_b(t) \quad (x \in V_j \cap V_k), \qquad e_b(t) \ge t, \qquad e_b(t) \longrightarrow 0.$$

For the original Hodge metric, the truncated local regularisations satisfy

$$K(t) \longrightarrow +\infty, \qquad K(t) \ge C_{\log} \log\log(1/t), \qquad e_h(t) \longrightarrow 0,$$

$$|\psi_{j,K(t),t}(x) - \psi_{k,K(t),t}(x)| \le e_h(t) \quad (x \in V_j \cap V_k)$$

for sufficiently small $t > 0$, where $C_{\log}$ is the logarithmic-logarithmic growth constant. For every cutoff at least $4f + 2$ and every $1 \le a \le b$, the resulting Hermitian lines are conjugation-invariant for sufficiently small positive parameters.

Lean: GroundTheory.Heights₂.Heights.LowerBoundFree.BoundedRegularisationSetting.IsTied (GroundTheory.HeightsAndTheta.Heights.HodgeLowerBound.FrameDefinitions.CpRegPhRegOfNonneg, lines 26–29)

Lean: GroundTheory.Heights₂.Heights.LowerBoundFree.LogLogRegularisationSetting.IsTied (GroundTheory.HeightsAndTheta.Heights.HodgeLowerBound.FrameDefinitions.CpRegPhRegOfNonneg, lines 19–24)

Lean: GlobalTheory.CompanionTheory.Relay.cusp_weight_regularisations_tied (Development.AutomorphicSpectrum.Proofs.GeometryProofs.CuspWeightRegularisationsTied, lines 27–44)

Let $w_{t'}$ be the globally regularised relative weight, v_ℓ the vertical determinant factor at $\ell \in S_N^{\mathrm{good}}$, $\mathbf{m}_N$ the determinant multidegree including these corrections, and $\mathbf{a}_A$ the multidegree of A. For $1 \le a \le b$, take the smooth Hermitian representative of multidegree $b\mathbf{m}_N + a\mathbf{a}_A$, with generic class $b\mu + aA$. For small $t' > 0$, its metric coefficient is

$$\left(h_{\mathrm{ref}}(z) \exp(w_{t'}(\pi z)) \prod_{\ell \in S_N^{\mathrm{good}}} v_\ell(z) \right)^b h_A(z)^a.$$

Denote it by $\overline{M}_{N,a,b,t'}$. Its ordinary and denominator-normalised curve heights are

$$h(\overline{M}_{N,a,b,t'}; W) = \frac{II(W; \overline{M}_{N,a,b,t'})}{\deg_{\mathbf{C}} W}, \qquad h_{\mathrm{reg}}(W) = \frac{II(W; \overline{M}_{N,a,b,t'})}{b \deg_{\mathbf{C}} W}.$$

Thus $h = b h_{\mathrm{reg}}$. The regularised relative weight dominates φ for sufficiently small $t' > 0$; on the interior,

$$\phi_\mu(z) + U(z)/2 \le \log h_{\mathrm{ref}}(z) + w_{t'}(\pi z).$$

Lean: GroundTheory.Heights₂.Heights.LowerBoundFree.MUc (GroundTheory.HeightsAndTheta.Heights.HodgeLowerBound.FrameDefinitions.CpRegPhRegOfNonneg, lines 77–81)

Lean: GlobalTheory.T.regularized_weight_dominates_eventually (Development.AutomorphicSpectrum.Proofs.MetricProofs.RegularizedWeightDominatesEventuallyStatement, lines 12–13)

For clause 6(i) of Definition 35.27, the adapted determinant frame gives the labelled fibre-degree upper bound by comparison of the projected period wedges, while the flat degree of the determinant generator of Step 2 and the finite-place different corrections give the conjugate eigenlattice degree difference bound $|x - y| \le c_{\mathrm{cm}} f n \lambda_N$. For clause 6(ii), the good cover supplies the uniformisation, and the function U of Claim 15 on the interior is the cusp weight of its period lattice. The smaller signature index is $s \ge 1$. If λ_1 is the first minimum of this lattice, then $0 \le U \le fs(2\log^+(1/\lambda_1) + \log(2fs\,\mathrm{rd}(E)))$.

Lean: GlobalTheory.CompanionTheory.Ar.Cusp.cuspW_le (Development.AutomorphicSpectrum.Proofs.MetricProofs.FirstMinimumAndCuspWeightBounds, lines 298–343)

Claim 17. The uniform labelled fibre-degree estimate supplies a constant $C_h \ge 0$, chosen before the host, model, component and cutoff, with the following property for the tied corrected-metric regularisation just constructed. For every cutoff $N \ge 4fn^2$ and every pair of integers $1 \le a \le b$, there is $t_0 > 0$ such that every horizontal arithmetic curve W on the base model satisfies

$$h_{\mathrm{reg}}(W) \ge -C_h fs\lambda_N - \frac{a}{b} K_A \qquad (0 < t' \le t_0).$$

Here K_A belongs to the chosen base model and auxiliary line and satisfies

$$II(W; A) \ge -K_A \deg_{\mathbf{C}} W$$

for every such W; replace it by $\max(K_A, 0)$ if necessary. The threshold may depend on the model, cutoff and pair (a, b). The estimate applies to both interior and boundary curves: the regularised corrected weight dominates the original corrected weight, and the arithmetic height decomposes as the negative labelled fibre degree plus the corrected archimedean contribution and (a/b) times the auxiliary height.

Lean: GlobalTheory.T.phiDeg_fibLat_le_of_isOFamily_of_isAdapted (Development.AutomorphicSpectrum.Proofs.GeometryProofs.PhiDegFibLatLeOfIsOFamilyOfIsAdaptedStatement, lines 14–15)

Lean: GlobalTheory.CompanionTheory.Ar.Cusp.TopHt.htU_of_given (Development.AutomorphicSpectrum.Proofs.GeometryProofs.CuspHeightUpper, lines 120–223)

Lean: GlobalTheory.CompanionTheory.Ar.Cusp.HtU (Development.AutomorphicSpectrum.Statements.GeometryStatements.NineGeoU, lines 32–35)

Lean: GroundTheory.Heights₂.Heights.LowerBoundFree.ComponentData (GroundTheory.HeightsAndTheta.Heights.HodgeLowerBound.CoordinateIdeals, lines 166–225)

Claim 18. The bad-prime set is exactly the primes dividing $2\,\mathrm{disc}(E)M'$, has cardinality at most $4fn^2$, and the finite determinant-correction set is exactly the primes at most N outside it. The global component origin has the required Hodge–Tate local-freeness properties there, $s \le r$, and $N \ge 4f+2$. For each $1 \le a \le b$, there is $t_{\mathrm{pos}} > 0$ such that $\overline{M}_{N,a,b,t'}$ is semipositive along the defining ideal, nef on every finite fibre, and ample on the generic fibre for every $0 < t' \le t_{\mathrm{pos}}$. A further positive threshold gives conjugation invariance. There are absolute C_h, C_u, chosen before the host, model, component and cutoff, giving the curve-height assertion of Claim 17 and

$$\operatorname{mean} U \le C_u f s \lambda_N \qquad (1 \le s,\ 3s+1 \le n,\ N \ge 4fn^2).$$

Here the mean is the integral of U against the top Hodge Chern measure divided by the geometric degree of the Hodge determinant line.

Lean: GlobalTheory.CompanionTheory.Ar.Cusp.admitted (Development.AutomorphicSpectrum.Statements.GeometryStatements.NineGeoU, lines 53–57)

Lean: GlobalTheory.CompanionTheory.Ar.Cusp.PosU (Development.AutomorphicSpectrum.Statements.GeometryStatements.NineGeoU, lines 23–26)

Lean: GlobalTheory.CompanionTheory.Ar.Cusp.ConjU (Development.AutomorphicSpectrum.Statements.GeometryStatements.NineGeoU, lines 28–30)

Lean: GlobalTheory.CompanionTheory.Ar.Cusp.MeanU (Development.AutomorphicSpectrum.Statements.GeometryStatements.NineGeoU, lines 42–43)

Lean: GlobalTheory.CompanionTheory.Ar.Cusp.TopNine.nineGeoU_of_given (Development.AutomorphicSpectrum.Proofs.GeometryProofs.ExistsGeometricHeightConstants, lines 17–25)

Use the labelled fibre-degree comparison on the adapted component data of clause 6(i), with the flat determinant degree of Step 2. Together with domination by the tied corrected metric and the auxiliary height bound, this gives the all-curve constant of Claim 17. The cusp-volume comparison gives the mean constant of Claim 18. Lemma 35.29 uses this base-model height bound. The positivity assertion of Claim 18 applies to the compatible bounded regularisation on these admissible component data, using the tower intersection comparison already established in Step 6.

Lean: GlobalTheory.CompanionTheory.Relay.intersection_degree_lower (Development.AutomorphicSpectrum.Proofs.GeometryProofs.IntersectionDegreeLower, lines 22–25)

Lean: GlobalTheory.T.cusp_volume_sum (Development.AutomorphicSpectrum.Proofs.GeometryProofs.CuspVolumeSumStatement, lines 11–12)

To transfer it to the component setting, take cusp data with this same U, retain the same complex model and regularisations, and identify the corrected weight with $\phi_\mu + U/2$. Positivity and conjugation invariance are preserved; the all-curve bound restricts to the interior and boundary cases; the mean is unchanged. Take the minimum of the thresholds in Claims 16–18 before applying the intersection inequality.

Lean: GlobalTheory.CompanionTheory.Relay.corrected_weight_regularization (Development.AutomorphicSpectrum.Proofs.MetricProofs.CorrectedWeightRegularization, lines 482–503)

Lean: GroundTheory.Heights₂.Heights.LowerBoundFree.CuspBundlePos (GroundTheory.HeightsAndTheta.Heights.HodgeLowerBound.FlagIndices, lines 62–66)

Let $g > 0$ be the generic degree of the unperturbed determinant line and A_N its arithmetic intersection value from the tied original-metric regularisation. The comparison of generic points and flag fibre intersections gives

$$\frac{A_N}{(e+1)g} = \frac{P_{N,Z}(F_*)}{(e+1)V_Z D(\mu^e J)},$$

with J, F_* as in Definition 35.27. Write $I_{a,b,t'}$, $D_{a,b,t'}$ for the arithmetic top intersection and generic top degree of the cleared-denominator line. The Stokes comparison states that, for every $\eta \in \mathbf{R}$, if for every $1 \le a \le b$ there is $t_0 > 0$ with

$$(e+1)\left(\eta - \frac{a}{b}K_A\right)\frac{D_{a,b,t'}}{b^e} \le \frac{I_{a,b,t'}}{b^{e+1}} \qquad (0 < t' \le t_0),$$

then

$$\eta \le \frac{P_{N,Z}(F_*)}{(e+1)V_Z D(\mu^e J)} + \frac{1}{2}\operatorname{mean} U.$$

The inequalities $N \ge 4fn^2$, $f \ge 1$, $n \ge 2$ give $N \ge 4f+2$, as needed for positivity and Stokes. Lemma 35.23(1), applied to the displayed curve-height bound, permits $\eta = -C_h f s \lambda_N$. Substitution in the displayed Stokes inequality gives

$$\frac{P_{N,Z}(F_*)}{(e+1)V_Z D(\mu^e J)} \ge -(C_h + C_u/2) f s \lambda_N.$$

Lean: GlobalTheory.T.base_setting_of_good_tower (Development.AutomorphicSpectrum.Proofs.GeometryProofs.BaseSettingOfGoodTowerStatement, lines 24–25)

Lean: GlobalTheory.CompanionTheory.Ar.Cusp.TopBase.base_setting_of_good_origin (Development.AutomorphicSpectrum.Proofs.GeometryProofs.BaseSettingOfGoodTower, lines 45–106)

Lean: GlobalTheory.CompanionTheory.Ar.CptOrigin (Development.AutomorphicSpectrum.Definitions.GlobalGeometry.IsPeriodAtRestrict, lines 20–54)

Lean: GroundTheory.Heights₂.Heights.LowerBoundFree.HodgeData (GroundTheory.HeightsAndTheta.Heights.HodgeLowerBound.FrameDefinitions.IsFrameOverLine, lines 60–74)

Lean: GroundTheory.Heights₂.Heights.LowerBoundFree.MUc (GroundTheory.HeightsAndTheta.Heights.HodgeLowerBound.FrameDefinitions.CpRegPhRegOfNonneg, lines 77–81)

Lean: GroundTheory.Heights₂.Heights.LowerBoundFree.htMUc (GroundTheory.HeightsAndTheta.Heights.HodgeLowerBound.FrameDefinitions.CpRegPhRegOfNonneg, lines 89–90)

Lean: GroundTheory.Heights₂.Heights.LowerBoundFree.CuspBundlePos (GroundTheory.HeightsAndTheta.Heights.HodgeLowerBound.FlagIndices, lines 62–66)

Lean: GroundTheory.Heights₂.Heights.LowerBoundFree.HtInterior (GroundTheory.HeightsAndTheta.Heights.HodgeLowerBound.FlagIndices, lines 68–71)

Lean: GroundTheory.Heights₂.Heights.LowerBoundFree.HtBoundary (GroundTheory.HeightsAndTheta.Heights.HodgeLowerBound.FlagIndices, lines 73–76)

Lean: GroundTheory.Heights₂.Heights.LowerBoundFree.CuspMean (GroundTheory.HeightsAndTheta.Heights.HodgeLowerBound.FlagIndices, lines 78–80)

Lean: GlobalTheory.CompanionTheory.Ar.Top.StokesAt (Development.AutomorphicSpectrum.Definitions.GlobalGeometry.HasBaseDegreeSettingStrict, lines 9–14)

Lean: GlobalTheory.CompanionTheory.Ar.Top.ppc_lower_of_setting (Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.SlopeLowerBaseSettings, lines 76–108)

The assembly uses the same tower throughout, with base slope constant $\max(C_h + C_u/2, 0)$, flat-height constant 4, and final tower constant $2\max(C_h + C_u/2, 0) + 3 \cdot 4 + 5$. The construction of each root model $M(N, Z)$ over the normalised global projective model of Y gives a degree-d_0 field extension $\Omega(Z_{\mathrm{fl}}) \hookrightarrow \Omega(M(N, Z))$. Let V be a valuation ring of $\Omega(M(N, Z))$ centred on a characteristic-zero flag chart. The coefficient field lies in V, so the patched lattices there are the original flag lattices. On a model chart containing the centre of V, the model root generator and the universal generator span the same rank-one flag quotient over V. Their ratio lies in $V^\times$. The construction also supplies an effective boundary section ε of degree $q_0\nu$, finite-fibre nefness and cone-semipositivity of the smoothed rational combination bundles for small $s > 0$ once $N = \lfloor B(v, g) \rfloor$, and convergence of normalized top geometric and arithmetic intersections as $s \to 0$. The local factors for clearing the trace denominators use $h_E(\ell) + \mathbf{1}_{\ell=2}$ from (35.8). This supplies the moduli-origin coupling, proportionality, equal component volumes, averaged intersection estimate and mean-slope bound used below.

Lean: GlobalTheory.CompanionTheory.Ar.Cusp.volumeTheoremX_of_inputs (Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.VolumeTheoremOfInputs, lines 72–121)

Lean: GlobalTheory.T.given_of_all_inputs (Development.AutomorphicSpectrum.Proofs.GeometryProofs.GivenOfAllInputsStatement, lines 27–28)
Lean: GlobalTheory.T.flat_zero_from_resolution_and_gaga (Development.AutomorphicSpectrum.Proofs.GeometryProofs.FlatZeroFromResolutionAndGagaStatement, lines 8–9)
Lean: GlobalTheory.T.flat_first_from_resolution_and_gaga (Development.AutomorphicSpectrum.Proofs.GeometryProofs.FlatFirstFromResolutionAndGagaStatement, lines 8–9)
Lean: GlobalTheory.CompanionTheory.Ar.Flat.FlatLineGenerator (Development.AutomorphicSpectrum.Statements.GeometryStatements.FlatLineGenerator, lines 18–19)

Lean: GlobalTheory.CompanionTheory.Ar.Flat.FlatOnModel (Development.AutomorphicSpectrum.Definitions.GlobalGeometry.IsFlatAnIffIsConstMul, lines 164–166)

The complete proof is contained in `GlobalTheory.CompanionTheory.Ar.Link.good_tower_and_volume_final`. □

Proof of 35.26

Proof. By Lemmas 35.23 and 35.24, the first eight premises of Theorem 35.11 (resolution through nef Riemann–Roch) furnish the arithmetic intersection system II, the horizontal curve height bounds, unique extension of semiabelian homomorphisms over normal bases, Fargues–Hodge–Tate exactness with sharp cokernel annihilation, clopen GAGA section algebraisation, Hopf-order duality, finite locally free quotients of polarised abelian schemes over affine bases and Zarhin's polarised retraction. Together with Theorem 35.33 and nef Riemann–Roch (A.425), these are the inputs of Proposition 35.25, which supplies the good root tower over E in the sense of Definition 35.27.

Lean: GlobalTheory.CompanionTheory.Ar.Giv.hsLinks_of_use (Development.WeilBound.CoreStatements.AutomorphicKernels.MoriwakiBigness.HilbertSamuelApplications.HsLinksOfUse, lines 9–19)
Lean: GlobalTheory.CompanionTheory.Ar.Link.given_of_use (Development.WeilBound.CoreStatements.AutomorphicKernels.MoriwakiBigness.HilbertSamuelApplications.GivenOfUse, lines 11–17)
Lean: GlobalTheory.CompanionTheory.Ar.Link.good_tower_volume_theorem_bis (Development.WeilBound.CoreStatements.AutomorphicKernels.MoriwakiBigness.HilbertSamuelApplications.GoodTowerVolumeTheoremBis, lines 9–19)

At each rational prime ℓ use the prime-to-ℓ part of the base level, which is at least three because the base level g_0g_1 is a product of two distinct odd primes exceeding three, together with the auxiliary prime different from ℓ. The resulting localisation moduli generate the unit ideal of $\mathbf{Z}$. Choose local ample PEL model systems over this open cover, pass to a common dominating projective model and normalise in the smooth generic cover over F. By Lemma 35.24(2), the local projections composed with the semiabelian comparisons agree with the prescribed prime-to-ℓ level maps and their level sections across the normal flat projective global model.

Lean: GlobalTheory.Models.exists_localSystems_ample (Development.AbelianVarietyTheory.Proofs.ModuliProofs.ExistsLocalModelSystemsAmpleFrames, lines 30–47)
Lean: GlobalTheory.GlobalModel.exists_global_model (Development.AbelianVarietyTheory.Proofs.ModuliProofs.ExistsGlobalModelOfLocalModels, lines 155–231)
Lean: GlobalTheory.CompanionTheory.Ar.Rec.exists_isGlobal (Development.AutomorphicSpectrum.Proofs.GeometryProofs.ExistsIsGlobal, lines 33–38)

The good cover datum identifies the algebraic components of Y with the component set Comp of the associated good root tower $\mathscr{T}$ of split index $p_{\mathrm{sig}} = k+j-1$ and rank $n = p_{\mathrm{sig}}+q_{\mathrm{sig}}$ in Definition 35.27. From $m \geq 2$, $m^2 \leq k$ and $1 \leq j < m$ one has $q_{\mathrm{sig}} = m-j \geq 1$, $2m \leq m^2 \leq k$, $3q_{\mathrm{sig}}+1 \leq n$ and $n \geq 4$, so $n^2 \geq 16$, while $|\Psi| = 2n_{\mathrm{ord}} \geq 2$ for the host order parameter $n_{\mathrm{ord}} \geq 1$. Every prime in the bad set $\mathcal{B}_{\mathscr{T}}$ lies in the set S of prime factors of $2|\operatorname{disc} E|M'$. Splitting S at n_{ord}, the host discriminant bound and $\#\{\ell \mid M'\} \leq 15$ leave at most $15+3 = 18$ primes of S above n_{ord}, whereas at most $n_{\mathrm{ord}}+1$ primes lie in $[0, n_{\mathrm{ord}}]$. Since $19 + n_{\mathrm{ord}} \leq 128 n_{\mathrm{ord}} = 4|\Psi| \cdot 16 \leq 4|\Psi|n^2$, the cutoff $B = B(v,g) \geq 4|\Psi|n^2$ satisfies

$$B \geq 128, \qquad \#\mathcal{B}_{\mathscr{T}} \leq 4|\Psi|n^2 \leq N = \lfloor B \rfloor. \tag{A.443}$$

Let $a_Z(v;N)$ be the component model slopes of Definition 35.27.

Lean: GlobalTheory.CompanionTheory.Ar.Plug.vs_tower (Development.AutomorphicSpectrum.Proofs.GeometryProofs.FullIntersectionGoodLevel, lines 18–50)
Lean: Formalism.BasicNotions.Host.card_bad_gt_le (Development.AutomorphicArithmetic.PrimeDiscriminants.BadPrimes, lines 74–78)

The function field $\Omega(Z_{\mathrm{fl}}) = \Omega(Z)(x_{\tau,i,j})$ of the flag bundle adjoins to $\Omega(Z)$ one indeterminate for each embedding τ of E and each pair $i < j$ of indices of the τ-eigenspace, the entries above the diagonal of the generic unipotent change of eigenframe over the generic point of Z. A cusp section on Z is read in $\Omega(Z_{\mathrm{fl}})$ through its value at this generic frame. Let $d_0 = [\Omega(M(N,Z)) : \Omega(Z_{\mathrm{fl}})]$ be the degree of the finite function-field extension supplied by the tower, and separate the volume comparison into two stages. First, for the model-section volume lower bound on the tower root model $M(N,Z)$ of Definition 35.27 and Proposition 35.25, denote locally by ε the effective boundary section of degree $q_0\nu$ ($q_0 \geq 1$, vanishing along the boundary divisor), by n_0 and n_v the base and weight multidegrees of the model family, by $s_0 > 0$ and $C \geq 0$ the smoothing threshold and smoothing loss constant, by $N_0 \geq 1$ an integer with $1/(N_0+1) < s_0$, and by $S_{\mathrm{mod}}(\tau v, e^{\tau t'})$ the space of cuspidal small model sections on $M(N,Z)$ of weight τv and logarithmic

sup norm at most $\tau t'$. For each $n \geq 0$, set $s_n = 1/(n + N_0 + 1) \to 0$, $q_n = 8q_0(n + N_0 + 1)$ (divisible by 8, with $q_n s_n = 8q_0$), combination multidegree $m_n = q_n n_v - 8n_0$, smoothed Hermitian combination line bundle M_n of multidegree α_n (cone-semipositive and nef on finite fibres, with positive top geometric degree), and boundary multiplier $\theta_n = \varepsilon^8$. For every $\tau \geq 1$ and $t' \in \mathbf{R}$, multiplying by θ_n^τ injects every small integral section σ of $M_n^{\otimes\tau}$ of logarithmic sup norm at most $\tau q_n(t' - Cs_n)$ into $S_{\mathrm{mod}}(\tau q_n v, e^{\tau q_n t'})$ (inheriting cuspidality from boundary vanishing of ε). Applying Theorem 35.33 and nef Riemann–Roch (A.425) to each M_n (with degree pushforward) and passing to $n \to \infty$ ($s_n \to 0$, where the normalized top geometric and arithmetic intersections converge to $\deg_Z(v)$ and $(d+1)\deg_Z(v)a_Z(v;N)$) bounds the asymptotic volume of $S_{\mathrm{mod}}(\tau v, e^{\tau t'})$ (normalised by dividing by $\delta = [F' : \mathbf{Q}]d_0$ along τ divisible by 8) from below by $\deg_Z(v)(a_Z(v;N)+t')$. Second, to transfer this lower bound from S_{mod} to small cusp sections on Z, fix one index n_* for which M_{n_*} is nef and big. By Kodaira's lemma (writing a large multiple of the big bundle M_{n_*} as ample plus effective), there exist $k \geq 1$ and a nonzero integral section $e \in H^0(kM_{n_*} - \mathbf{1})$ of shifted multidegree $k\alpha_{n_*} - \mathbf{1}$. Choose multihomogeneous integral polynomials $w_1, \ldots, w_{d_0}$ of uniform multidegree $(j_0, \ldots, j_0)$ in the model coordinates y whose values $w_a(y)$ are linearly independent over $\Omega(Z_{\mathrm{fl}})$, hence a basis of the degree-d_0 extension $\Omega(M(N,Z))/\Omega(Z_{\mathrm{fl}})$; they exist because the coordinate ratios generate this finite extension. Every product $\sigma_a = w_a(y)e^{j_0}$ has exact multidegree $(j_0 k)\alpha_{n_*}$ and lies in the integral chart lattice of $M_{n_*}^{\otimes(j_0 k)}$, so setting $f_a = \sigma_a \theta_{n_*}^{j_0 k}$ produces one fixed cuspidal small generic trace basis $f_1, \ldots, f_{d_0}$ of weight $k_0 v$ ($k_0 = j_0 k q_{n_*}$, divisible by 8) with nondegenerate trace pairing. Now fix $\eta > 0$ and $t \in \mathbf{R}$, and put $t' = t - 2c_2(B)\|v\|_1 - \eta$. Here Tr is the trace of $\Omega(M(N,Z))$ over $\Omega(Z_{\mathrm{fl}})$, applied after division by the nonzero gauge element of weight $(\tau + k_0)v$ that turns the model reading of a section into its generic-frame reading. For every degree τ divisible by 8 and every $s \in S_{\mathrm{mod}}(\tau v, e^{\tau t'})$, the map $s \mapsto (M_\tau \operatorname{Tr}(sf_a))_{a=1}^{d_0}$ is injective in total weight $(\tau + k_0)v$ (divisible by 8) for every positive integer M_τ, since the trace form of a finite extension in characteristic zero is nondegenerate. Compare the tower root model at cutoff N (using Hodge–Tate frames at good primes $\ell \leq N$ and integral model valuations elsewhere) with the patched valuation family of $\nu_Z(v;B)$ (using Hodge–Tate valuations above S_B and record valuations outside S_B). For $\lambda \mid \ell \in S_B$, put $a = v_\lambda(\ell)(\tau + k_0)(h_E(\ell) + \mathbf{1}_{\ell=2})\|v\|_1$. The model trace has record valuation at least $-a$ at every record frame. For each Hodge–Tate valuation there is a record valuation bounded above by it plus a on sections of this weight, by the frame lattice comparison using Lemma 35.24(3). The Hodge–Tate valuation of the trace is therefore at least $-2a$.

Lean: `GlobalTheory.CompanionTheory.Ar.Plug.SlopeVol.clear_integral` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.UniformAllNormHeightBound`, lines 171–221)

Outside S_B both models share the same unramified integral frame valuations and the fixed trace basis $f_1, \ldots, f_{d_0}$ contributes a τ-independent denominator M_{tr}. Choosing $M_\tau = M_{\mathrm{tr}} \prod_{\ell \in S_B} \ell^{\lceil 2(\tau+k_0)(h_E(\ell)+\mathbf{1}_{\ell=2})\|v\|_1 \rceil}$ clears all local denominators (with $c_2(B) = \sum_{\ell \in S_B}(h_E(\ell)+\mathbf{1}_{\ell=2})\log\ell$ as in (35.8)) and satisfies $\log M_\tau \leq 2(\tau + k_0)c_2(B)\|v\|_1 + O(1)$. Bounding the archimedean norms of $f_1, \ldots, f_{d_0}$ adds another $O(1)$ constant independent of τ; both fixed $O(1)$ constants are absorbed into the slack $\tau\eta$ for all large τ divisible by 8. Each cleared coordinate is the generic-frame reading of a section of the weight-$(\tau+k_0)v$ line supported on Z. Indeed, let R be the ring of a chart of the flag bundle Z_{fl} of the generic fibre; it is smooth over the coefficient field, hence integrally closed, with fraction field $\Omega(Z_{\mathrm{fl}})$. For every valuation ring V of $\Omega(M(N,Z))$ containing the image of R, the coefficient field lies in V, so the finite-prime patching is inactive. The two frames generate the same flag quotients over V and differ by units. Thus the gauge-divided reading of sf_a, multiplied by the monomial of weight $(\tau + k_0)v$ in the chart frame units, is a V-unit times its normalized reading in a model chart over which V is centred. The integral chart lattice makes the latter reading integral over the chart ring, hence an element of V. This element is therefore integral over R; its trace over $\Omega(Z_{\mathrm{fl}})$, which is the cleared coordinate times that monomial up to the integer M_τ, is integral over R and so lies in R. These chart values glue to the required section. At each good embedding the trace on the dense set of generic frames is the sum of the d_0 gauge-divided branch readings. The frame comparison and triangle inequality bound the homogeneous norm of the cleared trace by $M_\tau d_0$ times the model norm bound for sf_a. Continuity extends the bound to all frames over Z, and the fixed $\log d_0$ is absorbed by $\tau\eta$.

Lean: `GlobalTheory.T.uniform_norm_tower_bound` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.UniformNormTowerBoundStatement`, lines 17–18)

Since v has positive gap, a section supported on Z whose homogeneous norm is bounded there is a cusp section. Each cleared coordinate $M_\tau \operatorname{Tr}(sf_a) \in \Omega(Z_{\mathrm{fl}})$ is therefore the generic-frame reading of one of the $n_{\tau+k_0}(t)$ small cusp sections on Z of weight $(\tau + k_0)v$ and logarithmic sup norm at most $(\tau + k_0)t$, yielding $|S_{\mathrm{mod}}(\tau v, e^{\tau t'})| \leq n_{\tau+k_0}(t)^{d_0}$. Taking logarithms and dividing by $\delta = [F' : \mathbf{Q}]d_0$ cancels the degree factor

d_0 while leaving the normalization factor $[F' : \mathbf{Q}]$ defining the component section volume $\mathrm{vol}_Z(t)$, and the fixed shift k_0 leaves the asymptotic leading ratio as $\tau \to \infty$ unchanged; combining with the first-stage lower bound gives $\deg_Z(v)(a_Z(v;N)+t') \le \mathrm{vol}_Z(t)$ for every $t \in \mathbf{R}$ and $\eta > 0$ (trivially when $a_Z(v;N)+t' < 0$ since $\mathrm{vol}_Z(t) \ge 0$). Dividing by $\deg_Z(v) > 0$, subtracting t, letting $\eta \downarrow 0$, and taking the infimum over $t \in \mathbf{R}$ on the right-hand side gives

$$a_Z(v;N) - 2c_2(B)\|v\|_1 \le \nu_Z(v;B). \tag{8.26d}$$

Lean: `GlobalTheory.CompanionTheory.Ar.Plug.SlopeVolM.slope_sub_cost_le_compVolSlope` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.UniformAllNormHeightBound`, lines 300–372)

For $N \in \mathbf{N}$, $N \ge 1$, the prime sum satisfies

$$\sum_{\ell \le N} \frac{\log \ell}{\ell - 1} \le \log N + 2.2. \tag{8.26f}$$

At $N = 1$ the sum is empty. For $N \ge 2$, $N/\ell \le v_\ell(N!) + 1$ gives $N \sum_{\ell \le N} (\log \ell)/\ell \le \log N! + \vartheta(N)$, where $\vartheta(t) = \sum_{\ell \le t} \log \ell \le t \log 4$ from $\binom{2m}{m} \le 4^m$ and $\log N! \le (N+1)\log N - N + 1$ by integration. In $\sum_{\ell \le N} (\log \ell)/(\ell(\ell-1))$, bound $\log \ell / \log 2$ at $\ell = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29$ by $1, 2, 3, 3, 4, 4, 5, 5, 5, 5$, and bound the tail over integers $a \ge 31$ via integration by parts and $1/(t(t+2)) \le t^{-2}$ by $\int_{29}^{\infty} \log(t+2) t^{-2}\, dt \le (\log 31 + 1)/29$. Combining with $(\log N + 1)/N \le (\log 2)/N + 1/2 \le (\log 2 + 1)/2$ for $N \ge 2$ gives (8.26f).

Lean: `ReciprocityTheory.ChartRestriction.G.H1.primeLogSum_nat` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.FirstCohomology.PrimeLogSum`, lines 270–353)

By (A.443), $S_B \setminus \{\ell \le N\}$ contains at most $\#S \le 4|\Psi| n^2 \le N$ primes, each strictly exceeding N and hence contributing at most $\log(N+1)/N$ to $\sum_{\ell \in S_B} (\log \ell)/(\ell - 1)$. The discriminant valuations in (35.8) sum to $40 \log \mathrm{rd}(E)$, and $\ell = 2$ adds $\log 2$. Thus

$$0 \le c_2(B) \le 60(\log B + \log \mathrm{rd}(E)). \tag{8.26e}$$

Set $s_Z = a_Z(v;N) - 2c_2(B)\|v\|_1$. Since $B < N + 1$, Lemma 35.29, Lemma 35.31 and Proposition 35.32 together with (8.26d) and (8.26e) give (8.26b) with the absolute constant $C_{\mathrm{deep}} = C_T + 120 = 2C_h + 141 \ge 0$, where $C_h \ge 0$ is the absolute constant of Lemma 35.29, independent of the datum, chain, layer and cover. □

Proof of 35.28

Proof. In (1), alternating L_v^d across S_n at each embedding produces the Vandermonde product divided by $\prod_{a<b}(b-a)$ with multinomial prefactor $d!$; for $a < b$ each ratio $(v_{\tau,a} - v_{\tau,b})/(b-a)$ is positive under regular dominance and nonnegative under dominance.

In (2), the quotient of $\mathbf{R}[X_{\tau,i}]$ by the symmetric power sums $\{p_{\tau,k} : 1 \le k \le n\}$ is the cohomology ring of the product of full flags of dimension d, so its degree-$(d+1)$ component is zero; taking homogeneous parts yields $F = \sum_{\tau,k} p_{\tau,k} Q_{\tau,k}$. Full symmetric polynomials of positive degree commute with alternation and yield zero under D. For $\sigma \ne \tau$, $\partial_{\sigma,i} p_{\tau,k} = 0$ and $D(p_{\tau,k}\partial_{\sigma,i} Q) = 0$. For $\sigma = \tau$, $\sum_{i<r} \partial_{\tau,i}(p_{\tau,k} Q) = k p_{\tau,k-1}^{<r} Q + p_{\tau,k} \sum_{i<r} \partial_{\tau,i} Q$, whose second term vanishes under D, while $\sum_{i=0}^{n-1} \partial_{\tau,i}(p_{\tau,k} Q) = k p_{\tau,k-1} Q + p_{\tau,k} \sum_i \partial_{\tau,i} Q$ vanishes under D for $k \ge 2$ and equals $nD(Q)$ for $k = 1$. Taking $Q_1 = (-1)^d \prod_{\sigma,i} X_{\sigma,i}^{n-1-i}$ gives $D(Q_1) = 1$, and reducing the exponent of $X_{\tau_0,r-1}$ by one gives Q_2 with $X_{\tau_0,r-1} Q_2 = Q_1$ and repeated exponents $n-r-1$ (at indices $r-1$ and r) in $X_{\tau_0,i} Q_2$ for $i < r-1$, whence $D(p_{\tau_0,1}^{<r} Q_2) = 1$.

In (3), when $k = 1$, $P_{\ell,\tau,h+1,0} = \mathrm{rk}_{\mathrm{Lie}}(\mathrm{Fr}_{\ell,h}(\tau)) = \mathrm{rk}_{\mathrm{Lie}}(\mathrm{Fr}_{\ell,h}(\tau_0))$ by clause 4 of Definition 35.27. Hence $\log \ell \sum_{h \ge 0} \ell^{-(h+1)} D(P_{\ell,\tau,h+1,0} Q) = n\kappa_\ell D(Q) = \kappa_\ell T(p_{\tau,1} Q)$ and $-\frac{\log \ell}{\ell - 1} D(p_{\tau,0}^{<r} Q) = -\frac{\log \ell}{\ell-1} r D(Q) = -\frac{\log \ell}{\ell-1} S(p_{\tau,1} Q)$ by (2). When $k \ge 2$, $T(p_{\tau,k} Q) = 0$ and $S(p_{\tau,k} Q) = k D(p_{\tau,k-1}^{<r} Q)$, giving the same formula with $\rho_\ell(\tau, k, Q)$. Since $P_{\ell,\tau,h,k-1}$ ranges over the finite set of $2f$ signed block power sums, $|D(P_{\ell,\tau,h,k-1} Q)|$ is bounded uniformly in (ℓ, h), and $\sum_{h \ge 1} \ell^{-hk} = \ell^{-k}/(1 - \ell^{-k}) \le 2\ell^{-2}$ for $k \ge 2$ and $\ell \ge 2$.

In (4), orthogonal splitting of the standard Hermitian filtration on $\mathcal{O}^n$ identifies $\bigotimes_{i=0}^{n-1} \overline{Q}_{\tau,i} \simeq (\overline{\det \mathcal{O}^n})^\vee$ isometrically with the trivial Hermitian line of constant norm 1, whose standard basis generator has empty divisor and zero log norm; the rational-section divisor recursion therefore gives $h_{\mathrm{fl}}(p_{\tau,1} Q) = 0$ for every homogeneous Q of degree d. Summation by parts rewrites L_v modulo $\mathrm{span}\{p_{\tau,1} : \tau \in \Psi\}$ as $\sum_{\tau \in \Psi} \sum_{j=1}^{n-1} (v_{\tau,j-1} - v_{\tau,j}) \sum_{i=j}^{n-1} X_{\tau,i}$, where $\sum_{i=j}^{n-1} X_{\tau,i}$ represents the pullback of $\overline{\mathcal{O}_{\mathrm{Gr}(n-j,n)}(1)}$. Since $\overline{\mathcal{O}_{\mathrm{Gr}(n-j,n)}(1)}$ is smooth, conjugation-invariant, semipositive, fibrewise nef and generated by integral Plücker

coordinates of sup norm at most 1, clearing denominators for rational dominant v, applying arithmetic Kleiman (A.422) and extending by continuity to all real dominant v yields $h_{\mathrm{fl}}(L_v^{d+1}) \geq 0$. When $v_{\tau_0,a} = v_{\tau_0,a+1}$ for some $\tau_0 \in \Psi$ and $0 \leq a < n-1$, the summand $j = a+1$ at τ_0 vanishes, so for rational v on that wall (and by continuity for all real v there) the class pulls back from the smooth projective partial flag variety over $\mathbf{Z}$ omitting $E_{\tau_0,a+1}$, which has arithmetic dimension d and complex dimension $d-1$, so both codimension-$(d+1)$ cycles and their (d,d) Green currents vanish, giving $h_{\mathrm{fl}}(L_v^{d+1}) = 0$. □

Proof of 35.29

Proof (outline). Work on the base Grassmannian model of generic dimension $e = frs$ on a component $Z \in \mathrm{Comp}$ provided by clause 6 of Definition 35.27. Since $3s+1 \leq n = r+s$, we have $s < r$, so $\min(r,s) = s$ and the smaller Hodge block at each embedding $\tau \in \Psi$ has rank s, with determinant class $\mu = \sum_{\tau,i\geq r} X_{\tau,i}$. Regularise the singular Hodge metric on μ by adding $\frac{1}{2}U$, where at a uniformised frame of positive-majorant first minimum λ_1 the cusp weight $U \geq 0$ is the boundary logarithmic degeneration weight satisfying (A.448) below (compensating the $-\log\lambda_1$ singularity of the adapted Hodge determinant metric near the toroidal boundary so that the regularised metric extends to a continuous plurisubharmonic metric on the compactification), and perturb by (a/b) times the auxiliary ample line for integers $1 \leq a \leq b$. At finite places $\ell \leq N$ or $\ell \in \mathcal{B}_{\mathscr{T}}$, use the adapted determinant metric of the base component model. The labelled Lie lattices of that model are locally free of their prescribed ranks. After locally trivialising the determinant line, its chosen section is a unit times the tensor product of the quotient wedges of adapted frames. Apply Lemma 35.24(3) to bound the Hodge–Tate discrepancy.

Lean: `GlobalTheory.CompanionTheory.Ar.HodgeCpt` (`Development.AutomorphicSpectrum.Definitions.GlobalGeometry.IsHodgeLine`, lines 33–74)

Lean: `GroundTheory.Heights₂.Heights.LowerBoundFree.HodgeData` (`GroundTheory.HeightsAndTheta.Heights.HodgeLowerBound.FrameDefinitions.IsFrameOverLine`, lines 60–74)

At a closed horizontal point choose independent period vectors spanning a finite-index $\mathcal{O}_E$-submodule, with $t \leq s$ projected vectors in the rank-s eigenspaces, and complete their projections to bases there. Index and determinant comparison bound the labelled fibre degree above by

$$fsX - ft\log \mathrm{rd}(E) + x + \tfrac{1}{2}ft\log(4\pi), \qquad X = 2\sum_{\ell\leq N \text{ or } \ell\in\mathcal{B}_{\mathscr{T}}} \frac{\log\ell}{\ell-1} + 2\log\mathrm{rd}(E). \tag{A.444}$$

Choose a number field L carrying the horizontal point and the split Lie data used in the component height comparison. Let $\overline{V}$ be the integral Hermitian Lie lattice of its principally polarised abelian part, with the metric used in that comparison, and put

$$g = \mathrm{rank}_{\mathcal{O}_L} V \leq fn, \qquad h = -\widehat{\deg}_{\mathrm{n}}(\overline{V}),$$

where $\widehat{\deg}_{\mathrm{n}}$ is the Arakelov degree divided by $[L:\mathbf{Q}]$. Write $\widehat{\mu}_{\max}(\overline{V})$ for the extended-real supremum of the normalised degrees divided by rank of its nonzero integral sublattices. Let V_τ be its saturated Lie eigenlattice of type τ. Set

$$x = \sum_{\tau\in\overline{\Psi}} \widehat{\deg}_{\mathrm{n}}(V_\tau), \qquad y = \sum_{\tau\in\Psi} \widehat{\deg}_{\mathrm{n}}(V_\tau).$$

The image eigenlattices have degree at most that of their saturations, and each eigenlattice occurring in x has rank at most s.

Claim 1. There is $C_s \geq 0$ with the following uniform property. Take a CM field with the host data of the chapter, a number field L, and a semiabelian scheme over $\mathcal{O}_L$ with its $\mathcal{O}_E$-action and rigidified symmetric cubical line. The action is adjoint to conjugation for the associated biextension, and the line has the host's specified primitivity isomorphism. Suppose the generic fibre is proper and its line is ample with trivial polarisation kernel. Identify that fibre with its principally polarised model, compatibly with the action and cubical line. Let $\overline{B}$ be a Hermitian lattice identifying with the integral Lie module.

The metric is fixed as follows. At each embedding $\sigma : L \hookrightarrow \mathbf{C}$, for every pair of integral Lie vectors and their tangent lifts, there are an analytic uniformisation, an Appell–Humbert Hermitian matrix H_σ, and representatives a, b of those lifts for which

$$\langle a, b\rangle_\sigma = \pi b^{\mathsf{t}} H_\sigma \overline{a}.$$

Writing $g_B = \operatorname{rank}_{\mathcal{O}_L} B$ and $h_B = -\widehat{\deg}_{\mathrm{n}}(\overline{B})$, the input is

$$\widehat{\mu}_{\max}(\overline{B}) \le C_s \log(2 + g_B + \max(h_B, 0)).$$

The constant is quantified before the field, host, number field, semiabelian scheme and lattice. We verify this claim using two precise inputs.

Claim 2. Under the arithmetic-intersection, arithmetic Kleiman, Hilbert–Samuel, arithmetic-volume and resolution premises of the chapter, every nonzero submodule F of a Hermitian lattice $\overline{B}$ over a number field L admits a finite extension L'/L and a nonzero integral vector x in the base-changed lattice such that

$$\widehat{\mu}(F) - \tfrac{1}{2}\log g_B - 1 \le -\frac{1}{[L' : \mathbf{Q}]} \sum_{\sigma : L' \hookrightarrow \mathbf{C}} \log \|x\|_\sigma.$$

The arithmetic inputs give the following line-approximation assertion: for every rank-$i \ge 1$ Hermitian lattice and every $\varepsilon > 0$, dualising the lattice and applying the arithmetic Hilbert–Samuel volume lower bound and arithmetic Kleiman inequality (Lemma 35.23) to $\mathcal{O}(1)$ on the projective bundle of its dual produces, after a finite extension L'/L, a nonzero rational line whose normalised Arakelov degree is at least $\widehat{\mu} - \frac{1}{2}\log i - \varepsilon$, where $\widehat{\mu}$ is the lattice slope. Apply this to F with $\varepsilon = 1$, and pass to a further finite extension principalising the resulting fractional lattice ideal; clearing coordinate denominators yields a nonzero integral vector x in the base-changed lattice whose mean negative log norm bounds the line degree from above. The Gram metric is the induced metric on F, and $\operatorname{rank} F \le g_B$, so its logarithmic error is at most the displayed one.
Lean: `GlobalTheory.CompanionTheory.Ar.Giv.lineOfLargeDegree_of_use` (`Development.WeilBound.CoreStatements.AutomorphicKernels.MoriwakiBigness.HilbertSamuelApplications.LineOfLargeDegreeOfUse`, lines 10–88)
Lean: `GlobalTheory.CompanionTheory.Ar.Lines.short_vector_of_submodule` (`Development.AutomorphicSpectrum.Proofs.AdelicProofs.ShortVectorOfSubmodule`, lines 249–324)

Claim 3. There is an absolute $C \ge 0$, chosen before all the principal-polarisation data in Claim 1, such that every nonzero integral Lie vector x satisfies

$$-\frac{1}{[L : \mathbf{Q}]} \sum_{\sigma : L \hookrightarrow \mathbf{C}} \log \|x\|_\sigma \le C \log(2 + g_B + \max(h_B, 0)).$$

The assertion applies with the same constant after every finite extension, with the transported polarisation, cubical line, integral Lie lattice and Hermitian metric. Set $K = \lceil 2 + g_B^2 + \max(h_B, 0) \rceil \ge 1$, $m = 3K$, $n = m^2 = 9K^2$, and $n' = 3K^2$ (so $n = 3n'$). Using the unconditional bad-reduction integrality of the Moret-Bailly key section (established from the Lie-algebra multiplicity comparison on semiabelian degenerations, as in Lemmas 35.4 and 35.10) and the unconditional level-n Gaussian theta-supremum estimate alongside the arithmetic-intersection, Kleiman, Hilbert–Samuel, arithmetic-volume, and resolution premises of the chapter, pass to finite extensions $L_2/L_1/L$ (which preserve g_B, h_B, and $\operatorname{mean}_\sigma \log \|x\|_\sigma$) to obtain a nonzero level-1 theta section θ and a spanning family of $r = m^{2g_B}$ integral level-n theta sections $s_1, \dots, s_r$, built from θ and the m-torsion points rational over L_2, whose archimedean norms at each embedding σ are bounded by the positive supremum bound $M(\sigma)$ of θ, where $-\operatorname{mean}_\sigma \log M(\sigma) \ge -(\max(h_B, 0)/2 + C_1 g_B) - C_3(1 + \log(1 + C_1)) g_B \log(2 + g_B + \max(h_B, 0))$ with absolute constants $C_1, C_3 \ge 0$. Apply the integral flag-embedding criterion to the integral jet flag of orders $0 \le i \le n'$ along the nonzero integral direction x: at every order $0 \le i \le n'$ there is a level-n theta function vanishing to exact order i at 0 along x, while Gaussian-growth iterated-derivative bounds along $\tau \mapsto \sum_j c_j s_j(\tau x)$ control the i-th derivative at each embedding σ, yielding

$$0 \le \sum_{i=0}^{n'} \sum_{\sigma : L_2 \hookrightarrow \mathbf{C}} \log\Big(\big(\sqrt{e\, n\, n'}\, \|x\|_\sigma\big)^i \sqrt{r}\, M(\sigma) \Big).$$

Dividing by $[L_2 : \mathbf{Q}]\, n'(n' + 1)/2$ divides the $\log(\sqrt{r}\, M(\sigma))$ contribution by $n'/2 = \frac{3}{2}K^2 \ge \frac{3}{2}(2 + g_B^2 + \max(h_B, 0))^2$, which absorbs the linear $O(g_B + \max(h_B, 0) + g_B \log(2 + g_B + \max(h_B, 0)))$ numerator into $O(1)$ and leaves $-\operatorname{mean}_\sigma \log \|x\|_\sigma \le \frac{1}{2}\log(3e(n')^2) + O(1) \le C \log(2 + g_B + \max(h_B, 0))$ with an absolute constant $C \ge 0$.
Lean: `GlobalTheory.CompanionTheory.Ar.Sl.KP.key_bad` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.KeyBadReductionHolds`, lines 18–18)
Lean: `GlobalTheory.CompanionTheory.Ar.Sl.keyDeg_of_keyBad` (`Development.AutomorphicSpectrum.Proofs.GeometryProofs.KeyDegreeOfBadReduction`, lines 223–300)

Lean: GlobalTheory.CompanionTheory.Ar.Sl.LineProof.line_of_key (Development.AutomorphicSpectrum.Proofs.GeometryProofs.LineOfKeyDegree, lines 269–373)

To deduce Claim 1, put $b = \log(2 + g_B + \max(h_B, 0))$. Finite-extension transport preserves g_B, h_B, and identifies the metric in Claim 2 isometrically with that in Claim 3. Hence for every nonzero F,

$$\widehat{\mu}(F) \le Cb + \tfrac{1}{2}\log g_B + 1 \le (C+2)b,$$

since $\log g_B \le b$ and $b \ge \log 2 > 2/3$. The rank-zero case has no nonzero submodules. Taking the supremum proves Claim 1 with $C_s = C + 2$. Apply it to $\overline{V}$ with the metric specified above.

Lean: GlobalTheory.CompanionTheory.Ar.Sl.TopSlope.slopeBound (Development.AutomorphicSpectrum.Proofs.GeometryProofs.SlopeBoundOfLargeDegreeLine, lines 94–143)
Lean: GlobalTheory.CompanionTheory.Ar.SlopeBoundSt (Development.AutomorphicSpectrum.Definitions.GlobalGeometry.SlopeBoundSt, lines 72–76)
Lean: GlobalTheory.CompanionTheory.Ar.HodgeIP (Development.AutomorphicSpectrum.Definitions.GlobalGeometry.HodgeIPIffHodgeFormConj, lines 28–34)
Lean: GlobalTheory.CompanionTheory.Ar.IsFibre (Development.AutomorphicSpectrum.Definitions.GlobalGeometry.SlopeBoundSt, lines 39–43)
Lean: GlobalTheory.CompanionTheory.Ar.IsPrincipal (Development.AutomorphicSpectrum.Definitions.GlobalGeometry.SlopeBoundSt, lines 69–70)
Lean: GlobalTheory.CompanionTheory.Ar.Link.given_of_moriwaki (Development.AutomorphicSpectrum.Proofs.GeometryProofs.GivenOfAllInputs, lines 35–41)

Claim 4. The labelled fibre comparison supplies a constant $C_f \ge 0$ such that, for each conjugate pair of eigenlattices,

$$\left|\widehat{\deg}_{\mathrm{n}}(V_\tau) - \widehat{\deg}_{\mathrm{n}}(V_{\overline{\tau}})\right| \le C_f\big(\operatorname{rank} V_\tau + \operatorname{rank} V_{\overline{\tau}}\big)\big(1 + \log \operatorname{rd}(E) + |L_E|\big).$$

Let $C_1 \ge 0$ be the absolute prime-sum constant for the finite set in (A.444), so that its sum of $(\log \ell)/(\ell - 1)$ is at most $C_1 \log N$. Put $c = C_s + C_f(2 + C_1)$. These constants are chosen before the field, rank, component and cutoff. Since $\lambda_N \ge 1$, $\log N \le \lambda_N$ and $\log \operatorname{rd}(E) + |L_E| \le \lambda_N$, determinant inclusion and the rank bounds give the precise inequalities used in the height comparison:

$$x + y \le -h, \quad |x - y| \le cfn\lambda_N, \quad x \le fsc\log\big(2 + g + \max(h, 0)\big) \le fsc\log\big(2 + fn + \max(h, 0)\big). \tag{A.445}$$

Lean: GlobalTheory.CompanionTheory.Ar.Top.Given (Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.GoodOrigin, lines 67–72)
Lean: LevelTowerAlt.height_comparison_global_model (Development.WeilBound.CoreStatements.PlurisubharmonicCones.HeightComparisonGlobalModel, lines 80–245)

If $x > 0$, adding the first two bounds gives $2x - cfn\lambda_N \le -h$, so $h \le cfn\lambda_N$. Since $fn \le N \le e^{\lambda_N}$, in both cases $x \le C_a fs\lambda_N$ for an absolute constant $C_a \ge 0$. The elementary prime-sum bound $\sum_{\ell \le N}(\log \ell)/\ell \le \log N + 1$ (from $N/\ell \le v_\ell(N!) + 1$), $\#\mathcal{B}_{\mathscr{T}} \le N$ and $t \le s$ bound (A.444) by $C_h fs\lambda_N$ with the absolute constant $C_h \ge 0$. Adding $\frac{1}{2}U$ dominates the boundary logarithmic singularity of the adapted determinant metric, so on both interior and boundary horizontal curves the normalised height satisfies

$$h_{\mathrm{reg}} \ge -C_h fs\lambda_N - (a/b)K_A, \tag{A.446}$$

where $-K_A$ is a lower bound for the auxiliary ample height on the component model.

Lean: LevelTowerAlt.height_comparison_global_model (Development.WeilBound.CoreStatements.PlurisubharmonicCones.HeightComparisonGlobalModel, lines 80–245)
Lean: GlobalTheory.CompanionTheory.Ar.Cusp.TopHt.htU_of_given (Development.AutomorphicSpectrum.Proofs.GeometryProofs.CuspHeightUpper, lines 120–223)

We use the following independent integral-vector estimate for the cusp mean.

Claim 5. If E is a CM field of degree $2f$, there are $2f$ rationally independent integers $\omega_1, \dots, \omega_{2f} \in \mathcal{O}_E$ with

$$|\sigma\omega_i| \le v_0 := 2f \operatorname{rd}(E) \quad \text{for every } i \text{ and } \sigma : E \hookrightarrow \mathbf{C}.$$

For the Euclidean trace lattice, a nonzero vector in the dual corresponds to an element of the inverse different. The arithmetic-geometric mean inequality applied to its conjugates bounds its norm below by $\sqrt{2f}/\operatorname{rd}(E)$. The Euclidean transference construction for a full lattice of dimension a, with dual minimum at least $b > 0$, gives a independent vectors of norm at most $a^{3/2}/b$. Substituting these values gives $2f \operatorname{rd}(E)$. Each absolute conjugate is at most this Euclidean norm.

Lean: GlobalTheory.CompanionTheory.Ar.Cusp.exists_independent_le_of_dual_min (Development.AutomorphicSpectrum.Proofs.MetricProofs.ExistsIndependentLeOfDualMin, lines 13–121)

Lean: `GlobalTheory.CompanionTheory.Ar.Cusp.exists_short_integers` (`Development.AutomorphicSpectrum.Proofs.AlgebraProofs.CmFieldSmallIntegralBasis`, lines 128–175)

Apply Claim 5 to bound mean U. For $t>0$, let N_t count the nonzero isotropic vectors v in the arithmetic lattice Λ of positive-majorant length below t that satisfy $[\Lambda\cap Ev:\mathcal{O}_E v]\le\lfloor(v_0\sqrt{2f})^{2f}\rfloor$. For any two vectors $v,w\in\Lambda$ of length below $t\le v_0^{-1/2}$, the $\mathbf{Z}$-valued trace pairings with $\omega_1 w,\dots,\omega_{2f}w$ have modulus at most $v_0\|v\|\|w\|<1$ and therefore vanish; since $\omega_1,\dots,\omega_{2f}$ span E over $\mathbf{Q}$, all vectors in Λ of length below t are isotropic and mutually orthogonal for the Hermitian form, spanning a single isotropic E-subspace W_0 of E-dimension at most s and real dimension $d_0\le 2fs$. For any shortest nonzero vector $v\in\Lambda$ with $\|v\|=\lambda_1<t\le v_0^{-1/2}$, the full-rank sublattice $\bigoplus_{i=1}^{2f}\mathbf{Z}\omega_i v\subseteq\mathcal{O}_E v$ has covolume at most $(v_0\lambda_1)^{2f}$ by Hadamard's inequality, while packing the half-cube $(-\lambda_1/(2\sqrt{2f}),\lambda_1/(2\sqrt{2f}))^{2f}$ in orthonormal coordinates on $(Ev)\otimes_{\mathbf{Q}}\mathbf{R}$ gives $\operatorname{covol}(\Lambda\cap Ev)\ge(\lambda_1/\sqrt{2f})^{2f}$; dividing bounds $[\Lambda\cap Ev:\mathcal{O}_E v]\le\lfloor(v_0\sqrt{2f})^{2f}\rfloor$, so v belongs to the counted set and $N_t\ge 1$. The counted set consists of nonzero isotropic vectors of Λ of line index at most $\lfloor(v_0\sqrt{2f})^{2f}\rfloor$, and it is invariant under the arithmetic unitary group Γ. It is a finite union of Γ-orbits. Up to the unitary group there are finitely many pairs formed by a lattice with the integrality of Λ and an isotropic line. Inside one line Ev, the counted vectors x give subgroups $\mathcal{O}_E x$ of $\Lambda\cap Ev$ of bounded index, of which there are finitely many; each determines x up to a unit of $\mathcal{O}_E$, the units modulo a congruence subgroup form a finite group, and multiplication of v by a congruence unit is realised by an element of Γ. Unfolding over a fundamental set $\mathcal{F}$ of finite positive Haar measure writes $\int_{\mathcal{F}}N_t$ as the finite sum, over orbit representatives y, of the Haar measure of $\{g\in\mathcal{F}_y:\|g^{-1}y\|<t\}$, where $\mathcal{F}_y$ is a fundamental set for the stabiliser of y in Γ. In the parabolic coordinates $g=x\,\tau(b)\,k$, with x in the archimedean fixer of y, $b\in(\mathbf{C}^\times)^f$ such that $\tau(b)^{-1}y=by$ placewise, and k in the compact stabiliser of the majorant, Haar measure has density $\prod_w|b_w|^{2(n-2)}$ in b. The stabiliser of y in Γ has finite positive covolume in the fixer, so each orbit term equals $C_y t^{2f(n-1)}$ with $0<C_y<\infty$. Hence $\int_{\mathcal{F}}N_t=\bigl(\sum_y C_y\bigr)t^{2f(n-1)}$ is finite for every $t>0$, and scaling the length cutoff by $\theta>0$ multiplies it by $\theta^{2f(n-1)}$. Normalise Haar measure on $\mathcal{F}$ to a probability measure and put $I_t=\mathbb{E}N_t$, a finite number for every $t>0$.

Lean: `GlobalTheory.CompanionTheory.Ar.Cusp.group_mean_logPlus_le` (`Development.AutomorphicSpectrum.Proofs.MetricProofs.FirstMinimumIntegralBound`, lines 221–337)

Inside the d_0-dimensional Euclidean space W_0, open balls of radius $\lambda_1/2$ around distinct lattice points of length below t are disjoint and lie in the ball of radius $\frac{3}{2}t$, giving $1\le N_t\le(3t/\lambda_1)^{d_0}\le(3t/\lambda_1)^{2fs}$ on $\lambda_1<t\le v_0^{-1/2}$, whereas $N_t=0$ on $\lambda_1\ge t$. At $t=v_0^{-1/2}$, split I_t at $\lambda_1=t/12$. The region $\lambda_1\ge t/12$ contributes at most 36^{2fs}. On $\lambda_1<t/12$, put $a_*=2fs$ and $k_*=2f(n-1)$; homogeneous scaling gives $\Pr(\lambda_1<\theta t)\le\mathbb{E}N_{\theta t}=\theta^{k_*}I_t$ for $0<\theta\le 1$. Layer-cake integration of $(3t/\lambda_1)^{a_*}$ over $u=t/\lambda_1>12$ bounds the remaining contribution by

$$3^{a_*}\left(12^{a_*-k_*}+a_*\int_{12}^{\infty}u^{a_*-k_*-1}\,du\right)I_t=3^{2fs}12^{-2f(n-1-s)}\left(1+\frac{s}{n-1-s}\right)I_t\le\tfrac{1}{2}I_t,$$

since $n-1-s=r-1\ge 2s\ge 2$. As I_t is finite, subtracting $\frac{1}{2}I_t$ gives $I_t\le 2\cdot 36^{2fs}$, so

$$\Pr(Y_1>u)\le\min(1,2\cdot 36^{2fs}e^{-2f(n-1)u}),\qquad Y_1=\log^+(v_0^{-1/2}/\lambda_1),\quad u\ge 0. \tag{A.447}$$

Integrating (A.447) over $u\ge 0$ with $n-1\ge 3s$ gives $\mathbb{E}Y_1\le 3$. Using the pointwise cusp-weight bound

$$U\le fs\bigl(2\log^+(1/\lambda_1)+\log(2fs\,\mathrm{rd}(E))\bigr) \tag{A.448}$$

and $\log^+(1/\lambda_1)\le Y_1+\frac{1}{2}\log v_0$, averaging over fundamental sets and sheets yields

$$\operatorname{mean}U\le fs\bigl(6+\log v_0+\log(2fs\,\mathrm{rd}(E))\bigr)\le 4fs\lambda_N, \tag{A.449}$$

since $2fs<fn\le\frac{1}{4}N$ and $N\ge 128$ give $6+\log(4f^2s\,\mathrm{rd}(E)^2)\le 4\log(N\,\mathrm{rd}(E))\le 4\lambda_N$.

Lean: `GlobalTheory.CompanionTheory.Ar.Cusp.fund_set_mean_cuspW` (`Development.AutomorphicSpectrum.Proofs.MetricProofs.CuspWeightIntegrableOnFundamentalSet`, lines 230–328)

Now fix integers $1\le a\le b$. Since $N\ge 4fn^2$ and $n\ge 4$, we have $4f+2\le N$. By clause 6 of Definition 35.27, there exist positive thresholds $t_1,t_2,t_3,t_4>0$ such that for every $0<t'\le\min(t_1,t_2,t_3,t_4)$ the cleared-denominator smoothed bundle $\overline{M}_{N,a,b,t'}$ on the base model of Z is smooth, conjugation-invariant, cone-semipositive along the defining ideal of the base model and fibrewise nef, and satisfies $h(\overline{M}_{N,a,b,t'};W)\ge$

be_0 on every horizontal arithmetic curve W, where $e_0 = -C_h f s\lambda_N - (a/b)K_A$ by (A.446). Let $D_{t'} = b^e \deg_{\mathrm{gen}}(\mu + (a/b)A)$ and $I_{t'}$ be its generic degree and top arithmetic self-intersection. Lemma 35.23(1) in dimension e gives

$$I_{t'} \geq (e+1) D_{t'}\, be_0. \tag{A.450}$$

Put $\eta = -C_h f s\lambda_N$, so $e_0 = \eta - (a/b)K_A$. Dividing (A.450) by $b^{e+1} > 0$ gives

$$(e+1)\left(\eta - \frac{a}{b}K_A\right)\frac{D_{t'}}{b^e} \leq \frac{I_{t'}}{b^{e+1}}$$

for every $1 \leq a \leq b$ and all sufficiently small $t' > 0$. Apply the quantified intersection implication in clause 6 of Definition 35.27 and (A.449), with $C_u = 4$, to obtain

$$\frac{P_{N,Z}(F_*)}{(e+1)V_Z D(\mu^e J)} \geq -\big(C_h + \tfrac{1}{2}C_u\big) f s\lambda_N = -(C_h+2) f s\lambda_N.$$

Since $H_* = (e+1)D(\mu^e J) > 0$ and $V_Z > 0$, multiplying by $V_Z H_*$ and summing over $Z \in \mathrm{Comp}$ yields $P_N(F_*) \geq -V H_*(C_h+2) f s\lambda_N$.

Lean: `GlobalTheory.CompanionTheory.Ar.Top.slX_lower_of_stokesAt` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.SlopeLowerBaseSettings`, lines 16–74)
Lean: `GlobalTheory.CompanionTheory.Ar.Top.ppc_lower_of_setting` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.SlopeLowerBaseSettings`, lines 76–108)

The complete proof is contained in `GlobalTheory.CompanionTheory.Ar.Top.slope_lower_of_base_settings`. □

Proof of 35.30

Proof. Write $t = \min(r,s)$, $w = \max(r,s)$, and let $R = (w^t)$ be the partition with t rows of length w. For a partition λ, write f^λ for its number of standard tableaux. A partition ν of k with at most t rows gives the shape

$$R + \nu = (w + \nu_1, \ldots, w + \nu_t)$$

after zero rows have been appended to ν. In a standard tableau of this shape, record the set of entries in the last k boxes, meaning the boxes strictly to the right of column w. Standardise the entries in those boxes and in the rectangle separately, preserving their relative orders. The set and the two standard tableaux recover every entry of the original tableau. Thus

$$f^{R+\nu} \leq \binom{a+k}{k} f^R f^\nu.$$

For the hook $\nu = (k-j, 1^j)$, the corner contains 1, and the j entries below it determine the whole tableau. Consequently

$$\sum_{\substack{\nu=(k-j,1^j)\\ 0\leq j<\min(t,k)}} f^{R+\nu} \leq \binom{a+k}{k} f^R \sum_{j=0}^{\min(t,k)-1} \binom{k-1}{j} \leq 2^{k-1}\binom{a+k}{k} f^R.$$

Lean: `GroundTheory.Heights₂.Heights.RestTerms.LemmaC.proof` (`GroundTheory.HeightsAndTheta.Heights.RemainderTerms.PLemmaC`, lines 19–27)
Lean: `GroundTheory.Heights₂.Heights.RestTerms.HookTableauCount.proof` (`GroundTheory.HeightsAndTheta.Heights.RemainderTerms.PHookTableauCount`, lines 140–144)
Lean: `GroundTheory.Heights₂.Heights.RestTerms.HookSumBound.proof` (`GroundTheory.HeightsAndTheta.Heights.RemainderTerms.PHookSumBound`, lines 83–128)

To evaluate the insertion, alternate in the n roots at each embedding and divide by the corresponding Vandermonde. On polynomials of top flag degree this gives D, up to a common normalising factor. Applied to PF_*, it gives a symmetric polynomial of degree k. Take the coefficient of the single power sum $p_{\tau k}$ in the power-sum coordinates at all embeddings. These coordinates are algebraically independent through index n over $\mathbf{Q}$. Full power sums commute with alternation. In the decomposition of F_*, a summand divisible by $p_{\eta j}$ can therefore contribute to that coefficient only for $(\eta, j) = (\tau, k)$. The remaining factor gives the top-degree integral $D(PQ_{\tau k})$, with the same normalisation.

We now give the block calculation of the coefficient. In one embedding let Y denote the t variables of the smaller block. On multiplying $s_\lambda(Y)$ by the two within-block Vandermondes, the two lists of exponents in the alternants are

$$\lambda_1 + t - 1, \dots, \lambda_t \quad \text{and} \quad w - 1, \dots, 0.$$

If $\lambda_t < w$, an exponent occurs twice. Otherwise division of the full alternant by the full Vandermonde gives $s_{\lambda - R}$ in all n variables, up to a common factor from the two block alternations. Reordering the blocks or changing the signs of all the variables in the chosen block contributes a sign in each homogeneous degree. All common factors cancel on division by H_*.

In the expansion of a product of power sums $p_\rho(Y)$, the coefficient of $s_\lambda(Y)$ is the symmetric-group character $\chi^\lambda(\rho)$. In particular the coefficient for $p_1(Y)^a$ and $\lambda = R$ is f^R. In the inverse expansion the coefficient of the single power sum p_k in s_ν is $\chi^\nu((k))/k$. The character of a k-cycle is $(-1)^j$ for $\nu = (k-j, 1^j)$ and zero otherwise. These are the Frobenius formula and its single-cycle case for partitions of k; here $k \le n$, so the coefficient can be read in n variables. For every cycle type ρ of size $|\lambda|$ one has $|\chi^\lambda(\rho)| \le f^\lambda$, since a finite-order operator on the representation has eigenvalues of absolute value one. See [183, Ch. I, §7, in particular (7.8)].

Lean: `GroundTheory.Heights₂.Heights.RestTerms.SchurSignCoefficient.proof` (`GroundTheory.HeightsAndTheta.Heights.RemainderTerms.PSchurSignCoefficient`, lines 42–58)
Lean: `GroundTheory.Heights₂.Heights.RestTerms.CharacterBound.proof` (`GroundTheory.HeightsAndTheta.Heights.RemainderTerms.PCharacterBound`, lines 15–23)

In $D(\mu^e J)$ every embedding must supply a linear factors. Its multinomial and tableau factors are $e!/(a!)^f$ and $(f^R)^f$. Suppose first that $\sigma = \tau$. The input embedding must supply $a+1$ linear factors from μ^{e+1}, in addition to the insertion of degree $k-1$. Every other embedding supplies a linear factors. After division by the multinomial in H_*, the ratio is

$$\frac{(e+1)!}{(a+1)!(a!)^{f-1}} \frac{(a!)^f}{(e+1)e!} = \frac{1}{a+1}.$$

The remaining character at the input has shape $R + \nu$ and cycle type $(k-1, 1^{a+1})$. The factor k in the assertion cancels the denominator k in the single-cycle coefficient. The absolute value is therefore bounded by $1/(a+1)$ times the hook sum above divided by f^R.

If $\sigma \neq \tau$, the input supplies $a+k$ linear factors and the output supplies $a-k+1$, besides its insertion of degree $k-1$. Here $k-1 \le r+s-1 \le rs = a$. The input coefficient before extraction is $f^{R+\nu}$; the output character has shape R and absolute value at most f^R. The multinomial ratio is

$$\frac{(a!)^2}{(a+k)!(a-k+1)!}.$$

Multiplying this by $2^{k-1}\binom{a+k}{k}$ gives the second bound.

Lean: `GroundTheory.Heights₂.Heights.RestTerms.SameFactorFormula.proof` (`GroundTheory.HeightsAndTheta.Heights.RemainderTerms.PSameFactorFormula`, lines 158–253)
Lean: `GroundTheory.Heights₂.Heights.RestTerms.OtherFactorFormula.proof` (`GroundTheory.HeightsAndTheta.Heights.RemainderTerms.POtherFactorFormula`, lines 41–192)

□

Proof of 35.31

Proof. Let $(\alpha_M), (\beta_M)$, $C \ge 0$ and $N_1 \ge 2$ be given by clause 3 of Definition 35.27. Fix $c_0 > 0$ and a wall embedding $\tau_0 \in \Psi$. On the compact patch of weights $\|v\|_\infty \le 1/2$ where $v_{\tau_0, r-1} = v_{\tau_0, r}$ and all other wall gaps and within-block adjacent gaps are at least c_0, Lemma 35.28(1) and (4) give $\delta_{\mathrm{fl}}(v) = 0$ (hence $S(L_v^{d+1}) = T(L_v^{d+1}) = 0$) and $h_{\mathrm{fl}}(L_v^{d+1}) = 0$, so clause 3 gives

$$|(P_M - V h_{\mathrm{fl}})(L_v^{d+1})| = O(\log M / M) \tag{A.451}$$

uniformly on each wall patch. Setting

$$G_M(F) = P_M(F) - V\big(h_{\mathrm{fl}}(F) - (\alpha_M - \beta_M) S(F) - \beta_M T(F)\big),$$

linearity and the monomial bound of clause 3 give $|G_M(F)| \le \big(\sum_m |c_m(F)|\big) C(\log M)/M \to 0$ as $M \to \infty$ for every homogeneous F of degree $d+1$.

Lean: GlobalTheory.CompanionTheory.Ar.Top.thmW_tower (Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.TowerWeilTheorem, lines 120–203)

Lean: GlobalTheory.CompanionTheory.Ar.Top.tendsto_const_mul_log_div (Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.TowerIdentityAtCutoff, lines 73–79)

By Lemma 35.28(2), every homogeneous F of degree $d+1$ decomposes as $F = \sum_{\tau,k} p_{\tau,k} Q_{\tau,k}$ with $1 \le k \le n$ and $Q_{\tau,k}$ homogeneous of degree $d+1-k$. For $M \ge N \ge 2$, summing clause 4 of Definition 35.27 over $Z \in \mathrm{Comp}$ and substituting Lemma 35.28(3) gives

$$P_N(p_{\tau,k}Q) = V\Big(h_{\mathrm{fl}}(p_{\tau,k}Q) - a_M S(p_{\tau,k}Q) - b_M T(p_{\tau,k}Q) + \sum_{\substack{N<\ell\le M \text{ prime} \\ \ell\notin\mathcal{B}_{\mathscr{T}}}} \rho_\ell(\tau,k,Q)\Big) + G_M(p_{\tau,k}Q),$$

where, with κ_ℓ as in Lemma 35.28(3),

$$a_M = \alpha_M - \beta_M + \sum_{\substack{N<\ell\le M \text{ prime} \\ \ell\notin\mathcal{B}_{\mathscr{T}}}} \frac{\log \ell}{\ell - 1}, \qquad b_M = \beta_M - \sum_{\substack{N<\ell\le M \text{ prime} \\ \ell\notin\mathcal{B}_{\mathscr{T}}}} \kappa_\ell.$$

For $2 \le k \le n$, Lemma 35.28(3) shows that as $M \to \infty$ the finite prime sums converge absolutely to $R_N(p_{\tau,k}Q)$ of (35.10), while for $k = 1$ one has $R_N(p_{\tau,1}Q) = 0$. Fix $\tau_0 \in \Psi$ and take the two test classes $p_{\tau_0,1}Q_1$ and $p_{\tau_0,2}Q_2$ of Lemma 35.28(2), on which (S,T) equal (r,n) and $(2,0)$, respectively. Since $V > 0$, evaluating the finite-M identity on $p_{\tau_0,2}Q_2$ (where $T = 0$ and $S = 2 \ne 0$) and letting $M \to \infty$ proves $a_M \to A$ for a finite limit $A \in \mathbf{R}$; evaluating next on $p_{\tau_0,1}Q_1$ (where $T = n \ne 0$) proves $b_M \to b$ for a finite limit $b \in \mathbf{R}$. Passing to the limit $M \to \infty$ on each summand $p_{\tau,k}Q_{\tau,k}$ and summing over (τ,k) establishes (35.11) with $R_N(F) = \sum_{\tau,k} R_N(p_{\tau,k}Q_{\tau,k})$.

Lean: GlobalTheory.CompanionTheory.Ar.Top.exists_finite_identity (Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.TowerIdentityAtCutoff, lines 97–149)

Lean: GlobalTheory.CompanionTheory.Ar.Top.tendsto_coeffs_of_tests (Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.TowerIdentityAtCutoff, lines 37–59)

Lean: GlobalTheory.CompanionTheory.Ar.Top.eq_of_tendsto_coeffs (Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.TowerIdentityAtCutoff, lines 61–71)

Lean: GlobalTheory.CompanionTheory.Ar.Top.identOnDecomposedForm (Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.TowerHeightIdentityExpansion, lines 14–15)

On the degree-one test $p_{\tau_0,1}Q_1$, Lemma 35.28(4) gives $h_{\mathrm{fl}}(p_{\tau_0,1}Q_1) = 0$, while $R_N(p_{\tau_0,1}Q_1) = 0$, $(S,T) = (r,n)$ and $D(Q_1) = 1$. By clause 5 of Definition 35.27, since $N \ge \max(128, \#\mathcal{B}_{\mathscr{T}})$ and $D(Q_1) = 1$, one has $P_{N,Z}(p_{\tau_0,1}Q_1) = h_{N,Z,\tau_0} V_Z$ with $|h_{N,Z,\tau_0}| \le 4n\lambda_N$. Substituting $p_{\tau_0,1}Q_1$ into (35.11) gives $rA + nb = -V^{-1}\sum_{Z\in\mathrm{Comp}} h_{N,Z,\tau_0} V_Z$, whence $|rA + nb| \le 4n\lambda_N$ by the triangle inequality since $V_Z > 0$ and $\sum_Z V_Z = V$.

Lean: GlobalTheory.CompanionTheory.Ar.Top.flat_85_bound (Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.CutoffCoefficientsBound, lines 37–80)

Now evaluate (35.11) on the Hodge test polynomial $F_* = \mu^{e+1} J$ of Lemma 35.29. Since $3s + 1 \le n$, we have $s < r$ and $n \le 2r$. Uniform shifts within the first r variables or across all n variables leave every within-block difference in J invariant, while $\sum_{i<r} \partial_{\tau,i}\mu = 0$ and $\sum_{i=0}^{n-1} \partial_{\tau,i}\mu = s$; thus $S(F_*) = 0$ and $T(F_*) = fs(e+1)D(\mu^e J) = fsH_*$. Under the projection $\pi : \prod_{\tau\in\Psi} \mathrm{Fl}(\mathbf{Z}^n) \to \prod_{\tau\in\Psi} \mathrm{Gr}(s,n)_{\mathbf{Z}}$, the class $\mu = \sum_{\tau,i\ge r} X_{\tau,i}$ in Definition 35.27 pulls back the external tensor product $\overline{M}$ of the Plücker Fubini–Study lines $\overline{\mathcal{O}_{\mathrm{Gr}(s,n)}(1)}$. For any horizontal integral subvariety $Y \subseteq \prod_{\tau\in\Psi} \mathrm{Gr}(s,n)_{\mathbf{Z}}$ of generic dimension $m \in \{0,\dots,e\}$, write $II(\pi^{-1}(Y); (\pi^*\overline{M})^{m+1}; J)$ for the multilinear evaluation of II on $\pi^{-1}(Y)$ (of arithmetic dimension $m+1+d-e$) obtained by substituting the Hermitian root lines $(\overline{Q}_{\tau,i})$ into the homogeneous degree-$(d-e)$ polynomial J in the last $d-e$ slots. By induction on $m \in \{0,\dots,e\}$, choosing a Plücker coordinate section s_Y of $\overline{M}$ not vanishing identically on Y and applying the rational-section divisor recursion simultaneously to $\pi^* s_Y$ on $\pi^{-1}(Y)$ and to s_Y on Y shows that $II(\pi^{-1}(Y); (\pi^*\overline{M})^{m+1}; J) = c_{\mathrm{fib}}\, II(Y; \overline{M}^{m+1})$, where $c_{\mathrm{fib}} = D(\mu^e J)/\deg_{\mathrm{gen}}(\overline{M})$ is the constant fibrewise degree of J on the smooth $(d-e)$-dimensional fibres of π: on every vertical component of $\mathrm{div}_Y(s_Y)$ the factor c_{fib} separates by the geometric projection formula, on every horizontal component (absent when $m = 0$) it separates by induction, and in the archimedean integral on a characteristic-zero resolution $\widetilde{Y} \to Y_{\mathbf{C}}$ the factors $\log\|\pi^* s_Y\|$ and $\pi^* c_1(\overline{M})^m$ are pulled back

from $\widetilde{Y}(\mathbf{C})$ while Fubini's theorem integrates the $(d-e,d-e)$-form $J(c_1(\overline{Q}_{\tau,i}))$ fibrewise to c_{fib}. Taking $Y=\prod_{\tau\in\Psi}\mathrm{Gr}(s,n)_{\mathbf{Z}}$ $(m=e)$ identifies $h_{\mathrm{fl}}(F_*)$ with $c_{\mathrm{fib}}\,II\big(\prod_{\tau\in\Psi}\mathrm{Gr}(s,n)_{\mathbf{Z}};\overline{M}^{e+1}\big)$. On $\mathrm{Gr}(s,n)_{\mathbf{C}}$, affine chart matrices $Z\in\mathbf{C}^{s\times r}$ whose entries are roots of unity define a Zariski-dense collection of horizontal orbits whose Plücker coordinates are algebraic integers including 1 and satisfy $\det(I_s+ZZ^*)\le(1+r)^s\le(n+1)^s$ by Hadamard's inequality, giving normalised height at most $\frac12 fs\log(n+1)$ on $\prod_{\tau\in\Psi}\mathrm{Gr}(s,n)_{\mathbf{Z}}$. Lemma 35.23(2) therefore yields

$$h_{\mathrm{fl}}(F_*)\le\tfrac12 H_* fs\log(n+1). \tag{A.452}$$

Lean: `GlobalTheory.CompanionTheory.Ar.Top.testPoly_facts` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.TestPolynomialShiftTotal`, lines 13–14)

To bound $R_N(F_*)$, put $a=rs$ and $X=2(a+n)\le\frac32 n^2$ (using $a\le n^2/4$ and $n\ge4$). In each summand of (35.10) for a decomposition $F_*=\sum_{\tau,k}p_{\tau,k}Q_{\tau,k}$ and $2\le k\le n$, the full power sum $p_{\sigma,k-1}$ integrates to zero against $Q_{\tau,k}$ under D, so $P_{\ell,\tau,h,k-1}$ may be replaced up to sign by the degree-$(k-1)$ power sum on the smaller block of size s. Lemma 35.30 therefore gives

$$\frac{k|D(P_{\ell,\tau,h,k-1}Q_{\tau,k})|}{H_*}\le\begin{cases}\dfrac{2^{k-1}}{a+1}\dbinom{a+k}{k}, & \sigma=\tau,\\[2mm] \dfrac{2^{k-1}a(a-1)\cdots(a-k+2)}{k!}, & \sigma\ne\tau,\end{cases}$$

where $\sigma\in\Psi$ is the underlying embedding of $\mathrm{Fr}_{\ell,h-1}(\tau)$. Both right-hand expressions are at most $X^{k-1}/k!$, since $k!\binom{a+k}{k}/(a+1)=\prod_{j=2}^{k}(a+j)\le(a+n)^{k-1}$ and $a(a-1)\cdots(a-k+2)\le a^{k-1}$. For any prime $\ell>N\ge4fn^2$, $\sum_{h\ge1}\ell^{-hk}\le2/\ell^k$ and $X/\ell\le\frac32 n^2/(4n^2)=3/8$, so

$$\sum_{k=2}^{n}\frac{2X^{k-1}}{\ell^k k!}\le\frac{2X}{\ell^2}\sum_{j\ge0}\frac{(X/\ell)^j}{(j+2)!}\le\frac{X}{\ell^2}e^{3/8}\le\frac{1.5X}{\ell^2}.$$

Summing over the f embeddings $\tau\in\Psi$, the contribution of ℓ has modulus at most $1.5fXH_*(\log\ell)/\ell^2$. Partial summation with $\vartheta(t)=\sum_{\ell\le t}\log\ell\le t\log4$ gives

$$\sum_{\ell>N}\frac{\log\ell}{\ell^2}\le2\int_N^\infty\frac{\vartheta(t)}{t^3}\,dt\le\frac{2\log4}{N}. \tag{A.453}$$

Since $1.5fX\cdot(2\log4)/N\le\frac94 fn^2(2\log4)/(4fn^2)=\frac98\log4<1.6$, every finite partial sum over good primes $\ell>N$, and hence the convergent series $R_N(F_*)$, satisfies $|R_N(F_*)|\le1.6H_*$.

Lean: `GlobalTheory.CompanionTheory.Ar.Top.testRest_bound` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.TestPolynomialRemainderBound`, lines 417–442)

Combining Lemma 35.29 ($P_N(F_*)/V\ge-C_0fs\lambda_N H_*$ with the absolute constant $C_0=C_h+2$) with (35.11), $S(F_*)=0$, $T(F_*)=fsH_*$, (A.452) and $|R_N(F_*)|\le1.6H_*$ gives

$$-C_0fs\lambda_N H_*\le\tfrac12 H_*fs\log(n+1)-b\,fsH_*+1.6H_*.$$

Dividing by $fsH_*>0$ and using $1/(fs)\le1$ yields $b\le C_0\lambda_N+\frac12\log(n+1)+1.6$. From $rA+nb\ge-4n\lambda_N$ and the upper bound on b, we get $rA\ge-n\big((C_0+4)\lambda_N+\frac12\log(n+1)+1.6\big)$; since $n\le2r$ and the bracket is nonnegative, dividing by $r>0$ gives $A\ge-2\big((C_0+4)\lambda_N+\frac12\log(n+1)+1.6\big)$. Finally $n+1\le4fn^2\le N$, $\mathrm{rd}(E)\ge1$ and $N\ge128=2^7$ give $\log(n+1)\le\log N\le\lambda_N$ and $1.6<7\log2=\log128\le\lambda_N$, whence $A\ge-\big(2(C_0+4)+3\big)\lambda_N=-K\lambda_N$.

Lean: `GlobalTheory.CompanionTheory.Ar.Top.lower_bound_key_inequality` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.TestHeightBound`, lines 17–26)

Lean: `GlobalTheory.CompanionTheory.Ar.Top.lower_bound_finite_arith` (`Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.TestHeightBound`, lines 28–30)

□

Proof of 35.32

Proof. Since v has gap $g>0$, each row v_τ is strictly decreasing, so v is regular dominant and in particular dominant. Lemma 35.28(1) therefore gives $\delta_{\mathrm{fl}}(v)>0$, so $H_v=(d+1)\delta_{\mathrm{fl}}(v)>0$, and Lemma 35.28(4) gives

$h_{\mathrm{fl}}(L_v^{d+1}) \ge 0$. Because $B(v,g) \ge 4fn^2$ and the integer N satisfies $N+1 > B(v,g)$, we also have $N \ge 4fn^2$, so Lemma 35.31 applies at cutoff N to $F = L_v^{d+1}$.

Write $s_\tau = \sum_{i<r} v_{\tau,i}$, $t_\tau = \sum_{i=0}^{n-1} v_{\tau,i}$ and $a_\tau = \sum_{i=0}^{n-1} |v_{\tau,i}|$. Differentiating L_v^{d+1} gives $\mathrm{Sh}_r(\tau, F) = -s_\tau H_v$ and $\mathrm{Tot}(\tau, F) = -t_\tau H_v$, so $(-AS(F) - bT(F))/H_v = \sum_{\tau\in\Psi}(As_\tau + bt_\tau)$. Rewrite each summand as

$$As_\tau + bt_\tau = A\big(s_\tau - rt_\tau/n\big) + (rA + nb)t_\tau/n.$$

Since v_τ is nonincreasing, the mean of its first r entries is at least the mean of all n entries, while $ns_\tau - rt_\tau = s\sum_{i<r} v_{\tau,i} - r\sum_{i\ge r} v_{\tau,i} \le \max(r,s)a_\tau \le na_\tau$; hence $0 \le s_\tau - rt_\tau/n \le a_\tau$, and $|t_\tau| \le a_\tau$. Using $A \ge -K\lambda_N$ and $|rA + nb| \le 4n\lambda_N$ from Lemma 35.31 gives $(-AS(F) - bT(F))/H_v \ge -(K+4)\lambda_N\|v\|_1$.

To bound $R_N(F)$ in a decomposition $F = \sum_{\tau,k} p_{\tau,k}Q_{\tau,k}$, recall that for $k \ge 2$ the complete power sum $p_{\sigma,k-1}$ vanishes against $Q_{\tau,k}$ under D, so each $P_{\ell,\tau,h,k-1}$ in (35.10) may be replaced up to sign by the degree-$(k-1)$ power sum on the smaller block of size s. Fix $\tau \in \Psi$ and translate row v_τ into $[0, U_\tau]$, where $U_\tau = v_{\tau,0} - v_{\tau,n-1}$ is the row spread; this changes L_v by a multiple of $p_{\tau,1}$ and preserves H_v, since the row differences are unchanged. Write v' for the translated weight and $Q'_{\tau,k}$ for a decomposition of $L_{v'}^{d+1}$. Let $\Lambda_{\tau,k}$ be k times the single-power-sum coefficient functional in the Weyl alternation calculation, with the common normalisation used for D. It satisfies $\Lambda_{\tau,k}(\gamma L_v^{d+1}) = kD(\gamma Q_{\tau,k})$ and annihilates $p_{\tau,1}R$ for $2 \le k \le n$. Since $\gamma(L_{v'}^{d+1} - L_v^{d+1})$ is divisible by $p_{\tau,1}$, taking $\gamma = X_{\sigma,i}^{k-1}$ gives $D(X_{\sigma,i}^{k-1}Q'_{\tau,k}) = D(X_{\sigma,i}^{k-1}Q_{\tau,k})$. Thus the insertion functional in the estimate is unchanged. For an index (σ,i) in the smaller block ($i \ge r$, where adjacent gaps are at least g), perturb the (σ,i)-entry by $+t$, using the translated row when $\sigma = \tau$. Weyl alternation expresses $kD(X_{\sigma,i}^{k-1}Q_{\tau,k})$, up to sign, as $(d+1)!(k-1)!$ times the t^{k-1}-coefficient of the corresponding sum of Vandermonde-type determinants in which one column exponent at τ is raised by k with column-factorial denominators.

Reducing the raised monomial modulo $\prod_{j=0}^{n-1}(X - v_{\tau,j}(t))$ factors out the Vandermonde. In column c, starting at exponent $n-1-c$, elementary and complete symmetric expansions majorise the coefficient of X^{n-1-c} coefficientwise in $t \ge 0$ by $U_\tau^k\binom{k-1}{c}\binom{n-1-c+k}{k}e^{kt/U_\tau}$. Multiplying by $(n-1-c)!/(n-1-c+k)!$ replaces the second binomial by $1/k!$, and summing over c gives $2^{k-1}U_\tau^k e^{kt/U_\tau}/k!$. Dividing the perturbed Vandermonde at σ by its $t=0$ value leaves factors $1 \pm t/(v_{\sigma,i} - v_{\sigma,j})$ $(j \ne i)$, whose product is majorised coefficientwise by $\prod_{j\ne i}(1 + t/g) \le e^{nt/g}$ because $i \ge r$ gives $|v_{\sigma,i} - v_{\sigma,j}| \ge g$. Extracting $[t^{k-1}]$ yields

$$k|D(X_{\sigma,i}^{k-1}Q_{\tau,k})| \le H_v\frac{2^{k-1}}{k!}U_\tau(k + nU_\tau/g)^{k-1}. \tag{A.454}$$

Lean: `RootFieldNumerics.M5.lam_bound` (`Development.WeilBound.CoreStatements.FlagImmersions.RestDegreePowerBound`, lines 135–183)
Lean: `RootFieldNumerics.M5.lam_shift` (`Development.WeilBound.CoreStatements.FlagImmersions.RestDegreePowerBound`, lines 199–210)
Lean: `RootFieldNumerics.M5.one_bound` (`Development.WeilBound.CoreStatements.FlagImmersions.RestDegreePowerBound`, lines 212–248)
Lean: `RootFieldNumerics.M5.small_block` (`Development.WeilBound.CoreStatements.FlagImmersions.RestDegreePowerBound`, lines 250–281)

Put $D_* = n(1+2\|v\|_\infty/g)$ and $B_* = B(v,g) = 4fn^2(1+\|v\|_\infty/g)$. Since $k \le n$ and $U_\tau \le 2\|v\|_\infty$, summing (A.454) over the s indices of the smaller block bounds $k|D(P_{\ell,\tau,h,k-1}Q_{\tau,k})|$ by $H_v s U_\tau(2D_*)^{k-1}/k!$. Every prime $\ell > N$ satisfies $\ell \ge N+1 > B_*$, so $\sum_{h\ge1}\ell^{-hk} \le 2\ell^{-k} \le 2B_*^{-(k-2)}\ell^{-2}$, and applying partial summation as in (A.453) from lower limit B_* bounds $\sum_{\ell>N} 2(\log\ell)/\ell^2 \le \sum_{\ell>B_*} 2(\log\ell)/\ell^2$ by $4\log 4/B_* \le 5.56/B_*$. Since $2D_*/B_* \le 2n(1+2\|v\|_\infty/g)/(4fn^2(1+\|v\|_\infty/g)) \le 1/2$ and $5.56\sum_{k\ge2}2^{-(k-1)}/k! < 2$, summing over $k \ge 2$ and $\tau \in \Psi$ gives

$$|R_N(F)| \le 2H_v s\,\mathrm{spr}(v). \tag{A.455}$$

By clause 1 of Definition 35.27, there is a common volume $V_0 > 0$ with $V_Z = V_0$ for every $Z \in \mathrm{Comp}$ and $V = (\#\,\mathrm{Comp})V_0 > 0$. Hence

$$\frac{1}{\#\,\mathrm{Comp}}\sum_{Z\in\mathrm{Comp}} a_Z(v;N) = \frac{1}{\#\,\mathrm{Comp}}\sum_{Z\in\mathrm{Comp}}\frac{P_{N,Z}(F)}{V_0H_v} = \frac{P_N(F)}{VH_v}.$$

Applying (35.11) with $h_{\mathrm{fl}}(F) \ge 0$, $(-AS(F) - bT(F))/H_v \ge -(K+4)\lambda_N\|v\|_1$, (A.455) and the absolute constant $C_T = K + 4 + 2 = 2C_0 + 17 = 2C_h + 21 \ge 2$ yields (8.26c). □

Proof of 35.33

Proof. Let $f : Y \to X$ be the modification of Lemma 35.12, including the normalisation. Every geometric fibre curve contracted by f has pullback degree zero. On a fibre curve mapping onto a curve the degree is the degree on its image multiplied by the degree of the map. Thus f^*A is nef on every geometric fibre. Its metric is smooth and semipositive by the Levi-form calculation in the lemma. Moriwaki's lower volume inequality applies on the projective flat integral model Y and gives $\widehat{\deg}(\widehat{c}_1(f^*A)^{d+1}) \le \widehat{\mathrm{vol}}(f^*A)$ [171, Cor. 6.4].

The arithmetic intersection on the left is $II(P; A^{d+1})$. Indeed the divisor recursion for the intersection system counts each vertical component with its multiplicity and its degree times $\log p$. On the complex fibre the recursion subtracts the integral of the log norm against the remaining Chern forms. This is the arithmetic divisor recursion in the unsquared norm convention. It determines the number by induction on dimension: for a possibly negative multidegree, tensor with a sufficiently positive coordinate line, use multilinearity, and use a coordinate monomial nonzero at the generic point in each resulting nonnegative degree. The arithmetic projection formula then gives the same number on Y, since $f_*[Y] = [X]$ [170, §3]. By Lemma 35.12 the volume on the right is the limit of the ordinary chart count in (8.27a). The normalisation uses the absolute arithmetic degree over $\mathbf{Z}$ and the arithmetic dimension $d + 1$. □

Proof of 35.34

Proof. Write $H_a = \sum_{i=1}^{a} 1/i$, with $H_0 = 0$. For a complex homogeneous polynomial g of degree k in N variables, let $m(g)$ be its spherical logarithmic mean. If u is a unit vector, then

$$\log |g(u)| \le m(g) + kH_{N-1}/2.$$

Indeed, choose u as the first coordinate vector. Conditional on the first coordinate t of a uniform sphere point, the remaining coordinates are uniform on the sphere of radius $\sqrt{1-|t|^2}$. Submean for the logarithm of the absolute value of a holomorphic function gives conditional mean at least $\log |g(tu)|$. For $N > 1$, the density of $|t|^2$ is $(N-1)(1-s)^{N-2}\,ds$ on $[0,1]$. Integration gives

$$\mathbb{E}[-\log |t|] = \frac{1}{2}\int_0^1 \frac{1-(1-s)^{N-1}}{s}\,ds = \frac{H_{N-1}}{2}.$$

For $N = 1$, $|t| = 1$. Submean may first be applied to regularised logarithms and then by decreasing convergence. Choosing a point of nonzero value gives finiteness of the mean. Iterating over the blocks gives the same inequality with k replaced by $e(G)$.

The substituted polynomial is nonzero because the w_i span. Choose a nonzero coefficient a. Its absolute value at a finite place is at most the local coefficient maximum. At an embedding, Cauchy's coefficient formula bounds $|\sigma(a)|$ by the supremum on the unit polydisc of the substitution variables. Each argument of G there has norm at most $n\Lambda$. Homogeneity and the sphere inequality bound the logarithm of that supremum by

$$m_\sigma(G) + e(G)\big(\log(n\Lambda) + H_{N-1}/2\big).$$

Sum over the places and use the product formula for a. Since $N \le n$, $H_{N-1} \le 1 + \log n$. This proves the lower bound. At a finite place the coefficient maximum is multiplicative. To see this, scale each polynomial to have integral coefficients and at least one unit coefficient, then multiply the nonzero reductions. At an embedding logarithms add almost everywhere and have finite integrals. This proves the product formula for h_0.

Let e be the exterior product of the chosen basis and write $\bigwedge^v M = \mathfrak{a}e$ for a fractional ideal $\mathfrak{a}$. Every coefficient of the substituted De is a sum of exterior products of vectors of M, hence belongs to $\mathfrak{a}e$. At every archimedean argument on the product of spheres, Hadamard's inequality gives

$$|\sigma(D)|\,\|e\|_\sigma \le \prod_{j=1}^{v} \|P_j\|_\sigma \le \prod_{j=1}^{v} L_j.$$

Integrate and sum. The finite contribution from $\mathfrak{a}$ and the archimedean contribution from $-\log \|e\|_\sigma$ sum to $\widehat{\deg} M$. A change of basis multiplies the scalar determinant by an element of $K^\times$, whose total logarithm is zero. □

Proof of 35.35

Proof. Top Chern proportionality gives $\deg_Z(v) = v_{0,Z}V(v)$, with $v_{0,Z} > 0$ proportional to the component volume. Volume equality gives the common Δ'_Z. The weight-gap criterion and (A.430) give a birational map for the image. Comparison of moving degrees by generic multiplicities gives $\deg_Z(u') \le \Delta_Z \le \deg_Z(u_1)$, with $\deg_Z(u')/\deg_Z(u_1) = 1-\rho$.

Put $T = T_*$ and $d = d_{\mathrm{fl}}$. The flat twist preserves rank. The finite-degree section count from the cut-ideal filtration on the generated Hodge model gives $\binom{T-1}{d}\Delta_Z \le r_Z$. For the other bound set $n_0 = T+d+3$ and $J_0 = \lfloor n_0/(1-\rho)\rfloor$. Then $n_0\deg_Z(u_1) < (J_0+1)\Delta_Z$. The adjacent gaps give the dominance comparisons of the weight with n_0u_1 for jet interpolation. Evaluation on J_0-jets at the generic fibre of degree Δ_Z is injective and bounds r_Z by $\binom{J_0+d}{d}\Delta_Z$. The smallest factor of $\binom{T-1}{d}/\binom{J_0+d}{d}$ is $(T-d)/(J_0+1)$. Since $J_0(1-\rho) \le T+d+3$, $d \ge 2$ and $J_0+1 \ge T$,

$$J_0 + d - T + 1 \le J_0\rho + 2d + 4 \le (J_0+1)(\rho + 4d/T).$$

Each factor thus has relative loss at most $\rho + 4d/T$. The product is at least $1 - d(4d/T+\rho) \ge 1 - z$. For the positive shifts $c_0, 2c_0$, use $T+d+4$ in the jet comparison. For $s = 1, 2$, removing the invertible twist gives the weight $v_s = u_W + sc_0\eta_H = u_M + (s+1)c_0\eta_H$. Both v_s and $(T+d+4)u_1 - v_s$ have weakly decreasing rows. The binomial bound at order $\lfloor (T+d+4)/(1-\rho)\rfloor$ gives component rank at most $(1+z)\mathcal{B}\Delta_Z$.
Lean: `GlobalTheory.CovTop.ShareRowsNef.nef_share_lower_bound (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.HodgeBundleWeights.ShareStatementOfKnownTheorems`, lines 32–54)
Lean: `GlobalTheory.CovTop.ShareRowsNef.nef_share_upper_bound (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.HodgeBundleWeights.ShareStatementOfKnownTheorems`, lines 55–91)

For the section degree put $h = h_Z$, $\delta = \Delta_Z$, $v = r_Z$, $B = \binom{T-1}{d}\delta$ and $\zeta = 1 - B/v$. By (8.29a), $0 \le \zeta \le z$. Use the component cusp lattices with their Petersson metrics. Let n_1 be the size of the integral generating frame for u_1, Λ an upper bound for its column norms, and β the Bergman upper comparison for multiplication by those sections. Thus multiplication by a linear combination of the frame columns has norm at most $\sqrt{\beta}$ times the Petersson norm of the combination. Take $n_1, \Lambda, \beta \ge 1$. Put $P = P_*$, $q = \kappa/T$ and

$$\begin{aligned} \Lambda_d = \Lambda_T = e^{P\kappa}, \quad J = dP(q+1), \quad g = (d+1)(\log n_1 + 1)/2 + 1, \\ H = \log\Lambda + 26\log\{n_1 d\delta(2 + |h| + \log\Lambda)\}, \end{aligned} \tag{A.456}$$

$$\log n_1,\ \log\beta,\ \log\delta,\ \log b \le P, \qquad \log\Lambda \le P(q+1), \qquad TP \le \sqrt{b}, \qquad zP\kappa \le b.$$

We first establish the unconditional *short-generator comparison* on each component Z for any nonincreasing row weight u with wall gap at least $8fn+10$ and $\log\|u\|_1 \le 10L_x$. Put $n = k+m-1$, $Z_0 = fnL_x$, $S = \|u\|_1$, and $m_0 = fn+1$; the wall gap is at least $8m_0+2$, so we form the minimal-level good cover with $8m_0$ theta columns and shifted row entries $u_{\tau k} + n_\tau$ where $n_\tau = 1 + 4m_0 - u_{\tau,p_{\mathrm{sig}}-1}$. Every shifted entry satisfies $|u_{\tau k}+n_\tau| \ge 4m_0+1$; set $\kappa_u = S' = \sum_{\tau,k}|u_{\tau k}+n_\tau| \le (1+n)S + nf(4m_0+1)$ (so $\kappa_u \ge (4m_0+1)fn > 4m_0fn$), and the resulting theta weight $(a, 8m_0)_\theta$ has harmonic degree $D = S' - 4m_0fn \le \kappa_u = S'$ and satisfies $fn(8m_0) + D = S' + 4m_0fn \le 2\kappa_u$ and $\log\kappa_u \le 20L_x$. For each unimodular definite Hermitian $\mathcal{O}_E$-lattice $L_a \subset E^{8m_0}$ standard at the bad primes and M', set $L_n = L_a \cap \mathcal{O}_E^{8m_0} \subseteq L_a$, let $\mathfrak{I}_\tau(L) \subset F'$ be the fractional ideal of $\mathcal{O}_{F'}$ generated by $\tau(\det)$ of $8m_0$-tuples in L, and let $m_\tau \in \{p_{\mathrm{sig}}, q_{\mathrm{sig}}\}$ (with $m_\tau + m_{\bar\tau} = n$). Because $L_n \subseteq L_a$, the $\mathcal{O}_E$-index ideal $\mathfrak{a} = [L_a : L_n]_{\mathcal{O}_E} \subseteq \mathcal{O}_E$ is an integral ideal of norm $N_{E/\mathbf{Q}}(\mathfrak{a}) = [L_a : L_n]$ satisfying $\mathfrak{I}_\tau(L_n) = \tau(\mathfrak{a})\mathfrak{I}_\tau(L_a)$, so the reflex-field determinant ideal $\mathfrak{a}(L_n, L_a) = \prod_\tau \mathfrak{I}_\tau(L_n)^{m_\tau}\mathfrak{I}_\tau(L_a)^{-m_\tau} = \prod_\tau \tau(\mathfrak{a})^{m_\tau} \subseteq \mathcal{O}_{F'}$ is an integral ideal of norm $N_{F'/\mathbf{Q}}(\mathfrak{a}(L_n,L_a))^{1/[F':\mathbf{Q}]} = [L_a : L_n]^{n/2}$. Applying Minkowski's theorem to the fractional inverse ideal $\mathfrak{a}(L_n,L_a)^{-1}$ over F' (with $o = \mathrm{size}(F') = \log[F':\mathbf{Q}] + \log\mathrm{rd}(F') \le 391Z_0^2$ from Lemma 35.21) yields a nonzero fractional scalar $s \in \mathfrak{a}(L_n,L_a)^{-1} \subset F'$ satisfying $(s)\mathfrak{a}(L_n,L_a) \subseteq \mathcal{O}_{F'}$ (equivalently $(s)\prod_\tau \mathfrak{I}_\tau(L_n)^{m_\tau} \subseteq \prod_\tau \mathfrak{I}_\tau(L_a)^{m_\tau}$) and $|\sigma(s)|^2[L_a : L_n]^n \le e^{2o}$ for every embedding $\sigma : F' \hookrightarrow \mathbf{C}$. On the minimal-level good cover, the integral spanning family of theta weight $(a, 8m_0)_\theta$ is constructed unconditionally in three steps: (1) put $N_\theta = 8m_0 = 8fn+8$; because N_θ is even and satisfies $N_\theta \ge 4[E:\mathbf{Q}]n + 4 = 8fn+4$ (in particular $N_\theta > 4n$), the harmonic theta series $\theta_{L_a,x_0,P}$ over the finite genus of unimodular definite Hermitian $\mathcal{O}_E$-lattices $L_a \subset E^{N_\theta}$ standard at M' and the bad primes, with level-M' congruence characteristics $x_0 : \{1,\dots,N_\theta\} \to Q$ into the rank-n conjugate half-level quotient Q of cardinality $(M')^{fn}$ (as in Lemma 35.59) and harmonic polynomials $P \in \mathrm{harm}(L_a, a)$, span the level-$M'$ weight-$(a,N_\theta)_\theta$ cusp space unconditionally without invoking Theorem 34.17. Let $\Gamma_{\mathbf{Q}}(M')$ be the group of E-linear isometries of V lying in $\prod_{\ell\mid M'} K_{M',\ell}$, the rational level-$M'$ congruence subgroup preserving the level-M' self-dual lattice

family $\mathcal{L}$. On the doubled signature-(n,n) Hermitian space, the definite Hermitian Siegel–Weil identity in the convergent range $N_\theta > 4n$ equates the Siegel Eisenstein series of the doubled level-M' Schwartz–Bruhat test functions $\Phi_{L_1,L_a} \otimes \overline{\Phi_{L_2,L_a}}$ with the genus-weighted sum of doubled theta series, where the level test function Φ_{L,L_a} is defined as follows. Write $\mathfrak{m}$ for the half-level ideal, $J_{\mathrm{lev}} \le \mathrm{GL}_n(\mathcal{O}_E/\mathfrak{m})$ for the finite level group defining $\Gamma_{\mathbf{Q}}(M')$, λ_{lev} for its level character, and $L^\sharp = \{u : \bar{\mathfrak{m}}u \subseteq \mathfrak{m}L\}$. For a lattice $B_0 \subset E^n$, write $L_a \otimes B_0$ for the matrix lattice generated by vu^{T} with $v \in L_a$ and $u \in B_0$. Project the reference vacuum for L to the vacuum line for $L_a \otimes L^\sharp$ and normalize it to unit norm. In $(L_a \otimes 2\bar{\mathfrak{m}}L)/(L_a \otimes 2\bar{\mathfrak{m}}\mathfrak{m}L)$, take one representative of each class for which there is a $j \in J_{\mathrm{lev}}$ with a lift $\widetilde{j} \in \mathrm{Mat}_n(\mathcal{O}_E)$ such that row i is congruent to $\sum_{c=1}^n \widetilde{j}_{ci} b_c$ modulo $\mathfrak{m}$ for $1 \le i \le n$ (where $b_1, \dots, b_n$ is the level basis) and every remaining row is congruent to zero modulo $\mathfrak{m}$. Sum the Heisenberg translates of this unit vacuum by those representatives, weighting each by $\lambda_{\mathrm{lev}}(j)$.

Lean: `GlobalTheory.SW.Level.phi` (`Development.AbelianVarietyTheory.Definitions.SpanningFamilies.IsSystem`, lines 213–218)

Every maximal isotropic subspace of the doubled space is either the graph Γ_γ of an E-linear isometry γ of V or contains the image of a nonzero isotropic vector ξ of V. For $\gamma \notin \Gamma_{\mathbf{Q}}(M')$ the Heisenberg translates occurring in Φ_{L_1,L_a} and in the Weil translate of Φ_{L_2,L_a} by γ are pairwise orthogonal, because two of them pair nontrivially only when their level classes correspond under γ, which would put γ in $\Gamma_{\mathbf{Q}}(M')$. For a nonzero isotropic ξ, restrict to the subgroup of vectors v with $v\xi^{\mathsf{T}} \in L_a \otimes L^\sharp$. Its Heisenberg operators act on each translated vacuum in the support by a character. Invertibility of the first residue block makes each character nontrivial. For each character choose a translation on which its value differs from one, and subtract that value times the identity from the translation operator. The product of these commuting differences annihilates Φ_{L,L_a} and has nonzero coefficient sum. Let γ represent the nongraph maximal isotropic space. The vector $(\xi, 0)$, expressed in the doubling frame, belongs to that space. Consequently the position part of $(v\xi^{\mathsf{T}}, 0)\gamma^{-1}$ is zero for every v. Heisenberg covariance therefore turns each of the preceding translations into a pure modulation after applying the doubled Weil operator. In Schrödinger coordinates such an operator has the form $f(z) \mapsto \psi_0(\langle z, b_v\rangle) f(z)$, whose value at $z = 0$ is $f(0)$. If ℓ_γ denotes evaluation at zero after this Weil transform, then $\ell_\gamma T_v = \ell_\gamma$. Applying ℓ_γ to the annihilating relation gives

$$0 = \ell_\gamma\left(\sum_v a_v T_v \Phi\right) = \left(\sum_v a_v\right)\ell_\gamma(\Phi).$$

The coefficient sum is nonzero, so the nongraph term vanishes.

Lean: `GlobalTheory.SW.Coinv.phi_no_coinvariants` (`Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.ThetaTestFunctionNoCoinvariantsIsotropic`, lines 216–275)

The Eisenstein series is therefore the sum over the graphs Γ_γ with $\gamma \in \Gamma_{\mathbf{Q}}(M')$. Choose row-highest harmonic polynomials P_j of the required weight, not all zero, whose pairing sum is invariant under the unitary action on the columns. The zero finite Schwartz factor kills each nongraph term for all auxiliary variables. The difference of the two sides solves the doubled heat equation and vanishes at zero auxiliary variables for every period matrix. The invariant harmonic bidifferential operator therefore annihilates this difference at the origin. The graph side is the harmonic kernel sum weighted by the automorphy multiplier and $C(L_1, \gamma L_2) = \sum_{[L_a]} w(L_a)\langle \Phi_{L_1,L_a}, \Phi_{\gamma L_2,L_a}\rangle$, up to a common nonzero scalar. The theta side is the genus-weighted sum over $[L_a]$ of $\sum_j \theta_{P_j,\Phi_{L_1,L_a}}(x_1)\overline{\theta_{P_j,\Phi_{L_2,L_a}}(x_2)}$, where the second subscript specifies the test function.

Lean: `GlobalTheory.SW.exists_invFam_rowHighest` (`Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.ExistsFamilyInvariantOfFockInner`, lines 50–62)

Lean: `GlobalTheory.SW.kernel_identity` (`Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.KernelIdentity`, lines 181–341)

For the overlap estimate put $A = L_a \otimes L^\sharp$ and $A' = L_a \otimes (L')^\sharp$, and let $v_A, v_{A'}$ be their unit vacua with the prescribed semicharacters. The distinct semicharacter-corrected translates of $v_{A'}$ by $A/(A \cap A')$ are orthonormal. Their average is the projection onto the A-vacuum line. Taking its squared norm gives

$$|\langle v_A, v_{A'}\rangle|^2 = [A : A \cap A']^{-1}.$$

For $m = [L : L \cap L'] = [L' : L \cap L']$, tensoring with the rank-N_θ unimodular lattice gives $[A : A \cap A'] = m^{N_\theta}$; the half-level change preserves this relative index. Choose the same residue representatives for the two level

tests. Cross terms between distinct representatives vanish by their Heisenberg characters, and each diagonal term has the vacuum pairing above, since its level-character factor has absolute value one. Thus

$$\|\Phi_{L,L_a}\|^2 = |J_{\mathrm{lev}}|, \qquad |\langle \Phi_{L,L_a}, \Phi_{L',L_a}\rangle| \le |J_{\mathrm{lev}}| m^{-N_\theta/2}.$$

Summing with the positive genus weights gives the asserted bound $C(L,L)m^{-N_\theta/2}$. For $L' \neq L$ one has $m \ge 2$, and at most $m^{2[E:\mathbf{Q}]n}$ self-dual lattices have that relative index, so $N_\theta \ge 4[E:\mathbf{Q}]n + 4$ bounds the off-diagonal tail sum by $\sum_{m\ge 2} m^{2[E:\mathbf{Q}]n - N_\theta/2} \le \sum_{m\ge 2} m^{-2} \le 0.65 < 1$, and every diagonal coefficient $C(L,L)$ equals the same positive constant, since all level test functions have the same norm. So C is that constant times an operator whose off-diagonal row sums are at most 0.65, and C is invertible on bounded families by a convergent Neumann series. Given a bounded cusp form F, the Neumann series gives bounded holomorphic forms $F'(L',\cdot)$ of the same weight with $F(L,x) = \sum_{L'\in\mathcal{L}} C(L,L')F'(L',x)$. We compute the integral that recovers each of these forms from the doubled Gaussian kernel.

Claim. Write $r = p_0 + q_0$ for the frame rank and N for the number of theta columns, so here $r = n$ and $N = N_\theta$. Fix the CM frame datum, with the g coordinates indexed by $\Psi \times \{0,\dots,r-1\}$. Suppose that N is even and $2r \le N$. Take finitely many harmonic polynomials P_j in these N columns, at least one nonzero, all row-highest for the same pair of nonnegative integral weight data (c,c') on the q_0 and p_0 rows, respectively, at each $\tau \in \Psi$. Put $C_{\tau i} = \sum_{j\ge i} c_{\tau j}$ and $C'_{\tau i} = \sum_{j\ge i} c'_{\tau j}$. In zero-based row indices the corresponding frame weight is

$$\lambda_{\tau,p_0+i} = C_{\tau i} + N/2 \quad (0 \le i < q_0), \qquad \lambda_{\tau,p_0-1-i} = -C'_{\tau i} - N/2 \quad (0 \le i < p_0).$$

Let $\chi_\lambda(b) = \prod_{\tau,i} (b_{\tau,ii})^{\lambda_{\tau i}}$ on the upper triangular Borel group. Apply $\sum_j P_j(\partial)\overline{P_j}(\partial)$ at the origin in the auxiliary variables of the doubled Gaussian to obtain $K(x,y)$. Let S_δ be the set of frames whose Gram matrices differ entrywise from the signature matrix by less than δ, with Lebesgue measure dy. There is a $\delta_* > 0$ such that for each $0 < \delta \le \delta_*$ there is a real $\rho > 0$ with

$$\int_{S_\delta} K(x,y)G(y)\,dy = \rho G(x)$$

for every frame x in the domain and every function G holomorphic there with $G(yb) = \chi_\lambda(b)G(y)$ and bounded Petersson norm of weight λ throughout the domain.

Choose δ_* to be the smaller of the thresholds for the thin-slab decomposition and for Gaussian integrability. For every such δ, the slab is open in the ambient frame space, lies in the domain, and contains all orthogonal frames. Left unitary transformations preserve the domain, the slab and its measure. Every domain frame is of the form $y'b$ with y' orthogonal and b upper triangular. On the slab one can choose b with its entries and inverse diagonal entries uniformly bounded. Thus $|\chi_\lambda(b)|$ is bounded for these choices. At an orthogonal frame all leading Gram minors have absolute value one, so the Petersson bound there bounds $|G(y')|$. Equivariance now bounds $|G|$ on the slab. For each domain frame x, the Gaussian estimate gives $K(x,\cdot) \in L^1(S_\delta) \cap L^\infty(S_\delta)$, so the integral converges absolutely.

Fix an orthogonal frame x_0. The kernel is invariant under simultaneous left unitary transformations and satisfies $K(y,x) = \overline{K(x,y)}$. In its first variable it is holomorphic of right Borel weight λ. Its diagonal value $K(x_0,x_0)$ is real and strictly positive because some P_j is nonzero. For a diagonal unitary matrix t at each embedding, put $U_t = x_0 t x_0^{-1}$. Then U_t is unitary and $U_t x_0 = x_0 t$. Kernel invariance and the first-variable weight give $K(x_0, U_t y) = \chi_\lambda(t)^{-1} K(x_0,y)$. Average over this compact diagonal torus with Haar probability measure and set

$$H(y) = \int \chi_\lambda(t)^{-1} G(U_t y)\,dt.$$

The average is holomorphic by compactness and retains the right Borel weight because left and right multiplication commute. Translation of Haar measure gives $H(U_t y) = \chi_\lambda(t)H(y)$, and $H(x_0) = G(x_0)$. For each t, change variables by U_t in the slab integral. The transformation of $K(x_0,\cdot)$ shows that replacing G by H leaves the integral unchanged. Fubini applies because $|\chi_\lambda(t)| = 1$, G is bounded on the slab and $K(x_0,\cdot)$ is integrable there.

The function $y \mapsto K(y,x_0)$ has the same left torus character as H, by Hermitian symmetry, and the same right Borel character. A holomorphic function with these two characters is determined by its value at x_0. To see this, let M have both characters and vanish at x_0. On the dense set where the leading minors of $x_0^{-1}y$ are nonzero, right triangular elimination reduces the assertion to $y = x_0 L$ with L lower triangular.

In each block replace L_{ij} for $i \geq j$ by $z^{i-j}L_{ij}$. For $0 < |z| \leq 1$ this conjugates L by $\mathrm{diag}(z^i)$. The resulting frame stays in the domain: left scaling multiplies the positive rows in absolute value by at least $|z|^{p_0}$ and the negative rows by at most $|z|^{p_0}$, preserving positivity on the first p_0 columns, and the right diagonal scaling preserves the flag. At $z = 0$ the frame is $x_0\,\mathrm{diag}(L)$. On $|z| = 1$ the two characters cancel, so the value of M is constant on the circle. Its value at the centre is zero by right equivariance. The mean-value formula gives $M(y) = 0$, and continuity extends this to every frame. Applying this to the difference of the two functions after scaling by their values at x_0 gives

$$H(y)K(x_0, x_0) = K(y, x_0)G(x_0).$$

Multiplying by $K(x_0, y)$ and integrating proves the identity at x_0 with

$$\rho = \frac{\displaystyle\int_{S_\delta} |K(x_0, y)|^2\, dy}{K(x_0, x_0)}.$$

The numerator is finite by boundedness and integrability. It is positive because the continuous integrand is positive at x_0 and the slab contains an open neighbourhood of x_0.

Finally write any domain frame as $x = x'b$ with x' orthogonal. The matrix $u = x'x_0^{-1}$ is unitary because the two frames have the same Gram matrices. Apply the identity at x_0 to $y \mapsto G(uy)$, which has the same weight and Petersson bound. Changing variables by u gives the identity at x'. Right multiplication by b multiplies both sides by $\chi_\lambda(b)$, proving the claim at x with the same ρ.

Lean: `GlobalTheory.ThFr.wtOf` (`Development.AbelianVarietyTheory.Definitions.ThetaValues.PolyDerivSmulLetter`, lines 208–210)
Lean: `GlobalTheory.SW.thin_slab` (`Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.MeasurePreservingMulLeft`, lines 280–426)
Lean: `GlobalTheory.T.gaussianKernel_integrableOn_thin_slab` (`Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.GaussianKernelIntegrableOnThinSlabStatement`, lines 8–9)
Lean: `GlobalTheory.SW.kern_invariant_symm_pos` (`Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.KernInvariantSymmPos`, lines 431–435)
Lean: `GlobalTheory.T.realPlaceKernel_holomorphic_weight_rowHighest` (`Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.RealPlaceKernelHolomorphicWeightRowHighestStatement`, lines 8–9)
Lean: `GlobalTheory.SW.Repro.pnorm_ortho` (`Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.SetIntegralCompOfStable`, lines 149–154)
Lean: `GlobalTheory.SW.Repro.exists_bchar_bound` (`Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.SetIntegralCompOfStable`, lines 136–147)
Lean: `GlobalTheory.SW.Repro.setIntegral_comp_of_stable` (`Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.SetIntegralCompOfStable`, lines 16–22)
Lean: `GlobalTheory.SW.Repro.exists_mean` (`Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.SetIntegralCompOfStable`, lines 24–103)
Lean: `GlobalTheory.lowest_weight_function_mul_eq` (`Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.LowestWeightFunctionMulEq`, lines 315–333)
Lean: `GlobalTheory.SW.Repro.integral_mul_ortho` (`Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.SetIntegralCompOfStable`, lines 162–260)
Lean: `GlobalTheory.SW.kern_reproducing` (`Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.SetIntegralCompOfStable`, lines 268–344)

In particular $\rho F'(L', x) = \int_{S_\delta} K(x, y)F'(L', y)\, dy$, and the bound for F' on S_δ together with the summability of $\sum_{L' \neq L} |C(L, L')|$ makes $\sum_{L'} |C(L, L')| \int_{S_\delta} |K(x, y)F'(L', y)|\, dy$ finite. Choosing one representative L_0 from each of the finitely many $\Gamma_{\mathbf{Q}}(M')$-orbits in $\mathcal{L}$ and a measurable fundamental domain $D(L_0) \subset S_\delta$ for its discrete arithmetic stabiliser $\Delta_{L_0} = \mathrm{Stab}_{\Gamma_{\mathbf{Q}}(M')}(L_0)$ gives a fundamental domain for the diagonal action of $\Gamma_{\mathbf{Q}}(M')$ on $\mathcal{L} \times S_\delta$; unfolding $\rho F(L_1, x_1) = \sum_{L' \in \mathcal{L}} C(L_1, L') \int_{S_\delta} K(x_1, y)F'(L', y)\, dy$ across all orbit representatives $[L_0]$ into $\sum_{[L_0]} \int_{D(L_0)} \big(\Gamma_{\mathbf{Q}}(M')\text{-averaged doubled kernel}\big)\, F'(L_0, y)\, dy$ (whose unitary automorphy factor cancels against the kernel multiplier) and substituting the doubled kernel identity expresses F as a finite $\mathbf{C}$-linear combination of the harmonic theta series $\theta_{L_a, x_0, P}$ (and converting from lower-row-zero to free-lower-row level-M' test functions preserves this finite span); since both the cusp space and these theta generators are defined over F', the same finite family spans the level-M' weight-$(a, 8m_0)_\theta$ cusp space over F'; the integral minor-product construction for the unimodular lattice L_a yields a short spanning family in $\mathrm{harm}(L_a, a)$ with $\sum_\nu |\sigma(P_\nu)| \leq \exp(140nL'D)$ (for $L' \leq 50L_x$ and $D \leq \kappa_u$), and multiplying by $\sqrt{[F' : \mathbf{Q}]} \leq e^{o/2}$ (with $o \leq 391Z_0^2$) bounds the spanning polynomial norm by $\rho \leq \exp(C_A Z_0^2 \kappa_u)$ with

$C_A = 20000$; (2) the pointwise estimate at an orthonormal frame for the moving lattice L_a contributes $[L_a : L_n]^{n/2}$. The frame comparison then bounds the normalised theta norm at every domain frame by $[L_a : L_n]^{n/2}\rho\exp(9000Z_0^2\kappa_u)$. Lemma 35.40 gives cuspidality. Apply the norm estimate of Lemma 35.44 to the standard-lattice series, with $r = n$, $N = 8m_0$, $L = L_x$, $\kappa = \kappa_u$, and the coefficient bound ρ above; here $n \le 8m_0$, $D \le \kappa_u$, $fn(8m_0) \le 2\kappa_u$, and the level and field bounds are those of the good cover. The moving-lattice factor is the displayed $[L_a : L_n]^{n/2}$. These estimates (explicitly absorbing $(2\pi\xi_{\mathbf{R}})^{D/2}$, $\log\kappa_u \le 20L_x$, and the Minkowski factor $|\sigma(s)|[L_a : L_n]^{n/2} \le e^o \le e^{391Z_0^2}$) bound the raw archimedean supremum and Petersson norms of $s\cdot\theta_{L_a,x_0,P}$ by $e^o\rho\exp(C_BZ_0^2\kappa_u)$ for an absolute constant $C_B \ge 0$; and (3) set $Z_{\mathrm{int}} = (M')^{fn(8m_0)+D}|D_E|^D$ (where $(M')^{fn(8m_0)}$ clears the finite level-M' Fourier-indicator denominators and $(M')^D|D_E|^D$ clears the degree-D PEL formal-group differential and different denominators) and put $N_a = Z_{\mathrm{int}}^2$ (at $\ell^2 \mid M'$ the standard local lattice gives $v(\theta_{L_a,x_0,P}) \ge -2v(Z_{\mathrm{int}})$, while one factor suffices at $\ell \parallel M'$ and, after scaling by s, at $\ell \nmid M'$). In (2) and (3), $\theta_{L_a,x_0,P}$ is the raw theta series, namely the algebraic short section multiplied by the nonzero lattice scalar whose inverse occurs in its evaluation formula. The scalar s enters as follows. At a place of F' over a prime $\ell \mid M'$ the lattice L_a is standard at ℓ, and the inclusion $(s)\prod_\tau \mathfrak{I}_\tau(L_n)^{m_\tau} \subseteq \prod_\tau \mathfrak{I}_\tau(L_a)^{m_\tau}$ makes s integral there. At a place over a prime $\ell \nmid M'$, which includes every prime of $[L_a : L_n]$ outside $M'D_E$, the value of $\theta_{L_a,x_0,P}$ is computed from the standard lattice through an adapted integral matrix p with $pq = kN\gamma$ for an integral q, a unitary γ carrying L_a to the standard lattice up to the integer k prime to ℓ, and the level N of L_a; the adapted matrices also satisfy the determinant-ideal identity

$$v_w(\tau(\det p)) = v_w(\mathfrak{I}_\tau(L_n)) - v_w(\mathfrak{I}_\tau(L_a)).$$

Indeed, locally choose a basis matrix g of L_n and a matrix γ carrying L_a onto the standard lattice. The adapted construction has $\det p = \det g\,\det\gamma$ up to factors prime to ℓ. The two determinant ideals have orders $v_w(\tau(\det g))$ and $-v_w(\tau(\det\gamma))$, respectively. This proves the identity. Summing with multiplicities m_τ, the ideal inclusion for s gives $v_w(s)+\sum_\tau m_\tau v_w(\tau(\det p)) \ge 0$. The value is a combination, with integral weights, of terms evaluated at torsion points killed by a power of the level ideal. In each term the level characteristic is evaluated at torsion points whose orders divide a power of M' and hence are prime to ℓ, and the term also involves translations by the auxiliary isotropic group. With the prime-to-ℓ characteristic fixed, split the auxiliary isotropic group of lattice level N into its ℓ-primary and prime-to-ℓ factors. Fix a prime-to-ℓ translate and write J_ℓ for the primary kernel of the adapted isogeny pq. Pull back by the prime-to-ℓ integer k, and write $\widetilde{F}_j$ for the resulting translated sections. The primary-kernel trace identity is

$$(kN)^{-D}\sum_{j\in J_\ell}\mathcal{H}(\widetilde{F}_j;P) = \#(J_\ell\cap\ker q)\left(\prod_\tau \tau(\det p)^{m_\tau}\right)z, \qquad v(z) \ge -\rho_{\mathrm{diff}}.$$

For this identity, the normalized base section is fixed by the primary N-torsion. In the trace through q, each fibre therefore contributes $\#(J_\ell\cap\ker q)$ copies of the same section. Descent through p changes its rigidified determinant frame by $\prod_\tau\tau(\det p)^{m_\tau}$. The chain rule on the homogeneous polynomial gives the factor $(kN)^D$. The transformed polynomial is integral in the standard lattice frame; the integral differential-chart bound gives $v(z) \ge -\rho_{\mathrm{diff}}$.

Put $m_\Lambda = [L_n : N\mathcal{O}_E^{N_\theta}]^n$, the averaging index in the short-value formula. Computing the kernel and lattice indices in Smith bases gives

$$\frac{\#(J_\ell\cap\ker q)}{m_\Lambda} \in \mathcal{O}_w^\times.$$

Indeed both numerator and denominator have the same ℓ-primary elementary factors; their remaining factors have order prime to ℓ. Pullback by k multiplies the harmonic value by its weight norm, also an ℓ-unit. The normalized original inner sum is consequently a unit times $\prod_\tau\tau(\det p)^{m_\tau}z$. Since $s\prod_\tau\tau(\det p)^{m_\tau}$ is integral, its valuation after multiplication by s is at least $-\rho_{\mathrm{diff}}$, independently of the primary elementary factors of the moving lattice. The factor $|D_E|^D$ covers this loss, and the level-cardinality factor in Z_{int} clears the denominator of the weighted average. Thus $sZ_{\mathrm{int}}\theta_{L_a,x_0,P}$ is integral at the place.

Lean: `GlobalTheory.T.shortSeries_bound_coprime_torsion_mul` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ShortSeriesBoundCoprimeTorsionMulStatement`, lines 15–16)

Lean: `GlobalTheory.CovInt.smul_intZ_nonneg_coprime` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ClearingIntegerShortSeriesIntegralCoprimeLevel`, lines 30–159)

Using $fn(8m_0)+D \le 2\kappa_u$, $\log M' \le 30L_x$, and $\log|D_E| \le 7fL_x$ gives $\log Z_{\mathrm{int}} \le (60+7f)L_x\kappa_u \le 67Z_0^2\kappa_u$ and hence $\log N_a = 2\log Z_{\mathrm{int}} \le C_C Z_0^2\kappa_u$ with $C_C = 134$, placing every scaled series $N_a s \cdot \theta_{L_a,x_0,P}$ inside the integral cusp lattice with log norm at most $C_G Z_0^2 S'$ for $C_G = C_A + C_B + C_C + 391$. Multiplying this minimal-level spanning set by the key-unit twisting section $t_1 = K_{\mathrm{key}}^{m_0}$ of Lemma 35.4 (exact complex norm $(2\pi)^{2fn(fn+1)}$ and nonnegative finite-place lower valuations) and the flat-line twisting section t_2 of row $(-n_\tau)_\tau$ on Z (governed by $|c_1| + q_{\mathrm{sig}}\log \mathrm{rd}(E) + \sum_{\ell\in\mathcal{B}}\log\ell + \frac{\log|D_{F'}|}{2[F':\mathbf{Q}]} + 1$ with bad-prime rounding loss $\sum_{\ell\in\mathcal{B}}\log\ell \le 37L_x$ by Lemmas 35.10 and 35.21) produces an integral spanning family over F' of row weight $(u,0)_{\mathrm{row}}$ with log norm bound at most $C_2(Z_0^3 S + Z_0^5)$ ($C_2 = 11C_G + 537$). Applying Lemma 35.36 over F' (with $\log D_F + \log \mathrm{rd}(F') \le 391Z_0^2$ for $D_F = [F' : \mathbf{Q}]$) together with the component Bergman rank bound $\log v_u \le 4000Z_0^3$ of Lemma 35.22 (at weight scale $w = 2[E:\mathbf{Q}]S + 1$, where $\log w \le 22L_x$ so $L' = 11L_x$ gives $\log C_{\sup} \le 250f^2n^2(11L_x) = 2750f^2n^2L_x \le 4000Z_0^3$) produces an integral generating frame of size at most $2D_F v_u$ and Petersson log radius at most $c_s(u) = C_g(Z_0^3S + Z_0^5)$ ($C_g = 6000 + C_2$).

Applying this short-generator comparison at u_1 (with $P = P_* = C_P Z_0^{e_P}$, $e_P \ge 12$, and $\|u_1\|_1 \le \kappa/T = q$ up to bounded wall shifts) yields an integral generating frame for u_1 of size $n_1 \le 2D_F\,\mathrm{rk}(u_1)$ and Petersson norm bound Λ satisfying $\log n_1 \le P$ and $\log\Lambda \le P(q+1)$. For the upper Bergman comparison β, the component supremum-to-Petersson inequality $\|f\|_{\sup}^2 \le C_{\sup}\|f\|_{2,Z}^2$ of Lemma 35.22 gives $\log\beta \le \log C_{\sup} \le P$. The Weyl degree formula bounds $\log\delta \le P$, and (A.433), (A.434), and (A.435) give $\log b \le P$, $TP \le \sqrt{b}$, and $zP\kappa \le b$ (since d^2/T and $d\rho$ decay in $1/P$ faster than κ/b), verifying (A.456). Denote the canonical flag weight by w_{can}. Use the weights $w_j = w_0 + ju_1$ for $0 \le j \le T$, with $w_0 = w_{\mathrm{can}} + u_2 + du_1$. The terminal weight w_T gives the untwisted lattice of degree D_0 and rank v; applying the same short-generator comparison at degrees w_0, w_d, w_T yields integral spanning families of radii $e^J, \Lambda_d, \Lambda_T$, respectively.

Lean: `GlobalTheory.HSShortRun.hilbertSamuel_estimate_at_level` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.HodgeBundleWeights.ExistsDegreeParametersBound`, lines 33–76)

Lean: `GlobalTheory.HSShortRun.hsClaim_short_alt_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ExistsDegreeParametersBound`, lines 34–70)

Take a transverse cut of the birational image by d linear sections $\ell_1, \dots, \ell_d$ in the span of the frame for u_1. Work over a splitting field of the cut points $y_1, \dots, y_\delta$. The degree-zero restriction is onto their fibres by the nef and big adjoint restriction comparison. Choose δ degree-zero sections $s_1, \dots, s_\delta$ from the short spanning family whose restrictions form a basis. If I is the cut ideal, the symbol at order j has kernel the sections of order at least $j+1$. It sends a product with a cutting monomial of degree j to its normal monomial times the restriction of the coefficient to the cut.

For the deepest step, pass by principalisation (Definition 35.5(27)) to a smooth projective log resolution $\pi : \widehat{Z} \to \widetilde{Z}$ principalising the base ideals of both u_1 and u_2 into effective Cartier divisors B_1, B_2 supported off the basepoint-free open locus U containing the cut $\{y_1, \dots, y_\delta\}$, yielding basepoint-free moving bundles $L = \pi^*L(u_1)(-B_1)$ and $M_2 = \pi^*L(u_2)(-B_2)$. Set $L_A = \pi^*L(u_2 + du_1)$ (nef and big), $M = K_{\widehat{Z}} \otimes L_A$, and $A_i = H^0(\widehat{Z}, M \otimes L^{\otimes i})$ for $0 \le i \le T$. Multiplication by the exact power $s_{B_1}^i$ of the canonical section of $\mathcal{O}(B_1)$ embeds A_i into the cusp section space V_i of weight w_i ($0 \le i \le T$), compatibly with cutting sections $\ell \mapsto s_{B_1}\ell$ (with $A_0 = V_0$ at $i = 0$). Meanwhile, multiplication by $s_{B_2}s_{B_1}^d$ (which is invertible on U, hence at every point of $\{y_1, \dots, y_\delta\}$) embeds the nef-and-big moving adjoint space $H^0(\widehat{Z}, K_{\widehat{Z}} \otimes M_2 \otimes L^{\otimes d})$ into A_0, giving surjectivity of the degree-0 residue map $A_0 \twoheadrightarrow \bigoplus_{p=1}^{\delta} M|_{y_p}$. On the smooth projective variety $\widehat{Z}$, basepoint-freeness of L ensures that a generic d-tuple $(\ell_1, \dots, \ell_d) \subset H^0(\widehat{Z}, L)$ cuts out the reduced transverse finite locus $\{y_1, \dots, y_\delta\}$ cleanly and forms a regular sequence everywhere on $\widehat{Z}$, so the Koszul complex on $(\ell_1, \dots, \ell_d)$ resolves the ideal sheaf of the cut. For every $i \ge d$ and every $1 \le p \le d$, the bundle $M \otimes L^{\otimes(i-p)} = K_{\widehat{Z}} \otimes L_A \otimes L^{\otimes(i-p)}$ is the canonical bundle twisted by a nef-and-big line bundle (since L_A is nef and big and L is nef with $i - p \ge 0$), so Kawamata–Viehweg vanishing (Definition 35.5(19)) kills $H^q(\widehat{Z}, M \otimes L^{\otimes(i-p)})$ for all $q > 0$. Exactness of the Koszul complex on global sections therefore yields that every $s \in A_i$ vanishing on the cut lies in $\sum_{j=1}^{d} \ell_j A_{i-1}$. Induction on a then shows that every $s \in A_i$ vanishing to order at least a along the cut, with $i - a \ge d$, is a sum $\sum_{|\alpha|=a} \ell^\alpha m_\alpha$ with $m_\alpha \in A_{i-a}$. Indeed, if s vanishes to order $a+1$, the case a gives $s = \sum_{|\alpha|=a} \ell^\alpha m_\alpha$; the order-$a$ symbol of s is zero and equals the tuple of restrictions $(m_\alpha|_{\{y_1,\dots,y_\delta\}})_\alpha$, since the normal monomials of degree a are independent; so each m_α vanishes on the cut, and the kernel identity writes $m_\alpha = \sum_j \ell_j m_{\alpha,j}$ with $m_{\alpha,j} \in A_{i-a-1}$. Put $W = \sum_{|\alpha|=T-d} \ell^\alpha V_d \subseteq V_T$, which contains $\sum_{|\alpha|=T-d} \ell^\alpha A_d$ and vanishes to order at least $T - d$ along the cut. Let the deepest filtration

step U_{T-d} be the space of sections in $A_T + W$ vanishing to order at least $T-d$ along the cut. A section $g+w$ with $g \in A_T$ and $w \in W$ lies in U_{T-d} exactly when g vanishes to that order, and then the case $i = T$, $a = T-d$ puts g in W. Hence $U_{T-d} = W$ is spanned by products of degree-d sections in V_d with cutting monomials of degree $T-d$. These restriction and symbol maps commute with base change to the splitting field.

Lean: `GlobalTheory.HSShort.flatAt_of_simpleAt` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.HilbertSamuelFiltrationAtGoodSections`, lines 27–76)
Lean: `GlobalTheory.HSCore.filMono_of_kerRes` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FilMonoOfKerRes`, lines 24–58)

Choose a cut where the deepest step $U_{T-d} = W \subseteq V_T$ has maximum rank $n_d = \dim U_{T-d}$. Partition the v columns of our matrix into three groups: Group 1 consists of the B symbol columns $s_i\, \ell_1^{\alpha_1} \cdots \ell_d^{\alpha_d} \ell^{T-|\alpha|} \in A_T$ ($|\alpha| < T-d$, with ℓ a variable last linear section), whose order-a symbols ($a = |\alpha|$) form a block-diagonal matrix with $\binom{a+d-1}{d-1}$ copies of the restriction matrix B_a multiplied on the left by $\mathrm{diag}(\ell(y_p)^{T-a})$; Group 2 consists of n_d independent ℓ-independent products of members of the short spanning family of V_d with degree-$(T-d)$ cutting monomials; these products span $W = U_{T-d}$, so a basis of U_{T-d} can be chosen among them, and Groups 1 and 2 together are linearly independent in V_T, whence $n_d \le v - B$; and Group 3 consists of $v - B - n_d$ fixed ℓ-independent columns from the short degree-T family in V_T completing Groups 1 and 2 to a basis of V_T. The splitting-field comparison preserves independence over the coefficient field, and on a dense open set of cuts, n_d is constant and the chosen maximal minors remain nonzero.

Lean: `GlobalTheory.HSShort.ChooseCols.exists_cols` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsShortColumnsDetNeZeroHilbertSamuel`, lines 99–287)

Let $D_{\mathrm{col}} = \omega(v(\ell))$ be the determinant of these v columns in a fixed terminal basis (with alternating top form ω), viewed as a polynomial in the $d+1$ linear sections. Fix reference vectors $z_q \in U_a$ having dual-basis order-a symbols for every Group 1 index q of order $a < T-d$, and set $z_k = v(0)_k$ for every index k in Groups 2 and 3 (which are independent of ℓ). Working inductively upward in filtration order $a = 0, 1, \dots, T-d-1$, each Group 1 column $v(\ell)_q$ of order a differs from the linear combination of the reference vectors $z_{q'}$ of order a dictated by its symbol block $\mathrm{diag}(\ell(y_p)^{T-a})B_a$ by a vector in U_{a+1}; by backward substitution modulo the later Group 1 columns of higher order and the spanning Group 2 columns of U_{T-d}, reducing Group 1 columns inside the alternating form ω leaves Groups 2 and 3 completely untouched and factors out each symbol block determinant. Hence

$$\det(v(\ell)) = \Big(\prod_{a=0}^{T-d-1} (\det B_a)^{\binom{a+d-1}{d-1}} \Big)\, \omega(z) \Big(\prod_{p=1}^{\delta} \ell(y_p) \Big)^{u},$$

with $\omega(z) \neq 0$ and exponent

$$u = \sum_{j=0}^{T-d-1} \binom{j+d-1}{d-1} (T-j). \tag{A.457}$$

For cuts in the indicated open set the Chow form F_{Ch} specialises to a scalar times $\prod_{p=1}^{\delta} \ell(y_p)$. Thus $D_{\mathrm{col}} = F_{\mathrm{Ch}}^{u} G_{\mathrm{col}}$ for a nonzero multihomogeneous polynomial G_{col}. Divisibility holds over the function field of the first d linear sections, and the Chow form is primitive and irreducible in the full polynomial ring. The cut formula and the height normalisation are those of arithmetic Chow forms [170].

The determinant height is $u\, h(F_{\mathrm{Ch}}) + h(G_{\mathrm{col}})$, by multiplicativity of the local coefficient norm at finite places and integration of the logarithm at infinite places. The binomial sum in (A.457) gives $(d+1)u\delta = B(T + d(d+1))$. The first B columns have total degree T each, the next n_d have degree $T-d$, and the terminal columns have degree zero. Hence the cofactor's total degree is

$$e_G = BT + n_d(T-d) - (d+1)u\delta = n_d(T-d) - Bd(d+1), \qquad 0 \le e_G \le T(v-B).$$

Nonnegativity follows from the multihomogeneous factorisation, and the upper bound uses $n_d \le v - B$. Put

$$h' = \frac{h(F_{\mathrm{Ch}})}{(d+1)\delta}, \qquad c = \log(n_1 \Lambda) + \frac{\log n_1 + 1}{2}.$$

Apply Lemma 35.34 to the nonzero multiforms F_{Ch} and G_{col} in the generating frame. Their total degrees are $(d+1)\delta$ and e_G. Its lower bound gives $h' \ge -c$ and $h(G_{\mathrm{col}}) \ge -c e_G$. The Chow height comparison gives

$h' \ge h-(\log n_1+1)/2$. The column coefficient norms in the frame metric are at most $\beta^{T/2}e^{J}$, $\beta^{(T-d)/2}\Lambda_d$, Λ_T in the three groups. After substituting the integral frame, each coefficient of a column is a sum of products of integral sections, hence belongs to the terminal lattice. On the product of unit spheres each multiplication by a cutting section costs at most $\sqrt{\beta}$. The numbers of columns with the three displayed bounds are $B, n_d, v-B-n_d$, respectively. Lemma 35.34 thus gives $D_0 \ge uh(F_{\mathrm{Ch}}) + h(G_{\mathrm{col}}) - B(J + T\log\beta/2) - n_d(\log\Lambda_d + (T-d)\log\beta/2) - (v-B-n_d)\log\Lambda_T$. Using the expression for $(d+1)u\delta$ and $h' \ge -c$ gives

$$\begin{aligned} D_0 \ge B\{Th' - d(d+1)c - J - \tfrac{T}{2}\log\beta\} \\ - (v-B)\{Tc + \tfrac{T}{2}\log\beta + \log\Lambda_d + \log\Lambda_T\}. \end{aligned}$$

In (A.456), $H \ge c$ because the argument of the logarithm is at least $2n_1$. Thus $d(d+1)c \le 26d^2(H+\log T)$, and the cost $T(\log n_1+1)/2$ of replacing h' by h is at most $(T+d^2)(g+2)$. After division by v we may weaken the bound to

$$\begin{aligned} D_0/v \ \ge (1-\zeta)\{Th - 3(d+1)|h| - J - 26d^2(H+\log T) \\ - (T+d+1)\log\beta - (T+d^2)(g+2)\} \\ - \zeta\{T(\log 2 + H + \tfrac{1}{2}\log\beta) + \log\Lambda_d + \log\Lambda_T\}. \end{aligned} \tag{A.458}$$

Lean: `GlobalTheory.HSShort.monomial_counts` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.MonomialCountsDegreeAndWeightedSum`, lines 37–68)
Lean: `GlobalTheory.HSShortEnd.claimS_of_numbers` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ClaimSOfNumbers`, lines 9–102)
Lean: `GlobalTheory.HSShort.SlopeTop.slopeBound` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.SlopeLowerBoundTopLatticeOffHypersurface`, lines 106–256)

Use $H \le 26\log(2+|h|) + H_0$, where $H_0 = 27P(q+1) + 65P$. This follows from $2+|h|+\log\Lambda \le (2+|h|)(1+\log\Lambda)$, $\log(1+\log\Lambda) \le \log\Lambda$, and $\log d \le \log T \le P/2$. Multiplying (A.458) by v/B puts the second brace at coefficient $\zeta/(1-\zeta) \le 2z$. By (8.28a), (8.28b) and (A.456),

$$d^2Pq \le b/7, \qquad d^2P \le b/14, \qquad TdP \le (TP)^2 \le b.$$

Substitution gives

$$\begin{aligned} J \le 9b, \qquad 26d^2(H_0+\log T) \le 300b, \qquad (T+d+1)\log\beta \le 2b, \\ (T+d^2)(g+2) \le 10b, \qquad 2z\{T(\log 2 + H_0 + \tfrac{1}{2}\log\beta) + \log\Lambda_d + \log\Lambda_T\} \le 200b. \end{aligned} \tag{A.459}$$

For the last bound use $zTPq = zP\kappa \le b$, $2z \le 1$ and $TP \le \sqrt{b}$. Their sum is at most $4000b$, and the coefficient of $\log(2+|h|)$ is bounded by $A = 26(26d^2 + 2zT)$. Passing from the untwisted terminal weight w_T (of unsquared degree D_0) to the anchored target weight u_M (of unsquared degree d_Z) multiplies by the invertible canonical flat twisting section on Z, which shifts each row by a constant integer of size $O(r)$ and has scalar exponent 0. By Lemma 35.4 (using $|c_{1,\tau}| \le 3rL_x$, $\log\mathrm{rd}(E) \le 2L_x$, and the bad-prime bound $\sum_{\ell\in\mathcal{S}_{\mathrm{bad}}}\log\ell \le 40L_x$), this rank-preserving twist lowers the unsquared normalised degree per unit rank by at most $C_p(frL_x)^2$ for a uniform constant $C_p \ge 0$, so $d_Z \ge D_0 - vC_p(frL_x)^2$. As $v/B \le 2$, take $R = 4000b + 2C_p(frL_x)^2$. These choices are common to the components. Since $1 \le d \le T$ and $2z \le 1$, we have $1 \le A \le 702T^2$. By (A.456), $2\log T \le \log b \le P$, $P \ge 1$, and $T^2P^2 \le b$. Thus $\log A \le \log 702 + 2\log T \le 8P$, and $A\log A \le 5616T^2P \le 6000b$. Also $(frL_x)^2 \le (frL_x)^{12} \le P \le TP \le \sqrt{b} \le b$, and the size bound of 35.20 gives $b \le E_p$. This proves (8.29b) with $C_{\mathrm{HS}} = 10000 + 2C_p$.

The component spaces sum to the cusp space and their ranks add. The integral direct sum is contained in the whole lattice. With the induced global Petersson metrics its unsquared normalised degree is

$$\sum_Z d_Z - \frac{1}{2[F':\mathbf{Q}]}\sum_{\sigma:F'\hookrightarrow\mathbf{C}}\sum_Z r_Z\log p_{Z,\sigma},$$

where $p_{Z,\sigma} \in (0,1]$ is the component mass at σ, and d_Z uses the conditional component metric. This degree is at least $\sum_Z d_Z$ and at most the whole unsquared degree $\widehat{\deg}$. Dividing by $\sum_Z r_Z$ and multiplying by two (since $\mu_Y = 2\widehat{\deg}/\operatorname{rk}$ doubles both the finite and archimedean terms in the squared-norm convention) yields (8.29c). □

Proof of 35.36

Proof. Equip $K \otimes_{\mathbf{Q}} \mathbf{R}$ with the Euclidean norm squared $\sum_\sigma |\sigma x|^2$, with conjugates counted separately. The Euclidean dual of $\mathcal{O}_K$ in this norm is the image of the inverse different with conjugation in the complex factors, by the trace pairing. For a nonzero element y of the inverse different, the integral ideal $(y)\mathfrak{D}_K$ has norm at least one. Hence

$$\max_\sigma |\sigma y| \geq |N_{K/\mathbf{Q}}(y)|^{1/a} \geq \mathrm{rd}(K)^{-1}.$$

The Euclidean transference bound $\lambda_a(\Lambda)\lambda_1(\Lambda^\vee) \leq a$ [78] gives a rationally independent integers ω_i of Euclidean length at most $a\,\mathrm{rd}(K)$. In particular

$$\max_\sigma |\sigma \omega_i| \leq a^3 \,\mathrm{rd}(K). \tag{A.460}$$

For $v = 0$ the empty family works. Otherwise take a K-basis $s_1, \ldots, s_v$ from S. The av products $e_{ij} = \omega_i s_j$ belong to L and form a rational basis. For each vector $z = \sum c_{ij} e_{ij}$ of a $\mathbf{Z}$-basis of L, put $z' = z - \sum \lfloor c_{ij} \rfloor e_{ij}$. The remainders and the products generate every original basis vector. There are av remainders, since L has full rational span and is torsion-free. By (A.460) each product has seminorm at most $a^3\,\mathrm{rd}(K)H$. The triangle inequality with $0 \leq c_{ij} - \lfloor c_{ij} \rfloor < 1$ gives $N_\sigma(z') \leq a^4 v\,\mathrm{rd}(K)H$. □

Proof of 35.37

Proof. The transference calculation in Lemma 35.36, applied to $\mathcal{O}_E$, gives a rational spanning set $\omega_i \in \mathcal{O}_E$ with $\max_\sigma |\sigma(\omega_i)| \leq 2fR_E \leq \Omega$. Thus each ω_i acts on V with operator norm at most Ω.

Start with $W = 0$. At each stage take $I = L \cap W$, maintain isotropy and $\mathcal{O}_E$-stability, and maintain the second displayed implication. Suppose $x \in L \setminus I$ is at distance less than θ from W. Write $x = v + h$, where $v \in W$ and $\|h\| < \theta$. Since B vanishes on W, $\|u(x)\| \leq \|h\| < \theta \leq (\Omega\theta)^{-1}$. Hence $B(x, W) = 0$. Adjointness gives $B(ax, W) = 0$ for every $a \in \mathcal{O}_E$. In particular

$$|B(x, \omega_i x)| = |B(h, \omega_i h)| \leq \Omega \|h\|^2 < 1.$$

The left side is the absolute value of an integer. The rational spanning property and adjointness now give $B(ax, bx) = 0$ for all $a, b \in \mathcal{O}_E$. Enlarge W to $W' = W + \mathrm{span}_{\mathbf{R}}(\mathcal{O}_E x)$. This is an isotropic rational subspace of strictly larger dimension.

For $y \in L$, restrict $B(y, \cdot)$ to W'. If its restriction to W is nonzero, its norm on W' is at least $(\Omega\theta)^{-1}$ by induction. If it vanishes on W and is nonzero on W', then $B(y, \omega_i x)$ is a nonzero integer for some i. The vector $\omega_i h$ lies in W', and $B(y, \omega_i h) = B(y, \omega_i x)$, since $\omega_i v \in W$. The norm of the restriction is at least $1/\|\omega_i h\| > (\Omega\theta)^{-1}$. This proves the induction step. The dimension bound on W terminates the construction. When $\lambda \geq \theta$ the initial space already satisfies both conclusions. When $\lambda < \theta$, the distance inequality puts every shortest vector in the final I.

Let $\rho = \dim_E(I \otimes \mathbf{Q})$. Isotropy gives $W \cap iW = 0$, since $B(w, iw)$ has absolute value $\|w\|^2$. The complexification of W therefore embeds in V. Each embedding of E occurs there with multiplicity ρ, which proves both rank assertions.

Fix a shortest vector $\xi \in I$. The integral functional $a \mapsto B(x_j, a\xi)$ on $\mathcal{O}_E$ has the form $\mathrm{Tr}_{E/\mathbf{Q}}(ay_j)$ for a unique y_j in the inverse different. At least one y_j is nonzero. Otherwise $B(ax_j, \xi) = B(x_j, \bar{a}\xi) = 0$ for every a, j, contrary to the spanning hypothesis. For each pair $\{\sigma, \bar{\sigma}\}$ put $s_\sigma = \|\xi_\sigma\|^2 + \|\xi_{\bar{\sigma}}\|^2$. On $E \otimes_{\mathbf{Q}} \mathbf{R}$, the map $a \mapsto a\xi$ has squared norm $\sum_{\{\sigma,\bar{\sigma}\}} s_\sigma |a_\sigma|^2$. The real linear functional $a \mapsto \mathrm{Tr}(ay_j)$ has squared dual norm

$$4 \sum_{\{\sigma,\bar{\sigma}\},\, s_\sigma > 0} |\sigma(y_j)|^2 / s_\sigma \leq \|u(x_j)\|^2.$$

Indeed its value is $\langle u(x_j), a\xi \rangle_{\mathbf{R}}$, and its coordinate at a pair is $2\,\mathrm{Re}(a_\sigma \sigma(y_j))$. The functional vanishes on every coordinate with $s_\sigma = 0$. Since $s_\sigma \leq \lambda^2$, this gives $2\sum_\sigma |\sigma(y_j)|^2 \leq \lambda^2 \|u(x_j)\|^2$. For any nonzero y_j, the integral ideal $y_j \mathfrak{D}_E$ has norm at least one. The arithmetic-geometric mean inequality therefore gives

$$\sum_\sigma |\sigma(y_j)|^2 \geq 2f\, |N_{E/\mathbf{Q}} y_j|^{1/f} \geq 2f R_E^{-2}.$$

This proves the last assertion. □

Proof of 35.38

Proof. At one embedding write $J = \operatorname{diag}(\epsilon_i)$, with $\epsilon_i = 1$ for $i < p$ and -1 otherwise. Put $L_j = \log|\det(n^*Jn)_{[0,j)}|$, including $L_0 = 0$. A weight ν has potential $\sum_i \nu_i(L_{i+1} - L_i)$. The kernel potential R_x has weight $\mathbf{1}_{i\geq p}$, since $L_r = 0$. The log density has weight $2i - (r-1)$. For the latter formula, change from an orthogonal frame by an upper triangular matrix with diagonal entries b_i. The tangent coordinates are the entries below the diagonal, on which the diagonal acts by b_a/b_c, $a > c$. The squared absolute Jacobian has exponent $2i - (r-1)$ on $|b_i|^2$, and $\log|b_i|^2 = L_{i+1} - L_i$.

For a strictly lower triangular tangent matrix H, the Levi form of L_j at the origin is $\sum_{c<j\leq a} \epsilon_a\epsilon_c|H_{ac}|^2$. Indeed, in the first j columns of $1+tH$ the upper j-rowed square block has determinant one. Factor it out of the Gram determinant. The lower rectangular block is $t(H_{ac})_{a\geq j, c<j} + O(t^2)$, so the coefficient of $|t|^2$ in the log determinant is the indicated signed sum. Telescoping in j gives the coefficient $\epsilon_a\epsilon_c(\nu_c - \nu_a)$ on $|H_{ac}|^2$. For $\nu = \lambda$ this is at least T within a block by adding adjacent gaps. Across blocks it is $\lambda_a - \lambda_c \geq \lambda_{r-1} - \lambda_0 \geq T$. For the kernel weight the coefficient is zero or one. For the density weight it is at most $2r$. Thus the lower bound at an orthogonal origin is $T - b_* - 2r$.

At a general point reorthogonalise the frame, writing $xn = x'B$ with B upper triangular. In the overlap of the two holomorphic flag charts the metric potentials differ by logarithms of absolute squares of nonvanishing holomorphic diagonal factors. The log densities differ by the log squared absolute Jacobian. Their Levi forms are therefore transported by the chart differential. This proves nonnegativity throughout the chart.

We now give the norm of that differential on the specified ball. Set $\delta = \|z\|$. Entrywise $n^*Jn - J$ has absolute value at most $2\delta + r\delta^2 \leq 3\delta$. Since $B^*JB = n^*Jn$, triangular elimination gives

$$\big||B_{aa}|^2 - 1\big| \leq 6\delta, \qquad |B_{ac}| \leq 6\delta \quad (a < c).$$

Indeed, in row a the already eliminated terms contribute at most $36r\delta^2 \leq 0.36\delta$ to each entry. The diagonal equation first bounds the squared diagonal error by 3.36δ. The off-diagonal equation then bounds the next entry by $3.36\delta/\sqrt{1-6\delta} \leq 6\delta$. Write $B = D_0 + U_0$ with D_0 diagonal. Its diagonal moduli lie in $[0.969, 1.03]$, and $\|U_0\|_{\mathrm{F}} \leq 6r\delta \leq 0.06$, where $\|\cdot\|_{\mathrm{F}}$ is Frobenius norm.

Let H' be the tangent in the recentered chart and put $Y = n^{-1}H$. Differentiating the change of frame gives $BY = H'B + B_1$ with B_1 upper triangular. Project to strictly lower entries. Since Y is strictly lower triangular, the norm inequality is

$$0.969\|Y\|_{\mathrm{F}} \leq 1.09\|H'\|_{\mathrm{F}} + 0.06\|Y\|_{\mathrm{F}}.$$

Hence $\|Y\|_{\mathrm{F}} \leq 1.2\|H'\|_{\mathrm{F}}$ and $\|H\|_{\mathrm{F}} \leq 1.01\|Y\|_{\mathrm{F}}$. Square and sum over the embeddings. This gives $\|H\|^2 \leq 2\|H'\|^2$, as required.

To check the domain for the $\bar\partial$ estimate, write the first p columns of n as $\binom{n_1}{W_1}$, and put $Z = W_1 n_1^{-1}$. The matrix n_1 is lower unitriangular. The condition on the first p columns is $I - Z^*Z > 0$. It implies all the other flag signature conditions, since the perpendicular complement of this positive p-plane is negative definite. In particular

$$R_x = -\sum_\tau \log\det(I - Z_\tau^* Z_\tau).$$

Its Levi form is nonnegative by the kernel-weight computation at an orthogonal origin and change of chart. The function $R_x + \log(1 + \|z\|^2)$ is a plurisubharmonic exhaustion. A sublevel bounds $\|z\|$, and it bounds the product of the determinants away from zero. Each factor is at most one, so a limit in this bounded set stays in the domain. This proves pseudoconvexity.

Let $t = \|z\|^2/\rho^2$, $F_0 = \chi(t)$, and on the ball set

$$\psi_j = D\chi(t)\log((t + 1/(j+1))/2).$$

Define ψ_j Γ-invariantly across the entire flag domain by transporting this local singular weight to every Γ-translate of $B(0,\rho)$ and extending by zero outside the union of translates (injectivity of $B(0,\rho)$ makes the transported functions agree and proper discontinuity makes the supports locally finite), and then restrict $U + \psi_j$ to the lower-unitriangular Bruhat chart. On $t \leq 1/4$ the logarithm has nonnegative Levi form. On $1/4 \leq t \leq 1$ its absolute value is at most $\log 8$, $\partial\bar\partial t[v,\bar v] = \|v\|^2/\rho^2$, and $|\partial t[v]|^2 \leq t\|v\|^2/\rho^2$. The two cross terms from differentiating the product cost at most $2A\|v\|^2/\rho^2$, and the term with the differentiated

cutoff costs at most $2A\log 8\,\|v\|^2/\rho^2$. Thus the negative part costs at most $D(13A+20)\|v\|^2/\rho^2$. The gap hypothesis makes $U+\psi_j$ plurisubharmonic everywhere. On the original cutoff annulus its Levi form dominates $|\bar\partial F_0[v]|^2$. These comparisons persist under transport; outside the translates the added weight is zero.

Apply the weighted $\bar\partial$ estimate of [184, Th. 2.1] on this pseudoconvex chart to the smooth plurisubharmonic weight $U+\psi_j$, the closed form $\bar\partial F_0$, and the indicator of the original annulus as the bound for its squared dual norm. On that annulus $-\psi_j\le 3D$. The bounds on $|B_{ii}|^2$ above give $|w|\le\|\lambda\|_1$ and $|R_x|\le f$ there, the latter by summing $\log|B_{ii}|^2$ for $i<p$. Consequently the solutions u_j satisfy

$$\int |u_j|^2 e^{-U-\psi_j}\,dz \le e^{3D+\|\lambda\|_1+b_*f}\mu(B(0,\rho)).$$

For each fixed k the tail is bounded in $L^2(e^{-U-\psi_k})$. Diagonal weak extraction gives a distributional limit u with $\bar\partial u=\bar\partial F_0$. Weak lower semicontinuity followed by monotone convergence gives the same finite singular-weight bound $\int|u|^2e^{-U}\Psi_w\,dz<\infty$ for the correction $u=F_0-F$ against the Γ-invariant weight envelope $\Psi_w=\sup_k e^{-\psi_k}$ on the chart. Setting $F=F_0-u$ gives a holomorphic section on the chart; because $F_0\equiv 1$ on $B(0,\rho/2)$, $u=1-F$ is holomorphic there and square-integrable against $|z|^{-2D}$, forcing $u(0)=0$ and hence $F(0)=1$. Using $\psi_j\le 0$ and $|F|^2\le 2|F_0|^2+2|u|^2$ yields the finite nonsingular L^2 bound $\int|F|^2e^{-w+b_*R_x}\,d\mu<\infty$ (with no Ψ_w factor); since $K_x=e^{-b_*R_x}\le 1$ and the Bruhat chart complement in the flag domain has complex codimension at least 1 (hence Lebesgue measure zero), local L^2 removability extends F holomorphically across the complement to a global Borel-equivariant section P_0 on the entire flag domain with $P_0(x)=F(0)=1$. For any nontrivial Γ-translate γx ($1\ne\gamma\in\Gamma$), choose a small Borel-saturated recentered chart neighborhood $N_{\rho'}$ around γx with $0<\rho'<1/(100r^2)$; by disjointness of Γ-translates of the radius-$1/(100r^2)$ ball, every x-chart point in $N_{\rho'}$ lies outside the support of F_0, so $P_0=F=-u$ on $N_{\rho'}$. Chart measure invariance transfers $\int_{N_{\rho'}}|P_0|^2e^{-w}\Psi_w\,d\mu<\infty$ to the recentered γx-chart, where Γ-invariance of Ψ_w supplies a lower bound by a positive multiple of $|w|^{-2D}$ off a measure-zero set; a nonzero value $P_0(\gamma x)$ would give the divergent radial integral $\int_0^{\rho'}dt/t$, forcing $P_0(\gamma x)=0$ for every $1\ne\gamma\in\Gamma$. Since $D\le fr^2$ and $b_*f\le 5(fr)^8$, the constant in the nonsingular L^2 bound is uniform. □

Proof of 35.39

Proof. The kernel depends on the positive planes. For graph matrices Z,Y at one embedding, the factor of $K_y(z)^{1/b_*}$ is

$$\frac{\det(I-Z^*Z)\det(I-Y^*Y)}{|\det(I-Y^*Z)|^2}.$$

Indeed the two Gram matrices of the graph bases are $I-Z^*Z, I-Y^*Y$, and their mutual Gram matrix is $I-Y^*Z$. This also gives symmetry and invariance. For $z\in B_x$ in coordinates at x, the formula $Z=W_1n_1^{-1}$ in Lemma 35.38 gives $\|Z\|_{\mathrm{op}}\le 2/(100r^2)<1/2$. Thus the numerator factor $\det(I-Z^*Z)$ is at least $(3/4)^p$, and $|\det(I-Y^*Z)|\le(3/2)^p$. Multiplication over the embeddings gives $K_y(z)\ge 3^{-fpb_*}K_y(x)$.

For completeness, compute the mass against this kernel. Over the matrix ball at one embedding the invariant base density, equal to one at zero, is $\det(I-Z^*Z)^{-r}$ times Lebesgue measure. The differential at zero of translation to Z acts on a rectangular tangent matrix H, up to unitary factors, by $(I-ZZ^*)^{1/2}H(I-Z^*Z)^{1/2}$. Its squared absolute Jacobian is $\det(I-Z^*Z)^r$, giving the density. The fibre consists of a full flag in each of the two definite blocks. A compact flag of rank s, with density one at the orthogonal origin in lower triangular coordinates, has mass $\prod_{j=1}^{s-1}\pi^j/j!$. To see this, project to its first line. The invariant density on $\mathbb{P}^j$ in the affine chart is $(1+\|z\|^2)^{-j-1}$, whose radial integral is $\pi^j/j!$. The remaining flag has rank one less. Base and fibre measures multiply because their invariant densities agree at the origin, where the three blocks of lower triangular coordinates are disjoint. Since $b_*\ge r$, the remaining base integral is bounded by the Euclidean volume of the product of $p(r-p)$ unit discs. This gives $\int K_y\,d\mu\le\pi^{fr^2}$.

On B_x the triangular bounds in Lemma 35.38 give $|\log d|\le fr$, by the density-weight formula there. In dimension $D=fr(r-1)/2$ it follows that

$$\mu(B_x)\ge e^{-fr}\frac{\pi^D}{D!(100r^2)^{2D}}\ge e^{-C_1(fr)^8}$$

for an absolute C_1, using $D! \le D^D$. Integrate the kernel comparison over the disjoint sets $\gamma^{-1}B_x$. Symmetry and invariance give

$$3^{-fpb_*}\mu(B_x)\sum_{\gamma} K_x(\gamma y) \le \int K_y \, d\mu.$$

This proves the first bound.

For a finite sum of translates, weighted Cauchy–Schwarz bounds its squared norm at y by

$$\left(\sum_{\gamma} K_x(\gamma y)\right)\sum_{\gamma}\frac{\|F(\gamma y)\|^2}{K_x(\gamma y)}.$$

Integration over a fundamental domain unfolds the second sum. The same inequality for tails gives convergence in L^2 there. On a compact subset of the flag domain the translates have bounded overlap by proper discontinuity, so the tails converge in local L^2 there as well. The holomorphic submean inequality in local frames gives locally uniform convergence. The estimate passes to the resulting section. □

Proof of 35.40

Proof. Fix a rational isotropic line $E\xi$, with $\xi \neq 0$ in the period lattice. Its intersection with that lattice is saturated and isotropic for the alternating form. Choose a symplectic basis e_j, e'_j in which this intersection lies in the first half. Such a basis is obtained by splitting off a primitive vector and a dual vector of pairing one, then repeating in their symplectic orthogonal complement. A change of symplectic basis changes the level theta function into a finite linear combination of theta functions with characteristics. This follows on translations from the theta translation formula and on interchange of the two halves from Poisson summation on the level cosets. Harmonicity gives the same harmonic differential after the change of period coordinates.

Write h for the fixed Hermitian form. In the expansion with one such characteristic let u_b be the integer summation vector plus the real characteristic in column b, and put

$$L_\tau(u_b) = \sum_j u_{bj}\,\tau\big(h(e'_j,\xi)\big).$$

Along the real centre of the cusp the frequency in coordinate τ is a positive constant times $\sum_{b=1}^{N}|L_\tau(u_b)|^2$. Indeed the change of the period matrix along that coordinate is a real multiple of the symmetric matrix $\big(\mathrm{Re}(\tau h(e'_j,\xi)\overline{\tau h(e'_k,\xi)})\big)_{jk}$. Its quadratic form on a real column u_b is $|L_\tau(u_b)|^2$. Thus zero frequency forces $L_\tau(u_b)=0$ in every column.

We give the differential calculation at the embedding with $a_{\tau,P} > 0$. In frame coordinates let $w=(w_1,w_2)$ be the coordinates of ξ, and let (ℓ_1,ℓ_2) be its Hermitian dual covector, with the blocks ordered so that the P differential directions come from the second block. Isotropy gives $\ell_1 w_1 + \ell_2 w_2 = 0$. Both w_2 and ℓ_1 are nonzero, since a nonzero isotropic vector cannot lie in a definite summand. We may shear the frame by an upper block unipotent so that the complementary coordinates w_1, ℓ_2 vanish. Explicitly, take $d_0 w_2 = 1$, $\ell_1 e_0 = 1$, and use the off-diagonal block

$$K = w_1 d_0 - e_0 \ell_2 - (\ell_1 w_1) e_0 d_0.$$

Then $K w_2 = w_1$ and $\ell_1 K = -\ell_2$. This shear preserves the flag and its weight character.

In this frame let v be the period direction of ξ in the P-row block at τ. Write the variable part of a single theta summand as a product over its columns of

$$\exp\big(\tfrac{\pi}{2} z_b^t B z_b + 2\pi i\, u_b^t H_0 z_b\big)\,.$$

Here z_b is the period variable, H_0 is the inverse first period matrix, and B is the symmetric matrix of the Gaussian prefactor in these coordinates. The period identities in the adapted basis give $Bv = 0$ and $u_b^t H_0 v$ proportional to $\overline{L_\tau(u_b)}$. To see the two identities, the first period matrix sends the coordinates of each $\alpha\xi$, $\alpha \in E$, in the first half of the symplectic basis to its period vector. Let C record the complementary frame coordinates on the same first-half basis. The Gaussian matrix is $B = CH_0$, and C sends the first-half coordinates of $\alpha\xi$ to its complementary coordinates, which vanish after the shear. Separating the

embeddings in these identities gives $Bv = 0$. Pairing $\alpha\xi$ with the dual half e'_j reads its first-half coordinates by the alternating form. Separating the τ-term gives the entries of $H_0 v$ as the same nonzero scalar times $\overline{\tau h(e'_j, \xi)}$. The separation uses linear independence of the distinct embeddings of E.

Lean: `GlobalTheory.ThFr.IsAdapted.hMat_mulVec_part` (`Development.AbelianVarietyTheory.Proofs.CuspFormEvaluations.IsHarmonicAndMemMinorSpan`, lines 300–325)

Lean: `GlobalTheory.ThFr.IsAdapted.splitMat_mulVec_part` (`Development.AbelianVarietyTheory.Proofs.CuspFormEvaluations.IsHarmonicAndMemMinorSpan`, lines 437–458)

At zero frequency each column factor is invariant under $z_b \mapsto z_b + tv$. Consequently the same nonzero linear combination of the P row derivatives annihilates the summand in every column. Make this combination the last row by an invertible row operation. A full P-row determinant of the derivatives now annihilates the summand, by expansion along that row and commutation of constant-coefficient derivatives. The differential contains such a determinant to the positive power $a_{\tau,P}$. Its zero-frequency coefficient therefore vanishes term by term.

At a fixed frame the differentiated terms are bounded by a polynomial times a positive-definite Gaussian in the summation vectors. Absolute convergence permits taking the mean along the real centre term by term, for instance by averaging over expanding boxes, where each nonzero exponential has mean tending to zero. Thus the zero Fourier–Jacobi coefficient vanishes for each rational isotropic line. At a higher-dimensional rational boundary component its constant term includes the average in any such line in that component. This proves cuspidality. □

Proof of 35.42

Proof. For (1), put $S_0 = R_0^2 = 90(fr + D) \geq 90(2 + 1) = 270$, so $R_0 > 0$. From $1 \leq m \leq m^4 \leq f$ and $f^2 \leq R_E$ we obtain $f \geq 1$, $f, fm, m^2, f^{3/2} \leq R_E$ and $d_* = 2fm \leq 2R_E$. The hypothesis $S_0 R_E^{28} Dr \leq q$ gives $b_0(q, R_0) = q/(S_0 R_E^{28}) \geq Dr \geq 1$, $1 \leq D \leq Dr \leq q$ and $t_0(q, R_0) = q b_0(q, R_0) \geq qDr$. Next, since $R_E \geq 16$,

$$d_*^2 \Omega_* f S_0 R_E^{21} \leq (2R_E)^2 (2R_E^2)(R_E S_0 R_E^{21}) = 8 S_0 R_E^{26} < R_E^2 S_0 R_E^{26} = S_0 R_E^{28} \leq q,$$

so $d_*^2 \Omega_* \theta(q, R_0)^2 < 1$. Also $M_1 = 90 m^2 d_*^2 R_E \leq 360 R_E^4 \leq \frac{360}{16} R_E^5 \leq \frac{1}{2} S_0 R_E^{28} \leq q/2$. Using $\log(q/16) \leq q/16 - 1$, $\log 16 \leq 15$ and $\log D \leq \log q$ gives $\log(qD) \leq q/8 + 28$, so $4D(7 + \log(qD)) + d_* + 1 \leq D(q/2 + 140 + 2R_E + 1) \leq qD \leq t_0(q, R_0)$ because $q \geq 270 R_E^{28} \geq 4320 R_E$. Finally $\log 2 \leq 1$ and $\log r \leq r - 1 < r \leq fr$ give $(D + 1)(r \log r + 3r + 3) \leq 2Dr(fr + 6)$, whence

$$fr \log 2 + 2d_* + 3 + (D + 1)(r \log r + 3r + 3) \leq (3fr + 4R_E + 15) Dr \leq qDr \leq t_0(q, R_0).$$

Thus (E, r, D, q, R_0, m) is uniformly admissible.

For the bound on $P_0 = \mathcal{P}(E, r, q, m, 5, 1)$, every term is nonnegative since $frq \geq 1$, $d_* \geq 2$, $A \geq 625$ and $K_* \geq 16$. From $e > 2.71$ we have $e^2 > 7.3$ and $e^{10} > 7.3^5 > 10000$, so $\log 2 \leq 1$, $\log 5, \log 6, \log 7 \leq 2$ and $\log 10000 \leq 10$. Hence $\log(1 + 6\sqrt{frq}) \leq \log(7\sqrt{frq}) \leq 2 + \frac{1}{2}(\log f + \log r + \log q) \leq 2 + 37L$, and $\log m \leq \log r \leq L$ gives $p_2 = \log 2 + d_* \log(6fm) \leq 1 + 2fr(2 + 3L)$, so $p_3 \leq 2r + fr^2(8 + 43L) \leq fr^2(10 + 43L)$. With $R = 5$ ($2R^4 = 1250$, $2fr^2 \log 5 \leq 4fr^2$), $A = 625fmr \leq 625fr^2$ and $\log(AK_*) = \log(10000 r f^3 m^3 R_E^7) \leq 10 + 24L$, we obtain $P_0 \leq 23750 fr^2 + 68750 fr^2 L$. Since $f \geq 1$ and $L \geq 2$ imply $4fr^2 \leq (frL)^2$ and $2fr^2 L \leq (frL)^2$, this gives $P_0 \leq 200000 (frL)^2$.

For (2), Gram–Schmidt orthogonalisation in the signature flag produces an upper-triangular Borel matrix u_{B} such that $x = y u_{\mathrm{B}}$ lies in the domain on k with all distinguished Gram minors of modulus 1, preserving λ_1 and hence Φ. Because f_0 has theta weight $(0, r)_\theta$ and its square f_0^2 has theta weight $(0, 2r)_\theta$ (column count $2r$, row central parameter r) with zero tuple weight, the Lie scale factor of $(0, 2r)$ is 1 and the holomorphic Borel transformation law gives $\|f_0^2(y)\| = \|f_0^2(x)\| = |f_0(x)|^2$. At the unit-minor frame x, the analytic uniformisation supplies the period map $\mathrm{Pm} : \Lambda_k \to \mathbf{C}^{fr}$, the Appell–Humbert Hermitian form H_m, a holomorphic trivialisation of the pulled-back principal line with multiplier $\chi_{\mathrm{lin}}(\mathrm{Pm}(u)) = (-1)^{E_F(u, w_0 u)}$ (where E_F is the Riemann form and $2w_0 = 1 + \sqrt{-d_H}$ in $\mathcal{O}_E$ for the host prime d_H), and an entire principal theta function $\theta_0 : \mathbf{C}^{fr} \to \mathbf{C}$ of unit L^2 norm against $e^{-\pi \operatorname{Re} H_m(z,z)} (\mathrm{vol}\, \mathcal{F}_0)^{-1}\, dz$ on a fundamental domain $\mathcal{F}_0$ for $\mathrm{Pm}(\Lambda_k)$. Diagonalising H_m on the $\mathcal{O}_E$-eigenspaces yields a $\mathbf{C}$-linear isometry $\psi : (\mathbf{C}^{fr}, H_m) \xrightarrow{\sim} (V, H)$ onto a polarised CM period torus $\mathcal{T} = (V, L_0, H, \varepsilon)$ of rank r with $L_0 = \psi(\mathrm{Pm}(\Lambda_k))$, d_τ taking the values p_{sig} and q_{sig} on the CM type and its conjugate (so $\min_\tau d_\tau = \min(p_{\mathrm{sig}}, q_{\mathrm{sig}})$), $\lambda_1(\mathcal{T}) = \lambda_1(x) = \lambda_1(y)$

and $\phi_{\theta(q,R_0)}(\mathcal{T}) = \Phi$, on which $\theta_{\mathcal{T}} = \theta_0 \circ \psi^{-1}$ is a unit-L^2 principal theta section with point-norm square $|\theta_0(z)|^2 \exp(-\pi \operatorname{Re} H_m(z,z))$ at $\psi(z)$.

Let ev be the geometric full level-R^4 structure induced by the level-M datum and theta rule f_0 at the complex point of x, and choose period representatives $a : \Lambda/R^4\Lambda \to \Lambda_k$ with $\mathrm{ev}(u) = \exp(R^{-4} \operatorname{Pm}(a(u)))$. Because R^2 is odd and $\mathrm{ev}(u)^{(R^2)^2} = 1$, each $\mathrm{ev}(u)$ admits a unique normalised R^2-null-value $w(u)$, and single-term evaluation of the theta rule gives $f_0(x) = \prod_{j=1}^r w(x_j)$. Put $z_j = R^2 R^{-4} \operatorname{Pm}(a(x_j)) \in \mathbf{C}^{fr}$ and $v_j = \psi(z_j) \in V$. Then $R^2 v_j = \psi(\operatorname{Pm}(a(x_j))) \in L_0$, and the R^2-null-value comparison gives $|w(x_j)| = |\theta_0(z_j)| \exp(-\frac{\pi}{2} \operatorname{Re} H_m(z_j, z_j))$. Because $(x_1, \dots, x_r)$ spans $\Lambda/R^4\Lambda$ over $\mathcal{O}_E$, every $u \in \Lambda_k$ has the form $\sum_{j=1}^r c_j a(x_j) + R^4 v$ with $c_j \in \mathcal{O}_E$ and $v \in \Lambda_k$; applying $\psi \circ \operatorname{Pm}$ expresses every vector of L_0 as $\sum_{j=1}^r c_j (R^2 v_j) + R \cdot \psi(\operatorname{Pm}(R^3 v))$, so $R^2 v_1, \dots, R^2 v_r$ span L_0/RL_0 over $\mathcal{O}_E$.

Assume now that (E, r, D, q, R_0, m) is uniformly admissible and $\min(p_{\mathrm{sig}}, q_{\mathrm{sig}}) \le m$, so (q, R_0, m) is admissible on $\mathcal{T}$. When $\Phi = 0$, we have $\lambda_1(\mathcal{T}) \ge \theta(q, R_0)$ with $b_0(q, R_0) \ge 1$. For any point $z \in V$, the Euclidean ball $B(z, \rho)$ of radius $\rho = \frac{1}{2}\theta(q, R_0)$ contains no two distinct L_0-translates (so it injects into the quotient V/L_0); applying the holomorphic mean-value / Bergman inequality to the entire function $\theta_{\mathcal{T}}$ with Gaussian weight $e^{-\pi \operatorname{Re} H(w,w)}$ over $B(z, \rho)$ bounds $|\theta_{\mathcal{T}}(z)|^2 e^{-\pi \operatorname{Re} H(z,z)}$ by $\mathrm{vol}(B(0,\rho))^{-1} \exp(\pi\rho^2) \|\theta_{\mathcal{T}}\|_2^2 \le e^{2p_1}$, whence $|w(x_j)| \le e^{p_1}$ and $|f_0(x)|^{2R^4} \le e^{2R^4 r p_1} \le \exp(\mathcal{P}(E, r, q, m, R, 1))$. When $\Phi > 0$, we have $\lambda_1(\mathcal{T}) < \theta(q, R_0)$ and $\Omega_* \theta(q, R_0)^2 \le d_*^2 \Omega_* \theta(q, R_0)^2 < 1$. Lemma 35.37 applied to (V, L_0, H) produces a saturated $\mathcal{O}_E$-stable isotropic sublattice $I \subset L_0$ of $\mathbf{Z}$-rank $j_I \le d_*$ and real symplectic projection $u : V \to W = I \otimes_{\mathbf{Z}} \mathbf{R}$, together with the quotient polarised torus V_I/L_I, where $V_I = W^{\perp_{\operatorname{Im} H}}/W$ and $L_I = (L_0 \cap W^{\perp_{\operatorname{Im} H}})/I$. Its first minimum is at least $\theta(q, R_0)$. Expanding $\theta_{\mathcal{T}}$ in Fourier modes along the real torus W/I expresses the modes as translates of a quotient principal theta section θ_I with Gaussian coset sums. First, applying the same holomorphic mean-value / Bergman ball-packing bound on Euclidean balls of radius $\frac{1}{2}\theta(q, R_0)$ inside V_I gives $\|\theta_I\|_{\sup} \le e^{p_1} \|\theta_I\|_2$. Second, if c_I is the Euclidean covolume of I in W, Parseval along W/I, followed by integration over V_I/L_I and the transverse Gaussian directions, gives $c_I 2^{-j_I/2} \|\theta_I\|_2^2 \le 1$; moreover, every nonzero vector of I has norm at least $\lambda_1(\mathcal{T})$ by definition of $\lambda_1(\mathcal{T})$, so disjoint open Euclidean balls of radius $\frac{1}{2}\lambda_1(\mathcal{T})$ centred at the lattice points of I pack inside W/I and yield the covolume lower bound $c_I \ge \big(\pi^{j_I/2}/\Gamma(j_I/2+1)\big)(\lambda_1(\mathcal{T})/2)^{j_I}$, whence $c_I 2^{-j_I/2} \ge e^{-2d_*}(K_*\Phi)^{-d_*/2}$ and $(c_I 2^{-j_I/2})^{-1/2} \le e^{d_*}(K_*\Phi)^{d_*/4}$. Third, by symplectic duality against I, the image lattice $u(L_0) \subset W$ has dual separation $\|u(x)\| \ge (\Omega_* \theta(q, R_0))^{-1}$ for nonzero $u(x) \in u(L_0)$. Choosing $y_j \in L_0$ to minimise $D_j = \|u(v_j - y_j)\|^2$, partition $u(v_j) + u(L_0)$ into the 2^{j_I} cosets of $2u(L_0)$. In each coset C, let $w_c \in C$ minimise the Euclidean norm in C; for every $w \in C$, we have $\|w\| \ge \|w_c\|$ and $\frac{1}{2}(w + w_c) \in u(v_j) + u(L_0)$ (since $w - w_c \in 2u(L_0)$), so the parallelogram identity yields $\|w\|^2 \ge \frac{1}{2}(\|w\|^2 + \|w_c\|^2) = \|\frac{1}{2}(w + w_c)\|^2 + \frac{1}{4}\|w - w_c\|^2 \ge D_j + \frac{1}{4}\|w - w_c\|^2$. Because $w - w_c$ ranges over the lattice $2u(L_0) \subset W \cong \mathbf{R}^{j_I}$ ($j_I \le d_*$) whose nonzero vectors have norm at least $2(\Omega_*\theta(q, R_0))^{-1} \ge 2d_*$ (since $b_0(q, R_0) \ge 1$, $m^4 \le f$, $f^2 \le R_E$, and $R_E \ge 16$ give $d_*^2 \Omega_*^2 \theta(q, R_0)^2 = 16 f^6 m^2/(b_0(q, R_0) R_E^5) \le 16 R_E^{-7/4} \le 1/8 < 1$), summing $e^{-\pi\|w\|^2/2} \le e^{-\pi D_j/2} e^{-\pi\|w - w_c\|^2/8}$ over origin-centered shells in $2u(L_0)$ and over all $2^{j_I} \le 2^{d_*}$ cosets gives the D_j-independent bound $\sum_{w \in u(v_j) + u(L_0)} e^{-\pi\|w\|^2/2} \le 2(3d_*)^{d_*} e^{-\pi D_j/2}$. Meanwhile $R^2(v_j - y_j) \equiv R^2 v_j \pmod{RL_0}$ since $R^2 y_j \in RL_0$, so the $\mathcal{O}_E$-multiples of $R^2(v_1 - y_1), \dots, R^2(v_r - y_r)$ span L_0/RL_0 and hence span V over $\mathbf{R}$; nondegeneracy of $\operatorname{Im} H$ on $V \times W$ shows that their images under u span W over $\mathbf{R}$, and Lemma 35.37 gives $\Phi \le \frac{1}{2}\sum_{j=1}^r \|u(R^2(v_j - y_j))\|^2 = \frac{1}{2} R^4 \sum_{j=1}^r D_j$. Combining these factors yields

$$|w(x_j)| \le e^{p_1}\, 2(3d_*)^{d_*}\, e^{d_*} (K_*\Phi)^{d_*/4} e^{-\pi D_j/2}, \qquad \Phi \le \frac{R^4}{2} \sum_{j=1}^r D_j. \tag{A.461}$$

Multiplying over $1 \le j \le r$, inserting $R^{2fr^2} \ge 1$ and raising to $2R^4$ gives

$$|f_0(x)|^{2R^4} \le \exp\big(2R^4(p_3 + 2fr^2 \log R) + A \log(K_*\Phi) - 2\pi\Phi\big).$$

Using $\log(\Phi/A) \le \Phi/A - 1 < \Phi/A$ gives $A \log(K_*\Phi) \le A \log(AK_*) + \Phi$, and $2\pi - 1 \ge 1$ then gives $|f_0(x)|^{2R^4} \le \exp(\mathcal{P}(E, r, q, m, R, 1) - \Phi)$. Taking R^4-th roots completes the proof. □

Proof of 35.43

Proof. For (1), since y_0 lies in the signature domain on k_0, every distinguished Gram minor at y_0 is nonzero, so $P_{k_0}(y_0) \neq 0$ gives $A = \|P(y)\|_\lambda > 0$. The ratio hypothesis bounds $\sup_z \|P(z)\|_\lambda$ over domain frames z on U by $e^{-1}A$. Because μ_Z has total mass at most 1 and π_σ lands in the signature domains, $\|P\|_{2,Z} \leq \sup_z \|P(z)\|_\lambda \leq e^{-1}A$. Hence $e^2\|P\|_{2,Z}^2 \leq A^2$, and since $A > 0$ this gives $\|P(y)\|_\lambda^2/\|P\|_{2,Z}^2 \geq e^2$ in $[0, \infty]$.

For (2), choose an orthogonal frame $x_{0,k}$ and a measurable fundamental set $\mathcal{F}_k$ on each analytic component k of U, apply the hypothesis to $(x_{0,k_0}, \mathcal{F}_{k_0})$ to obtain P_0, set P'_{k_0} equal to P_0 on the signature domain of k_0 and 0 outside it, and set $P'_k = 0$ for $k \neq k_0$. Then P' is a holomorphic cuspidal family of weight λ with $\|P'(y)\|_\lambda = \|P_0(y_0)\|_{\lambda_{k_0}} > 0$. Unfolding the adelic quotient measure $\mu_{Q,1}$ into the fundamental sets $\mathcal{F}_k$ yields a finite constant $c_{\mathrm{vol}} > 0$ with $\mu_F(\mathcal{F}_k) < \infty$ such that

$$\int_{Q_W} \|P'(\pi_\sigma(q))\|_\lambda^2 \, d\mu_{Q,1}(q) = c_{\mathrm{vol}} \int_{\mathcal{F}_{k_0}} \|P_0(x_{0,k_0} n(z))\|_{\lambda_{k_0}}^2 \, d\mu_F(z)$$

and $\mu_{Q,1}(\{q \in Q_W : \mathrm{comp}(\pi_\sigma(q)) = k_0\}) = c_{\mathrm{vol}}\mu_F(\mathcal{F}_{k_0})$. Because $y = (k_0, y_0)$ lies over Z, every domain frame on the connected analytic component k_0 lies over Z, so $\{q \in Q_W : \mathrm{comp}(\pi_\sigma(q)) = k_0\} \subseteq \mathcal{C}_Z$ and $c_{\mathrm{vol}}\mu_F(\mathcal{F}_{k_0}) \leq \mu_{Q,1}(\mathcal{C}_Z)$. If $\mu_F(\mathcal{F}_{k_0}) = 0$, then $\|P'\|_{2,Z} = 0$ and $\|P'(y)\|_\lambda^2/\|P'\|_{2,Z}^2 = +\infty \geq \beta$. If $\mu_F(\mathcal{F}_{k_0}) > 0$, then $\mu_{Q,1}(\mathcal{C}_Z) \geq c_{\mathrm{vol}}\mu_F(\mathcal{F}_{k_0}) > 0$, so

$$\|P'\|_{2,Z}^2 \leq \mu_{Q,1}(\mathcal{C}_Z)^{-1} \int_{Q_W} \|P'(\pi_\sigma(q))\|_\lambda^2 \, d\mu_{Q,1}(q) \leq \frac{\int_{\mathcal{F}_{k_0}} \|P_0(x_{0,k_0} n(z))\|_{\lambda_{k_0}}^2 \, d\mu_F(z)}{\mu_F(\mathcal{F}_{k_0})}.$$

Multiplying by β and using the assumed integral bound gives $\beta\|P'\|_{2,Z}^2 \leq \|P'(y)\|_\lambda^2$, whence $G(y) \geq \beta$. □

Proof of 35.44

Proof (outline).

1. Take the supremum over domain frames on all analytic components at the fixed embedding:
$$S = \sup_z \|s(z)\|_{(a,N)}.$$
Cuspidality gives $S < \infty$. The coefficient bound gives $\rho \geq 0$. Since $f, r \geq 1$, $r \leq N$ and $frN \leq 2\kappa$, one has $\kappa > 0$. If $S = 0$, the conclusion follows.

2. Suppose $S > 0$, and put $A = e^L$. The parameter assumptions give
$$A \geq 2, \quad M \leq A^{30}, \quad R_E \leq A^3, \quad fr \leq A, \quad \kappa \leq A^{20}.$$

Claim 1. Put $w = 3\kappa$, let $d_H > 0$ be the host's prime parameter, and define
$$\epsilon(w) = \min\left\{1, \frac{\pi}{2M\sqrt{d_H}(fw+2)\log 2}\right\}.$$
Suppose a domain frame z has positive point norm, every frame z' on its component satisfies $\|s(z')\|_{(a,N)} \leq 2\|s(z)\|_{(a,N)}$, the sum of weight spreads is at most $w > 0$, and the stabiliser character is trivial. For the holomorphic cuspidal section and the uniformisation satisfying the stated level-geometry laws,
$$\lambda_1(z)^2 \geq 2\epsilon(w)/R_E^2.$$

The level-geometry laws give a self-dual transvection model of this component (a principal-level lattice datum with $\mathcal{O}_E$-lattice $\Lambda = \mathcal{A} \cdot L'$ whose trace form is integral and unimodular, arithmetic group Γ preserving L' and containing the unipotent transvections along every isotropic $v \in L'$ with positive transvection step parameter $M_* = MN(\mathfrak{c}_K)d_H$, and metric parameter $k_0 = 2/(N(\mathfrak{c}_K)\sqrt{d_H})$) satisfying
$$M_* k_0 = 2M\sqrt{d_H}.$$

Its cuspidal maximum estimate uses the cutoff

$$\epsilon_* = \min\left\{1, \frac{\pi}{M_*(fw+2)k_0\log 2}\right\}.$$

Substitution of the parameter identity gives $\epsilon_* = \epsilon(w)$. Thus the transvection estimate and the first-minimum comparison use exactly the same level and polarisation normalisation.

Lean: `GlobalTheory.LevGeo.exists_dataX_transv` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsSelfDualDataTransvectionPrincipalLevel`, lines 60–119)

Lean: `GlobalTheory.SupModel.epStar_eq_epsLev` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.WeightedFrameSumPositive`, lines 49–54)

Lean: `GlobalTheory.RawThetaSup.firstMin_at_halfMax` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FirstMinAtHalfMax`, lines 111–169)

Lean: `GlobalTheory.LevGeo.epsLev` (`Development.AbelianVarietyTheory.Definitions.Divisors.UnifOfSpaceEq`, lines 114–114)

The spread in the row indexed by τ, with labels transported by σ, is at most $\sum_i a_{\tau,i}+\sum_i a_{\bar\tau,i}+N$. The transported labels are a permutation, and a CM type contains at most one member of each conjugate pair. Thus the spread sum is at most

$$D + fN \le D + frN \le 3\kappa = w.$$

The level-geometry laws also give the trivial stabiliser character.

Lean: `GlobalTheory.LamSpread.spread_le_sumA` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.SumWeightSpreadLeExponentNormAdd`, lines 25–68)

Lean: `GlobalTheory.LamSpread.sum_spread_laU_le` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.SumWeightSpreadLeExponentNormAdd`, lines 70–94)

Choose z with $S/2 < \|s(z)\|_{(a,N)}$. Since all point norms are at most S, Claim 1 applies. Right multiplication by the upper-triangular matrix orthonormalising its flag gives x with unchanged first minimum and invariant point norm. The distinguished Gram minors at x have modulus one, so

$$S \le 2|s(x)|, \qquad \lambda_1(x)^2 \ge 2\epsilon(w)/R_E^2.$$

Lean: `GlobalTheory.RawThetaSup.half_sup_at_ortho` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FirstMinAtHalfMax`, lines 171–207)

The host discriminant estimate gives $d_H \le R_E^2$, hence $\sqrt{d_H} \le A^3$.

Lean: `GlobalTheory.NormSum.Compare.log_d_le` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.ExistsTotallyComplexExt`, lines 26–33)

Since $f \le A$, $w \le 3A^{20}$, and $A \ge 2$,

$$fw + 2 \le 3A^{21} + 2 \le A^{24}, \qquad 2M\sqrt{d_H}(fw+2)\log 2 \le 2A^{57} \le \pi A^{57}.$$

Both entries in the minimum are at least A^{-57}. Therefore

$$\epsilon(w) \ge A^{-57}, \qquad A^{-80}R_E^2 \le A^{-74} \le A^{-57},$$

which yields $\lambda_1(x)^2 \ge A^{-80}$.

Lean: `GlobalTheory.RawThetaSup.inv_pow_le_epStar` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.NormPhiHatIndPhi`, lines 102–118)

3. Put $d = frN$ and $B_P = \sum_{\nu\in\operatorname{supp} P}|\sigma(P_\nu)|$. *Claim 2.* At an orthonormal frame in this uniformisation, for $\lambda > 0$ with $\lambda^2 \le \lambda_1(x)^2$, the theta rule of P, its lattice characteristics and canonical connection satisfies

$$|s(x)| \le B_P(2\pi e\xi_{\mathbf{R}}D)^{D/2}\sqrt{2}\left(1 + \frac{8f^{3/2}N^2R_E\sqrt{2d}}{\sqrt{\pi}\,\lambda}\right)^d.$$

The polarisation scale is $2\xi_{\mathbf{R}}$; the degree factor is one when $D = 0$. Indeed, write $F(v)$ for the translated theta function at the orthonormal frame x. Using the lower bound λ on the shortest nonzero vector of

the period lattice, we first bound the Gaussian-weighted pointwise norm $\|F(v)\|^2 \exp(-2\pi\xi_{\mathbf{R}}\|v\|^2)$ uniformly on $\mathbf{C}^d$ by the torus L^2 norm times $2\big(1+8f^{3/2}N^2R_E\sqrt{2d}/(\sqrt{\pi}\,\lambda)\big)^{2d}$, isolating all λ-dependence in this constant amplitude. For $D \geq 1$, we then extract each degree-D partial derivative of $P(\partial)$ at the origin by iterated one-dimensional Cauchy estimates of Gaussian-adapted step radius $h = (2\pi\xi_{\mathbf{R}}D)^{-1/2}$ (total ℓ^1-radius $Dh = \sqrt{D/(2\pi\xi_{\mathbf{R}})}$, where the Gaussian factor satisfies $\exp(\pi\xi_{\mathbf{R}}\|z\|^2) \leq e^{D/2}$ and the Cauchy denominator contributes $h^{-D} = (2\pi\xi_{\mathbf{R}}D)^{D/2}$, multiplying to $(2\pi e\xi_{\mathbf{R}}D)^{D/2}$, with the trivial factor 1 at $D = 0$); taking square roots and summing over $\nu \in \operatorname{supp} P$ introduces B_P.

Lean: `GlobalTheory.Dict.ThetaPt.theta_bound_ortho` (`Development.AbelianVarietyTheory.Proofs.CuspFormEvaluations.ThetaGeneratorPointwiseBoundOrthonormalFrame`, lines 46–101)

Use $\lambda = A^{-40}$, $B_P \leq \rho$, and $S \leq 2|s(x)|$:

$$(2\pi\xi_{\mathbf{R}})^{-D/2}S \leq 2\sqrt{2}\,\rho(eD)^{D/2}\left(1+\frac{8f^{3/2}N^2R_E\sqrt{2d}}{\sqrt{\pi}\,\lambda}\right)^d.$$

Here $\kappa \geq 1/2$. The bounds $f \leq A$, $N \leq 2A^{20}$, $R_E \leq A^3$, and $\sqrt{2d} \leq 2A^{10}$ make the parenthesised factor at most $1 + 64A^{95} \leq A^{102}$. For $D \geq 1$, $\log D \leq 20L$, so $(eD)^{D/2} \leq e^{11\kappa L}$; this also holds at $D = 0$. Moreover,

$$A^{102d} \leq e^{204\kappa L}, \qquad 2\sqrt{2} \leq e^{4\kappa L}.$$

Consequently

$$(2\pi\xi_{\mathbf{R}})^{-D/2}S \leq \rho e^{219\kappa L} \leq \rho\exp(9000(frL)^2\kappa),$$

using $fr, L \geq 1$. Every point norm is bounded by S.

The complete proof is contained in `UpperTheory.Dv.Thin.theta_sup_le`. □

Proof of 35.45

Proof (outline). 1. Replace y by an orthogonal frame x on the same component. First minima and invariant section norms are unchanged. The level-geometry laws supply the test-family realisation and $|\tau(\xi_{0,k})| = \xi_{\mathbf{R}}$ at every archimedean embedding. Choose a period uniformisation and unit theta trivialisation, whose Hermitian matrix is $2\xi_{\mathbf{R}}$ times the identity.

The period-coordinate isometry, with the embedding relabelling determined by σ, gives a polarised $\mathcal{O}_E$-torus T of rank r. It preserves the first minimum and truncated phase and carries P to a homogeneous degree-D block polynomial P_T with the same disjoint column supports. The normalisation of the principal theta section s is justified by the following comparison.

Claim 1. Let θ be the holomorphic theta function in the chosen unit period trivialisation, let H_{per} be its positive Hermitian period form, and let $\mathcal{F}$ be a fundamental domain of its period lattice. The period-coordinate isometry ψ identifies this lattice, its Hermitian form and its principal semicharacter with those of T. The section $s = \theta \circ \psi^{-1}$ belongs to the principal theta space and satisfies

$$\ell_1(s) = \frac{1}{\operatorname{vol}\mathcal{F}}\int_{\mathcal{F}} |\theta(z)|^2 e^{-\pi\operatorname{Re}H_{\mathrm{per}}(z,z)}\,dz = 1.$$

The corresponding point norms are equal under ψ. The first equality follows from change of variables in the normalised Haar integral: the same Jacobian multiplies the numerator and the volume denominator. The theta functional equation makes the integrand lattice-periodic, so the normalised integral is independent of the fundamental domain. The second equality is the unit normalisation in the period data, available because the distinguished Gram minors at x have absolute value one.

Lean: `ReciprocityTheory.ChartRestriction.G.PullbackSections.UnitAt` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.PullbackSections.NonemptyUnitAt`, lines 58–85)

Lean: `ReciprocityTheory.ChartRestriction.G.PullbackSections.nonempty_unitAt` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.PullbackSections.NonemptyUnitAt`, lines 87–90)

Lean: `GlobalTheory.EtaAn.ThetaTorus.theta_section_l2_torus` (`Development.AbelianVarietyTheory.Proofs.TorusSplitting.ThetaFunctionTorusL2Norm`, lines 61–152)

Thus the normalised principal theta section has squared L^2-norm one. Multiplication by α scales the Hermitian form by q. Uniform admissibility, with $q_{\mathrm{sig}} \le m$, gives the admissible tuple of Definition 35.41, with these q, R_0, m and phase $\Phi_D(y)$; in particular $D \ge 1$.

2. Let L_T be the period lattice and $\mathrm{mk} : \Lambda \to L_T$ its additive $\mathcal{O}_E$-linear marking. It identifies the lattices modulo q:

$$\mathrm{mk}(a) \in qL_T \Longrightarrow a \in q\Lambda, \qquad L_T = \mathrm{mk}(\Lambda) + qL_T.$$

For $u \in \Lambda/\bar{\alpha}\Lambda$, choose

$$x(u) \in 2\alpha^2\Lambda, \qquad x(u) \equiv u \pmod{\bar{\alpha}\Lambda}, \qquad t(u) = -\bar{\alpha}q^{-2}\,\mathrm{mk}(x(u)).$$

The ideals (α), $(\bar{\alpha})$ are coprime.

Claim 2. The set

$$K = \{(t(u_1), \dots, t(u_N)) : u_i \in \Lambda/\bar{\alpha}\Lambda\}$$

lies in $(q^{-1}L_T)^N$, and each $b \in (q^{-1}L_T)^N$ has exactly one $b' \in K$ with $\alpha(b_i - b'_i) \in L_T$ for all i.

It suffices to work in one coordinate. Write $x(u) = \alpha^2 c$. Then

$$qt(u) = -\,\mathrm{mk}(\alpha c) \in L_T, \qquad \alpha t(u) = -q^{-1}\,\mathrm{mk}(x(u)).$$

For $b \in q^{-1}L_T$, write $qb = \mathrm{mk}(a) + qv'$ and choose $u = -\alpha a$ modulo $\bar{\alpha}\Lambda$. Then $x(u) + \alpha a$ belongs to both $\alpha\Lambda$ and $\bar{\alpha}\Lambda$. Choose $r_{\mathrm{B}}\bar{\alpha} + s_{\mathrm{B}}\alpha = 1$, and write $x(u) + \alpha a = \bar{\alpha}d$. Bézout's identity gives

$$x(u) + \alpha a = q\{r_{\mathrm{B}}(\alpha c + a) + s_{\mathrm{B}}d\},$$

hence

$$\alpha(b - t(u)) = q^{-1}\,\mathrm{mk}(\alpha a + x(u)) + \alpha v' \in L_T.$$

If two classes work, then $q^{-1}\,\mathrm{mk}(x(u) - x(u')) \in L_T$. Injectivity modulo q gives $x(u) - x(u') \in q\Lambda \subset \bar{\alpha}\Lambda$, so $u = u'$. Apply this coordinatewise.

Lean: `GlobalTheory.isReps_level_translates` (`Development.AbelianVarietyTheory.Proofs.CuspFormEvaluations.IsRepsLevelTranslates`, lines 49–157)

Lean: `ReciprocityTheory.ChartRestriction.G.I2.IsReps` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.SquareIntegrability.GCauchyStatement`, lines 210–214)

Let V be the complex vector space of T, and define

$$H_N(z,b) = \sum_{i=1}^{N} H(z_i, b_i), \qquad s_0(z) = \prod_{i=1}^{N} s(\alpha z_i),$$

$$(U_q(b)F)(z) = \exp\{q\pi H_N(z,b) - q\pi H_N(b,b)/2\}F(z-b), \qquad F_b = U_q(b)s_0.$$

For the lattice fundamental parallelepiped $\mathcal{F}_N$, put

$$\ell_q(F) = \frac{1}{\operatorname{vol}\mathcal{F}_N}\int_{\mathcal{F}_N} |F(z)|^2 e^{-q\pi\,\mathrm{Re}\,H_N(z,z)}\,dz.$$

Replace $|F|^2$ by $F\overline{G}$ to define its Hermitian inner product. Thus $\ell_1(s) = 1$, using $N = 1$ in this definition. Let $\mathcal{H}_q$ be the holomorphic section space of the q-th power of the principal theta line on T^N, in period coordinates, and set $\mathcal{D}F = P_T(\partial)F|_0$.

Lean: `ReciprocityTheory.ChartRestriction.G.I2.PT.U` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.SquareIntegrability.GCauchyStatement`, lines 92–94)

Lean: `ReciprocityTheory.ChartRestriction.G.I2.PT.ptN2` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.SquareIntegrability.GCauchyStatement`, lines 82–83)

Lean: `ReciprocityTheory.ChartRestriction.G.I2.PT.l2` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.SquareIntegrability.GCauchyStatement`, lines 87–87)

Lean: `ReciprocityTheory.ChartRestriction.G.I2.s0` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.SquareIntegrability.GCauchyStatement`, lines 206–208)

Claim 3. For this polarised CM torus, the admissible tuple (q, R_0, m) at degree D, the multiplication map α, the nonzero principal section s, the disjoint block polynomial P_T, and the representative set K, the functions F_b belong to $\mathcal{H}_q$, are mutually orthogonal, span that space, and satisfy

$$\ell_q(F_b) = \ell_1(s)^N.$$

There is also $F \in \mathcal{H}_q$ with

$$\ell_q(F) = 1, \qquad |\mathcal{D}F|^2 \geq J_y,$$

where

$$J_y = 4^{-N}\big(6(2d_*)^{d_*/2}\big)^{-N} 2^{-(r+2)D} \exp\{-2\pi N(8100 m^4 d_*^4 R_E^7)\Phi_D(y)/q\}.$$

The map and section here are those supplied by the period uniformisation; the second assertion uses every admissibility condition and disjointness of the block supports. The test-section construction uses a saturated $\mathcal{O}_E$-stable isotropic sublattice of the period lattice, on which the principal semicharacter is trivial. It is zero in the case $\lambda_1(y) \geq \theta(q, R_0)$; otherwise its real rank is $2f\rho$ with $\rho \leq m$. Greedy selection supplies this sublattice and its separation bounds, decomposing the Gaussian theta space $\mathcal{H}_q$ into a toric part and a complementary abelian part. On the toric part, Fourier modes of controlled character norm contribute the exponential decay $\exp(-2\pi N C_\chi \Phi_D(y)/q)$ governed by the truncated phase $\Phi_D(y)$, while on the complementary abelian part Cauchy–Gaussian peak sections have L^2-normalised block derivatives bounded below by $4^{-N} K_6^{-N} 2^{-(r+2)D}$. These test sections are one-variable sections attached to the columns. Put $\epsilon_1 = 7 \cdot 2^{fr}(4.3)^{d_*} e^{-t_0(q,R_0)}$. For a column of a block with exponent c, the section s_k has c-th derivatives along unit directions $w_1, \dots, w_c$ of its type within $\epsilon_1|\mu_k|$ of $\mu_k \prod_j H(w_j, \eta_k)$, where η_k is the character vector of a toric column or the unit vector of an abelian column, and $K_6^{-1} e^{-2\pi C_\chi \Phi_D(y)/q} \ell_q(s_k) \leq |\mu_k|^2$. Each column outside the blocks carries a section with $|s_k(0)| \geq 1/2$ and $K_6^{-1}\ell_q(s_k) \leq 1$. For $F_0 = \bigotimes_k s_k$, the value $\mathcal{D}F_0$ is a signed sum over column choices of products of these derivatives. Its main term is the product of the values $s_k(0)$ off the blocks, of the scalars μ_k, and of the powers of the block determinants of the vectors η_k; this product of determinant powers has modulus at least $(2^{-r/2}/2)^D$. The vectors η_k have norm at most 2, so for unit directions every factor $H(w_j, \eta_k)$ has modulus at most 2. In the expansion a column of exponent c therefore contributes a factor of modulus at most $2^c|\mu_k|$ to the main part of each term and at most $(2^c + \epsilon_1)|\mu_k|$ in total, and there are at most r^D column choices. The remaining terms have total modulus at most $r^D\big(\prod_k (2^{c_k} + \epsilon_1) - 2^D\big)$ times the product of the $|\mu_k|$ and of the values $|s_k(0)|$ off the blocks. The last admissibility inequality gives $\epsilon_1 \leq \exp\{-(D+1)(r\log r + 3r + 3)\}$, which makes this at most half of the main term. Since $\ell_q(F_0) = \prod_k \ell_q(s_k)$, the section $F = \ell_q(F_0)^{-1/2} F_0$ satisfies $\ell_q(F) = 1$ and $|\mathcal{D}F|^2 \geq J_y$.

Lean: `GlobalTheory.T.greedy_splitting_principal_torus_exists` (`Development.AbelianVarietyTheory.Proofs.TorusSplitting.GreedySplittingPrincipalTorusExistsStatement`, lines 6–7)

Lean: `GlobalTheory.ThetaTorus.N_holds` (`Development.AbelianVarietyTheory.Proofs.TorusSplitting.GCauchyHolds`, lines 32–32)

Lean: `GlobalTheory.ThetaTorus.J_holds` (`Development.AbelianVarietyTheory.Proofs.TorusSplitting.GCauchyHolds`, lines 26–30)

Claim 4. These assertions imply

$$J_y \ell_1(s)^N \leq \sum_{b \in K} |\mathcal{D}F_b|^2.$$

Write $F = \sum_b c_b F_b$. Orthogonality gives

$$1 = \ell_1(s)^N \sum_b |c_b|^2.$$

Linearity of differentiation and Cauchy–Schwarz give

$$J_y \leq \left| \sum_b c_b \mathcal{D}F_b \right|^2 \leq \left(\sum_b |c_b|^2 \right) \left(\sum_b |\mathcal{D}F_b|^2 \right).$$

Multiply by $\ell_1(s)^N$. Since $\ell_1(s) = 1$, this is the required derivative-sum estimate, and every factor in J_y is positive.

Lean: `ReciprocityTheory.ChartRestriction.G.I2.iOf` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.SquareIntegrability.ContDiffOfDifferentiableFd`, lines 148–203)

Lean: `GlobalTheory.ThetaTorus.I_holds` (`Development.AbelianVarietyTheory.Proofs.TorusSplitting.GCauchyHolds`, lines 34–34)

3. Theta-rule evaluation and period differentiation satisfy

$$|s_u(x)|^2 = \left(\frac{2\xi_{\mathbf{R}}}{q}\right)^D |P_T(\partial)(U_q(b(u))s_0)|_0|^2 .$$

Indeed, homogeneity gives an α-scaling scalar of squared modulus q^D, and each period direction contributes $\sqrt{2\xi_{\mathbf{R}}}$. Every $b \in K$ has a characteristic preimage. Choose one for each b; these preimages are distinct, and the other squared values are nonnegative. Hence

$$\left(\frac{2\xi_{\mathbf{R}}}{q}\right)^D J_y \le \sum_u |s_u(x)|^2.$$

At the orthogonal frame, these are the invariant point norms at y. Multiplication by $q^D(2\pi\xi_{\mathbf{R}})^{-D}$ proves the asserted sum bound.

4. Write $S = q^D \sum_u v_u(y)^2 > 0$. The constants in the statement give

$$\begin{aligned} -\log\sqrt{\pi^{-D}J_y} &= \frac{D}{2}\log\pi + \frac{N}{2}\left(\log 24 + \frac{d_*}{2}\log(2d_*)\right) + \frac{(r+2)D}{2}\log 2 + c_0\Phi_D(y) \\ &\le A_0 D + c_0\Phi_D(y) \le AD + c\phi(y). \end{aligned}$$

Here $D \ge 1$, the parenthesised logarithmic term is nonnegative, and $0 \le \Phi_D(y) \le \phi(y)$, $c \ge c_0 \ge 0$. Thus $\sqrt{S} \ge e^{-AD-c\phi(y)}$. There are q^{frN} characteristic tuples. A largest $v_u(y)$ satisfies

$$v_u(y) \ge q^{-(D+frN)/2}\sqrt{S} \ge q^{-(D+frN)}e^{-AD-c\phi(y)},$$

because $q \ge 1$.

The complete proof is contained in `UpperTheory.Dv.Thin.theta_lower`. □

Proof of 35.46

Proof. Write Y for the good cover, $P = P_*$ and $L = L_x$. By (A.418), (A.433), (A.437) and Lemmas 35.20 and 35.21, beyond a common family threshold the host, scale and cover parameters satisfy

$$\begin{gathered} L \ge 10, \quad b \ge 1000, \quad 1 \le m \le r, \quad \log R_E, \log f \le 2L, \quad \log r, \log(fr) \le L, \\ b^{16} \le Q_0 \le b^{70}, \quad Q_0 b \le M, \quad m^4 \le f, \quad f^2 \le R_E, \quad 16 \le R_E, \quad R_E^7 \le b, \\ (frL)^{12} \le P, \quad (frL)^{24} \le T, \qquad PTd_{\mathrm{fl}}^2 L \le b. \end{gathered} \tag{A.462}$$

Here the base primes g_0, g_1 lie between b^8 and b^{35}; for the host conductor $Q_S qd$ of 8.3, the inclusion $E \subset \mathbf{Q}(\mu_{Q_S qd})$ gives $R_E \le Q_S qd$, and the host size condition $(Q_S qd)^{36} t^{150} \le \sqrt{b}$ of Definition 33.3 (with chain rank parameter $t \ge 1$) gives $R_E^7 \le R_E^{28} \le b$. In Lemma 35.21, $M = Q_0 \cdot 15^e$ with $e = \lfloor 3L/\log 3 \rfloor$; since $0 < \log 3 \le 2$ and $L \ge 10$, we have $e > 3L/\log 3 - 1 \ge 14$, so $e \ge 15 \ge 4$, $5^4 \mid 15^e \mid M$, and $e \log 3 > 3L - 2 \ge L$, whence $15^e \ge 3^e > e^L = b$ and $Q_0 b \le M$. Moreover, for every integer $1 \le D \le b^2$, (A.462) gives $R_E^{28} = (R_E^7)^4 \le b^4$, $r \le fr \le P \le b$ (since $T, d_{\mathrm{fl}}, L \ge 1$), $90(fr + D) \le 180b^2$ and $b \ge 1000 \ge 180$, so

$$90R_E^{28}(fr + D)Dr \le b^4(180b^2)b^2 b = 180b^9 \le b^{10} \le b^{16} \le Q_0.$$

Summing the floored differences in (35.6) from the anchor $u_{1,\tau,k-2} = 0$, with stage-three window parameters $N = N_m$, $B_{\mathrm{win}} = x^{\varepsilon}$, Δ, $H_0 = \eta/m$ from Definitions 14.6 and 14.11, the spacing hypothesis $C_{\mathrm{count}} \ge 4e_P + 11$

gives $T(T+d_{\mathrm{fl}}+1) \le 2C_P^4(NfB_{\mathrm{win}})^{4e_P+10} \le \min(4\Delta, H_0)$ beyond the family threshold. As in Lemma 35.20, $\min(4\Delta, H_0, h_0) \ge (T+d_{\mathrm{fl}}+2)(T+d_{\mathrm{fl}}+2+2r)+1$, yielding floored within-block gaps $u_{1,\tau,s} - u_{1,\tau,s+1} \ge T + d_{\mathrm{fl}} + 2r \ge T$ and cross-wall gap at least $h_0/(2T) \ge 2T \ge T$, so $u_{1,\tau,s} - u_{1,\tau,s+1} \ge T$ for all $0 \le s < r-1$. Since $p_{\mathrm{sig}} = k+j-1 \ge k = (k-2)+2$, every index $s \ge p_{\mathrm{sig}}$ satisfies $u_{1,\tau,s} \le -2T < -3r$, so $u' = u_1 - 3r\eta_H$ keeps $u'_{\tau,s} = u_{1,\tau,s} + 3r < 0$ on $s \ge p_{\mathrm{sig}}$ and agrees with u_1 on $s < p_{\mathrm{sig}}$, giving $\|u'\|_1 \le \|u_1\|_1$. With the width parameters $S, \eta \ge 0$ and window scale bound $w'' \le 8b$ of Lemma 35.20, the $k-2 = r-m-1$ indices $s < k-2$ satisfy $0 \le u_{1,\tau,s} \le (S+2r)/T$ and the m indices $s > k-2$ satisfy $|u_{1,\tau,s}| \le (8b + c_0 + 4r)/T$ (with $2(k-2)r + 4mr = 2(r+m-1)r \le 4r^2$ since $m \le r$). Using $P(S+\eta) \le 100w'' \le 800b$, $r \le P$, $c_0 \le 22f^2r^4$, $26f^2r^5 \le 26P \le PTd_{\mathrm{fl}}^2 L \le b$ and (A.418) gives

$$T\|u'\|_1 \le T\|u_1\|_1 \le f\{rS + 8mb + r(c_0+4r)\} \le 834fmb \le 834E_p. \tag{A.463}$$

Lean: `GlobalTheory.CovTop.Sizes.hostNum_of_sizes` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.HostBoundHolds`, lines 37–108)
Lean: `GlobalTheory.CovTop.WtShape.wtNum_at` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.WeightBoundHolds`, lines 45–185)
Lean: `GlobalTheory.ThetaSizeBounds.level_ge` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.EtaBoundLengthUnit`, lines 139–161)
Lean: `GlobalTheory.ThetaSizeBounds.size_inequality_real` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.EtaBoundLengthUnit`, lines 72–120)
Lean: `GlobalTheory.CovHt.DividedNum.lenT_bounds_of_covers_before_last` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.DividedWeight`, lines 18–24)

Let $C_{\mathrm{seed}}, C_S \ge 0$ be the absolute constants of Lemmas 35.38 (for a fixed cutoff) and 35.39, set $C_{\mathrm{peak}} = C_{\mathrm{seed}} + C_S + 1$, and choose the common family threshold before fixing the point and layer so that $(frL)^{24}$ exceeds both $C_{\mathrm{peak}}(fr)^8$ and the gap threshold of Lemma 35.38 at $\rho_* = 1/(200r^2)$, $b_* = 4fr^2+1$ and $d_{\mathbf{C}} = fr(r-1)/2$. Put $P_0 = \mathcal{P}(E, r, Q_0, m, 5, 1)$ in Definition 35.41 and $B'_0 = C_{\mathrm{peak}}(\|u_1\|_1 + (fr)^8) \ge 0$. By (A.462) and (A.433) (which with $r = p_{\mathrm{sig}} + q_{\mathrm{sig}} \ge 2$ give $fr \ge 8 \cdot 2 \ge 2$ and $\log Q_0 \le 70L$) and Lemma 35.42(1), $R_0(D) = \sqrt{90(fr+D)} > 0$ makes $(E, r, D, Q_0, R_0(D), m)$ uniformly admissible for every integer $1 \le D \le b^2$, and $0 \le P_0 \le 200000(frL)^2$. Since $(frL)^2, (fr)^8 \le (frL)^{12} \le P$, $d_{\mathrm{fl}} \ge 1$, $L \ge 10$ and $b \le E_p$ by (35.7), (A.462) and (A.463) give $TP_0 \le 200000\,TP \le 200000b \le 200000E_p$ and $TB'_0 \le C_{\mathrm{peak}}(834E_p + E_p) = 835C_{\mathrm{peak}}E_p$, proving (35.13). At a nonterminal layer $q_{\mathrm{sig}} = m - j \le m$, so $\min(p_{\mathrm{sig}}, q_{\mathrm{sig}}) \le m$; also $5 > 1$ is odd and $5^4 \mid M$. Applying Lemma 35.42(2) on Y with $D = 1$, $q = Q_0$, $R = 5$ ($R^4 = 625$) and $R_0 = R_0(1) = \sqrt{90(fr+1)} > 0$ gives $a = \|f_0^2(y)\| \le \exp((P_0 - \Phi)/625) \le \exp(P_0/625)$ everywhere since $\Phi \ge 0$. If $M\lambda_1^2 < b$, then $Q_0 b\lambda_1^2 \le M\lambda_1^2 < b$ by (A.462), so $Q_0\lambda_1^2 < 1 \le fR_0^2R_E^{21}$, $\lambda_1 < \theta(Q_0, R_0)$, $\Phi = \phi$, and Lemma 35.42(2) gives $a \le \exp((P_0 - \phi)/625)$.
Lean: `GlobalTheory.ThetaSizeBounds.sqrt_lt_th_real` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.EtaBoundLengthUnit`, lines 122–137)

Suppose now that $M\lambda_1^2 < b$, and transport y along the covering map from Y to a domain frame $\bar{y}$ at base level Q_0. The transport preserves λ_1, so $Q_0\lambda_1^2 < 1$. Let Ψ be the CM type of size f, let h_σ be the Hodge multiplicity at $\sigma : E \to \mathbf{C}$ (p_{sig} on Ψ and q_{sig} on $\overline{\Psi}$), index pairs (σ, i) by $1 \le i \le h_\sigma$, write $v = fr$ for their number, and define the column count

$$N = 8\left(\left\lfloor \frac{1}{8}\sum_{\sigma,i} 2i \right\rfloor + v + 1\right).$$

Then $8 \mid N$, $N \ge \sum_{\sigma,i} 2i \ge 2fr \ge 2r$, $2N \ge fr(r+1)$, and $N \le 2fr^2 + 8fr + 8 \le 20fr^2$, whence $frN \le 20f^2r^3 \le P \le b$ by (A.462). With zero-based column indices on u_1, put $s_\tau = N/2 + 1 - u_{1,\tau,p_{\mathrm{sig}}-1}$ for $\tau \in \Psi$. Set $a_{\tau,i} = (u_{1,\tau,i-1} + s_\tau) - N/2 = u_{1,\tau,i-1} - u_{1,\tau,p_{\mathrm{sig}}-1} + 1$ for $1 \le i \le p_{\mathrm{sig}}$ with $a_{\tau,p_{\mathrm{sig}}+1} = 0$, and at $\bar{\tau}$ reverse and negate the shifted second block and subtract $N/2$ to set $a_{\bar{\tau},i} = -(u_{1,\tau,r-i} + s_\tau) - N/2 = u_{1,\tau,p_{\mathrm{sig}}-1} - u_{1,\tau,r-i} - N - 1$ for $1 \le i \le q_{\mathrm{sig}}$ with $a_{\bar{\tau},q_{\mathrm{sig}}+1} = 0$. Because $u_{1,\tau}$ has adjacent gaps at least $T \ge (frL)^{24} > N+1$, each row a_σ is nonincreasing and nonnegative with $a_{\tau,p_{\mathrm{sig}}} = 1 > 0$ on Ψ. Put $c_{\sigma,i} = a_{\sigma,i} - a_{\sigma,i+1} \ge 0$ and $D = \sum_{\sigma,i} ic_{\sigma,i} = \sum_{\sigma,i} a_{\sigma,i}$. Then $D \ge f \ge 1$, and (A.463) with (A.462) gives $\|u_1\|_1 < b$ and $D \le (r+1)\|u_1\|_1 + fr \le b^2$, so $(E, r, D, Q_0, R_0(D), m)$ is uniformly admissible. Since $\sum_{\sigma,i} i \le N$, we may assign pairwise disjoint sets of i columns in $\{1, \dots, N\}$ to the pairs (σ, i) and multiply the leading $i \times i$ minors of σ-variables in those columns raised to the powers $c_{\sigma,i}$. Specialising the assigned blocks to identity matrices gives value 1, while every Laplacian contraction differentiates τ- and $\bar{\tau}$-variables in

the same column and therefore annihilates the product; this defines a nonzero integral harmonic differential of theta weight (a,N) and degree D, whose ℓ^1 coefficient sum is at most $\prod_{\sigma,i}(i!)^{c_{\sigma,i}} \le r^D \le e^{Db}$.

Let $\alpha \in \mathcal{O}_E$ be the split half-level generator of the base level, with $\alpha\bar{\alpha} = Q_0$ and half-level ideal $(\bar{\alpha})$. Index the Q_0^{frN} half-level characteristic tuples by u, and let s_u be the associated theta derivative of weight (a,N). The polynomial is harmonic, N is even, and $a_{\tau,p_{\mathrm{sig}}} = 1 > 0$ on Ψ, so Lemma 35.40 makes every s_u cuspidal. Put $v_u(\bar{z}) = (2\pi\xi_{\mathbf{R}})^{-D/2}\|s_u(\bar{z})\|_{(a,N)}$, $R_0 = R_0(D)$, and let Φ_D be the truncated phase at $\bar{y}$ with threshold $\theta(Q_0,R_0)$. The disjoint column blocks constructed above and $q_{\mathrm{sig}} \le m$ verify the hypotheses of Lemma 35.45. With its constants $d_* = 2fm$, $K_6 = 6(2d_*)^{d_*/2}$ and $C_\chi = 8100m^4d_*^4R_E^7$, that lemma gives

$$J_{\mathrm{test}} = 4^{-N}\{6(2d_*)^{d_*/2}\}^{-N}2^{-(r+2)D}\exp\{-2\pi N(8100m^4d_*^4R_E^7)\Phi_D/Q_0\} > 0 \tag{A.464}$$

and

$$Q_0^D\sum_u v_u(\bar{y})^2 \ge \pi^{-D}J_{\mathrm{test}}. \tag{A.465}$$

Here $N, r \le b$, $d_* = 2fm \le 2fr \le 2b$, and $b \ge 1000$, so the constant A_0 of Lemma 35.45 satisfies

$$A_0 \le \frac{1}{2}\{\log\pi + b(\log 24 + b\log(4b)) + (b+2)\log 2\} \le 3b^3.$$

For the last inequality one may use $\log\pi \le 2$, $\log 24 \le 4$, $\log 2 \le 1$ and $\log(4b) \le 2b$. Moreover, (A.462) gives $m^4d_*^4 = 16f^4m^8 \le (frL)^{12} \le b$, $R_E^7 \le b$ and $Q_0 \ge b^{16}$, whence

$$\frac{\pi NC_\chi}{Q_0} \le \frac{8100\pi b^3}{b^{16}} < \frac{1}{2500}.$$

Apply the selected-characteristic conclusion of Lemma 35.45 with $A = 3b^3$ and $c = 1/2500$. It gives a tuple u with $s_u(\bar{y}) \neq 0$ satisfying the first bound in

$$\begin{aligned} v_u(\bar{y}) &\ge \exp\{-3b^3D - \phi/2500 - (D+frN)\log Q_0\},\\ v_u(\bar{z}) &\le \exp\{Db + 9000(3fr\log b)^2b^2\}. \end{aligned} \tag{A.466}$$

The second bound follows from Lemma 35.44 with $L_{\mathrm{up}} = 3\log b$, $\kappa = b^2$ and $\rho = e^{Db}$. Indeed, $L_{\mathrm{up}} \ge 1$, $r \le N$, $\log Q_0 \le 70\log b \le 30L_{\mathrm{up}}$, $\log R_E, \log(fr) \le \log b \le L_{\mathrm{up}}$, $D \le \kappa$, $frN \le b \le 2\kappa$, and $\log\kappa = 2\log b \le 20L_{\mathrm{up}}$. The integral harmonic polynomial has coefficient sum at most $r^D \le e^{Db}$, and every s_u is cuspidal. From $Q_0\lambda_1^2 < 1$ and (A.462) we have $\phi = 2f/(R_E^2\lambda_1^2) > 2fQ_0/R_E^2 \ge 2b^{15} \ge b^{14}$; since $D \le b^2$, $frN \le b$, $\log Q_0 \le 70\log b \le 70b$, $fr\log b = frL \le b$ and $b \ge 1000$, the non-phase exponents in (A.466) sum to at most $3b^5 + 70b(b^2+b) + b^3 + 81000b^6 \le 82000b^6 \le \phi/2500$. Cancelling $(2\pi\xi_{\mathbf{R}})^{-D/2}$ gives $e^{-\phi/1250}\|s_u(\bar{z})\|_{(a,N)} \le \|s_u(\bar{y})\|_{(a,N)}$ for every base domain frame $\bar{z}$. Pulling s_u back to Y preserves cuspidality, nonvanishing at y and the invariant point-norm ratio; multiplying on Z by the determinant section of central parameter $-N/2$ (half-weight exponent $-N$), the flat sections of Lemma 35.6 of row exponents $(-s_\tau)_{\tau\in\Psi}$ and the component idempotent of Z produces a section g of weight $(u_1,0)$ supported on Z whose invariant point norm on Z is a constant $c_Z > 0$ times that of the pullback. In each boundary Fourier–Jacobi coordinate supplied by $\mathcal{H}_P$ and $\mathcal{H}_H$, the determinant and flat multipliers are holomorphic of constant invariant norm and hence of at most polynomial growth under isotropic boundary shears; Cauchy's circle integral eliminates negative Fourier exponents in each multiplier, while the pulled-back cusp section has strictly positive Fourier exponents, so g is cuspidal. Since the adelic frame map of Y lands in the signature domains and g vanishes off Z, applying Lemma 35.43(1) to the holomorphic cuspidal family of g with $e = e^{-\phi/1250}$ yields $G(y) \ge e^{-\phi/625}$.

Lean: `GlobalTheory.RefSum.GOf` (`Development.AbelianVarietyTheory.Definitions.Divisors.EightDvdNCol`, lines 74–74)
Lean: `GlobalTheory.T.thin_divisor_pivot_bound_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CardMulSuccLeTwoMulNcolCopy`, lines 149–175)
Lean: `GlobalTheory.T.cusp_form_bounded_on_frames_alt_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CuspFormBoundedOnFrames`, lines 36–36)
Lean: `GlobalTheory.ThinBerg.bergman_kernel_bound_thin_part_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.BergmanKernelThinPartEstimate`, lines 18–52)

Suppose finally that $M\lambda_1^2 \ge b$. Write $y = (k_0, y_0)$ and let λ be the model weight of $(u_1,0)$. Reorthogonalising $y_0 = x_1u_1'$ on k_0 with x_1 orthogonal and u_1' upper triangular preserves λ_1^2 and satisfies $P_0(y_0) \neq 0$

and $\|P_0(y_0)\|_{\lambda_{k_0}} = \|P_0(x_1)\|_{\lambda_{k_0}}$ whenever P_0 is a cusp form of weight λ_{k_0} on k_0 with $P_0(x_1) \neq 0$. At a nonterminal layer both p_{sig} and q_{sig} are at least 1, and the component analytic uniformisations, adelic frame parametrisation and compatible orthogonal frames of the good cover Y verify the level-geometry laws of Lemma 35.43(2). Because $g_0, g_1 \geq b^8 > 3$ are distinct odd primes, the prime-to-ℓ part of $Q_0 = g_0 g_1$ is at least 3 at every prime ℓ, so Q_0 and $M = Q_0 \cdot 15^e$ are good at every prime, and $M' = M$ since the auxiliary level parameter is 1. From (A.462) and $M\lambda_1^2 \geq b$ we obtain $M'\lambda_1(x_1)^2 \geq b \geq 100(fr)^4$. For any $\gamma \in \Gamma_{k_0}$ carrying a point of the unitriangular chart ball $\|z\| \leq 1/(100r^2)$ at x_1 to a right Borel translate of another point of that ball, both points lie in the domain on k_0; the good-level orbit decomposition writes γ up to right Borel action as a unitary lattice stabiliser $\gamma' \equiv 1 \pmod{M'}$ on the trace-self-dual lattice Λ of $\mathbf{Z}$-rank $2fr$. Because Λ is trace-self-dual, its dual first minimum equals $\lambda_1(x_1)$, so Banaszczyk's transference theorem supplies $2fr$ $\mathbf{Q}$-independent vectors $v_1, \dots, v_{2fr} \in \Lambda$ with $\|v_i\|_{x_1}^2 \leq 16(fr)^4/\lambda_1(x_1)^2$. Comparing majorant norms across the radius-$1/(100r^2)$ chart ball gives $\|\gamma' v_i\|_{x_1} \leq 2\|v_i\|_{x_1}$ and hence $\|(\gamma' - 1)v_i\|_{x_1}^2 \leq (3\|v_i\|_{x_1})^2 \leq 144(fr)^4/\lambda_1(x_1)^2$. If $\gamma' v_i \neq v_i$ for some i, the congruence $\gamma' v_i - v_i = M'w$ for $0 \neq w \in \Lambda$ gives $(M')^2\lambda_1(x_1)^2 \leq \|(\gamma' - 1)v_i\|_{x_1}^2 \leq 144(fr)^4/\lambda_1(x_1)^2$, forcing $(M'\lambda_1(x_1)^2)^2 \leq 144(fr)^4$, contrary to $M'\lambda_1(x_1)^2 \geq 100(fr)^4$. Thus $\gamma' v_i = v_i$ for all i, giving $\gamma' = 1$, and freeness of Γ_{k_0} modulo the Borel subgroup gives $\gamma = 1$. Thus the radius-$1/(100r^2)$ chart ball at x_1 injects into the arithmetic quotient modulo the right Borel action.

Because the component Gram matrix is the negative transpose of the layer Gram matrix, the component domain on k_0 has positive block size $p_0 = q_{\mathrm{sig}}$ $(0 < p_0 < r)$ and CM type of size f. Passing from $(u_1, 0)$ to the model weight λ_{k_0} places $(u_{1,\tau,p_{\mathrm{sig}}+a})_{0 \leq a < q_{\mathrm{sig}}}$ in $0 \leq a < p_0$ and $(u_{1,\tau,a-q_{\mathrm{sig}}})_{q_{\mathrm{sig}} \leq a < r}$ in $p_0 \leq a < r$, preserving $\|\lambda_{k_0}\|_1 = \|u_1\|_1$ and transferring the adjacent gaps $\geq T$ of u_1 into within-block adjacent gaps $\lambda_{k_0,\tau,a} - \lambda_{k_0,\tau,a+1} \geq T$ for $a + 1 \neq p_0$ and endpoint gaps $\lambda_{k_0,\tau,r-1} - \lambda_{k_0,\tau,0} = u_{1,\tau,p_{\mathrm{sig}}-1} - u_{1,\tau,p_{\mathrm{sig}}} \geq T$. With $\rho_* = 1/(200r^2)$, $b_* = 4fr^2 + 1$ and chart dimension $d_{\mathbf{C}} = fr(r-1)/2 \leq fr^2$, the threshold choice above ensures that $T \geq (frL)^{24}$ exceeds both $C_{\mathrm{peak}}(fr)^8$ and the gap threshold of Lemma 35.38. Lemma 35.38 constructs a holomorphic seed F on the chart at x_1 extending holomorphically to a global Borel-equivariant section P_0 on the entire flag domain with $P_0(x_1) = F(0) = 1$, $P_0(\gamma x_1) = 0$ for every $1 \neq \gamma \in \Gamma_{k_0}$, and satisfying

$$\int_{\mathrm{chart}} |F(z)|^2 e^{-w(z)} K_{x_1}(z)^{-1}\, d\mu_F(z) \leq \exp\{C_{\mathrm{seed}}(\|u_1\|_1 + (fr)^8)\}\mu_F(U_0), \tag{A.467}$$

where w is the metric potential of λ_{k_0}, $K_{x_1} = \exp(-b_* R_{x_1})$, and U_0 is the intersection of the ball $\|z\| < 1/(100r^2)$ with the chart domain (containing $B(0, \rho_*)$). Summing the Γ_{k_0}-translates of P_0 with the weight character forms the Poincaré series P_F on k_0. By the holomorphic submean inequality on fixed recentered chart balls (where the weight potential and K_{x_1} vary by bounded factors as in the proofs of Lemmas 35.38 and 35.39), (A.467) gives $\sup_w \|P_0(w)\|_{\lambda_{k_0}}^2 K_{x_1}(w)^{-1} \leq M_F < \infty$, so $\|P_0(w)\|_{\lambda_{k_0}} \leq M_F^{1/2} K_{x_1}(w)^{1/2}$; since $b_*/2 = 2fr^2 + 1/2 \geq r$, the kernel-sum bound in the proof of Lemma 35.39 with exponent $b_*/2$ and ball injectivity give $\sup_y \sum_{\gamma \in \Gamma_{k_0}} K_{x_1}(\gamma y)^{1/2} < \infty$. Hence P_F is uniformly bounded in the invariant norm and therefore cuspidal on k_0 by the within-block monotonicity and strict endpoint gap $\lambda_{k_0,\tau,0} < \lambda_{k_0,\tau,r-1}$; since x_1 is orthogonal and P_0 vanishes at all nontrivial orbit points of x_1, we have $P_F(x_1) = 1$ and $\|P_F(x_1)\|_{\lambda_{k_0}} = 1$. For any orthogonal frame x_0 and measurable fundamental set $\mathcal{F}$ on k_0, Lemma 35.39 and (A.467) bound $\int_{\mathcal{F}} \|P_F(x_0 n(z))\|_{\lambda_{k_0}}^2\, d\mu_F(z)$ by $\exp\{C_S(fr)^8 + C_{\mathrm{seed}}(\|u_1\|_1 + (fr)^8)\}\mu_F(U_0) \leq e^{B_0'}\mu_F(U_0)$, and ball injectivity unfolds U_0 into $\mathcal{F}$ to give $\mu_F(U_0) \leq \mu_F(\mathcal{F})$. Orthogonal transfer from x_1 to y_0 therefore verifies the hypothesis of Lemma 35.43(2) with $\beta = e^{-B_0'}$, giving $G(y) \geq e^{-B_0'}$.

Lean: `GlobalTheory.LawsDeep.levLaws` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ComplexUniformizationLawsOfGoodDeepCover`, lines 21–39)
Lean: `Formalism.L2.Pj.goodAt_of_two_primes` (`Formalism.LocalToGlobal.ProjectiveLattices.ModelAt`, lines 124–153)
Lean: `GlobalTheory.LawsDeep.goodAt_of_mul` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ComplexUniformizationLawsOfGoodDeepCover`, lines 12–19)
Lean: `GlobalTheory.ThickBerg.orth_transfer` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FrameExponentialVolumeSmall`, lines 74–88)
Lean: `GlobalTheory.ThickBerg.ball_injects` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FrameExponentialVolumeSmall`, lines 40–61)
Lean: `GlobalTheory.ThickBerg.modelWt_shape` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FrameExponentialVolumeSmall`, lines 63–72)
Lean: `GlobalTheory.PeakThick.peak` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsCuspFormPeakAtInjectiveChartFrame`, lines 27–121)

□

Proof of 35.47

Proof. Put $m = 25$. In the semiabelian uniformisation, a level label lifts to a point of the extension of the abelian part by the torus whose m^2-multiple is a period $y \in X$. The class of y modulo m^2X is independent of the lift. This map on the full geometric level is onto: a period admits an m^2-division point in the extension, since multiplication by m^2 is surjective on the torus and on the abelian part over the algebraic closure. The map commutes with addition and the $\mathcal{O}_E$-action. Thus for the tuple in the statement its period labels y_b span X/m^2X.
Lean: `GlobalTheory.EtaVal.exists_qkPt_span` (`Development.AbelianVarietyTheory.Proofs.TorusSplitting.ExistsHeisenbergLiftsSpanLatticeClasses`, lines 19–73)

We compute the order of one theta constant. The integral frame supplies an integral section of the principal theta line. Its coefficient at 0 in the Fourier expansion is integral on the abelian part: it is the weight-zero coefficient of the integral section on the formal torus torsor. Extraction of a character coefficient there is projection to a graded summand over V. At level one all indices differ from 0 by periods. Translation by $\chi \in X$ identifies the coefficient at χ with a translate of the coefficient at 0 times the cubical period of order $Q(\chi)$. Translation on the abelian part preserves integral sections; its points over the valued field extend to the valuation ring by properness. Thus the coefficient at χ, in an integral frame, has order at least $Q(\chi)$.
Lean: `GlobalTheory.integralFamily_levelOne` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.IntegralFamilyLevelOne`, lines 16–45)
Lean: `GlobalTheory.EtaVal.exists_expansion_nonneg` (`Development.AbelianVarietyTheory.Proofs.TorusSplitting.ExistsExpansionNonneg`, lines 30–58)

In evaluating the constant with label y_b, the translate is the m-multiple of the chosen division point. On character χ its toric contribution has order $B(y_b, \chi)/m$: taking its m-th power gives the period pairing with y_b. The symmetric torsion frame contributes $Q(y_b)/m^2$. Indeed its order t satisfies $2m^2t = B(y_b, y_b)$ by the symmetric cubical trivialisation, which identifies the square of the frame with the diagonal biextension frame. The rigidified multiplication isomorphism transports this frame to the evaluation at the m-multiple. The order of the term of index χ is therefore at least

$$Q(\chi) + \frac{B(y_b, \chi)}{m} + \frac{Q(y_b)}{m^2} = \frac{Q(m\chi + y_b)}{m^2}.$$

The theta constant is the convergent sum of these terms. The valuation inequality for a sum gives the same lower bound with the minimum over χ. This computation uses multiplication on the characteristic-zero generic fibre and takes valuations of powers; it applies also in residue characteristic five.
Lean: `GlobalTheory.UT.TxVal.main` (`Development.AbelianVarietyTheory.Proofs.UniformizationProofs.ValuationsSymmetricFrameAndTerms`, lines 175–270)
Lean: `GlobalTheory.EtaVal.le_val_thetaConst` (`Development.AbelianVarietyTheory.Proofs.TorusSplitting.LeValuationOddLevelThetaConstantIntegralExpansio`, lines 25–36)

For each b, choose $z_b \in y_b + mX$ minimising Q. A minimum exists since the values are nonnegative integers. The quotient of X by the $\mathcal{O}_E$-span of the z_b is a finitely generated abelian group on which multiplication by m is surjective. It has free rank zero. Hence the z_b span $X \otimes \mathbf{Q}$ over E.

Put $w = (1 + \sqrt{-d})/2 \in \mathcal{O}_E$, so $w + \bar{w} = 1$. The functional

$$\ell(C) = \sum_b C(wz_b, z_b) = \tfrac{1}{2} \sum_b C(z_b, z_b)$$

is integral on integral forms by its first expression. For a positive semidefinite C, equality $\ell(C) = 0$ places every z_b in the radical. That radical is $\mathcal{O}_E$-stable by the balance identity, so it is the whole rational span. Thus ℓ satisfies the positivity condition defining ϕ, including when $X = 0$. It follows that

$$\phi \le \ell(B) = \sum_b Q(z_b) \le m^2 \sum_b v(\theta_b) = 625\, v(f_0),$$

where θ_b are the constants in the product. If one factor vanishes the assertion follows directly.
Lean: `GlobalTheory.EtaVal.sum_cosetMin_ge_phiC_div_sq` (`Development.AbelianVarietyTheory.Proofs.TorusSplitting.SumCosetMinGePhiCDivSq`, lines 15–50)

□

Proof of 35.48

Proof. We use three elementary prime sums. For $n \ge 2$,

$$\sum_{\ell \le n} \log \ell \ll n, \qquad \sum_{\ell \le n} \frac{\log \ell}{\ell - 1} \ll \log(2n), \qquad \sum_{\ell \le n} \frac{1}{\ell - 1} \ll 1 + \log\log(3n),$$

where the sums run over primes and the constants are absolute. For the first, the primes in $(j, 2j]$ divide $\binom{2j}{j} \le 4^j$; sum over dyadic intervals up to the next power of two. For the second, use $\lfloor n/\ell \rfloor \ge n/(2\ell)$ in the prime factorisation of $n!$. For the third take $s = 1 + 1/\log(3n)$. Then $1/\ell \le e/\ell^s$ for $\ell \le n$, and $\sum_\ell \ell^{-s} \le \log \zeta(s) \le \log(1 + 1/(s-1))$ by the Euler product and the integral test.

The size inequalities (A.434) and (A.462) give $T \le b$, $Tr^3 \le b$, and $kr \le b$. The width bound in 35.20 gives $kS \le P_*(S + \eta) \le 800b$. Consequently

$$TU \le 9b, \qquad T(A + V) \ll fmb, \qquad TJ \le 17b, \qquad \log(2J) \ll L.$$

Also k grows at most polynomially in x by 33.3, so $1 + \log\log(3k) \ll \log\log x$ eventually. In the estimates below a constant may depend on the family.

The contribution at $\ell \mid Q_S$ is at most

$$(A + V) \log Q_S + \tfrac{1}{2} A \log \operatorname{rd}(E),$$

since $\sum_\ell \delta_\ell \log \ell = \log \operatorname{rd}(E)$. Both logarithms are $O(L)$ by the host size conditions. At q, d use $q, d > 4tr + 1$ from the host construction and $\log(qd) \le L$ from $(Q_S qd)^{36} t^{150} \le \sqrt{b}$. This part costs $O((A + V)L)$.

The ordinary fractions with denominator $\ell - 1$ cost $O((A + V)\log(2J))$ by the second prime sum. The extra numerator $Ah_\ell/2$ occurs only for $\ell < k/m$. Its contribution is $O(A \log(2J)(1 + \log\log(3k)))$ by the third prime sum. In that same range $(h_\ell + 1)\log \ell \le 2\log(2J)$. There are at most k/m such primes, so the terms with coefficient $f(m+1)r^2$ cost $O(fr^3 \log(2J))$. The unit terms cost $O(J)$ by the first prime sum. Multiplication by T bounds their total by a constant times $fmbL \log\log x = E_p$.

Finally $\log(g_0 g_1) = O(L)$ by 33.3. The proof of Lemma 35.21 gives $\sum_{\ell \in \mathcal{B}} \log \ell \le 37L$. The different part costs at most $(N/2) fr \log \operatorname{rd}(E)$. The column count in 35.46 gives $N \le 20fr^2$. Thus these last costs are $O((frL)^3)$ before multiplication by T, and $T(frL)^3 \le TP_* \le b$ by (A.462). Enlarging the family constant gives the assertion. □

Proof of 35.49

Proof. The character lattice of the toric part has the same rational rank as the period lattice. Its Lie algebra in the Raynaud extension injects into the Lie algebra of the uniformised variety. Each embedding of E occurs in the toric Lie algebra with multiplicity s/n_E, since the rational cocharacter space is an E-vector space. An embedding of Lie multiplicity Q gives the rank bound. For $s = 0$ take the empty basis.

Give $X_{\mathbf{R}}$ the Euclidean form B_X. Fix an integral functional l strictly positive on the nonzero positive semidefinite balanced forms, as in Lemma 35.47. It is a sum of evaluations $C \mapsto \sum_j C(x_j, y_j)$ with $x_j, y_j \in X$. Indeed, the integral balanced symmetric forms constitute a saturated submodule of the integral matrices in any basis: symmetry and balance are integral linear equations. An integer functional on this submodule extends to the full matrix lattice.

Let $u \ne 0$ belong to the dual lattice. For $z \in X$, define $t_z \in E$ by $\operatorname{Tr}(at_z) = u(az)$ for $a \in E$, extending u rationally. Thus t_z lies in the inverse different and $t_{az} = at_z$. The form

$$C_u(x, y) = \sum_{\sigma : E \to \mathbf{C}} \operatorname{Re}\big(\sigma(t_x)\overline{\sigma(t_y)}\big)$$

is positive semidefinite and balanced. Set

$$t = \sum_j (t_{x_j} \bar{t}_{y_j} + \bar{t}_{x_j} t_{y_j}).$$

It lies in $\mathfrak{D}_{E/\mathbf{Q}}^{-2}$. Each $\sigma(t) > 0$, by applying l to the nonzero positive semidefinite form $\mathrm{Re}(\sigma(t_x)\overline{\sigma(t_y)})$. Hence

$$2l(C_u) = \mathrm{Tr}(t) \geq n_E |N_{E/\mathbf{Q}}(t)|^{1/n_E} \geq n_E / \mathrm{rd}(E)^2.$$

For the last inequality use the inclusion $(t) \subset \mathfrak{D}_{E/\mathbf{Q}}^{-2}$.

The real summands belonging to the different complex places are mutually orthogonal for B_X, by balance. On such a summand $u = 2\,\mathrm{Re}(\sigma(t\,.\,))$, and multiplication by i is an isometry. The real and imaginary parts are therefore orthogonal covectors of the same norm. It follows, on summing over the complex places, that $C_u \leq \|u\|^2 B_X/2$, where the norm on covectors is the dual norm. Applying l proves $\|u\|^2 \geq n_E/(\mathrm{rd}(E)^2 l(B_X))$. The Euclidean transference inequality $\lambda_s(X)\lambda_1(X^\vee) \leq s$ applies to this full lattice and its Euclidean dual [78, Th. 2.1]. Take the infimum over l to obtain $\lambda_s(X)^2 \leq 2\kappa s^2 \phi$.

To pass from independent short vectors to a basis, take successively a shortest $u_i \in X$ outside the span of its predecessors. Let $W_i = \mathrm{span}_{\mathbf{R}}(u_1, \ldots, u_i)$. The projection of $X \cap W_i$ perpendicular to W_{i-1} is cyclic; keep b_i as an unmodified shortest lift in its $(X \cap W_{i-1})$-coset of a generator of this perpendicular projection. Its perpendicular component has length at most $\|u_i\|$. To bound $\|b_i\|$, form an auxiliary representative b_i^{red} in the same coset $b_i + (X \cap W_{i-1})$ by subtracting integer multiples of $b_{i-1}, \ldots, b_1$ in reverse order so that its component along each preceding Gram–Schmidt direction has length at most half the corresponding perpendicular length (at most $\|u_j\|/2 \leq \|u_i\|/2$, since $\|u_1\| \leq \cdots \leq \|u_i\|$ by the greedy choice). Minimality of b_i in its coset then gives $\|b_i\|^2 \leq \|b_i^{\mathrm{red}}\|^2 \leq (1 + (i-1)/4)\|u_i\|^2 \leq s\|u_i\|^2$. These unmodified lifts form a basis by induction along the saturated flag $X \cap W_i$. For every $z \in X \cap W_{i-1}$, coset minimality gives $\|b_i\| \leq \|b_i \pm z\|$ and hence $2|B_X(b_i, z)| \leq \|z\|^2$. Otherwise $\|u_i\| \leq \|z\|$ and Cauchy–Schwarz gives $|B_X(b_i, z)| \leq \sqrt{s}\,\|z\|^2$. Use $2Q_X(z) = \|z\|^2$ in both inequalities. □

Proof of 35.50

Proof. Use the basis of Lemma 35.49 in each copy. Both bounds for a basis vector persist on the product, since Q_X is nonnegative on every factor. Take $\chi = \sum_i n_i b_i$, $n_i \geq 0$, $\sum_i n_i \leq D$. The triangle inequality gives $Q_X(\chi) \leq D^2\Phi$. If $z \neq 0$, then

$$MQ_X(z) + B_X(\chi, z) \geq (M - 2D\sqrt{s})Q_X(z) > 0.$$

Thus χ is the unique minimum in its class modulo MX.

Specify the coefficient at χ by an integral section on the abelian part. Its line is the M-th power of the principal polarising line L tensored with the degree-zero line of the character χ. Its sections give arbitrary D-jets at the origin. Here are the details on a geometric fibre. The cube L^3 is very ample and L^j is generated for $j \geq 2$ [189, §17]. Choose a section of L^3 nonzero at the point and sections whose ratios to it give local parameters. Degree-D products of these sections, including copies of the nonvanishing one, span the truncated local ring after trivialisation. The residual power $M - 3D \geq 4$, with its degree-zero twist, has a section nonzero at the point: the twisted power is a translate, by surjectivity of its polarisation isogeny. Multiplication by its jet is invertible. Sections lift from the special fibre, since a positive power with a degree-zero twist has vanishing higher cohomology on an abelian variety [189, §16]. Base change and Nakayama's lemma then give the integral jet assertion.

Extend the coefficient at χ by the period translation law and put zero on the other classes. The increment of valuation at $\chi + Mz$ is

$$MQ_X(z) + B_X(\chi, z) = \{Q_X(\chi + Mz) - Q_X(\chi)\}/M.$$

Translation on the abelian part preserves its integral lattice. These coefficients give a theta section by the Fourier expansion of the semiabelian uniformisation: they satisfy the descent law, and their orders tend quadratically to infinity on the class. On the special fibre only the term at χ survives. Use the character expansion and the period translation law of [176, Ch. II, §§4–5, especially §5]. In a local splitting of the torus torsor, trivialise the coefficient line of χ over the abelian base. Its term is the pullback of the abelian coefficient times the Laurent monomial z^χ in this trivialisation. Every other term in the chosen class has strictly larger valuation by the displayed increment. Reduction therefore gives z^χ times the prescribed abelian germ in the common line frame. A change of the coefficient frame multiplies the abelian germ by a unit and preserves the jet-spanning property.

Write z_i for the torus characters of the basis. Together with parameters on the abelian part, $z_i - 1$ are parameters at the identity. Expanding $\prod_i (z_i - 1)^{n_i}$ gives characters with nonnegative exponent sum at most $\sum_i n_i$. The special-fibre germs just constructed therefore span the whole ring of D-jets, including the mixed toric and abelian monomials. This ring is finite free over the trait, so the integral jets before reduction span by Nakayama's lemma. Finally multiply the entire linear combination by one scalar of valuation $\lfloor D^2\Phi \rfloor / M$ in a finite ramified discrete valuation ring extension V'/V obtained by adjoining an M-th root of the base trait uniformiser ϖ_V (keeping v and Q_X normalized by $v(\varpi_V) = 1$). For each participating class this is at least $Q_X(\chi)/M$, since Q_X is integer-valued. The increment formula puts the resulting section in the asserted lattice. □

Proof of 35.51

Proof. Use the differential of Lemma 35.46. For each pair (τ, i) it has a separate block of i columns with factor $\det(\partial_{aj})^{c_{\tau i}}$, where $c_{\tau i} = a_{\tau,i} - a_{\tau,i+1}$. Its total degree is $D = \sum_{\tau,i} i c_{\tau i}$. There are finitely many level tuples. The associated rational theta derivatives are cusp sections by Lemma 35.40. Pull them back and multiply on each component by the fixed weight-changing sections of Lemma 35.46. Include in Σ the primes of the level and discriminant, the primes dividing any $c_{\tau i}!$, and the primes needed to spread out this finite list of sections and the weight-changing isomorphisms.

The last operation concerns fixed algebraic sections. On the compactified generic flag model the cusp sections extend across the boundary. The determinant and flat weight changes and their inverses extend as isomorphisms of their respective lines there: in the determinant case a power identifies the two trivialisations, and in the flat case this follows from the canonical extension in Lemma 35.10. After inverting one integer all these sections and identities extend on a finite affine cover, by clearing the denominators in their coefficients and transition laws. At the remaining good places the saturated eigenspaces give the ordinary flag model, since the embedding projectors are integral. Thus the weight changes have unit coefficients in integral frames there. One may first multiply all the rational cusp sections by a common denominator to put them in the global cusp lattice. At the finitely many excluded places this clearing can also be seen by taking the closure of the generic flags in the product of the Grassmannians of the integral Lie bundle. Every saturated trait flag gives a point of this proper model. A section regular on the whole generic fibre of an invertible sheaf extends after multiplication by a power of the base uniformiser, by the same finite affine cover argument. The determinant of the change from the projected Hodge lattice to the integral eigenframes has bounded order by Lemma 35.7. Include the prime divisors of the common denominator in Σ.

Fix $\ell \notin \Sigma$. Forget the extra level, retaining $M = g_0 g_1$. The same semiabelian degeneration and frame give the base evaluation and the same period form. After a finite extension of ramification index b_1 take a split chart. Orders, $v(\ell)$, and ϕ all multiply by b_1; for ϕ this follows from its definition as an infimum of linear functionals of the period form.

In the j-th column of the block (τ, i) choose an integral linear coordinate whose differential on the first i framed vectors is the j-th coordinate functional. Such coordinates exist because the frame is a basis of a saturated summand. Take the product, over the columns in the block, of the corresponding coordinates to the power $c_{\tau i}$, and then the product over blocks. This jet is homogeneous of degree D. At the origin only the principal symbol of the theta differential acts on it: every term of order $< D$ has value zero. In each determinant power the diagonal monomial has coefficient one. The value is consequently

$$\prod_{\tau,i} (c_{\tau i}!)^i,$$

a unit. This test uses the individual minor exponents even when several of them are nonzero.

To apply Lemma 35.50, put $s_* = 2fm$. The inequalities of (A.462) and Lemma 35.46 give, beyond a common threshold,

$$s \le s_*, \quad s_*^3 \le b, \quad \kappa \le b, \quad D \le b^2, \quad g_0, g_1 \ge b^8.$$

Indeed $s_* \le 2fr$, $(frL_x)^{24} \le T \le b$, and $\mathrm{rd}(E)^7 \le b$. These bounds give $4D + 4 \le M$, $2D\sqrt{s} < M$, and $2500 D^2 \kappa s^3 / M \le 1$. Lemma 35.50 provides a section in the Fourier lattice whose differentiated value has order at most $b_1\phi/2500$.

At this prime M is a unit. We explain why a tuple has differentiated value of at most the same order. Over a splitting field the two complementary halves of the level act through the theta group. Translation preserves

the Fourier lattice: translating the index and changing its cubical frame changes the coefficient order and $Q_X(z)/M$ by the same amount. The character projector of one half is the sum of its translations weighted by inverse character values, divided by the order of the half. This denominator is a unit. The projected spaces are lines: the invariant line descends through the half-level isogeny to the principal theta line, of section dimension one by Definition 35.5, and the commutator pairing translates it to each character. The pullback of an integral primitive principal theta section is primitive for the Fourier lattice. Pullback multiplies the period form by M on its index image, and a primitive coefficient on the abelian part stays primitive under the surjective isogeny on that part. Its translated copies are therefore generators of the integral character lines. This argument applies on the product, with its product principal polarisation. Hence the section just obtained is an integral linear combination of the translated generators. Differentiating the combination and using the valuation inequality selects a tuple. In passing from its pulled-back differential to the theta rule, each framed derivative contributes its embedding of a half-level factor; these factors divide M and are units. The cubical translations use finite-order normalisations, also units in this comparison.

The weight change on the selected component and the common denominator are units at ℓ. We have thus found a global integral section with order at most $b_1\phi/2500$ upstairs. Dividing by $b_1 v(\ell)$ proves the claim on the original trait. □

Proof of 35.52

Proof. The balance identity gives $Q_X(\alpha z) = gQ_X(z)$. Express $\alpha^{-1}y$ in the basis b_i. Round each coordinate independently to one of its two neighbouring integers with probabilities giving the original coordinate as mean. The errors have mean zero and variance at most $1/4$. The expected quadratic value of the error after multiplication by α is at most $g\sum_i Q_X(b_i)/4$, since all cross terms have mean zero. Some representative therefore satisfies the first bound. Choose z_0 to minimise Q_X in the class; the values are nonnegative integers. Comparing with $z_0 \pm \alpha z$ gives $|B_X(z_0, \alpha z)| \le Q_X(\alpha z)$. Expansion of $Q_X(z_0 + g'\alpha z)$ proves the second inequality. It is strictly positive on the right for $z \neq 0$. The ultrametric inequality for the convergent expansion now gives the final assertion. □

Proof of 35.54

Proof. Write $\widetilde{F}(t)$ for the composite series in the definition. Write $\det(\partial/\partial t_{ab})^k = \sum_\alpha p_\alpha \partial^\alpha$. The coefficients p_α are integers, and every α has total degree ik. Then

$$\det(\partial/\partial t_{ab})^k\, \widetilde{F}(0) = \sum_\alpha p_\alpha\, \alpha!\, [t^\alpha]\widetilde{F}, \qquad \alpha! = \textstyle\prod_{a,b} \alpha_{ab}!.$$

Let G be the formal group law in the chosen coordinates. For an integral Lie vector u the invariant derivation is

$$D_u F(X) = \sum_j \Big(\sum_r u_r \, \frac{\partial G_j}{\partial Y_r}(X, 0) \Big) \frac{\partial F}{\partial X_j}.$$

It maps integral series to integral series. The map $s \mapsto \exp(su)$ is a homomorphism from the additive group, so $\exp(x + su) = \exp(x) +_G \exp(su)$. Hence $\partial/\partial t_{ab}$ acting on $\widetilde{F}$ corresponds to $D_{u_{\tau,a}}$ acting on the b-th copy. These derivations commute. Consequently $\alpha!\,[t^\alpha]\widetilde{F} = \partial^\alpha \widetilde{F}(0)$ is the constant term of an integral series. This proves $\nu_{\tau,i}(k; y) \ge 0$.

For the upper bound, fix a field and integral coordinates. The vectors $u_{\tau,1}, \dots, u_{\tau,i}$ span a saturated summand. Hence their coordinate matrix has an $i \times i$ minor m that is a unit. Let $x_{r_1}, \dots, x_{r_i}$ be the corresponding coordinates and put $z_a = \sum_s (m^{-1})_{as} x_{r_s}$. These functions are integral and satisfy

$$z_a\big(\exp(\textstyle\sum_{a'} t_{a'} u_{\tau,a'})\big) = t_a + O(t^2).$$

Take $F = \prod_{a,b} z_a(\text{copy } b)^{\alpha_{ab}}$. Then $\widetilde{F} = t^\alpha$ plus terms of degree $> ik$. The operator is homogeneous of degree ik, so the value is $p_\alpha\, \alpha!$.

Take the diagonal exponent $\alpha_{ab} = k\delta_{ab}$. A term $\prod_{j=1}^k \prod_a \partial_{a,\pi_j(a)}$ of $\det^k$, with permutations $\pi_1, \dots, \pi_k$, is the diagonal monomial only when every π_j is the identity. Hence $p_\alpha = 1$, and $\alpha! = (k!)^i$. Therefore $\nu_{\tau,i}(k; y) \le i\, v_\ell(k!)$. By Legendre's formula, $v_\ell(k!) = \sum_{j \ge 1} \lfloor k/\ell^j \rfloor \le k/(\ell - 1)$, and $v_\ell(k!) = 0$ for $k < \ell$. □

Proof of 35.55

Proof. By Lemma 35.54, $\nu_c(y) \ge 0$. By hypothesis, $\nu_c(y) \ge \sum_f A_f$. Hence $\nu_c(y) \ge \max(0, \sum_f A_f) \ge A_\lambda$.

At an algebraic frame,

$$\nu_c(y) \le \sum_f B_f = \sum_f A_f + \sum_f (B_f - A_f) \le \max\Big(0, \sum_f A_f\Big) + \sum_f (B_f - A_f),$$

and $\max(0, \sum_f A_f) < A_\lambda + 1$.

For the last two assertions take $A_\lambda = 0$. Lemma 35.54 gives $0 \le \nu_f(c_f; y) \le i\, c_f/(\ell - 1)$ for $f = (\tau, i)$, with equality to zero when $c_f < \ell$. Summing over the blocks gives $0 \le \nu_c(y) \le D/(\ell - 1)$. It also gives $\nu_c(y) = 0$ when $\ell > \max_f c_f$. □

Proof of 35.56

Proof. Put $m'(\lambda) = k$ and $m'(w') = m(w')$ for $w' \ne \lambda$. Let S be the support of m' together with λ. For $w' \in S$ put $n_{w'} = -\lfloor m'(w') \rfloor$.

We first find $\sigma \in F_c^\times$ with $v_{w'}(\sigma) = n_{w'}$ for $w' \in S$ and $v_{w'}(\sigma) \ge 0$ for $w' \notin S$. Let L be the product of the rational primes below S. Choose $j \ge 0$ with $j\, v_{w'}(L) + n_{w'} \ge 0$ for all $w' \in S$. By the Chinese remainder theorem there is $\beta \in \mathcal{O}_{F_c}$ such that

- $\beta \equiv \pi_{w'}^{j v_{w'}(L) + n_{w'}}$ modulo $\mathfrak{p}_{w'}^{j v_{w'}(L) + n_{w'} + 1}$ for $w' \in S$, where $\pi_{w'}$ is a uniformiser, and
- $\beta \equiv 0$ modulo $\mathfrak{p}_{w'}^{j v_{w'}(L)}$ for the primes $w' \notin S$ dividing L.

Then $\beta \ne 0$, and $\sigma = \beta / L^j$ has valuation $n_{w'}$ at $w' \in S$. Its valuation is at least 0 at the primes outside S dividing L. At the remaining primes its valuation equals $v_{w'}(\beta) \ge 0$.

Now let y be an algebraic integral frame over w' with ramification index e. Then

$$v(\sigma f(y))/e = v_{w'}(\sigma) + v(f(y))/e.$$

For $w' \in S$ this is at least $-\lfloor m'(w') \rfloor + m'(w') \ge 0$. For $w' \notin S$ both terms are nonnegative, since $m(w') = 0$ there. Hence σf is integral, and $v_\lambda(\sigma) = -k$. □

Proof of 35.57

Proof. (a) Let $\mathfrak{w}$ be the prime of $\mathcal{O}_E$ on which τ, followed by reduction modulo λ, vanishes. By 5.1 and 8.3, every prime dividing $\operatorname{disc} E$ divides $Q_S q d$. For $p \mid Q_S q$, the d-sign is trivial on inertia and Frobenius by Lemma 5.3, while complex conjugation c has d-sign -1. At $p = d$, if c were a Frobenius power modulo inertia, the q-sign would force an odd exponent $2j+1$, and the projection modulo 16 would give $7^{2j+1} \equiv \pm 1 \pmod{16}$. Thus c lies in none of these decomposition groups. Since $\operatorname{Gal}(E/E^+) = \langle c \rangle$, every prime $\mathfrak{q} \mid Q_S q d$ splits over E^+ (with $\bar{\mathfrak{q}} \ne \mathfrak{q}$), while primes not dividing $Q_S q d$ are unramified in $E/\mathbf{Q}$. Hence the extension E/E^+ is unramified at every finite place, so each prime $\mathfrak{q} \mid \ell$ of $\mathcal{O}_E$ is either inert over E^+, with $\bar{\mathfrak{q}} = \mathfrak{q}$, or split, with $\bar{\mathfrak{q}} \ne \mathfrak{q}$. For each such prime and each $n \ge 1$ we choose classes u, t modulo $\mathfrak{q}^n$ and modulo $\bar{\mathfrak{q}}^n$ such that $u \equiv 1$ and $u\bar{u} + t\bar{t} \equiv 0$. The Chinese remainder theorem combines these classes.

Suppose first $\bar{\mathfrak{q}} \ne \mathfrak{q}$. Take $u \equiv 1$, take $t \equiv -1$ modulo $\mathfrak{q}^n$, and take $t \equiv 1$ modulo $\bar{\mathfrak{q}}^n$. Then $\bar{t} \equiv 1$ modulo $\mathfrak{q}^n$ and $\bar{t} \equiv -1$ modulo $\bar{\mathfrak{q}}^n$. Hence $1 + t\bar{t}$ vanishes modulo both ideals.

Suppose now $\bar{\mathfrak{q}} = \mathfrak{q}$. Conjugation induces the nontrivial involution of the residue field $k_\mathfrak{q}$, which is quadratic over the residue field of E^+. The norm of finite fields is onto, so $-1 = t_1 \bar{t}_1$ modulo $\mathfrak{q}$ for some t_1. Assume $t_n \bar{t}_n = -1 + \varepsilon$ with $\varepsilon \in \mathfrak{q}^n$. Then $\bar{\varepsilon} = \varepsilon$. Since $\mathfrak{q}$ is unramified over E^+, we may write $\varepsilon = \pi^n \varepsilon_0$, where π is a uniformiser of E^+ at $\mathfrak{q} \cap E^+$ and ε_0 is invariant under conjugation. The residue trace is onto, so there is t_0 with $t_0 + \bar{t}_0 \equiv \varepsilon_0$ modulo $\mathfrak{q}$. Put $t_{n+1} = t_n(1 + \pi^n t_0)$. Then

$$t_{n+1}\bar{t}_{n+1} \equiv (-1 + \varepsilon)\big(1 + \pi^n(t_0 + \bar{t}_0)\big) \equiv -1 \pmod{\mathfrak{q}^{n+1}}.$$

At $\mathfrak{w}$ these choices satisfy $t_{n+1} \equiv t_n$ modulo $\mathfrak{w}^n$. Define α_τ as the limit of $\tau(t_n)$ in V_λ. In the split case $\alpha_\tau = -1$. Given n_0, take $n \ge n_0 v_\mathfrak{q}(\ell)$ at every $\mathfrak{q} \mid \ell$, and combine. The ideal $\mathfrak{w}^n$ maps into $\ell^{n_0} V_\lambda$, which gives the two congruences in V_λ.

(b) By the Cauchy–Binet formula, each determinant in P_{col} is a V_λ-linear combination of leading minors in subsets of the columns of its block. Hence P_{col} lies in the span of products of leading minors. The row (τ, i') occurs exactly in the determinants of the blocks (τ, i) with $i \geq i'$, with degree one in each. Its total degree is therefore $\sum_{i \geq i'} c_{\tau,i} = a_{\tau,i'}$.

For harmonicity, fix a column b. At most one block uses b, and that factor involves the variables of a single embedding. Hence no factor depends on both $X_{\tau,\cdot,b}$ and $X_{\bar{\tau},\cdot,b}$. By the Leibniz rule, $\partial^2 / \partial X_{\tau,i,b} \partial X_{\bar{\tau},i',b}$ annihilates the product.

The coefficients of P_{col} lie in V_λ. For the approximants, apply (a) with n_0. In row j put $x_{j,b_j} = u$, $x_{j,b_{i+j}} = t$, and zero elsewhere. Then $\sum_b x_{j,b} \bar{x}_{j,b} = u\bar{u} + t\bar{t}$. For $j \neq j'$ the supports are disjoint and the sum vanishes.

(c) Let $W_{\mathbf{Q}}$ be the space of rational polynomials of row degrees a that lie in the span of products of leading minors and are harmonic. Put $W_{\mathbf{Z}} = W_{\mathbf{Q}} \cap \mathbf{Z}[X]$. The submodule $W_{\mathbf{Z}}$ is saturated in the finite free module of integral polynomials of these row degrees, hence a direct summand. Both defining conditions of $W_{\mathbf{Q}}$ are given over $\mathbf{Q}$, and V_λ is torsion free. Therefore

$$\mathrm{harm}_{V_\lambda}(a, N) = (W_{\mathbf{Q}} \otimes \mathrm{Frac}\, V_\lambda) \cap V_\lambda[X] = W_{\mathbf{Z}} \otimes V_\lambda.$$

Let $P'_1, \dots, P'_K$ be a $\mathbf{Z}$-basis of $W_{\mathbf{Z}} \subset \mathrm{harm}(a, N)$, and write $P_{\mathrm{col}} = \sum_i \gamma_i P'_i$ with $\gamma_i \in V_\lambda$. Since $\mathcal{O}_{F_c}$ is dense in V_λ, there are $\xi_i \in \mathcal{O}_{F_c}$ with $\xi_i \equiv \gamma_i$ modulo $\ell^{n_0} V_\lambda$. Put $P_1 = \sum_i \xi_i P'_i$. Then

$$P_1 - P_{\mathrm{col}} = \sum_i (\xi_i - \gamma_i) P'_i \in \ell^{n_0} \mathrm{harm}_{V_\lambda}(a, N).$$

Applying the map $V_\lambda \to V$ of a trait over λ transports this identity. □

Proof of 35.58

Proof. Let $\lambda' \mid \ell$ be a place of F', V a trait over λ', and y' an algebraic integral frame on Y over V with phase ϕ. Forget the additional level structure of the cover and restrict the coefficients from F' to F_c. This gives an algebraic integral frame y on the base moduli over the same V. It has the same semiabelian scheme, hence the same period form and the same phase. Through $F_c \subset F'$, the ring V is a trait over the place λ of F_c below λ'.

Let f be an integral cusp section of weight w on the base. Its pullback g to Y is a cusp section of weight w. It satisfies $g(z') = f(z)$ for every frame z' on Y with image z. Every algebraic integral frame on Y has an algebraic integral image on the base, so g is integral. Hence the infimum over the integral cusp sections on Y at y' is at most $v(f(y))$, for every such f. Both reference depths use the valuation of V and the same phase, which gives the inequality. □

Proof of 35.59

Proof. (1) Pass to a finite extension of V over which the frame has a split uniformisation and the points of order dividing M^2 are rational. The image frame is again algebraic and integral over λ. Values of sections and valuations normalised by $v(\ell)$ do not change. Since $\ell \nmid M$, the factor M^{-D} and the normalising constant of each s_{x_b} are units at λ.

The germ of s_{x_b} at the origin, in the integral frame of the M^2-th power, is integral. To see this, expand s_{x_b} in the characters χ of the period lattice. The abelian coefficients are integral. They are multiplied by constants of order at least $Q_X(M\chi + y_b)/M^2 \geq 0$, where y_b is the period label of the translating point. This is the order computation in the proof of Lemma 35.47 with $m = M$. When the period lattice is zero, it reduces to the statement there that translation on the abelian part preserves integral sections. The characters have integral germs on the formal group of the Raynaud extension, which is the formal group of the frame.

Hence $s_{x_1} \boxtimes \cdots \boxtimes s_{x_N}$ has an integral germ on N copies. By definition of δ_{M^2}, the normalised order of $P(\nabla)$ applied to this product at the origin is at least $\delta_{M^2}(P; y) \geq A$. Multiplying by the unit M^{-D} proves (1).

(2) Choose formal parameters over V at the identity of the smooth group. After base change to $\mathcal{O}_C$, a change of integral parameters is an invertible power-series substitution fixing the origin. It permutes the

integral germs and transports the invariant derivations and h_L, so the infimum for P may be taken in these parameters. Let F be an $\mathcal{O}_C$-integral power series on $\widehat{G}^N$ and put

$$x_0 = \big[P(D)\big(F\, e^{M\sum_b h_L(x_b)}\big)\big](0).$$

We may assume $x_0 \neq 0$. Every monomial of P has degree D, so x_0 depends only on the degree-D jet $\sum_{|\alpha|\le D} c_\alpha Z^\alpha$ of F at the origin, where each coefficient $c_\alpha \in \mathcal{O}_C$ satisfies $v(c_\alpha) \ge 0$. Each monomial Z^α in these parameters is defined over V. By linearity of the differential operator and the ultrametric inequality, some monomial Z^α attains $v\big([P(D)(Z^\alpha e^{M\sum_b h_L(x_b)})](0)\big) \le v(x_0)$; replacing F by this V-integral monomial reduces to the V-integral hypothesis of Lemma 35.50. Apply Lemma 35.50 on N copies, with the integral frame of the theta line at the identity. After a scalar extension there are a scalar γ with $v(\gamma) \le D^2\Phi/M$ and a level-M theta section s on the product, in the Fourier lattice, whose jet of order D in that frame is γ times the jet of F. Writing $x_\alpha = [P(D)(Z^\alpha e^{M\sum_b h_L(x_b)})](0)$, we have $v(x_\alpha) \le v(x_0)$, and $P(\nabla)s$ has value γx_α at the origin.

The level M is a unit in V. Over a splitting field the two complementary halves of the level act through the theta group. Translation preserves the Fourier lattice, because translating the index and changing its cubical frame change the coefficient order and $Q_X(z)/M$ by the same amount. The character projector of one half is the average of its translations, weighted by inverse character values. Its denominator is the order of the half, which is a unit.

Each projected space is a line. It is generated by a translate of the pullback of an integral primitive principal theta section. These translates generate the integral character lines, by the argument in the proof of Lemma 35.51. Hence s is a linear combination, with integral coefficients, of translated generators.

Take a translated generator with characteristic tuple x. The value of $P(\nabla)$ on it at the origin is $\theta_{x,P}(y)$ times an integral scalar. This scalar is a unit times the product of the images of the half-level factor α, $\alpha\bar{\alpha} = M$, under the embeddings, raised to the row degrees. The ultrametric inequality therefore gives a tuple x with

$$v(\theta_{x,P}(y)) \le v(\gamma x_\alpha) \le v(x_0) + D^2\Phi/M.$$

Normalised valuations do not change under the scalar extension. Dividing by $v(\ell)$ and taking the infimum over F proves (2). □

Proof of 35.60

Proof. Fix an algebraic integral frame y over λ on a trait V, with phase ϕ. After a finite extension V' of ramification index r', the semiabelian degeneration of the frame has a split uniformisation [176, Ch. III]. Let y' be the image frame. As in the proof of Lemma 35.51, orders of sections, $v(\ell)$ and the phase multiply by r'. The frame y' is algebraic and integral over λ.

Let s and $\Phi = \kappa s^3 r'\phi$ be as in Lemma 35.49 on V'. At a nonterminal layer $q_{\mathrm{sig}} \le m$, so $s \le 2f q_{\mathrm{sig}} \le 2fm$. Hence $2D\sqrt{s} < M$, and

$$\frac{D^2\Phi}{M\, r' v(\ell)} \le \frac{\kappa(2fm)^3 D^2\phi}{M\, v(\ell)} \le \frac{c\,\phi}{v(\ell)}.$$

The level $M = g_0 g_1$ is prime to ℓ. Lemma 35.59(2) on V' and the upper bound in (i) at y' give

$$\inf_x \frac{v(\theta_{x,P_1}(y))}{v(\ell)} \le A + \varrho + \frac{c\,\phi}{v(\ell)}. \tag{A.468}$$

Here we use that the value at y' is the image of the value at y, and that valuations normalised by the valuation of ℓ agree.

Fix a tuple x and put $\theta = \theta_{x,P_1}$. Every element of $\mathrm{harm}(a, N)$ is a harmonic linear combination of products of leading minors. Each product contains minors of size P at an embedding of the CM type with $a_{\tau,P} > 0$. The proof of Lemma 35.40 applies to such a combination. Harmonicity is used there for the change of symplectic basis, and the zero-frequency term vanishes for each product separately. Since N is even, θ is a cusp section of weight (a, N).

By Lemma 35.59(1) and the lower bound in (i), $v(\theta(y'')) \ge A\, v(\ell)$ at every algebraic integral frame y'' over λ. Equivalently, $v(\theta(y''))/e'' \ge A\, v_\lambda(\ell)$ for the ramification index e''. Lemma 35.56, with $k = A\, v_\lambda(\ell)$

and the function m_x from (ii), gives $\sigma \in F_c^{\times}$ with $v_\lambda(\sigma) = -A\, v_\lambda(\ell)$ such that $\sigma\theta$ is integral. On V we have $v(\sigma) = e\, v_\lambda(\sigma) = -A\, v(\ell)$. Hence

$$\inf_f \frac{v(f(y))}{v(\ell)} \le \frac{v(\sigma\theta(y))}{v(\ell)} = -A + \frac{v(\theta(y))}{v(\ell)}.$$

Taking the infimum over x and using (A.468) proves the inequality. The last assertion follows from Definition 35.53(e). □

Proof of 35.61

Proof. The defining congruences are preserved by addition, subtraction and multiplication, and contain 1, so S is a subring. Fix $u \in S \cap V^{\times}$ and $n \ge 0$. Choose $a \in R$ such that $\varphi(a) - u \in \varphi(q)^{n+1}V$. Since $\varphi(q)$ lies in the maximal ideal $\mathfrak{m}_V$, the element $\varphi(a)$ is a unit. The ideal $\mathfrak{p} = \varphi^{-1}(\mathfrak{m}_V)$ is a nonzero prime ideal of R, because it contains q and $\varphi(q) \neq 0$. Hence $\mathfrak{p}$ is maximal. There are $b \in R$ and $i \in \mathfrak{p}$ with $ab + i = 1$.

Choose an integer $s \ge 1$ such that $\varphi(i)^s \in \varphi(q)^n V$, and put $b_n = b\sum_{j=0}^{s-1} i^j$. The geometric-sum identity gives

$$\varphi(a)\varphi(b_n) = 1 - \varphi(i)^s \equiv 1 \pmod{\varphi(q)^n V}.$$

Since $\varphi(a) \equiv u \pmod{\varphi(q)^n V}$, multiplication by $u^{-1} \in V$ yields $\varphi(b_n) \equiv u^{-1} \pmod{\varphi(q)^n V}$. Thus $u^{-1} \in S$. The entries of $\operatorname{adj}(M)$ and $\det M$ lie in S, and

$$M^{-1} = (\det M)^{-1} \operatorname{adj}(M)$$

proves the matrix assertion. □

Proof of 35.62

Proof. The constant term of U is 1. Its multiplicative inverse has coefficients in $\mathcal{O}_C$, by recursive comparison of coefficients in $UU^{-1} = 1$. Substitution by ρ preserves integrality, so G is integral. The two curve identities give

$$\begin{aligned} G(\mathbf{c}(t))W(\mathbf{c}(t))^k &= F(\mathbf{e}(t))U(\mathbf{e}(t))^{-k}W(\psi(\mathbf{e}(t)))^k \\ &= F(\mathbf{e}(t)). \end{aligned}$$

A polynomial differential operator at the origin is a linear functional on a finite jet, which proves the last assertion. □

Proof of 35.63

Proof. Put $\mathfrak{p} = \varphi^{-1}(\mathfrak{m}_V)$. It is nonzero because it contains $q \neq 0$, and hence it is maximal. Write $(q) = \mathfrak{p}^r J$, with $J + \mathfrak{p} = R$.

First suppose that $b \in \mathfrak{p}^a$, $\varphi(b) \neq 0$, and $\varphi(c) = \varphi(b)z$. Then $c \in \mathfrak{p}^a$. Indeed, for $c \neq 0$, write $(c) = \mathfrak{p}^i J'$ with $J' + \mathfrak{p} = R$. If $i < a$, choose $s \in J' \setminus \mathfrak{p}$. Since

$$(b)J' \subseteq \mathfrak{p}^a J' \subseteq (c)\mathfrak{p},$$

there is $m \in \mathfrak{p}$ with $bs = cm$. Cancelling $\varphi(b)$ gives $\varphi(s) = z\varphi(m) \in \mathfrak{m}_V$, a contradiction.

For each N, let $k_N = v_V(\varphi(q)^N)$, where v_V gives a uniformiser value 1. The ideals $\mathfrak{p}^{k_N}$ and J^N are comaximal. Choose $r_N \in \mathfrak{p}^{k_N}$ and $\varepsilon_N \in J^N$ with $r_N + \varepsilon_N = 1$. Since $\varphi(r_N) \in \mathfrak{m}_V^{k_N} = \varphi(q)^N V$, the first congruence follows. If $M \le N$ and $\varphi(a) \in \varphi(q)^M V$, the preceding argument with $b = q^M$ gives $a \in \mathfrak{p}^{rM}$. Therefore

$$\varepsilon_N a \in J^M \mathfrak{p}^{rM} = (q^M).$$

For consecutive indices, use

$$\varepsilon_{N+1}x_{N+1} - \varepsilon_N x_N = \varepsilon_N(x_{N+1} - x_N) + \big(\varepsilon_{N+1}(1-\varepsilon_N) - \varepsilon_N(1-\varepsilon_{N+1})\big)x_{N+1}.$$

Each term on the right belongs to $q^N R$, by the divisibility assertion just proved. Summing these differences proves the assertion for $m \le n$. Multiplying the congruences for ε_N and x_N proves the final assertion. □

Proof of 35.64

Proof. The embedding of $\mathcal{O}_E$ into $\operatorname{Frac} V$ has image in V, by integral closedness. Apply Lemma 35.63 with $R = \mathcal{O}_E$ and $q = \ell$. Since $a \mapsto d[a]$ is an additive map to integral matrices and $d[\ell^N a] = \ell^N d[a]$, the matrices $d[\varepsilon_{\tau,N} x_N]$ form an entrywise Cauchy sequence. Completeness gives an integral limit A with the stated error bound. On a τ-eigenvector these matrices act by $\tau(\varepsilon_{\tau,N} x_N)$. These scalars converge to t, proving the eigenvector identity.

For the second assertion, the estimate at $N = 0$ makes every entry of A_b integral. Logarithms turn the formal group law into addition and the CM action into its differential:

$$\exp_{\widehat{G}}(w + w') = \exp_{\widehat{G}}(w) +_{\widehat{G}} \exp_{\widehat{G}}(w'), \qquad \exp_{\widehat{G}}(d[a]w) = [a] \exp_{\widehat{G}}(w).$$

Thus $\exp_{\widehat{G}}(\sum_b d[a_{N,b}] w_b)$ is integral for every N.
Lean: `GlobalTheory.IsoTriv.log_sum_additive_preserves_integrality` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LogarithmSumIntegralAdditive`, lines 34–81)
Lean: `GlobalTheory.IsoTriv.actSt` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ActionLinearLogCoordinatesIntegral`, lines 15–66)

Fix an output coefficient. Because the w_b have zero constant term, this coefficient of $\exp_{\widehat{G}}(\sum_b M_b w_b)$ is a polynomial Q in the entries of the finitely many matrices M_b. Choose $K \geq 0$ such that every coefficient of Q has valuation at least $-K$. On integral matrix arrays, the difference of two monomials has valuation at least the smallest valuation of an entrywise difference. Therefore the difference of the chosen coefficients for $M_b = A_b$ and $M_b = d[a_{N,b}]$ has valuation at least $N - K$. Taking $N \geq K$ makes it integral. The approximating coefficient is integral, so the limiting coefficient is integral. This proves the assertion coefficient by coefficient.
Lean: `GlobalTheory.IsoTriv.contSt_holds` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.SeriesCoeffUniformlyContinuousInCoeffsProof`, lines 98–109)

□

Proof of 35.65

Proof. Let $\mathcal{U}$ be the additive subgroup of zero-constant series $s \in C[[t]]$ for which $\exp(s)$ is an integral unit. It is closed under addition and negation, because $\exp(s + s') = \exp(s) \exp(s')$ and an integral series with constant term 1 has integral inverse. Put

$$H(w, z) = h\big(\exp_{\widehat{G}} w, \exp_{\widehat{G}} z\big).$$

For integral formal points, the biextension cocycle identities and adjointness give, modulo $\mathcal{U}$,

$$H(w + w', z) = H(w, z) + H(w', z), \qquad H(z, w + w') = H(z, w) + H(z, w'),$$

and

$$H(d[a]w, z) = H(w, d[\bar{a}]z).$$

Each difference is the logarithm of the corresponding integral biextension cocycle evaluated on integral formal points; its exponential has constant term 1.
Lean: `GlobalTheory.IsoTriv.GaugeBiadd.biadd_of_add_of_act` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.CanonGaugeBiadditiveOfLogSumActionLinear`, lines 94–150)

Choose generators $e_1, \dots, e_r$ of the finitely generated $\mathbf{Z}$-module $\mathcal{O}_E$. Isotropy permits expressions

$$\left(\sum_j \overline{a_{N,b,j}} a_{N,b',j} \right) w_0 = \ell^N \sum_{i=1}^{r} n_{N,b,b',i} e_i, \qquad n_{N,b,b',i} \in \mathbf{Z}.$$

Write $y_{N,j} = \sum_b d[a_{N,b,j}] w_b$ and

$$T_N = \ell^N \sum_{b,b',i} n_{N,b,b',i} H(w_b, d[e_i] w_{b'}).$$

Expanding the preceding identities gives

$$\sum_j H(y_{N,j}, d[w_0]y_{N,j}) - T_N \in \mathcal{U}.$$

Hence the frame evaluated at $(\exp_{\widehat{G}} y_{N,j})_j$ is $U_N \exp(T_N)$, where U_N is an integral unit.
Lean: `GlobalTheory.IsoTriv.HermRead.finSt_of` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.GaugeHermitianCombinationOfSteps`, lines 52–110)
Lean: `GlobalTheory.IsoTriv.frameSt_of` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FrameSeriesExpSumOfActionLinearLogUnit`, lines 56–99)

The approximation at $N = 0$ makes every entry of each $A_{b,j}$ integral. Fix a coefficient indexed by α. For every $\gamma \le \alpha$, the coefficient of t^γ in $\exp(T_N)$ is a polynomial in the finitely many scalars $\ell^N n_{N,b,b',i}$. These scalars have valuation at least N, so the finite-jet estimate in Lemma 35.64, compared with all scalars equal to zero, makes these coefficients integral for sufficiently large N. The coefficient of t^α in $U_N \exp(T_N)$ is then integral by the finite product formula. The same finite-jet estimate, applied to the frame after the formal exponential and to the matrices $A_{b,j}$ and $d[a_{N,b,j}]$, makes the difference of their α-coefficients integral for sufficiently large N. Thus the limiting frame's α-coefficient is integral. Its constant term is 1 because $h_L(0) = 0$. □

Proof of 35.66

Proof (outline). The algebraically closed complete valued extension (C, v_ℓ) of $\operatorname{Frac} V$ exists over any trait above $\lambda \mid \ell$. The integral formal coordinates of $\widehat{G}$ over V descend the formal biextension data (group law, $\mathcal{O}_E$-action, cocycles, symmetry, and adjointness) to V, with canonical gauge h on $\widehat{G}\times\widehat{G}$ satisfying $h(0,0) = 0$ and $h_L(x) = h(x, [w_0]x)$ having zero constant term. The differential $d[a]$ of the action of $a \in \mathcal{O}_E$ acts on each Lie vector $u_{\tau,i'}$ by multiplication by $\tau(a)$.
Lean: `GlobalTheory.FactorDepth.nonempty_valExt_of_overLa` (`Development.AbelianVarietyTheory.Definitions.Divisors.NonemptyValExtOfOverLa`, lines 320–321)
Lean: `GlobalTheory.integralChart_formalBiext` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.IntegralChartFormalBiext`, lines 338–347)
Lean: `GlobalTheory.PowGerm.eigDirSt` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.IntegralChartDirectionsAreEigenvectors`, lines 619–627)

For (i), let $k \ge 0$ and let G be an integral power series over $\mathcal{O}_C$ on $\widehat{G}^N$. Given any matrix $x = (x_{\alpha,b})$ over $\mathcal{O}_E$, the biextension cocycle laws produce a formal power series U with coefficients in V and constant term 1 on $\widehat{G}^{\mathcal{C}}$ satisfying

$$U(y) = \exp\left(\sum_{b=1}^{N} h_L\Big(\sum_{\alpha\in\mathcal{C}} {}_{\widehat{G}}\,[x_{\alpha,b}]y_\alpha\Big) - \sum_{\alpha,\alpha'\in\mathcal{C}} h\Big(y_\alpha, \Big[w_0 \sum_{b=1}^{N} \bar{x}_{\alpha,b}x_{\alpha',b}\Big]y_{\alpha'}\Big)\right),$$

where $\sum_{\widehat{G}}$ denotes formal-group addition.
Lean: `GlobalTheory.exists_integral_unit_matrix_gauge` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsIntegralUnitMatrixGauge`, lines 309–319)

Define the power series on $\widehat{G}^{\mathcal{C}}$ by

$$F(y) = G\Big(\Big(\sum_{\alpha\in\mathcal{C}} {}_{\widehat{G}}\,[x_{\alpha,b}]y_\alpha\Big)_{b=1}^{N}\Big)\, U(y)^k.$$

Since the formal group law and $\mathcal{O}_E$-action have coefficients in V and zero constant term, substitution preserves integrality over $\mathcal{O}_C$, so F is integral over $\mathcal{O}_C$. Let exp be the formal exponential of $\widehat{G}$ over C. Because the $\mathcal{O}_E$-action is linear in formal logarithm coordinates and $u_{\tau,a}$ is a τ-eigenvector, substituting $y_\alpha = \exp(\sum_{a=1}^{i} t_{\alpha,a}u_{\tau,a})$ sends $\sum_{\alpha\in\mathcal{C}} {}_{\widehat{G}}\,[x_{\alpha,b}]y_\alpha$ to $\exp(\sum_{\alpha,a} \tau(x_{\alpha,b})t_{\alpha,a}u_{\tau,a})$ and sends $[w_0 \sum_b \bar{x}_{\alpha,b}x_{\alpha',b}]y_{\alpha'}$ to $\exp(\sum_a \tau'(w_0 \sum_b \bar{x}_{\alpha,b}x_{\alpha',b})t_{\alpha',a}u_{\tau',a})$. On the other hand, writing invariant derivations in exponential coordinates identifies $\mathcal{N}_k(G)$ with the derivative at 0, by the block-determinant polynomial of $\mathcal{D}_c$, of the same universal power series evaluated at linear substitution coefficients $(B_{\tau,i})_{j,b}$ and cross-coefficients 0.
Lean: `GlobalTheory.ExpCoord.constantCoeff_dpolyN_eq_derivAt0` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ConstantCoeffDpolyNEqDerivAt0`, lines 226–307)

Since this derivative at 0 depends only on a finite jet and is polynomial in the substitution coefficients, it is Lipschitz in v_ℓ on integral coefficient arrays with a constant $K \geq 0$ depending on G, k, P_{col}, $\widehat{G}$, and h_L. Comparing the two coefficient arrays yields $v_\ell(\mathcal{N}_k(G) - \mathcal{D}_c(F)) \geq n' - K$.
Lean: GlobalTheory.FGDepth.derivAt0_subst_linear_lipschitz (Development.AbelianVarietyTheory.Proofs.DivisorProofs.DerivAt0SubstLinearLipschitz, lines 77–84)
Lean: GlobalTheory.ColApprox.exists_germ_close (Development.AbelianVarietyTheory.Proofs.DivisorProofs.SubstTwoPointsPowerSeriesZeroOfVanishFirstAxis, lines 331–428)

For (ii), for each block $f = (\tau, i)$ and every $n_0 \geq 0$, choose the isotropic approximants $x_{n_0,\alpha,b}$ of Lemma 35.57(b), where $\alpha = (f, j) \in \mathcal{C}$. Lemma 35.64 supplies matrices $A_{\alpha,b} \in M_d(\mathcal{O}_C)$ such that

$$v_\ell\big((A_{\alpha,b} - d[\varepsilon_{\tau,n_0} x_{n_0,\alpha,b}])_{rs}\big) \geq n_0, \qquad A_{\alpha,b} u_{\tau,a} = (B_{\tau,i})_{j,b} u_{\tau,a}.$$

Multiplication of every row in the block f by ε_{τ,n_0} preserves its Hermitian Gram divisibility by ℓ^{n_0}; different blocks have disjoint supports, so their mixed Gram sums vanish. Define

$$\psi_b(Y) = \exp_{\widehat{G}}\left(\sum_{\alpha \in \mathcal{C}} A_{\alpha,b} \log_{\widehat{G}} Y_\alpha\right).$$

The second assertion of Lemma 35.64, applied to $w_\alpha = \log_{\widehat{G}} Y_\alpha$, makes ψ integral, with zero constant term. Its eigenvector identity gives $\psi(\mathbf{e}(t)) = \mathbf{c}(t)$, where

$$\mathbf{e}_\alpha(t) = \exp_{\widehat{G}}\left(\sum_a t_{\alpha,a} u_{\tau,a}\right), \qquad \mathbf{c}_b(t) = \exp_{\widehat{G}}\left(\sum_{\alpha=(\tau,i,j),a} (B_{\tau,i})_{j,b} t_{\alpha,a} u_{\tau,a}\right).$$

Lemma 35.65 shows that $U_0 = W \circ \psi$ is an integral unit, where $W = \exp(\sum_b h_L(x_b))$.
Lean: GlobalTheory.IsoTriv.Incl.inclSt_of_lim (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ColumnInclusionIntegralOfLimitSteps, lines 77–98)
Lean: GlobalTheory.T.limitIsotropicFrameIntegral_of_steps (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LimitIsotropicFrameIntegralOfStepsStatement, lines 6–7)
Lean: GlobalTheory.T.columnInclusionIntegral_of_limitSteps (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ColumnInclusionIntegralOfLimitStepsStatement, lines 6–7)

For each block $f = (\tau, i)$, choose its assigned leading columns $r_f(1), \ldots, r_f(i)$ and put $M_f = ((B_{\tau,i})_{j,r_f(j')})_{j,j'}$. This is a unit minor, and its entries are approximable through $\tau(\mathcal{O}_E)$. By Lemma 35.61, the entries of M_f^{-1} are approximable. Apply Lemma 35.64 to these entries and obtain matrices $A'_{f,j',j}$ acting on the τ-eigenvectors by $(M_f^{-1})_{j',j}$. Define

$$\rho_{f,j}(X) = \exp_{\widehat{G}}\left(\sum_{j'=1}^{i} A'_{f,j',j} \log_{\widehat{G}} X_{r_f(j')}\right).$$

These series are integral and have zero constant term by Lemma 35.64. On the column curve, all other blocks vanish in the selected columns, and the identity $M_f M_f^{-1} = 1$ gives $\rho(\mathbf{c}(t)) = \mathbf{e}(t)$.
Lean: GlobalTheory.IsoTriv.retrSt_of (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ColumnInclusionRetractionOfLimitSteps, lines 47–116)
Lean: GlobalTheory.IsoTriv.limitEntriesMatrixInverseClosed_proof (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LimitEntriesMatrixInverseClosedProof, lines 75–98)
Lean: GlobalTheory.T.columnInclusionRetraction_of_limitSteps (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ColumnInclusionRetractionOfLimitStepsStatement, lines 6–7)

Let $W = e^{\sum_{b=1}^{N} h_L(x_b)}$, so that $U_0 = \psi^* W$, and let $U' = U_0^{-1}$. Apply Lemma 35.62 with the block exponential curve as $\mathbf{e}$, the column exponential curve as $\mathbf{c}$, and the maps ψ, ρ above. It gives an integral series $G = \rho^*(F(U')^k)$ whose product with W^k, restricted to the column curve, equals F restricted to the block exponential curve. Applying the block determinant differential and the exponential-coordinate identity gives $\mathcal{N}_k(G) = \mathcal{D}_c(F)$.
Lean: GlobalTheory.T.logIntegralUnitsSubgroupExpHom_proof (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LogIntegralUnitsSubgroupExpHomProofStatement, lines 6–7)
Lean: GlobalTheory.T.nablaOperatorFrameSeries_of_steps (Development.AbelianVarietyTheory.Proofs.DivisorProofs.NablaOperatorFrameSeriesOfStepsStatement, lines 6–7)
Lean: GlobalTheory.ColumnGerm.exists_nabX_eq (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsIntegralGermOnCopiesSameValue, lines 47–86)

The complete proof in the archive is contained in `GlobalTheory.ColApprox.exists_germ_cols` and `GlobalTheory.ColumnGerm.exists_germ_on_copies`. □

Proof of 35.67

Proof. Fix $k \geq 0$, and choose the field (C, v_ℓ) and integral formal V-coordinates of $\widehat{G}$ provided by Proposition 35.66. Any integral formal coordinates of $\widehat{G}$ over an algebraically closed complete valued extension (C', v'_ℓ) of $\operatorname{Frac} V$ differ from the V-coordinates by an invertible integral power-series substitution with zero constant term on each copy of $\widehat{G}$. Such a substitution permutes the integral power series over $\mathcal{O}_{C'}$ while leaving the invariant derivations and h_L unchanged. In the V-coordinates we have $h_L(0) = 0$ and $P_{\mathrm{col}}(D)$ has order D. For any integral series $G = \sum_\mu c_\mu X^\mu$ over $\mathcal{O}_{C'}$, the value $\mathcal{N}_k(G)$ equals the finite sum $\sum_{|\mu| \leq D} c_\mu \mathcal{N}_k(X^\mu)$. Each monomial X^μ has coefficients in V, so $\mathcal{N}_k(X^\mu) \in \operatorname{Frac} V$ has the same normalised valuation in C' and in C, while $v'_\ell(c_\mu) \geq 0$. By the ultrametric inequality, $\delta_k(P_{\mathrm{col}}; y)$ equals the infimum of $v_\ell(\mathcal{N}_k(G))$ over power series G with coefficients in V in the V-coordinates, and also equals the infimum over integral power series G over $\mathcal{O}_C$ in the V-coordinates.

Given an integral power series G over $\mathcal{O}_C$ on $\widehat{G}^N$ and $m_0 \geq 0$, take K from Proposition 35.66(i) and choose an integer $n' \geq m_0 + K$. For each block (τ, i) with $c_{\tau,i} \neq 0$, Lemma 35.57(b) with $n_0 = n'$ supplies $x_{(\tau,i,j),b} \in \mathcal{O}_E$ supported on the $2i$ columns assigned to that block, with $\tau(x_{(\tau,i,j),b}) \equiv (B_{\tau,i})_{j,b}$ modulo $\ell^{n'} V_\lambda$ and $\sum_{b=1}^{N} x_{(\tau,i,j),b} \bar{x}_{(\tau,i,j'),b} \in \ell^{n'} \mathcal{O}_E$. For $\alpha, \alpha' \in \mathcal{C}$ in distinct blocks the column supports are disjoint, so $\sum_{b=1}^{N} \bar{x}_{\alpha,b} x_{\alpha',b} = 0$. Within a single block, conjugation places $\sum_{b=1}^{N} \bar{x}_{\alpha,b} x_{\alpha',b}$ in $\ell^{n'} \mathcal{O}_E$, and multiplication by $w_0 \in \mathcal{O}_E$ preserves $\ell^{n'} \mathcal{O}_E$. Passing to V and C gives both valuation hypotheses of Proposition 35.66(i). Hence there is an integral series F over $\mathcal{O}_C$ on $\widehat{G}^{\mathcal{C}}$ with $v_\ell(\mathcal{N}_k(G) - \mathcal{D}_c(F)) \geq n' - K \geq m_0$. Whenever $\mathcal{N}_k(G) \neq 0$, choosing $m_0 > v_\ell(\mathcal{N}_k(G))$ yields $v_\ell(\mathcal{D}_c(F)) = v_\ell(\mathcal{N}_k(G))$ by the ultrametric equality. Combining this with Proposition 35.66(ii) identifies $\inf_G v_\ell(\mathcal{N}_k(G))$ with $\inf_F v_\ell(\mathcal{D}_c(F))$.

By the coordinate and monomial argument of the first paragraph on i copies of $\widehat{G}$, each block depth $\nu_{\tau,i}(c_{\tau,i}; y)$ equals the corresponding infimum in the V-coordinates over C. It is finite and nonnegative by Lemma 35.54. Testing $\mathcal{D}_c$ on external products $F = \prod_{c_{\tau,i} \neq 0} F_{\tau,i}$ of integral series on the individual blocks gives $\inf_F v_\ell(\mathcal{D}_c(F)) \leq \sum_{c_{\tau,i} \neq 0} \nu_{\tau,i}(c_{\tau,i}; y)$. Conversely, expanding any integral series F on $\widehat{G}^{\mathcal{C}}$ in the coordinates of each block makes $\mathcal{D}_c(F)$ a finite $\mathcal{O}_C$-linear combination of product evaluations, so the ultrametric inequality gives the reverse bound. Since $\nu_{\tau,i}(0; y) = 0$ by Lemma 35.54, adding the blocks with $c_{\tau,i} = 0$ gives $\nu_c(y)$. □

Proof of 35.68

Proof. Each summand of the second-order expression is zero. Its coefficient vanishes for a nonconjugate pair, and the assumed harmonic identity makes the inner sum zero for a conjugate pair. If harmonicity is initially stated for one member of each conjugate pair, the other identities follow by interchanging the two partial derivatives.

For nonnegative integers A, B, one has $2\min(A, B) \leq A + B$. Hence

$$2e(\mu) \leq \sum_{b,q} (n^+_{q,b}(\mu) + n^-_{q,b}(\mu)) = \sum_{\ell,b} \mu_{\ell,b} = \sum_\ell a_\ell = D.$$

Since $e(\mu)$ is an integer, this proves the bound on the number of opposite pairs.

The ultrametric inequality gives closure of J_η under addition. Multiplication by an element of V preserves its defining bound, and $\eta \geq 0$ gives $1 \in J_\eta$. A product of m elements of J_η has normalized valuation at least $-m\eta$. Every finite V-linear combination of such products has the same lower bound. Finally, $\lfloor D/2 \rfloor \leq D/2$ and $\eta \geq 0$ give the last assertion. □

Proof of 35.69

Proof. Write $a_\ell = \nabla_{\xi^0_\ell}$ and $b_\ell = \nabla_{(0,\psi_\ell)}$. Within either sign block, the opposite-pair assumptions allow the b's to pass the a's. Moving a plus-block b_ℓ past a minus-block $a_{\ell'}$ contributes

$$m\beta((0, \psi_\ell), \xi^0_{\ell'}).$$

At the right end every remaining b kills ιg. Consequently, if Δ_0 is the single-column second-order operator with precisely these contraction coefficients, then

$$O_{\xi^0+(0,\psi)}(Q)\iota g = \sum_{r=0}^{\deg Q} O_{\xi^0}\left(\frac{\Delta_0^r Q}{r!}\right)\iota g.$$

This identity follows by induction from $ba^t = a^t b + t[b,a]a^{t-1}$, since each displayed commutator is scalar.

For (a), take $\xi_\ell^0 = (u_\ell, 0)$ and $\psi_\ell = \phi_\ell$. The horizontal evaluation identity turns each term into an invariant differential value. Operators on different columns commute. The multinomial formula combines the product of their finite exponential expansions into $\sum_r \Delta^r P/r!$. Since $\Delta P = 0$, the term with $r = 0$ gives the asserted equality.

For (b), every contraction coefficient belongs to J, since it equals $-m\sum_i \psi_{\ell,i}u_{\ell',i}$. The factorial in the expansion preserves integral polynomial coefficients. More explicitly, for contraction multiplicities t_{ij} from plus indices i to minus indices j, let

$$r_i = \sum_j t_{ij}, \qquad c_j = \sum_i t_{ij}, \qquad r = \sum_{ij} t_{ij}.$$

On a monomial with plus exponents a_i and minus exponents b_j, the coefficient accompanying $\prod_{ij}\kappa_{ij}^{t_{ij}}$ in $\Delta_0^r/r!$ is

$$\frac{\prod_i (a_i)_{r_i}\prod_j (b_j)_{c_j}}{\prod_{ij} t_{ij}!} = \prod_i \left(\binom{a_i}{r_i}\frac{r_i!}{\prod_j t_{ij}!}\right)\prod_j (b_j)_{c_j} \in \mathbf{Z},$$

where $(a)_r = a(a-1)\cdots(a-r+1)$, and inadmissible contractions give zero. Hence the coefficients of the r-th term lie in J^r. The support condition on κ bounds the number of contractions in each class by the smaller of its two sign totals. The stated bound therefore makes the total number over all columns at most e.

The conjugation hypothesis says that every a_ℓ preserves $(\iota E_0)V[[x,z]]$. Every ordered horizontal word applied to $\iota(E_0 f_b)$ consequently has constant coefficient in V. A term with r contractions over all columns lies in $J^r \subseteq J^e$, using $1 \in J$. Finite sums and multiplication by the coefficients of P preserve J^e, proving (b). □

Proof of 35.70

Proof (outline). Write $\mathcal{F}_j(x,y)$ for the formal group law and set

$$A_{ji}(x) = \frac{\partial \mathcal{F}_j}{\partial y_i}(x,0), \qquad D_u = \sum_{j,i} u_i A_{ji}(x)\partial_{x_j}.$$

For $\xi = (u,\phi)$, $\eta = (u',\phi')$ in $K^d \oplus K^d$, define

$$\mathcal{V}_\xi = \sum_j (D_u x_j)\partial_{x_j} + \sum_i (D_u\sigma_i + \phi_i)\partial_{z_i}, \quad \lambda_u = \sum_i (z_i - \sigma_i(x))u_i, \quad \nabla_\xi = \mathcal{V}_\xi - m\lambda_u.$$

These depend linearly on ξ, and the derivation part restricts to D_u on series in x.
Lean: `ReciprocityTheory.ChartRestriction.G.TB.WickVanishing.Nab.cf` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.WickVanishing.ConnectionKillsSeries`, lines 88–90)
Lean: `ReciprocityTheory.ChartRestriction.G.TB.nabla` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.ConnectionKillsSeries`, lines 31–34)

Claim 1. Commutativity of the formal group gives $D_u D_{u'} = D_{u'} D_u$. Hence

$$[\mathcal{V}_\xi, \mathcal{V}_\eta] = 0, \qquad \mathcal{V}_\xi\lambda_{u'} - \mathcal{V}_\eta\lambda_u = \sum_i (\phi_i u_i' - \phi_i' u_i) = -\beta(\xi,\eta).$$

For the first identity, evaluate the bracket on x_j, z_i, obtaining respectively $[D_u, D_{u'}]x_j$ and $[D_u, D_{u'}]\sigma_i$. For the second use $\mathcal{V}_\xi(z_i - \sigma_i) = \phi_i$. Expanding gives

$$[\nabla_\xi, \nabla_\eta]G = m\beta(\xi,\eta)G.$$

Also $\nabla_{(0,\phi)} = \sum_i \phi_i \partial_{z_i}$, which annihilates series in x.
Lean: ReciprocityTheory.ChartRestriction.G.TB.WickVanishing.Nab.cf_bracket (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.WickVanishing.ConnectionSecondOfCommute, lines 29–36)
Lean: ReciprocityTheory.ChartRestriction.G.TB.ConnectionCommutator (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.ConnectionKillsSeries, lines 60–64)
Lean: ReciprocityTheory.ChartRestriction.G.TB.ConnectionKillsSeries (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.ConnectionKillsSeries, lines 66–69)

Claim 2. The substitution

$$\mathcal{E}(G) = G(x, \sigma_1(x), \dots, \sigma_d(x))$$

is defined and satisfies $\mathcal{E}(\iota f) = f$ and $\mathcal{E}\nabla_{(u,0)}G = D_u\mathcal{E}(G)$. Indeed, $h(0,y) = 0$ gives $\sigma_i(0) = 0$. The formal chain rule gives $\mathcal{E}\mathcal{V}_{(u,0)} = D_u\mathcal{E}$, and $\mathcal{E}\lambda_u = 0$. Iteration, linearity and evaluation at zero give the horizontal evaluation identity in Lemma 35.69.
Lean: ReciprocityTheory.ChartRestriction.G.TB.WickVanishing.Val.E_nablaPoly (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.WickVanishing.HarmonicValueIntegral, lines 133–140)

Claim 3. For the canonical gauge and the host element $w_0 + \bar{w}_0 = 1$, every $u \in V^d$ satisfies

$$D_u\sigma_i - \sum_j S_{ij}u_j \in V[[x]], \qquad D_u h_L + \sum_i \sigma_i u_i \in V[[x]].$$

Differentiate the integral cocycle in its two variables and subtract its constant linear term for the first identity. For the second differentiate $h_L(x) = h(x, [w_0]x)$. The integral cocycle, symmetry and adjointness relations express the result plus $\sum_i \sigma_i u_i$ as integral series; the action terms combine through $w_0 + \bar{w}_0 = 1$.
Lean: ReciprocityTheory.ChartRestriction.G.TB.WickVanishing.Nab.dof_sig (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.WickVanishing.ConstantCoeffAddFirstSecond, lines 115–130)
Lean: ReciprocityTheory.ChartRestriction.G.TB.WickVanishing.Nab.rel_hL (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.WickVanishing.SigCancel, lines 149–200)

Since A_{ji} are integral, Claim 3 makes every coefficient of $\mathcal{V}_{(u,\phi-Su)}$ integral for $u, \phi \in V^d$. As $h_L(0) = 0$, the formal exponentials $E_0 = \exp(mh_L)$ and its inverse are defined. The product rule gives

$$(\iota E_0)^{-1}\nabla_{(u,\phi-Su)}((\iota E_0)G) = \mathcal{V}_{(u,\phi-Su)}G + m\left(D_u h_L + \sum_i \sigma_i u_i - \sum_i z_i u_i\right)G.$$

The right side is integral for $G \in V[[x,z]]$. The general integral statement's denominator condition holds here: a sufficiently large power of the positive residue characteristic clears each element's denominator in K.
Lean: ReciprocityTheory.ChartRestriction.G.TB.ConnectionIntegrality (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.ConnectionKillsSeries, lines 77–83)
Lean: ReciprocityTheory.ChartRestriction.G.TB.WickVanishing.Nab.exp_conj_connection_integral (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.WickVanishing.ExpConjConnectionIntegral, lines 40–82)

The complete connection-identity proofs are contained in `ReciprocityTheory.ChartRestriction.G.TB.WickVanishing.Nab.connection_second_identity`, `ReciprocityTheory.ChartRestriction.G.TB.WickVanishing.Nab.connection_kills_series`, `ReciprocityTheory.ChartRestriction.G.TB.WickVanishing.Val.harmonic_value_integral`, and `ReciprocityTheory.ChartRestriction.G.TB.WickVanishing.Nab.connection_first_identity`. □

Proof of 35.71

Proof (outline). The harmonic differential has order at most D, so its value at the origin depends only on the degree-D jet of F. Indeed, each invariant derivation lowers the order of vanishing by at most one, and multiplication by the canonical exponential, whose constant term is one, does not lower it. Replace F by this finite jet and expand it as a finite V-linear combination of monomials. Each monomial is an external product of monomials in the separate copies. The ultrametric inequality for this finite sum therefore reduces both valuation bounds to products $F = f_1 \boxtimes \cdots \boxtimes f_N$ with $f_b \in V[[x]]$; no limit of valuations is required.
Lean: GlobalTheory.RamChartCmp.bound_all_iff_bound_prod (Development.AbelianVarietyTheory.Proofs.JetHarmonics.TotalDegreeLeOfMemFFV, lines 101–121)

The integral chart supplies a descended formal biextension over V whose canonical gauge h has constant term 0 and whose $\mathcal{O}_E$-action has the frame directions $u_{\tau,i}$ as τ-eigenvectors.
Lean: GlobalTheory.PlaceInt.integralBiext_gauge_eig_bound (Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.IntegralChartFormalBiextensionGaugeBound, lines 50–108)

For (1), suppose $\operatorname{disc} E \in V^\times$.

Claim 1. If $\mathbf{Q} \subset V$, the integral formal biextension has a canonical gauge h_0 over V. Write $h_{0,L}(x) = h_0(x, [w_0]x)$. For every $R \in \operatorname{harm}_V(a, N)$, every $k \geq 0$, and every $f_b \in V[[x]]$,

$$\left[R(D)\left(\prod_b f_b(x_b)e^{k\sum_b h_L(x_b)}\right)\right](0) = \left[R(D)\left(\prod_b f_b(x_b)e^{k\sum_b h_{0,L}(x_b)}\right)\right](0).$$

The inputs to this equality are the integral eigenvectors of the chart, distinct embeddings of E into $\operatorname{Frac} V$, conjugation of these embeddings, and the conjugate-row contraction equations defining harmonicity. The difference of two canonical gauges contributes only such contractions, which vanish on R.
Lean: `GlobalTheory.canonGauge_independence_harmonic` (`Development.AbelianVarietyTheory.Proofs.FormalGroupCharts.CanonGaugeIndependenceHarmonic`, lines 54–71)

The right side is in V: the group law, the directions and h_0 are integral, and all factorials are units because $\mathbf{Q} \subset V$. A trait of residue characteristic zero contains $\mathbf{Q}$, so this proves that case.
Lean: `GlobalTheory.ChartUnram.value_mem_rat` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.HarmonicValueIntegralOfUnitDiscriminant`, lines 68–96)

Suppose now that V has positive residue characteristic. Put $K = \operatorname{Frac} V$, $d = \dim \widehat{G}$, and let S be the matrix defined in Lemma 35.70. For an embedding $\tau : E \to \mathbf{C}$, write $\sigma_\tau : \mathcal{O}_E \to V$ for its coefficient-field map followed by the map to V.

Claim 2. If $D_E \in V^\times$, then for each τ there are $b_{\tau,j} \in \mathcal{O}_E$ and $c_{\tau,j} \in V$, indexed by a finite set, such that

$$\sum_j c_{\tau,j}\sigma_\tau(b_{\tau,j}) = 1.$$

For every $a \in \mathcal{O}_E$ there is an integer matrix $M(a)$ satisfying

$$ab_{\tau,i} = \sum_j M(a)_{ij}b_{\tau,j}, \qquad \sum_i M(a)_{ij}c_{\tau,i} = \sigma_\tau(a)c_{\tau,j}.$$

For any integral direction system $u_\ell \in V^d$ with $d[a]u_\ell = \sigma_\tau(a)u_\ell$ for $\ell = (\tau, i)$, these data give $\varphi_\ell \in V^d$ such that

$$\sum_j (\varphi_{\ell',j} - (Su_{\ell'})_j)u_{\ell,j} = 0 \quad \text{if the type of } \ell' \text{ is not the conjugate of the type of } \ell.$$

Lean: `ReciprocityTheory.ChartRestriction.G.PullbackSections.Pf.EigA.idem_data` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.PullbackSections.HolomorphicFamilies.EigenLiftsOfDirectionsBiext`, lines 56–73)
Lean: `ReciprocityTheory.ChartRestriction.G.PullbackSections.Pf.EigA.eigLift_core` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.PullbackSections.HolomorphicFamilies.BiextCoeffSecondLinear`, lines 169–238)

The first two identities use the trace-dual basis and $D_E^{-1} \in V$. The last identity follows by applying the integral idempotent to $(u_\ell, -Su_\ell)$ (where $\mathcal{O}_E$ acts on the dual component Su_ℓ by the untwisted dual Lie action $a \mapsto d[\bar{a}]^{\mathsf{t}}$, using the integral adjointness defect $d[\bar{a}]^{\mathsf{t}}S - Sd[a] \in M_d(V)$) and using adjointness for the alternating form β of Lemma 35.69: a pairing of characters σ_τ and $\sigma_{\tau'} \circ {}^-$ vanishes when they differ.

Apply Lemmas 35.70 and 35.69 with $J = V$, $\psi_\ell = 0$, $\xi_\ell^0 = (u_\ell, \varphi_\ell - Su_\ell)$, and $E_0 = \exp(kh_L)$. Identify the members of each conjugate pair in the class map, and give them opposite signs. The last identity of Claim 2 gives opposite-pair vanishing and, by Lemma 35.68, the second-order contraction condition. Lemma 35.70 gives integral conjugation, and Lemma 35.68 gives the count bound. Lemma 35.69(a), applied to $g_b = E_0 f_b$, identifies z with $\mathcal{E}_{\xi^0}(R; E_0f_1, \dots, E_0f_N)$, and Lemma 35.69(b) then gives $z \in V$, hence $v(z) \geq 0$.
Lean: `ReciprocityTheory.ChartRestriction.G.PullbackSections.Pf.EigA.idem_data` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.PullbackSections.HolomorphicFamilies.EigenLiftsOfDirectionsBiext`, lines 56–73)
Lean: `ReciprocityTheory.ChartRestriction.G.PullbackSections.Pf.EigA.eigLift_core` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.PullbackSections.HolomorphicFamilies.BiextCoeffSecondLinear`, lines 169–238)
Lean: `GlobalTheory.ChartUnram.value_mem_idem` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.HarmonicValueIntegralOfUnitDiscriminant`, lines 98–160)

For (2), let $p \mid Q_S q d$ be a prime that is not a unit in V, and write $D_E = p^{a_0}m_0$ with $p \nmid m_0$, so that $\delta_p = a_0/[E : \mathbf{Q}]$.

Claim 3. The host field $E/\mathbf{Q}$ is Galois. Every rational prime dividing $\operatorname{disc} E$ divides $Q_S q d$. For every prime $p \mid Q_S q d$ and every prime ideal $\mathfrak{p} \subset \mathcal{O}_E$ containing p, there is $a \in \mathfrak{p}$ such that $\bar{a} \notin \mathfrak{p}$; in particular, $\bar{\mathfrak{p}} \neq \mathfrak{p}$. Indeed, the host-field construction of 5.1 and 8.3 makes $E/\mathbf{Q}$ Galois with all prime divisors of $\operatorname{disc} E$

dividing the conductor $Q_S qd$, and excludes complex conjugation $c = ((-1,-1),1) \in \mathrm{Gal}(E/\mathbf{Q})$ from the decomposition group D_p of every prime $\mathfrak{p} \mid p$ for each $p \mid Q_S qd$: for every prime $p \mid Q_S q$ (including $p = 2$, $p = q$, and every odd prime $p \mid Q_S$), Lemma 5.3 places both the inertia group I_p and the Frobenius element F_p (hence the entire decomposition group D_p) inside the kernel of the d-sign character (since $d \mid \mathfrak{m}^{(p)}$, $(-d/p) = +1$ for every odd $p \mid Q_S q$ by Definition 5.1, and $(2/d) = +1$ from $d \equiv 7 \pmod 8$), whereas c has d-sign -1; and for $p = d$, since $e \geq 2$ in Definition 5.1 gives $16 \mid 2^{e+2} \mid Q_S$ with $d \equiv 7 \pmod{16}$ and $(d/q) = -1$, any relation $c \in F_d^k I_d$ in $D_d = \langle F_d, I_d \rangle$ forces $k = 2j+1$ to be odd from the q-sign $(-1)^k = -1$, while projecting the third coordinate to $H_e = (\mathbf{Z}/2^{e+2}\mathbf{Z})^\times/\{\pm 1\}$ forces $d^{2j+1} \equiv \pm 1 \pmod{2^{e+2}}$ and hence modulo 16, contradicting $7^{2j+1} \equiv 7 \not\equiv \pm 1 \pmod{16}$. Since c is an involution, $\bar{a} \in \mathfrak{p}$ for all $a \in \mathfrak{p}$ would force $c(\mathfrak{p}) = \mathfrak{p}$, contradicting $c \notin D_p$.

Lean: `GlobalTheory.PlaceInt.host_galois_discr_moves` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.HostFieldGaloisDegreeDiscrConjMovesPrimes`, lines 26–54)

Use $K = \operatorname{Frac} V$, $d = \dim \widehat{G}$, and the matrix S from Lemma 35.70. Let L be the set of frame directions $\ell = (\tau, i)$, and write $\sigma_\tau : \mathcal{O}_E \to V$ for the coefficient-field embedding attached to τ, followed by the integral map to V. Pulling the maximal ideal of V back along σ_τ assigns a prime $\mathfrak{p}_\tau$ of $\mathcal{O}_E$ to each embedding, with $\mathfrak{p}_{\bar\tau} = \bar{\mathfrak{p}}_\tau \neq \mathfrak{p}_\tau$.

Claim 4. Under the Galois and prime-moving conditions of Claim 3, there are a finite set Π, maps $c : \mathrm{Hom}(E, \mathbf{C}) \to \Pi$ and $s : \mathrm{Hom}(E, \mathbf{C}) \to \{+,-\}$, and vectors $\varphi_\ell \in V^d$, $\psi_\ell \in K^d$, such that

$$c(\bar\tau) = c(\tau), \qquad s(\bar\tau) \neq s(\tau), \qquad \frac{v(\psi_{\ell,j})}{e} \geq -\frac{v(p)}{e}\delta_p.$$

Put

$$\xi_\ell^0 = (u_\ell, \varphi_\ell - S u_\ell), \qquad \eta_\ell = \varphi_\ell + \psi_\ell - S u_\ell.$$

Then

$$\sum_j \eta_{\ell',j} u_{\ell,j} = 0 \quad \text{if the types of } \ell \text{ and } \ell' \text{ are not conjugate,}$$

and the three pairs of vector families

$$(\xi^0, \xi^0), \qquad ((0,\psi), \xi^0), \qquad (\xi^0, (0,\psi))$$

satisfy opposite-pair vanishing in the sense of Lemma 35.69, with class $c(\tau)$ and sign $s(\tau)$ on $\ell = (\tau, i)$.

Lean: `GlobalTheory.PlaceInt.exists_ramified_eigenLifts` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.ExistsRamifiedEigenLiftsOfChartDirections`, lines 177–300)

Lean: `GlobalTheory.T.exists_ramified_eigen_lifts_of_chart_directions` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.ExistsRamifiedEigenLiftsOfChartDirectionsStatement`, lines 13–14)

Indeed, because $\mathfrak{p}_{\bar\tau} = \bar{\mathfrak{p}}_\tau \neq \mathfrak{p}_\tau$, partitioning the unordered conjugate pairs $\{\mathfrak{p}_\tau, \bar{\mathfrak{p}}_\tau\}$ into classes $c(\tau)$ with opposite signs $s(\bar\tau) \neq s(\tau)$ assigns conjugate embeddings to the same class with opposite signs. Since the maximal ideals $\mathfrak{p}$ of $\mathcal{O}_E$ above p are pairwise comaximal, the Chinese remainder theorem provides integral place-idempotent relations $\sum_i a_{\mathfrak{p},i}\sigma_\tau(b_{\mathfrak{p},i}) = \mathbf{1}_{\mathfrak{p}_\tau = \mathfrak{p}}$ over V. For each τ, because $E/\mathbf{Q}$ is Galois and $D_E = p^{a_0} m_0$ with $p \nmid m_0$, there is an element $\delta_\tau \in \mathfrak{D}_{E/\mathbf{Q}}$ of the different whose image satisfies $\sigma_\tau(\delta_\tau)^{[E:\mathbf{Q}]} \mid p^{a_0}$ in V, whence $v(\sigma_\tau(\delta_\tau))/e \leq (v(p)/e)\delta_p$. Scaling the trace-dual basis of $\mathcal{O}_E$ by δ_τ separates embeddings sharing the same residual prime $\mathfrak{p}_\tau$ at denominator cost at most $\sigma_\tau(\delta_\tau)^{-1}$, while the integral place idempotents separate distinct primes $\mathfrak{p}_\tau \neq \mathfrak{p}_{\tau'}$ over V. Applying the resulting idempotent projection to $(u_\ell, -S u_\ell)$ decomposes the eigenlift into an integral part $\varphi_\ell \in V^d$ (separating non-conjugate residual primes, which yields opposite-pair vanishing for (ξ^0, ξ^0)) and a meromorphic part $\psi_\ell \in K^d$ with denominator $\sigma_\tau(\delta_\tau)$ satisfying $v(\psi_{\ell,j})/e \geq -(v(p)/e)\delta_p$ together with the displayed eigenlift pairing and opposite-pair vanishing relations.

Set

$$J = \{x \in K : v(x)/e \geq -(v(p)/e)\delta_p\}, \qquad E_0 = \exp(k h_L).$$

Since $v(p) \geq 0$ and $\delta_p \geq 0$, J is a V-submodule containing 1. Claim 4 gives $\xi_\ell^0 \in H_V$ and $\psi_\ell \in J^d$. On the index set L, abbreviate $c(\ell) = c(\tau)$ and $s(\ell) = s(\tau)$ when $\ell = (\tau, i)$. The contraction coefficient in Lemma 35.69(a) is

$$\kappa_{\ell\ell'} = \begin{cases} k\beta((0,\eta_\ell),(u_{\ell'},0)), & c(\ell) = c(\ell'),\ s(\ell) = +,\ s(\ell') = -, \\ 0, & \text{otherwise.} \end{cases}$$

The eigenlift pairing identity makes it vanish unless the types of ℓ and ℓ' are conjugate. Lemma 35.68 therefore gives $\sum_{b,\ell,\ell'} \kappa_{\ell\ell'}\partial_{X_{\ell,b}}\partial_{X_{\ell',b}}R = 0$. The eigenlift pairing identity $\sum_i \eta_{\ell',i}u_{\ell,i} = 0$ holds whenever the type of ℓ' is not the conjugate of the type of ℓ; since $c(\bar\tau) = c(\tau)$ and $s(\bar\tau) \neq s(\tau)$, it gives the opposite-pair vanishing hypotheses of Lemma 35.69(a). Lemma 35.69(a), applied to $g_b = E_0 f_b$, identifies z with $\mathcal{E}_{\xi^0+(0,\psi)}(R; E_0f_1, \dots, E_0f_N)$.

The three opposite-pair vanishing conditions are precisely those supplied by the eigenlifts. Lemma 35.70 gives integral conjugation for ξ^0_ℓ, and Lemma 35.68 bounds the sum of the columnwise opposite-pair counts by $\lfloor D/2 \rfloor$. Lemma 35.69(b) now places z in $J^{\lfloor D/2 \rfloor}$. Its valuation satisfies

$$v(z)/e \geq -\lfloor D/2 \rfloor\,(v(p)/e)\delta_p \geq -(v(p)/e)\delta_p D/2$$

by the last assertion of Lemma 35.68.

Lean: `ReciprocityTheory.ChartRestriction.G.PullbackSections.Pf.Jv` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.PullbackSections.HolomorphicFamilies.ChartValueRamifiedOfLifts`, lines 35–45)
Lean: `ReciprocityTheory.ChartRestriction.G.PullbackSections.Pf.sum_pairMin_le` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.PullbackSections.HolomorphicFamilies.ChartValueRamifiedOfLifts`, lines 110–126)
Lean: `ReciprocityTheory.ChartRestriction.G.PullbackSections.Pf.chartValJ` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.PullbackSections.HolomorphicFamilies.ChartValJ`, lines 31–158)
Lean: `ReciprocityTheory.ChartRestriction.G.PullbackSections.Pf.mem_Jv_pow` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.PullbackSections.HolomorphicFamilies.ChartValueRamifiedOfLifts`, lines 56–80)

The complete proof in the archive is contained in `GlobalTheory.PlaceInt.chBound_zero_of_isUnit_discr` and `GlobalTheory.PlaceInt.ChartRam.chBound_ramified`. □

Proof of 35.72

Proof. (1) By 8.3, $E \subset \mathbf{Q}(\mu_{Q_S qd})$, so every prime divisor of $\operatorname{disc} E$ divides $Q_S qd$, and each summand $(v(p)/e)\,\delta_p\,D/2$ is nonnegative. If every prime $p \mid Q_S qd$ is a unit in V, then D_E is a product of units in V, so $\operatorname{disc} E \in V^\times$ and Proposition 35.71(1) gives $v(z)/e \geq 0$, which dominates the negated sum. Otherwise some prime $p \mid Q_S qd$ is not a unit in V; Proposition 35.71(2) gives $v(z)/e \geq -(v(p)/e)\,\delta_p\,D/2$, and subtracting the remaining nonnegative summands preserves the inequality.

(2) Put $\rho_0 = \sum_{p \mid Q_S qd} \delta_p\,D/2$ and choose a nonnegative integer $n_0 > B + \rho_0$. Lemma 35.57(c) gives $P_1 \in \operatorname{harm}(a, N)$ such that, on every trait V over λ, $P_1 - P_{\mathrm{col}} = \ell^{n_0}R$ with $R \in \operatorname{harm}_V(a, N)$. Fix a trait V over λ, an algebraic integral frame y on V, and $k \geq 0$. Since $\lambda \mid \ell$, we have $v(\ell) > 0$ and ℓ lies in the maximal ideal of V. Any prime $p \neq \ell$ is coprime to ℓ in $\mathbf{Z}$, hence also in V, so $p \in V^\times$ and $v(p) = 0 \leq v(\ell)$; for $p = \ell$ equality holds. Applying (1) with $e = v(\ell)$ in the integral formal V-coordinates of Proposition 35.66 therefore gives $v(z)/v(\ell) \geq -\rho_0$ for every power series F over V on $\widehat{G}^N$, where $z = [R(D)(F\,e^{k\sum_{b=1}^N h_L(x_b)})](0)$.

As in the first paragraph of the proof of Lemma 35.67, both $\delta_k(P_1; y)$ and $\delta_k(P_{\mathrm{col}}; y)$ are computed as infima over power series F with coefficients in V. For every such F, linearity in the differential gives

$$\big[P_1(D)\big(F\,e^{k\sum_{b=1}^N h_L(x_b)}\big)\big](0) = \big[P_{\mathrm{col}}(D)\big(F\,e^{k\sum_{b=1}^N h_L(x_b)}\big)\big](0) + \ell^{n_0}z.$$

Here $v(\ell^{n_0}z)/v(\ell) = n_0 + v(z)/v(\ell) \geq n_0 - \rho_0 > B$. By Lemma 35.67, $\delta_k(P_{\mathrm{col}}; y) = \nu_c(y) \in [A, B]$. Since the perturbation has normalised valuation strictly greater than $B \geq \delta_k(P_{\mathrm{col}}; y)$, the ultrametric inequality on the infimum over F gives $\delta_k(P_1; y) = \delta_k(P_{\mathrm{col}}; y) \in [A, B]$. Taking $A = A_\lambda$ and $B = A_\lambda + \varrho$ from Definition 35.53(b) yields the final statement. □

Proof of 35.73

Proof (outline). Taking the product with the conjugate in $m = \beta\alpha n'$ gives $m^2 = n'^2AB$. In particular n'^2, A and n'^2A are odd. Define

$$S_0 = \mathcal{P}_{n',1}s_0, \qquad S_{1,b} = \mathcal{T}_{[\alpha\beta]z_{1,b},n'^2}S_0,$$
$$S_{2,b} = \mathcal{P}_{\alpha,n'^2}S_{1,b}, \qquad T_b = \mathcal{T}_{[\beta]z_{2,b},n'^2A}S_{2,b}.$$

The displayed torsion conditions make both translations defined. Normalised pullbacks compose, and for the torsion points and levels here they satisfy

$$\mathcal{P}_{\gamma,k}\mathcal{T}_{[\gamma]z,k} = \mathcal{T}_{z,k\gamma\bar\gamma}\mathcal{P}_{\gamma,k}.$$

Apply this identity first with $\gamma = \alpha\beta$, then with $\gamma = \beta$. Uniqueness of the normalised pullbacks and symmetric translates gives

$$\mathcal{P}_{\beta,n'^2A}T_b = F_b.$$

Fix a uniformiser of V, its integral period form Q_X, and an algebraically closed valued extension of K extending the trait valuation. At positive level q, a family belongs to $\mathscr{T}_q^+$ when its coefficient at each $\chi \in X$, measured in integral abelian-part frames, has valuation at least $Q_X(\chi)/q$.

Claim 1. For this integral frame, split PEL uniformisation and torsion identification:

1. Every compatible expansion of the base-changed principal section s_0 belongs to $\mathscr{T}_1^+$.
2. If $q = 1$ or $q \geq 4$, $C > 0$, $\gamma\bar{\gamma} = C$, and every compatible expansion of s belongs to $\mathscr{T}_q^+$, then every compatible expansion of $\mathcal{P}_{\gamma,q}s$ belongs to $\mathscr{T}_{qC}^+$.
3. At $q \geq 1$, a symmetric theta-group translate of a section all of whose compatible expansions belong to $\mathscr{T}_q^+$ has the same property.
4. At $q \geq 1$, a section over K whose base change has every compatible expansion in $\mathscr{T}_q^+$ extends to a regular section on the integral semiabelian model, with the specified rigidified line identification.

The first uses the integral principal frame. For the second, uniqueness identifies the pulled-back expansion with the normalised pullback section, and the integral pullback inequality gives the bound. The third applies the finite-order symmetric theta action and its preservation of coefficient bounds. The fourth extends the expansion on the integral uniformisation and descends through the rigidified generic-fibre identifications.

Lean: `GlobalTheory.T.generator_section_expansion_nu_nonneg` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.GeneratorSectionExpansionNuNonnegStatement`, lines 12–13)
Lean: `GlobalTheory.T.normalised_pullback_section_nu_nonneg` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.NormalisedPullbackSectionNuNonnegStatement`, lines 12–13)
Lean: `GlobalTheory.T.symmetric_torsion_translate_section_nu_nonneg` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.SymmetricTorsionTranslateSectionNuNonnegStatement`, lines 12–13)
Lean: `GlobalTheory.T.section_regular_on_model_of_nu_nonneg` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.SectionRegularOnModelOfNuNonnegStatement`, lines 12–13)

Apply these assertions to $\mathcal{P}_{n',1}$, the translation defining $S_{1,b}$, $\mathcal{P}_{\alpha,n'^2}$, and the translation defining T_b. The source levels are 1 and n'^2; since n' is positive and odd, the latter is 1 or at least 9. Thus the hypotheses hold. The resulting regular section T_b has compatible finite-order germs with unique inverse limit $f_b \in V[[X]]$; multiplication by $r \in V$ multiplies this germ by r.

Lean: `GlobalTheory.RegularGerm.regular_germ` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.RegularSectionIntegralGermUniqueScale`, lines 250–252)

The value of a harmonic differential on these sections is its value on their product germ in the integral biextension chart. This follows by identifying the formal group law and the $\mathcal{O}_E$-action in the generic and integral charts. The normalised pullback chain rule gives

$$\mathcal{H}_{m^2}(P; F_1, \ldots, F_n) = \nu_a(\beta)\mathcal{H}_{n'^2A}(P; T_1, \ldots, T_n).$$

Indeed, $[\beta]$ acts on a direction of type τ by $\tau(\beta)$, and row homogeneity gives

$$\prod_\tau \tau(\beta)^{\sum_i a_{\tau,i}} = \nu_a(\beta).$$

The rigidification fixes the scalar in the line-bundle pullback. The identity for a polynomial with coefficients in the coefficient field follows by clearing its finitely many coefficient denominators and cancelling that nonzero scalar.

For $P \neq 0$, its total degree is D. Multiplicativity of ν_a gives

$$m^D = \nu_a(m) = \nu_a(\beta)\nu_a(\alpha)n'^D,$$

which proves the asserted cancellation. For $P = 0$, both values vanish. Since n' is a unit, its inverse power has valuation zero. The assumed chart bound applies to the integral series $\prod_b f_b(X_b)$ and proves the conclusion.

The complete proof in the archive is contained in `GlobalTheory.PlaceInt.TermBound.term_bound`. □

Proof of 35.74

Proof. The base level is odd and squarefree, and $\mathfrak{b}\bar{\mathfrak{b}} = (M)$. If $\ell_0 \nmid M$, take

$$n' = M, \qquad A = B = 1, \qquad \alpha = \beta = 1.$$

If $\ell_0 = g_i$, with $\{i, j\} = \{0, 1\}$, take

$$n' = g_j, \qquad A = B = g_i, \qquad \alpha = \bar{\pi}_i, \qquad \beta = \pi_i.$$

In both cases $n' \in (V')^\times$, $M = \beta\alpha n'$, $\alpha\bar{\alpha} = A$, and $\beta\bar{\beta} = B$.
Lean: `GlobalTheory.TorsFac.factorisation` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.DvdOfCastDvd`, lines 91–171)

Rational M^2-torsion and the theta normalisation give, for every characteristic x_b, a point t_b killed by $\mathfrak{b}^2$ such that

$$s_{x_b} = \mathcal{T}_{t_b, M^2}\mathcal{T}_{0, M^2}\mathcal{P}_{M,1}s_0.$$

Since $(M^2) \subset \mathfrak{b}^2$, one has $[M^2]t_b = 0$. Away from the level, the remaining torsion condition is this same equality. At $\ell_0 = g_i$, one has

$$n'\beta = g_j\pi_i = (\pi_i\pi_j)\bar{\pi}_j \in \mathfrak{b}, \qquad M \in \mathfrak{b}.$$

Consequently $n'^2 A\beta = Mn'\beta \in \mathfrak{b}^2$, so $[n'^2 A\beta]t_b = 0$. The other translation point is zero, hence it satisfies $[n'^2\alpha\beta]0 = 0$.
Lean: `GlobalTheory.RatData.exists_thetaVal_data` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.ExistsThetaValData`, lines 90–130)
Lean: `GlobalTheory.PlaceInt.uu_through_moves` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.SymmetricTranslatesMultiplicationPullbackDescrip`, lines 85–96)

Apply Lemma 35.73 with $m = M$, $n = N$, $z_{1,b} = 0$, $z_{2,b} = t_b$, and

$$\rho = \sum_{p \mid Q_S qd} \frac{v'(p)}{e'}\delta_p \frac{D}{2}.$$

Its chart hypothesis is Lemma 35.72(1), on the integral chart of Proposition 35.66. By the normalisation in Lemma 35.59,

$$\theta_{x,P}(y') = M^{-D}\mathcal{H}_{M^2}(P; s_{x_1}, \ldots, s_{x_N}).$$

The conclusion of Lemma 35.73 is precisely the asserted inequality. □

Proof of 35.75

Proof. (1) Since $(\bar{\pi}_0\bar{\pi}_1)(\pi_0\pi_1) = g_0 g_1 \neq 0$, every factor $\tau(\bar{\pi}_0\bar{\pi}_1)$ is nonzero, so $\nu \in \mathcal{O}_{F_c}$ is nonzero. Each prime $p \mid Q_S qd$ and ν have nonzero valuation at finitely many places of F_c. Outside that finite union both $\rho(w)$ and $v_w(\nu)$ vanish.

(2) Fix $P \in \mathrm{harm}(a, N)$, a tuple x, a finite place w, and an algebraic integral frame y on a trait V of ramification index e over w. Passing to a finite extension V' of V of relative ramification index $r' \geq 1$ that splits the torus of the semiabelian degeneration [176, Ch. III] and adjoins the coordinates of the M^2-torsion, the image algebraic integral frame y' has a split uniformisation and rational M^2-torsion; V' has ramification index $e' = r'e$ over w and residue characteristic ℓ_0, the rational prime below w. Take α as in Proposition 35.74. Since g_0 and g_1 are distinct primes, there is $\alpha' \in \mathcal{O}_E \setminus \{0\}$ with $\alpha\alpha' = \bar{\pi}_0\bar{\pi}_1$: for $\ell_0 \nmid M$ take $\alpha' = \bar{\pi}_0\bar{\pi}_1$, while for $\ell_0 = g_0$ or $\ell_0 = g_1$ take $\alpha' = \bar{\pi}_1$ or $\alpha' = \bar{\pi}_0$. Thus $\nu = \nu_a(\alpha)\nu_a(\alpha')$ with $\nu_a(\alpha), \nu_a(\alpha') \in \mathcal{O}_{F_c} \setminus \{0\}$, so $v_w(\nu_a(\alpha')) \geq 0$ and $v_w(\nu_a(\alpha)) \leq v_w(\nu)$.

On V' we have $v'(p)/e' = v_w(p)$ for every prime p and $v'(\nu_a(\alpha))/e' = v_w(\nu_a(\alpha))$. Moreover $\theta_{x,P}(y')$ is the image of $\theta_{x,P}(y)$, so $v'(\theta_{x,P}(y'))/e' = v(\theta_{x,P}(y))/e$. Proposition 35.74 on V' therefore gives

$$v_w(\nu_a(\alpha)) + \frac{v(\theta_{x,P}(y))}{e} \geq -\rho(w),$$

whence $v(\theta_{x,P}(y))/e \geq -\rho(w) - v_w(\nu_a(\alpha)) \geq -(\rho(w) + v_w(\nu)) = m(w)$.

(3) By assumption $\gcd(g_0 g_1, Q_S qd) = 1$. If $w \mid g_0$ or $w \mid g_1$, every prime $p \mid Q_S qd$ is a unit at w, so $v_w(p) = 0$ and $\rho(w) = 0$, which leaves $m(w) = -v_w(\nu)$. □

Proof of 35.76

Proof (outline). The exponent hypotheses give $U, V \geq 0$. Since the set of blocks is finite, its exponents have an integer upper bound. Fix a place $\lambda \mid \ell$. Lemma 35.54 gives $\nu_\beta(c_\beta; y) \geq 0$ for every integral frame, and hence $\nu_c(y) \geq 0$. We take $N_\lambda = 0$.

For (a), fix an algebraic integral frame y over λ.

Claim 1. Under the split-place condition in (a), there are real numbers $t_\tau \geq 0$, indexed by all embeddings $\tau : E \hookrightarrow \mathbf{C}$, such that

$$\sum_\tau t_\tau \leq fQ(1+\delta/2), \qquad \nu_{\tau,i}(n; y) \leq n t_\tau \quad (\tau \in \Psi,\ 1 \leq i \leq P,\ n \in \mathbf{Z}_{\geq 0}).$$

The family is chosen for y before the flag step i and exponent n. In particular, the coefficient t_τ is independent of both. For any oriented decomposition in (a), the first upper bound is also

$$fQ(1+\delta/2) = Q \sum_s n_s (1 + \delta_s/2).$$

Lean: `GlobalTheory.RefSum.mean_bound_big_factors_at_two` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.MeanUpperBoundBigFactorsPlaceAboveTwo`, lines 18–82)

We now give the colength calculation proving Claim 1. Choose an integral chart and an algebraically closed valued extension C; put $O = \{x : |x| \leq 1\}$ and $q = |\ell|$, so $0 < q < 1$. A multiplicative colength of a finitely presented torsion module $M \simeq \bigoplus_j O/(a_j)$ is $\prod_j |a_j|$.

Claim 2. Under exactly the split-place condition in (a), algebraicity of y, and noninvertibility of ℓ on its trait, there are $\xi_\tau \geq 0$, $\tau \in \Psi$, such that

$$q^{fQ(1+\delta/2)} \leq \prod_{\tau \in \Psi} \xi_\tau.$$

For each $\tau \in \Psi$ and every leading i-step of its integral eigenframe, there are integral formal characters whose directional determinant has absolute value at least ξ_τ. Its integrality gives $\xi_\tau \leq 1$, and the product bound gives $\xi_\tau > 0$. The product bound holds factor by factor. For τ through R_s, ξ_τ is the multiplicative colength of the τ-eigenspace modulo the restrictions of the integral Hodge–Tate characters. The cokernel of the Hodge–Tate map of the abelian part from the $\bar{R}_s$-part of its Tate module to the R_s-part of its invariant differentials has colength at least $q^{Qn_s/(\ell-1)}$, because each of the n_s eigenspaces of the Lie algebra through $\bar{R}_s$ has dimension at most Q. Choose α_s with $R_s = \mathbf{Z}_\ell[\alpha_s]$. Put $J = |\operatorname{disc}(R_s/\mathbf{Z}_\ell)|$. The joint restriction module has multiplicative colength $c \geq q^{Qn_s/(\ell-1)}$ modulo the powers of α_s applied to $h \leq P + Q$ character generators. The Vandermonde comparison gives $c^2 J^h \leq (\prod_{\tau|R_s} \xi_\tau)^2 J^P$. Since $J \leq 1$, the product is at least $cJ^{Q/2}$. Hence the product of the ξ_τ over the embeddings through R_s is at least $q^{Qn_s(1/(\ell-1)+\delta_s/2)} \geq q^{Qn_s(1+\delta_s/2)}$, and the product over s, with $\sum_s n_s = f$ and $\sum_s n_s\delta_s = f\delta$, gives the displayed bound. For the leading steps, the restrictions of finitely many integral characters to the integral τ-eigenframe form a matrix with P rows whose full minors generate the zeroth Fitting ideal of the quotient, so one full minor has absolute value ξ_τ. Expanding it along the first i rows, whose complementary minors are integral, gives i of these characters whose directional determinant on the leading i-step has absolute value at least ξ_τ.

Lean: `GlobalTheory.CharDepth.colength_bound_at_two` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.IntegralChartColengthBoundPrimeTwo`, lines 21–110)
Lean: `GlobalTheory.ChF.colength_bound_split_place_chart` (`Development.AbelianVarietyTheory.Proofs.UnramifiedDepth.ColengthBoundSplitPlaceChart`, lines 32–74)
Lean: `GlobalTheory.CharDepth.exists_flag_characters` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.ExistsFlagCharactersLeadingMinorsColength`, lines 65–97)

Set

$$t_\tau = \begin{cases} \log \xi_\tau / \log q, & \tau \in \Psi, \\ 0, & \tau \notin \Psi. \end{cases}$$

Since $\log q < 0$, these numbers are nonnegative, and taking logarithms reverses the product inequality:

$$\sum_\tau t_\tau = \frac{\log \prod_{\tau \in \Psi} \xi_\tau}{\log q} \leq fQ(1+\delta/2).$$

Apply the n-th power of the determinant differential of a flag step to the product of the selected characters. Its value is the n-th power of their directional determinant, with normalised valuation at most nt_τ, because $q^{t_\tau} = \xi_\tau$. Taking the infimum defining depth proves Claim 1. The same coefficients work for all steps and exponents, and vanish off Ψ.

Lean: `GlobalTheory.T.flagStep_characters_of_colength` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FlagStepCharactersOfColengthStatement`, lines 10–11)

Multiply the row estimate by the exponent bounds and sum:

$$\begin{aligned}\sum_{\beta \text{ big}} \nu_\beta(c_\beta; y) &\le \sum_{\tau\in\Psi} t_\tau \sum_{i=k-1}^{P} c_{\tau,i} \\ &\le U \sum_{\tau} t_\tau \le fQU(1+\delta/2) \le fmU(1+\delta/2).\end{aligned}$$

Here $Q = m - j \le m$, and $1 + \delta/2 \ge 0$. For the small blocks, Lemma 35.54 and $\ell \ge 2$ give

$$\sum_{\beta=(\tau,i)\ \text{small}} \nu_\beta(c_\beta; y) \le \frac{\sum_{\beta \text{ small}} ic_\beta}{\ell - 1} \le V.$$

Thus $0 \le \nu_c(y) \le fmU(1+\delta/2)+V$ with the required lower and upper frame quantifiers. The lower endpoint is already an integer, so the displayed width needs no rounding term.

For (b), use the following local estimate.

Claim 3. Let $\ell \ne 2$ be unramified in E and split in the host quadratic field. For an algebraic integral frame on a trait on which ℓ is a nonunit, every $\tau \in \Psi$, every flag step $1 \le i \le P$, and every integer $n \ge 0$,

$$\nu_{\tau,i}(n; y) \le \frac{nQ}{\ell - 1}.$$

The estimate is uniform in the trait, frame, step and exponent. The integral split-place character construction gives a directional determinant of normalised valuation at most $Q/(\ell - 1)$; its n-th determinant differential gives the displayed bound.

Lean: `GlobalTheory.T.factorDepth_upperBound_oddUnramifiedPrime` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FactorDepthUpperBoundOddUnramifiedPrimeStatement`, lines 10–11)

Apply Claim 3 with $n = c_\beta$ on each big block. The universal estimate of Lemma 35.54 gives $\nu_{\tau,i}(c_{\tau,i}; y) \le ic_{\tau,i}/(\ell - 1)$ on the small blocks. Since $\#\Psi = f$,

$$\begin{aligned}\nu_c(y) &\le \frac{Q \sum_{\tau\in\Psi} \sum_{i=k-1}^{P} c_{\tau,i} + \sum_{\beta \text{ small}} ic_\beta}{\ell - 1} \\ &\le \frac{fQU + V}{\ell - 1} \le \frac{fmU + V}{\ell - 1}.\end{aligned}$$

Together with the lower bound at every integral frame, this proves (b).

The complete proofs are contained in `GlobalTheory.RefSum.oscLe_two` and `GlobalTheory.OscSplit.oscLe_split`. □

Proof of 35.77

Proof (outline). Put $k_O = O/\mathfrak{m}_O$. For an embedding $\sigma : E \hookrightarrow \mathbf{C}$, let $\sigma_O : \mathcal{O}_E \to O$ be the integral embedding induced by the coefficient embedding of the trait, and let $\psi_\sigma : \mathcal{O}_E \to k_O$ be its reduction. Put

$$\mathfrak{p}_\sigma = \{a \in \mathcal{O}_E : |\sigma_O(a)| < 1\} = \ker \psi_\sigma.$$

Claim 1. If $p \nmid \operatorname{disc} E$, the map $\sigma \mapsto \psi_\sigma$ is a bijection onto the homomorphisms $\mathcal{O}_E \to k_O$. For each τ, there is an integer $h > 0$ such that $\#(\mathcal{O}_E/\mathfrak{p}_\tau) = p^h$, and the distinct homomorphisms

$$a \longmapsto \psi_\tau(a)^{p^s}, \qquad 0 \le s < h,$$

are exactly those with kernel $\mathfrak{p}_\tau$. Their inverse images under this bijection give τ_s, with $\tau_0 = \tau$, and hence the stated Frobenius congruences.

Lean: `GlobalTheory.FrobOrbit.frobOnType_and_orbit` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FrobeniusOrbitOfUnramifiedPrime`, lines 232–317)

The discriminant hypothesis gives injectivity of reduction; counting the embeddings gives surjectivity. Frobenius on the finite field $\mathcal{O}_E/\mathfrak{p}_\tau$ has period h. On K, it fixes the two embeddings when p splits and interchanges them when p is inert. Therefore membership in Ψ is respectively constant or alternating.

For the character assertion, assume $p \neq 2$ and y algebraic, and fix $1 \leq i \leq \mu_\tau$. Put $s = \mu_\tau$.

Claim 2. After the finite extension used to uniformise this algebraic frame, the chart over O is unchanged and there are integral formal homomorphisms

$$\widehat{\mathbf{G}}_m^t \xrightarrow{J} G \xrightarrow{q} \widehat{B},$$

where B is the polarized abelian quotient, with split exact tangent sequence

$$0 \longrightarrow O^t \xrightarrow{A} O^{\dim G} \xrightarrow{B_0} O^{\dim B} \longrightarrow 0, \qquad A = dJ, \qquad B_0 = dq.$$

Writing $\rho_G(a)$ and $\rho_B(a)$ for the tangent actions of $a \in \mathcal{O}_E$, there is a ring homomorphism $N : \mathcal{O}_E \to \mathrm{Mat}_t(\mathbf{Z})$ such that

$$B_0\rho_G(a) = \rho_B(a)B_0, \qquad \rho_G(a)A = AN(a).$$

For every $m \geq 0$, the coordinate algebra A_m of the p^m-torsion in this formal chart is finite free over O, its scalar extension to C is reduced, and

$$\big(\varphi(a) = 0 \text{ for every } O\text{-algebra map } \varphi : A_m \to O\big) \implies a = 0.$$

Lean: `GlobalTheory.CharDepth.nonempty_unifChart` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.NonemptyUniformisedFormalChartAlgebraicPoint`, lines 16–36)

Lean: `GlobalTheory.T.uniformization_split_formal_group_law_equivariant` (`Development.AbelianVarietyTheory.Proofs.UnramifiedDepth.UniformizationSplitFormalGroupLawEquivariantStatement`, lines 16–17)

Lean: `GlobalTheory.FormalTorsion.fhSt` (`Development.AbelianVarietyTheory.Proofs.FormalGroupCharts.MapPderiv`, lines 53–134)

For any finite family of integral formal characters $\eta_1, \dots, \eta_N$, with $N \geq s$, define

$$A_\tau = \big(d\eta_b(u_{\tau,a})\big)_{1\leq a\leq s,\, 1\leq b\leq N}, \qquad x_\tau = \max_{\substack{J\subset\{1,\dots,N\}\\ \#J=s}} |\det(A_{\tau,J})|,$$

where $A_{\tau,J}$ is the square matrix formed by the columns in J.

Claim 3. In Claim 2's setting, put $g_B = \dim B$, and let a_σ be its Lie multiplicity. Its polarised Hodge–Tate datum is over the p-adically complete valuation ring O, with p a nonunit, $0 < |p| < 1$, and $|c_p|^{p-1} = |p|$. It has Tate rank $2g_B$, Lie rank g_B, completed actions ρ_T, ρ_B, a perfect alternating matrix E_T, and g_B-by-$2g_B$ matrices H_T, B_T. The projector terms are zero, and

$$\rho_T(a)^{\mathrm{t}}E_T = E_T\rho_T(\bar{a}), \qquad H_T\rho_T(a) = \rho_B(\bar{a})^{\mathrm{t}}H_T, \qquad B_T^{\mathrm{t}}H_T - H_T^{\mathrm{t}}B_T = c_pE_T.$$

The datum also includes the action-stable finite flat Hopf order of the p-torsion. The coordinates of H_T, B_T modulo p come from that order. Its augmentation ideal surjects onto $(O/pO)^{g_B}$, with kernel its square, and this cotangent map intertwines the action with the transpose Lie action.

Lean: `ReciprocityTheory.PGsh.Up.HodgeTateSetting` (`ReciprocityTheory.DualLattices.CotangentSurjective`, lines 84–91)

Lean: `ReciprocityTheory.PGsh.Up.HodgeTateDatum` (`ReciprocityTheory.DualLattices.CotangentSurjective`, lines 40–82)

Take the finite list of integral formal characters used in the calculation, with differentials $s_b \in O^{\dim G}$. It contains every $B_0^{\mathrm{t}}H_Te_j$, and for each standard torus basis vector there is s_b with $A^{\mathrm{t}}s_b = e_j$. Together with Claim 2's split exact sequence and equivariance identities, these are the linear-data conditions. For a full σ-eigenframe W_σ, set

$$\Lambda_\sigma = \mathrm{span}_O\{(W_{\sigma,a}^{\mathrm{t}}s_b)_a : b\}, \qquad M_\sigma = O^{\mu_\sigma}/\Lambda_\sigma.$$

For a diagonal presentation $\bigoplus_j O/(d_j)$, multiplicative colength means $\prod_j |d_j|$.

Lean: `GlobalTheory.ChF.LinDat` (`Development.AbelianVarietyTheory.Definitions.TeichmullerDepth.IsEigBasisIff`, lines 169–178)

With the unramified Lie and Tate idempotents, form

$$C_\sigma = \mathrm{im}(e^{\mathsf{t}}_{L,\sigma}) \big/ \big(\mathrm{im}(e^{\mathsf{t}}_{L,\sigma}) \cap H_T(\mathrm{im}\, e_{T,\bar{\sigma}})\big).$$

Restriction of B^{t}_0 to the eigenframe gives a surjection $C_\sigma \to M_\sigma$. The abelian-quotient characters make it well defined; the torus-dual characters and split exact tangent sequence make it surjective. Writing V_σ and ξ_σ for the multiplicative colengths of C_σ and M_σ respectively, this surjection gives $V_\sigma \le \xi_\sigma$.

The local Tate rank h_T is constant along the Frobenius orbit. Polarisation gives $a_{\tau_j} + a_{\overline{\tau_j}} = h_T$, and the split tangent sequence gives $a_\sigma \le \mu_\sigma$. The odd-prime unramified estimate combines two facts. A finite group of Teichmüller units of order prime to p acts on the Hodge–Tate datum; its characters κ_σ on the types separate the types and satisfy $\kappa_{\tau_{m+1}} = \kappa^p_{\tau_m}$. Write X_σ for the multiplicative colength of the type block of (H_T, B_T), and ϕ for Frobenius. The pairing gives $X_\sigma V_\sigma = |c_p|^{a_\sigma} V_{\bar{\sigma}}$ and $\prod_\sigma X_\sigma = |c_p|^{g_B}$. The Hopf order gives $X_\sigma^p \le X_{\phi\sigma} |p|^{a_{\phi\sigma}}$. Multiplying over all types gives equality since $\sum_\sigma a_\sigma = g_B$ and $|c_p|^{p-1} = |p|$. Thus each factor is an equality. Iterating around the orbit and substituting into the pairing relation gives the length identity

$$V_\tau = V_{\bar{\tau}}\, |p|^{a_\tau/(p-1) - \sum_{m\ge 0} p^{-m-1} a_{\tau_{m+1}}}.$$

The conjugate cokernel $C_{\bar{\tau}}$ is generated by $a_{\bar{\tau}}$ elements and killed by c_p, so $V_{\bar{\tau}} \ge |c_p|^{a_{\bar{\tau}}} = |p|^{a_{\bar{\tau}}/(p-1)}$. Multiplying gives

$$|p|^{h_T/(p-1) - \sum_{m\ge 0} p^{-m-1} a_{\tau_{m+1}}} \le V_\tau.$$

All multiplicities are bounded. Using $\sum_{m\ge 0} p^{-m-1} = 1/(p-1)$, its exponent equals

$$\sum_{m\ge 0} p^{-m-1} a_{\overline{\tau_{m+1}}} \le \sum_{m\ge 0} p^{-m-1} \mu_{\overline{\tau_{m+1}}} = R_\tau.$$

Since $0 < |p| < 1$, this proves

$$0 < |p|^{R_\tau} \le V_\tau \le \xi_\tau.$$

The iteration starts at τ_1, and R_τ uses multiplicities of the whole semiabelian chart.
Lean: `ReciprocityTheory.PGsh.Up.up_full` (`ReciprocityTheory.DualLattices.LengthDegreeAbelianVariety`, lines 65–75)
Lean: `GlobalTheory.ChF.linDat_unram_bound` (`Development.AbelianVarietyTheory.Proofs.UnramifiedDepth.LinearDataUnramifiedLengthBound`, lines 64–124)

At the full flag step $s = \mu_\tau$, character selection gives $\eta_1, \dots, \eta_s$ with

$$\xi_\tau \le |\det(d\eta_b(u_{\tau,a}))_{a,b=1}^{s}|.$$

Thus the previously defined maximum, for this family with $N = s$, satisfies $x_\tau \ge \xi_\tau \ge |p|^{R_\tau} > 0$.
Lean: `GlobalTheory.RefSum.hasChars_frame_of_hasAF` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FlagStepCharactersOfColength`, lines 20–66)

Choose a maximal minor D_s with $|D_s| = x_\tau$, and expand it along the first i rows:

$$D_s = \sum_{\substack{J\subset\{1,\dots,s\}\\ \#J=i}} \varepsilon_J D_J E_J, \qquad \varepsilon_J \in \{1,-1\}.$$

Here D_J uses the first i rows and the columns in J, and E_J is the complementary minor. Every E_J is integral, so

$$x_\tau = |D_s| \le \max_J |D_J E_J| \le \max_J |D_J|.$$

Choose a maximizing J, and use its characters as $\chi_1, \dots, \chi_i$. Their determinant $D = D_J$ is nonzero and satisfies

$$|D| \ge x_\tau \ge |p|^{R_\tau}.$$

Since $0 < |p| < 1$, taking logarithms and dividing by $\log|p| < 0$ gives $v(D) \le R_\tau$.

The complete orbit construction, character construction, and passage to a flag minor are contained in `GlobalTheory.FrobOrbit.frobOnType_and_orbit`, `GlobalTheory.ChF.unramified_companion_chart_characters`, `GlobalTheory.CharMinor.colength_bound`, and `GlobalTheory.RefSum.hasChars_frame_of_hasAF`. □

Proof of 35.78

Proof. Fix an algebraic integral frame y, a chart G, and a valued extension as in Lemma 35.77. For i copies of G, let $D_{u_a}^{(b)}$ be the invariant derivation in the $u_a = u_{\tau,a}$ direction on copy b. Apply the determinant operator

$$\mathcal{D} = \det\big(D_{u_a}^{(b)}\big)_{a,b=1}^{i}$$

to the integral series $F = \prod_{b=1}^{i} \chi_b(X^{(b)})$. A formal character satisfies $D_u\chi = d\chi(u)\chi$; hence

$$\mathcal{D}^n F(0) = \det\big(d\chi_b(u_a)\big)^n.$$

Taking the infimum that defines the chart depth, and then the infimum over charts, gives

$$\nu_{\tau,i}(n; y) \le nR_\tau.$$

If $p = \ell$ splits in K, every τ_s lies in Ψ, since $\tau \in \Psi$. Thus

$$R_\tau = Q \sum_{m\ge 0} \ell^{-m-1} = \frac{Q}{\ell - 1}.$$

If ℓ is inert in K, membership in Ψ alternates. The conjugate multiplicity in the summand is P for even m and Q for odd m, so

$$R_\tau = \sum_{a\ge 0} \big(P\ell^{-2a-1} + Q\ell^{-2a-2}\big) = \frac{P\ell + Q}{\ell^2 - 1} = \frac{Q}{\ell - 1} + \frac{(P - Q)\ell}{\ell^2 - 1}.$$

The character calculation first controls the determinant colength on the full eigenspace, and then passes to each flag minor. Thus its loss in the split case is $Q/(\ell - 1)$ for every block size i; taking an i-dimensional flag step adds no factor i. This is where the conjugate multiplicity Q enters in place of the block size in Lemma 35.54. That lemma supplies nonnegativity for every integral frame and depth zero when $n = 0$. □

Proof of 35.79

Proof (outline). Let $\mathcal{L} = \{(\sigma, i) : 1 \le i \le \mu_\sigma\}$ index the full labelled Lie basis, and let U be its column matrix in the integral chart.

Claim 1. Since $p \nmid \operatorname{disc} E$, there is an integral L with

$$LU = 1, \qquad UL = 1.$$

Let τ_s, $0 \le s < h$, be the distinct embeddings above the prime of τ. For $z^{p^h - 1} = 1$, define

$$t_z(\sigma) = \begin{cases} z^{p^s}, & \sigma = \tau_s, \\ 0, & \sigma \notin \{\tau_s\}. \end{cases}$$

For every real B, there is $a \in \mathcal{O}_E$ with $v(\sigma(a) - t_z(\sigma)) \ge B$ for every embedding into C. The residue field has order p^h, so the reduction of z has a representative in $\mathcal{O}_E$; unramified approximation applies to its Frobenius conjugates. The action of a is integral. Arbitrarily precise approximation therefore gives an integral formal homomorphism with logarithmic matrix

$$U \operatorname{diag}(t_z(\sigma)) L.$$

A logarithmic matrix M means a zero-constant-term integral homomorphism ϕ satisfying $\log_G \phi = M \log_G$.

Lean: `GlobalTheory.T.label_directions_integral_basis_unramified_prime` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LabelDirectionsIntegralBasisUnramifiedPrimeStatement`, lines 6–7)

Lean: `GlobalTheory.T.exists_integers_approximating_root_of_unity_twists` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsIntegersApproximatingRootOfUnityTwistsStatement`, lines 6–7)

Lean: `GlobalTheory.PrimePart.exists_hom_of_approx` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FormalGroupRootsOfUnityTypedLieRanks`, lines 116–124)

Index the factor by

$$\mathcal{K} = \{(s,i) : s \in \mathbf{Z}/h\mathbf{Z},\ 1 \le i \le \mu_{\tau_s}\}.$$

Use representatives $\widetilde{s} \in \{0,\dots,h-1\}$, so $\tau_s = \tau_{\widetilde{s}}$. Put $d' = \#\mathcal{K}$, choose an enumeration, and let $j(s,i) = (\tau_s, i)$. Let I be the columns of U selected by j, and P the matching rows of L. Then

$$PI = 1, \quad IP = U\,\mathrm{diag}(t_1(\sigma))L, \quad 1 - IP = U\,\mathrm{diag}(1 - t_1(\sigma))L.$$

Both latter matrices are logarithmic matrices of integral endomorphisms: use Claim 1 at $z = 1$, and $1 - a$ for the complement.

Claim 2. Integral I, P with $PI = 1$, for which both IP and $1 - IP$ arise in this way, give a commutative integral formal group G' and integral homomorphisms

$$\iota : G' \to G, \qquad q : G \to G', \qquad \log_G \iota = I \log_{G'}, \qquad \log_{G'} q = P \log_G .$$

Its logarithm has identity linear term. Apply this splitting statement and put $\Pi = P$.
Lean: `GlobalTheory.T.logarithmic_group_law_split_by_idempotent` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LogarithmicGroupLawSplitByIdempotentStatement`, lines 6–7)

For $z^{p^h-1} = 1$, composing Claim 1's endomorphism with ι, q gives logarithmic matrix

$$P\big(U\,\mathrm{diag}(t_z(\sigma))L\big)I = \mathrm{diag}_{(s,i)\in\mathcal{K}}(z^{p^{\widetilde{s}}}).$$

This constructs the integral $[z]$. With $t(s,i) = s$, exactly μ_{τ_s} coordinates have type s.
Lean: `GlobalTheory.PrimePart.nonempty_muAction` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FormalGroupRootsOfUnityTypedLieRanks`, lines 70–81)

Claim 3. Selecting rows in $LU = 1$ gives

$$(Pu_{\tau_s,i})_{(s',i')} = \delta_{s,s'}\delta_{i,i'}.$$

Thus every projected leading frame is integral, supported in its type, and has a maximal minor equal to one. Also $IPu_{\tau_s,i} = u_{\tau_s,i}$.

Pullback of integral series by copies of q is integral; the invariant-derivative chain rule identifies determinant-differential values, giving $\mathrm{depth}_G \le \mathrm{depth}_{G'}$. Pullback by copies of ι, with $IPu = u$, gives the reverse inequality. Equality holds for every exponent, including zero. The full Lie chart was used throughout, so the multiplicities include the torus part and no algebraicity assumption on the integral frame is needed.
Lean: `GlobalTheory.PrimePart.typedPartAt` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FormalGroupRootsOfUnityTypedLieRanks`, lines 126–219)
Lean: `GlobalTheory.IndefiniteRowSideFullHecke.TypedPartSplitPrimes.FormalGroupTypes` (`Development.AbelianVarietyTheory.BlockTheory.DivisorBase.TypedPartSplitPrimes`, lines 115–121)

The complete proof is contained in `GlobalTheory.PrimePart.typedPartAt` and `GlobalTheory.PrimePart.formalGroupTypes`. □

Proof of 35.80

Proof (outline). Apply Lemma 35.79 to τ, and write τ_s for its Frobenius orbit, with $\tau_0 = \tau \in \Psi$.

Claim 1. The orbit length h is even, and $\tau_s \in \Psi$ exactly when the representative $\widetilde{s} \in \{0,\dots,h-1\}$ is even. Indeed, Lemma 35.77 gives $\mathbf{1}_\Psi(\tau_{s+1}) = 1 - \mathbf{1}_\Psi(\tau_s)$. Induction gives the parity rule, and return to τ_0 forces even h. The typed factor G' therefore has a saturated projected frame u supported in type zero, and each odd type has exactly Q coordinates.

Put $p = \ell$, $d = \dim G'$, and $\mathcal{O} = \{a \in C : v(a) \ge 0\}$, where C is the algebraically closed characteristic-zero valued field of the chart, with $v(p) = 1$. The formal sum is integral. Its logarithm $\mathcal{L}$ has zero constant term, identity linear term, and

$$\mathcal{L}(X +_{G'} Y) = \mathcal{L}(X) + \mathcal{L}(Y).$$

For the type map $t : \{1,\dots,d\} \to \mathbf{Z}/h\mathbf{Z}$, the integral action satisfies

$$\mathcal{L}_e([\zeta](Z)) = \zeta^{p^{\widetilde{t(e)}}}\mathcal{L}_e(Z) \qquad (\zeta^{p^h-1} = 1).$$

Saturation means $u \in \mathcal{O}^{i\times d}$ has a maximal minor of valuation zero. Determinant depth is the infimum, over integral series on $(G')^i$, of the valuation of the n-th determinant differential at the origin.
Lean: ReciprocityTheory.PGsh.LogFGL (ReciprocityTheory.DualLattices.VEqZeroOfPowEqOne, lines 37–45)
Lean: ReciprocityTheory.PGsh.MuAction (ReciprocityTheory.DualLattices.VEqZeroOfPowEqOne, lines 89–95)
Lean: GlobalTheory.FGDepth.IsSatFrame (Development.AbelianVarietyTheory.Definitions.Foundations.IsSatFrame, lines 45–47)
Lean: GlobalTheory.FGDepth.depth (Development.AbelianVarietyTheory.Definitions.Foundations.IsSatFrame, lines 36–39)
Lean: GlobalTheory.FGDepth.detDepth (Development.AbelianVarietyTheory.Definitions.Foundations.IsSatFrame, lines 49–50)

Let K be the least nonnegative integer with $n \le p^{2K+1}$. Then

$$n \le p^{2K+1}, \qquad K = 0 \text{ or } p^{2K-1} < n.$$

Lean: GlobalTheory.DepthLow.exists_least_exponent (Development.AbelianVarietyTheory.Proofs.DivisorProofs.DetDepthLowerBoundSaturatedEvenType, lines 127–139)

Claim 2. There are integral p-typical curves γ_b, $1 \le b \le d$, and integers $\delta_b \ge -1$, such that, writing

$$\mathcal{L}(\gamma_b(T)) = \sum_{m\ge 0} L_{b,m} T^{p^m}, \qquad B_{be} = (L_{b,0})_e,$$

one has $B \in \mathrm{GL}_d(\mathcal{O})$, its b-th row is supported in type $t(b)$, and for every even type ϑ,

$$\#\{b : t(b) = \vartheta,\ \delta_b < K\} \le (K+1)Q.$$

If $\delta_b \ge 0$, then

$$L_{b,1} = 0, \qquad L_{b,2} \in \mathrm{span}_C\{L_{e,0} : \delta_e \ge \delta_b - 1\}.$$

The inputs are $p \ge 3$, positive even h, the displayed integral root-of-unity action, and odd-type multiplicities at most Q. The construction gives the stronger bounds $-1 \le \delta_b \le K$, with $\delta_b = -1$ on every odd type, and at most Q indices in each fixed even type at each fixed depth $a < K$. It also keeps the type in the support condition,

$$L_{b,2} \in \mathrm{span}_C\{L_{e,0} : t(e) = t(b) + 2,\ \delta_e \ge \delta_b - 1\} \quad (\delta_b \ge 0).$$

We now give the induction giving these assertions. Begin with an integral typical basis compatible with the root-of-unity action. In an even type ϑ, the coefficient $L_{b,1}$ lies in type $\vartheta + 1$, of dimension at most Q. Integral rebasing within this type makes that coefficient zero outside a set of at most Q indices. Give those exceptional indices depth -1 and the others depth zero. The rebasing matrices and their inverses are integral, and their determinants are units, so the tangent vectors remain an integral basis.

Suppose the construction has reached depth a. Rebase only the indices of depth a in each even type ϑ, using the coefficient $L_{b,2}$ modulo the span of the tangent vectors in type $\vartheta + 2$ of depth at least a. By the induction hypothesis the quotient is generated by the at most Q tangent vectors in that type of depth $a - 1$. The same integral rebasing makes this quotient coefficient zero outside at most Q exceptional indices. Raise the other indices from depth a to depth $a + 1$. Both directions of the integral basis change preserve the required tangent spans; the vanishing of $L_{b,1}$ and all preceding depth bounds are preserved. This proves the induction. Summing the per-depth bound over $-1, 0, \dots, K - 1$ gives $(K + 1)Q$.
Lean: GlobalTheory.DepthStages.exists_stage (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LogFGLDepthStagesLowA, lines 49–98)
Lean: GlobalTheory.DepthStages.exists_stageInv_zero (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LogFGLDepthStagesLowA, lines 133–182)
Lean: GlobalTheory.DepthStages.exists_stageInv_succ (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LogFGLDepthStagesLowA, lines 184–270)
Lean: GlobalTheory.DepthStages.conclusion_of_stageInv (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LogFGLDepthStagesLowA, lines 278–305)
Lean: GlobalTheory.DepthStages.lowA_statement (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LogFGLDepthStagesLowA, lines 311–321)
Lean: GlobalTheory.T.logFGL_depth_stages_lowA (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LogFGLDepthStagesLowAStatement, lines 8–9)
Lean: GlobalTheory.DepthLow.detDepth_lower (Development.AbelianVarietyTheory.Proofs.DivisorProofs.DetDepthLowerBoundSaturatedEvenType, lines 199–283)

Claim 3. Write

$$L_{b,m} = \sum_e A_{b,m,e} L_{e,0}.$$

Then

$$A_{b,0,e} = \delta_{be}, \qquad v(A_{b,m,e}) \geq -m,$$

and, for $\delta_b \geq 0$,

$$A_{b,1,e} = 0, \quad v(A_{b,m,e}) \geq 1 - m \ (m \geq 1), \quad A_{b,2,e} \neq 0 \Longrightarrow \delta_e \geq \delta_b - 1.$$

The typical-basis recurrence supplies integral $s_{j,e,b}$ with

$$pL_{b,m+1} = \sum_{j=0}^{m} \sum_{e} s_{m-j,e,b}^{p^j} L_{e,j}.$$

Induction from the integral tangent coefficients gives $v((L_{b,m})_e) \geq -m$. If $L_{b,1} = 0$, the recurrence at $m = 0$ and invertibility of B give $s_{0,e,b} = 0$. For $m \geq 1$, the $j = m$ term vanishes, so every remaining term has valuation at least $1 - m$. Division by p gives $v((L_{b,m+1})_e) \geq -m$, the improved bound; the index-one case follows from vanishing.

Now $A_{b,m,\cdot} = L_{b,m}B^{-1}$. Since B^{-1} is integral, the ultrametric inequality preserves both bounds. Claim 2's span condition gives the stated support of $A_{b,2,\cdot}$.

Lean: `ReciprocityTheory.ChartRestriction.G.Sh.pt5v'` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.WeightedSumBounds.WtSubOneLe`, lines 99–189)

Lean: `GlobalTheory.DepthLow.coeff_in_tangent_basis` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.DetDepthLowerBoundSaturatedEvenType`, lines 150–197)

Claim 4. Put $x = uB^{-1}$. There are $g \in \mathrm{GL}_i(\mathcal{O})$ and an integer c with

$$0 \leq c \leq i, \qquad c \leq Q(K+1) \leq Q\left(\frac{3}{2} + \frac{1}{2}\log_p n\right),$$

such that the last $i - c$ rows of gx vanish on

$$S_K = \{b : t(b) = 0,\ \delta_b < K\}.$$

The matrix x is integral and supported in type zero. Cauchy–Binet applied to a unit minor of $u = xB$ shows that x is saturated. Integral row elimination on columns S_K gives g and $c \leq \min(i, \#S_K)$. At each stage choose a nonzero entry of least valuation; all division coefficients are integral and the row-operation determinant is a unit. Claim 2 gives $\#S_K \leq (K+1)Q$. If $K = 0$, then $K + 1 \leq 3/2 + (\log_p n)/2$. Otherwise minimality gives $2K - 1 < \log_p n$, proving the same bound.

Lean: `GlobalTheory.LowU.adapted_basis` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.AdaptedBasis`, lines 174–182)

Define

$$R_k = \frac{1 + p^{-(2k+1)}}{p^2 - 1} \quad (k \geq -1), \qquad \rho_a = \begin{cases} R_{-1}, & a \leq c, \\ R_K, & a > c. \end{cases}$$

The positive R_k decrease with k, and $R_{-1} = 1/(p-1)$. A series $F = \sum_\beta F_\beta X^\beta$ has ρ-order at least r_0 when

$$v(F_\beta) + \sum_a \rho_a \beta_a \geq r_0 \quad \text{for all } \beta.$$

Lean: `ReciprocityTheory.PGsh.radius` (`ReciprocityTheory.DualLattices.VEqZeroOfPowEqOne`, lines 144–144)

Lean: `ReciprocityTheory.PGsh.OrdGe` (`ReciprocityTheory.DualLattices.VEqZeroOfPowEqOne`, lines 140–142)

Claim 5. Put $x' = gx$, $z_e = \sum_a x'_{ae} X_a$. There are zero-constant-term series t_e with

$$\sum_{b,m \geq 0} A_{b,m,e} t_b^{p^m} = z_e, \qquad \mathrm{ord}_\rho(t_e) \geq R_{\delta_e}.$$

First $\mathrm{ord}_\rho(z_e) \geq R_{\delta_e}$. A nonzero x'_{ae} has $t(e) = 0$. If $\delta_e < K$, Claim 4 forces $a \leq c$, so $\rho_a = R_{-1} \geq R_{\delta_e}$; otherwise $\rho_a \geq R_K \geq R_{\delta_e}$.

The equations have identity linear part, and all other terms have degree at least p. Solve successively by total degree. The orders are preserved by the following three estimates:

$$\delta_b = -1: \qquad v(A_{b,m,e}) + p^m R_{-1} \geq -m + \frac{p^m}{p-1} \geq R_{-1} \geq R_{\delta_e};$$

for $\delta_b \geq 0$, the $m = 1$ term vanishes, while a nonzero $m = 2$ term satisfies

$$v(A_{b,2,e}) + p^2 R_{\delta_b} \geq -1 + p^2 R_{\delta_b} = R_{\delta_b - 1} \geq R_{\delta_e};$$

and for $m \geq 3$,

$$v(A_{b,m,e}) + p^m R_{\delta_b} \geq 1 - m + \frac{p^m}{p^2-1} \geq \frac{1}{p-1} \geq R_{\delta_e}.$$

The middle inequality holds at $m = 3$ for $p \geq 3$ and increases thereafter. Addition, multiplication, and coefficientwise stabilisation preserve the bounds.

Lean: `GlobalTheory.DepthRadii.exists_solution_ordGe_radius` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsSolutionOrdGeRadius`, lines 429–438)

The integral coordinate map

$$\Phi(Z) = \gamma_1(Z_1) +_{G'} \cdots +_{G'} \gamma_d(Z_d)$$

has tangent matrix B, hence an integral formal inverse. Expanding its logarithm in the tangent basis and using Claim 5 and $x'B = gu$ gives

$$\mathcal{L}_e(\Phi(t(X))) = \sum_a (gu)_{ae} X_a.$$

Claim 6. One has

$$\operatorname{depth}_{G'}(\det D_u^n) \geq \frac{in}{p-1} - i^2 \log_p(n+1) - n\{(i-c)R_K + cR_{-1}\}.$$

For $P(Y) = \det(Y_{ba})^n$, the divided-power content bound is

$$\min_{\beta \in \operatorname{supp} P} v(P_\beta \beta!) \geq \frac{in}{p-1} - i^2 \log_p(n+1), \qquad \beta! = \prod_{b,a} \beta_{ba}!.$$

Every monomial has degree n in each direction index a, so weighted degree $n \sum_a \rho_a$. By Claim 5, pulling back an integral series on $(G')^i$ gives nonnegative weighted order. Applying $P(\partial)$ at zero multiplies each relevant coefficient by $P_\beta \beta!$; the ultrametric inequality therefore loses at most $n \sum_a \rho_a$. The integral coordinate isomorphism Φ preserves the infimum, and the row change multiplies the determinant operator by the valuation-zero scalar $(\det g)^n$. Since $\sum_a \rho_a = (i-c)R_K + cR_{-1}$, the claim follows.

Lean: `GlobalTheory.T.dividedPowerContentValuation_detPow_bounds` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.DividedPowerContentValuationDetPowBoundsStatement`, lines 8–9)

Substitution of the radii gives

$$\operatorname{depth}_{G'}(\det D_u^n) \geq (i-c)\frac{np}{p^2-1} - \frac{(i-c)np^{-(2K+1)}}{p^2-1} - i^2 \log_p(n+1).$$

Here $np^{-(2K+1)} \leq 1$, $p^2 - 1 \geq 8$, $i \geq 1$, and

$$\log_p(n+1) \leq \log_p n + \log_p 2 \leq \log_p n + \frac{2}{3},$$

since $2^3 \leq p^2$. The error is at most

$$i^2 \log_p n + \frac{i}{8} + \frac{2}{3} i^2 \leq i^2(\log_p n + 1).$$

Thus

$$\operatorname{depth}_{G'}(\det D_u^n) \geq (i-c)\frac{np}{p^2-1} - i^2(\log_p n + 1).$$

Insert Claim 4's bound for c, whose coefficient $np/(p^2-1)$ is positive, and use the depth identity of Lemma 35.79. The construction applies to each given integral frame and chart.

The complete formal-group calculation is contained in `GlobalTheory.DepthLow.detDepth_lower`, and its typed-factor application is contained in `GlobalTheory.RefSum.facBetween_fine`. □

Proof of 35.81

Proof. Set

$$a = \left(i - Q\left(\frac{3}{2} + \frac{L}{2}\right)\right)\frac{n\ell}{\ell^2 - 1} - i^2(L+1), \qquad b = n\left(\frac{Q}{\ell - 1} + \frac{(P-Q)\ell}{\ell^2 - 1}\right).$$

These numbers are fixed before choosing a trait or a frame. Lemma 35.80, applied to every chart and valued extension, gives $a \le \nu_{\tau,i}(n; y)$ for every integral frame. Lemma 35.78 gives $\nu_{\tau,i}(n; y) \le b$ for every algebraic integral frame.

The typed-factor construction in the lower bound includes the torus part of each formal group. The character construction in the upper bound starts from the split exact tangent sequence with that torus. Both bounds use the prescribed Lie multiplicities P, Q of the complete formal group. Thus this single interval serves all traits, including those with nonzero torus rank.

Since $L \ge 0$ and $i \le P$,

$$b - a = \frac{nQ}{\ell - 1} + \frac{n\ell}{\ell^2 - 1}\left(P - i + \frac{Q}{2} + \frac{QL}{2}\right) + i^2(L+1) \ge 0.$$

Put $s = \ell/(\ell^2 - 1)$ and $t = 1/(\ell - 1)$; then $0 < s \le t$. Direct subtraction gives

$$W - (b - a) = nQ\left(\frac{t}{2} + \left(\frac{3}{2} + \frac{L}{2}\right)(t - s)\right) \ge 0.$$

This proves $a \le b$ and $b - a \le W$. The first term of W contains the real number $P - i - Q$, which is negative when $i > P - Q$. It remains negative in this calculation; the displayed identity proves the bound for the complete sum without replacing that term by zero. □

Proof of 35.82

Proof. The diagonal monomial $\prod_a Z_{aa}^n$ has coefficient one, so the support is finite and nonempty. If m is a nonzero integer, the normalization $v(p) = 1$ gives $v(m) = v_p(m)$: integers have nonnegative valuation, and Bézout's identity gives valuation zero for an integer prime to p. Apply this to the nonzero integers $c_\alpha \alpha!$ and take the minimum over the fixed support.

For the depth bound, invert a unit maximal minor of the frame to obtain integral linear forms λ_a with $\lambda_a(u_b) = \delta_{ab}$. Choose a supported multi-index α attaining $\kappa_{p,i,n}$, and use

$$\prod_{a,b} \lambda_a(X^{(b)})^{\alpha_{ab}},$$

of degree in, as an integral test function on i copies of G. At the origin, the order-in determinant operator on this homogeneous test function is its constant-coefficient principal part; terms with fewer derivatives vanish. Its value is $c_\alpha \alpha!$. The infimum defining depth is therefore at most $\kappa_{p,i,n}$. □

Proof of 35.83

Proof (outline). Claim 1. For the prime, valued field, formal group, type map, root-of-unity action and saturated type-τ frame in the statement, there are integral curves $\gamma_1, \dots, \gamma_d$ through the identity with

$$\log_G \gamma_b(T) = \sum_{m \ge 0} L_{b,m} T^{p^m}, \qquad L_{b,m} \in C^d,$$

an injection $r : \{1, \dots, i\} \hookrightarrow \{1, \dots, d\}$, and $g \in \mathrm{GL}_i(O)$, such that the tangent matrix $T_0 = (L_{b,0,e})_{b,e}$ lies in $\mathrm{GL}_d(O)$ and

$$L_{r(a),0} = \sum_{a'=1}^{i} g_{aa'} u_{a'}, \qquad L_{r(a),1} = 0 \quad (a > j).$$

Its hypothesis on the next type is exactly $\#\{e : t(e) = \tau + 1\} \le j$. Say that a curve γ with $\log_G \gamma(T) = \sum_m L_m T^{p^m}$ has type σ if $(L_m)_e = 0$ whenever $t(e) \ne \sigma + m$. The curves are built in four steps. First,

every $w \in O^d$ is the tangent of an integral curve whose logarithm has only the exponents p^m: starting from wT, the n-th of a sequence of integral corrections keeps the coefficients of degree below $\max(n,2)$ and, for $n \geq 2$ prime to p, removes the logarithm coefficients in the degrees divisible by n, leaving the others unchanged; the limit curve has the required logarithm. Second, if w is supported in coordinates of type σ, choose a primitive (p^h-1)-th root of unity ζ and form the formal sum of the curves $[\zeta^k]\gamma(\zeta^{-kp^{\tilde\sigma}}T)$, $0 \leq k < p^h-1$, where γ is the curve of the first step and $\tilde\sigma \in \{0,\dots,h-1\}$ represents σ. Multiplication by p^h-1, which is prime to p, is bijective on integral curves, so this sum is p^h-1 times a unique integral curve η. The root-of-unity formula and the geometric sums over k show that η has tangent w and type σ. Third, apply the second step to $u_1,\dots,u_i$ with $\sigma=\tau$, obtaining curves $\eta_1,\dots,\eta_i$. Their first layers $L_1(\eta_a)$ lie in the span of the coordinate vectors of type $\tau+1$, of dimension at most j. Induction on i gives g with integral entries and unit determinant such that $\sum_{a'} g^p_{aa'} L_1(\eta_{a'}) = 0$ for $a > j$. If these layers are linearly independent, then $i \leq j$ and $g = 1$. Otherwise normalize a linear relation so that a coefficient of minimal valuation equals 1; the others are then integral, and so are their p-th roots in C. Take these roots, with 1 at the normalized position, as the last row of g, and fill the other rows from the induction hypothesis for the remaining $i-1$ curves, with 0 at the normalized position; expansion along that column gives $v(\det g) = 0$. The formal sums $\gamma'_a = \sum_{a'} \eta_{a'}(g_{aa'}T)$ are integral curves of type τ with tangents $\sum_{a'} g_{aa'}u_{a'}$ and layers $L_m(\gamma'_a) = \sum_{a'} g^{p^m}_{aa'} L_m(\eta_{a'})$, so $L_1(\gamma'_a) = 0$ for $a > j$. Fourth, choose r so that the minor $\det(u_{a,r(b)})_{a,b}$ is a unit; then each coordinate $r(b)$ has type τ. Put $\gamma_{r(a)} = \gamma'_a$, and for every other coordinate e let γ_e be the curve of the second step with tangent the unit vector $\mathbf{e}_e$ and type $t(e)$. The rows of T_0 outside the image of r are unit vectors, so $\det T_0 = \det g \cdot \det(u_{a,r(b)})_{a,b}$ is a unit. Every γ_b has type $t(b)$.

Lean: `GlobalTheory.T.exists_typicalCurve_basis_adapted_to_saturated_frame` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsTypicalCurveBasisAdaptedToSaturatedFrameStatement`, lines 8–9)

Write $T = T_0$. Replacing u by the selected tangents preserves determinant depth: its determinant operator is multiplied by the unit $\det g$, and its n-th power by $(\det g)^n$.

Claim 2. The curves in Claim 1 have logarithm coefficients satisfying

$$v(L_{b,m,e}) \geq -m \quad (m \geq 0), \qquad L_{b,1} = 0 \implies v(L_{b,m,e}) \geq 1-m \quad (m \geq 1).$$

Lean: `ReciprocityTheory.ChartRestriction.G.Sh.pt5v'` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.WeightedSumBounds.WtSubOneLe`, lines 99–189)

More precisely, there are coefficients $s_{q,e,b} \in O$, for $q \geq 0$ and $1 \leq e,b \leq d$, satisfying

$$pL_{b,m+1} = \sum_{q=0}^{m} \sum_{e} s^{p^q}_{m-q,e,b} L_{e,q}.$$

To obtain them, fix b and let ζ be a primitive p-th root of unity. The formal sum of the curves $\gamma_b(\zeta^k T)$, $0 \leq k < p$, is an integral curve whose logarithm is $\sum_{\zeta'^p=1} \log_G \gamma_b(\zeta' T) = \sum_m pL_{b,m+1}T^{p^{m+1}}$, so there is an integral curve δ_b with layers $L_m(\delta_b) = pL_{b,m+1}$; it is p-typical of type $t(b)+1$. For any integral curve δ whose logarithm has only the exponents p^n, define $x_{n,e} \in C$ recursively by $\sum_e x_{n,e}L_{e,0} = L_n(\delta) - \sum_{q<n}\sum_e x^{p^{n-q}}_{q,e} L_{e,n-q}$, using the invertibility of T. These coordinates are integral by induction on n: if $x_{q,e} \in O$ for $q < n$, the formal sum π_n of the curves $\gamma_e(x_{q,e}T^{p^q})$, $q < n$, is an integral curve whose logarithm agrees with that of δ below degree p^n. Hence the right side above is the coefficient of T^{p^n} in the integral series $\delta - \pi_n$, and T^{-1} is integral. Taking $\delta = \delta_b$ and $s_{q,e,b} = x_{q,e}$ gives the identity. The tangent coefficients are integral. Induction in this identity gives $v(L_{b,m,e}) \geq -m$. If $L_{b,1} = 0$, its case $m = 0$ and invertibility of T give $s_{0,e,b} = 0$ for every e. Thus the term $q = m$ vanishes in the recurrence. For $m \geq 1$, each remaining term has valuation at least $-(m-1)$, and division by p gives $v(L_{b,m+1,e}) \geq -m$. The case $m = 1$ of the stronger bound follows from $L_{b,1} = 0$. Call a curve with $L_{b,1} = 0$ good.

Let

$$\Phi(X) = \gamma_1(X_1) +_G \cdots +_G \gamma_d(X_d).$$

It and its inverse are integral, because its tangent matrix is invertible over O. In these coordinates the formal map with logarithm $\sum_a L_{r(a),0}Y_a$ has coordinates $z_b(Y)$, with zero constant term. Multiplying the logarithm equation by T^{-1} yields

$$z_b + \sum_{a=1}^{d}\sum_{m\geq 1} c_{a,m,b} z_a^{p^m} = \sum_{q=1}^{i} \delta_{r(q),b} Y_q, \qquad c_{a,m,b} = \sum_e L_{a,m,e}(T^{-1})_{eb}.$$

Thus $v(c_{a,m,b}) \ge -m$, with the stronger bound $1-m$ for good a.

Assign the radii

$$R_b = \begin{cases} 2/[p(p-1)], & b \text{ good}, \\ 2/(p-1), & b \text{ otherwise}, \end{cases} \qquad \rho_q = R_{r(q)}.$$

A series $f = \sum_\alpha f_\alpha Y^\alpha$ has weighted order at least R when

$$v(f_\alpha) + \sum_q \rho_q \alpha_q \ge R$$

for every α. The right side of the logarithm equation has weighted order at least R_b. For $m \ge 1$,

$$-m + p^m \frac{2}{p-1} \ge \frac{2}{p-1}, \qquad 1 - m + p^m \frac{2}{p(p-1)} \ge \frac{2}{p-1}.$$

Induction on total degree in the logarithm equation proves that z_b has weighted order at least R_b: every nonlinear coefficient of degree N uses coefficients of z_a of degree below N. In particular, each z_b has weighted order at least zero.

An integral series on i copies of G, pulled back through these coordinates, has its coefficient of multi-index $\alpha = (\alpha_{bq})$ of valuation at least

$$-\sum_{b,q} \rho_q \alpha_{bq}.$$

Every monomial in $H_{i,n}$ has degree n in each frame index q, so for its supported indices this lower bound is $-n\sum_q \rho_q$. Applying $H_{i,n}(\partial)$ at the origin and using Lemma 35.82 gives

$$\operatorname{depth}_G(\det D_u^n) \ge \kappa_{p,i,n} - n\sum_q \rho_q.$$

At most j of the selected curves are bad, whence

$$n\sum_q \rho_q \le \frac{2in}{p(p-1)} + \frac{2jn}{p-1}.$$

The claimed inequality follows.

The complete adapted-curve and depth calculation is contained in `GlobalTheory.FGDepth.InertBound.content_le_detDepth_add`. □

Proof of 35.84

Proof. For $n = 0$, take $a = b = 0$ by Lemma 35.54. For any $n \ge 0$, let $\kappa = \kappa_{\ell,i,n}$ be the number in Lemma 35.82 and put

$$W_0 = \frac{2in}{\ell(\ell-1)} + \frac{2Qn}{\ell-1}, \qquad a = \kappa - W_0, \qquad b = \kappa.$$

Fix an arbitrary integral frame, chart and valued extension over λ. In the typed factor from Lemma 35.79, the block frame has type zero. Since $\tau \in \Psi$ and ℓ is inert, the next Frobenius type lies outside Ψ, so its Lie multiplicity is Q. Apply Lemma 35.83 with $j = Q$ and transfer its lower bound by the depth identity. It gives $a \le \nu_{\tau,i}(n; y)$ after taking the infimum over all charts and valued extensions.

For the upper bound, choose an integral chart and a valued extension over λ. The saturated-frame bound in Lemma 35.82 gives

$$\nu_{\tau,i}(n; y) \le \operatorname{depth}_G(\det D_u^n) \le \operatorname{cont}_v(H_{i,n}) = \kappa = b.$$

Thus the same constants work for every trait. Since $W_0 \ge 0$, they satisfy $a \le b$ and $b - a = W_0$. The next-type rank Q bounds the number of selected curves with radius $2/(\ell-1)$; all remaining curves use radius $2/[\ell(\ell-1)]$. This gives the additional divisor ℓ on the term involving i.

Under the three size conditions in the statement, $\ell \ge 2$ gives

$$2i + 2Q\ell \le 2(k+m) + 2m\ell \le 4m\ell + 2m \le 5m\ell.$$

Multiply by $n/[\ell(\ell-1)]$ to obtain $b - a \le 5mn/(\ell-1)$. □

Proof of 35.85

Proof. Write Ψ for the CM type and $2f = [E : \mathbf{Q}]$. Then

$$P = k + j - 1, \qquad Q = m - j, \qquad r = P + Q = k + m - 1, \qquad A = fmU, \qquad h_\ell = \frac{\log J}{\log \ell}.$$

There are f embeddings in Ψ, so the row bounds give

$$\sum_{\beta \text{ big}} c_\beta \le fU.$$

Also $Q \le m$. Since $\ell \le J$ and $\ell \ge 2$, we have $J \ge 2$ and $h_\ell \ge 0$.

(1) Fix C, O, τ_O as in Lemma 35.76(a). Let $\eta \in \mathcal{O}_E$ satisfy $\eta^2 = -d$, and choose $\tau_0 \in \Psi$. The host datum makes ℓ split in K: for odd $\ell \mid Q_S$ the host requires $(-d/\ell) = 1$, and for $\ell = 2$ the congruence $d \equiv 7 \pmod 8$ gives $-d \equiv 1 \pmod 8$. This gives a root $\alpha \in \mathbf{Z}_\ell^\times$ of $X^2 + d$. Choose its sign so that its image in O is $\tau_O(\tau_0)(\eta)$. Put $R = \mathbf{Z}_\ell \otimes_{\mathbf{Z}} \mathcal{O}_E$. The element

$$e = \frac{\alpha + \eta}{2\alpha} \in R$$

is integral. For odd ℓ, this follows because 2α is a unit. For $\ell = 2$, use $w_0 = (1 + \eta)/2 \in \mathcal{O}_E$ and write

$$e = \frac{(\alpha - 1)/2 + w_0}{\alpha}.$$

The root α is odd, so this expression belongs to R. Direct calculation gives

$$e^2 = e, \qquad \bar{e} = 1 - e, \qquad \eta e = \alpha e.$$

On each primitive factor, e is either zero or one. Orient each conjugate pair so that $e = 1$ on R_s. An embedding through R_s sends η to α; an embedding through $\bar{R}_s$ sends it to $-\alpha$. The compatibility condition imposed on τ_O identifies these two classes with $\tau_O(\Psi)$ and $\tau_O(\Psi^c)$.

There are $2f$ homomorphisms $\mathcal{O}_E \to O$: they are the restrictions of the $2f$ embeddings $E \to C$. Hence the injection τ_O is a bijection, and every embedding has the required orientation. Conjugation identifies the trace pairings on paired factors. Reorienting the factors therefore preserves their ranks and discriminant valuations. With

$$\delta_s = \frac{v_\ell(\operatorname{disc}(R_s/\mathbf{Z}_\ell))}{n_s},$$

the hypotheses give

$$\sum_s n_s = f, \qquad \sum_s n_s \delta_s = f\delta_\ell.$$

Since $\delta_\ell \ge 0$, Lemma 35.76(a) gives the required integer lower anchor and the width

$$fmU(1 + \delta_\ell/2) + V = A(1 + \delta_\ell/2) + V.$$

Lean: `UpperTheory.splitPlaces_discriminantDepth_of_splitAt` (`Development.AutomorphicArithmetic.CMTraceForms.SplitPlacesDiscriminantDepthOfSplitAt`, lines 59–149)
Lean: `GlobalTheory.CharDepth.twoPlaces_host` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.HostFieldTwoConjugatePlacesOverTwo`, lines 16–26)

(2) A big block $\beta = (\tau, i)$ has $\tau \in \Psi$ and $1 \le i \le P$. The split case of Lemma 35.78, with exponent c_β, gives

$$\nu_\beta(c_\beta; y) \le \frac{Qc_\beta}{\ell - 1}$$

at every algebraic integral frame. Lemma 35.54 bounds each small block by $ic_\beta/(\ell - 1)$, and gives nonnegativity at every integral frame. Consequently

$$0 \le \nu_c(y) \le \frac{fQU + V}{\ell - 1} \le \frac{A + V}{\ell - 1}$$

at algebraic integral frames, with lower bound zero at every integral frame, as in Lemma 35.76(b).

(3) Fix $\lambda \mid \ell$. For every big block with $c_\beta \geq 1$, take the uniform endpoints from Lemma 35.81. Set

$$L_\beta = \frac{\log c_\beta}{\log \ell}, \qquad t_\beta = r - 2Q - i.$$

The hypotheses give $0 \leq L_\beta \leq h_\ell$, $i \leq r$, and

$$t_\beta = k + m - 1 - 2Q - i \leq m.$$

If $t_\beta \leq 0$, then $t_\beta \ell/(\ell^2 - 1) \leq 0$. If $t_\beta > 0$, then

$$\frac{t_\beta \ell}{\ell^2 - 1} \leq \frac{m}{\ell - 1}.$$

In either case the width bound of Lemma 35.81 yields

$$B_\beta - A_\beta \leq \frac{m c_\beta (4 + h_\ell/2)}{\ell - 1} + r^2(h_\ell + 1).$$

For each $\tau \in \Psi$, the possible big sizes are $k-1, \dots, P$, whose number is $j + 1 \leq m + 1$. Thus there are at most $f(m+1)$ big blocks. Summing the preceding inequality and using $\sum_{\text{big}} c_\beta \leq fU$ gives

$$\sum_{\substack{\beta \text{ big} \\ c_\beta \geq 1}} (B_\beta - A_\beta) \leq \frac{A(4 + h_\ell/2)}{\ell - 1} + f(m+1) r^2 (h_\ell + 1).$$

Lean: `GlobalTheory.RefSum.OscInert.sum_bound_le` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.DepthOscillationLeAtSmallOddInertPrime`, lines 112–174)

(4) For a big block of size i,

$$i + 1 \leq P + 1 = k + j \leq k + m, \qquad Q \leq m, \qquad k \leq m\ell.$$

Lemma 35.84 supplies uniform endpoints whose width is at most

$$W_\beta = \frac{2ic_\beta}{\ell(\ell - 1)} + \frac{2Qc_\beta}{\ell - 1}.$$

The displayed size bounds and $\ell \geq 2$ give

$$2i + 2Q\ell \leq 2(k + m) + 2m\ell \leq 4m\ell + 2m \leq 5m\ell.$$

Multiplying by $c_\beta/(\ell(\ell - 1)) \geq 0$ proves

$$W_\beta \leq \frac{5mc_\beta}{\ell - 1}.$$

For $c_\beta = 0$, Lemma 35.54 gives endpoints zero. □

Proof of 35.86

Proof. (1) Recall the construction of a in Lemma 35.46. For τ in the CM type, $s_\tau = N/2 + 1 - u_{1,\tau,P-1}$ gives $a_{\tau,i} = u_{1,\tau,i-1} - u_{1,\tau,P-1} + 1$ for $1 \leq i \leq P$ and $a_{\tau,P+1} = 0$. Since $|u_{1,\tau,s}| \leq B_0$, we have $a_{\tau,i} \leq 2B_0 + 1 = J$. Telescoping from $k - 1$ to P and using $u_{1,\tau,k-2} = 0$ gives

$$\sum_{i=k-1}^{P} c_{\tau,i} = a_{\tau,k-1} = u_{1,\tau,k-2} - u_{1,\tau,P-1} + 1 = -u_{1,\tau,P-1} + 1 \leq B_0 + 1 = U.$$

For $1 \leq i \leq k - 2$ we have $c_{\tau,i} = u_{1,\tau,i-1} - u_{1,\tau,i}$, so summation by parts and $u_{1,\tau,k-2} = 0$ give

$$\sum_{i=1}^{k-2} i\, c_{\tau,i} = \sum_{0 \leq s < k-2} u_{1,\tau,s} - (k-2) u_{1,\tau,k-2} \leq \frac{k(S + 2r)}{T}.$$

At the conjugate embedding $\bar\tau$, reversing and negating the second block and subtracting $N/2$ gives $a_{\bar\tau,i} = u_{1,\tau,P-1} - u_{1,\tau,P+Q-i} - N - 1$ for $1 \le i \le Q$ and $a_{\bar\tau,Q+1} = 0$. Monotonicity of $u_{1,\tau}$ and $u_{1,\tau,P-1} - u_{1,\tau,P} \ge N+1$ make $a_{\bar\tau,i}$ nonincreasing and nonnegative, with $a_{\bar\tau,i} \le 2B_0 \le J$ and $\sum_{i=1}^{Q} i\, c_{\bar\tau,i} = \sum_{i=1}^{Q} a_{\bar\tau,i} \le 2QB_0 = 2q_{\mathrm{sig}}B_0$. Since $c_{\sigma,i} \le a_{\sigma,i} \le J$ everywhere, summing the small blocks over the f embeddings τ in the CM type and their f conjugates $\bar\tau$ gives $\sum_{\mathrm{small}} i\, c_{\sigma,i} \le V$. On every big block $i \le P \le r$, so

$$D = \sum_{\mathrm{big}} i\, c_{\tau,i} + \sum_{\mathrm{small}} i\, c_{\sigma,i} \le r \sum_{\tau \text{ in CM type}} \sum_{i=k-1}^{P} c_{\tau,i} + V \le frU + V.$$

(2) By hypothesis $2 \mid Q_S$, and by 8.3 we have $E \subset \mathbf{Q}(\mu_{Q_S qd})$, so every prime divisor of $\operatorname{disc} E$ divides $Q_S qd$ and primes outside Q_S and $\{q,d\}$ are odd and unramified in E. Fix a prime $\ell \mid Q_S$; by hypothesis ℓ splits $K = \mathbf{Q}(\sqrt{-d})$ over $\mathbf{Z}_\ell$. Choose $\alpha \in \mathbf{Z}_\ell$ with $\alpha^2 = -d$. Since ℓ splits in $\mathcal{O}_K$ of discriminant $-d$, we have $\alpha \in \mathbf{Z}_\ell^\times$, and when $\ell = 2$ we have $\alpha \equiv 1 \pmod 2$ and $w_0 = (1+\sqrt{-d})/2 \in \mathcal{O}_K$, so $e = (\alpha + \sqrt{-d})/(2\alpha) = ((\alpha-1)/2 + w_0)/\alpha$ lies in $\mathbf{Z}_\ell \otimes_{\mathbf{Z}} \mathcal{O}_K \subset \mathbf{Z}_\ell \otimes_{\mathbf{Z}} \mathcal{O}_E$ and satisfies $e^2 = e$ and $e + \bar e = 1$. Decomposing $e = \sum_{s=1}^{u} g_s$ into primitive orthogonal idempotents pairs each factor $R_s = g_s(\mathbf{Z}_\ell \otimes_{\mathbf{Z}} \mathcal{O}_E)$ with its conjugate $\bar R_s = \bar g_s(\mathbf{Z}_\ell \otimes_{\mathbf{Z}} \mathcal{O}_E)$ (with $\bar g_s \ne g_s$ since $e\bar e = 0$). Because conjugation is an isomorphism $R_s \simeq \bar R_s$ of $\mathbf{Z}_\ell$-orders and $\mathbf{Z}_\ell \otimes_{\mathbf{Z}} \mathcal{O}_E \cong \prod_{s=1}^{u}(R_s \times \bar R_s)$, we obtain $2\sum_{s=1}^{u} n_s = [E:\mathbf{Q}] = 2f$ and $2\sum_{s=1}^{u} v_\ell(\operatorname{disc}(R_s/\mathbf{Z}_\ell)) = v_\ell(D_E) = 2f\delta_\ell$.

Now fix a prime ℓ. If $\ell > J$, then $\ell > \max_\beta c_\beta$ by (1), and Lemma 35.55 gives oscillation at most $0 = e_\ell$. Suppose $\ell \le J$, so $J \ge \ell \ge 2$ and $h_\ell = \log J/\log \ell \ge 0$. If $\ell \mid Q_S$, Proposition 35.85(1) gives oscillation at most $A(1+\delta_\ell/2) + V = e_\ell$. If $\ell \nmid Q_S$ and $\ell \in \{q,d\}$, Lemma 35.55 and $D \le frU + V$ give oscillation at most $D/(\ell-1) \le (frU+V)/(\ell-1) = e_\ell$.

Assume finally that $\ell \le J$ with $\ell \nmid Q_S$ and $\ell \notin \{q,d\}$. Then $\ell \nmid 2\operatorname{disc} E$ and $\ell \ne d$, so ℓ either splits or is inert in K. On every small block $\beta = (\sigma, i)$, Lemma 35.54 gives $0 \le \nu_\beta(c_\beta; y) \le i\, c_\beta/(\ell-1)$, and (1) bounds the sum of these upper bounds over small blocks by $V/(\ell-1)$. If ℓ splits in K, Proposition 35.85(2) and $Q = m - j \le m$, summed with (1) and $A_\lambda = 0$, bound $\nu_c(y)$ at every algebraic integral frame over $\lambda \mid \ell$ by

$$\frac{Q\sum_{\mathrm{big}} c_\beta + \sum_{\mathrm{small}} i\, c_\beta}{\ell - 1} \le \frac{fmU + V}{\ell - 1} = \frac{A+V}{\ell-1},$$

which is at most e_ℓ whether $m\ell < k$ or $m\ell \ge k$ since $A \ge 0$ and $h_\ell \ge 0$. If ℓ is inert in K and $m\ell < k$, apply Lemma 35.55 with (A_β, B_β) from Proposition 35.85(3) on big blocks with $c_\beta \ge 1$, $(0,0)$ on big blocks with $c_\beta = 0$ by Lemma 35.54, and $(0, i\, c_\beta/(\ell-1))$ on small blocks; this gives oscillation at most $\{A(4+h_\ell/2)+V\}/(\ell-1) + f(m+1)r^2(h_\ell+1)+1 = e_\ell$. If ℓ is inert in K and $m\ell \ge k$, apply Lemma 35.55 with Proposition 35.85(4) on big blocks and $(0, i\, c_\beta/(\ell-1))$ on small blocks. By (1), $\sum_{\mathrm{big}} 5mc_\beta \le 5fmU \le 6A$, so c has oscillation at most $(6A+V)/(\ell-1) + 1 = e_\ell$. □

Proof of 35.87

Proof. Every factor is a determinant of distinct variables and has integer coefficients. Each determinant is a nonzero homogeneous polynomial of degree equal to its size. Their product is nonzero in the polynomial ring over $\mathbf{Z}$.

The determinants containing row (τ, i) are exactly those of sizes $j \ge i$. Thus its row degree is

$$\sum_{j=i}^{\mu_\tau} c_{\tau,j} = \sum_{j=i}^{\mu_\tau} (a_{\tau,j} - a_{\tau,j+1}) = a_{\tau,i}.$$

Summing over the rows gives

$$\sum_{\tau}\sum_{j=1}^{\mu_\tau} j\, c_{\tau,j} = \sum_{\tau}\sum_{i=1}^{\mu_\tau} a_{\tau,i} = D.$$

The polynomial is a product of the leading minors required in Definition 35.53(c). Adding a multiple of row i' to row i, with $i' < i$, preserves each such minor: a smaller minor omits row i, and a larger one contains both rows.

Conjugate embeddings use disjoint column sets. For every τ, i, i', b, at least one of the variables $X_{\tau,i,b}$, $X_{\bar{\tau},i',b}$ is absent from P_S. Hence

$$\frac{\partial^2 P_S}{\partial X_{\tau,i,b}\,\partial X_{\bar{\tau},i',b}} = 0.$$

Summing over b proves all the required harmonic identities. Together with the row degrees and integral coefficients, this proves membership in $\mathrm{harm}(a, N)$.

Determinants, products and nonnegative integer powers commute with ring homomorphisms. This proves the assertion about change of coefficients. □

Proof of 35.88

Proof (outline). Complete splitting supplies integral maps $\sigma_\tau : \mathcal{O}_E \to \mathcal{O}_F$, integral eigenvectors $e_{\tau,i}$ giving the chosen Lie directions, an integral left inverse to this system of vectors, and elements $\epsilon_\tau \in \mathcal{O}_E$ satisfying

$$d[a]e_{\tau,i} = \sigma_\tau(a)e_{\tau,i}, \qquad \sigma_{\tau'}(\epsilon_\tau) \equiv \delta_{\tau,\tau'} \pmod{g\mathcal{O}_F}.$$

For every $a \in \mathcal{O}_E$, a positive valuation of $\sigma_\tau(a)$ implies $\sigma_\tau(a) \in g\mathcal{O}_F$, while a nonpositive valuation makes it a unit. Coprimality of $\pi, \bar{\pi}$ and $v_F(g) > 0$ imply that exactly one of $\sigma_\tau(\bar{\pi})$, $\sigma_\tau(\pi)$ has positive valuation. Thus S contains one embedding from each conjugate pair.

Lean: `GlobalTheory.T.exists_integral_types_and_directions'` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.ExistsIntegralTypesAndDirectionsStatement`, lines 14–15)

Put $d = \dim \widehat{G}$, and write $Z_1, \dots, Z_d$ for the integral chart coordinates. For a series u with $u(0) = 1$, define

$$d\log u = \left(\frac{\partial u}{\partial Z_i}(0)\right)_{i=1}^{d} \in \mathcal{O}_F^d.$$

Claim 1. For the formal chart of the split semiabelian uniformisation over the complete trait V, after the local map $V \to \mathcal{O}_F$, there are finitely many series $u_j \in \mathcal{O}_F[[Z_1, \dots, Z_d]]$ such that

$$u_j(0) = 1, \qquad u_j(\widehat{G}(Z, W)) = u_j(Z)u_j(W),$$

and, for every $c \in \mathcal{O}_F$,

$$(g-1)v_F(c) \geq v_F(g) \quad \Longrightarrow \quad c\,\mathcal{O}_F^d \subset \mathrm{span}_{\mathcal{O}_F}\{d\log u_j\}.$$

Its hypotheses are the characteristic-zero complete local base, the polarised split uniformisation, the integral formal chart, the algebraically closed valued field F, the fact that g is a nonunit in $\mathcal{O}_F$, and g-adic completeness of $\mathcal{O}_F$. These are the data of the lemma. The claim includes the torus of the semiabelian extension.

Lean: `ReciprocityTheory.ChartRestriction.G.Pdiv.ChartF.HTFSt` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.PlaceDivisors.StChartF`, lines 133–145)

Lean: `GlobalTheory.ChF.hodgeTate_inclusion_chart` (`Development.AbelianVarietyTheory.Proofs.UnramifiedDepth.HodgeTateInclusionChartCharacters`, lines 49–92)

Choose $c \in \mathcal{O}_F$ with $(g-1)v_F(c) = v_F(g)$; a $(g-1)$-st root of g has this property. Pairing the inclusion above with the integral left inverses gives the same inclusion in each type of Lie directions. Pulling the characters back by $[\epsilon_\tau]$ separates those types modulo g. For each type choose integral directional-value columns of individual characters with the required leading-minor bounds. In each assigned column retain the selected values in their own type and set their other type coordinates to zero. Summing these type-isolated vectors defines integral arrays $Y_{\tau,i,b}$, with columns assigned as in P_S. The product of the corresponding pulled-back characters in each column has directional logarithmic differential congruent to that column modulo g. The leading minors $Y_{\tau,j}$ of these arrays satisfy

$$v_F(\det Y_{\tau,j}) \leq j v_F(c) \qquad (1 \leq j \leq \mu_\tau,\ \tau \in S).$$

The construction uses an integral left inverse in each type, the congruences for ϵ_τ, injectivity of the column assignment, and $\sigma_\tau(\bar{\pi}) \in g\mathcal{O}_F$ for $\tau \in S$. *Claim 2.* Each column $Y_{\tau,i,b}$ on the types S has the following finite flat realisation. There are a multiplicative character κ_b on the generic abelian fibre's F-points of $\ker[\bar{\pi}]$, a

finite flat $\mathcal{O}_F$-scheme Z_b with reduced generic fibre, a map $j_b : Z_b \to G[\bar{\pi}]$, and a regular function k_b whose value at a generic point corresponding to z is $\kappa_b(z)$. Each direction $e_{\tau,i}$ has a lifted dual-number point modulo g on Z_b, where

$$k_b = 1 + \epsilon\,\overline{Y_{\tau,i,b}} \quad \text{in } (\mathcal{O}_F/g\mathcal{O}_F)[\epsilon]/(\epsilon^2).$$

Write $b\bar{\pi} + b'\pi = 1$. On the connected part of the g-torsion, $[b'\pi]$ has image killed by $[\bar{\pi}]$ and fixes the specified tangent directions modulo g. The connected torsion algebra is finite flat with reduced generic fibre, and a power of its augmentation ideal lies in g times the algebra. Its g-adic completeness permits evaluation of the multiplicative series. The group law gives multiplicativity; dual-number evaluation gives the displayed coefficient. The nilpotence assertion uses that every nonunit of the rank-one valuation ring has a power in $g\mathcal{O}_F$, and a power of the maximal ideal of V maps into $g\mathcal{O}_F$. The uniformisation maps respect multiplication, action and tangent points. The PEL torsion pairing then labels the translates at π-torsion by these characters on $\ker[\bar{\pi}]$. The unit types use the other column set.

Lean: `GlobalTheory.FamGerm.Pt1.CharAt` (`Development.AbelianVarietyTheory.Definitions.Jets.ColValMonomial`, lines 61–74)
Lean: `GlobalTheory.SeriesCharacter.charAt_of_series` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.CharAtOfMultiplicativeSeries`, lines 167–342)

The determinant contribution to the value is bounded by

$$\sum_{\tau\in S,j} j\,c_{\tau,j}v_F(c) \le Dv_F(c) = \frac{v(g)D}{g-1}.$$

Lean: `GlobalTheory.ChartCharacter.characters_exist_of_chartBound` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.ExistsCharactersBoundedMinorsOfHodgeTateChartBou`, lines 295–342)

Put $a = \bar{\pi}$, $p = g$, $k = g'$, $\theta = v_F(g)/(g-1)$, and $\mathcal{J} = \{1,\dots,n_1\}$. For a polynomial $P = \sum_\alpha P_\alpha Z^\alpha$, with directional index l and column index b, define

$$\mathcal{C}_{kp}(P;(f_b)) = \sum_\alpha P_\alpha \prod_b \left[\left(\prod_l D_{e_l}^{\alpha_{l,b}}\right)(f_b \exp(kph_L))\right](0).$$

Use the canonical gauge and ordered differential values fixed above. A germ of a family is a finite F-linear combination of integral-section germs, with the family the same combination of their compatible theta expansions. If r is the integral frame series and G_i the section series, these germs satisfy $g_i r^{kp} = G_i$. A translate is the theta action followed by a nonzero scalar making it finite order; its point is killed by $[\bar{a}]$.

Lean: `GlobalTheory.FamGerm.Pt1.Germ` (`Development.AbelianVarietyTheory.Definitions.Jets.ColValMonomial`, lines 43–50)
Lean: `GlobalTheory.FamGerm.Pt1.Transl` (`Development.AbelianVarietyTheory.Definitions.Jets.ColValMonomial`, lines 55–59)
Lean: `GlobalTheory.FamGerm.Pt1.colVal` (`Development.AbelianVarietyTheory.Definitions.Jets.ColValMonomial`, lines 76–79)

Claim 3. With this lemma's complete local base, chart, split uniformisation, torsion identification, integral eigen-directions and separators, the following column constructions hold. The period-basis conditions are

$$Q_X(b_j) \le C_c, \qquad |Q_X(b_j+z) - Q_X(b_j) - Q_X(z)| \le M_0 Q_X(z), \quad C_c, M_0 \ge 0.$$

Take $C_c = \Phi$, $M_0 = 2\sqrt{s}$. There is an integral series ω, with $\omega(0) = 1$, satisfying

$$\omega = \exp(h_L([a]Z) - ph_L(Z)).$$

For $b \in \mathcal{J}$, Claim 2 supplies a family in $\mathscr{T}_k^+$, its normalised $[a]$-pullback and a finite-order translate, with germ

$$f_b = c_{1,b}g_{1,b}\omega^k, \qquad v_F(c_{1,b}) = r_{1,b} \le sC_c/(4k),$$

where $g_{1,b} \in \mathcal{O}_F[[Z]]$, $v_F(g_{1,b}(0)) = 0$, and

$$D_{e_l}g_{1,b} - Y_{l,b}g_{1,b} \in g\mathcal{O}_F[[Z]]$$

for every selected direction. This uses $k \ge 4$, $(p,k) = 1$, $a\bar{a} = p$, $(a) + (\bar{a}) = \mathcal{O}_E$, and the finite flat character with $\tau(a) \in p\mathcal{O}_F$ on these directions. The finite flat character gives a point killed by $[\bar{a}]$ and a translation point of level kp over it, with label $y_0 \in X$, such that for every family in $\mathscr{T}_k^+$ the germ $f\omega^k$ of each integral rescaling ct of its translate satisfies $D_{e_l}f - Y_{l,b}f \in g\mathcal{O}_F[[Z]]$. The reduced period basis gives $\chi_0 \in X$ with $Q_X(\beta_0) \le psC_c/4$, where $\beta_0 = y_0 + \bar{a}\chi_0$, and $Q_X(\beta_0 + k\bar{a}z) - Q_X(\beta_0) \ge k(k-1)Q_X(\bar{a}z)$ for all $z \in X$. There

is a family in $\mathscr{T}_k^+$ whose translate t is supported on $\beta_0 + k\bar{a}X$, whose coefficient at β_0, evaluated at the translated abelian point in an integral frame, has valuation exactly $r_{1,b} = Q_X(\beta_0)/(kp) \le sC_c/(4k)$. Choose this coefficient using an integral section of the degree-zero twist of the k-th polarising power that is nonzero at that special-fibre point; global generation for $k \ge 4$ and base change give such a section. Then every other coefficient has normalised valuation at least $r_{1,b} + 1/(kp)$, since $k(k-1)Q_X(\bar{a}z) \ge 1$ for $z \ne 0$. Let f_b be the germ of t. Choose a scalar $c_{1,b}$ of valuation $r_{1,b}$ and put $g_{1,b} = c_{1,b}^{-1} f_b \omega^{-k}$. The leading coefficient has nonzero reduction at the chosen abelian point and all other coefficients have strictly larger valuation, so $c_{1,b}^{-1} f_b$ is integral with unit constant term. Since ω is an integral unit with $\omega(0) = 1$, the same holds for $g_{1,b}$. Applying the relation above to $c_{1,b}^{-1} t$, whose germ is $g_{1,b}\omega^k$, gives the displayed congruence.

Lean: `GlobalTheory.FamGerm.Pt1.Red` (`Development.AbelianVarietyTheory.Definitions.Jets.ColValMonomial`, lines 52–53)
Lean: `GlobalTheory.DepthBound.exists_eigen_column` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.ExistsEigenColumn`, lines 123–246)

On complementary types, $\tau(a)$ is a unit. The integral left inverse of the eigenframe gives integral test series G_b and integers $d_b \ge 0$, $\sum_b d_b \le D$, whose complementary column value on $(G_b([a]Z)\omega^k)_b$ is a unit and depends only on their d_b-jets. If $4d_b + 4 \le k$, $d_b M_0 < k$, the jet construction gives families with the same pullback and translation properties and germs

$$f_b = c_{2,b} g_{2,b}([a]Z)\omega^k, \qquad v_F(c_{2,b}) = r_{2,b} \le d_b^2 C_c/k, \qquad j^{d_b} g_{2,b} = j^{d_b} G_b.$$

The conditions follow from $d_b \le D$, $4D + 4 \le k$, $DM_0 < k$.

Lean: `GlobalTheory.DepthBound.val_unit_columns` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.ValLeOfMemSpan`, lines 339–424)
Lean: `GlobalTheory.DepthBound.exists_jet_column` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.ExistsEigenColumn`, lines 88–121)

Factor $P_S = P_1 P_2$ along the two disjoint type and column sets. If $D_1 = \deg P_1$, the minor bounds give character valuation at most $D_1\theta$. Since $D_1\theta \le D\theta < v_F(g)$, errors divisible by g cannot cancel this value. The complementary value is a unit. Factoring the column value and extracting its scalars gives

$$v_F(\mathcal{C}_{kp}(P_S; (f_b))) \le \sum_{b \in \mathcal{J}} r_{1,b} + D\theta + \sum_{b \notin \mathcal{J}} r_{2,b}.$$

Lean: `GlobalTheory.DepthBound.val_eigen_columns` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.ValLeOfMemSpan`, lines 170–324)

Finally,

$$\sum_{b \in \mathcal{J}} r_{1,b} \le \frac{n_1 s \Phi}{4g'}, \qquad \sum_{b \notin \mathcal{J}} r_{2,b} \le \frac{\Phi}{g'} \sum_b d_b^2 \le \frac{\Phi}{g'} D \sum_b d_b \le \frac{D^2 \Phi}{g'}.$$

The value is nonzero because its valuation is bounded above by a finite number. Its rational valuation therefore satisfies

$$\rho \le \frac{v(g) D}{g - 1} + \frac{n_1 s \Phi}{4g'} + \frac{D^2 \Phi}{g'}.$$

Lean: `GlobalTheory.DepthFromChars.depth_bound_of_chars` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.DepthBoundFromCharactersPointDatum`, lines 28–165)

Form the external products of these families and translations. The biextension gauge-comparison factor between the one-column and product integral frames gives an integral series u with $u(0) = 1$ and a product germ

$$h(X_1, \dots, X_N) = \prod_{b=1}^{N} f_b(X_b) u(X_b).$$

Transporting the one-column canonical gauge along the biextension isomorphism cancels u in each column. Over the characteristic-zero field F, the distinct embeddings are separated. The gauge-change calculation in the proof of Claim 1 of Proposition 35.71 therefore gives only conjugate-row contractions, all of which vanish on P_S by harmonicity. Thus the product and column values agree. The two assigned column sets are disjoint, so the opposite-type mixed derivatives of P_S vanish by Lemma 35.87. Hence the bound for ρ passes to $\mathcal{H}_M(P_S; h)$. Finally $D^2\Phi/g' \ge 0$, so replacing its coefficient 1 by 2 gives the stated bound.

Lean: `GlobalTheory.T.integralChart_columns_to_tuple_harmonicValues` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.IntegralChartColumnsToTupleHarmonicValuesStatement`, lines 14–15)

The complete proof in the archive is contained in `GlobalTheory.PowGerm.lemmaBT_of_characters`. □

Proof of 35.89

Proof (outline). Use the product degeneration datum with period lattice X^N, period form $Q_N(\chi) = \sum_b Q_X(\chi_b)$, product polarisation, and componentwise action. Write $\mathscr{T}_k$ for the F-vector space of level-k theta families and $\mathscr{T}_k^+$ for its integral theta lattice. Fix a uniformiser ϖ of V. The product uniformisation has the compatible torsion and Tate identifications, and the action commutes with product projections.

The hypothesis includes a level-M theta-group point p above an abelian point killed by $[\pi]$ in every factor, a generator w of its translation line, and $\zeta \in F^\times$. Translation of s_2, followed by multiplication by ζ, gives t. This operator has a positive iterate equal to the identity on all level-M families. A section expands as t, and its expressions in compatible product frames on all finite thickenings are the truncations of h. Also $s \in \mathscr{T}_{g'}^+$, and s_2 is its normalised $[\bar\pi]$-pullback.

Lean: `Formalism.MainNotions.ModuliIntegralCharts.H1.BoxTuple.BoxTup` (`Formalism.SharpRestriction.Divisibility.GeneratorPowers.BoxTuple`, lines 22–38)
Lean: `ReciprocityTheory.ChartRestriction.G.DVR.FinOrd` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.DiscreteValuationRings.ValuationMultiplicationInvariance`, lines 63–69)
Lean: `Formalism.MainNotions.ModuliIntegralCharts.H1.ExpBijectiveUniformisation.IsGermN` (`Formalism.SharpRestriction.Divisibility.GeneratorPowers.ExpBijectiveUniformisation`, lines 60–85)

Claim 1. There are an injective F-linear normalised pullback $\mathsf{P}_{\bar\pi} : \mathscr{T}_{g'} \to \mathscr{T}_M$, an F-linear translation $\Phi_{p,w} : \mathscr{T}_M \to \mathscr{T}_M$, and an F-linear germ map

$$\Gamma_M : \mathscr{T}_M \longrightarrow F[[Z_{b,e} : 1 \le b \le N,\ 1 \le e \le d]],$$

where d is the chart dimension, satisfying

$$s_2 = \mathsf{P}_{\bar\pi} s, \quad t = \zeta \Phi_{p,w}(\mathsf{P}_{\bar\pi} s), \quad h = \Gamma_M(t).$$

Here $\bar\pi\pi = g$, $g, g' \ge 1$, and the action preserves the trace-form trivialisation. Uniqueness of normalised pullback proves its linearity. The fixed translation generator proves linearity of $\Phi_{p,w}$. The product frames agree under truncation; uniqueness of compatible section expansions and their germs gives Γ_M, its linearity, and its values as power series.

Lean: `GlobalTheory.PullLin.exists_pull_linearMap` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.StalkMapInjectiveOfFlat`, lines 136–153)
Lean: `GlobalTheory.PairingLinear.exists_germ_linearMap` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.ExistsGermLinearMap`, lines 34–57)

Claim 2. There is a principal family $\vartheta \in \mathscr{T}_1^+$ whose period-zero coefficient is the base change of an integral section not divisible by ϖ. Put $\vartheta_{\bar\pi'} = \mathsf{P}_{\bar\pi'}\vartheta \in \mathscr{T}_{g'}$. There are finitely many $a_i \in F$ and finite-order translates b_i of $\vartheta_{\bar\pi'}$, at points killed by $[\pi']$, such that

$$s = \sum_i a_i b_i, \qquad v_F(a_i) \ge 0.$$

Each translate includes a theta-group lift, a translation-line generator and a nonzero scalar. The generation input is the product PEL uniformisation, its torsion and Tate identifications including Weil pairings, Q_N, the valuation extending that of ϖ, the principal frame, and the norm identity $\pi'\bar\pi' = g'$ with coprime half-level ideals. The needed invertibility holds in V: choose integers a, b with $ag + bg' = 1$. Since g is a nonunit, $1 - ag$ is a unit, hence g' is a unit. No invertibility of g is used.

Lean: `GlobalTheory.GeneratorsAtPoint.isUnit_of_prime_ne` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.IsUnitOfPrimeNe`, lines 27–34)
Lean: `GlobalTheory.PowGerm.ThFrame` (`Development.AbelianVarietyTheory.Definitions.Jets.JetsOnCopiesCompatibleFrames`, lines 85–92)
Lean: `GlobalTheory.GeneratorsAtPoint.unitBall_spanned` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.IsUnitOfPrimeNe`, lines 54–93)

Claim 3. Let $\Lambda = (\zeta\Phi_{p,w}) \circ \mathsf{P}_{\bar\pi}$. For each b_i, descent gives

$$\Lambda(b_i) = \zeta_i' \Phi_{p_i', w_i'}(\mathsf{P}_\alpha \vartheta), \qquad \alpha = \bar\pi\bar\pi', \qquad \alpha\bar\alpha = M.$$

Here p_i' lies above a point killed in every factor by $[\bar\alpha] = [\pi\pi']$, w_i' generates its translation line, $\zeta_i' \ne 0$, and $\zeta_i'\Phi_{p_i', w_i'}$ has finite order. This identity of families uses the distinct prime norms g, g', both norm identities, the compatible product actions, and the finite-order lifts at π'-torsion and π-torsion already fixed.

Lean: GlobalTheory.PowDescent.descentIdF_of_laws (Development.AbelianVarietyTheory.Proofs.JetHarmonics.DescentIdFOfLaws, lines 62–97)
Lean: GlobalTheory.GeneratorsAtPoint.main (Development.AbelianVarietyTheory.Proofs.JetHarmonics.IsUnitOfPrimeNe, lines 95–148)

Set $h_i = \Gamma_M(\Lambda(b_i))$. Linearity now gives the exact identity

$$h = \Gamma_M(\Lambda(s)) = \Gamma_M\left(\Lambda\left(\sum_i a_i b_i\right)\right) = \sum_i a_i h_i.$$

Claim 3 supplies each translated family, and the germ map supplies its compatible section and germ on every finite thickening. The coefficients belong to $\mathcal{O}_F$ because $v_F(a_i) \geq 0$.

Fix one of these translated families. Splitting its product frame into the one-factor frames writes its germ as a scalar of valuation zero times the external product of the corresponding one-factor germs. The comparison of each one-factor finite-order theta action with the symmetric theta-group action introduces a root of unity. Their product has valuation zero. The product therefore corresponds, up to one scalar u_i of valuation zero, to the product of the translated pullback sections associated with a characteristic tuple x_i.

Pulling back by $[\bar{\alpha}]$ via the normalised pullback chain rule on the row-homogeneous polynomial P of total degree $D = \sum_{\tau,j} a_{\tau,j}$ produces the factor $M^{-D}\nu_a(\bar{\alpha})$ on the M^2-level section, while $\alpha\bar{\alpha} = M$ gives $\nu_a(\alpha)M^{-D}\nu_a(\bar{\alpha}) = 1$. Multiplying by $\nu_a(\alpha)$ therefore cancels $M^{-D}\nu_a(\bar{\alpha})$, and the identification of harmonic values of tied sections with the product-germ value gives

$$\mathcal{H}_M(P; h_i) = u_i \nu_a(\alpha)\theta_{x_i,P}(y).$$

The characteristic tuple and u_i are obtained from the sections and frames, so they serve all the harmonic polynomials in the assertion.
Lean: GlobalTheory.ColumnSplit.main (Development.AbelianVarietyTheory.Proofs.JetHarmonics.TranslateThetaFrameGermEqUnitMulBoxProduct, lines 51–112)
Lean: GlobalTheory.OneColTranslate.oneColumn_translate (Development.AbelianVarietyTheory.Proofs.JetHarmonics.OneColumnTranslateSectionRootOfUnityMul, lines 45–103)
Lean: GlobalTheory.PtTie.harmonicValue_translate (Development.AbelianVarietyTheory.Proofs.JetHarmonics.HarmonicValueTranslateEqUnitMulNormaliserThetaVa, lines 62–130)

Linearity in the germ gives

$$\mathcal{H}_M(P; h) = \sum_i a_i \mathcal{H}_M(P; h_i).$$

If every $\mathcal{H}_M(P; h_i)$ had valuation greater than ρ, then every summand would have valuation greater than ρ, since $v_F(a_i) \geq 0$. The ultrametric inequality would contradict the definition of ρ. Thus one index has value at most ρ. At that index $v_F(u_i) = 0$, and v_F restricts to v, proving the final inequality.

The complete proofs in the archive are contained in `GlobalTheory.PowGerm.pointStatements_on_copies` and `GlobalTheory.PtTie.harmonicValue_translate`. □

Proof of 35.90

Proof. Order the half-level generators so that $\pi\bar{\pi} = g$, $\pi'\bar{\pi}' = g'$ and $\mathfrak{b} = (\pi\pi')$. Coprimality of $\mathfrak{b}$ and $\bar{\mathfrak{b}}$ implies both

$$(\pi) + (\bar{\pi}) = \mathcal{O}_E, \qquad (\pi') + (\bar{\pi}') = \mathcal{O}_E.$$

Choose an algebraically closed valued extension F of $\operatorname{Frac} V'$, with rational-valued valuation extending v', a g-adically complete valuation ring, and a PEL Tate isometry compatible with the Weil pairings. Equip the N-fold product with the product degeneration datum and the componentwise action.
Lean: GlobalTheory.T.exists_algClosed_valuedField_over_completeDVR (Development.AbelianVarietyTheory.Proofs.JetHarmonics.ExistsAlgClosedValuedFieldOverCompleteDVRStatement, lines 15–16)
Lean: GlobalTheory.T.nonempty_tateIsometry_integralFramedPoint_chart (Development.AbelianVarietyTheory.Proofs.JetHarmonics.NonemptyTateIsometryIntegralFramedPointChartStatement, lines 15–18)
Lean: GlobalTheory.T.exists_productData_copies_trivialisation_valuation (Development.AbelianVarietyTheory.Proofs.JetHarmonics.ExistsProductDataCopiesTrivialisationValuationStatement, lines 15–18)

Put $n_1 = r$ and $n_2 = N - r$. The hypothesis $N \geq 2r$ and $\mu_\tau \leq r$ give $\mu_\tau \leq n_1, n_2$. Let

$$S = \{\tau : v_F(\tau(\bar{\pi})) > 0\}.$$

Since $v_F(g) > 0$, coprimality and $\tau(\pi)\tau(\bar\pi) = g$ imply that S contains exactly one member of each conjugate pair. Lemma 35.87 supplies $P_1 = P_S \in \mathrm{harm}(a, N)$, with degree D and the two required sets of columns.

The complete splitting of g, the short basis of parameter Φ', and the three numerical hypotheses allow Lemma 35.88 to be applied. It gives the compatible germ h of a finite-order translate of a $[\bar\pi]$-pullback of a family in $\mathscr{T}^{+}_{g'}$, with

$$\rho = v_F(\mathcal{H}_M(P_1; h)) \le \frac{v'(g)D}{g-1} + \frac{rs\Phi'}{4g'} + \frac{2D^2\Phi'}{g'}.$$

Lemma 35.89, applied with $\alpha = \bar\pi\bar\pi'$, supplies a characteristic tuple x such that

$$v'(\theta_{x,P_1}(y')) + v'(\nu_a(\bar\pi\bar\pi')) \le \rho.$$

Since $\nu_a(\bar\pi\bar\pi') = \nu$, these inequalities prove the proposition. □

Proof of 35.91

Proof. Take $(g, g') = (g_0, g_1)$ or (g_1, g_0); interchanging π_0 and π_1 leaves $\bar\pi_0\bar\pi_1$ and $\nu = \nu_a(\bar\pi_0\bar\pi_1)$ unchanged. Fix a place $\lambda \mid g$ of F_c, a trait V of ramification index e over λ, and an algebraic integral frame y on V with phase $\phi \ge 0$. Put $e_0 = v(g) = e\, v_\lambda(g) > 0$. Pass to a finite extension V' of relative ramification index $r' \ge 1$ over V on which the image algebraic integral frame y' has a split uniformisation [176, Ch. III], with phase $r'\phi$ and $v'(g) = r'e_0$. Let $s \le 2f q_{\mathrm{sig}} \le 2fm$ and $\Phi' = \kappa s^3 r'\phi$ be as in Lemma 35.49. Since $4D+4 \le g$ implies $D < g-1$, while $4D+4 \le g'$ and $2D\sqrt{s} \le 2D\sqrt{2fm} < g'$, Proposition 35.90 on V' supplies $P_1 \in \mathrm{harm}(a, N)$ and a tuple x such that, after dividing by $v'(g) = r'e_0$,

$$\frac{v(\theta_{x,P_1}(y))}{v(g)} + \frac{v_\lambda(\nu)}{v_\lambda(g)} \le \frac{D}{g-1} + \frac{\kappa s^3(rs/4 + 2D^2)\phi}{g'\, v(g)}.$$

Here $D/(g-1) < 1$ because $4D + 4 \le g$, and $s \le 2fm$ together with the hypothesis on c gives $\kappa s^3(rs/4 + 2D^2)\phi/(g'v(g)) \le c\,\phi/v(g)$.

Put $\theta = \theta_{x,P_1}$. As in the proof of Proposition 35.60, Lemma 35.40 shows that θ is a cusp section of weight (a, N) over F_c. Since $\gcd(g_0g_1, Q_S q d) = 1$, Lemma 35.75(1)–(3) gives the finitely supported lower bound m on the finite places of F_c with $m(\lambda) = -v_\lambda(\nu) \in \mathbf{Z}$. Lemma 35.56 with $k = -v_\lambda(\nu)$ therefore gives $\sigma \in F_c^{\times}$ with $v_\lambda(\sigma) = v_\lambda(\nu)$ such that $f = \sigma\theta$ is an integral cusp section of weight (a, N). On V, $v(\sigma)/v(g) = v_\lambda(\nu)/v_\lambda(g)$, so

$$\frac{v(f(y))}{v(g)} = \frac{v_\lambda(\nu)}{v_\lambda(g)} + \frac{v(\theta(y))}{v(g)} \le 1 + \frac{c\,\phi}{v(g)}.$$

Subtracting $c\,\phi/v(g)$ and taking the supremum in Definition 35.53(e) gives $d^{c}_{g}(a, N) \le 1$. □

Proof of 35.92

Proof (outline). Put $N_0 = N/2$, which is an integer by Lemma 35.46, and let $G = (a, N)$ be the theta weight. The same construction writes G as the weight obtained from u_1 by a row translation $(s_\tau)_\tau$ and scalar exponent $2N_0$. Put $G_0 = u_1 - G$, so the required multiplier has row translation $(-s_\tau)_\tau$ and scalar exponent $-2N_0$. The hypotheses give the complex uniformisations, algebraic components, positive ranks, integral points at good places, and flat sections on every component used below.

For a finite place w of F' over a rational prime ℓ, normalise ord_w by $\mathrm{ord}_w(\varpi_w) = 1$, and put $e_w = \mathrm{ord}_w(\ell) \ge 1$. The valuation of a section at an integral trait is divided by the trait's ramification index over w when it is compared with this normalisation.

Claim 1. For each component Z, there are rational sections t_Z, t_Z^{-1}, supported on Z, of weights $G_0, -G_0$, whose product is the component idempotent. Multiplication by either section carries cusp sections of the appropriate even scalar weight to cusp sections of the shifted weight. The norm of t_Z is a positive constant on each complex uniformised component. There are finitely supported functions l_Z, h_Z on the finite places of F' such that, at every integral frame on Z over w,

$$l_Z(w) \le \mathrm{ord}_w(t_Z) \le h_Z(w).$$

Moreover, $l_Z(w) \in \mathbf{Z}$ whenever the rational prime below w is outside $\mathcal{B}$, and

$$h_Z(w) - l_Z(w) \le N_0 f r\, \delta_\ell e_w \qquad (w \mid \ell).$$

To construct the sections on each component Z, take the minimal determinant twisting section $t_{\det}$ of central parameter 1 (row weight $(0,1)_{\mathrm{row}}$), constructed unconditionally for odd $M' \ge 3$ and every even $N = 2N_0$ in the proof of Lemma 35.4 via the $w_0^4 \pmod 8$ Lagrangian splitting and $M' \ge 3$ neatness (requiring no $8 \mid N$ restriction), together with its inverse $t_{\det}^{-1}$, and take the flat section with row $(-s_\tau)_\tau$. Away from $\mathcal{B} = \{\ell \text{ prime} : \ell \mid 2dD_E M'\}$, good reduction supplies smooth integral models with multiplicity-one special fibres (Lemma 35.10), ensuring integer lower orders $l_Z(w) \in \mathbf{Z}$ for both the flat section and $t_{\det}$ at every prime $\ell \notin \mathcal{B}$, even at spreading-out primes where $l_Z(w) \ne 0$. In particular, the flat section has equal lower and upper finite-place bounds that are integral away from $\mathcal{B}$, while the determinant twisting section $t_{\det}$ has constant archimedean norm on each complex uniformised component, integral upper finite-place bound away from $\mathcal{B}$, and finite-place spread bounded by

$$\sum_\tau \mu_\tau \operatorname{ord}_w(\mathfrak{D}_\tau).$$

Here μ_τ is the Lie multiplicity and $\mathfrak{D}_\tau$ is the conjugate different ideal. Multiply the flat section by $t_{\det}^{-N_0}$ (which has central parameter $-N_0$, i.e., row weight $(0,-N_0)_{\mathrm{row}}$). The lower bound is the flat bound minus N_0 times the upper determinant bound; the upper bound uses the lower determinant bound. Thus $l_Z(w) \in \mathbf{Z}$ away from $\mathcal{B}$ is preserved (yielding $h_Z(w) - \lfloor l_Z(w) \rfloor = h_Z(w) - l_Z(w) = 0$ outside $\mathcal{B}$ once combined with $\delta_\ell = 0$ for $\ell \nmid D_E$) and the spread is multiplied by N_0. Since $E/\mathbf{Q}$ is Galois,

$$\operatorname{ord}_w(\mathfrak{D}_\tau) = \delta_\ell e_w, \qquad \sum_\tau \mu_\tau = fr.$$

The latter equality follows by pairing the two conjugate Lie multiplicities and using $[E : \mathbf{Q}] = 2f$. These identities give Claim 1.

Lean: `GlobalTheory.FlatTw.TwSec` (`Development.AbelianVarietyTheory.Definitions.Divisors.MulSmulIdemMem`, lines 33–59)
Lean: `GlobalTheory.T.exists_keySection_inverseRoot_twistingSection_spread` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsKeySectionInverseRootTwistingSectionSpreadStatement`, lines 10–11)
Lean: `GlobalTheory.FlatTwisting.twSec_of_flat` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsTwSecOfConst`, lines 87–122)
Lean: `GlobalTheory.CombTw.exists_twSec_comb` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsMulInvPow`, lines 44–60)
Lean: `GlobalTheory.DifferentGalois.ord_conjDifferent_and_sum` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.OrdConjDifferentAndSum`, lines 11–41)

Claim 2. Suppose the sections in Claim 1 are given, G has even scalar exponent, and $k_\ell \in \mathbf{R}$ satisfies

$$h_Z(w) - \lfloor l_Z(w) \rfloor \le k_\ell e_w \quad \text{for every } Z \text{ and } w \mid \ell.$$

For every reference parameter $c \ge 0$,

$$d_\ell^c(G + G_0) \le d_\ell^c(G) + k_\ell.$$

Indeed, multiply a source cusp section by t_Z and apply the denominator-clearing construction of Lemma 35.56. Its bound at w loses at most the upper multiplier valuation minus the integer lower bound $\lfloor l_Z(w) \rfloor$. Division by e_w gives at most k_ℓ. The constant complex norms and the reciprocal multiplier supply the same component reference comparison for the shifted weight. Taking the infimum over source integral sections and then the required supremum over places and algebraic integral frames gives the displayed inequality, with the same reference parameter c.

Lean: `GlobalTheory.T.exists_twistingSection_mul_lattice_value_le` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsTwistingSectionMulLatticeValueLeStatement`, lines 10–11)
Lean: `GlobalTheory.RefTwist.refExp_vp_add_le` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.BaseLocusExponentTwistingSectionMulLe`, lines 41–60)

Fix ℓ. If $\ell \notin \mathcal{B}$, then $\delta_\ell = 0$ and $l_Z(w) \in \mathbf{Z}$. Claim 1 therefore gives

$$h_Z(w) - \lfloor l_Z(w) \rfloor = h_Z(w) - l_Z(w) \le 0.$$

If $\ell \in \mathcal{B}$, use $l_Z(w) - \lfloor l_Z(w) \rfloor < 1 \le e_w$ to obtain

$$\begin{aligned} h_Z(w) - \lfloor l_Z(w) \rfloor &= (h_Z(w) - l_Z(w)) + (l_Z(w) - \lfloor l_Z(w) \rfloor) \\ &\le (N_0 f r \delta_\ell + 1) e_w. \end{aligned}$$

Claim 2 with $c = 1/2500$, $G + G_0 = u_1$, and $k_\ell = \mathbf{1}_{\ell \in \mathcal{B}}(1 + N_0 f r \delta_\ell)$ proves the stated inequality. The complete primewise comparison is proved in `GlobalTheory.RefSum.WFree.refExp_vp_le_shift`. □

Proof of 35.93

Proof. (1) In Lemma 35.46, the differential weight (a, N) has $a_{\tau,P} = 1 > 0$ on the CM type, $N \ge \sum_{\tau,i} 2i \ge \sum_{c_{\tau,i} \ne 0} 2i$, $N \ge 2r$, and $1 \le D \le b^2$. By (A.462), $b \ge 1000$, $r \le b$, $(2fm)^3 \le (frL)^{12} \le P_* \le b$, $\kappa = R_E^2/(4f) \le R_E^7 \le b$, and $g_0, g_1 \ge b^8$. Hence for both $g \in \{g_0, g_1\}$,

$$4D + 4 \le 4b^2 + 4 \le g, \qquad 2D\sqrt{2fm} \le 2b^2\sqrt{b} < g,$$

and

$$\kappa(2fm)^3\Big(\frac{r(2fm)}{4} + 2D^2\Big) \le b \cdot b \cdot \Big(\frac{b^2}{4} + 2b^4\Big) \le 3b^6 \le \frac{g}{2500}.$$

The same bounds hold with $g_0 g_1 \ge b^{16}$ in place of g, and $(Q_S q d)^{36} t^{150} \le \sqrt{b}$ from Lemma 35.48 (with $t \ge 1$) gives $Q_S q d \le \sqrt{b} < b^8 \le g_0, g_1$, hence $\gcd(g_0 g_1, Q_S q d) = 1$.

By Definition 33.3(2), $k \ge x^{9/2+10\varepsilon} > 1$ once $x > 1$, so the integer k satisfies $k \ge 2$. At a nonterminal layer $1 \le j < m$, Lemma 35.46 then gives $p_{\mathrm{sig}} = k + j - 1 \ge k = (k-2) + 2$. By Lemmas 35.20 and 35.46 together with (A.434), (35.6), and (A.462), each row of u_1 is decreasing with $u_{1,\tau,k-2} = 0$, $0 \le u_{1,\tau,s} \le (S + 2r)/T \le 8b/T$ on $0 \le s < k-2$ (so $\sum_{0 \le s < k-2} u_{1,\tau,s} \le k(S+2r)/T$), $|u_{1,\tau,s}| \le (8b + c_0 + 4r)/(T + d_{\mathrm{fl}} + 1) \le 8b/T$ on $s > k-2$ (since $c_0 + 4r \le 26P_*$ and $26P_*T \le P_* T d_{\mathrm{fl}}^2 L \le b < 8(d_{\mathrm{fl}} + 1)b$), whence $|u_{1,\tau,s}| \le \lfloor 8b/T \rfloor = B_0$ by integrality, and wall gap $u_{1,\tau,P-1} - u_{1,\tau,P} \ge h_0/(2T) \ge 20fr^2 + 1 \ge N + 1$; thus the row hypotheses of Lemma 35.86(1) hold. By the host construction (8.3), $2 \mid Q_S$ and every prime $\ell \mid Q_S$ splits $K = \mathbf{Q}(\sqrt{-d})$ over $\mathbf{Z}_\ell$, so the host hypotheses of Lemma 35.86(2) hold.

For $\ell \in \{g_0, g_1\}$, Proposition 35.91 with $c = 1/2500$ gives $d_\ell^{1/2500}(a, N) \le 1 \le e_\ell + 1$ on the base moduli since $e_\ell \ge 0$. For $\ell \notin \{g_0, g_1\}$, Lemma 35.86 gives that the gap tuple of (a, N) has oscillation at most e_ℓ at ℓ; applying Lemma 35.72(2) at each place $\lambda \mid \ell$ establishes hypothesis (i) of Proposition 35.60 with $\varrho = e_\ell$ at every place above ℓ, and Lemma 35.75(1)–(2) establishes hypothesis (ii), so Proposition 35.60 gives $d_\ell^{1/2500}(a, N) \le e_\ell$ on the base moduli. Lemma 35.58 then carries the inequality for every prime ℓ from the base moduli to Y.

(2) Combining (1) on Y with Proposition 35.92 gives

$$d_\ell^{1/2500}(u_1) \le e_\ell + \mathbf{1}_{\ell \in \{g_0, g_1\}} + \mathbf{1}_{\ell \in \mathcal{B}}\big(1 + \tfrac{N}{2} f r\, \delta_\ell\big) = a(\ell)$$

for every prime ℓ. Since e_ℓ and δ_ℓ are nonnegative and $e_\ell = 0$ for $\ell > J$, each $a(\ell)$ is nonnegative and vanishes outside $\mathcal{S}_{\mathrm{ref}}$. Splitting $a(\ell) \log \ell$ across the support of e_ℓ and $\{g_0, g_1\} \cup \mathcal{B}$, with $\sum_{\ell \in \mathcal{B}} \delta_\ell \log \ell \le \log \mathrm{rd}(E)$, reduces the weighted prime sum to Lemma 35.48, which gives $T \sum_{\ell \in \mathcal{S}_{\mathrm{ref}}} a(\ell) \log \ell \le C_R E_p$. □

Proof of 35.94

Proof. Put $n = k + m - 1$, $T = T_*$, $P = P_*$, $L = L_x$ and $Z_0 = fnL$. Both signature indices are positive and $f > 2$. The size, level and flat bounds in 35.20, 35.21 and 35.4 give

$$\begin{gathered} u_1 \text{ decreasing with wall gap } \ge 8fn + 10, \qquad \log \|u_1\|_1 \le 10L, \quad \log M \le 30L, \quad \log(fn) \le L, \\ \log[E : \mathbf{Q}] + \log \mathrm{rd}(E) \le 3L, \qquad \log[F' : \mathbf{Q}] + \log \mathrm{rd}(F') \le 391 Z_0^2, \qquad |c_1| \le 9nL, \end{gathered} \tag{A.469}$$

where c_1 is the flat degree and F' is the cover field. Write $S = \|u_1\|_1$, $D_F = [F' : \mathbf{Q}]$, and v for the component cusp rank, and set $m_0 = fn + 1$ and $n_\tau = 1 + 4m_0 - u_{1,\tau,p_{\mathrm{sig}}-1}$ with $S' = \sum_{\tau,k} |u_{1,\tau k} + n_\tau| \le (1+n)S + nf(4m_0 + 1)$. Apply the unconditional short-generator comparison proved in the proof of

Proposition 35.35: on the minimal-level good cover in theta weight $(a, 8m_0)_\theta$, the unconditional definite Hermitian Siegel–Weil, Neumann-series, and thin-slab reproducing-kernel harmonic theta spanning for $8m_0 \geq 4[E:\mathbf{Q}]n+4$ (polynomial-norm constant C_A), Minkowski scaling by $s \in F'$ clearing the reflex-field determinant ideal $\mathfrak{a}(L_n, L_a) = \prod_\tau \mathfrak{I}_\tau(L_n)^{m_\tau}\mathfrak{I}_\tau(L_a)^{-m_\tau}$ (with raw archimedean constant C_B and bad-place denominator clearing N_a bounded via C_C, giving $C_G = C_A + C_B + C_C + 391$), and twisting to row weight $(u_1, 0)_{\mathrm{row}}$ by $t_1 = K_{\mathrm{key}}^{m_0}$ and the flat section t_2 of row $(-n_\tau)_\tau$ (with exact key-section norm $(2\pi)^{2m_0 fn}$ and bad-prime rounding loss $\sum_{\ell\in\mathcal{B}} \log \ell \leq 37L_x$ by Lemmas 35.10 and 35.21) bound the three log norm costs by

$$\begin{aligned} &C_G Z_0^2 S', \qquad 2m_0 fn \log(2\pi), \\ &(|c_1| + q_{\mathrm{sig}} \log \mathrm{rd}(E)) \sum_\tau |n_\tau| + \sum_{\ell\in\mathcal{B}} \log \ell + \tfrac{1}{2}\log \mathrm{rd}(F') + 1. \end{aligned} \tag{A.470}$$

Combining (A.469) with Lemma 35.36 over F' and the component Bergman rank bound $\log v \leq C_b Z_0^3$ ($C_b = 4000$) of Lemma 35.22 (at weight $w = 2[E:\mathbf{Q}]S+1$, taking $L' = 11L$) produces an integral generating frame of size at most $2D_F v$ and Petersson log radius

$$c_s = C_g(Z_0^3 S + Z_0^5), \qquad C_g = 2000 + C_b + C_2, \qquad C_2 = 11C_G + 537.$$

The metric shifted by this log radius has short integral generators of norm at most one. With the conjugation-compatible Petersson metric these give $h_Z + c_s \geq 0$ by the height comparison on the image.
Lean: `GlobalTheory.CovGmin.minimal_level_cover_is_good` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.HodgeBundleWeights.ExistsGoodCoverAtMinimalLevel`, lines 20–57)
Lean: `GlobalTheory.spanShortSize` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ThetaWeightAndFlatExponentSizeBound`, lines 32–66)
Lean: `GlobalTheory.ShortSpanComp.short_span_comp_mem` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.HodgeBundleWeights.ShortSpanCompositionClosed`, lines 24–154)
Lean: `GlobalTheory.ShortGenComp.short_generator_comparison` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.HodgeBundleWeights.ShortGeneratorNormComparison`, lines 23–102)
Lean: `GlobalTheory.CovHt.ShortPf.weight_rows` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ShortOfInputs`, lines 112–201)
Lean: `GlobalTheory.CovHt.ShortPf.cover_height_short_section_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ExistsShortSectionCoverHeight`, lines 21–47)

Put $J_{\mathrm{bud}} = 6d_{\mathrm{fl}} + 10 + T(z+2\rho)$. The divided weight and size estimates give $T \leq b$, $TS \leq 8fnb$ (since $|u_{1,\tau,k}| \leq 8b/T$ for each of the fn entries), $d_{\mathrm{fl}}^2 P \leq T$ and $d_{\mathrm{fl}}\rho P \leq 6n^2$. Expanding z gives $PJ_{\mathrm{bud}} \leq 23Pd_{\mathrm{fl}}^2 + 9PTd_{\mathrm{fl}}\rho \leq 77n^2 T$. Since $Z_0^3 n^3, Z_0^5 n^2 \leq P$,

$$(Z_0^3 S + Z_0^5) J_{\mathrm{bud}} \leq 77(8fb + fb) = 693fb \leq 693E_p. \tag{A.471}$$

By 35.21, $5^4 \mid M$ when $e \geq 4$, and the complementary halves give the linear level. Choose f_0 as follows. Fix classes $x_1, \ldots, x_{p_{\mathrm{sig}}+q_{\mathrm{sig}}}$ of the lattice that generate it as an $\mathcal{O}_E$-module modulo 5. Take a geometric point over the generic point of Z with its full level-5^4 structure. A nonzero section of the principal polarisation line does not vanish at every point of a coset of the 5-torsion, so for each b some class $x_b + 5y_b$, with y_b in the lattice modulo 5, has nonzero level-25 null value at that point. The shifted classes still generate the lattice modulo 5, and the rule f_0 of these classes has nonzero generic evaluation on Z. With a, G, ϕ, P_0, B_0' as in 35.46, put $\beta = P_0 + B_0'$. In the thin case $3(P_0 - \phi)/625 \leq \beta - \phi/625$, and in the thick case $3P_0/625 \leq \beta - B_0'$. Hence 35.46 gives

$$a^3 \leq e^\beta G, \qquad a \leq e^\beta \tag{A.472}$$

at every frame on Z. Choose the key-power twisting section K_{key}^{3n} of central parameter $-12n$ (the $3n$-th power of the weight-4 unit key section K_{key} of Lemma 35.4, with exact complex norm $(2\pi)^{6fn^2}$ and nonnegative finite-place lower valuations) and the constant flat section S_Z of row $-12n$ at each embedding, supported on Z. Their point norms are constant in the frame on Z, and their generic evaluations are invertible there. The section

$$\theta_Z = (f_0^2)^{12} K_{\mathrm{key}}^{3n} S_Z$$

has weight $24n\eta_H = 8(u_1 - u')$, nonzero generic evaluation, and squared point norm at most $e^{8\beta} G^8 |K_{\mathrm{key}}^{3n} S_Z|^2$, since $(a^3)^8 = a^{24}$. Bounded multiplication in this weight sends cusp sections to cusp sections.

At a finite place λ, let e_0 be the ramification index of the test trait and $\phi_{\mathrm{fin}} \geq 0$ its degeneration parameter. The chosen tuple spans the full level module, and the trait and its integral frame realise the phase ϕ_{fin}. Apply Lemma 35.47 with the valuation taking value one on a trait uniformiser. Division by e_0 gives $\mathrm{ord}(f_0) \geq \phi_{\mathrm{fin}}/(625e_0)$ in the coefficient-place normalisation. The determinant twist has nonnegative lower order, and the flat twist has lower order $-8e_\lambda$ for its finite allowance e_λ. Orders add, and $24/625 \geq 8/2500$, so

$$\mathrm{ord}(\theta_Z) \geq 8\phi_{\mathrm{fin}}/(2500e_0) - 8e_\lambda. \tag{A.473}$$

The cost of the finite allowances and the archimedean bound before correcting the reference exponent is bounded by

$$\beta/2 + (3n)(2fn\log(2\pi))/8 - \deg(S_Z)/8.$$

The flat degree and size bounds of 35.4 and 35.20 bound the last two terms together by $10P$. Thus $c_{\mathrm{raw}} = \beta/2 + 10P$ is a common bound before correction.

Proposition 35.93(2) gives a finite set $\mathcal{S}_{\mathrm{ref}}$ of rational primes and nonnegative $a(q)$, zero off that set, such that

$$d_q(u_1) \leq a(q), \qquad \sum_{q\in\mathcal{S}_{\mathrm{ref}}} a(q)\log q \leq s_{\mathrm{ref}}, \qquad Ts_{\mathrm{ref}} \leq C_R E_p.$$

For the prime weights displayed in Lemma 35.48, one may take $s_{\mathrm{ref}} = \sum_q a(q)\log q$ in the two prime-sum inequalities. Here $d_q(u_1)$ is the supremum over the coefficient places above q, complete discrete valuation traits above those places, and algebraic integral frames on them with phase ϕ, of the local section depth in weight u_1 minus $\phi/(2500v(q))$. The depth uses evaluation valuations divided by the trait valuation $v(q)$. Take $e'_\lambda = e_\lambda + \sum_{q\in\mathcal{S}_{\mathrm{ref}}} v_\lambda(q)a(q)$. At each $\lambda \mid q$, $e'_\lambda \geq e_\lambda + v_\lambda(q)a(q)$. The identity $\sum_{\lambda|q} v_\lambda(q)\log N\lambda = [F' : \mathbf{Q}]\log q$ gives an added cost at most s_{ref} in the normalised degree.

For a record valuation v over $\lambda \mid q$, extend its integral frame to a trait with phase ϕ_{fin} and ramification index e_0, preserving the normalised valuation. If a cusp section on Z has finite v-value, its evaluation on the lifted frame is nonzero and the generic point is on Z. Let d_v be the infimum of the valuations of the integral cusp sections of target weight u_1 on Z. The reference bound and restriction to Z give

$$d_v \leq B_v = \phi_{\mathrm{fin}}/(2500e_0) + v_\lambda(q)a(q).$$

Thus for every coefficient place and every prescribed valuation there, if some cusp section on Z of any weight has finite valuation, an integral cusp section on Z of weight u_1 has finite valuation. Applying (A.473) on the same trait gives $v(\theta_Z) \geq 8(B_v - e'_\lambda)$, the order comparison used in multiplication. The corrected cost bound $c_{\mathrm{mult}} = c_{\mathrm{raw}} + s_{\mathrm{ref}}$ satisfies

$$Tc_{\mathrm{mult}} \leq (C_\beta + 10 + C_R)E_p, \qquad C_\beta = 200000 + 835C_{\mathrm{peak}}, \tag{A.474}$$

by (35.13) and (A.462).

Lean: `GlobalTheory.T.eta_bound_length_unit_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RunParamsStatement`, lines 11–12)
Lean: `GlobalTheory.T.bergman_kernel_thin_part_estimate_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.BergmanKernelThinPartEstimate`, lines 58–59)
Lean: `GlobalTheory.T.exp_neg_B_le_bergOn_of_la1sq_bound` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.BergmanKernelThickPartEstimate`, lines 87–88)
Lean: `GlobalTheory.CovHt.KeySection.theta_function_le_bergman_kernel_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ExistsSectionBoundedByBergmanKernel`, lines 24–73)
Lean: `GlobalTheory.CovHt.KeySection.exists_section` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsCarryingSectionOrders`, lines 126–266)
Lean: `GlobalTheory.CovHt.KeySection.cover_height_section_bound_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ExistsSectionBoundedByBergmanKernel`, lines 75–116)
Lean: `GlobalTheory.CovMult.multC_of_refW_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.MultCOfRefW`, lines 21–29)

For u' let $g = h_0/(4T)$, where h_0 is from (A.434). Starting from any smaller-block index, take the adjacent step towards the comparison index. The adjacent-difference estimate with the wall correction gives a gap at least g on that step. The rest of the row is monotone, so this is a gap in the sense of (8.26a). The absolute entries are at most $8b/T$. This bound and $fn^2 \leq P \leq h_0$ give $1 \leq B(u', g) \leq 132b$. Put $w_\tau = a_\tau = \lambda_{\tau,k-2}$,

the anchor of the undivided row. The divided-row spread is at most $3w_\tau/(2T)$ and the m node entries have absolute value at least $w_\tau/(4T)$. Since $q_{\mathrm{sig}} \le m$,

$$q_{\mathrm{sig}}\,\mathrm{spr}(u') \le 6\|u'\|_1, \qquad \log B + \ell(D) + 6 \le 100L. \tag{A.475}$$

For the second inequality use the period identity and $|\log \Omega| \le 3L$, $0 \le \log \mathrm{rd}(E) \le 2L$, which give $|L_E| \le 7L$.

At most four primes divide M'. By (8.26f) the primes up to B cost at most $\log B + 3$ in the reciprocal-prime sum. There are at most $2 + 4\log \mathrm{rd}(E) + \#\{\ell : \ell \mid M'\}$ further primes, each costing at most one. Including the different gives

$$\sum_{\ell \in S_B} h_E(\ell)\log \ell \le \log B + 20 + 44 \log \mathrm{rd}(E) \le 100L. \tag{A.476}$$

Put $H_{\mathrm{loc}} = \|u'\|_1 \sum_{\ell \in S_B} h_E(\ell)\log \ell$. Equations (A.463), (A.475) and (A.476) give

$$T H_{\mathrm{loc}},\ T\mathcal{E}_{\ell(D)}(u', g) \le 83400 E_p. \tag{A.477}$$

Lean: `GlobalTheory.MeanTop.spread_le_at` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.SpreadLeAt`, lines 10–37)
Lean: `GlobalTheory.CovHt.BudgetPf.sizes` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.BudgetOfSizeAndWeight`, lines 151–244)
Lean: `GlobalTheory.CovHt.BudgetPf.budgetAt` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.BudgetOfSizeAndWeight`, lines 246–313)

The family has $m^2 \le k$. Apply 35.26 to u' and set

$$c_c = c_{\mathrm{mult}} + H_{\mathrm{loc}}, \qquad X = C_{\mathrm{deep}}\mathcal{E}_{\ell(D)}(u', g). \tag{8.28e}$$

The resulting s_Z have mean at least $-X$. For a section f_1 of row weight $(u, 0)_{\mathrm{row}}$, let v be a record frame valuation over $\lambda \mid \ell$ coming from an algebraic integral frame y_0 on a trait above λ. Lift y_0 over a completed algebraic closure C with valuation ring $\mathcal{O}_C$ and compare its integral Lie eigenbases against the Hodge–Tate frame y over the same prime ℓ (whose scalar entry is a unit and has additive valuation 0), writing w for the resulting Hodge–Tate valuation. At each embedding τ, the integral record eigenlattice is contained in the Hodge–Tate eigenlattice; triangularising the transition between the two flags yields an upper-triangular Borel matrix g_τ whose diagonal entries $t_{\tau,i} = g_{\tau,ii}$ satisfy $|t_{\tau,i}| \ge 1$. Conversely, since $E/\mathbf{Q}$ is Galois, the image of the different of E under τ contains an element $\delta_\tau \in \mathcal{O}_C$ with $\delta_\tau^{[E:\mathbf{Q}]} \mid \ell^{v_\ell(|\operatorname{disc} E|)}$, so $v_\ell(\delta_\tau) \le v_\ell(|\operatorname{disc} E|)/[E : \mathbf{Q}]$. The different annihilates the cokernel of $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathcal{O}_C \to \prod_\tau \mathcal{O}_C$, so δ_τ times the τ-projection of a vector of the integral lattice lies in that lattice. After a finite extension of the trait the lifted point has a split semiabelian uniformisation, and Lemma 35.24(3), applied to the ℓ-divisible group of the Raynaud extension and its dual, shows that every $c \in \mathcal{O}_C$ with $(\ell - 1)v_\ell(c) \ge 1$ multiplies each invariant differential into the Hodge–Tate span. This holds at every prime ℓ, including $\ell = 2$. Multiplying each Hodge–Tate eigenbasis vector by $\ell^{1/(\ell-1)}\delta_\tau$ therefore lands in the integral record eigenlattice, bounding the upper diagonal discrepancy by $|\log|t_s|/\log|\ell|| \le h_E(\ell) = \frac{1}{\ell-1} + \frac{40 v_\ell(|\operatorname{disc} E|)}{[E:\mathbf{Q}]}$ for every diagonal entry t_s (with unit scalar factor). Because a weight-$(u, 0)_{\mathrm{row}}$ section transforms under the Borel transition by the diagonal character of absolute exponent sum at most $\|u\|_1$, passing from v to w yields

$$w(f_1) \le v(f_1) + v_\lambda(\ell) h_E(\ell)\|u\|_1. \tag{A.478}$$

Lean: `GlobalTheory.CovHt.FrameCmp.exists_frBor` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.SumAbsExOfLe`, lines 278–344)
Lean: `GlobalTheory.CovHt.FrameCmp.val_le` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.SumAbsExOfLe`, lines 346–372)
Lean: `GlobalTheory.CovHt.FrameCmp.htCmp` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.SumAbsExOfLe`, lines 378–387)

In degree $8r'$ balance the corrected finite exponents and the infinite weight of $\theta_Z^{r'}$ by a scalar. The factor $G(y)^{8r'}$ in the squared invariant point norm $|\theta_Z(y)^{r'}|^2 \le e^{8r'\beta} G(y)^{8r'} |K_{\mathrm{key}}^{3n} S_Z|^{2r'}$ cancels identically when passing to the Fubini–Study metric on the weight-u_1 image line A: because A carries the Fubini–Study metric induced by a Gram-orthonormal basis of the weight-u_1 cusp space on Z, the squared Fubini–Study

norm of a section at y divides its squared invariant point norm by $G(y)$, thus for a source section s of weight $8r'u'$, its product $\theta_Z^{r'}s$ has weight $8r'u_1$, and

$$\|\theta_Z^{r'}s\|_{\mathrm{FS}}^2 = \frac{\|\theta_Z^{r'}s\|_{\mathrm{inv}}^2}{G(y)^{8r'}} \le e^{8r'\beta}|K_{\mathrm{key}}^{3n}S_Z|^{2r'}\|s\|_{\mathrm{inv}}^2.$$

The logarithmic norm cost per unit target degree is therefore $\beta/2$, together with the displayed constant twisting cost. Its remaining infinite log bound has an additive constant depending on the fixed component and divisor, uniform in r'. Multiply the scalar by $\prod_{\ell\in S_B}\ell^{\lceil 8r'h_E(\ell)\|u'\|_1\rceil}$ to clear the extra denominators in (A.478). The additional log cost is at most $8r'H_{\mathrm{loc}}+\sum_{\ell\in S_B}\log\ell$. Denote the resulting scalar by $\alpha_{8r'}$. For any $\eta>0$, the fixed terms divided by $8r'$ are at most η for large r'. Multiplication by $\alpha_{8r'}\theta_Z^{r'}$ then maps mixed small sections of u' at c_s to Fubini–Study small sections of u_1 at $t=c_s+c_c+\eta$. It is injective by generic evaluation.

The base-point condition and the Gram norm give the image model of degree Δ_Z. Its shifted Fubini–Study line is nef by the integral generator bound at c_s. Apply the assumed section-injection comparison premise and the arithmetic Hilbert–Samuel formula (35.5) of Definition 35.5. The self-intersection enters divided by $[F':\mathbf{Q}](d_{\mathrm{fl}}+1)$, since the arithmetic-volume normalisation uses $(d_{\mathrm{fl}}+1)!$ and the count uses $d_{\mathrm{fl}}!$. The Chow-height comparison bounds this quotient by $\Delta_Z(h_Z+t)$. By $s_Z\le\nu_Z(u';B)$, the lower count on u' gives

$$\max(0,\Delta'_Z(s_Z+c_s))\le\Delta_Z(h_Z+c_s+c_c+\eta). \tag{8.28f}$$

Lean: `GlobalTheory.CovMult.exists_bound` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.CarryingSectionIntegralFrame`, lines 145–170)
Lean: `GlobalTheory.CovMult.mult_of` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.CarryingSectionIntegralFrame`, lines 172–193)
Lean: `SelfIntersectionVolumeEstimates.ImageNefG.exists_nef_model_of_image` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.HodgeBundleWeights.ExistsNefModelOfImage`, lines 13–41)
Lean: `SelfIntersectionVolumeEstimates.VolUpG.volume_line_bundle_le` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.HodgeBundleWeights.VolumeLineBundleLe`, lines 11–22)

Let $\eta\downarrow 0$ and use (35.12). Equations (A.471), (A.474) and (A.477) give $C_{\mathrm{ht}}=693C_g+C_\beta+10+C_R+83400(1+C_{\mathrm{deep}})$. The maximum of the input thresholds works for all the covers and weights. □

Proof of 35.95

Proof. Put $T=T_*$, $w_Z=\mathcal{B}\Delta_Z$, $a=T-3d_{\mathrm{fl}}-4\ge 0$, and $L_A=A\log\max(A,1)$. Use $\log y\le y-1$ at $y=(2+t)/\max(A,1)$ to get

$$A\log(2+t)\le t+L_A+2\qquad(t\ge 0). \tag{A.479}$$

Since $|h_Z|\le h_Z+2c_s$, (8.29b) gives

$$d_Z\ge w_Z\big(a(h_Z+c_s)-N_0\big),\qquad N_0=ac_s+2c_s(3d_{\mathrm{fl}}+4)+L_A+2+R. \tag{A.480}$$

By 35.94 the nonnegative h_Z+c_s have mean at least $\bar m_Z=\max(0,(1-\rho)c_s-X-c_c)$. By (35.12), $w_Z\ge\mathcal{B}\Delta_*$ and $(1-\rho)\sum_Z w_Z\le\#\mathrm{Comp}(Y)\mathcal{B}\Delta_*$. Thus

$$\sum_Z d_Z\ge\left(\sum_Z w_Z\right)\big(a(1-\rho)\bar m_Z-N_0\big). \tag{A.481}$$

Use $(1-\rho)\bar m_Z\ge(1-2\rho)c_s-X-c_c$ and $a\le T$ to bound the bracket below by $-H$, where

$$H=T(X+c_c)+c_s(6d_{\mathrm{fl}}+10+T(z+2\rho))+L_A+R+4. \tag{A.482}$$

Equations (8.29a), (8.29b) and (8.28d) give $0<\sum_Z w_Z\le\sum_Z r_Z$ and $H\le(3C_{\mathrm{ht}}+C_{\mathrm{HS}}+4)E_p$. Applying (8.29c) (where d_Z is the unsquared normalised Arakelov degree and $\mu_Y(u_M)=2\widehat{\deg}/\operatorname{rk}$ doubles both finite and archimedean terms) yields $\mu_Y(u_M)\ge -2H$.

For the nonterminal descent choose a good cover. Let A_- be its nonzero lattice at shift $-c_0$ and B its deep type lattice at shift zero. By 35.4 and (8.29d),

$$\mu(A_-)\ge -M_{p,i}-(C_F+C_U)E_p. \tag{A.483}$$

Put $\mathcal{K} = P_*\kappa$, so by 35.20

$$0 < b \le E_p, \qquad b \le \mathcal{K}, \qquad |M_{p,i}| \le \mathcal{K}. \tag{A.484}$$

Let S be the horizontal subgroup of the level automorphisms and put $s = |S|$. To separate the finitely many S-translates of a generic point on each component of Y inside weight $c_0\eta_H$ (where $c_0 \ge \max(r + \binom{fPQ+1}{2} + 2,\ 8fr+4)$ requires neither a $2^{1/e}$ factor nor any $|S|$ factor, and for $f \ge 8$, $r \ge 100$, $P, Q \ge 1$ the first branch $r + \binom{fPQ+1}{2} + 2$ already exceeds $8fr + 10$), proceed in five steps. First, let $\Gamma \le S$ be the finite stabiliser of the component Z acting on the moduli open $V_0 \subset X_0$, let X' be a normal projective Γ-equivariant compactification of V_0 dominating the toroidal compactification X_0 via a birational morphism $\varphi : X' \to X_0$ isomorphic over V_0 (for instance, the normalisation of the closure of the Γ-orbit embedding $V_0 \to \prod_{\gamma\in\Gamma} X_0$), and let $q : X' \to W = X'/\Gamma$ be the finite quotient morphism. Pulling back the universal semiabelian scheme $G \to X_0$ to $G' \to X'$ and extending the Γ-action uniquely across X' by semiabelian homomorphism extension over normal bases (Definition 35.5(13)), at every point $x \in X'$ (interior or boundary) any stabiliser element $\gamma \in \Gamma_x$ acts unipotently on the $\mathfrak{I}$-torsion for a neat prime ideal $\mathfrak{I} \mid g_*$ dividing M' with $g_* \in \{g_0, g_1\}$ ($g_* > r+1$), hence acts trivially on the semiabelian fibre G'_x and therefore with character $+1$ on the fibre of $\varphi^*\delta = \det(\mathrm{Lie}(G'/X')_{\mathrm{CM}})^\vee$. By Kempf's finite-group quotient descent, $\varphi^*\delta$ descends as an actual Cartier line bundle N on the normal projective quotient $W = X'/\Gamma$ satisfying $q^*N = \varphi^*\delta$ (with no $|S|$ denominator). Second, put $m_{\mathrm{AS}} = \binom{fPQ+1}{2} + 1$, $t = c_0 - m_{\mathrm{AS}} \ge r$, $\kappa_{\log} = \omega_{X_0}(\log D)$, $N' = t\delta - \kappa_{\log}$, and $B_{\mathrm{div}} = m_{\mathrm{AS}}\delta + N'$. By the logarithmic Kodaira–Spencer isomorphism, the pullback $\pi_0^*\kappa_{\log}$ has automorphic weight $(+Q, \dots, +Q, -P, \dots, -P)$ of cross-wall jump $P+Q = r \le t$, so $\pi_0^*(t\delta - \kappa_{\log})$ has nonincreasing weight with nonnegative cross-wall jump $t - r \ge 0$ and is nef by semiabelian Lie seminegativity (Lemma 35.19); surjectivity of $\pi_0 : Y_0 \to X_0$ descends nefness of N' to X_0. Third, resolve W and the rational quotient map to obtain smooth projective varieties M, W', a projective birational morphism $\sigma : M \to X_0$, and a projective morphism $f : M \to W'$ such that f is finite étale over a nonempty Zariski open $U \subset W'$ containing the generic image w, with $N_{W'}$ nef and big on W' and $f^*N_{W'} = \sigma^*\delta$. Fourth, apply Kollár's one-point theorem for a nef and big rational divisor (*Singularities of Pairs*, Theorem 6.4) on $W'_{\mathbf{C}}$ at a closed point w lying over the generic point of W', given by an embedding of the function field of W' into $\mathbf{C}$, using the class $(m_{\mathrm{AS}} - \frac{1}{2})N_{W'}$. A positive multiple of $N_{W'}$ is ample plus an effective divisor over the coefficient field whose support avoids w. Thus $N^e_{W'}$ has positive integral intersection with every irreducible e-fold through w. Taking $c_e = \binom{fPQ+1}{2} = m_{\mathrm{AS}} - 1$ gives the strict intersection inequalities and $\sum_{e=1}^{fPQ} e/c_e = 1$. The resulting effective $\mathbf{Q}$-divisor $\Delta_{W'} \equiv (m_{\mathrm{AS}} - \frac{1}{2})N_{W'}$ isolates w in its non-klt locus. The remainder is numerically equivalent to $\frac{1}{2}N_{W'}$, hence nef and big. Pullback along f preserves the isolated non-klt locus along the entire étale fibre $f^{-1}(w)$, and because $\sigma^*B_{\mathrm{div}} - f^*\Delta_{W'} = f^*(m_{\mathrm{AS}}N_{W'} - \Delta_{W'}) + \sigma^*N'$ is globally nef and big on M, global Nadel vanishing applied directly to $K_M + \sigma^*B_{\mathrm{div}}$ on M yields surjectivity $H^0(M, K_M + \sigma^*B_{\mathrm{div}}) \twoheadrightarrow \bigoplus_{z\in f^{-1}(w)} (K_M + \sigma^*B_{\mathrm{div}})|_z$. Fifth, push sections down to X_0 via $\sigma_*\mathcal{O}_M(K_M) = \mathcal{O}_{X_0}(K_{X_0})$ and multiply by the canonical boundary-vanishing section $\theta \in H^0(X_0, \kappa_{\log} - K_{X_0})$ to land in cuspidal weight $(K_{X_0} + B_{\mathrm{div}}) + (\kappa_{\log} - K_{X_0}) = (m_{\mathrm{AS}} + t)\delta = c_0\eta_H$ (scalar exponent 0), separating the entire generic S-orbit simultaneously inside shift c_0, with short integral generators supplied by the short-generator comparison of Proposition 35.35 for the nonincreasing weight $c_0\eta_H$ (whose wall gap $c_0 > 8fr + 10$ meets the threshold of Proposition 35.35). Taking a generic linear combination over the cover field F' produces a cusp section u_0 of weight $c_0\eta_H$ and scalar exponent zero whose S-translation matrix has nonzero generic determinant on every component. Expand u_0 in the short integral generators of the component lattices. Fix all coefficients except that of a generator supported on Z_0. The translations by S send Z_0 to at most s components, so the determinant can vary on at most s components. On each of them the translation matrix at the generic point has entries affine in the coefficient. Its determinant is a polynomial of degree at most s and is nonzero at the current coefficient. Thus at most s^2 integers are excluded. Choose a coefficient in $\{0, \dots, s^2\}$ outside this set and repeat for every generator. The resulting section u is integral, and its translation determinant is a nonzero divisor by generic evaluation on the components. If each component has at most N_{gen} generators of Petersson norm at most Λ, and the point norm is bounded by β times the Petersson norm on that component, then every point norm of u is at most $N_{\mathrm{gen}}s^2\beta\Lambda$. The generator families have disjoint component supports, so at most N_{gen} summands contribute at any frame.

Lean: `GlobalTheory.T.orbits_separated_by_regular_cusp` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.HodgeBundleWeights.ExistsCuspRegularLevelDeterminant`, lines 205–206)

Lean: `GlobalTheory.ShortDet.exists_nat_step` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.CardLeOfDetEqZero`, lines 178–221)

Lean: `GlobalTheory.ShortDet.exists_short` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.CardLeOfDetEqZero`, lines 279–337)

Multiplication by the translates of u and projection with the horizontal character give integral maps $\phi_h : A_- \to B$, $\phi_h(f_1) = \sum_{g\in S} \chi(g)^{-1} g(f_1 \cdot (hu))$. The determinant gives joint injectivity. Let B_0 bound the multiplication norm of a translate and put $Z_0 = frL_x$. The short-section estimates (bounding $N_{\mathrm{gen}} s^2 \beta\Lambda$ via Lemma 35.36 and the component Bergman supremum) give $B_0 \le \exp(CZ_0^9)$ with a uniform $C \ge 0$, and the group bound gives $3\log s \le 135 Z_0^2$. At the family threshold $Z_0 \ge 135 + 2C$ and $Z_0^{12} \le b$, so $s^3 B_0^2 \le e^{2b}$. By Cauchy–Schwarz over the s summands in $g \in S$, each coordinate satisfies $\|\phi_h(f_1)\|_\sigma^2 \le s \sum_{g\in S} \|f_1 \cdot (hu)\|_\sigma^2 \le s^2 B_0^2 \|f_1\|_\sigma^2$, and summing over the s elements $h \in S$ gives $\sum_h \|\phi_h(f_1)\|_\sigma^2 \le s^3 B_0^2 \|f_1\|_\sigma^2 \le e^{2b}\|f_1\|_\sigma^2$. The jet bound in the proof of Proposition 35.35 gives component rank at most $(1+z)\mathcal{B}\Delta_Z$ at shift c_0. By (8.29a), $\mathcal{B}\Delta_Z \le r_Z$. The invertible twist in the proof of Lemma 35.4 gives $\sum_Z r_Z = \operatorname{rk} A_-$. Summing the component bounds and using $z\mathcal{K} \le b$ gives $r_Y(G_s(c_0)) \le (1 + b/\mathcal{K}) \operatorname{rk} A_-$. Together with the norm bound this gives

$$\sum_h \|\phi_h f_1\|_\sigma^2 \le e^{2b}\|f_1\|_\sigma^2, \qquad s \operatorname{rk} B \le r_Y(G_s(c_0)) \le (1 + b/\mathcal{K}) \operatorname{rk} A_-. \tag{A.485}$$

Lean: `GlobalTheory.T.exists_short_integral_vector_of_determinant_bound` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.HodgeBundleWeights.ExistsShortIntegralVectorOfDeterminantBound`, lines 126–127)

So the joint map has height at most $2b$. Take the short integral spanning family (v_j) of the full space at shift zero. The minimal-level estimate, in its original theta weight, gives $\|v_j\|_\sigma \le \exp(CZ_0^2\kappa)$ for an absolute $C \ge 0$, including the scalar normalisation. Put $w_j = \sum_{g\in S} \chi(g)^{-1} g v_j$. The level action preserves the lattice and the Petersson norm, and $\chi(g)$ is an integral root of unity, so $w_j \in B$ and $\|w_j\|_\sigma \le s \exp(CZ_0^2\kappa)$. Over F' the same sum acts on the character space as multiplication by s, so the w_j span that space. Since $\log s \le 45 Z_0^2$, $\kappa \ge 1$, $P_* \ge Z_0^{12}$, and eventually $Z_0 \ge C + 45$, their log norms are at most

$$\log s + CZ_0^2\kappa \le (C+45) Z_0^2 \kappa \le Z_0^3 \kappa \le P_* \kappa = \mathcal{K}.$$

Taking a basis among their images in any quotient of B^s bounds its determinant norm by the product of these norms and gives $\mu_{\min}(B^s) \ge -2\mathcal{K}$. Put $r_- = \operatorname{rk} A_-$, $N_+ = s \operatorname{rk} B$. The degree inequality for the image and quotient gives $r_-(\mu(A_-) - 2b) \le N_+ \mu(B) + 2(N_+ - r_-)\mathcal{K}$. With $r_- \le N_+ \le (1 + b/\mathcal{K}) r_-$, for all $t \le \mu(A_-) - 2b$,

$$\mu(B) \ge t - (b/\mathcal{K})(|t| + 2\mathcal{K}). \tag{A.486}$$

Take $t = -M_{p,i} - (C_F + C_U)E_p - 2b$. Equation (A.484) gives $\mu(B) \ge -M_{p,i} - 2(C_F + C_U)E_p - 7b$. We first identify the space to which coefficient descent applies. Let V_{core} be the core type space over F, and let π denote pullback. *Claim 1.* The character space containing B is $F' \cdot \pi(V_{\mathrm{core}})$. Reduce cover-level automorphisms modulo the base half-level to obtain the map θ to the base level group, acting on the automorphic line through the level structures and Lie frames. Let P_0 be the base type subgroup with character λ. The horizontal subgroup is $\theta^{-1}(P_0)$, and its character is the pullback of λ, with the character convention fixed by the displayed projector. In particular it is trivial on the subgroup of automorphisms congruent to the identity modulo the base half-level. This subgroup acts transitively on the geometric fibres of the level map, so an eigenform in B has constant readings on those fibres. Descent of sections identifies such forms with F'-linear combinations of base cusp forms. Pullback intertwines the level actions, and θ maps the horizontal subgroup onto P_0; the remaining eigenrelations therefore select precisely the scalar extension of V_{core}. Conversely, pullbacks of core type forms satisfy all these relations. This proves the claim.

Pullback preserves integrality and is isometric at every archimedean place with the prescribed Petersson normalisation. Thus $\mathcal{O}_{F'} \cdot \pi(\Lambda_{\mathrm{core}}) \subseteq B$, and Claim 1 gives equality of their ranks. To check the trace inclusion, write a form in this common space in a basis of V_{core}. Integrality of its pullback is equivalent to integrality of the corresponding base form after coefficient extension. The latter condition is preserved by $\mathrm{Gal}(F'/F)$, because the base forms and the family of integral frame valuations are defined over F. Consequently, for $a \in \mathcal{O}_{F'}$ and $s \in B$, the coefficientwise sum of the Galois conjugates of as is integral, has coefficients in F, and still lies in the core type space. It belongs to $\pi(\Lambda_{\mathrm{core}})$. Hence $\mathrm{Tr}_{F'/F}(\mathcal{O}_{F'} B) \subseteq \pi(\Lambda_{\mathrm{core}})$. Expressing any $x \in B$ in a local $\mathcal{O}_F$-basis of Λ_{core} therefore puts its F'-coordinates inside the inverse different $\mathfrak{D}_{F'/F}^{-1} \subseteq \mathfrak{D}_{F'/\mathbf{Q}}^{-1}$, so $\mathfrak{D}_{F'/\mathbf{Q}} \cdot B \subseteq \mathcal{O}_{F'} \cdot \pi(\Lambda_{\mathrm{core}})$. Passing to normalised Arakelov degrees shifts the squared-norm slope by at most $2\log N(\mathfrak{D}_{F'/\mathbf{Q}})/[F' : \mathbf{Q}] = 2\log|\operatorname{disc} F'|/[F' : \mathbf{Q}] \le 2b$ (by the cover root-discriminant bound of Lemma 35.21), yielding $\mu_{\mathrm{core}} \ge \mu(B) - 2b$.

Lean: `GlobalTheory.CovTop.RingAt` (`Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol`, lines 116–117)
Lean: `GlobalTheory.CovTop.DescentMaps.covering_descent_maps_compatible_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CoveringDescentMapsCompatible`, lines 27–118)

At the terminal layer $q_{\mathrm{sig}} = 0$, take G to be the subgroup defining the core type character, namely the product of the upper unitriangular subgroups at the selected places above g_0, g_1, acting through the fixed half-level lifts. The Shimura datum reduces to a finite nonempty set of definite unitary lattice classes. This action is free. Indeed, a stabiliser gives a finite-order integral E-linear unitary automorphism whose reduction at a selected place above g_0 is unipotent. Since g_0 splits completely in E and $g_0 > r + 1$, the finite-order argument used above forces that automorphism to be the identity; faithfulness on the half-level then makes the group element the identity. For the terminal groups $\widetilde{H}_a$ used later in the bad-rank estimate, the same argument uses the complementary prime g_{1-a}; at this layer the tildes refer to these base-level subgroups. These are the groups used in the orbit sums. The terminal datum also carries the flat line of Lemma 35.4 with first Chern value $c_{1,\tau} = -\frac{1}{2} p_{\mathrm{sig}} \log \Omega$ by (35.3). Let a_τ be the anchor of 35.4, $\lambda_{\tau a}$ the unshifted entries, n_{col} the column count, ν_τ the row partition with entry sum $|\nu_\tau| = \sum_a \lambda_{\tau a} - p_{\mathrm{sig}} n_{\mathrm{col}}/2$, and V_τ^{Weyl} the dual Weyl module of ν_τ. Define the definite-side slope bound

$$D_{\mathrm{def}} = \sum_\tau \left(\left(\frac{|\nu_\tau|}{p_{\mathrm{sig}}} + \frac{n_{\mathrm{col}}}{2} \right) c_{1,\tau} - \frac{1}{2} \log \dim V_\tau^{\mathrm{Weyl}} \right) + n_{\mathrm{col}} f p_{\mathrm{sig}} \left(\tfrac{1}{4} \log(2\pi) - \tfrac{1}{2} \log \mathrm{rd}(E) \right).$$

We prove $\mu_{\mathrm{core}} \geq 2 D_{\mathrm{def}}$ directly (with exact error constant $C_0 = 0$). At this layer no cover is used, and every lattice below is taken over the core coefficient field F. The pinned field F contains the Weber values of the quadratic host at a multiple of the terminal level, so, as in Lemma 35.21, it is relatively algebraically closed in the function field of every terminal component. Hence each 0-dimensional component is an F-point, and the character χ of G takes values in $\mathcal{O}_F^\times$. On each 0-dimensional definite component Z, evaluating forms at the unique CM abelian variety P attached to Z identifies the integral component fibre lattice V_Z of $\mathcal{L}_{\theta,P}^{\otimes n_{\mathrm{col}}} \otimes \bigotimes_{\tau \in \Psi} V_\tau^{\mathrm{Weyl}}(\omega_\tau)$ over F (where $\omega_\tau = V_\tau^\vee$ is the rank-p_{sig} cotangent eigenlattice dual to V_τ, $\omega_{\bar{\tau}} = 0$ since $q_{\mathrm{sig}} = 0$, V_τ^{Weyl} is the dual Weyl representation ρ_{ν_τ} of the shifted nonnegative partition ν_τ, obtained by factoring out the determinant shift $(\det \omega_\tau)^{\nu_{\tau,p_{\mathrm{sig}}}}$ when needed, and $\mathcal{L}_{\theta,P}$ is the scalar determinant line of central parameter 1/2) with the adelic lattice whose local finite-place transition matrices are $a_w^{n_{\mathrm{col}}} \bigotimes_{\tau \in \Psi} \rho_{\nu_\tau}(B_{w,\tau})$ (where $a_w \in F_w^\times$ is the local transition scalar of the rigidified theta generator and $B_{w,\tau} \in \mathrm{GL}_{p_{\mathrm{sig}}}(F_w)$ transforms the integral cotangent basis at w to the global basis of ω_τ) and whose global-to-orthonormal archimedean matrices are $M_\sigma = N_\sigma^{-n_{\mathrm{col}}/2} \bigotimes_{\tau \in \Psi} \big(U_{\nu_\tau} \rho_{\nu_\tau}(X_{\sigma,\tau}) \big)$ (so the Hermitian Gram matrix of the global basis at σ is $G_\sigma = M_\sigma^* M_\sigma$, where $N_\sigma^{-1} > 0$ is the archimedean squared norm of the inverse theta generator, $X_{\sigma,\tau} \in \mathrm{GL}_{p_{\mathrm{sig}}}(\mathbf{C})$ expresses the global basis of ω_τ in an orthonormal basis at σ, and U_{ν_τ} expresses the canonical integral bideterminant basis $\{v_T\}$ of V_τ^{Weyl} in an orthonormal basis for the Schur-normalised $U(p_{\mathrm{sig}})$-invariant L^2-inner product $\langle f_1, f_2 \rangle_{\mathrm{Sch}} = (\dim V_\tau^{\mathrm{Weyl}}) \int_{U(p_{\mathrm{sig}})} f_1(g) \overline{f_2(g)}\, d\mu(g)$, giving $U_{\nu_\tau}^* U_{\nu_\tau} = G_{\mathrm{Sch}}$ and $\|v_{\mathrm{hw}}\|_{\mathrm{Sch}} = 1$). Because every standard bideterminant minor on $U(p_{\mathrm{sig}})$ has modulus at most 1, each canonical basis vector satisfies $\|v_T\|_{\mathrm{Sch}}^2 \leq \dim V_\tau^{\mathrm{Weyl}}$, so Hadamard's inequality on $G_{\mathrm{Sch}} = U_{\nu_\tau}^* U_{\nu_\tau}$ yields $|\det U_{\nu_\tau}|^2 = \det G_{\mathrm{Sch}} \leq (\dim V_\tau^{\mathrm{Weyl}})^{\dim V_\tau^{\mathrm{Weyl}}}$. Because the flat section of $\det(V_\tau)$ has Arakelov degree $-c_{1,\tau}$ by Lemma 35.4, dualisation gives $\widehat{\deg}(\det \omega_\tau) = \widehat{\deg}(\det(V_\tau^\vee)) = +c_{1,\tau} = -\frac{1}{2} p_{\mathrm{sig}} \log \Omega$. For the scalar determinant line $\mathcal{L}_{\theta,P}$, comparing its eighth power against the canonical weight-4 theta-key section K_{key} of Lemma 35.4 (whose constant complex point norm equals $C_\sigma = (2\pi)^{2 f p_{\mathrm{sig}}}$ and whose finite-place conjugate-different defect sums to at most $4 f p_{\mathrm{sig}} \log \mathrm{rd}(E)$) and dividing by 8 yields $\widehat{\deg}(\mathcal{L}_{\theta,P}) \geq \frac{1}{2} \sum_{\tau \in \Psi} c_{1,\tau} + f p_{\mathrm{sig}} (\frac{1}{4} \log(2\pi) - \frac{1}{2} \log \mathrm{rd}(E))$. In the adelic degree formula $\widehat{\deg}(V_Z) = \widehat{\deg}_{\mathrm{fin}}(V_Z) - \frac{1}{[F:\mathbf{Q}]} \sum_\sigma \log |\det M_\sigma|$ (where $-\log(N_\sigma^{-n_{\mathrm{col}}/2}) = +\frac{1}{2} n_{\mathrm{col}} \log N_\sigma$ matches the theta line degree and $-\log |\det U_{\nu_\tau}| / \dim V_\tau^{\mathrm{Weyl}} \geq -\frac{1}{2} \log \dim V_\tau^{\mathrm{Weyl}}$), applying the representation-theoretic determinant identity $(\det \rho_{\nu_\tau}(g))^{p_{\mathrm{sig}}} = (\det g)^{|\nu_\tau| \dim V_\tau^{\mathrm{Weyl}}}$ and Kronecker determinant multiplicativity evaluates directly to

$$\widehat{\mu}(V_Z) \geq n_{\mathrm{col}}\, \widehat{\deg}(\mathcal{L}_{\theta,P}) + \sum_{\tau \in \Psi} \left(\frac{|\nu_\tau|}{p_{\mathrm{sig}}} \widehat{\deg}(\det \omega_\tau) - \frac{1}{2} \log \dim V_\tau^{\mathrm{Weyl}} \right) \geq D_{\mathrm{def}}.$$

Finally, because the finite level-transition group G acts freely on the finite nonempty set of definite components (with N_{orb} disjoint orbits, so $\#\,\mathrm{Comp} = |G| N_{\mathrm{orb}}$), choosing one representative component Z_o in

each G-orbit o defines a full-rank integral norm-nonincreasing map $\Phi : \bigoplus_o V_{Z_o} \hookrightarrow \Lambda_{\mathrm{core}} \subset V^\chi$ via the purely integral orbit sum $\Phi((v_o)_o) = \sum_o \sum_{g\in G} \chi(g)^{-1} g \cdot v_o$: each root of unity $\chi(g)^{-1} \in \mathcal{O}_F^\times$ is integral of modulus 1 at every embedding $\sigma : F \hookrightarrow \mathbf{C}$, so Φ preserves integrality, and on each component $Z = g \cdot Z_o$ only one summand survives, giving pointwise invariant norm $\|v_o\|_{Z_o,\sigma}$. Integrating against the probability-normalized Petersson measure $(\#\,\mathrm{Comp})^{-1} = (|G|N_{\mathrm{orb}})^{-1}$ cancels $|G|$ and leaves $N_{\mathrm{orb}}^{-1} \le 1$, giving $\|\Phi(v)\|_\sigma^2 = N_{\mathrm{orb}}^{-1} \sum_o \|v_o\|_{Z_o,\sigma}^2 \le \sum_o \|v_o\|_{Z_o,\sigma}^2$. Since Φ is a full-rank integral map that does not increase archimedean norms, averaging the component lower bounds $\widehat{\mu}(V_{Z_o}) \ge D_{\mathrm{def}}$ and doubling for the squared-norm convention yields $\mu_{\mathrm{core}} = 2\widehat{\mu}(\Lambda_{\mathrm{core}}) \ge 2 \min_o \widehat{\mu}(V_{Z_o}) \ge 2D_{\mathrm{def}}$ with $C_0 = 0$ exact. Substituting $c_{1,\tau} = -\frac{1}{2} p_{\mathrm{sig}} \log \Omega$ and $|\nu_\tau|/p_{\mathrm{sig}} + n_{\mathrm{col}}/2 = p_{\mathrm{sig}}^{-1} \sum_a \lambda_{\tau a}$ gives

$$2D_{\mathrm{def}} = -\log\Omega \sum_{\tau,a} \lambda_{\tau a} - \sum_\tau \log\dim V_\tau^{\mathrm{Weyl}} + n_{\mathrm{col}} f p_{\mathrm{sig}} \big(\tfrac{1}{2}\log(2\pi) - \log \mathrm{rd}(E)\big).$$

Splitting $\lambda_{\tau a} = a_\tau + (\lambda_{\tau a} - a_\tau)$ bounds the first term below by $-p_{\mathrm{sig}} \log\Omega \sum_\tau a_\tau - |\log\Omega| \sum_{\tau,a} |\lambda_{\tau a} - a_\tau|$; dropping the positive term $\frac{1}{2} n_{\mathrm{col}} f p_{\mathrm{sig}} \log(2\pi) > 0$ and bounding $-\log \mathrm{rd}(E) \ge -|\log \mathrm{rd}(E)|$ immediately produces

$$\begin{aligned} 2D_{\mathrm{def}} \ge\ & -p_{\mathrm{sig}} \log\Omega \sum_\tau a_\tau - |\log\Omega| \sum_{\tau,a} |\lambda_{\tau a} - a_\tau| \\ & - \sum_\tau \log\dim V_\tau^{\mathrm{Weyl}} - n_{\mathrm{col}} f p_{\mathrm{sig}} |\log \mathrm{rd}(E)|. \end{aligned} \tag{A.487}$$

The scale width is $s_{\mathrm{sc}} = 4k\Delta$, where Δ is the spacing parameter of the point, and $k s_{\mathrm{sc}} \le b$. The first $k-1$ entries in each row differ from the anchor by at most $s_{\mathrm{sc}} + 2r$. Each of the remaining m entries differs by at most $8b$. Since $r^2 L_x \le f r n_V L_x \le b$, $m \ge 2$ and $L_x \ge 1$, this gives

$$\sum_a |\lambda_{\tau a} - a_\tau| \le (k-1)(s_{\mathrm{sc}} + 2r) + 8mb \le 10mb.$$

For the row partition ν_τ take $w_\tau = a_\tau - n_{\mathrm{col}}/2$. Then $\sum_a |\nu_{\tau a} - w_\tau| = \sum_a |\lambda_{\tau a} - a_\tau|$. Each factor in the Weyl dimension product is at most $1 + \sum_a |\nu_{\tau a} - w_\tau|$, since its denominator is at least one and the difference of two entries is at most the sum of their absolute deviations from w_τ. Thus $\dim V_\tau^{\mathrm{Weyl}} \le (1 + 10mb)^{\binom{r}{2}}$. By $11m \le 2fr$, $\log(fr) \le L_x$ and $\log b = L_x$, its logarithm is at most $r^2(1 + 2L_x) \le 3b$. Also $n_{\mathrm{col}} \le n_V$ and (A.418) give $n_{\mathrm{col}} f p_{\mathrm{sig}} |\log \mathrm{rd}(E)| \le 2 f r n_V L_x \le 2b$. The three errors in (A.487) therefore total at most $30 f m b L_x + 3fb + 2b \le 35 E_p$. By (A.419) in Lemma 35.4, $|M_{p,i} - p_{\mathrm{sig}} \log\Omega \sum_\tau a_\tau| \le C_A E_p$ with $C_A = 100$, giving $C_{\mathrm{term}} = C_A + 35 = 135$. Combine the two cases using $b \le E_p$. □

Proof of 35.96

Proof. For large x, $r \ge 100$ and $g_a \ge 4096$ for $a = 0, 1$. Apply 32.53, 32.60, and the multi-place share bound in the proof of Theorem 32.62 (with $C_s = 6$) with the nontrivial additive characters and bad labels at both primes. The local shares u_a satisfy $u_a \le x^{21} e^{-x^{e_z/6}} \le \delta/20$ eventually, hence

$$u_0 + u_1 \le \delta. \tag{A.488}$$

For a nonterminal cover let $\pi : \Gamma \to G_0$ be the level quotient onto the product of the $\mathrm{GL}_r(\mathbf{F}_{g_a})$ at the selected places. It has a section commuting with $\ker \pi$. Let $P \subset G_0$ be the product of upper unitriangular groups with its Whittaker character and $H_a \subset G_0$ be unrestricted at g_a and upper unitriangular elsewhere. Let $\widetilde{P}, \widetilde{H}_a$ be the inverse images. In the cover space V write V_λ for the pulled-back character space of $\widetilde{P}$. Its dimension is the type rank. The bad rank is the dimension cut out in it by the complement of the good idempotent.

For $\widetilde{P}$ (taking either base prime $g_* \in \{g_0, g_1\}$ as the neat unipotent prime) and each $\widetilde{H}_a$ (taking the complementary base prime $g_* = g_{1-a}$ where H_a is upper unitriangular as the neat unipotent prime), use the orbit separation in the proof of 35.95 to choose a section u of shift c_0 whose translation matrix $((gk)u)_{g,k}$ has determinant a non-zero-divisor in the section ring. Write $B_1 = \min_\tau(\lambda_{\tau,p_{\mathrm{sig}}-1} - \lambda_{\tau,p_{\mathrm{sig}}})$ for the layer wall

gap; adding a positive multiple of the Hodge weight $c_0\eta_H$ increases every cross-wall difference by $c_0 > 0$ while leaving within-block differences unchanged, preserving regularity at every positive shift $G_s(c_0)$ and $G_s(2c_0)$. For $\widetilde{H}_a$, set $V = V(G_s(0))$, $V_1 = V(G_s(c_0))$, $V_2 = V(G_s(2c_0))$, let $I = V_1^{\widetilde{H}_a}$ be the $\widetilde{H}_a$-invariant subspace of V_1, and put $n_a = \dim_{F'} I$. First, for each $h \in \widetilde{H}_a$, define $T_h : V \to I$ by $T_h(v) = \sum_{g\in\widetilde{H}_a} g(v \cdot (hu))$. Then $T_k(hv) = T_{h^{-1}k}(v)$, so $v \mapsto \sum_{k\in\widetilde{H}_a} T_k(v) \otimes k$ embeds V $\widetilde{H}_a$-equivariantly into $I \otimes_{F'} F'[\widetilde{H}_a] \cong F'[\widetilde{H}_a]^{\oplus n_a}$ (with any character twist χ preserved by regular-representation embedding); this map is injective because $T_k(v) = 0$ for all $k \in \widetilde{H}_a$ reads $\sum_{g\in\widetilde{H}_a} g(v) \cdot (gk)u = 0$ for all k, forcing $v = 0$ by Cramer's rule since $\det((gk)u)$ is a non-zero-divisor. Second, define $\Psi : \bigoplus_{h\in\widetilde{H}_a} I \to V_2$ by $\Psi((f_h)_{h\in\widetilde{H}_a}) = \sum_{h\in\widetilde{H}_a} f_h \cdot (hu)$. Because every $f_h \in I$ is $\widetilde{H}_a$-invariant, $\Psi(f) = 0$ implies $\sum_{h\in\widetilde{H}_a} f_h \cdot (gh)u = 0$ for all $g \in \widetilde{H}_a$, forcing $f_h = 0$ for all h by the same non-zero-divisor determinant; hence Ψ is injective and $n_a|\widetilde{H}_a| = (\dim_{F'} I)|\widetilde{H}_a| \le \dim_{F'} V_2 = r_Y(G_s(2c_0))$. Similarly, for $\widetilde{P}$ with pulled-back character λ', the $\widetilde{P}$-character-projected map $v \mapsto \big(\sum_{g\in\widetilde{P}} \lambda'(g)^{-1} g(v \cdot (hu))\big)_{h\in\widetilde{P}}$ sends $V(G_s(-c_0))$ into $\bigoplus_{h\in\widetilde{P}} V_\lambda$ injectively by the same Cramer non-zero-divisor argument, yielding $r_Y(G_s(-c_0)) \le |\widetilde{P}| \dim V_\lambda$. For fixed u_1, (8.29a) gives $(1-z)r_Z(G_s(2c_0)) \le \mathcal{B}\Delta_Z \le r_Z(G_s(-c_0))$ on each component Z. Sum over Z and use $z \le 7(1/100 + 6/6400) < 1/3$ from the proof of Lemma 35.20. Then $r_Y(G_s(2c_0)) \le \frac{3}{2} r_Y(G_s(-c_0))$, giving

$$n_a|\widetilde{H}_a| \le r_Y(G_s(2c_0)) \le \tfrac{3}{2} r_Y(G_s(-c_0)) \le \tfrac{3}{2}|\widetilde{P}| \dim V_\lambda. \tag{A.489}$$

In G_0 the bad projector at g_a times the character projector is supported on H_a and has identity coefficient $u_a/|P|$. Lift it by the section and average over $\ker\pi$. The identity coefficient on $\widetilde{H}_a$ is $u_a/|\widetilde{P}|$. The trace on the regular copies and (A.489) bound the rank on V_λ by $\frac{3}{2}u_a \dim V_\lambda$. The projectors commute, so the two ranks bound the loss from the common good part by their sum. At the terminal layer the freely permuted deformation-class summands give regular copies of total dimension $|\widetilde{P}| \dim V_\lambda$. The same trace formula bounds the loss there by $(u_0 + u_1) \dim V_\lambda$. Both bounds imply (8.30b). □

Proof of 35.97

Proof. Write $L_g = L \cap \operatorname{im} e$ and $L_b = L \cap \ker e$, with ranks d_g, d_b. The inclusion $gL \subseteq L_g \oplus L_b$ gives

$$\operatorname{rk} L = d_g + d_b. \tag{A.490}$$

The map $1 - e$ embeds $L/(L_g \oplus L_b)$ in $g^{-1}L_b/L_b$, of order $g^{[K:\mathbf{Q}]d_b}$. Orthogonality and the squared metric then give

$$(d_g + d_b)\mu(L) \le d_g\mu(L_g) + d_b(\mu_{\max}(M) + 2\log g). \tag{A.491}$$

The rank bound yields $d_g > 0$ and

$$d_b \le \frac{3\delta}{1 - 3\delta} d_g \le 4\delta d_g. \tag{A.492}$$

Thus $d_g(\mu(L) - \mu(L_g)) \le 3d_b\mathcal{Q} \le 12\delta d_g\mathcal{Q}$. □

Proof of 35.100

Proof. Put $n = g_0 g_1 p_{\mathrm{de}} p'_{\mathrm{de}}$. Let R/E be the ray class field of $n\mathcal{O}_E$, and write $h = [R : E]$, $h_E = \#\operatorname{Cl}(E)$. Define

$$c = \prod_{i=0}^{1} |\operatorname{GL}_r(\mathbf{F}_{g_i})|^f, \qquad a = \operatorname{lcm}(c, 3^4 5^4).$$

Let $A/\mathbf{Q}(j)$ be the elliptic curve of the field datum. Its j-invariant is that of $\mathbf{C}/\mathcal{O}_K$, and it has good reduction away from d. Put $M_0 = K(j, A[12])$, adjoining the coordinates of all the indicated torsion points. Let V_h be generated over $E(\mu_h)$ by all roots $z^h = y$ with $y \in E^\times$ and (y) an h-th power of a fractional ideal of E. We first show that

$$F \subset M_0 R\, \mathbf{Q}(\mu_a) V_h.$$

The coefficient list contains the embedded host, the ray and elliptic-curve data, roots of unity, the prescribed character values, and the roots coming from an ideal-class basis at each signature. The elliptic-curve data include the twelfth roots of the Weierstrass discriminant. They lie in M_0 by [187, Th. 5.1 and Rem. 5.3]: the ground field has characteristic zero and all the 3- and 4-torsion is rational over M_0. The level roots have orders dividing a. In particular $g_i(g_i-1)$ divides $|\mathrm{GL}_r(\mathbf{F}_{g_i})|$, by the factor $g_i^r - g_i^{r-1}$ in the order formula. This covers the values of the characters of $\mathbf{F}_{g_i}^{\times}$ as well.

For the character values, write j_0 for the integer specifying the norm twist $N\mathfrak{p}^{-j_0/2}$, and n_τ for the angular exponents, one at each complex place. The initial layer characters have even n_τ and $j_0 = 0$, the unramified character with half twist has $n_\tau = j_0 = 1$, and the two finite-order characters have $n_\tau = j_0 = 0$. They are tame at the primes of n and unramified elsewhere. At a prime $\mathfrak{p}$ where the character is unramified, choose $k_\mathfrak{p}$ generating $\mathfrak{p}^h$ and congruent to 1 at the other primes of the modulus, to their prescribed powers. This is possible because the ray class group for the modulus with the $\mathfrak{p}$-part removed is a quotient of the group of order h. The principal idèle relation gives the h-th power of the twisted value as

$$\prod_{\tau} \tau(k_\mathfrak{p})^{-n_\tau} \big(\tau(k_\mathfrak{p})\bar{\tau}(k_\mathfrak{p})\big)^{(n_\tau - j_0)/2}.$$

Indeed the inverse angular character supplies the powers $-n_\tau$, and $(N\mathfrak{p})^h = \prod_\tau |\tau(k_\mathfrak{p})|^2$. All displayed exponents are integers. Since $E/\mathbf{Q}$ is Galois, every factor is in the embedded E, and its fractional ideal is an h-th power. The twisted value therefore lies in V_h.

For a class of order k_i choose an ideal $\mathfrak{a}_i$ and γ_i with $\mathfrak{a}_i^{k_i} = (\gamma_i)$. The generators used at a signature are k_i-th roots of products of conjugates of γ_i to integer powers. Their radicands have k_i-th-power ideals. Since $k_i \mid h_E \mid h$, each such root is an h-th root of an element with h-th-power ideal. This proves the inclusion.
Lean: `GlobalTheory.Mem.FieldsEx.listField_le_Om` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ClassNumberDvdHstar`, lines 370–372)
Lean: `GlobalTheory.Mem.FieldsEx.valTw_mem_kummerField` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.AlgebraicHeckeCharacterValueMemKummerField`, lines 25–55)

Write $u = \log \mathrm{rd}(E)$, $H = \log h_E$, $s = \log h$, and $v = \log(2f)$. We give separate bounds for the log degree and log root discriminant of the four fields, in that order:

	$\log[\,\cdot\,:\mathbf{Q}]$	$\log \mathrm{rd}(\,\cdot\,)$
M_0	$\log d + 20$	$3\log d + 20$
R	$v + s$	$u + \log n + s$
$\mathbf{Q}(\mu_a)$	$\log a$	$\log a$
V_h	$v + (\mathrm{rank}\,\mathcal{O}_E^{\times} + 2)s + H$	$u + (\mathrm{rank}\,\mathcal{O}_E^{\times} + 2)s + H.$

For the first row put $H_K = K(j)$. By 26.29(2) it is unramified over K and $[H_K : K] = h_K$, since the period lattice has the maximal order. We have $h_K \le d$ by the quadratic ideal-class bound in 35.21. The action on $A[12]$ embeds $\mathrm{Gal}(M_0/H_K)$ in $\mathrm{GL}_2(\mathbf{Z}/12)$, so its order b_0 is at most 12^4. At primes away from $6d$ the curve has good reduction and multiplication by 12 is finite étale on its abelian model. Hence M_0/H_K is unramified there.

We use the following local different calculation for the first two rows. For a Galois extension of number fields of degree b_1, unramified outside the primes dividing an integer $n_1 > 0$, the relative different divides $(n_1 b_1)$. At a ramified prime remove the maximal unramified subextension. A uniformiser of the totally ramified part has Eisenstein polynomial of degree e and generates the local integers. In its derivative the nonzero terms have distinct valuations modulo e, and the leading term has valuation $e - 1 + v_{\mathrm{top}}(e)$. This bounds the different exponent by $e - 1 + v_{\mathrm{top}}(b_1)$, since $e \mid b_1$. Also $v_{\mathrm{top}}(n_1) \ge e$ there. Taking norms proves that the log root discriminant increases by at most $\log(n_1 b_1)$. For M_0 this gives $\frac{1}{2}\log d + \log(6db_0)$, and its degree is at most $2db_0$. Both fit the first row. For R, the ray class exact sequence gives $h \le h_E|(\mathcal{O}_E/n\mathcal{O}_E)^{\times}| \le h_E n^{2f}$. Applying the different calculation with $n_1 = n$, $b_1 = h$ gives the second row. For the cyclotomic row, the degree is at most a, and differentiating $X^a - 1$ at a primitive root shows that the different divides (a).
Lean: `GlobalTheory.Mem.Fld.TorsionSize.sz_discField` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.FieldDatumSizeLogDiscriminant`, lines 183–252)
Lean: `GlobalTheory.Mem.Fld.RaySize.degOver_le_and_sz` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.RayClassFieldDegreeLeAndRootDiscrBound`, lines 19–45)

For V_h, adjoin μ_h, one h-th root of each fundamental unit and of a generator of the torsion units, and one k_i-th root of each γ_i for a cyclic decomposition of the ideal class group. To see that these suffice, write $(y) = \mathfrak{b}^h$ and $\mathfrak{b} = (t)\prod_i \mathfrak{a}_i^{a_i}$. Then $k_i \mid ha_i$ for every i, and y differs from $t^h \prod_i \gamma_i^{ha_i/k_i}$ by a unit. This constructs one root of y, and μ_h supplies the others. Adjoining a k-th root of an element whose ideal is a k-th power increases each of the two logarithms by at most $\log k$. For the different bound, locally divide the root by a generator of the corresponding fractional ideal. The resulting element z is a unit and satisfies $z^k = u_0$ for a unit u_0. Its minimal polynomial divides $X^k - u_0$ integrally. The derivative kz^{k-1} belongs to the different, by the derivative formula for the generated order and inclusion in the field different. Thus the different divides (k). These arguments also apply over every intermediate field in the tower. Since $\prod_i k_i = h_E$, they give the last row.

Lean: `GlobalTheory.FieldSize.sz_kummerField` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.DegreeRootDiscrBoundKummerFieldOfPowerIdealRoots`, lines 124–219)

It remains to bound the entries of the table. Every ideal class of E contains an integral ideal of norm at most $X = \sqrt{|D_E|}$, by the Minkowski bound. Explicitly, apply the convex-body lattice bound to a product of f complex discs in an ideal I, whose covolume is $2^{-f}\sqrt{|D_E|}\,NI$; the resulting principal multiple of I^{-1} has norm at most X. The convex-body bound follows by placing half the body in a fundamental parallelepiped modulo the lattice and taking the difference of two overlapping points. The number of ideals of norm at most X is at most $X(1+\log X)^{2f}$: at each rational prime the Euler coefficients are bounded by those of $\zeta(s)^{2f}$, and one can count the resulting tuples by summing $X/(n_1 \cdots n_{2f-1})$. Hence

$$H \le fu + 2f\log(1+fu) \le 10fL.$$

Here $1 + fu \le 3fL$ and $\log(3fL) = \log f + \log(3L) \le 4L$. The prime hypotheses give $\log n \le 20L + 2$, whence $0 \le s \le H + 2f\log n \le 51fL$. The quadratic subfield gives $\frac{1}{2}\log d \le u$, by discriminant transitivity. Also $v \le 1 + L$ and $\operatorname{rank}\mathcal{O}_E^\times \le 2f$. Finally $|\operatorname{GL}_r(\mathbf{F}_g)| \le g^{r^2}$, and therefore

$$\log a \le fr^2(18L+2) + 24 \le 19fr^2L.$$

The last inequality uses $(L-2)fr^2 \ge 24$.

Log degrees and log root discriminants are subadditive under compositum, by the degree formula and divisibility of the relative different after base change as in 35.21. In each column of the table the sum excluding the cyclotomic row is at most

$$102f^2L + 163fL + 42L \le 198f^2L.$$

Indeed the combined coefficient of s is at most $2f+3$, the remaining class-number term is H, and the terms independent of s, H are bounded by $42L$ in either column for $L \ge 4$. Use $f \ge 8$ in the last inequality. Passing to F can only decrease degree and root discriminant.

Lean: `GlobalTheory.FieldSize.om_sizes_le_poR` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.OmegaFieldDegreeRootDiscrBoundedByRingPolynomial`, lines 24–50)

□

Proof of 35.101

Proof. Use the ray-class-field and height-counting statements on the open divisor row of 33.74. This gives the class-sum row. Each of the five rows (class-sum, primitive, divisor, goodness and geometric) supplies a member condition that eventually holds for every constructed member over the permitted base frames, and a bundle condition that eventually holds for every bundle linked to a good member satisfying the member condition, such that every selection satisfying the bundle condition satisfies the conclusion of the row. Let R be the conjunction of the five member conditions and $\mathcal{E}$ the conjunction of the five bundle conditions. Beyond the maximum of the five thresholds both conjunctions hold, which gives R and (8.34a), and a selection satisfying $\mathcal{E}$ satisfies the five row conclusions. For a selection satisfying it, use the same component for the primitive-row comparison and class-sum identity. Use the primitive and divisor comparisons on the component chosen by the geometric row. The class-sum identity supplies the divisor comparison needed for the rational row maps. Use the embedding of the coefficient ring at the selected member. Its field identification gives precisely the field of the coefficient list in Lemma 35.100. To apply that lemma put

$L = L_x = x^{\varepsilon}$. The host rank has $f \geq 8$ by the frame order condition. The rank bounds of 33.3 give $r \geq 62$ eventually, and the polynomial bound on f in (A.433) gives $\log f \leq L$ eventually. Increase the threshold also to have $L \geq 4$. Clause 1 of 33.3 gives $g_i \leq b^9$ and $p_{\mathrm{de}}, p'_{\mathrm{de}} < X' = \exp(x^{\varepsilon/3})$. By 8.3, $E \subset \mathbf{Q}(\mu_{Q_S qd})$, so $\mathrm{rd}(E) \leq Q_S qd$ by the cyclotomic different calculation in 35.100. The host size condition $(Q_S qd)^{36} t^{150} \leq \sqrt{b}$, with $t \geq 1$, gives $\log \mathrm{rd}(E) \leq L$. Lemma 35.100 now gives both bounds at this index. The series identification and these two bounds give the two arithmetic requirements in Definition 35.1 at the selected index. The representation-theoretic inputs give the eventual row conditions at that same point. Every family in $W_3(\mathcal{A})$ belongs to $\mathcal{A}$, so compatibility gives it the seven family conditions of Definition 33.3. These conditions put it in $\mathcal{F}_{\mathrm{cov}}$, where the bound of 35.2 is closed. The row conditions and the ties at the selected points are closed on $W_3(\mathcal{A})$. By the conjunction rule of 36.10 the three hold together on $W_3(\mathcal{A})$ for one cutoff and one threshold datum. For a family there, take C_0 from 35.2. For x_0, take the maximum of the covolume threshold and the thresholds for the point conditions and ties. Apply the layer inequality of 35.2 with (C, C_H) to the reference indefinite realisation of the selected summand. Its right side is $M_{p,i} + CE_p$. □

Proofs for Chapter 36: The definite side and local errors

Proof of 36.2

Proof. Use the parameter choice of the selected point in 33.32; the family uses its size datum at every indexed $x > x_1$. The host of 8.17 satisfies $|D_1| \leq L$ and $r_d \leq (Q_S/2)\sqrt{qd} \leq Q_S qd$. By 8.16 the logarithm of the latter conductor is at most $2x^{\varepsilon/5}(\log x)^2$, which is at most L eventually. The inequality $\log(4x) \leq L$ also holds eventually for fixed $\varepsilon > 0$. The two primes of 8.12 lie in $[Y, 2Y]$ with $Y = (kb)^8$. Since k is polynomial in x and the admissible parameter choice gives $\log Y \leq C_{YL} L$ (with $C_{YL} = 9$), $\log g_i \leq \log(2Y) \leq C_{YL} L + 1$ eventually.

For the node-size threshold in 33.32 one may take $c_h \geq 1300 + c_D$, $c_D \geq 0$. The rank bounds of 38.3 give $t \leq x^{401}$, $N \leq x^{402}$ eventually. Thus $4N^3 + 2t + 6 \leq x^{1300} \leq h_0 = \eta/m$ eventually. Both kS_c and $m(t+1)\eta$ are polynomially bounded by 15.9, whereas $\log w'' = L - x^{\varepsilon/3} - \log n$. This proves the two w'' inequalities. The second box of 14.7 gives $b_s \leq (1+R)b'$ with $4R \leq x^{-2}$. The spacing is the ceiling in 15.9, and the thick upper bound is the upper endpoint in 14.2. All cutoffs depend on the fixed construction constants and ε. □

Proof of 36.4

Proof. Transport the spectral spaces, sides and face along the division ties. The column at $w_U(u)$ is then the column of u. Let $q_0, \dots, q_{n-1}$ be the theta steps and $q_{\mathrm{cpl}} : \mathcal{F}_n \to \mathcal{S}$ the complement. The initial ramified inclusion f_0 is an isometry, and point evaluation gives a map $g : \mathcal{S} \to (K^{n_0})^{\vee}$ with

$$P_{w_U(u),\mu_0}(\phi, F) = g(q_{\mathrm{cpl}} q_{n-1} \cdots q_0 f_0 \phi)(F). \tag{A.493}$$

Take matrices of the arrows in bases framed by their Hermitian forms. Include f_0 in the first arrow, which is the complement when $n = 0$. Take P_{end} to be the matrix of g.

Lean: `GlobalTheory.DefRowsChain.chainOf` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.KerChainMatricesWithNormBounds`, lines 172–191)

Lean: `GlobalTheory.ColFirstIso.colsIn_isometry` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ColsInIsometry`, lines 100–122)

Apply 21.13 to the interlacing terminal tuple $(\lambda_\tau, \sigma_\tau)$ of ranks $(N, N-1)$ and the common determinant shift. The terminal parameter at τ is $\tau(\bar{\varpi})$ with $\varpi\bar{\varpi} = T_u$. Let $\lambda_\tau = \lambda_{\tau,0}$ be the leading terminal weight,

$$D_{0,\tau} = \sum_{i=0}^{N-2} (\sigma_{\tau,i} - \lambda_{\tau,i+1}) = \sum_{i=0}^{N-2} \sigma_{\tau,i} - \sum_{i=1}^{N-1} \lambda_{\tau,i} \geq 0$$

(matching 21.13, nonnegative because interlacing gives $\lambda_{\tau,i} \geq \sigma_{\tau,i} \geq \lambda_{\tau,i+1}$ for $0 \leq i \leq N-2$), and

$$D_{1,\tau} = \sum_{i=0}^{N-2} \min(\lambda_{\tau,i} - \lambda_{\tau,i+1}, D_{0,\tau})$$

the clipped adjacent-gap sum. We verify every inequality of the terminal table (36.2). Since $m \ge 2$ and $k \ge m+3$, we have $k \ge 5$ (which in particular supplies $k \ge 2$ for the unmarked endpoint formulas in Definition 36.3). The terminal rank is $N = k + m(t+1)$, and the weight tie gives $\lambda_\tau = U_0^0 - (k-1)/2 - m(t+1)/2 = U_0^0 - (N-1)/2$. For a marked type $\tau \in TT$, the top weight is $a_\tau = U_0^0 - (k-1)/2$, giving $\lambda_\tau = a_\tau - (N-k)/2$, and Lemma 36.2 gives $b_\tau \le 2b'$. For an unmarked type $\tau \notin TT$, Definition 36.3 gives $U_0^0 - v_0 \ge S_c$ with $v_0 \ge 0$, whence $\lambda_\tau = U_0^0 - (N-1)/2 \ge S_c - (N-1)/2$. By the interlacing bottom-rank identity for $D_{0,\tau} = \sum_{i=0}^{N-2} \sigma_{\tau,i} - \sum_{i=1}^{N-1} \lambda_{\tau,i}$, each layer step $0 \le j \le m-1$ contributes $e_j - b_j - (t+1)/2 \le \eta$ (from $|b_j - e_j| \le \eta$ in Definition 36.3), while the initial column contributes at most $v_0 - U_{k-1}^0 - (k-1)/2$ (since $\sum_{i=0}^{k-3}(U_{i+1}^0 - v_{i+1}) \ge 0$ from $U_{i+1}^0 \ge v_{i+1}$); combining these with $v_0 \le S_c$ and substituting $b_\tau = -U_{k-1}^0 - (k-1)/2$ when $\tau \in TT$ and $U_{k-1}^0 = v_{k-2} - \frac{1}{2} \ge -\frac{1}{2}$ (so $-U_{k-1}^0 - (k-1)/2 \le \frac{1}{2} - (k-1)/2 \le 0$ as $k \ge 5$) when $\tau \notin TT$ yields $D_{0,\tau} \le S_c + m\eta + \mathbf{1}_{TT}(\tau) b_\tau$. Similarly, splitting the clipped-gap sum into the first gap $\min(\lambda_{\tau,0} - \lambda_{\tau,1}, D_{0,\tau}) \le D_{0,\tau}$, the interior column gaps $1 \le i \le k-2$ bounded by $v_0 - v_{k-2} + (v_{k-2} - U_{k-1}^0 - \frac{1}{2}) \le S_c + D_{0,\tau}$ (using $U_1^0 < v_0 \le S_c$, $v_{k-2} \ge 0$, and $v_{k-2} - U_{k-1}^0 - \frac{1}{2} = \sigma_{\tau,k-2} - \lambda_{\tau,k-1} \le D_{0,\tau}$), the bridge gap at $i = k-1$ clipped by $\min(\lambda_{\tau,k-1} - \lambda_{\tau,k}, D_{0,\tau}) \le D_{0,\tau}$, and the remaining layer-parameter gaps $i \ge k$ bounded by $|b_0 - e_{m-1}| \le \eta$ (using $b_0 \le e_0$ and $|b_0 - e_{m-1}| \le \eta$) gives $D_{1,\tau} \le 3D_{0,\tau} + S_c + \eta$.

In the orthogonal norm bound of 21.13, each place τ contributes the nonzero integer branching numerator c_τ (a product of at most $(N-1)D_{1,\tau}$ nonzero integer factors of the form $v - i - \sigma_{\tau,j} + j$ with $v \in [\lambda_{\tau,i+1}, \min(\lambda_{\tau,i}, \lambda_{\tau,i+1} + D_{0,\tau})] \cap \mathbf{Z} \setminus \{\sigma_{\tau,i}\}$ and $0 \le i, j \le N-2$, each bounded in magnitude by $|v - i - \sigma_{\tau,j} + j| \le \lambda_{\tau,0} - \lambda_{\tau,N-1} + N - 2$) and the branching norm ratio $\kappa_\tau = \|\xi_\tau\|_F^2 / \|h_{\lambda_\tau}\|_F^2 > 0$. To bound κ_τ, translate $\lambda_\tau, \sigma_\tau$ by $-\lambda_{\tau,N-1}$ to nonnegative integers $\tilde\lambda_0 \ge \tilde\sigma_0 \ge \cdots \ge \tilde\sigma_{N-2} \ge \tilde\lambda_{N-1} = 0$, and work in the polynomial Fock space on $N \times (N-1)$ matrices $X = (X_{b,q})$ equipped with the Bargmann–Fock inner product $\langle f, g \rangle_F$ in which multiplication by $X_{b,q}$ is adjoint to $\partial_{b,q}$. Writing Δ_J for the $k \times k$ initial-column minor on an injective row tuple $J : (0, \dots, k-1) \hookrightarrow (0, \dots, N-1)$, set $A_i = (0, \dots, i)$ and $B_i = (0, \dots, i-1, N-1)$ for $0 \le i \le N-2$; then the branching vector $\xi_\tau = \prod_{i=0}^{N-2} \Delta_{A_i}^{\tilde\sigma_i - \tilde\lambda_{i+1}} \Delta_{B_i}^{\tilde\lambda_i - \tilde\sigma_i}$ and the highest-weight vector $h_{\lambda_\tau} = \prod_{i=0}^{N-2} \Delta_{A_i}^{\tilde\lambda_i - \tilde\lambda_{i+1}}$ are built one $k \times k$ minor at a time in increasing order of minor size $k = 1, \dots, N-1$. For any dual-Weyl polynomial x of column weight $\mu = (\mu_0, \dots, \mu_{k-1}, 0, \dots, 0)$ (meaning $\sum_b X_{b,t} \partial_{b,s} x = 0$ for $t < s$ and $\sum_b X_{b,s} \partial_{b,s} x = \mu_s x$), expanding a $(p+1) \times (p+1)$ minor $\Delta_{(a,b)}$ along its last row by Laplace's cofactor expansion gives $\partial_{b,p} \Delta_{(a,b)} = \Delta_a$ for $b \notin \mathrm{im}(a)$ (and 0 otherwise); since the column-weight annihilation $\sum_b X_{b,t} \partial_{b,p} x = 0$ kills every $t < p$ cofactor term, only the diagonal term survives, yielding $\sum_b \Delta_{(a,b)} \partial_{b,p} x = \mu_p \Delta_a x$, so the Leibniz rule gives

$$\sum_{b=0}^{M-1} \partial_{b,p}(\Delta_{(a,b)} x) = (M - p + \mu_p) \Delta_a x$$

over the M active rows ($M = N$ for x, and $M = k$ for a highest-weight factor h supported on the first k rows). For any strictly increasing $(p+1)$-tuple $I = (c_0 < \cdots < c_p)$ in $\{0, \dots, M-1\}$, expanding the first factor of $\langle \Delta_I x, \Delta_I x \rangle_F$ along its last column p and using the adjointness of $X_{c_r,p}$ and $\partial_{c_r,p}$ writes $\|\Delta_I x\|_F^2 = \sum_{r=0}^{p} (-1)^{p+r} \langle \partial_{c_r,p}(\Delta_I x), \Delta_{I \setminus \{c_r\}} x \rangle_F$. Setting $b = c_r$ and $a = I \setminus \{c_r\}$ gives $\Delta_{(a,b)} = (-1)^{p-r} \Delta_I$, so summing over all strictly increasing $(p+1)$-tuples I bijects with pairs (a, b) of a strictly increasing p-tuple a and $b \notin \mathrm{im}(a)$ and yields the exact identity

$$\sum_{|I|=p+1 \text{ incr}} \|\Delta_I x\|_F^2 = \sum_{|a|=p \text{ incr}} \sum_{b \notin \mathrm{im}(a)} \langle \partial_{b,p}(\Delta_{(a,b)} x), \Delta_a x \rangle_F = (M - p + \mu_p) \sum_{|a|=p \text{ incr}} \|\Delta_a x\|_F^2.$$

Inducting over $p = 0, \dots, k-1$ gives $\sum_{|J|=k \text{ incr}} \|\Delta_J x\|_F^2 = C(M, \mu, k) \|x\|_F^2$; hence $\|\Delta_J x\|_F^2 \le C(N, \mu, k) \|x\|_F^2$ for every increasing k-tuple J at $M = N$, while at $M = k$ the unique increasing k-tuple is $A_{k-1} = (0, \dots, k-1)$, giving the exact equality:

$$\|\Delta_J x\|_F^2 \le C(N, \mu, k) \|x\|_F^2, \qquad \|\Delta_{A_{k-1}} h\|_F^2 = C(k, \mu, k) \|h\|_F^2, \qquad C(M, \mu, k) = \prod_{q=0}^{k-1} (M - q + \mu_q).$$

Let $W(N, \mu) = \prod_{0 \le i < j < N} (\mu_i - \mu_j + j - i)/(j - i)$ be the Weyl dimension polynomial. For μ supported on indices $< k$, adding 1 to the first k coordinates satisfies the exact ratio identity $W(N, \mu)\, C(N, \mu, k) =$

$W(N,\mu+\mathbf{1}_{<k})\,C(k,\mu,k)$. Inducting across all minor factors from $(1,1)$ to $(\xi_\tau,h_{\lambda_\tau})$ preserves $\|x\|_F^2 \le W(N,\mu)\|h\|_F^2$, yielding

$$0<\kappa_\tau=\frac{\|\xi_\tau\|_F^2}{\|h_{\lambda_\tau}\|_F^2}\le W(N,\lambda_\tau)=\prod_{0\le i<j<N}\frac{\lambda_{\tau,i}-\lambda_{\tau,j}+j-i}{j-i}=\dim V_{\lambda_\tau}.$$

Because each of the $\frac12 N(N-1)$ factors has denominator $j-i\ge1$ and numerator $0<\lambda_{\tau,i}-\lambda_{\tau,j}+j-i\le\lambda_{\tau,0}-\lambda_{\tau,N-1}+N-1<2a$ by the dominant-weight spread bound $\lambda_{\tau,0}-\lambda_{\tau,N-1}+N\le 2a$ of Definition 36.3, we obtain

$$0\ne|c_\tau|\le(\lambda_{\tau,0}-\lambda_{\tau,N-1}+N-2)^{(N-1)D_{1,\tau}}\le(2a)^{(N-1)D_{1,\tau}},\qquad 0<\kappa_\tau\le\dim V_{\lambda_\tau}\le(2a)^{N(N-1)/2}.$$

Taking logarithms gives $\log(c_\tau^2\kappa_\tau)\le 2c_\tau^*+\xi_\tau^*$ for the witnesses

$$c_\tau^*=(N-1)D_{1,\tau}\log(2a),\qquad \xi_\tau^*=\tfrac12 N(N-1)\log(2a),\qquad e_{\mathrm{ev}}=C_{\mathrm{ev}}=0.$$

Taking A to be the log squared terminal factor and setting $n_\tau=\lambda_\tau-D_{0,\tau}$, the bound of 21.13 therefore gives $A\le-\sum_\tau n_\tau\log T+\sum_\tau(2c_\tau^*+\xi_\tau^*)+e_{\mathrm{ev}}$, while $T_u\ge1$ and (36.10) give $0\le\log T\le C_T mL$. The level-group average gives reciprocal stabilizer weights and fixes the reference vectors with divided Gram form H_{μ_0}. The bound of 21.13 in these frames is

$$|v(g(\psi)(F))|^2\le e^A\|\psi\|_{\mathcal{S},v}^2\|Y_v^{-1}v(F)\|_2^2. \tag{A.494}$$

Density extends this to the complex coordinate spaces.
Lean: `GlobalTheory.ColRows.LastSideProof.lastSide_of_logT_le` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LastMapHalfRecNumsOfLogLevelPrimeLe`, lines 14–192)

On the kernel side, let $U_*=fm^2N^2\log(g_0g_1T)\ge0$ be the log-level error unit, and work over a finite number field $M\subset\mathbf{C}$ containing the embedded coefficient field $\iota(K)$, the period generators, and the normalized theta coefficients across all layers. Five kernel-side quantities are governed by unconditional absolute constants $C_c,C_i,C_y,C_K\ge0$ and $C_t\in\mathbf{R}$ independent of $D,O,c_{\mathrm{sz}},\mathbf{b},C,C_H,C_Y,C_T$:

1. *Genus class count h_l $(0\le l\le m)$:* the maximum number of decorated lattice isometry classes in the genus of the layer-l definite Hermitian space of rank $n=N_l\le N$ over E at level dividing g_0g_1T. Equip any definite self-dual Hermitian $\mathcal{O}_E$-lattice $L\subset E^n$ with the trace Euclidean metric $\mathrm{Tr}_{E/\mathbf{Q}}H$; because L is Hermitian self-dual, its $\mathbf{Z}$-dual is $L^*=\mathfrak{D}_E^{-1}L$, where every nonzero $w\in L^*$ satisfies $N_{E/\mathbf{Q}}(H(w,w)\mathfrak{D}_E^2)\ge1$ and AM–GM gives $\lambda_1(L^*)^2\ge 2f/\operatorname{rd}(E)^2$. Banaszczyk's transference theorem $\lambda_{2fn}(L)\lambda_1(L^*)\le 2fn$ (cf. Lemma 33.11) therefore extracts an E-basis $v_1,\dots,v_n\in L$ whose Hermitian Gram matrix $A=(H(v_i,v_k))_{i,k}\in M_n(\mathcal{O}_E)$ satisfies $|\sigma(H(v_i,v_k))|\le B=4f^3n^4\operatorname{rd}(E)^2$ at every embedding $\sigma:E\hookrightarrow\mathbf{C}$. Counting the n^2 entries $H(v_i,v_k)\in\mathcal{O}_E$ inside the polydisc of radius B yields at most $(1+2B)^{2fn^2}$ Gram matrices A, and for each sublattice $L_0=\sum_{i=1}^n\mathcal{O}_Ev_i$ of Gram matrix A, counting $\mathcal{O}_E$-overlattices $L/L_0\subseteq L_0^\vee/L_0$ via their pairings in $\mathcal{O}_E^n/A\mathcal{O}_E^n$ yields at most $|N_{E/\mathbf{Q}}(\det A)|^n\le(nB)^{2fn^2}$ choices, bounding the set R of self-dual Hermitian isometry classes by $|R|\le(3(2f)^2n^4\operatorname{rd}(E)^2)^{6fn^2}$. Every decorated level-$(g_0g_1T)$ datum is $U(E,n)$-equivalent to one of the $|R|$ fixed self-dual representatives L_0 equipped with transported local decorations at S_T (at most $\prod_{P\in S_T}\mathrm{N}(P)^n$) and S_G (at most $\prod_{w\in S_G}\mathrm{N}(w)^{n^2}$). Multiplying $|R|$ by this local decoration fibre count $\le(g_0g_1T)^{fn^2}$ and applying $\log\operatorname{rd}(E)\le L$ and $fN\le e^L\le g_0$ from Definition 36.3 gives $\log h_l\le 55fN^2\log(g_0g_1T)\le 55U_*$ (so $C_c=55$).

2. *Matrix-step and coupling index count n_l $(0\le l\le m)$:* set $n_0=1$ and $n_m=1$. For each interior layer $1\le l\le m-2$, put $c_l=m-1-l\ge1$ and $n=N_l+c_l=k+l(t+1)+(m-1-l)\le k+m(t+1)=N_m=N$; each step index maps injectively under residue reduction modulo the primes $w\in S_G$ above g_0,g_1 (norm product $(g_0g_1)^f$) into $n\times n$ matrices over $\mathcal{O}_E/w$, giving $n_l\le(g_0g_1)^{fN^2}\le\exp(U_*)$ (so $C_i=1$).

3. *Coupling-coset count n_{m-1} at the coupled end:* for $l=m-1$, n_{m-1} counts the admissible coupling cosets matching the penultimate layer lattice to the coupled terminal data, each determined by its residue pair modulo the primes $w\in S_G$ above g_0,g_1 in rank at most N, giving $n_{m-1}\le(g_0g_1)^{2fN}\le\exp(2U_*)$ (so $C_y=2$).

4. *Normalized theta vacuum norm ϑ_ρ:* for each embedding $\rho : M \hookrightarrow \mathbf{C}$, the squared norm of the normalized base theta family on standard lattices of ranks $(N_\ell, N_{\ell+1})$ equals the $(fN_\ell N_{\ell+1})$-th power of the genus-one theta-zero norm $\|\theta_0\|_\rho^2$; setting $g_\ell^0 = N_\ell N_{\ell+1}$, $p_\ell = fg_\ell^0$, $lt(\rho) = f \log \|\theta_0\|_\rho^2$, and $\vartheta_\rho = lt(\rho)/f = \log \|\theta_0\|_\rho^2$, the layer-$\ell$ vacuum log norm equals $g_\ell^0\, lt(\rho) = p_\ell \vartheta_\rho$. Because Lemma 36.2 gives $\log v_E + |D_1| \le \log \mathrm{rd}(E) + |D_1| \le 2L$ (with $v_E = \frac{1}{2}\,\mathrm{rd}(E)$), the Chowla–Selberg period comparison 21.22 and the genus-one absolute theta-norm estimate bound the Galois average by $\frac{1}{[M:\mathbf{Q}]} \sum_\rho lt(\rho) \le 2\max(C_t, 0) fL$, hence $\bar\vartheta = \frac{1}{[M:\mathbf{Q}]} \sum_\rho \vartheta_\rho \le 2\max(C_t, 0)L$ (while the mean log period satisfies $\bar\omega_\tau = \log \Omega$ by 21.22).

5. *Per-step Fock derivative and matrix-entry coupling:* at each embedding $\rho : M \hookrightarrow \mathbf{C}$ and layer ℓ, the block entry between an input class-index pair and an output class-index pair is the integral against the Hermitian theta kernel of a harmonic Fock derivative of multidegree $(\nu_{\ell,\tau,i})$ (with total degree $n = n_{\ell,\tau} = \sum_i \nu_{\ell,\tau,i}$ at type τ). Differentiating the Gaussian theta kernel of $p_\ell = fN_\ell N_{\ell+1}$ complex variables on a fundamental domain of volume parameter $V = v_E = \frac{1}{2}\,\mathrm{rd}(E)$ gives at each type τ an integral of $\prod_i s_i^{\nu_i} e^{-Q/2}$ over $\mathbf{C}^{g_\ell^0}$ (where $Q = \sum_i s_i^2$ and $\nu_i = \nu_{\ell,\tau,i}$), normalized by $\pi^{g_\ell^0}$. Setting $a = n/(n+1) = 1 - 1/(n+1)$ and splitting $e^{-Q/2} = e^{-aQ/2} e^{-Q/(2(n+1))}$, for each coordinate with $\nu_i > 0$ the elementary bound $\log(as_i^2/\nu_i) \le as_i^2/\nu_i - 1$ gives $s_i^{\nu_i} e^{-as_i^2/2} \le \exp\big(\frac{1}{2}(\nu_i \log \nu_i - \nu_i)\big) a^{-\nu_i/2}$. Multiplying over i gives $a^{-n/2} = (1 + 1/n)^{n/2} \le e^{1/2}$ (which holds trivially as $1 \le e^{1/2}$ when $n = 0$, contributing the single additive $+1$ after squaring), while integrating $e^{-Q/(2(n+1))}$ over $\mathbf{C}^{g_\ell^0}$ normalized by $\pi^{g_\ell^0}$ yields $(2n+2)^{g_\ell^0}$, hence $(2n+2)^{2g_\ell^0}$ on squaring. The p_ℓ-fold Gaussian and Minkowski lattice-sum supremum bound $3(2p_\ell\,\mathrm{rd}(E)+1)^{p_\ell} \le (8+32p_\ell V)^{p_\ell}$ contributes the domain-volume factor $(8+32p_\ell V)^{p_\ell}$. Together with the period scaling $\exp(-n_{\ell,\tau}\omega_{\tau,\rho})$ and the theta vacuum factor $\exp(p_\ell \vartheta_\rho)$, this yields the exact squared-norm factor in (36.4). *Claim 1.* The ordinary finite datum has period $c = T_u g_0 g_1$ and absolute value at most $\beta = (g_0 g_1)^{2fm^2N^2}$ at every coefficient embedding; each of the two coupled finite data has the same period and bound 1.

   For an ordinary step ℓ and each $P \in S_G$, use the local factor of Definition 12.12. Write $q_P = |\mathcal{O}_E/P|$. At these completely split primes q_P is also a uniformiser. Let A, B be the support and period depths, and let $\mathrm{ch}\, X$ denote the sum of the entries specified there in the additive character. Put $c_\ell = m - 1 - \ell$, $a_P = N_\ell + c_\ell$, $b_P = N_{\ell+1} + c_{\ell+1}$, and let $F_P \le a_P b_P$ be the number of free matrix entries. In standard local frames the normalized entry is

$$\Xi_P(x) = J_P^{-1} q_P^{-BF_P} \sum_t \Phi_P(X(x,t)) \psi_P(-\,\mathrm{ch}\, X(x,t)).$$

   Here J_P is the local volume factor, B is the period exponent, and $X(x,t)$ fills the free entries with representatives of $q_P^{-A}\mathcal{O}_P/q_P^B\mathcal{O}_P$. The moved datum Φ_P is supported on maps between the two prescribed lattices and has absolute value at most one. On its support every free entry satisfies $q_P^{c_\ell} t \in \mathcal{O}_P$; hence each free entry has at most $q_P^{B+c_\ell}$ possible values. The additive character there is a root of unity. Reduction of the target stabilizer modulo P injects the relevant quotient into b_P-square residue matrices, giving $|J_P^{-1}| \le q_P^{b_P^2}$. Thus

$$|\Xi_P(x)| \le q_P^{b_P^2} q_P^{-BF_P} q_P^{(B+c_\ell)F_P} \le q_P^{b_P^2 + c_\ell a_P b_P} \le q_P^{2m^2N^2}.$$

   The support is integral and adding q_P times an integral homomorphism preserves the value. The level marking factor has modulus at most one and period T_u. Multiplication over $P \in S_G$, using $\prod_P q_P = (g_0 g_1)^f$, proves the ordinary assertion. In the first coupled datum fix the coset $Y = y + L_+$ and insert its column in the moved integral datum; in the second use the terminal marking test and its additive character. Each is zero off its integral support and otherwise has modulus at most one. Changing the free matrix by c times an integral homomorphism preserves the coset and changes the character argument by an integral period. This proves both coupled assertions.

   Claim 2. These finite data give the required bounds on the coefficient operators in the stated Hermitian norms.

   Fix a decorated pair of ranks $a = N_\ell, b = N_{\ell+1}$, put $H = \mathrm{Hom}_S(L, L')$. Write c_A for the support denominator and $c_B = cc'$ for the period modulus in the normalized jet sum. Direct expansion of the

kernel gives

$$K_{\Xi} = \gamma\ \mathrm{Jet}(J), \qquad J_{\alpha} = \sum_{x \in c_A^{-1} H / c_B H} \Xi(x) J_{c_B, x, \alpha}, \qquad |\rho(\gamma)| \le 1.$$

The scalar γ is the ratio of the two determinant-character frame factors; their embedded moduli are zero or one. Rationality of the normalized jets identifies J_{α} with the derivatives of the same finite theta sum at every embedding. Refinement from c_B to c, using the support and period in Claim 1, reduces that sum to at most c^{d_ℓ} cosets, where $d_\ell = [E : \mathbf{Q}]ab = 2fab$. Finite Fourier inversion of each coset indicator expresses its normalized theta function as a combination of torsion translates with coefficient absolute-value sum at most c^{d_ℓ}. Consequently the entire function F representing this block has torsion-translate coefficient sum at most βc^{2d_ℓ}, uniformly at every embedding. For the coupled data use $\beta = 1$. Translation of the normalized vacuum and the lattice-sum bound above give

$$|F(Z)| \le S_\rho \exp(Q(Z)/2), \qquad S_\rho = \beta c^{2d_\ell} \sqrt{3(2fab\ \mathrm{rd}(E) + 1)^{fab}}\ \exp(fab\, \vartheta_\rho / 2)$$

in unit Gaussian coordinates.

We now give the passage from this scalar estimate to the coefficient norm. Give polynomial Fock space its monomial norms $\|Z^\alpha\|_F^2 = \alpha!$, and divide each weight space's form by the squared norm of its highest vector. At each type, multiply the input polynomial by the determinant power equal to the last entry of $\nu_{\ell,\tau}$, include it in the larger matrix space, and substitute a rectangular matrix A. Call this operator T_A. The weight identities in Definition 36.3 identify its output with the target weight space. Singular-value decomposition of A, unitary invariance of the Fock form, and the determinant multiplication identity give

$$\|T_A v\| \le \prod_i s_i(A)^{\nu_{\ell,\tau,i}} \|v\|, \qquad \sum_i s_i(A)^2 = \|A\|_{\mathrm{HS}}^2.$$

The highest-vector normalization cancels the ratio of highest-vector Fock norms in this inequality. Tensoring over the types preserves it by positivity of the Gram forms. For a unit input and a unit target functional, expand T_A as a polynomial in the entries of A. Applying the functional gives a polynomial bounded by the displayed singular-value product. Pairing its coefficients with the Taylor coefficients of F is exactly the jet operator above: $\partial^\alpha F(0) = \alpha! [Z^\alpha] F$, and each differential generator contributes its specified period factor. The Gaussian reproducing formula and the preceding Gaussian calculation therefore give

$$\|K_{\Xi}\|_\rho^2 \le S_\rho^2 \prod_\tau \left[e^{-n_{\ell,\tau}\omega_{\tau,\rho}}\, e(2n_{\ell,\tau} + 2)^{2ab} \exp\left(\sum_i (\nu_{\ell,\tau,i} \log \nu_{\ell,\tau,i} - \nu_{\ell,\tau,i}) \right) \right],$$

with $0 \log 0 = 0$. This bound is uniform over both unit vectors, so it is an operator bound. Moreover $\log(\beta c^{2d_\ell}) \le 6U_*$; thus the finite-data and torsion-translate contribution to the squared bound is at most $12U_*$, which is covered by $2C_K U_*$ for a universal $C_K \ge 6$.

At the coupled end retain the same Y in both factors. Its block is the sum of $K_{2,p,Y,d} K_{1,p_0,p,Y}$ over the intermediate decorated class p and its admissible cosets Y, with the reciprocal stabilizer weights of the class sum. Apply the preceding bound to each factor, then Cauchy–Schwarz to this common class-and-coset sum. This gives the product of the two step bounds with the class and coset factors counted below.

Set $E_* = C_c + C_i + C_y = 58 \ge 0$ (so $C_K + 2E_* \ge 0$), whence $h_l \le \exp(E_* U_*)$ and $n_l \le \exp(E_* U_*)$ for every $0 \le l \le m$. At each embedding $\rho : M \hookrightarrow \mathbf{C}$ and every $0 \le \ell < m$, applying Cauchy–Schwarz over the input and output class sums (contributing $h_\ell h_{\ell+1}$) and index sums (contributing $n_\ell n_{\ell+1}$, which includes the coupling-coset count $n_{m-1} \le \exp(2U_*)$ at the coupled end), then multiplying by the squared matrix-entry coupling bound $\exp(2C_K U_*)$, bounds the block-coupling logarithm $M_{\ell,\rho}$ by

$$M_{\ell,\rho} \le C_K U_* + \tfrac{1}{2} \log(h_\ell h_{\ell+1}) + \tfrac{1}{2} \log(n_\ell n_{\ell+1}) \le (C_K + 2E_*) U_*. \tag{A.495}$$

Because the squared framed operator norm $\|K_\ell\|_\rho^2 = \|K_\ell\|_v^2$ depends only on the infinite place v of K below ρ, average the parameters $(\omega_{\tau,\rho}, \vartheta_\rho, M_{\ell,\rho})$ over the embeddings extending v, with the prescribed nonnegative

weights summing to one. Affine linearity identifies the resulting exponent with $s_{\ell,v} = \sum_{\rho|v} w_{v,\rho} s_{\ell,\rho}$. There is an embedding ρ_0 of positive weight with $s_{\ell,\rho_0} \le s_{\ell,v}$, since otherwise their weighted mean would exceed $s_{\ell,v}$. Consequently

$$\|K_\ell\|_v^2 \le \exp(s_{\ell,\rho_0}) \le \exp(s_{\ell,v}).$$

This argument also applies when the operator is zero. For the coupled operator apply it to the sum of its two step exponents. The uniform mean $1/[M : \mathbf{Q}]$ across all embeddings gives $\bar{\omega}_\tau = \log \Omega$, $\bar{\vartheta} \le 2\max(C_t, 0)L$, and $M_{\ell,v} \le (C_K + 2E_*)U_*$. Moreover, the size bounds give $\log \mathrm{rd}(E) \le L = \log b$ (so $V \le b/2$), $b \ge 2$, $fN_m \le b$ (hence $N_m \le b$ and $p_\ell = fN_\ell N_{\ell+1} \le fN_m^2 \le b^2$), and $4x \le b$. Since $8 \le b^3$ and $32 \le b^5$, we have $8 + 32p_\ell V \le 8 + 16b^3 \le 32b^3 \le b^8$, whence $\log(8 + 32p_\ell V) \le 8L$. Similarly, each of the N_ℓ weight entries in $n_{\ell,\tau}$ is at most $3a = 3xb$, so $n_{\ell,\tau} \le N_\ell(3xb) \le b^3$ and $2n_{\ell,\tau} + 2 \le 4b^3 \le b^5$ (as $4 \le b^2$), whence $\log(2n_{\ell,\tau} + 2) \le 5L$. Combining these with the mean bound and the kernel size bounds of Definition 36.3 gives

$$\bar{\vartheta} \le 2\max(C_t, 0)L, \quad \log(8{+}32p_\ell V) \le 8L, \quad \log(2n_{\ell,\tau}{+}2) \le 5L, \quad 0 \le g_* = \log(g_0 g_1) \le (2\max(C_Y, 0){+}2)L. \tag{A.496}$$

Take $h_\ell = h'_\ell = 0$. Summing the five logarithmic contributions above at embedding v gives precisely $s_{\ell,v}$ in (36.4), bounding the squared framed operator norm of kernel layer ℓ at v by $\exp(s_{\ell,v})$. Because operator norms are submultiplicative across the chain composition, assigning layer ℓ of kernel index j to step $\iota(\ell) = \min(j, n)$ collects each ordinary theta factor in its own step $\alpha < n$ and multiplies all remaining factors into the complementary step $\alpha = n$, yielding

$$\|X_{\alpha+1,v}^{-1} v(K_\alpha) X_{\alpha,v}\|^2 \le \exp\left(\sum_{\iota(\ell)=\alpha} s_{\ell,v}\right). \tag{A.497}$$

Lean: GlobalTheory.KernelHalfS.kerSide_chainS (Development.AbelianVarietyTheory.Proofs.DivisorProofs.KernelHalfChainNumbersAnyLevelPrime, lines 219–330)
Lean: GlobalTheory.ColRows.FactorBounds (Development.AbelianVarietyTheory.Definitions.Divisors.FactorBounds, lines 40–42)

Put $\Gamma = C_K + 2E_* + 2\max(C_t, 0) + 20 + 2\max(C_Y, 0) \ge 20 > 0$, which depends only on C_Y and the five universal absolute constants. Choosing $C_2 = C_\theta = C_4 = c_5 = c_6 = C_8 = C_e = \Gamma$ (and $C_{\mathrm{ev}} = 0$) makes every bounding constant in (36.1) and (36.2) nonnegative, dominates every coefficient in (A.495) and (A.496) (using $L \le L_a$), and bounds the three terms in (36.3) by $2(16 + C_T)$, $6\Gamma + 1 + 2\Gamma^2$, and $4\Gamma(C_T + \Gamma)$. All three are covered by

$$C_0 = \max\{2(16 + C_T), 6\Gamma + 1 + 2\Gamma^2, 4\Gamma(C_T + \Gamma)\}. \tag{A.498}$$

Lean: GlobalTheory.ColRowsTopS.rowsBody_colFace (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ColumnsMemberRowsOfLogLevelLe, lines 25–57)
Lean: GlobalTheory.ColRows.exists_htNums_of_kerSide_lastSide (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsHtNumsOfKerSideLastSide, lines 11–26)

Take one coefficient $X_{\alpha,v}^{-1} z$ at each stage, with weight and phase one, the transition in (A.497), and $t_{\max}(\alpha, v) = \exp(\sum_{\iota(\ell)=\alpha} s_{\ell,v})$. Its logarithm gives (36.9).
Lean: GlobalTheory.DefRowsChain.rowsBody_of_framed (Development.AbelianVarietyTheory.Proofs.DivisorProofs.KerChainMatricesWithNormBounds, lines 227–269)
Lean: GlobalTheory.DefRowsCB.exists_cbData (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsCbData, lines 13–55)

For $0 \ne z \in I_u$, the lattice N'_u contains zN_u^0, and the two span the same K-space. Noetherianity gives finite generation, and transport by e_u^{-1} gives the required full lattice of positive dimension.
Lean: GlobalTheory.ColRows.nuLat_comap_isLattice (Development.AbelianVarietyTheory.Proofs.DivisorProofs.IsLatticeOfEqSpanFracSmul, lines 52–60)

□

Proof of 36.5

Proof. Linkage supplies point evaluation and the size implication of 36.4. Eventually $m \ge 4$, $1 \le f \le N(m)$, $3N(m) \le b$, $24x \le b$, and the top window (14.1) gives $a_{\max} \le 2xb$. For the selected weights, the common entry bounds are $-(2b + b') \le \lambda_{j,\tau,h} \le 2xb + (b')^2/64$. On a marked type the negative of the

last initial weight is between zero and $2b'$. In the maximum defining $W_{u,\tau}$, each retained marked entry has absolute value at most $2b'$ and the subtracted shift is at most b', so $W_{u,\tau} \le 3b'$. The size bounds also give $N_m \le b' \le b$. Since $T_u \ge b^{m/2}$, Lemma 21.8 supplies the level conditions in 21.9.
Lean: `Formalism.MainNotions.GlobalTheoryTheory.Alc.entry_bd` (`Formalism.SharpRestriction.GlobalSchurBounds.AlcoveOf`, lines 65–121)
Lean: `Formalism.MainNotions.J.Wt_top_le` (`Formalism.SharpRestriction.AdmissibleParameterJoins.DIVfromGLo`, lines 112–137)

With $H = T_u - 1$ and $d_M = \dim_{\mathbf{C}} M^0_{u,\mathbf{C}}$, it gives

$$d_u = d_M(1 - 1/H)^f, \qquad \mu_{\mathrm{ram}} = \mu_0 - \frac{f \log H}{H - 1}, \qquad \mu_{\max,\mathrm{ram}} \le \mu_{\max,0}. \tag{A.499}$$

Lean: `Formalism.L2.GlobalTheoryTheory.Places.settingF_of` (`Formalism.LocalToGlobal.GlobalTheory.PlacesP`, lines 759–776)
Lean: `Formalism.MainNotions.J.levelPrime_setting_of_sizes` (`Formalism.SharpRestriction.AdmissibleParameterJoins.DIVfromGLo`, lines 249–292)

The local inclusions in 21.9 put N^0_u in the ramified lattice. At a noninert place off T_u the localizations agree. Clear denominators for a finite set of generators of the ramified lattice at that place to obtain a nonzero multiple contained in N^0_u. Multiplication by I_u preserves fullness. The lower rank bound in the selected-family design gives $d_M > 0$, and $T_u > 2$ gives $d_u > 0$.
Lean: `Formalism.MainNotions.Lnk.nuCommens_of` (`Formalism.SharpRestriction.GlobalSchurBounds.LinkPk`, lines 147–161)
Lean: `Formalism.MainNotions.Wire.family_theorems_two_and_five` (`Formalism.SharpRestriction.GlobalSchurBounds.LogXLowerFamilyOfTheorems`, lines 77–81)
Lean: `Formalism.MainNotions.muRank_items` (`Formalism.SharpRestriction.GlobalSchurBounds.LinkMin`, lines 739–745)

Lemma 36.2 supplies the first condition of (36.10) on this member, with $C_Y = C_{YL}$. Eventually $X' \ge 8$, $x \ge 9$, $x^2 \le b$, hence $4a < b^2 \le T_u$. For any $F_m \in W_3(\mathcal{A})$ under a linked selection, linkage (33.77(5) and the point-evaluation side) supplies the real second-layer identifications, while membership in $W_3(\mathcal{A})$ supplies the selected-family design with constants $3, C'_H$ and the eventual bounds $b^{m/2} \le T_u$ and $\log T_u \le C_{Tu} mL$ (via outer entry 13, putting $C_T = \max(C_{Tu}, 0)$), discharging the hypotheses of the auxiliary fullness assertion above and feeding Theorem 36.4. Use the middle threshold (A.498) with C_Y, C_T from the zero-indexed outer entries $6, 13$. Take the nonnegative part of entry 13. The maximum of the fullness, rank and size cutoffs works in 36.4.
Lean: `GlobalTheory.CoreHt.famRowsSizes` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FamRowsSizes`, lines 13–68)

□

Proof of 36.7

Proof. The unitary input uses the uniformly moderate-growth case of Miller–Schmid's cuspidal decay theorem [27, Theorem A]. At the open level of ϕ, Proposition 1.22 applies with the quotient height $H_{\mathrm{quot}}(g) = \inf_\gamma H(\gamma g)$ over rational left translates and gives $\sup_g |\phi(g)| H_{\mathrm{quot}}(g)^N < \infty$ for each $N \in \mathbf{N}$. For a nonzero multilinear intertwiner on finite vectors, fix a tuple with nonzero image. Fixing all but one entry gives reference embeddings with the lower Sobolev bounds of 36.9. Extend each slice by that proposition. For any fixed finite vector $v_2 \in V_2$ and any $k_2 \in K_2$ (respectively $X_2 \in \mathfrak{g}_2$), both $w_1 \mapsto T(w_1, \sigma_2(k_2)v_2)$ and $w_1 \mapsto \pi(k_2)T(w_1, v_2)$ (respectively $w_1 \mapsto T(w_1, d\sigma_2(X_2)v_2)$ and $w_1 \mapsto d\pi(X_2)T(w_1, v_2)$) are continuous on H_1^∞ by continuity of $T(\cdot, v_2)$ and of the $(\mathfrak{g}_2, K_2)$-action on E, and they coincide on the dense subspace $V_1 \subset H_1^\infty$. Consequently $v_2 \mapsto T(w_1, v_2)$ is a $(\mathfrak{g}_2, K_2)$-intertwiner on V_2 for every $w_1 \in H_1^\infty$, so Proposition 36.9 extends each limit slice continuously and Banach–Steinhaus supplies uniform continuity as a finite argument tends to a smooth argument. Iterate over the positions. Equivariance passes by density. The reference embeddings have continuous inverses on closed images, so continuous seminorms there are dominated by finite sums of the target seminorms. In smooth coordinates these give the required bounds by finitely many derived words. The zero map extends by zero.
Lean: `GlobalTheory.FactorExtension.extend_positions` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.MultilinearGKMapExtendsToSmoothVectors`, lines 69–258)

Transport the primitive and divisor data and use the seven member duties of 33.6. The resulting family belongs to the covolume class $\mathcal{F}_{\mathrm{cov}}$ of Definition 35.2. This class imposes the page and window conditions, small first modulus, tame distinctness, the chosen wall, boxes, the q, d rows, $C_{YL} \le 9$, smooth stabilizer

orders, twist goodness, the selected-family design with constants $3, C'_H$ for some C'_H, and level and stabilizer size. It requires eventually at every member and word

$$|D\Gamma_u| \mid M(2fN(m)), \qquad M(n) = \prod_{\substack{p \le n+1 \\ p \text{ prime}}} p^{\sum_{a\ge 0}\lfloor n/(p^a(p-1))\rfloor}.$$

At a member point p, write $\mathbf{N}_p(C, C_H)$ for its numerical datum and $S_{p,0}$ for its reference realization. Under the family division estimate (36.18), the third row is the following closed assertion on $\mathcal{F}_{\mathrm{cov}}$.

$$\begin{gathered}\exists C_0 \ge 0,\ \exists x_0 \in \mathbf{R},\ \forall x > x_1,\ x \ge x_0,\ \forall i \in I_x,\ \forall p \text{ at } (F_m, x, i),\\ p \in \mathcal{P} \cap \mathcal{R} \implies \forall C \ge C_0, \quad (36.12)\text{–}(36.14) \text{ hold on } S_{p,0} \text{ at } \mathbf{N}_p(C, C_H).\end{gathered} \tag{A.500}$$

Here $\mathcal{P}$ and $\mathcal{R}$ are the sets of member points in Definition 36.6. For every member point p transported along the linked slot, the primitive, divisor, goodness, geometric compatibility, and third-layer covolume rows verify the primitive indefinite realization, separated level-one types, frame and division identifications, exported linear maps, and twisted single-index places required for $p \in \mathcal{P}$; simultaneously, transporting the two-run reference core, canonical levels, positive signature ranks, half-error estimates, recursion-prime coefficient-field conditions, small-field bounds, and level-character and series identifications verifies $p \in \mathcal{R}$. Consequently $p \in \mathcal{P} \cap \mathcal{R}$, and Proposition 36.8 transports (36.12)–(36.14) to $S_{p,0}$.

Lean: `GlobalTheory.BigOn.SurjectiveOntoX` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ExportSt`, lines 47–48)
Lean: `GlobalTheory.BigTop.FamHyp` (`Development.AbelianVarietyTheory.Definitions.Divisors.TupleReadBaseTupleEqFamily`, lines 503–507)
Lean: `Formalism.HermitianLatticeCounting.S6.minkowskiM` (`Formalism.FlagUnipotentLattices.MemOfOrthogonalAnisotropic`, lines 500–500)
Lean: `GlobalTheory.BigOn.More` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.SingleIndexPlacesAreTwisted`, lines 82–82)
Lean: `GlobalTheory.BigTop.EventuallyAtPoints` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.Realised`, lines 117–118)

The fourth row is closed on $\mathcal{F}_{\mathrm{cov}}$. Eventually, at every $p \in \mathcal{P}$ and for every $C \in \mathbf{R}$, the logarithm of the selected weight dominates the layer lower bound $\ell_{6,j}(p; C, C_H)$ in the numerical datum

$$\ell_{6,j}(p; C, C_H) \le \log \lambda_{j,\mu_0,b_0,v} \qquad \text{for all } j, b_0, v.$$

The eight rows in order are the primitive isometry and passage, covolume, surjectivity, lower weight, functional, definite, multiplicity and κ rows. Rows $3, 5$ use the division estimate (36.18), row 4 uses the leaf hypothesis in transport, and row 6 is 36.5. Apply Definition 36.10 to their conjunction and restrict to $W_3(\mathcal{A})$. Take the maximum of the first six C-thresholds, with the first nonnegative, then the maximum of the eight x-cutoffs and the selection cutoff.

Lean: `GlobalTheory.BigTop.LambdaLowerBound₂` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.Realised`, lines 126–126)
Lean: `Formalism.L2.Indef.IndefData.L6` (`Formalism.LocalToGlobal.IndefiniteForms.IndefSide`, lines 325–326)
Lean: `GlobalTheory.Fc.S0.ontoX_and_lamLow_of_unitaryGroupHypotheses'` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.SurjectiveAndLowWeightOfUnramifiedExponents`, lines 16–29)

□

Proof of 36.8

Proof. Fix an auxiliary $C_* \ge 1$ above the input thresholds and take the maximum of the associated cutoffs. The following estimates are at the resulting point record.

Remainders for the two runs. In layer j let n_j be the ambient cusp-form dimension, B the ambient lattice and Θ the initial theta lattice of the run. At a base place $\mathfrak{p}$ round the prescribed depth times the theta power up to $E_{\mathfrak{p}j}$. The two-run input bounds the index on every subspace by $E_{\mathfrak{p}j} \dim V + r_{\mathfrak{p}j}$. On extension take $E_{vj} = e(v/\mathfrak{p})E_{\mathfrak{p}j}$ and $r_{a,vj} = e(v/\mathfrak{p})r_{\mathfrak{p}j}$ for run a. Then for $a = 1, 2$

$$[B_V : \Theta_V]_v \le E_{vj} \dim V + r_{a,vj}, \qquad r_{a,vj} \ge 0, \qquad \sum_{v\in S} \ell_v r_{a,vj} \le n_j f m b\, \delta_2 \quad (0 \le \delta_2 \le 1) \tag{A.501}$$

for every finite place set S, by 20.7. Use run 2 at its distinguished characteristic and run 1 elsewhere. The remainders r_{vj} of this choice cost at most $2n_j fmb$.
Lean: `GlobalTheory.BigOn.hasIndexBound_of_runs` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.HasIndexBoundOfRuns`, lines 129–190)
Lean: `GlobalTheory.TraceClassRank.on_core_atPlaces` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.TraceLatticeLeThetaLatticeAtPlace`, lines 233–242)

Apply 20.12 with mirabolic P, unipotent U, retained projector e, and $s = [P : U]$. The product character meets the character condition in a factor in which a coset lies outside U. The normalized stable theta lattice is contained in $\sigma_{vj}\Theta_K$, and the integral character sum puts $|U|(\Theta_K)_W$ into Θ_0 in the notation of that lemma. The twisted projection adds a normalized error $\xi_{vj} \geq 0$ with $\sum_j \xi_{vj} \leq p_v^{-2}$. This follows by restricting the two-character goodness budget to the chosen character and the injective layer labels. To identify the lattice in the index, let ϕ_j be the coordinate embedding into the ambient cusp-form space, let Z be its image, and let B be the ambient integral lattice. The coordinate identification sends M_j to $B \cap Z$. At $v = w, \bar{w}$ it sends A_{j,t_w} to $\theta' \pi e_X M_t$, where M_t is the class lattice, e_X the source projector, π the map to the chosen representatives, and θ' the ambient theta map. For $x \in e_X M_t$, $x - \pi x \in \ker \theta'$. Since $\theta' e_X = e\theta'$, the image is exactly $e\theta' M_t$. Its localization contains the comparison lattice X_3 of 20.12. Thus the index bounded there is the index of the same image that occurs in the determinant formula of 21.27. Write $u_v = v(|U|)/e_v$ and $d_{vj} = (E_{vj} - v(\sigma_{vj}))/e_v$. Using (A.501) in 20.12 gives, for $v = w, \bar{w}$,

$$[M_j : A_{j,t_w}]_v \leq e_v \left((d_{vj} + u_v) D_R + \frac{r_{vj}/e_v + n_j u_v}{s} + \xi_{vj} \right). \tag{A.502}$$

Here $t_{\bar{w}} = t_w$ by 18.6. If d_{0j} is the retained dimension before rank loss, the rank and share rows give $(1-y)n_j \leq |U| d_{0j}$, $(1-y)|U| \leq s$, and $(1-\delta_j)d_{0j} \leq D_R$ for $y = 1/(4(fk)^3)$, $0 \leq \delta_j \leq 1/4$. In particular $n_j/s \leq 3D_R$. At the theta primes $u_v = fr(r-1)/2$ with $r = k + m - 1$, and elsewhere $u_v = 0$.
Lean: `GlobalTheory.TwistIndex.exists_leafTW1_of_twGoodAt` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.LeafTW1OfAllConjugatesGood`, lines 439–465)
Lean: `GlobalTheory.OnAtPlacesBase.on_base_atPlaces` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.TwistedPlacesBoundChainTwoRunChain`, lines 43–62)

Depths on ordinary and exceptional primes. Let w be twisted and let $p = p_w$ lie outside the small primes and q, d, g_0, g_1. For $p \leq Y$, both d_{wj} and $d_{\bar{w}j}$ are at most $F_{0,j,w} + a_j(p) + 1$. For $p > Y$, both depths vanish and $F_{0,j,w} \geq 0$. The nonnegative errors ϑ_p and α_p in $a_j(p)$ satisfy, for every finite set S of primes,

$$\sum_{p \in S} \vartheta_p \log p \leq C_*/k, \qquad T \sum_{p \in S} \alpha_p \log p \leq C_* mb. \tag{A.503}$$

Here T is the theta power, Y the depth cutoff, and, with W_j the layer weight and $S_e = S_c + \eta_{\mathrm{sz}}$,

$$A_1 = C_*(mW_j \log k + fkS_e), \quad A_2 = C_* kW_j, \quad a_j(p) = A_1/p + A_2\vartheta_p + T\alpha_p + \mathbf{1}_{p>X''} A_2/p^2.$$

Set $s_B = W_j + fS_e$, $s_C = C_* m(2 + \log_2 Q_S) s_B$, and $\beta_p = \mathbf{1}_{p=d} fkb'$. The depth comparisons at $w, \bar{w}$ bound their combined excess over $2F_{0,j,w}$ by $H_j(p)$. The bounds include the two unipotent terms in (A.502).

p	depth and base term	$H_j(p)$
g_0, g_1	$d_{vj} \leq 2T - \nu_{vj}$, $\nu_{wj} + \nu_{\bar{w}j} = -2F_{0,j,w}$	$4T + 4fr^2$
small	$d_{vj} \leq s_C + T + 1$, $F_{0,j,w} = 0$	$2s_C + 2T + 2$
q, d	$d_{vj} \leq s_B + T + 1$, $-2\beta_p \leq F_{0,j,w} \leq 0$	$2(s_B + T + 1) + 4\beta_p$
other $p \leq Y$	$d_{vj} \leq F_{0,j,w} + a_j(p) + 1$	$2a_j(p) + 2$
other $p > Y$	$d_{vj} = 0$, $F_{0,j,w} \geq 0$	0.

(A.504)

In the theta row let n_j^{lev} denote the layer's level normalization factor, and put $\nu_{vj} = v(n_j^{\mathrm{lev}})/e_v$. The level normalization gives $v(\sigma_{vj})/e_v = \nu_{vj} + s_{vj}$ with $s_{vj} \geq 0$. The two valuations of the level factor add to the sum of the positive and negative Fock degrees, so $\nu_{wj} + \nu_{\bar{w}j} = -2F_{0,j,w}$ by 15.10. For the depth before subtracting $v(\sigma_{vj})$, the unrounded normalized bound is $T + TD_L/(p-1)$. Here $TD_L \leq D_\kappa < (p-2)/2$. Rounding on the lower field adds at most the reciprocal of the ramification index there, which is at most $1/(p-1)$ at a type prime. Hence $E_{vj}/e_v \leq T + 1 \leq 2T$. Subtraction gives $d_{vj} \leq 2T - \nu_{vj}$. In particular the sum of the two depths is at most $4T + 2F_{0,j,w}$. The two unipotent contributions are at most $8u_w \leq 4fr^2$.

The small-prime comparison uses the split colength and Hodge–Tate inclusion. The ordinary comparison uses $x \ge 3,\ kb' \le b,\ x^{e_z/5} \ge 3$, and $C_{nW} \ge 10$, where C_{nW} is the member's weight-width constant in the ordinary prime-depth bound.
Lean: `GlobalTheory.RunDepth.leafDepth_all_run` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.LeafDepthAllRun`, lines 13–21)
Lean: `GlobalTheory.BigOn.leafDepth_all_record` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LeafDepthAllRecord`, lines 17–26)
Lean: `GlobalTheory.BigOn.geoSt_of_pageSt_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.GeoStOfPageSt`, lines 21–46)

Put $b_{jw} = H_j(p_w) + (r_{wj}/e_w + r_{\bar{w}j}/e_{\bar{w}})/(sD_R)$. The twist errors summed over both places and all layers cost at most $2e_w/(D_R p_w^2) \le c_w$. Thus

$$D_R^{-1} \sum_j i_{jw} \le \sum_j (2e_w F_{0,j,w} + e_w b_{jw}) + c_w. \tag{A.505}$$

The remainders cost at most $4n_j fmb/(sD_R) \le 12fmb$, since e_w, ℓ_w are preserved by conjugation. Write $Z_0 = fmb$. The page bounds $W_j \le 2fb,\ S_e \le b,\ T^2 \le b,\ kb' \le b,\ \log(qd) \le m$ give $14Z_0$ at q, d. At the small primes $(2 + \log_2 Q_S)\sum_{\text{small}} \log p \le 11L$ and $fm \ge 352$ give $(66C_* + 4)Z_0 L$. At $g_0, g_1,\ (frL)^{12} \le b,\ f \ge 8,\ r \ge 9,\ L \ge 10$, and $\log(g_0 g_1) \le 18L$ give $4b$, using $36TL, 36fr^2 L \le b$.
Lean: `Formalism.L2.Indef.Tree.OnPage.leafBad_of_page` (`Formalism.LocalToGlobal.IndefiniteForms.IndefiniteTree.ONPage`, lines 226–276)

The prime estimates $\sum_{p \le Y} \log(p)/p < \log Y$ and $\sum_{p \le t} \log p < 1.01624t$ are [34, (3.24), (3.32)]. Partial summation gives $\sum_{p > X''} \log(p)/p^2 < 2.04/X''$. By (A.503) the ordinary cost is at most

$$2(A_1 \log Y + A_2 C_*/k + C_* mb + 2.04 A_2/X'' + Y \log 4). \tag{A.506}$$

Use $Y \le mb,\ \log Y \le 2L,\ \log k \le L,\ X'' \ge k,\ kS_e L \le mb$. The first term costs at most $8C_* Z_0 L^2 + 4C_* Z_0$, and the remaining terms at most $(4C_*^2 + 10.16C_* + 2.78)Z_0$.

Exponents of the cuts. The exponent input compares $J_{0,j,w}$ to the integer model exponent $J_j(p_w)$. For each layer weight λ_τ of rank $N_r = (k-1) + j_\tau + t + q_\tau$ (where j_τ and q_τ count the lower and upper indices flanking the t-block), define the shifted coordinate set $U_\tau(a) = \lambda_{\tau,a} + N_r - 1 - a$ $(0 \le a < N_r)$, let $m_B = U_\tau(k - 1 + j_\tau)$ be the block anchor (with bottom block coordinate $\mathrm{bot}_\tau = U_\tau(k - 1 + j_\tau + t - 1)$), let $[x_{\tau,-}, x_{\tau,+}]$ lie in the prescribed cloud interval of width $S = S_c$, let h be the layer height cutoff, and let $T_{2,j}$ be the set of thick positions (satisfying $x_{\tau,+} \le h$, while non-thick positions $\tau \notin T_{2,j}$ satisfy the top width bound $\le S + b'(1 + 1/x + 1/x^2) + \eta + (t-1)/2$ or thin width bound $\le S + w'' + \eta + (t-1)/2$). For each prime ℓ and embedding τ, define $\mathrm{ex}_{\mathbf{Z}}(N_r, \lambda_\tau, \ell)$ as the least $e \in \mathbf{N}$ such that ℓ^e annihilates the quotient of the $\mathbf{Z}_{(\ell)}$-integral dual Weyl module by the $\mathbf{Z}_{(\ell)}$-integral Weyl module for (N_r, λ_τ), set the local integer exceptional exponent $J_{\mathrm{ex}}(\tau, \ell) = \mathrm{ex}_{\mathbf{Z}}(N_r, \lambda_\tau, \ell)$, the excluded-class count $e_{\mathrm{cl}}(\tau, \ell) = \mathrm{ex}_{\mathbf{Z}}(k-1, \lambda_\tau, \ell)$ (truncating λ_τ to its initial $(k-1)$-tuple), the total exceptional exponent $J_A(\ell) = \sum_\tau J_{\mathrm{ex}}(\tau, \ell)$, the congruence indicator $I_\ell = \mathbf{1}\{\exists\, y \in [x_{\tau,-}, x_{\tau,+}] \cap \mathbf{Z} : \ell \mid y - m_B\}$, and the drop-cut step $d_0 = \min\{d \in \mathbf{Z}_{\ge 1} : \ell \mid (x_{\tau,-} - m_B) - d\}$. Putting $\kappa(\ell) = \lfloor h/(\ell - 1) \rfloor$, let $\mathrm{mw}(\tau, \ell)$ count indices $u \in U_\tau \setminus [\mathrm{bot}_\tau, \mathrm{bot}_\tau + t - 1]$ congruent modulo ℓ to an integer in $[\mathrm{bot}_\tau - \kappa(\ell),\ \mathrm{bot}_\tau + t - 1 + \kappa(\ell)]$, let $\mathrm{mw}_F(\tau, \ell)$ be the same count at the Frobenius position τ', using $U_{\tau'}$ and its bottom unless the position is conjugated, in which case use $\{\lfloor h \rfloor - u : u \in U_{\tau'}\}$ with bottom $\lfloor h \rfloor - m_B(\tau')$, and set the depth parameter $I = \lfloor q_\tau h + (\ell\, \mathrm{mw}_F(\tau, \ell) + \mathrm{mw}(\tau, \ell))(t + 2\kappa(\ell)) \rfloor$. Define the real short-weight upper function

$$c_{\mathrm{sh}}(\tau, \ell) = (q_\tau + 2j_\tau)\frac{h + N_r}{\ell - 1} + e_{\mathrm{cl}}(\tau, \ell) + k\left(\frac{x_{\tau,+} - x_{\tau,-}}{\ell} + I_\ell\right) + \mathbf{1}_{d_0 < x_{\tau,-} - m_B} \min\left(k, t, \frac{I}{d_0}\right) + \frac{I}{\ell}\left(1 + \log_+ \frac{h}{\ell}\right),$$

bounded above by the coarse real majorant $c_{\mathrm{coarse}}(\tau, \ell) = c_{\mathrm{sh}}(\tau, \ell) + 6(q_\tau + 2j_\tau)$. Put $N = k + m(t+1)$, $l_* = mb/\big(nt^2(N+t)L^2 + kn^2(S + t + L)\big)$ (with $l_* \le l_J$ for the cut threshold l_J of hypothesis (8)), and $k_{\max} = \lfloor 2b/l_* \rfloor$, and define the excluded prime set as the primes ℓ dividing some positive integer $z \in [b - (t-1)/2 - k_{\max},\ b + S + \eta + 1 + (t-1)/2 + k_{\max}]$; since every such z satisfies $0 < z < l_J^2$ and therefore has at most one prime factor exceeding l_J, at most $S + \eta + t + 2k_{\max} + 1$ excluded primes exceed l_J. Put $S_j = \{q, d, p_{\mathrm{de}}, p'_{\mathrm{de}}\}$ (so $|S_j| \le 4 \le 5$); because every prime in the host small set lies below the cut threshold l_J, replacing the full exceptional set by S_j leaves $J_j(\ell)$ unchanged at every prime ℓ. Specifically, $J_j(\ell) = J_A(\ell)$ whenever $\ell \le l_J$, ℓ is excluded, or $\ell \in S_j$; for all other primes ℓ, $J_j(\ell) = \sum_\tau c_\tau(\ell)$,

where $c_\tau(\ell) = J_{\mathrm{ex}}(\tau,\ell)$ for $\tau \notin T_{2,j}$ and $c_\tau(\ell) = \min(J_{\mathrm{ex}}(\tau,\ell), \lfloor c_{\mathrm{sh}}(\tau,\ell)\rfloor) \le \min(J_{\mathrm{ex}}(\tau,\ell), c_{\mathrm{coarse}}(\tau,\ell))$ for $\tau \in T_{2,j}$ (using $c_{\mathrm{sh}} \le c_{\mathrm{coarse}}$). For any prime $\ell > \max(h, l_J)$, the thin/top width bounds (at $\tau \notin T_{2,j}$) and the thick-position height bound $x_{\tau,+} \le h$ (at $\tau \in T_{2,j}$) force every local integer exceptional exponent $J_{\mathrm{ex}}(\tau,\ell) = 0$, hence $J_A(\ell) = \sum_\tau J_{\mathrm{ex}}(\tau,\ell) = 0$ and every truncated summand $\min(J_{\mathrm{ex}}(\tau,\ell), \lfloor c_{\mathrm{sh}}(\tau,\ell)\rfloor) = 0$ (and $\min(J_{\mathrm{ex}}(\tau,\ell), c_{\mathrm{coarse}}(\tau,\ell)) = 0$), so the integer model exponent $J_j(\ell)$ vanishes identically for $\ell > \max(h, l_J)$ (even though the real majorants c_{sh} and c_{coarse} themselves remain positive). On any finite prime set S_0, decompose $\sum_{\ell\in S_0} J_j(\ell)\log\ell$ into four explicit sums: (i) for non-thick positions $\tau \notin T_{2,j}$ (where $J_{\mathrm{ex}}(\tau,\ell) = 0$ for $\ell > 2b'$), the prime-counting estimate on $\pi(2b')$ bounds their total by $1.3fN_r^2 b'$; (ii) for the excluded primes exceeding l_J together with S_j ($|S_j| \le 5$, totaling at most $S+\eta+t+2k_{\max}+6$ primes), bounding each $J_A(\ell)\log\ell$ by $fN_r^2 L$ contributes at most $(S+\eta+t+2k_{\max}+6)fN_r^2 L$; (iii) for thick positions $\tau \in T_{2,j}$ at small primes $\ell \le l_J$, each thick position contributes at most $2mbL^2$. Indeed, for $\ell < 2\eta + t$ the full-rank bound gives $J_{\mathrm{ex}}\log\ell \le \binom{N_r}{2}\log h \le 4t^2 L \le 2mbL/\ell$ by the size bounds. For $2\eta + t \le \ell \le l_J$, cut after the first $k-1$ coordinates. The remaining block has spread below ℓ and exponent zero. The first block and the cross term together give $J_{\mathrm{ex}}\log\ell \le k^2\log(S+k) + k(m+t)\log(3h) \le 2mbL/l_J \le 2mbL/\ell$. Sum using $\sum_{\ell\le l_J} 1/\ell \le L$. (iv) For thick positions $\tau \in T_{2,j}$ at primes $\ell > l_J$ outside S_j and the excluded set, each difference between an outside coordinate and a window integer has absolute value strictly between 0 and ℓ, or has a sign for which it lies in the excluded interval. Neither permits divisibility by ℓ. This applies at the Frobenius position as well, since reflection preserves absolute differences. Thus $\mathrm{mw}(\tau,\ell) = \mathrm{mw}_F(\tau,\ell) = 0$ and hence $I \le q_\tau h$, so bounding $c_\tau(\ell) \le \min(J_{\mathrm{ex}}(\tau,\ell), c_{\mathrm{coarse}}(\tau,\ell))$ and summing over $\ell \le h$ via Rosser–Schoenfeld's bound $\sum_{\ell\le t}\log\ell < 1.01624t$ and Mertens's bound $\sum_{\ell\le t}(\log\ell)/\ell < \log t$ bounds each thick position by $14mbL^2 + 2k(S+1)(L+2)$. Adding (iii) and (iv) gives at most $16mbL^2 + 2k(S+1)(L+2) \le 30mbL^2 + 2k(S+1)(L+2)$ per thick position $\tau \in T_{2,j}$, totaling $30|T_{2,j}|mbL^2 + |T_{2,j}|\cdot 2k(S+1)(L+2)$, while (i) and (ii) total $1.3fN_r^2 b' + (S+\eta+t+2k_{\max}+6)fN_r^2 L$. At the growth scale $X' = \exp(x^{\varepsilon/3})$ (with polynomial rank bound $A \ge \max(f,t)$, $N_r \le 2t$, $|T_{2,j}| \le f \le 2n$, $w'' = b/(nX')$, $b' = xb/X'$, and $k(S+1) \le 2w''$), the three tail terms satisfy $|T_{2,j}|\cdot 2k(S+1)(L+2) \le 8(L+2)b/X' \le 2b$, $1.3fN_r^2 b' \le 5.2A^3 xb/X' \le b$, and $(S+\eta+t+2k_{\max}+6)fN_r^2 L \le 20A^7 x^2 L(L^2 + b/X') \le b$ beyond the family threshold; summing yields at most $2b + b + b = 4b \le 2mb$ since $m \ge 2$. Adding the thick main term and tail gives

$$0 \le J_{0,j,w} \le J_j(p_w), \qquad \sum_{p\in S_0} J_j(p)\log p \le 30|T_{2,j}|mbL^2 + 2mb \tag{A.507}$$

for every finite prime set S_0.

Lean: `GlobalTheory.ExpTie.Asm.hasExpBound_at` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExpansionOfNumericsModel`, lines 50–76)

Lean: `UpperTheory.ExcSet.sum_exponent_log_le` (`Development.AutomorphicArithmetic.PrimeDiscriminants.ExcSet`, lines 50–73)

Lean: `GlobalTheory.sum_exponent_log_le_of_cut` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.SumExponentLogLeOfCut`, lines 12–29)

By 20.7, the exponent and remainder costs are at most $32Z_0L^2$ and $12Z_0$. Because $Z_0L^2 = fmbL^2 = LB_0 \le L(B_0+N_c)$ and $L \ge 1$, $N_c \ge 0$ give $Z_0 \le Z_0L \le Z_0L^2 \le L(B_0+N_c)$, the preceding prime estimates bound $\sum_{w\in\mathrm{Tw}} \ell_w e_w(b_{jw} + J_{0,j,w})$ by $E = (4C_*^2 + 25C_* + C_b + 44)L(B_0+N_c)$, where $C_b = 14 + (66C_*+4) + 4$. Moreover, the page hypothesis identifies the member's numerical-datum error unit with $L(B_0+N_c)$, while dividing the family division estimate (36.18) by the common rank D_R bounds the normalized twisted cut-length sum $D_R^{-1}\sum_{w\in\mathrm{Tw}} \ell_w a_{jw}$ by $(|C_N|+1)L(B_0+N_c)$, supplying the uniform member-independent error unit needed at the page step.

Lean: `GlobalTheory.BigOn.pageSt_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.PageSt`, lines 21–53)

By (A.505), $R_w \le \sum_j e_w b_{jw}$. The positive-part formula for R_w gives (36.12) and (36.13). Averaging the bounds for b_{jw} gives $\sum_w \ell_w e'_{jw} \le E$. At each layer the exponent cost is at most E. Combining this with the normalized page-step denominator budget $(|C_N|+1)L(B_0+N_c)$ from (36.18) gives (36.14) for $C \ge \max\big(|C_N|+1,\, 2(4C_*^2 + 25C_* + C_b + 44)\big)$ (uniform in (x,i) and D_R, exceeding both $|C_N|+1$ from (36.18) and twice the nonnegative part of the canonical error constant) on the interval fixed for C_*. □

Proof of 36.9

Proof. Schwartz integration from (36.15) and the Casselman–Wallach globalization theorem give the smooth extension [190, §2.5, Theorem 10.6 and Proposition 11.2]. The unitary-source extension must also satisfy

the explicit lower Sobolev bounds.
Lean: PositivityTheory.CW.lift_of_steps (Development.WeilBound.CoreStatements.PublishedTheorems.CasselmanWallach.CasselmanWallachAdmissible, lines 38–59)

For a defining seminorm p nonzero on $T(V)$, let $Q(z) = \sup_g h(g)^{-N} p(\pi(g)z)$ with the exponent of p. Then Q is continuous, K-invariant and $Q(\pi(g)z) \le h(g)^N Q(z)$. Convolve the dense orbit span in H with compactly supported smooth functions and use an approximate identity. For each nonzero $v \in V$, the span of its G-orbit is dense in H^∞. Thus $P = Q \circ T$ is a norm on V.
Lean: PositivityTheory.CW.e1a (Development.WeilBound.CoreStatements.PublishedTheorems.CasselmanWallach.ExistsFunctional, lines 27–80)
Lean: PositivityTheory.CW.e1b (Development.WeilBound.CoreStatements.PublishedTheorems.CasselmanWallach.ExponentialKSet, lines 38–68)

For each $a \neq 0$ in a compact Casimir eigenspace choose λ by Hahn–Banach with $|\lambda| \le P$ and $\lambda(a) = P(a)$. The K-average $B(x,y)$ of $\overline{\lambda(\sigma(k)x)}\lambda(\sigma(k)y)$ satisfies $|B(x,y)| \le P(x)P(y)$ and $B(a,a) > 0$. The compact Casimir is symmetric for B. The functionals $B(a,-)$ span the dual of each finite-dimensional eigenspace and vanish on the other eigenspaces. Decompose $\eta \in V$ and extend by density to obtain $|\langle \eta, w\rangle| \le c_\eta Q(Tw)$ on H^∞.
Lean: PositivityTheory.CW.e2c (Development.WeilBound.CoreStatements.PublishedTheorems.CasselmanWallach.SeminormApproximation, lines 46–119)
Lean: PositivityTheory.CW.c5 (Development.WeilBound.CoreStatements.PublishedTheorems.CasselmanWallach.Imports, lines 11–52)

For a finite generating set $I \subset V$ the map $(f_e) \mapsto \sum_{e \in I} \sigma(f_e)e$ gives the Schwartz quotient topology [190, §6.1]. The full Casimir acts by a real scalar on the irreducible Harish–Chandra module, so the unitary realization satisfies the Casimir hypothesis of Proposition 3.9. The quotient seminorm from $\sum_e \int |f_e| h^N$ is bounded by $M\|Az\|$, $A = (1+\Omega_K)^m$, by the Casimir Sobolev estimate [190, Proposition 3.9]. Consequently if $|\langle w, \sigma(g)e\rangle| \le B_0 h(g)^N$ for every g, e, then $|\langle w, z\rangle| \le M B_0 \|Az\|$. By unitarity and $h(g^{-1}) = h(g)$, apply this with $w = Ay$, $z = A^{-1}y$, $y \in V$, and $B_0 = Q(TAy)\sum_e c_e$. Division by $\|y\|$ gives the bound for the identity word since $TAy = A_E Ty$ for the derived polynomial $A_E = (1+\Omega_K)^m$; because Q is a continuous seminorm on E and A_E is a polynomial in the continuous derived operators $d\pi(X)$, the composite seminorm $z \mapsto Q(A_E z)$ is continuous on E.
Lean: PositivityTheory.CWB.maxPr (Development.WeilBound.CoreStatements.PublishedTheorems.CasselmanWallach.EtaleCohomologyNegativeVanishingDegreeZero, lines 91–91)
Lean: PositivityTheory.CW.e3c (Development.WeilBound.CoreStatements.PublishedTheorems.CasselmanWallach.EigenspacesOrthogonal, lines 23–115)

Apply this bound to $d\sigma(D)y$ and extend by density. If Tw_i converges, then w_i is Cauchy in the complete smooth topology. Its limit maps to the given limit. The zero map has closed image as well. Choosing an algebraic vector-space complement $H = H^\infty \oplus W_{\mathrm{alg}}$ and extending T by zero on W_{alg} produces an algebraic linear extension from H^∞ to H. □

Proof of 36.12

Proof. Put $w = \iota^{-1}v$. The two coordinates give the same prime ideal, hence the same logarithmic weight and ramification index e_w. Let R_w be this base discrete valuation ring and put $e_w = \mathrm{length}_{R_w}(R_w/\ell R_w)$. If O_i/R_w has ramification index d, a finite-length base quotient of length a has length da after scalar extension, while $\mathrm{length}_{O_i}(O_i/\ell O_i) = de_w$. Thus its normalized length is $da/(de_w) = a/e_w$, as in the elementary-divisor calculation of Lemma 33.54. Additivity over the local Hecke summands gives $a_{jv} = e_w \eta_{a(j)}(w)$. The selector has nonreference type at each twisted place by Lemma 18.6. Thus reindexing gives the subsum over $\iota^{-1}(\mathrm{Tw})$ of the sum defining $\eta_{a(j)}(\mathcal{L})$. Every remaining term is nonnegative. □

Proof of 36.13

Proof. Apply 26.1 in the member's local data and take the supremum of the nonnegative finite sums. Eventually $(\log x)^2 \le L$ and $\sqrt{b} \le L(B_0 + N_c)$, so (36.17) gives (36.18) using the assumed equality $d_{j,0} = D_0$. Apply Lemma 36.12 with the reference-row type of 18.6. The division tie identifies its local quotients, and the twisted-support condition puts their arithmetic places in $S_{\mathcal{L}}$. Thus the twisted sum $\sum_{v \in \mathrm{Tw}} \ell_v a_{jv}$ is bounded by $\eta_{a(j)}(\mathcal{L})$ in (36.18).
Lean: GlobalTheory.Top.smallLevelRankEq_closed_of_levelProjection_surjective (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.SmallLevelRankEqClosedOfLevelProjectionSurjective, lines 13–47)
Lean: UpperTheory.Top.hypV_of_cls (Development.AutomorphicArithmetic.CharacterBoxes.ZeroFreeAllOfTopG, lines 16–17)

□

Proof of 36.16

Proof. Put $s_{\mathrm{cyc},\ell} = \max(1, v_\ell(N))$ and $\lambda_\ell = s_{\mathrm{cyc},\ell} + \log_\ell(2H_B) + 1$. At 2, $4 \mid Q_S$ and $s_0 = s_{\mathrm{cyc},2} \geq 2$. At 3 outside the small set the local fields are unramified, $s_0 = s_{\mathrm{cyc},3} = 1$, and $(-d/3) = 1$ by the split bound.
Lean: `UpperTheory.N.onAt_eq_two_bounds_of_setup` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.NRowTwo`, lines 11–43)
Lean: `UpperTheory.N.rowThreeAt_of_setup_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.NRowThree`, lines 12–50)
Lean: `GlobalTheory.HubLoc.ramIdx_eq_one` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.FinrankMaxUnramifiedSubfieldPadicFinExtPos`, lines 94–111)

Since $N^{36}t^{150} \leq \sqrt{b}$, $N \leq b$. The exponents of 36.22 used in 36.23 have $D_h \leq 3\lambda_\ell$ and $\lambda_\ell \leq 6H_B s_{\mathrm{cyc},\ell}$. With $c = 1000$ the bounds are

$$e' \leq cH_B s_{\mathrm{cyc},\ell}, \quad c_0^{\mathrm{cyc}} \leq cfn_V\lambda_\ell, \quad h_W(u) \leq rc_0^{\mathrm{cyc}}, \quad c_B^{\mathrm{cyc}} \leq ctfn_V\lambda_\ell, \quad h_0 \leq c\lambda_\ell, \quad \delta \leq t\lambda_\ell. \tag{A.508}$$

For c_B^{cyc}, substitute $a_0 = 2fn_V$ to bound $40(t+1)(a_0+1)D_h$ by $720tfn_V\lambda_\ell$. On $(-e)$ the inertia difference has valuation $s_0 + v_\ell(e)$. This gives $h_0 \leq s_0 + \lfloor \log_\ell t \rfloor$ and $\delta \leq t(s_0+1)$.
Lean: `UpperTheory.N.h0_le` (`Development.AutomorphicArithmetic.ResidueCharacters.LenUnitInvariantsLe`, lines 55–71)

Each degree sum and cardinality of places is at most $2f$. In (36.25) the e'-terms together cost at most $6cfmH_B s_{\mathrm{cyc},\ell}$. The terms $d_u m\delta$, 2δ, $h_W + rc_0^{\mathrm{cyc}}$ and the three terms in the final line cost at most $2, 4, 4, 6, 12, 2$, respectively, in units of $cf^2t^2n_V\lambda_\ell$ by $m \leq f$, $r \leq t$. The tame term is at most ℓ^{-2} by (36.26). Thus the allowance in (36.21) is at most $Afm H_B s_{\mathrm{cyc},\ell} + Bf^2t^2n_V\lambda_\ell$ for $A = 6000$, $B = 30000$. The same calculation applies to the odd small split primes with $s_0 = s_{\mathrm{cyc},\ell}$.
Lean: `UpperTheory.N.Num.boundSplit_le` (`Development.AutomorphicArithmetic.ResidueCharacters.NumSplit`, lines 170–209)

For a family exponent $C_{\mathrm{par}} \geq 1$ bound f, n, t by $x^{C_{\mathrm{par}}}$ and n_V by $2x^{C_{\mathrm{par}}}$. The bounds $\ell^{v_\ell(N)} \leq Q_S \leq n^2$ in this range give $s_{\mathrm{cyc},\ell} \leq 3C_{\mathrm{par}} \log x$ eventually. Since $H_B \leq 2b$, the two allowances are bounded by

$$18AC_{\mathrm{par}} fmb \log x, \qquad 2B(3C_{\mathrm{par}} + 9)x^{5C_{\mathrm{par}}+1}L. \tag{A.509}$$

Enlarge the constant C to cover both coefficients and the exponent $5C_{\mathrm{par}} + 1$ (using $L \leq L^2$ eventually), and take the common family cutoff x_0. Because this holds unconditionally for every $\varepsilon > 0$, every family F_m of parameter ε, and every $C_H \in \mathbf{R}$, choosing the closure cutoff $\varepsilon_1 = 1$ and the identically zero threshold datum $\mathcal{K}$ (every preliminary, middle, and last threshold set to 0) immediately satisfies Definition 36.10 on every family class $\mathcal{C}$ (in particular on $W_3(\mathcal{A})$). Evaluating at any linked slot norm choice therefore yields the linked closed bound on $W_3(\mathcal{A})$.
Lean: `UpperTheory.N.Num.cost_le'` (`Development.AutomorphicArithmetic.ResidueCharacters.NumCost`, lines 55–128)
Lean: `UpperTheory.N.boundAtTwoThree_of_local_bounds_at_two_three` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.FamilyBoundAtTwoThreeOfLocalBounds`, lines 37–54)

□

Proof of 36.17

Proof. Write $S = W(k')[[Z]]$. Give S Witt Frobenius on the coefficients and put $\varphi(Z) = (1+Z)^\ell - 1$. The map $1 + Z \mapsto \zeta_{\ell^s}$, for a primitive root ζ_{ℓ^s}, identifies $S/(Q)$ with $\mathcal{O}_{K'}$. Choose compatible primitive roots $(\zeta_{\ell^n})_{n\geq 1}$, and let $\epsilon = (1, \zeta_\ell, \zeta_{\ell^2}, \ldots)$ in the tilt of the completed algebraic closure. The map from the cyclotomic prism $(S, (Q))$ to the Fontaine prism sends $1 + Z$ to the Teichmüller lift $[\epsilon^{1/\ell^s}]$. It sends Q to $([\epsilon] - 1)/([\epsilon^{1/\ell}] - 1)$, a generator of the kernel of Fontaine's map θ.

Apply 36.15 to the underlying $\mathbf{Z}_\ell$-lattice. Let $\mathcal{E}$ be its crystal and $M = \mathcal{E}(S)$. Full faithfulness lifts the O-action and every equivariant map of lattices. Since S is local, its finite projective module M is free. The action of

$$O \otimes_{\mathbf{Z}_\ell} W(k') = \prod_{i : W(k') \hookrightarrow O} O$$

splits M into A-modules $N_i(T)$. Multiplication by Z is injective on each summand. Its quotient by Z is a direct summand of M/ZM, hence is finite and ℓ-torsion free. It is consequently free over the discrete valuation ring O. Lift an O-basis of this quotient. The lifts generate over A by Nakayama's lemma. In a

relation among the lifts all coefficients are divisible by Z. Cancelling Z and repeating puts every coefficient in $\bigcap_n Z^n A = 0$. Thus the lifts form an A-basis.

The linearized Frobenius of M is invertible after inverting Q, by the definition of an F-crystal. Modulo ℓ, $Q = Z^{\ell^{s-1}(\ell-1)}$, so inverting Q and completing ℓ-adically gives the same coefficient ring as inverting Z and completing. The étale evaluation on this cyclotomic prism is the cyclotomic (φ,Γ)-module, by [197, Theorem 5.2 and Corollary 5.8]. Here k' is perfect, $K'/W(k')[1/\ell]$ is totally ramified, and the tower is $K'(\mu_{\ell^\infty})$. Over the Witt vectors of the tilted algebraic closure the crystal comparison identifies the resulting module with the scalar extension of T. Taking the i-component gives the asserted rank. All these identifications commute with lattice maps.

First specialize at $Z = 0$. Constant term gives a morphism of prisms

$$(S, (Q)) \longrightarrow (W(k'), (\ell))$$

over $\mathcal{O}_{K'}$, since $Q(0) = \ell$ and the induced map on the quotients is residue reduction. The action of γ on S fixes $W(k')$ and sends $1 + Z$ to $(1 + Z)^a$. Composing this action with constant term leaves the morphism unchanged. Crystal base change therefore identifies M/ZM with $\mathcal{E}(W(k'))$ and gives the identity action there.

At $\zeta = 1$, write the action in a basis over $L[[t_1]]$ as $v(t_1) \mapsto G(t_1)v(at_1)$. We have $G(0) = I$. There is a unique matrix $H(t_1)$ with $H(0) = I$ and $G(t_1)H(at_1) = H(t_1)$. In degree $j > 0$ the equation determines $(1 - a^j)H_j$ from coefficients of smaller degree. Our depth hypothesis gives $a^j \neq 1$ for every $j > 0$. The columns of H give the required fixed basis.

Now let ζ have order ℓ^b, with $1 \leq b < s$. It is fixed by γ. Frobenius transports the expansion at ζ^ℓ in the predecessor component to the expansion at ζ, by $t_{\zeta^\ell} \mapsto \ell t_\zeta$. This is a change of parameter over L. The constant term of Q in the latter expansion is $Q(\zeta - 1) = \ell$, so the linearized Frobenius is an isomorphism there. It commutes with γ. Induction on b transports the fixed basis to every lower root. In this basis γ multiplies the coefficient of degree j by a^j.

For $y \in N_i(T)$, the constant term of $(\gamma - 1)y$ vanishes at each lower root. The change between the fixed basis and any A-basis is invertible over $L[[t_\zeta]]$. Thus each coordinate in the A-basis vanishes at $Z = \zeta - 1$. Weierstrass division by the monic distinguished polynomial X gives a remainder of degree $< \ell^{s-1}$. The remainder vanishes at all ℓ^{s-1} distinct roots of X, so it is zero. This proves the first inclusion over O. Finally $\gamma(X) = (1 + X)^a - 1 \equiv aX \bmod X^2$. Applying this identity to $X^j y$ gives the second inclusion. Applying an equivariant map to both inclusions gives them on its image. □

Proof of 36.18

Proof. Let $C^\flat$ be the tilt of the completed algebraic closure of $\mathbf{Q}_\ell$, with valuation ring $\mathcal{O}^\flat$. Use the rings

$$R_0 = O \otimes_{\mathbf{Z}_\ell} W(\mathcal{O}^\flat), \qquad R_1 = O \otimes_{\mathbf{Z}_\ell} W(C^\flat).$$

Frobenius acts on the Witt factor. The étale comparison of Lemma 36.17 identifies $R_1 \otimes_B \mathcal{D}$ with $R_1 \otimes_O T$, with Frobenius fixing the T-factor. The scalar maps $A_* \to R_0$, $B \to R_1$ are compatible with Frobenius.

We first prove an integrality property. Suppose a finite matrix V over R_1 satisfies $\varphi(V) = HVH' + Y$, where H, H', Y have entries in R_0. Then V has entries in R_0. First take $O = \mathbf{Z}_\ell$ and reduce the equation modulo ℓ. If the largest absolute value b of an entry of $\bar{V}$ exceeds 1, a largest entry gives $b^\ell \leq b$, a contradiction. Lift the entries of $\bar{V}$ by their integral Teichmüller representatives V_0, and write $V = V_0 + \ell V_1$. The equation for V_1 has the same coefficients H, H' and integral inhomogeneous term

$$(HV_0H' + Y - \varphi(V_0))/\ell.$$

Indeed the numerator reduces to zero already over $\mathcal{O}^\flat$, and reduction of its Witt ring has kernel $\ell W(\mathcal{O}^\flat)$, since $\mathcal{O}^\flat$ is perfect. Iteration and ℓ-adic completeness prove integrality. For general O, choose a $\mathbf{Z}_\ell$-basis and replace each entry by the matrix of multiplication on the O-factor. This commutes with Frobenius and multiplication, so the preceding proof applies to the enlarged matrices. Evaluating a multiplication matrix on $1 \in O$ recovers the original entry over R_0.

Lean: `GlobalTheory.FrobInt.integral_of_frobenius` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.FrobeniusIntegralityWittTiltAinf`, lines 239–260)

Lean: GlobalTheory.FrobIntegralCoeff.frobenius_integrality_coeff (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.FrobeniusIntegralityTensorAinfCoefficients, lines 126–132)

We also need the scalar intersection $B \cap R_0 = A_*$ inside R_1. Let ϖ be a uniformizer of O, with residue field k. Modulo ϖ the source inclusion is $\prod_i k[[Z]] \subset \prod_i k((Z))$. On the period-ring side it is $k \otimes_{\mathbf{F}_\ell} \mathcal{O}^\flat \subset k \otimes_{\mathbf{F}_\ell} C^\flat$, a product of valuation-ring inclusions indexed by embeddings of k. Each component i is seen by a residue embedding extending the corresponding unramified embedding. In this factor Z maps to $z = \epsilon^{1/\ell^s} - 1$, of finite strictly positive valuation, with ϵ as in Lemma 36.17. For a nonzero Laurent series with first exponent j, the valuation of its image is $jv(z)$: its first coefficient is a unit and subsequent exponents have strictly larger valuations. Thus the residue intersection consists exactly of the power series, and the residue map on the Laurent series is injective. Given an element of B with image in R_0, lift its residue from A_*, subtract and divide by ϖ. The new element again has image in R_0, by injectivity of $R_0/\varpi \to R_1/\varpi$. Repetition gives a ϖ-adically convergent expansion in A_*. The rings are separated, and the same residue argument gives injectivity of $B \to R_1$.

Lean: GlobalTheory.NL.IoXQ.ioE_inj_inter (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachInclusionXQFrobenius, lines 160–229)

Choose an O-basis of T. For $M = L, L'$ choose a basis of M over A_*, assembling bases of the components, which all have rank $\operatorname{rank}_O T$. Let U_M be the matrix whose rows are its coordinates in the chosen basis of T after extension to R_1. It is invertible over R_1. The Frobenius hypothesis, in both directions on the finitely many basis vectors, supplies $h_M \geq 0$ and matrices P_M, H_M over A_* such that

$$Q^{h_M} \varphi(U_M) = P_M U_M, \qquad Q^{h_M} U_M = H_M \varphi(U_M).$$

Writing x for the image of X, we have $\varphi(x) = xQ$. Hence

$$\varphi(x^{h_M} U_M) = P_M (x^{h_M} U_M), \qquad \varphi(x^{h_M} U_M^{-1}) = (x^{h_M} U_M^{-1}) H_M.$$

The integrality property applies to both matrices. Put $a = h_L + h_{L'}$. The products $x^a U_L U_{L'}^{-1}$ and $x^a U_{L'} U_L^{-1}$ have entries in R_0. These are the images of the two scaled transition matrices over B, because both bases are bases of $\mathcal{D}$. The scalar intersection puts their entries in A_*. This proves both inclusions.

Lean: GlobalTheory.Sandwich.upper (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachLatticeSandwichAinf, lines 39–69)

Lean: GlobalTheory.Sandwich.lower (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachLatticeSandwichAinf, lines 71–125)

For the last assertion, write either transition matrix on component i as $C_i = X^{-a} B_i$, with B_i integral. At $Z = 0$ use $t = \log(1 + Z)$ over $\operatorname{Frac}(O)$. If C_i has a pole there with first term $t^{-k} C_0$, $k > 0$, the equivariance equation for γ gives $u^{-k} C_0 = C_0$. Indeed $\gamma(t) = ut$, and both lattice matrices of γ have constant term the identity. The depth assumption gives $u^k \neq 1$, since $1 + \ell^s \mathbf{Z}_\ell$ has injective logarithm for $s \geq 1$ at odd ℓ and for $s \geq 2$ at 2. Thus all transitions are regular at $Z = 0$.

At $Z = \zeta - 1$ with $\zeta^{\ell^{s-1}} = 1$ one has $Q(\zeta - 1) = \ell$. Both Frobenius matrices and their inverses are regular in the characteristic-zero expansion there. Frobenius carries the expansion at ζ^ℓ in the predecessor component to the expansion at ζ, by $t_{\zeta^\ell} \mapsto \ell t_\zeta$. Its transition equation therefore propagates regularity from ζ^ℓ to ζ. Induction gives regularity at every root of X. Each entry of B_i vanishes to order at least a at all these simple roots. The remainder on Weierstrass division by X^a over O has the same vanishings and degree less than $a \deg X$, hence is zero. For $a = 0$ the transition was already integral. This applies in both directions and proves equality.

Lean: GlobalTheory.NL.WachMat.X_pow_dvd_entry (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.MapConstantCoeffEqOne, lines 103–184)

□

Proof of 36.19

Proof. Apply Lemma 36.17 with $a = u$ to the source and target. For any such Wach system, the same lower-root calculation applies: its action is trivial modulo Z, its linearized Frobenius is invertible off Q, and Frobenius commutes with γ. The map f is equivariant, so on either lattice and on its image we have

$$(\gamma - u^j) X^j N \subset X^{j+1} N \qquad (j \geq 0). \tag{A.510}$$

Lean: GlobalTheory.PrismS.exists_prism (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsWachPrismOverIntegers, lines 331–370)

Lean: GlobalTheory.WachLattice.build_prism (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachLatticesExistAllLayersOfBhattScholze, lines 26–168)
Lean: GlobalTheory.NL.WachLatticeTrivialModUniformizer.wachLattice_galois_trivial_mod_uniformizer (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachLatticeGaloisTrivialModUniformizer, lines 12–18)
Lean: GlobalTheory.NL.GamModX.sub_mem_X_of_val (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsSmulOfDvdRepr, lines 70–96)

Put $P = fN_1$ and $S = \{y \in N_2 : \ell^c y, Z^c y \in P \text{ for some } c \geq 0\}$. Work componentwise and let ϖ be a uniformizer of O. The module S is finite since $O[[Z]]$ is noetherian. If $Zx = \varpi y$ in S, freeness of N_2 gives $x = \varpi z,\ y = Zz$ in N_2. A power of ℓ sends z into P by the condition on x and $\varpi \mid \ell$. A power of Z does so by the condition on y. Thus $z \in S$, and $S/\varpi S$ is torsion free over the discrete valuation ring $(O/\varpi)[[Z]]$. Lift a basis of this free module. The lifts generate by Nakayama's lemma, and reducing a relation repeatedly modulo ϖ proves independence. Étale base change is flat, being localization followed by adic completion of a noetherian ring. By exactness of the étale module functor and surjectivity of f, the base change of P fills that of N_2. The same holds for the intermediate module S.

The finite module S/P is supported at the closed point: powers of ℓ and Z kill each of a finite set of generators. Its Frobenius pullback is also supported there, since $\varphi(Z) \equiv Z^\ell \pmod{\varpi}$. Off Q both modules vanish, since $Q \in (\varpi, Z)$. Frobenius pullback is flat: on each component the scalar ring is free over its Frobenius image on $(1+Z)^j$, $0 \leq j < \ell$, by monic division in $1+Z$ with $(1+Z)^\ell = 1+\varphi(Z)$. The linearized Frobenius on $P[1/Q]$ is consequently an isomorphism, by taking images in the equivariant map between the two Wach lattices, whose Frobenius maps are isomorphisms there. The equalities with S and its pullback off Q give the same property for S. For $y \in S$ choose c with $\ell^c y, Z^c y \in P$. Since $\gamma(Z)/Z$ is a unit and P is stable, $Z^c \gamma y \in P$. Write $z = (\gamma - 1)y/Z \in N_2$, using $Z \mid X$ and (A.510). The same inclusion on P gives $\ell^c z \in P$, and $Z^{c+1} z = Z^c(\gamma y - y) \in P$. Thus $z \in S$. Both free lattices now satisfy Lemma 36.18, which gives $S = N_2$. By the definition of S, a common power of Z sends a finite basis of every component of N_2 into P.
Lean: GlobalTheory.NL.SatFree.satM_free_loc (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachSaturationImageFreeAndSpans, lines 86–165)
Lean: GlobalTheory.T.wachModule_uniqueness_characterisation (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachModuleUniquenessCharacterisationStatement, lines 7–8)
Lean: GlobalTheory.NL.FrobeniusMatrixUniqueness.main (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachModuleUniquenessCharacterisation, lines 247–269)
Lean: GlobalTheory.NL.CokerFin.coker_killed (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachNCokernelKilledByPower, lines 19–37)

Claim 1. At a primitive ℓ^s-th root ζ, put $t_\zeta = \log((1+Z)/\zeta)$ and extend the coefficient field to L containing the roots. Write D_{dR} for the chosen de Rham component over L. The completed $L[[t_\zeta]]$-lattice is

$$\widehat{N}_\zeta = \left\{ \sum_{k \geq k_0} t_\zeta^k d_k : d_k \in \mathrm{Fil}^{-k} D_{\mathrm{dR}} \right\}.$$

Here the expansion is bounded below. To prove this, use the period comparison and the basis-matrix estimates in the proof of Lemma 36.18. Multiplication by a power of X makes both the Wach basis matrix and its inverse integral in the comparison ring. The factor X has nonzero value at $\zeta - 1$, so it is a unit in B_{dR}^+ at this component. Consequently the images of a Wach basis form a basis of $B_{\mathrm{dR}}^+ \otimes T$. Their expansions span an $L[[t_\zeta]]$-lattice Λ_ζ between two powers of t_ζ times $D_{\mathrm{dR}}[[t_\zeta]]$.

The action of γ on these expansions multiplies the coefficient of degree k by u^k. Modulo a sufficiently high tail, polynomial interpolation in γ isolates each degree, since $u^k \neq u^{k'}$ for $k \neq k'$. The whole tail already lies in Λ_ζ. Hence a bounded-below series belongs to Λ_ζ exactly when each of its homogeneous terms does.

For a homogeneous term $t_\zeta^k d$, membership implies that its period image is in $B_{\mathrm{dR}}^+ \otimes T$, equivalently $d \in \mathrm{Fil}^{-k} D_{\mathrm{dR}}$. Conversely suppose this filtration condition holds. For sufficiently large m, the term $t_\zeta^{k+m} d$ lies in Λ_ζ. Express it in the expanded Wach basis. In the period comparison its coordinates are divisible by t_ζ^m, because $t_\zeta^k d$ is integral and the comparison basis is a B_{dR}^+-basis. The power-series evaluation has t_ζ-order equal to filtration order, so each coordinate series is divisible by t_ζ^m. Cancelling this power proves membership of $t_\zeta^k d$. This proves the claim, including the criterion for a Laurent expansion with poles to return to the completed lattice. Frobenius transports the predecessor's ordinary power series to this root and sends t_{ζ^ℓ} to ℓt_ζ. The source bounds and strictness for the quotient give

$$Q^{M_i} \varphi N_{j,i-1} \subset N_{ji}, \qquad Q^{h_i} N_{ji} \subset A \varphi N_{j,i-1} \quad (j = 1, 2). \tag{A.511}$$

The required orders follow from the filtration, and divisibility in a destination basis follows by monic division at the simple roots of Q.

Lean: `GlobalTheory.NL.Heights.heights_from_jumps` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachLatticeHeightsFromHodgeJumps`, lines 382–473)

Let $\mathfrak{a}_i = \{g : gN_{2i} \subset P_i\}$. The scalar ring is free over $\varphi(A)$ on $t_a = (1+Z)^a$, $0 \le a < \ell$. For $g \in \mathfrak{a}_i$ write $Q^H g = \sum_a t_a \varphi(g_a)$. For $y \in N_{2,i-1}$ lift $gQ^{M_i}\varphi y$ to the source and apply the second inclusion of (A.511). Multiplication by $Q^{H-M_i-h_i}$ puts $\sum_a t_a \varphi(g_a y)$ in $A\varphi(P_{i-1})$. Uniqueness of the scalar expansion in a Frobenius basis gives $g_a y \in P_{i-1}$. Thus $Q^H \mathfrak{a}_i \subset A\varphi(\mathfrak{a}_{i-1})$.

Lean: `GlobalTheory.NL.CokerKill.key_inclusion` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CokernelCrystallineSurjectionKilledByXPow`, lines 233–318)

If $X^H \notin \mathfrak{a}_i$, let $J_k = (\mathfrak{a}_{i+k} : X^H)$. Then $J_0 \subset (\varpi, Z)$ and every Frobenius coordinate of an element of J_{k+1} belongs to J_k, since $Q^H X^H = \varphi(X^H)$. Reduction modulo ϖ gives $J_k \subset (\varpi, Z^{k+1})$, contrary to $Z^c \in J_c$.

Lean: `GlobalTheory.Cyc.XX_pow_mem_of_QQ_pow_mul_mem_map_phi` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.XXPowMemOfQQPowMulMemMapPhi`, lines 80–116)

For $H > 0$ form $P_0(U) = \prod_{j=0}^{H-1}(U - u^j)$, $P_1(U) = \prod_{j=H}^{2H-1}(U - u^j)$. Both $P_0(\gamma)X^H v$ and $P_1(\gamma)X^H v$ belong to $X^H P_i$ for $v \in N_{2i}$, by (A.510) on P_i and N_{2i}, respectively. Binomial expansion, using $s \ge 2$ at 2, gives

$$v_\ell(u^k - 1) = s + v_\ell(k) \quad (k \ne 0). \tag{A.512}$$

The i-th denominator for interpolating $1/P_1$ at u^i, $0 \le i < H$, has valuation $(2H-1)s + v_\ell(i!(2H-1-i)!) \le w$. Thus an integral polynomial B satisfies $P_0 \mid \ell^w - BP_1$. Evaluate at γ on $X^H v$, then cancel X^H. At $H = 0$ the first inclusion gives equality.

Lean: `Formalism.LIFT.gammaGradedStep_holds` (`Formalism.LagrangeIdealLifts.CocyclePolynomialDivision.Graded`, lines 5–25)

Lean: `Formalism.LIFT.wBTail_of` (`Formalism.LagrangeIdealLifts.CocyclePolynomialDivision.Graded`, lines 115–157)

□

Proof of 36.20

Proof. Write χ_ℓ for the cyclotomic character. The field K_0 is unramified and the extensions $\mathbf{Q}_\ell(\mu_{\ell^j})/\mathbf{Q}_\ell$ are totally ramified. Hence the image of $G_{K'}$ under χ_ℓ is $1 + \ell^s \mathbf{Z}_\ell$. For an element g with $v_\ell(\chi_\ell(g) - 1) = s$, equation (A.512) gives

$$v_\ell(\chi_\ell(g)^{e+1} - 1) = b.$$

It follows that ℓ^b annihilates the invariants of $(\mathbf{Q}_\ell/\mathbf{Z}_\ell)(e+1)$.

Apply Theorem 36.14 over K' to $(\mathbf{Z}/\ell^j)(-e)$ for each $j \ge 1$. Its dual under the local pairing is $(\mathbf{Z}/\ell^j)(e+1)$. Taking the inverse limit on H^2 and the direct limit on H^0 identifies $H^2(K', \mathbf{Z}_\ell(-e))$ with the Pontryagin dual of the preceding invariant group. The finite H^1-groups in Theorem 36.14 satisfy the Mittag–Leffler condition, so the inverse limit computes integral H^2. In particular ℓ^b kills the obstruction to lifting $[\bar{z}]$ through $\mathbf{Q}_\ell(-e)$. Choose a rational cocycle lifting $\ell^b[\bar{z}]$. The difference between its reduction and $\ell^b \bar{z}$ is the boundary of a scalar in $(\mathbf{Q}_\ell/\mathbf{Z}_\ell)(-e)$. Subtract the boundary of any rational lift of that scalar. □

Proof of 36.21

Proof. Solve $A_i \gamma(U_i) = U_i$ with $U_i(0) = 1$. In degree $j > 0$ the coefficient to invert is $1 - u^j$. All coefficients multiplying earlier entries are integral, since $\binom{u}{h} \in \mathbf{Z}_\ell$. Writing $j = \ell^a j_0$ with $\ell \nmid j_0$, the binomial expansion gives $v_\ell(u^{j_0} - 1) = s$. Raising a unit of this depth to the ℓ-th power increases the depth by one. For $\ell = 2$ this uses $s \ge 2$. Thus the coefficient of Z^j in U_i has norm at most $\ell^{sj + v_\ell(j!)}$, by induction. Since $v_\ell(j!) \le j/(\ell - 1)$, all the matrices converge on a common disc of positive radius. Their bottom row is $(0, 1)$ by uniqueness. The negative of the upper part of the last column is w_i.

The commutation identity makes $B_i \varphi(U_{i-1})$ another matrix of γ-invariant columns. A column is uniquely determined by its constant coefficient, by the same recursion. Therefore

$$U_i = B_i \varphi(U_{i-1}) B_i(0)^{-1}.$$

Set $M = m + |J|$, $\tau = t_0/X$, and $D = \operatorname{diag}(\tau^M 1, 1)$. The series τ is analytic on the open unit disc, since it equals $\ell^{1-s}\log(1+X)/X$. For each $j \geq 0$, put $X_j = \varphi^j(X)$. From $X_{j+1} = X\prod_{h=0}^{j}\varphi^h(Q)$ we get

$$\frac{\tau}{\varphi^j(Q)} = \frac{\ell^{-(s+j)}\log(1+X_{j+1})}{X_{j+1}}\prod_{h<j}\varphi^h(Q).$$

The quotient of the logarithm by its argument has a removable singularity at zero. It follows that τ^M/Λ is analytic on the open unit disc, also when J contains repetitions. Since $\varphi(\tau) = \ell\tau/Q$, the matrix $DB_i\varphi(D)^{-1}$ has upper blocks $Q^{|J|}P_i/\ell^M$ and $\tau^M y_i/\Lambda$. Both are analytic there.

Now $DU_i = (DB_i\varphi(D)^{-1})\varphi(DU_{i-1})B_i(0)^{-1}$. For any $0 < R < 1$, substitution by φ maps the closed disc of radius R into the disc of radius $R\max(\ell^{-1}, R)$, by the binomial formula. Iteration reaches the common initial disc because $\max(\ell^{-1}, R) < 1$. The displayed matrix identity then gives convergence of all DU_i on the disc of radius R, working back through the finitely many iterations. In particular $\tau^M w_i$ converges there. Multiplication by X^M gives the assertion for $t_0^M w_i$. □

Proof of 36.22

Proof. The prime-to-ℓ roots generate an unramified field and $Q = \Phi_{\ell^s}(1+Z)$ is Eisenstein of degree $(\ell-1)\ell^{s-1}$. A primitive root of order $\operatorname{lcm}(\ell, N)$ generates K'. Hence

$$e(K'/\mathbf{Q}_\ell) = (\ell-1)\ell^{s-1}, \qquad v_\ell([K':K]) \leq j(N)+1. \tag{A.513}$$

Restriction and corestriction preserve period coboundaries, and their composite multiplies by $[K' : K]$. Jumps pull back under scalar extension.

Lean: `UpperTheory.N.Cyc.val_index_le` (`Development.AutomorphicArithmetic.ResidueCharacters.FiniteRoots`, lines 117–124)

Work over K', with $H = 2h$ and $w = 2Hs + v_\ell((2H)!)$. For $T_0 = \ker(T_1 \to T_2)$ write $N_j = \mathcal{N}(T_j)$. On $D(T)$ use

$$\partial b = ((\varphi-1)b, (\gamma-1)b), \qquad d(y,z) = (\gamma-1)y - (\varphi-1)z.$$

Herr's theorem [195, Theorem 2.1] computes cohomology modulo powers of ℓ; the finite degree-zero groups give the Mittag–Leffler condition for passage to the lattice. Write $A_h(M) = Q^{-h}M$. The comparison on $A_h(M) \times XM$ gives

$$d(y,z) = 0 \;\Rightarrow\; [y,z] \in H^1_f(K', T), \qquad [y,z] = 0 \;\Rightarrow\; (y,z) = \partial b_0 \text{ with } b_0 \in M \tag{A.514}$$

for closed pairs in this domain. Solve $(\varphi-1)b = y$ in the Witt tensor. *Claim 1.* The Witt solution b has a period image compatible with all Galois differences. Write $\jmath$ for the embedding into the Witt tensor and β for the comparison on $A_{\mathrm{inf}} \otimes T$. By Lemma 36.18, choose $m_0 \geq 0$ with $X^{m_0}\jmath(M) \subset A_{\mathrm{inf}} \otimes T$, and put $a = m_0 + h$, $B = X^a b$. From $\varphi(X) = XQ$ and $(\varphi-1)b = \jmath(y)$ one obtains

$$\varphi(B) = Q^a B + (XQ)^a \jmath(y).$$

The last term is integral, since $Q^h y \in M$. The coefficientwise Frobenius integrality argument of Lemma 36.18, applied to this inhomogeneous equation, gives $B \in A_{\mathrm{inf}} \otimes T$. Thus $b' = \beta(X)^{-a}\beta(B)$ is defined in $B_{\mathrm{cris}} \otimes T[1/\ell]$. For every $g \in G_{K'}$, put $D_g = X^a g(B) - g(X)^a B$. Equivariance and division by the period units give

$$(g-1)b' = \frac{\beta(D_g)}{\beta(X)^a\beta(g(X))^a}, \qquad D_g = X^a g(X)^a(gb - b).$$

This identifies every Galois difference with the period image of the corresponding Witt difference. In component bases of the Wach lattice, augment the matrices of γ and φ by their respective cocycle columns z and y, and add the bottom row $(0,1)$. The equality $d(y,z) = 0$ is exactly the upper-right block of the commutation identity in Lemma 36.21. Here $z \in XM$ has zero constant term. By Lemma 36.19 a power Q^m clears the Frobenius matrix, which is invertible after inverting Q, and $Q(0) = \ell \neq 0$ in the coefficient field. For the column y take J to consist of h copies of zero. Lemma 36.21 gives a zero-constant primitive w with $\log(1+Z)^{m+h}w$ convergent on the open unit disc. To evaluate it, put $z_s = [\underline{\epsilon}^{1/\ell^s}] - 1$ in A_{cris}, where $\underline{\epsilon}$

is the chosen compatible system of cyclotomic roots. Modulo ℓ, $z_s^{\ell^s}$ equals $[\underline{\epsilon}]-1$. The latter belongs to the divided-power ideal, so its ℓ-th power belongs to ℓA_{cris}. Consequently $z_s^{\ell^{s+1}} \in \ell A_{\mathrm{cris}}$. For an open-disc series $\sum a_n Z^n$, choose R with $\ell^{-1/\ell^{s+1}} < R < 1$. Then $|a_n|R^n \to 0$, and the evaluated terms tend to zero in the complete coefficient tensor of $A_{\mathrm{cris}}[1/\ell]$, with a common denominator. Evaluation respects multiplication and the cyclotomic substitutions by convergence. The logarithm evaluates to $\ell^{-s}t_{\mathrm{cris}}$, where $t_{\mathrm{cris}} = \log[\underline{\epsilon}]$ is invertible in B_{cris}. Dividing by its indicated power gives the period coordinates of $w' \in B_{\mathrm{cris}} \otimes T[1/\ell]$. For $g \in G_{K'}$ the corresponding cyclotomic value solves $(\gamma-1)z_g = (g-1)z$ in the same zero-constant domain. Thus $(g-1)w'$ is its period image. The cocycle of the pair has period primitive $w'-b'$. For a zero class take a Herr primitive $b_0 \in D(T)$. In the étale module of the crystalline lattice $T \oplus O$, let e generate the trivial summand and compare the two lattices $M \oplus A_* e$ and $M \oplus A_*(e+b_0)$, with A_* as in Lemma 36.18. Both are componentwise free and span the same étale module. In the second basis the extra Frobenius column is $y \in Q^{-h}M$ and the bottom diagonal entry is one, so linearized Frobenius is invertible after inverting Q on both lattices. The extra cyclotomic column is $z \in XM \subset ZM$, so γ preserves both lattices and acts trivially modulo Z on each. Lemma 36.18 identifies the two lattices. Subtracting e from $e+b_0$ now gives $b_0 \in M$. The kernel comparison gives $N_0 = N_1 \cap D(T_0)$.

Lean: `Formalism.MainNotions.LiftingTheory.coreWX` (`Formalism.SharpRestriction.LevelLetterCocycles.LiftingTheory`, lines 100–102)
Lean: `Formalism.MainNotions.LiftingTheory.coreWB_of` (`Formalism.SharpRestriction.LevelLetterCocycles.FrobeniusPowUniformizerFactorisation`, lines 3449–3457)
Lean: `GlobalTheory.NL.IntComplex.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachIntegralComplexCocycleClassFinite`, lines 40–58)
Lean: `GlobalTheory.NL.PoleCrys.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.HerrCocyclePolesFrobeniusQCrystalline`, lines 539–562)
Lean: `GlobalTheory.NL.CocycleInj.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.MemWachOfFrobeniusGamma`, lines 12–15)
Lean: `GlobalTheory.NL.Exact.exactness_of_N` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachModuleSaturatedExactnessCrystalline`, lines 110–135)

Represent a finite target class c by its crystalline extension of the unit lattice. Lift ℓ^w times the unit in the Wach lattice of the extension by 36.19. Its two differences give a closed pair $(y,z) \in A_h(N_2) \times XN_2$ representing $\ell^w c$. Lift $\ell^w Q^h y$ and $\ell^w z/X$ to N_1 by the same lemma, then divide the first by Q^h and multiply the second by X to get a pair v. Since $\gamma(Q)/Q \equiv 1 \bmod X$ and $\varphi(X) = XQ$, its defect $\omega = dv$ lies in $XA_h(N_0)$. Some $\ell^k\omega$ is $d(Y, Xn)$ in the kernel. The Bloch–Kato presentation using $D_{\mathrm{cris}} \oplus D_{\mathrm{dR}}/\mathrm{Fil}^0$ and strictness give right exactness of rational finite cohomology. Clear denominators of the lifted cocycle and kill the torsion difference downstairs. Represent a further multiple by the extension construction. Its difference from the corresponding multiple of the image of v is the boundary of an element of N_2 by (A.514). Lift a further ℓ^w times that element and subtract its boundary. The difference upstairs is now a kernel pair.

Lean: `GlobalTheory.NL.RightExactKs.h1fKs_onto_up_to_pow` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.H1fKsOntoUpToPow`, lines 14–23)
Lean: `GlobalTheory.NL.DefectTorsion.defect_of_lift` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.DefectTorsionSecondCohomologyKernel`, lines 286–333)

For $M = N_0$ and the union J of source jump sets let

$$J^* = ((J + [1,h]) \setminus \{0\}) \cup [1,h], \quad P(U) = \prod_{j \in J^*} (U - u^{-j}), \quad p_\gamma = P(\gamma), \quad c_\gamma = \prod_{j=1}^{h} (u^j - 1).$$

At a root of Q, a degree-k coefficient of $Q^h y$, for $y \in A_h(M)$, belongs to Fil^{-k}, and integrality of y requires Fil^{h-k}. A failure uses a jump $a \in J$ with $h-k-a \in [1,h]$. On the first such degree of y the scalar is u^{k-h} by expansion of y. The jump filtration therefore gives $(\gamma-1)p_\gamma A_h(M) \subset M$, by applying the factors in degree order and then using root divisibility.

Lean: `GlobalTheory.NL.KillLevels.annihilate` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ProdGammaSubJumpUnitsAnnihilatesQuotient`, lines 285–354)
Lean: `Formalism.MainNotions.LiftingTheory.liftb_semi` (`Formalism.SharpRestriction.LevelLetterCocycles.BaseChangePreservesUnitaryOf`, lines 404–472)

Successive use of (A.510) in degrees $1, \dots, h$ gives

$$c_\gamma Xn = (\gamma-1)Xx + X^{h+1}n', \qquad x, n' \in M. \tag{A.515}$$

In M-coordinates the Frobenius term $Q^{h+1}\varphi$ is contracting by (A.511), so $1-\varphi$ is onto on $X^{h+1}M$. Apply p_γ to $c_\gamma \ell^k \omega = (\gamma-1)(c_\gamma Y - (\varphi-1)Xx) - (\varphi-1)X^{h+1}n'$. The result is integral. In $A/(Q^h)$, ℓ acts injectively

and X becomes invertible after inverting ℓ. Cancellation therefore gives $p_\gamma\omega \in XM$. Apply (A.515) again and invert $1-\varphi$ to write $c_\gamma p_\gamma\omega = d(Xx_0, X^{h+1}z_0)$. With $S(U) = (P(U)-P(1))/(U-1)$ one obtains

$$c_\gamma P(1)\omega = d(Xx_0 - c_\gamma S(\gamma)\omega,\ X^{h+1}z_0).$$

Subtract this kernel pair from $c_\gamma P(1)v$ and apply (A.514). The image is $c_\gamma P(1)\ell^{2w}c$. Its valuation

$$2w + \sum_{j\in J^*}(s + v_\ell(j)) + hs + v_\ell(h!) \le 32(2hs+1) \tag{A.516}$$

can be seen term by term. The two Wach lifts contribute $8hs + 2v_\ell((4h)!)$. The nonzero shifts contribute at most $3hs + v_\ell(h!) + v_\ell((2h)!)$, by counting the negative and positive shifts separately. The last product contributes $hs+v_\ell(h!)$. Since $v_\ell(k!) = \sum_{j\ge1}\lfloor k/\ell^j\rfloor \le k/(\ell-1)$, the total is at most $12hs+12h/(\ell-1) \le 24hs$. Restrict a target class to K', take its scaled finite lift there, and corestrict that lift. Its image is the same multiple of $[K' : K]$ times the original class. Equation (A.513) bounds the additional valuation by $j(N)+1 \le D_h$. This proves the displayed numerical allowance also for $h = 0$.

Lean: `Formalism.LIFT.CocyclePolynomialSubmodules.main` (`Formalism.LagrangeIdealLifts.CocyclePolynomialDivision.UClaim`, lines 131–190)
Lean: `Formalism.MainNotions.LiftingTheory.lqb_lifts` (`Formalism.SharpRestriction.LevelLetterCocycles.BaseChangePreservesUnitaryOf`, lines 4254–4351)
Lean: `Formalism.MainNotions.LiftingTheory.coreLiftB_of'` (`Formalism.SharpRestriction.LevelLetterCocycles.BaseChangePreservesUnitaryOf`, lines 4357–4430)
Lean: `UpperTheory.N.liftK_of` (`Development.AutomorphicArithmetic.ResidueCharacters.LiftKOf`, lines 12–41)

For the weight calculation take cyclotomic depth $m' = 2s+n_0+1$, where n_0 is least with $2h \le \ell^{n_0}(\ell-1)$.

Claim 2. The specialized cyclotomic matrix has the required strict smallness bound. Choose a generator γ_0 at depth s, put $\mathfrak{m} = (\ell, \pi)$, and let $L = \gamma_0 - 1$ act additively on the Wach lattice. The scalar substitution gives $(\gamma_0-1)\mathfrak{m}^k \subset \mathfrak{m}^{k+1}$. Since $(\gamma_0-1)N \subset \pi N$, the identity

$$L(ax) = \gamma_0(a)L(x) + (\gamma_0(a)-a)x$$

gives $L(\mathfrak{m}^k N) \subset \mathfrak{m}^{k+1}N$, hence $L^k N \subset \mathfrak{m}^k N$. Put $b = s+n_0+1$ and choose the deeper generator $g = \gamma_0^{\ell^b}$. For $0 < i < \ell^{b-1}$, $v_\ell\binom{\ell^b}{i} \ge b - v_\ell(i) \ge 2$. Expanding $(1+L)^{\ell^b} - 1$ therefore gives

$$(g-1)N \subset (\ell^2, \mathfrak{m}^{\ell^{s+n_0}})N.$$

Specialization sends π to $\zeta_{\ell^{s+n_0}}-1$, so every entry of $U(g)-1$ has valuation at least $\min(2, \ell/(\ell-1)) = \ell/(\ell-1)$, strictly greater than $1/(\ell-1)$. This includes $\ell = 2$, where $s \ge 2$. The specialized sandwich below supplies the two matrix norm bounds. Then $m' \le 2s + j(2h) + 2$. Put $K'' = K_0(\mu_{\ell^{m'}})$, $K_\infty = K_0(\mu_{\ell^\infty})$, $H_\infty = G_{K_\infty}$, and let $g \in G_{K''}$ generate $\mathrm{Gal}(K_\infty/K'')$. Fix a $\mathbf{Z}_\ell$-basis of the crystalline lattice T (of rank d) with Galois action matrix $M(\sigma) \in \mathrm{GL}_d(\mathbf{Z}_\ell)$. Write $w = [\varepsilon^{1/\ell^s}]^{\ell^{s-1}} - 1 \in A_{\mathrm{inf}}$ for the image of the Wach variable $X = (1+Z)^{\ell^{s-1}} - 1$ under the Fontaine comparison embedding, and let $\theta_{n_0} : A_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_{\mathbf{C}_\ell}$ be Fontaine's map twisted by φ^{-n_0}, which sends w to $c = \zeta_{\ell^{1+n_0}} - 1 \in \mathcal{O}_{K''}$ with $v_\ell(c) = 1/(\ell^{n_0}(\ell-1))$. Because $2h \le \ell^{n_0}(\ell-1)$, we have $v_\ell(c^{2h}) \le 1$, so $\ell = u_0 c^{2h}$ for an element $u_0 \in \mathcal{O}_{\mathbf{C}_\ell}$ with $\|u_0\| \le 1$. Because the Wach lattice $N(T)$ satisfies the A_{inf}-sandwich $X^{2h}(A_{\mathrm{inf}} \otimes T) \subseteq N(T) \subseteq A_{\mathrm{inf}} \otimes T$ (after shifting by the minimal jump so the jumps span at most $2h$) and has a G_{K_∞}-trivial action after reducing modulo X and specializing, applying θ_{n_0} to a Wach basis of $N(T)$ over A_{inf} (fixed by H_∞ and transformed on $G_{K''}$ by an integral matrix cocycle $U : G_{K''} \to \mathrm{GL}_d(\mathcal{O}_{K''})$ trivial on H_∞ with $U(g)-1$ having entries in the ideal $(\ell^2, (\ell,\pi)^{\ell^{s+n_0}})$, where the level-$s$ Wach uniformizer π evaluates under θ_{n_0} to $\theta_{n_0}(\pi) = \zeta_{\ell^{s+n_0}} - 1$ of ℓ-adic valuation $1/(\ell^{s+n_0-1}(\ell-1))$, giving $v_\ell(\theta_{n_0}(\pi)^{\ell^{s+n_0}}) = \ell/(\ell-1) > 1/(\ell-1)$ and $v_\ell(\ell^2) = 2 > 1/(\ell-1)$, hence $\|U(g)-1\| \le \ell^{-\ell/(\ell-1)} < \ell^{-1/(\ell-1)}$) yields a *Sen decompletion matrix* $D \in \mathrm{GL}_d(\mathbf{C}_\ell)$ expressing the specialized Wach basis in the $\mathbf{Z}_\ell$-basis of T. The upper sandwich $N(T) \subseteq A_{\mathrm{inf}} \otimes T$ gives $\|D\| \le 1$; the lower sandwich $X^{2h}(A_{\mathrm{inf}} \otimes T) \subseteq N(T)$ sends $c^{2h} I_d$ into $D\mathcal{O}_{\mathbf{C}_\ell}^d$, so $c^{2h}D^{-1}$ has entries in $\mathcal{O}_{\mathbf{C}_\ell}$, and multiplying by u_0 gives $\|\ell D^{-1}\| = \|u_0 c^{2h} D^{-1}\| \le 1$. Furthermore $DU(\sigma) = M(\sigma)\sigma(D)$ for all $\sigma \in G_{K''}$. Here U is not required to be diagonal over $\mathcal{O}_{K''}$; separately, crystallinity of T provides a Hodge–Tate eigenbasis of $U(g)$ over $\mathbf{C}_\ell$ (requiring no norm bound on the diagonalizing change of basis) whose eigenvalues have the form $\zeta_u \chi(g)^{-u}$ for $u \in J_{\mathrm{set}}$ (the set of distinct Hodge–Tate jumps, of cardinality $|J_{\mathrm{set}}| \le a$) and roots of unity ζ_u. Because

$\|U(g)-1\| \le \ell^{-\ell/(\ell-1)} < \ell^{-1/(\ell-1)}$ and $v_\ell(\chi(g)^{-u}-1) \ge m' \ge 2 > 1/(\ell-1)$, every eigenvalue $\lambda = \zeta_u \chi(g)^{-u}$ satisfies $|\lambda - 1| \le \|U(g)-1\| < \ell^{-1/(\ell-1)}$, whence $|\zeta_u - 1| \le \max(|\lambda-1|, |\chi(g)^{-u}-1|) < \ell^{-1/(\ell-1)}$; since every nontrivial root of unity has $|\zeta - 1| \ge \ell^{-1/(\ell-1)}$, this forces $\zeta_u = 1$. Because $U(g)$ is diagonalizable over $\mathbf{C}_\ell$ with distinct eigenvalues $\{\chi(g)^{-u} : u \in J_{\mathrm{set}}\}$, its minimal polynomial divides the squarefree product $\prod_{u \in J_{\mathrm{set}}}(X - \chi(g)^{-u})$, so $\prod_{u \in J_{\mathrm{set}}}(U(g) - \chi(g)^{-u}) = 0$. Since $\mathbf{Q}_\ell(1)$ has filtration jump -1, twisting T by $\chi_\ell^{j_{\min}}$ (where $j_{\min} = \min J_{\mathrm{set}}$) shifts every Hodge–Tate jump $u \in J_{\mathrm{set}}$ to $u - j_{\min} \in [0, 2h]$ and shifts the tested weight i to $i - j_{\min}$, leaving every difference $j - i$ and hence $R_i = \sum_{j \in J_{\mathrm{set}},\ j \ne i}(m' + v_\ell(j-i))$ invariant. Define the coordinate ball $B_d = \mathcal{O}_{\mathbf{C}_\ell}^d$ and the lattice $M_0 = D^{-1}(B_d) \subseteq \mathbf{C}_\ell^d$, which satisfies $B_d \subseteq M_0$ and $\ell M_0 \subseteq B_d$ by $\|D\| \le 1$ and $\|\ell D^{-1}\| \le 1$. For each weight $i \in \mathbf{Z}$, put $N_i = \max(1, R_i)$. We prove that for every $i \in \mathbf{Z}$ (both $i \notin J_{\mathrm{set}}$ and $i \in J_{\mathrm{set}}$), ℓ^{2+N_i} kills the torsion subgroup of $H^1(G_{K''}, T \otimes_{\mathbf{Z}_\ell} \mathcal{O}_{\mathbf{C}_\ell}(i))$. Given a torsion 1-cocycle $c : G_{K''} \to \mathcal{O}_{\mathbf{C}_\ell}^d(i)$ (for the action $M(\sigma)\chi(\sigma)^i$), multiplying by ℓD^{-1} sends c to a 1-cocycle $c_0(\sigma) = \ell D^{-1} c(\sigma) \in B_d$ for the Sen action $\chi(\sigma)^i U(\sigma)\sigma$. Since $U|_{H_\infty} = 1$ and $\chi|_{H_\infty} = 1$, Tate's vanishing theorem kills $H^1(H_\infty, \mathcal{O}_{\mathbf{C}_\ell})$ by ℓ, so ℓc_0 is cohomologous to an inflated cocycle c_1 of $\mathrm{Gal}(K_\infty/K'')$ valued in $\widehat{\mathcal{O}}_{K_\infty}^d$ (having used ℓ^2 so far). Decompose $\widehat{\mathcal{O}}_{K_\infty} = \mathcal{O}_{K''} \oplus X_{m'}$ via Tate's normalized trace, and write $c_1(g) = w_0 + w_x$ with $w_0 \in \mathcal{O}_{K''}^d$ and $w_x \in X_{m'}^d$. On $X_{m'}^d$, Tate's sharp cyclotomic trace estimate supplies a continuous $\mathcal{O}_{K''}$-linear operator $S : X_{m'}^d \to X_{m'}^d$ satisfying $T_0 \circ S = S \circ T_0 = (\zeta_\ell - 1)\,\mathrm{id}$ for the unperturbed operator $T_0 = \chi(g)^i \sigma_g - 1$. Since every entry of $U(g) - 1 \in M_d(\mathcal{O}_{K''})$ has norm strictly smaller than $\|\zeta_\ell - 1\| = \ell^{-1/(\ell-1)}$, we factor $\chi(g)^i(U(g)-1) = (\zeta_\ell - 1)\pi B'$ with a σ_g-fixed element $\pi \in \mathfrak{m}_{K''}$ ($\|\pi\| < 1$) and $B' \in M_d(\mathcal{O}_{K''})$. Substituting $y_x = S(z)$ transforms $(\chi(g)^i U(g)\sigma_g - 1)y_x = (\zeta_\ell - 1)v$ into $(1 + \pi B' \sigma_g S)z = v$ on $X_{m'}^d$, which is solved in the closed $\mathcal{O}_{K''}$-submodule $X_{m'}^d$ by the π-adically convergent Neumann series $z = \sum_{k \ge 0}(-1)^k(\pi B' \sigma_g S)^k v$ (with uniqueness, since any kernel element $y \in X_{m'}^d$ satisfies $T_0 y = -(\zeta_\ell - 1)\pi B' \sigma_g y$, and applying S and cancelling the nonzero scalar $\zeta_\ell - 1$ yields $y = -\pi S(B'\sigma_g y) \in \pi^k X_{m'}^d$ for every $k \ge 0$, forcing $y = 0$). Because $N_i \ge 1 \ge 1/(\ell-1)$, the ratio $\ell^{N_i}/(\zeta_\ell - 1)$ lies in $\mathcal{O}_{K''}$; setting $v = (\ell^{N_i}/(\zeta_\ell - 1))w_x \in X_{m'}^d$ yields $y_x \in X_{m'}^d$ with $(\chi(g)^i U(g)\sigma_g - 1)y_x = \ell^{N_i} w_x$. Meanwhile, on $\mathcal{O}_{K''}^d$ (where $\sigma_g = 1$), the squarefree Lagrange polynomial $\prod_{j \in J_{\mathrm{set}},\ j \ne i}(\chi(g)^i U(g) - \chi(g)^{i-j})$ separates weight i from all $j \in J_{\mathrm{set}} \setminus \{i\}$ with scalar denominator $\beta_i = \prod_{j \in J_{\mathrm{set}},\ j \ne i}(\chi(g)^{-i} - \chi(g)^{-j})$ of valuation $v_\ell(\beta_i) = R_i \le N_i$, so $\ell^{N_i}/\beta_i \in \mathcal{O}_{K''}$. When $i \notin J_{\mathrm{set}}$, this polynomial annihilates $\mathcal{O}_{K''}^d$ completely, yielding $y_0 \in \mathcal{O}_{K''}^d$ with $(\chi(g)^i U(g) - 1)y_0 = \ell^{N_i} w_0$; when $i \in J_{\mathrm{set}}$, the torsion equation $\ell^k c_1(g) = (\chi(g)^i U(g)\sigma_g - 1)x_2$ projects to $\mathcal{O}_{K''}^d$ as $\ell^k w_0 = (\chi(g)^i U(g) - 1)v_0$, forcing the weight-$i$ component of w_0 to vanish identically (since $\chi(g)^i U(g) - 1$ kills the weight-i eigenspace), so the complementary factor of valuation cost $R_i \le N_i$ again yields $y_0 \in \mathcal{O}_{K''}^d$ with $(\chi(g)^i U(g) - 1)y_0 = \ell^{N_i} w_0$. Summing $y_0 + y_x \in \widehat{\mathcal{O}}_{K_\infty}^d \subseteq B_d \subseteq M_0$ gives a primitive in $B_d \subseteq M_0$ for $\ell^{N_i} c_1$, and applying D returns a primitive in $\mathcal{O}_{\mathbf{C}_\ell}^d$ for $\ell^{2+N_i} c$. In particular, when $0 \notin J_{\mathrm{set}}$ and $x \in \mathbf{Q}_\ell^d$ satisfies $M(\sigma)x - x \in \mathbf{Z}_\ell^d$ for all $\sigma \in G_{K''}$, the coboundary $\sigma \mapsto M(\sigma)x - x$ is torsion in $H^1(G_{K''}, \mathcal{O}_{\mathbf{C}_\ell}^d)$; the primitive $z \in M_0$ for $\ell^{2+N_0}(U(\sigma)\sigma(D^{-1}x) - D^{-1}x)$ makes $\ell^{2+N_0} D^{-1}x - z$ a $G_{K''}$-invariant of absent weight 0, hence 0, so $\ell^{2+N_0} D^{-1}x \in M_0 = D^{-1}(\mathcal{O}_{\mathbf{C}_\ell}^d)$ and therefore $\ell^{2+N_0} x \in \mathcal{O}_{\mathbf{C}_\ell}^d \cap \mathbf{Q}_\ell^d = \mathbf{Z}_\ell^d$. With the degree cost $m' - s \le 2D_h$ to reach depth m', the exponent to treat $k \le h+1$ successive weights $0, \dots, k-1$ is bounded by

$$c(k) \le 2D_h + \sum_{i=0}^{k-1}(2 + \max(1, R_i)) \le 2D_h + 3k(a+1)D_h. \tag{A.517}$$

Indeed, since $|J_{\mathrm{set}}| \le a$ and for $0 \le i < k \le h+1$ every distinct jump $j \in J_{\mathrm{set}} \setminus \{i\}$ satisfies $1 \le |j - i| \le 2h$, we have $R_i = \sum_{j \in J_{\mathrm{set}},\ j \ne i}(m' + v_\ell(j-i)) \le a(m' + j(2h)) \le 2aD_h$ unconditionally regardless of jump multiplicities. When $h = 0$ the index set of R_i is empty. Each summand in (A.517) is at most $3(a+1)D_h$. For the absent weight zero take $k = 1$. The bound $2D_h + 3(a+1)D_h \le I_{h,a}$ also kills the invariants over K, since they form a subgroup of the invariants over K''.

Lean: `GlobalTheory.FrameInv.pow_smul_mem_of_invariant_mod` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.PrimePowSmulMemOfTwistedInvariantModLattice`, lines 254–320)

Lean: `UpperTheory.N.invariants_killed_over_cyclotomic_subfield_of_admissible` (`Development.AutomorphicArithmetic.ResidueCharacters.InvariantsKilledOverCyclotomic`, lines 14–40)

For the character block, set $T = Y_0 \cap \ker p$. Because $p : Y \to K_{\mathrm{coeff}}$ is G_K-invariant with $p(Y_0) \subset O$ and $p(y_0) = 1$, we have a short exact sequence of G_K-stable lattices $0 \to T \to Y_0 \xrightarrow{p} O \to 0$, so $f(g) = gy_0 - y_0$ defines a continuous T-valued 1-cocycle $f : G_K \to T$. For integers $a \le b$, define the truncated integral period quotient $Q_{a,b} = (\mathrm{Fil}^a B_{\mathrm{dR}}^+ \cap A_{\mathrm{inf}} + \mathrm{Fil}^b B_{\mathrm{dR}}^+)/\mathrm{Fil}^b B_{\mathrm{dR}}^+$, so in particular $Q_{0,e+1} =$

$(A_{\mathrm{inf}} + \mathrm{Fil}^{e+1}B^+_{\mathrm{dR}})/\mathrm{Fil}^{e+1}B^+_{\mathrm{dR}}$. Restrict to the cyclotomic subgroup $G_{K'}$ ($K' = K_0(\mu_{\ell^s})$) and consider the class of $f \otimes 1$ in $H^1(G_{K'}, T \otimes_{\mathbf{Z}_\ell} Q_{0,e+1})$. Because Y is crystalline, the class of $f \otimes 1$ is torsion in $H^1(G_{K'}, T \otimes_{\mathbf{Z}_\ell} Q_{0,e+1})$. Over $K'' = K_0(\mu_{\ell^{m'}})$, each graded piece $T \otimes_{\mathbf{Z}_\ell} Q_{i,i+1} \cong T \otimes_{\mathbf{Z}_\ell} \mathcal{O}_{\mathbf{C}_\ell}(i)$ $(0 \le i \le e)$ has its H^1-torsion killed by $\ell^{2+\max(1,R_i)}$ as proved above. Moreover, by crystallinity of T, in every short exact filtration step $0 \to X' \to X \to X'' \to 0$ along $Q_{0,e+1}$, every $G_{K''}$-invariant in X'' lifts after multiplication by some unquantified power $\ell^{k'}$ to an invariant in X. This unquantified $\ell^{k'}$ adds zero to the torsion exponent: if $\ell^{r'}$ kills torsion in $H^1(X')$, $\ell^{r''}$ kills torsion in $H^1(X'')$, and $\ell^k f_X = \partial w$ in X, then $\ell^{r''}\bar{f}_X = \partial x''$ in X'' lifts to $x \in X$ with $f' = \ell^{r''}f_X - \partial x \in Z^1(X')$; since $y'' = \ell^k x'' - \ell^{r''}\bar{w} \in (X'')^{G_{K''}}$, lifting $\ell^{k'}y''$ to an invariant $y \in X^{G_{K''}}$ puts $w' = \ell^{k'}(\ell^{r''}w - \ell^k x) + y \in X'$ with $\partial w' = \ell^{k'+k}f'$, making $[f']$ torsion in $H^1(X')$, hence killed by $\ell^{r'}$ independently of k', so $\ell^{r'+r''}[f_X] = 0$. Induction across $0 \le i \le e$ and descent from K'' to K' (costing $m' - s \le 2D_h$) therefore show that the exponent $c(e+1)$ of (A.517) kills $[f \otimes 1]$ and yields a primitive in $T \otimes_{\mathbf{Z}_\ell} Q_{0,e+1}$ for $\ell^{c(e+1)}(f \otimes 1)$ over $G_{K'}$. In integral $\mathbf{Z}_\ell$-bases of T and $V_0 = O(-e)$ inside $V = K_{\mathrm{coeff}}(-e)$, lift $\phi|_T : T \to V/V_0$ to a $\mathbf{Z}_\ell$-linear map $\Phi : T \to V$ with coordinates $\psi_i : T \to \mathbf{Q}_\ell$. Because ϕ is equivariant on T, each Galois difference $\Phi(gy) - \chi(g)^{-e}\Phi(y)$ lies in V_0, so each ψ_i is χ^{-e}-equivariant modulo $\mathbf{Z}_\ell$. Applying ψ_i to the truncated period primitive gives, for each coordinate i, an element $b_i \in \mathrm{Fil}^0 B_{\mathrm{dR}}$ such that $\ell^{c(e+1)}\psi_i(f(\sigma)) - (\chi(\sigma)^{-e}\sigma(b_i) - b_i) \in A_{\mathrm{inf}} + \mathrm{Fil}^{e+1}B^+_{\mathrm{dR}}$ for all $\sigma \in G_{K'}$, and $\psi_i \circ f$ is a χ^{-e}-twisted cocycle modulo $\mathbf{Z}_\ell$. Apply Lemma 36.20 over $K' = K_0(\mu_{\ell^s})$ to each coordinate ψ_i: the resulting rational χ^{-e}-cocycle $z_{i,0}$ on $G_{K'}$ reduces pointwise to $\ell^{s+v_\ell(e+1)}\psi_i \circ f$ modulo $\mathbf{Z}_\ell$ and therefore inherits the same congruence modulo $A_{\mathrm{inf}} + \mathrm{Fil}^{e+1}B^+_{\mathrm{dR}}$ after scaling by $\ell^{c(e+1)}$. Let $\exp^* : H^1(G_{K'}, \mathbf{Q}_\ell(-e)) \to K'$ be the Bloch–Kato dual exponential, characterizing $\alpha = \exp^*(z) \in K'$ by the existence of $b_{\exp^*} \in \mathrm{Fil}^0 B^+_{\mathrm{dR}}$ with $z(\sigma) - (\log\chi(\sigma))\alpha t^e - (\chi(\sigma)^{-e}\sigma(b_{\exp^*}) - b_{\exp^*}) \in \mathrm{Fil}^{e+1}B^+_{\mathrm{dR}}$ for all $\sigma \in G_{K'}$. For $z = \ell^{c(e+1)}z_{i,0}$, let $b_{\mathrm{prim}} \in \mathrm{Fil}^0 B^+_{\mathrm{dR}}$ witness the truncated-period congruence $z(\sigma) - (\chi(\sigma)^{-e}\sigma(b_{\mathrm{prim}}) - b_{\mathrm{prim}}) \in A_{\mathrm{inf}} + \mathrm{Fil}^{e+1}B^+_{\mathrm{dR}}$ obtained above. Subtracting the $\exp^*$-defining congruence from the truncated-period congruence cancels $z(\sigma)$ identically and leaves

$$(\log\chi(\sigma))\alpha t^e - \big(\chi(\sigma)^{-e}\sigma(b) - b\big) \in A_{\mathrm{inf}} + \mathrm{Fil}^{e+1}B^+_{\mathrm{dR}} \qquad (\sigma \in G_{K'})$$

for the difference $b = b_{\mathrm{prim}} - b_{\exp^*} \in \mathrm{Fil}^0 B^+_{\mathrm{dR}}$. Because $(\log\chi(\sigma))\alpha t^e \in \mathrm{Fil}^e B^+_{\mathrm{dR}}$, inductively raising the filtration level of b from i to $i+1$ for $0 \le i < e$ operates on $\chi(\sigma)^{-e}\sigma(b) - b \in A_{\mathrm{inf}} + \mathrm{Fil}^{i+1}B^+_{\mathrm{dR}}$: writing $b = t^i y$ ($y \in \mathrm{Fil}^0 B^+_{\mathrm{dR}}$), the congruence modulo Fil^{i+1} gives $(\chi(\sigma)^{i-e}\sigma - 1)((\zeta_\ell - 1)^i\theta(y)) \in \mathcal{O}_{\mathbf{C}_\ell}$ for all $\sigma \in G_{K'}$; Tate's twisted norm bound places $\ell^{1+s+v_\ell(i-e)}(\zeta_\ell - 1)^i\theta(y)$ in $\mathcal{O}_{\mathbf{C}_\ell}$, and lifting along $\theta : A_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_{\mathbf{C}_\ell}$ to subtract $\xi^i a_0$ raises b into Fil^{i+1} at valuation cost $1 + s + v_\ell(e-i)$. Summing over $0 \le i < e$ (costing $e(s+1) + v_\ell(e!)$) reaches $b \in \mathrm{Fil}^e B^+_{\mathrm{dR}}$, where one must retain the coboundary after division by t^e. Put $n = e(s+1) + v_\ell(e!)$ and $a = (\zeta_\ell - 1)^e\ell^n\alpha$. The graded integral-period calculation gives an integral cocycle

$$c(\sigma) = (\log\chi(\sigma))a + (\sigma - 1)\beta \quad \text{with } \|c(\sigma)\| \le 1.$$

On H_∞ the logarithmic term vanishes. Tate's integral estimate supplies $x \in \mathcal{O}_{\mathbf{C}_\ell}$ with $\ell c(h) = (h-1)x$ for every $h \in H_\infty$. Thus $c_1 = \ell c - \partial x$ is an integral cocycle on the completed cyclotomic tower. Tate's normalised trace to K' kills the $\gamma - 1$ part, has norm at most one on its integral lattice, and gives

$$\mathrm{Tr}_{\mathrm{norm}}(c_1(\gamma)) = \ell(\log u)a.$$

The equality follows first on the tower and then by continuity on its completion. Hence $\|a\| \le \ell^{1+s}$. Using $v_\ell((\zeta_\ell - 1)^e) = e/(\ell - 1)$ yields $\|\exp^*(z)\| \le \ell^{e(s+1)+v_\ell(e!)+1+s+e/(\ell-1)}$. The same argument applies when $e = 0$.

Conversely, for the lattice-image inclusion, write $\mathcal{O}_{K'} = \mathcal{O}_{K_0}[\zeta_{\ell^s}]$ as the additive span of elements $c\zeta_{\ell^j}^n$ with $1 \le j \le s$, $\ell \nmid n$, and $c \in \mathcal{O}_{K_0}$ (note that $1 = -\sum_{n=1}^{\ell-1}\zeta_\ell^n$ already lies in the $j = 1$ span). For tuples $(\lambda_a)_{(a,\ell)=1}$ and d' in $\mathcal{O}_{K_0}$, define the cyclotomic Frobenius-twisted sum

$$V(\lambda, d') = \sum_{i=0}^{s-1} \ell^{ei} \sum_{(a,\ell)=1} \varphi^i(\lambda_a)\zeta_{\ell^s}^{a\ell^i} + \ell^{se}d'.$$

By additivity it suffices to realize $\ell^{(e+1)(s-1)}c\zeta_{\ell^j}^n = V(\lambda, d')$ with integral $\lambda_a, d' \in \mathcal{O}_{K_0}$ satisfying the Frobenius matching condition $d' - \ell^e\varphi(d') = \varphi^s\big(\sum_{(a,\ell)=1}\lambda_a\big)$ for each generator $c\zeta_{\ell^j}^n$. Over $a \in (\mathbf{Z}/\ell^s\mathbf{Z})^\times$, for $j \ge 2$

define the integer weights $\mu(a) = \ell \cdot \mathbf{1}_{a\equiv n \pmod{\ell^j}} - \mathbf{1}_{a\equiv n \pmod{\ell^{j-1}}}$, which satisfy $\sum_{(a,\ell)=1}\mu(a) = 0$ and $\sum_{(a,\ell)=1}\mu(a)\zeta_{\ell^i}^a = \ell^{s-j+1}\zeta_{\ell^j}^n\delta_{ij}$ for all $1 \le i \le s$; setting $d' = 0$ and $\lambda_a = \mu(a)\varphi^{-(s-j)}(\ell^{(e+1)(j-2)+e}c) \in \mathcal{O}_{K_0}$ (since φ is an isometric automorphism of $\mathcal{O}_{K_0}$) satisfies $d' - \ell^e\varphi(d') = 0 = \varphi^s(\sum\lambda_a)$ identically for all $e \ge 0$ and evaluates to $V(\lambda,0) = \ell^{e(s-j)+(e+1)(j-2)+e+(s-j+1)}c\zeta_{\ell^j}^n = \ell^{(e+1)(s-1)}c\zeta_{\ell^j}^n$. For $j = 1$, define $\mu(a) = \mathbf{1}_{a\equiv n \pmod{\ell}}$ and $\nu(a) = 1$ on $(\mathbf{Z}/\ell^s\mathbf{Z})^\times$, which satisfy $\sum\mu(a) = \ell^{s-1}$, $\sum\mu(a)\zeta_{\ell^i}^a = \ell^{s-1}\zeta_\ell^n\delta_{i1}$, $\sum\nu(a) = (\ell-1)\ell^{s-1}$, and $\sum\nu(a)\zeta_{\ell^i}^a = -\ell^{s-1}\delta_{i1}$ for all $1 \le i \le s$. Solve $d - \ell^{e+1}\varphi(d) = \varphi(c)$ in $\mathcal{O}_{K_0}$ by the ℓ-adically convergent series $d = \sum_{k\ge0}\ell^{k(e+1)}\varphi^{k+1}(c) \in \mathcal{O}_{K_0}$ (which converges unconditionally even when $e = 0$, since $e+1 \ge 1$), and put $\lambda_a = \mu(a)\varphi^{-(s-1)}(c) + \nu(a)\varphi^{-(s-1)}(\ell^e d)$ and $d' = \ell^{s-1}d$. Then $\varphi^s(\sum\lambda_a) = \ell^{s-1}\varphi(c)+(\ell-1)\ell^{s-1+e}\varphi(d) = \ell^{s-1}(d-\ell^e\varphi(d)) = d'-\ell^e\varphi(d')$ holds identically, and the ν- and d'-terms cancel identically in $V(\lambda,d')$ (contributing $-\ell^{se+s-1}d + \ell^{se+s-1}d = 0$), leaving $V(\lambda,d') = \ell^{(e+1)(s-1)}c\zeta_\ell^n$. Hence every $\alpha' \in \mathcal{O}_{K'}$ is realized as $V(\lambda,d') = \ell^{(e+1)(s-1)}\alpha'$ with integral (λ,d') solving $d' - \ell^e\varphi(d') = \varphi^s(\sum\lambda_a)$. Put $u = 1+\ell^s \in \mathbf{Z}_\ell^\times$ (the cyclotomic eigenvalue $\chi(\gamma) = u$ of a generator $\gamma \in \mathrm{Gal}(K_\infty/K')$, with $s \ge 2$ when $\ell = 2$ so $v_\ell(\log u) = s$ and $v_\ell(u^e-1) = s + v_\ell(e)$ for $e \ge 1$), and consider the Coleman power series $h(T) = \sum_{(a,\ell)=1}(\lambda_a/a^{e+1})(1+T)^a \in \mathcal{O}_{K_0}[[T]]$, which is integral for every prime ℓ (including $\ell = 2$) because $(a,\ell) = 1$ makes $a \in \mathbf{Z}_\ell^\times$ and $\lambda_a \in \mathcal{O}_{K_0}$. For $e = 0$, the Frobenius matching condition reads $h'(0) = \sum_{(a,\ell)=1}\lambda_a = (1-\varphi)(\varphi^{-s}d')$, and the integral power-sum construction gives the cocycle as follows. *Claim 3.* The power-sum construction produces integral cocycles. We give the construction in weight $r \ge 0$. Take finitely many $c_i \in \mathcal{O}_{K_0}$, positive integers m_i prime to ℓ, and $d_j \in \mathcal{O}_{K_0}$ for $0 \le j \le r$, satisfying

$$d_j - \ell^j\varphi(d_j) = \varphi^s\left(\sum_i c_i m_i^j\right).$$

Put $E_s = [\underline{\epsilon}^{1/\ell^s}]$, $\mu = [\underline{\epsilon}] - 1$, $\pi_1 = \varphi^{-1}(\mu)$, $\xi = \mu/\pi_1$, and $U = u^{-r}$. Lift the c_i to the unramified Witt subring of A_{inf}, obtaining w_i. For the construction put $q_k = \varphi^k(\sum_i w_i E_s^{m_i})$ and $b_k = \sum_i \varphi^k(w_i)$, and use the unramified Witt lift $\widetilde{d_0}$ of d_0. The matching equation at $j = 0$ gives $\widetilde{d_0} - \varphi(\widetilde{d_0}) = b_s$. Define

$$y = \sum_{k=0}^{s-1} q_k + \widetilde{d_0} + \sum_{k=s}^{\infty}(q_k - b_k).$$

For $k \ge s$, each $q_k - b_k$ is divisible by $\varphi^{k-s}(\mu)$. The latter tends to zero in the weak topology of A_{inf}, which proves convergence. Applying $1-\varphi$ to the partial sums gives $\sum_i w_i E_s^{m_i} - (q_{N+1} - b_{N+1})$; the last term tends to zero. For $j > 0$, the coefficient contributed by the tail is

$$\frac{1}{j!}\sum_{h\ge0}\ell^{jh}\varphi^{s+h}\left(\sum_i c_i m_i^j\right) = \frac{d_j}{j!},$$

by the convergent geometric inverse of $1-\ell^j\varphi$ and the matching equation. The tail has constant coefficient zero, so adding $\widetilde{d_0}$ gives constant coefficient d_0 before the finite initial sum is included. Expanding that finite sum uses $E_s^{m_i\ell^k} = \zeta_{\ell^s}^{m_i\ell^k}\exp(m_i\ell^{k-s}t_{\mathrm{cris}})$. These expansions are compatible modulo every fixed Fil^{r+1}: after subtracting the displayed Taylor polynomial, Frobenius tails tend to zero in each integral period quotient, and separatedness gives the asserted congruence. Thus the construction gives

$$y - \varphi(y) = \sum_i w_i E_s^{m_i}, \qquad y \equiv \sum_{j=0}^{r} A_j t_{\mathrm{cris}}^j \pmod{\mathrm{Fil}^{r+1}},$$

where

$$A_j = \frac{\sum_{k=0}^{s-1}\ell^{jk}\sum_i \varphi^k(c_i)m_i^j\zeta_{\ell^s}^{m_i\ell^k} + \ell^{sj}d_j}{j!\ell^{sj}}.$$

For the primitive, start a new sequence $(q_k)_{k\ge0}$ with $q_0 = y - \varphi(y)$ and choose H_∞-fixed p_k and q_{k+1} recursively so that

$$U\gamma(p_k)\mu^{k+1} - \gamma(\mu)^{k+1}p_k = \gamma(\mu)^{k+1}(\mu q_k - \ell^s q_{k+1}).$$

We now give the integral recursion for each monomial. In $\mathbf{Z}_\ell[[Z]]$, set $v = ((1+Z)^u - 1)/Z$, let γ substitute Zv, and start with $F_0 = 1$. Solve

$$U(1+Z)^{m_i}\gamma(P_k) - v^{k+1}P_k = v^{k+1}(ZF_k - \ell^s F_{k+1}).$$

The diagonal coefficients of the operator on the left are divisible by ℓ^s, and its first subdiagonal coefficients are units congruent to m_i modulo ℓ. Successive coefficient elimination modulo ℓ^s gives integral P_k, F_{k+1}. Evaluation at $Z = \mu$, multiplication by $w_i E_s^{m_i}$, and summation give the required p_k, q_k. This uses $\ell^s \geq 3$, also valid at $\ell = 2$.

Solve $\varphi(\beta_k) - \xi^{k+1}\beta_k = p_k$ in A_{inf}, and form

$$b = \sum_{k\geq 0} \ell^{sk}\beta_k/\pi_1^{k+1} \quad \text{modulo } \mathrm{Fil}^{r+1}.$$

These terms converge in the integral period-lattice topology: for each n, all sufficiently late terms belong to $\ell^n A_{\mathrm{inf}} + \mathrm{Fil}^{r+1}$. For the telescoping identity also solve $\varphi(\eta_k) - \xi^k\eta_k = q_k$, taking $\eta_0 = -y$. The Frobenius equations show that the twisted differences of β_k/π_1^{k+1} equal $\eta_k/\pi_1^k - \ell^s\eta_{k+1}/\pi_1^{k+1}$ modulo $\mathbf{Z}_\ell$; for H_∞ they lie in $\mathbf{Z}_\ell$. Thus

$$(h-1)b \in \mathbf{Z}_\ell + \mathrm{Fil}^{r+1}\ (h \in H_\infty), \qquad y + (U\gamma - 1)b \in \mathbf{Z}_\ell + \mathrm{Fil}^{r+1}.$$

Now put

$$b^\flat = b + \sum_{j<r} \frac{A_j t^j_{\mathrm{cris}}}{u^{j-r} - 1}, \qquad \alpha = A_r/\log u.$$

All denominators in this finite sum are nonzero. Define $z_1(\sigma)$ as the unique element of $\mathbf{Z}_\ell$ congruent modulo Fil^{r+1} to

$$(\log\chi(\sigma))\alpha t^r_{\mathrm{cris}} + (\chi(\sigma)^{-r}\sigma - 1)b^\flat.$$

Such a representative exists on H_∞ and at γ by the preceding identities, then on their products by the twisted cocycle identity, and on all of $G_{K'}$ by density and closedness of $\mathbf{Z}_\ell + \mathrm{Fil}^{r+1}$. It is unique since $\mathbf{Q}_\ell \cap \mathrm{Fil}^{r+1} = 0$. The induced topology on these constant representatives is the ℓ-adic topology, so z_1 is continuous. Uniqueness also gives $z_1(\sigma\tau) = z_1(\sigma) + \chi(\sigma)^{-r}z_1(\tau)$. The defining congruence says $\exp^*(z_1) = \alpha$.

For $r = e = 0$, take $c_i = \lambda_i$, $m_i = i$, $d_0 = d'$. This gives $\exp^*(z_1) = V(\lambda, d')/\log u$, with valuation multiplier $-s$, and hence $\beta_0 = -1$. For $e \geq 1$ apply this same construction in weight $r = e$ to the paired coefficients and matching values below. For $e \geq 1$, applying the $(\gamma - 1)$ twist to $(1+T)h'(T) = \sum_{(a,\ell)=1}(\lambda_a/a^e)(1+T)^a$ (equivalently $u^{-1}(1+T)\frac{d}{dT}(\gamma - u)h(T)$, forming the paired Teichmüller coefficients $c_{a,+} = \lambda_a/a^e$ at exponent ua and $c_{a,-} = -\lambda_a/a^e$ at exponent a for $(a,\ell) = 1$, which remain integral in $\mathcal{O}_{K_0}$ with $\ell \nmid ua$ and $\ell \nmid a$ for all primes ℓ including $\ell = 2$) cancels the degree-0 sum identically (giving $d_0 = 0$), uniquely solves $d_j - \ell^j\varphi(d_j) = \varphi^s(\sum_a \lambda_a a^{j-e}(u^j - 1))$ in $\mathcal{O}_{K_0}$ by ℓ-adic geometric series for $1 \leq j < e$, and sets $d_e = (u^e - 1)d' \in \mathcal{O}_{K_0}$ at degree e (satisfying $d_e - \ell^e\varphi(d_e) = \varphi^s((u^e - 1)\sum_a \lambda_a)$). Because $\zeta_{\ell^s}^{ua\ell^i} = \zeta_{\ell^s}^{a\ell^i}$, the resulting twisted sum equals $(u^e - 1)V(\lambda, d')$, and the same integral power-sum construction, in the manner of the explicit reciprocity law of Kato and Colmez [198, 199], applied to the paired coefficients yields a $\mathbf{Z}_\ell$-valued cocycle $z_1 \in Z^1(G_{K'}, \mathbf{Z}_\ell(-e))$ with $\exp^*(z_1) = \rho \cdot V(\lambda, d')$ and $\rho = (u^e - 1)/(\log(u)\, e!\, \ell^{es})$ (valuation $-v_\ell((e-1)!) - es$, which multiplied by $\ell^{(e+1)(s-1)} = \ell^{es+s-1-e}$ cancels ℓ^{es} and yields valuation $s - 1 - e - v_\ell((e-1)!) = \beta_e$, up to an explicit $\mathbf{Z}_\ell^\times$-unit absorbed by rescaling α'). Setting $a_0 = (e+2)s + v_\ell(e) + \lfloor e/(\ell-1)\rfloor + 1$, we have $e(s+1) + v_\ell(e!) + 1 + s + e/(\ell-1) \leq a_0 - \beta_e$ (for $e = 0$ directly $s + 1 \leq 2s + 2 = a_0 - \beta_0$, and for $e \geq 1$ using $v_\ell(e!) = v_\ell(e) + v_\ell((e-1)!)$ together with $e/(\ell-1) \leq \lfloor e/(\ell-1)\rfloor + 1$), so $\|\ell^{a_0-\beta_e}\exp^*(z)\| \leq 1$. Taking $\alpha' = \ell^{a_0-\beta_e}\exp^*(z)$ yields a $\mathbf{Z}_\ell$-valued cocycle z_1 with $\exp^*(z_1) = \ell^{a_0}\exp^*(z)$, so $z_i = \ell^{a_0}z - z_1$ is a rational cocycle over K' with $\exp^*(z_i) = 0$ (hence admitting a B_{cris}-primitive) matching $\ell^{c(e+1)+(e+2)s+v_\ell(e)+\lfloor e/(\ell-1)\rfloor+1+s+v_\ell(e+1)}\psi_i \circ f$ modulo $\mathbf{Z}_\ell$ on $G_{K'}$. Recombining the basis coordinates shows that this scaled pushout is finite over K', and corestriction from K' to K adds at most $v_\ell([K' : K]) \leq j(N) + 1$, proving $\ell^{c_{\mathrm{TB}}}\phi(f) \in H^1_f$ for the summed exponent c_{TB}. Here $v_\ell(0) = 0$ in the scalar cost. Put $D = D_h$. The first cost is at most $2D + 3t(a+1)D$ by (A.517). The two scalar costs together are at most $(t+2)D + 3t + 1$, using $s \leq D$, $v_\ell(e) \leq e$, $v_\ell(e+1) \leq e + 1$, and $e < t$. The final degree costs at most D. Their sum gives $c_{\mathrm{TB}} \leq 40(t+1)(a+1)D = T_{h,a,t}$. For $t = 0$ there is no integer e in the stated range.

Lean: `GlobalTheory.Cyc.ClassBound.class_bound` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CrystallineExtensionClassBound`, lines 359–387)
Lean: `UpperTheory.N.TBLoc.tb_part1` (`Development.AutomorphicArithmetic.ResidueCharacters.LenLeTorsion`, lines 176–203)
Lean: `UpperTheory.N.LocNum.cLL_le` (`Development.AutomorphicArithmetic.ResidueCharacters.LocNum`, lines 48–69)
Lean: `UpperTheory.N.LocNum.cTB_le` (`Development.AutomorphicArithmetic.ResidueCharacters.LocNum`, lines 80–117)
Lean: `UpperTheory.N.tbK_of` (`Development.AutomorphicArithmetic.ResidueCharacters.TbKOf`, lines 19–77)

By (36.20) the rational finite condition is zero for $e > 0$ and unramified for $e = 0$. In the latter case Frobenius evaluation lifts every divisible unramified class. The quotient in (36.24) injects into the ℓ^{c_B}-torsion of H^1/H^1_f. Duality identifies its dual with the integral finite group on $O(e+1)$ modulo ℓ^{c_B}. Its free rank is at most $[K : \mathbf{Q}_\ell]$. Its torsion comes from the invariants in a rank-one divisible module and contributes at most c_B after reduction.
Lean: `UpperTheory.N.TBLoc.tbe_clauses` (`Development.AutomorphicArithmetic.ResidueCharacters.LenLeTorsion`, lines 83–112)

□

Proof of 36.23

Proof. The common embedding sign in the jump conditions determines A, A' before π is chosen. Conjugate places have opposite types. The jump dictionary gives crystalline W on columns and crystalline $W^\vee(1)$ on rows, with zero absent. The completions embed in K' of 36.22. Lemma 36.22 gives the cyclotomic upper bounds $c_0^{\mathrm{cyc}}, c_B^{\mathrm{cyc}}$ used in this split calculation, which are distinct from the canonical hub constants c_0, c_B fixed in the hub setup of this section.
Lean: `UpperTheory.N.Two.decl_split` (`Development.AutomorphicArithmetic.ResidueCharacters.TwoDecl`, lines 29–67)
Lean: `UpperTheory.N.decl_split` (`Development.AutomorphicArithmetic.ResidueCharacters.PlaceTypeConj`, lines 69–107)

For the column module $\mathcal{M}_{\mathrm{col}}$ on g generators over the integral matrix algebra $\mathcal{A} = T' \otimes_T A_0$, let $\mathrm{Rel} \subset \mathcal{A}^g$ be the module of row vectors annihilating the generators. Take finitely many square arrays with every row in Rel, including arrays chosen for each block below, and form their linear combination U with independent variables. Its spectral determinant has coefficients in T'. Indeed, write each entry as a finite T'-linear combination of group matrices, so that U is a polynomial linear combination of $E_{ij} \otimes r_0(h)$, where E_{ij} are the g-by-g matrix units. A word in these matrices is zero or has the form $E_{ij} \otimes r_0(h_1 \cdots h_s)$. When $i \neq j$ its square is zero and its reverse characteristic polynomial is 1; when $i = j$ its reverse characteristic polynomial is that of $r_0(h_1 \cdots h_s)$. These polynomials have coefficients in T, hence in T', by the Hecke characteristic-polynomial condition. Amitsur's integral formula for a sum, applied over the polynomial ring in the independent variables, therefore puts every coefficient of $\det U$ in T'. This argument involves no division by integers. As verified below by rank-nullity off $Z(i)$, at every spectral point $x' \notin Z(i)$ there is a nonzero $\mathcal{A}$-homomorphism $\mathcal{M}_{\mathrm{col}} \to V_{x'}$ (hence a nonzero solution to the evaluated relation equations), so the determinant of this universal matrix vanishes identically on the off-$Z(i)$ locus over T'. Because T'/T is finite free of degree $t!$ and étale over the finite generic spectrum X_i (where the roots of X_0 on $\mathrm{im}(e_B)$ are distinct), expanding each T'-coefficient of the universal determinant in a T-basis of T' produces coefficients in T vanishing at every $x' \in X_i \setminus Z(i)$, hence belonging to the Hecke ideal of T supported on $Z(i)$. At the target point π, the matrix action splits by blocks (on the rank-r block, evaluation at the point of T' over π with the chosen ordering of roots is an O_i-algebra homomorphism $\psi_W : \mathcal{A} \to M_r(O_i)$; it is surjective, since the residual realization of W is absolutely irreducible, Burnside's theorem makes the residual image all of M_r over the residue field, and Nakayama's lemma lifts this to $M_r(O_i)$; the rank-one idempotent $\psi_W(e_1)$ has corner ring $\psi_W(e_1)M_r(O_i)\psi_W(e_1) \simeq O_i$, and the block carries the corresponding left $M_r(O_i)$-module structure $R_W = L^{\oplus r}$) and the determinant factors as the product of the block determinants; moreover, any change of K_i-basis on a block conjugates the relation matrix by an element of $\mathrm{GL}_d(K_i)$ (fixing the determinant), and a change of O_i-basis of the fixed block lattice multiplies its basis matrix on the right by an element of $\mathrm{GL}_d(O_i)$, leaving the least coefficient valuation in the DVR O_i invariant across O_i-bases of the block lattice. In a DVR the least coefficient valuation is additive on products, by dividing out each least power and reducing to nonzero polynomials over the residue field. In each block the matrix span gives an array whose evaluated rows are an elementary-divisor basis of relations. Specializing to it bounds that block's least coefficient valuation by the length of its solution module in the corresponding power of K_i/O_i. This bounds the length of the single evaluated congruence ideal by the sum of the block solution lengths.
Lean: `Formalism.RepresentationBoundsCohomology.ringDataObjects.displayFree` (`Formalism.WachLatticeTwists.RamifiedBlockRings.RamifiedBlockProofs.CompositeOfChainHolds`, lines 60–80)

Lean: GlobalTheory.DetGen.length_quot_le_sum_length (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.DeterminantLemmaDiscreteValuationRingQuotientLen, lines 259–271)

To construct the integral matrix algebra $\mathcal{A}$ and projectors on the full normalized representation r_0 over the base Hecke ring T (with blocks W of rank r and $O_i(-e)$ for $0 \le e < t$ at the target point π, of total rank $n_V = r + t$), set $A_0 = \mathrm{span}_T(r_0(G_E))$. By Lemma 36.25, the residual realization of W is absolutely irreducible and its polynomial on the prescribed tame generator at $v \mid g_k \neq \ell$ is nonzero at $\bar{\chi}_{i,1}$. Because all $r_0(x)$ ($x \in X_i$) share the same residual characteristic polynomial, lifting residual orthogonal projectors in A_0 yields integral orthogonal idempotents $e_B, e_1 \in A_0$ such that $e_B|_W = 0$ and $e_B|_{\mathrm{char}} = I_t$, while e_1 projects onto a rank-one line in W (with $\mathrm{im}(e_1) \simeq K_i/O_i$ on $W \otimes_{O_i} K_i/O_i$) and kills all character blocks $O_i(-e)$ (and e_B has rank t on every V_x, $x \in X_i$). Choose $g_0 \in G_E$ realizing the cyclotomic depth $s_0(i)$. At each point $x \in X_i \setminus \{\pi\}$ of the finite generic spectrum X_i, Hecke separation distinguishes x from π, and the assumed multiplicity-free semisimplicity of $r_0(x)$ (pairwise non-isomorphic absolutely irreducible summands over a finite splitting extension of K_i) makes the evaluated algebra span the full product on the summands over the splitting field by Burnside's theorem. The discriminant on the corner is therefore a nonzero polynomial in the coefficients of a linear combination. Choose those coefficients in the infinite field K_i, clear denominators, and multiply by a Hecke scalar vanishing at π and nonzero at x. This gives an element $s_x \in e_B A_0 e_B$ vanishing at π whose evaluation on $\mathrm{im}(e_B)$ at x has t distinct eigenvalues; taking a generic O_i-linear combination $s = \sum_{x \neq \pi} c_x s_x \in e_B A_0 e_B$ (avoiding the finitely many discriminant zeros on X_i) ensures that $X_0 = e_B r_0(g_0) e_B + s$ has t distinct eigenvalues at every point of X_i. Let T'/T be the finite free splitting extension of degree $t!$ adjoining the t ordered eigenvalues $\alpha_0, \dots, \alpha_{t-1}$ of the characteristic polynomial of X_0 on $\mathrm{im}(e_B)$ (specializing at π to $\chi_{i,1}(g_0)\varepsilon_\ell(g_0)^{-e}$), and set $\mathcal{A} = T' \otimes_T A_0$ with extended representation $\tilde{r}_0 = 1 \otimes r_0$. Define the integral scaled projector

$$E = e_B \prod_{e=1}^{t-1} (X_0 - \alpha_e e_B) \in \mathcal{A},$$

which satisfies $E^2 = \Delta E$ with $\Delta = \prod_{e=1}^{t-1}(\alpha_0 - \alpha_e)$ (specializing at π to $v_\ell(\Delta) = \delta$ by (A.512)); because $e_B|_W = 0$ integrally, E kills W identically without introducing any discriminant factor δ_X even when $t = 1$, kills $O_i(-e)$ for $1 \le e < t$, and acts as Δ on $O_i(0)$. At each column place $u \in A$, the left modules $P_{e_1} = \mathcal{A}e_1$ and $P_E = \mathcal{A}E$ embed into finite sums of copies of the crystalline representation $r_0 \otimes \chi_{i,1}^{-1}$ and project equivariantly onto W and $\Delta O_i(0)$ at π. Applying Lemma 36.22 at lifting exponent $e' = E_{\lfloor H_B \rfloor}$ along the crystalline surjections $P_{e_1} \twoheadrightarrow W$ and $P_E \twoheadrightarrow \Delta O_i(0)$ lifts integral cocycles representing a basis of the free O_i-module $H^1_f(E_u, W)/\mathrm{tors}$ (of rank $j_u \le d_u m$ by (36.20)) to matrix cocycles $K_c(u, a, g) \in \mathcal{A}$ ($1 \le a \le j_u$) and lifts a generator of $H^1_{\mathrm{ur}}(E_u, \Delta O_i(0))$ to a finite matrix cocycle $k_0(u, g) \in Z^1_f(D_u, \mathcal{A}E)$ whose specialization on $O_i(0)$ is unramified (since $H^1_f = H^1_{\mathrm{ur}}$ on $O_i(0)$) and acts on $O_i(0)$ at some $g \in D_u$ by a unit times $\ell^{e'}\Delta$. Define the column module $\mathcal{M}_{\mathrm{col}}$ over $\mathcal{A}$ as the quotient of the module generated by the global cocycle elements $y_{\mathrm{coc}}(g) = (\chi(g)^{-1}\tilde{r}_0(g) - 1)E \in Y \subseteq \mathcal{A}$ ($g \in G_E$, satisfying the global 1-cocycle identity $y_{\mathrm{coc}}(gh) = y_{\mathrm{coc}}(g) + \chi(g)^{-1}\tilde{r}_0(g) y_{\mathrm{coc}}(h)$ identically in $\mathcal{A}$) together with, at each column place $u \in A$, auxiliary local generators $x_u^{(0)}$ (killed by E), $(x_{u,a})_{1 \le a \le j_u}$ (killed by $1 - e_1$), and $x_u^{(\mathrm{tr})}$ (killed by $\Delta - E$), modulo the local column relation on each D_u ($u \in A$):

$$y_{\mathrm{coc}}(g) + (\chi(g)^{-1}\tilde{r}_0(g) - 1)x_u^{(0)} = \Delta \sum_{a=1}^{j_u} K_c(u, a, g) x_{u,a} + \Delta k_0(u, g) x_u^{(\mathrm{tr})} \qquad (g \in D_u).$$

At every spectral point $x' \notin Z(i)$ with n_V-dimensional evaluation representation $V_{x'}$, first no nonzero vector $\eta \in \mathrm{im}(E) \subset V_{x'}$ is fixed by $\chi^{-1}\tilde{r}_0(G_E)$ (by absence of invariants off $Z(i)$); second, at each column place $u \in A$, define $L_u = \pi^{-1}(H^1_f(D_u, V_{x'})) \subseteq Z^1(D_u, V_{x'})$ as the preimage of the Bloch–Kato finite subspace under the quotient map $\pi : Z^1(D_u, V_{x'}) \twoheadrightarrow H^1(D_u, V_{x'})$. Because K_c and k_0 are constructed by crystalline lifting from P_{e_1} and P_E, L_u contains $B^1(D_u, V_{x'})$, contains $g \mapsto K_c(u, a, g)s$ for every $1 \le a \le j_u$ and $s \in \mathrm{im}(e_1)$, and contains $g \mapsto k_0(u, g)s$ for every $s \in \mathrm{im}(E)$. By the assumed constancy of Hodge–Tate jump multisets of $r_0(x')$ across all $x' \in X_i$, every $V_{x'}$ ($x' \in X_i$) has the same dimension $\dim(D_{\mathrm{dR}}(V_{x'})/\mathrm{Fil}^0) = j_u$ as V_π (since the character blocks $O_i(-e)$ for $0 \le e < t$ have jumps $+e \ge 0$, so $\mathrm{Fil}^0 D_{\mathrm{dR}}(O_i(-e)) = D_{\mathrm{dR}}(O_i(-e))$ contributes zero to $D_{\mathrm{dR}}/\mathrm{Fil}^0$, leaving the same count $j_u = \dim(D_{\mathrm{dR}}(W)/\mathrm{Fil}^0)$); hence (36.20)

gives $\dim(L_u/B^1(D_u,V_{x'}))=\dim H^1_f(D_u,V_{x'})=\dim H^0(D_u,V_{x'})+j_u$ uniformly for every $x'\in X_i$. Since $\dim B^1(D_u,V_{x'})=n_V-\dim H^0(D_u,V_{x'})$, this yields $\dim L_u=n_V+j_u$. Because E and e_1 have rank 1 on $V_{x'}$, solving the displayed D_u-relations for unknowns $\eta\in\mathrm{im}(E)$ and $(x_u^{(0)}\in\ker E,\ x_{u,a}\in\mathrm{im}(e_1),\ x_u^{(\mathrm{tr})}\in\mathrm{im}(E))_{u\in A}$ imposes a homogeneous linear system of at most $\sum_{u\in A}\dim L_u=\sum_{u\in A}(n_V+j_u)$ scalar equations in $1+\sum_{u\in A}((n_V-1)+j_u+1)=1+\sum_{u\in A}(n_V+j_u)$ scalar unknowns in $\bigoplus_{u\in A}L_u$. By rank-nullity, a nonzero solution exists, and the resulting map sending $y_{\mathrm{coc}}(g)\mapsto(\chi(g)^{-1}\tilde r_0(g)-1)\eta$ defines a nonzero element of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}_{\mathrm{col}},V_{x'})$ (if $\eta\neq0$, then $(\chi(g)^{-1}\tilde r_0(g)-1)\eta\neq0$ for some $g\in G_E$; if $\eta=0$, some local component is nonzero), justifying the off-$Z(i)$ determinant vanishing used above.

Evaluation on y_{coc} defines an O_i-linear map $\Psi:\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}_{\mathrm{col}},M\otimes_{O_i}K_i/O_i)\to Z^1(G_E,M\otimes_{O_i}K_i/O_i)$ for every block $M\in\{W,O_i(0),O_i(-e)\}$. For every block M and every column place $u\in A$, the restriction $\Psi(\varphi)|_{D_u}$ automatically lands in the finite subgroup $H^1_f(E_u,M\otimes_{O_i}K_i/O_i)$: on W, both E and $k_0(u,\cdot)\in\mathcal{A}E$ vanish, so the D_u-relation expresses $\Psi(\varphi)|_{D_u}$ as a coboundary plus a combination of the prescribed finite basis cocycles $K_c(u,a,\cdot)$; on $O_i(0)$, $e_1=0$ and $\chi^{-1}\tilde r_0=1$, so $\Psi(\varphi)|_{D_u}=\Delta k_0(u,\cdot)\varphi(x_u^{(\mathrm{tr})})$ is unramified; and on $O_i(-e)$ $(1\le e<t)$, $E=e_1=k_0(u,\cdot)=0$, so $\Psi(\varphi)(g)=-(\chi(g)^{-1}\tilde r_0(g)-1)\varphi(x_u^{(0)})$ is a local coboundary on D_u. Hence column places $u\in A$ incur no local nonfinite quotient H^1/H^1_f.

We bound the length of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}_{\mathrm{col}},M\otimes_{O_i}K_i/O_i)$ on each block M by separating four contributions:

1. *Column kernel* $\ker\Psi$ *at* $u\in A$. When $\varphi(y_{\mathrm{coc}})=0$, on W the D_u-relation makes $\sum_a K_c(u,a,\cdot)\Delta\varphi(x_{u,a})$ a coboundary on D_u, so linear independence of the basis lifts modulo coboundaries up to $\ell^{e'}$ (the basis stays independent over K_i/O_i in $H^1(E_u,W\otimes_{O_i}K_i/O_i)$, since $H^1_f(E_u,W)/\mathrm{tors}$ is saturated in $H^1(E_u,W)/\mathrm{tors}$, being the preimage of a K_i-subspace, and $H^1(E_u,W)\otimes_{O_i}K_i/O_i$ injects into $H^1(E_u,W\otimes_{O_i}K_i/O_i)$) forces $\ell^{e'}\Delta\varphi(x_{u,a})=0$ in $\mathrm{im}(e_1)\simeq K_i/O_i$, costing $j_u(e'+\delta)$, while the residual freedom in $\varphi(x_u^{(0)})$ lies in $H^0(E_u,W\otimes_{O_i}K_i/O_i)$ and costs $h_W(u)$. On $O_i(0)$, evaluating $\Delta k_0(u,g)\varphi(x_u^{(\mathrm{tr})})=0$ at the test element $g\in D_u$ gives $\ell^{e'}\Delta^2\varphi(x_u^{(\mathrm{tr})})=0$, costing $e'+2\delta$ per column $u\in A$. On $O_i(-e)$ $(1\le e<t)$, $\varphi(x_{u,a})=0$ and $\varphi(x_u^{(0)})\in H^0(E_u,(K_i/O_i)(-e))$, costing at most h_0 per column $u\in A$.

2. *Global coboundaries and* Δ*-torsion.* Write $E=\sum_{g\in G_E}t_E(g)\tilde r_0(g)$ in $\mathcal{A}$ and set $y_1=\sum_{g\in G_E}t_E(g)\chi(g)y_{\mathrm{coc}}(g)=\big(E-\sum_g t_E(g)\chi(g)\big)E$, which satisfies $Ey_1=\Delta y_1$ by $E^2=\Delta E$ and evaluates to $\sum_g t_E(g)\chi(g)=\Delta$ on the trivial-character block $O_i(0)$. If $\Psi(\varphi)(g)=(\chi(g)^{-1}\tilde r_0(g)-1)v$ is a global coboundary on a block M of rank d, then $\varphi(y_1)=(E-\sum_g t_E(g)\chi(g))v$; on W and $O_i(-e)$ $(1\le e<t)$, where $E=0$, this gives $\varphi(y_1)=-\Delta v$ and hence $0=E\varphi(y_1)=\varphi(Ey_1)=\Delta\varphi(y_1)=-\Delta^2v$, bounding $\lambda_i(\mathrm{im}\,\Psi\cap B^1)\le 2d\delta$, whereas on $O_i(0)$ we have $\chi^{-1}\tilde r_0=1$ and $B^1=0$. Meanwhile, at each column place $u\in A$, the relations $Ex_u^{(0)}=0$ and $(\Delta-E)x_u^{(\mathrm{tr})}=0$ give $\Delta\varphi(x_u^{(0)})=0$ on $O_i(0)$ (where $E=\Delta$) and $\Delta\varphi(x_u^{(\mathrm{tr})})=0$ on W and $O_i(-e)$ (where $E=0$), contributing at most $|A|d\delta$ across the auxiliary column slots. Together these contribute at most $(2+|A|)d\delta\le 4d(|A|+1)\delta$ on block M, and summing over all blocks of total rank $n_V=r+t$ yields $4n_V(|A|+1)\delta$.

3. *Global Selmer classes and* Q_i*-terms.* Classes in $H^1(G_E,M\otimes_{O_i}K_i/O_i)$ satisfying the local conditions at $A\cup A'\cup S_i\cup Q_i$ contribute s_π on W, $v_\ell(h_E)$ on $O_i(0)$ by reciprocity, and $s_{\ell,e}(E)$ of Definition 36.11 on $O_i(-e)$ $(1\le e<t)$. Inertia exclusion gives the local conditions at S_i, while at each $v\in Q_i$ the character images are killed by $\zeta_{i,v}-1$, contributing $t\sum_{v\in Q_i}\lambda_i(O_i/(\zeta_{i,v}-1))$.

4. *Local nonfinite quotients at row places* $w\in A'$. On W, put $F_w=H^1_f(E_w,W\otimes_{O_i}K_i/O_i)$ and define the image submodule $B_w=F_w+\mathrm{im}(\mathrm{res}_w\circ[\Psi])\subseteq H^1(E_w,W\otimes_{O_i}K_i/O_i)$. Even when $E_w/\mathbf{Q}_\ell$ is ramified, $W^\vee(1)$ is crystalline with zero absent and reflected jumps $j\mapsto-1-j$ (since $|j|\le H_B-1$ on W, every jump of $W^\vee(1)$ satisfies $|-1-j|\le H_B$), and forming the crystalline pullback lattice $T_1\twoheadrightarrow W^\vee(1)$ from the dual of the crystalline ambient lattice Y gives a D_w-equivariant surjection onto $W^\vee(1)$. Applying Lemma 36.22 at lifting exponent $e'=E_{\lfloor H_B\rfloor}$ along $T_1\twoheadrightarrow W^\vee(1)$ lifts $\ell^{e'}H^1_f(E_w,W^\vee(1))$ into $H^1_f(E_w,T_1)$. By local Tate orthogonality of H^1_f with itself, the local Tate pairing of B_w/F_w with $H^1_f(E_w,W^\vee(1))$ vanishes on $\ell^{e'}H^1_f(E_w,W^\vee(1))$ (equivalently, $\ell^{e'}\Psi(\varphi)|_{D_w}\in F_w$), embedding B_w/F_w into the Pontryagin dual of $H^1_f(E_w,W^\vee(1))/\ell^{e'}H^1_f(E_w,W^\vee(1))$. Because $H^1_f(E_w,W^\vee(1))$ has O_i-free rank d_wm by (36.20) and torsion length at most rc_0^{cyc} by Lemma 36.22, this bounds the image

quotient by $\lambda_i(B_w/F_w) \le d_w m e' + r c_0^{\mathrm{cyc}}$. Pairing $w = \bar{u} \in A'$ with $u \in A$ (using $d_{\bar{u}} = d_u$) combines this with the column terms on W into $\sum_{u\in A}(d_u m(2e'+\delta) + h_W(u) + r c_0^{\mathrm{cyc}})$. On each of the t character blocks $O_i(-e)$ $(0 \le e < t)$, form the G_E-stable extension lattice $Y_0 = Y \oplus O_i$ with action $g\cdot(y,c) = (\chi(g)^{-1}\tilde{r}_0(g)y + c y_{\mathrm{coc}}(g), c)$, G_E-invariant projection $p(y,c) = c$, section $y_0 = (0,1)$ (so $gy_0 - y_0 = (y_{\mathrm{coc}}(g), 0)$), and the O_i-linear map $\phi(y,c) = \varphi(y) \in (K_i/O_i)(-e)$, which is G_E-equivariant on the kernel lattice $T = \ker(p|_{Y_0}) = Y \oplus 0$ (as required by Lemma 36.22, without assuming equivariance on all of Y_0); since Y_0 embeds into a finite sum of copies of $r_0 \otimes \chi_{i,1}^{-1}$ (crystalline at $w \in A'$), Lemma 36.22 applies directly to (Y_0, p, y_0, ϕ) and makes $\ell^{c_B^{\mathrm{cyc}}}$ times $\Psi(\varphi)|_{D_w}$ finite at $w \in A'$. By the local Euler characteristic formula (36.19) and Tate duality, $H^1(E_w, (K_i/O_i)(-e))/H^1_f$ has divisible corank at most d_w plus finite invariant and dual-invariant contributions ($h^0 + h^2 \le 2$, combining the $(d_w+1)c_B^{\mathrm{cyc}}$ bound of (36.24) with at most c_B^{cyc} from the remaining rank-one dual invariant), so its $\ell^{c_B^{\mathrm{cyc}}}$-torsion has length at most $(d_w+2)c_B^{\mathrm{cyc}}$ at w, totaling $t c_B^{\mathrm{cyc}} \sum_{w\in A'}(d_w+2)$.

Summing these block bounds (and bounding the $(t-1)|A|h_0$ positive-e column kernel terms by $t|A|h_0$) yields (36.25). More generally, the same determinant, projector, and Bloch–Kato congruence-ideal assembly applies at every odd prime ℓ and every disjoint pair (A, A') of subsets of $\{w \mid \ell\}$ such that, at every place $w \mid \ell$ outside $A \cup A'$, the restrictions to D_w of the classes in question satisfy the local conditions used for the global Selmer, Sha and class-number terms (in particular, both for $A = A' = \varnothing$ when $\ell > Z$ and for $A = \varnothing$, $A' = \{w \mid \ell\}$ at the ramified primes $\ell \in \{q, d\}$): the global Selmer, Sha, class-number, Q_i, and $4n_V(|A|+1)\delta$ projector terms combine with the column-kernel terms over $u \in A$ and the row nonfinite-quotient terms over $w \in A'$, using the local row lifting bounds available at each $w \in A'$.

Lean: `Formalism.MainNotions.EmbeddingTypes.length_bound_of_admitted` (`Formalism.SharpRestriction.LevelLetterCocycles.JumpAndLengthOfAdmitted`, lines 961–993)
Lean: `Formalism.MainNotions.EmbeddingTypes.sum_lenW_split` (`Formalism.SharpRestriction.LevelLetterCocycles.JumpAndLengthOfAdmitted`, lines 688–740)
Lean: `Formalism.NG.collectAlt` (`Formalism.CharacterColumnKernels.IdempotentNormalForms.CharacterColumnKernelsOfBlockChiCOfBlockChiOne`, lines 33–99)

□

Proof of 36.25

Proof. Choose a good point with its good list. Its squarefree degree-r label f_v at $v \mid g_k$ factors as $\prod_j f_{v,j}^{d_j}$ by the parameter parts of ranks a_j and multiplicities d_j. Since $\deg f_{v,j} \le a_j$ and $\sum_j d_j a_j = r$, all degrees agree. Squarefreeness gives $d_j = 1$. Two or more parts would give a common divisor degree $1 \le a_j \le r/2$ at every $v \mid g_k$, contrary to the divisor-degree condition in Definition 33.5.

Lean: `Formalism.MainNotions.AdmissibleThreePrime.cusp_of` (`Formalism.SharpRestriction.LevelLetterCocycles.ResidualIrreducibleOfTheoremThreePrime`, lines 29–98)

The sets $B(g_0, r)$ and $B(g_1, r)$ are disjoint, so choose $g = g_k$ with $\ell \notin B(g_k, r)$. Then $g \ne \ell$. Since ρ realizes the list, its reduced eigenvalues at the tame generator t_v come from the prime-to-ℓ components $z_{\ell'}$ of roots z of f_v, via the fixed torsion embedding. They are distinct by the prime-to-ℓ condition in Definition 33.5. A stable subspace is a sum of eigenlines. The relation $s_v t_v s_v^{-1} = t_v^g$ in this realization makes the corresponding roots stable under $z \mapsto z^g$. The chosen roots and their complement give divisors of f_v over $\mathbf{F}_g$. A proper nonzero subspace of dimension d gives degree $\min(d, r-d)$ at every $v \mid g$, which the good labels exclude.

Lean: `Formalism.CyclotomicNorms.CycleTypeFiniteOrder.t1` (`Formalism.CyclotomicNorms.NonsquarefreeBounds.IrreducibilityOfRestriction`, lines 427–636)

At either $g_k \ne \ell$, a prescribed character eigenvalue would identify $z_{\ell'}$ with the prime-to-ℓ component of the corresponding character value or its inverse. Both are excluded by the root test. Twisting by $\bar{\xi}$ preserves stable subspaces and leaves these inertia matrices unchanged.

Lean: `Formalism.MainNotions.AdmissibleThreePrime.not_root_of` (`Formalism.SharpRestriction.LevelLetterCocycles.ResidualIrreducibleOfTheoremThreePrime`, lines 129–171)

Since $\rho \otimes \xi$ realizes x, cancel the common monic character factor in its reduced Frobenius polynomials and those of the residual realization. Split density extends equality to their common finite quotient. Brauer–Nesbitt identifies the semisimplifications over $\overline{\mathbf{F}}_\ell$ [193]. The irreducible side has the full dimension r, so the other side is irreducible. Polynomial equality transfers the inertia exclusion.

Lean: `Formalism.MainNotions.N.residual_irreducible_of_theorem` (`Formalism.SharpRestriction.LevelLetterCocycles.JointConditionsZpowers.ResidualIrreducibleOfTheorem`, lines 101–198)

□

Proof of 36.26

Proof. In the dependency order of the chapter, Part (I) is established first and depends on none of the prime-cost or multiplicity lemmas that follow; those lemmas invoke only Part (I) and its cut constants C_T, C', and Part (II) then combines their prime-cost sum (36.41) and multiplicity bound (36.28) to restore Definition 36.11 on F_m.

Proof of (I). Take X_i to consist of points having a type vector F with $e_0F \neq 0$, where e_0 is the good paired-character idempotent. The eigenspan tie gives the required inclusions. For a central idempotent e of the product type group Γ, put $M_0 = |\Gamma|$. Then $(h^{-1}e)^{M_0+1} = h^{-1}e$. Its regular trace is $M_0[h]e$ and its eigenvalues are zero or M_0-th roots of unity. Thus M_0e has integral cyclotomic coefficients constant on conjugacy classes. Evaluate at the chosen primitive root, and sum the resulting integral lifts over good tuples to obtain α_i.

Lean: `GlobalTheory.CentralIdem.centralIdemCoeff` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TraceMemAdjoin`, lines 49–61)

Lean: `UpperTheory.GalC.exists_unit_one_add_generator` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.HeckeGaloisCriteria.ExistsUnitOneAddGenerator`, lines 17–56)

For a lift with $e_0F \neq 0$, some summand of the good-tuple projector acts nontrivially. At a type prime g_k take its finite-type label f. Goodness gives a squarefree polynomial with nonzero constant term and excludes the prescribed character value and its inverse from its roots. The assumed depth-zero type identity gives zero monodromy and the label $f(X)(X-u)^t$, with $u \in \mathbf{F}_{g_k}^{\times}$ the character value supplied by the identity. At the conjugate place the root labels are inverted. The tame characters attached to these roots are trivial on wild inertia. On a tame generator the label map is injective and multiplicative. Every repeated label therefore comes from u, or from u^{-1} on the conjugate side, and its character value has order dividing $g_k - 1$. Complete splitting gives $q_w = g_k$, and the parameter dimension satisfies $a_w \leq n_V \leq kb$ while $(kb)^8 \leq g_k$ forces $16a_w^2 < g_k$, so the cut multiplicity threshold is 2. These are the inertia conditions of the cut. The first provision identifies the other lists at the same spectral point up to permutation.

Lean: `UpperTheory.GalS.nc_of_label` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.TypeConditions.CardRootsBar`, lines 92–144)

Lean: `UpperTheory.GalC.nc_of_type` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.HeckeGaloisCriteria.ExistsUnitOneAddGenerator`, lines 143–150)

At such a place these tests pass to each cuspidal summand in the parameter list. Monodromy is a direct sum, the inertia polynomials multiply, and unramified twists preserve both tests. A repeated root of a factor is a repeated root of the product. For a cuspidal automorphic summand the normalized local component is generic, as in 39.8, and $E_w \simeq \mathbf{Q}_{g_k}$ by complete splitting. Lemma 40.11 gives depth-zero cuspidal support. Consequently the value of the weak correspondence used for this comparison is independent of its choice by 40.10.

Lean: `UpperTheory.GalS.nc_sum` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.TypeConditions.AutomorphicGaloisAdmissible`, lines 91–93)

Lean: `UpperTheory.GalS.nc_tw` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.TypeConditions.AutomorphicGaloisAdmissible`, lines 49–50)

Lean: `UpperTheory.GalC.exists_good_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ExistsGood`, lines 15–54)

At $w \mid \ell_i$ use the cut clause at type primes and unramified parameters elsewhere. Local compatibility and the dictionaries give admissible jumps, and the good spectrum gives residual existence. At S_i the type-prime exclusion is 36.25. At the auxiliary prime cancel the common character factor from $(U-1)^r(U-\bar{\chi}_{i,1}(\gamma_{i,w}))^t$ for the specified inertia element. The rank-r polynomial is $(U-1)^r$ and $\bar{\chi}_{i,1}(\gamma_{i,w}) \neq 1$.

Lean: `UpperTheory.GalS.eigPd_holds_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EigPdHolds`, lines 17–65)

For the fifth provision: at each $w \in Q_i$ and each auxiliary-prime place $w \mid p_*(i)$ off ℓ_i, the admissibility conditions give zero monodromy of the local parameter at w and its inertia polynomial $(U-1)^r(U-\chi_{i,1}(\gamma))^t$ for every $\gamma \in I_w$. Local-global compatibility at w transfers both to $r_0(x)$: the image $r_0(x)(I_w)$ is finite, and $r_0(x)(\gamma)$ has characteristic polynomial $(U-1)^r(U-\chi_{i,1}(\gamma))^t$. Since $t \geq 1$, this polynomial determines $\chi_{i,1}(\gamma)$ from $r_0(x)(\gamma)$, so $\mathrm{diag}(1^r, \chi_{i,1}^t)$ also has finite image on I_w. Both representations of I_w are therefore semisimple over the characteristic-zero field K_i and have the same characteristic polynomials, so Brauer–Nesbitt conjugates $r_0(x)|_{I_w}$ over K_i to $\mathrm{diag}(1^r, \chi_{i,1}^t)$, and the quadratic relation $(r_0(x)(\gamma)-1)(r_0(x)(\gamma)-$

$\chi_{i,1}(\gamma)) = 0$ follows from this diagonal form; at each type-prime place $w \in S_i$, Lemma 36.25 excludes the character eigenvalue on the residual rank-r block. The common residual characteristic polynomial thus has that root with multiplicity exactly t throughout X_i. Tame local-global compatibility splits off the corresponding degree-t character factor in characteristic zero. Dividing the Hecke characteristic polynomial by this monic factor gives the complementary annihilating polynomial $P_p \in T_i[U]$ with $P_p(\chi_{i,1}(\gamma_{i,w})) \in T_i^{\times}$, which together with the auxiliary-prime exclusion forces character-block cocycle functionals to vanish on I_w; and every residual cyclotomic twist of the normalized rank-r block and of its Tate dual has zero invariants at S_i by the exclusion, whence local duality and (36.19) give $H^0 = H^1 = H^2 = 0$. This establishes the fifth provision. For the sixth provision, a nonzero G_E-invariant vector in $r_0(x) \otimes \chi_{i,1}^{-1}$ places $\chi_{i,1}$ in a centred string at x. Purity and the residual constituent dimensions $r, 1, \dots, 1$ force rank 1, length t and shift zero for this occurrence. Character identification gives the prescribed character part, so $x \in Z(i)$. For any parameter list Σ at x, its augmentation by that part accounts for all t residual characters. Cancelling them leaves a single residual constituent of dimension r for Σ, forcing one cuspidal part of rank r and length 1. Combining the fifth and sixth provisions with the Z-multiplicity identity gives the second Galois provision. The first and the four ties restrict to the cut. Density, coefficient largeness and the jump and irreducibility conditions give its coefficient duties. Under $\mathscr{H}_2$, fixing any integer $C_T \geq 40$ and real $C' \geq 8C_T$ (for instance $C_T = 40$, $C' = 8C_T$) before selecting cuts depends only on $\mathscr{H}_2$, completing Part (I).

Lean: `UpperTheory.GalS.prvG5_of_admissible_cutRec` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.AdmissibleJOfRestricted`, lines 28–44)
Lean: `GlobalTheory.StringRepresentationDecomposition.HOneTwo` (`Development.AbelianVarietyTheory.Definitions.GaloisGeometry.HeckeCharEqOfLAdicRealisation`, lines 129–135)
Lean: `Formalism.MainNotions.SumRepresentations.HStarZero_iff` (`Formalism.SharpRestriction.LevelLetterCocycles.ExistsInertiaCharpolyFactorOfTameType`, lines 1479–1485)
Lean: `UpperTheory.GalC.facts_of_galois_preconditions_of_cutRec` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CFacts`, lines 45–57)
Lean: `UpperTheory.GalC.compatibilityTies_eigenRestriction` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.HeckeGaloisCriteria.CompatibilityTiesEigenRestriction`, lines 86–88)

Proof of (II). Fix the type threshold $C_T \geq 40$ and paired-character constant $C' \geq 8C_T$ from Part (I) before choosing the cuts. Apply Part (I) at every $x > x_1$ and $i \in I_x$ to obtain a good cut $c(x, i)$, and use the full cut for $x \leq x_1$. The transported cut family is twist-good.

Lean: `GlobalTheory.ExtraFromFacts.main_alt_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ImportsExtraAlt`, lines 18–27)

Use (36.27) and prime factorization for h_E, M_0 to sum (36.40) for this family. The square term satisfies

$$\sum_{\ell \text{ prime}} \frac{2 \log \ell}{\ell^2} \leq \sum_{n \geq 2} \frac{2 \log n}{n^2} \leq 6. \tag{A.518}$$

Thus (36.41) gives (36.17) on the cut, under its finiteness and summability hypotheses.

Lean: `GlobalTheory.NLinkClose.main_alt_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ImportsFurtherTail`, lines 22–109)
Lean: `GlobalTheory.LinkFam.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LinkNAlongFamilyOfLocalBounds`, lines 11–17)

Project U_i/L_i to its cut counterpart. For a kernel representative u, lift its cut image to an integral lattice vector m_0. Then $\alpha_i(u - m_0) = 0$, hence $M_0 u = \alpha_i m_0 \in L_i$. The preimage of this kernel is free of rank at most the dimension of the good sum. Indices without good points contribute zero. Consequently

$$\eta_j(w) \leq \eta_j^c(w) + v_{\ell(w)}(M_0) \sum_{i \in I_{j,w}} \sum_{\pi \in Z_{\mathrm{good}}(i)} \mathrm{mult}(i, \pi). \tag{A.519}$$

By (36.28), the rank row and 20.7, this gives

$$\eta_j(\mathcal{L}) \leq \eta_j^c(\mathcal{L}) + d_{j,0} \log M_0. \tag{A.520}$$

Lean: `UpperTheory.GalC.Alg.length_le` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.HeckeGaloisCriteria.LengthLeOfSpan`, lines 186–292)
Lean: `UpperTheory.GalC.etaJ_le` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.HeckeGaloisCriteria.IsHeckeGaloisIff`, lines 73–115)
Lean: `UpperTheory.GalC.sum_val_le` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.HeckeGaloisCriteria.EvPowLeExp`, lines 16–45)

Lean: `UpperTheory.GalC.etaSum_le_add` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.HeckeGaloisCriteria.EvPowLeExp`, lines 48–57)

The cut preserves $N_c, d_{j,0}, v_\ell(D)$. Eventually $\log M_0 \le C_1 x^{C_1} L \le C_1 b L^2$, since the member scale gives $\varepsilon > 0$. Absorb this in the family constant.

Lean: `UpperTheory.GalC.linkN_of_gd` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.HeckeGaloisCriteria.LinkNOfGd`, lines 38–93)

□

Proof of 36.27

Proof. For a good point take the parameter list and its good triple s. Write $f : O_0 \to O_i$ for the coefficient map. Cancel the common character factors in the tied Hecke polynomials, and use 36.24 in the finite quotients modulo every power of the maximal ideal. Separation gives trace equality of ρ_π and $f_*\rho_s$ on G_E. Absolute residual irreducibility by 36.25 gives the full matrix span over the residue field by matrix-algebra density [194, Chapter II], and over O_i by Nakayama.

Let ϕ, ϕ' be their group-algebra maps and s_1 an O_i-linear section of ϕ. Put $\theta = \phi' s_1$. The map $T(A)_{ab} = \operatorname{tr}(A\theta(E_{ba}))$ satisfies $T\phi' = \phi$ by trace equality applied to $x s_1(E_{ba})$, hence $T\theta = 1$. Finite equal ranks give invertibility, and $\phi' = \theta\phi$ makes θ an algebra isomorphism. Take a unit diagonal entry $u = \theta(E_{11})_{yy}$ using its trace 1. The matrices $U_{xa} = \theta(E_{a1})_{xy}$ and $V_{bx} = \theta(E_{1b})_{yx}$ satisfy $VU = uI$ and $\theta(A)U = UA$. The character tie now gives the integral identification

$$\chi_{i,1} = f_*(\chi_0)\psi_L, \qquad \rho_\pi \otimes \chi_{i,1}^{-1} \simeq f_*(\rho_s \otimes \chi_0^{-1}) \otimes \psi_L^{-1}.$$

Lean: `GlobalTheory.TraceConjugacy.exists_conj_of_surjective` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.ExistsConjOfTraceEqResiduallyAbsIrreducibleDVR`, lines 10–78)
Lean: `GlobalTheory.TraceConjugacy.exists_conj_of_trace_eq_local` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.ExistsConjOfTraceEqResiduallyAbsIrreducibleDVR`, lines 93–133)
Lean: `GlobalTheory.PtCore.twistedRho_conj_latTpsi` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TwistedRhoConjLatTpsi`, lines 33–43)

The auxiliary twist ψ_L^{-1} is trivial on decomposition groups above ℓ and preserves the required unramifiedness. The signature budget in the twist-good condition gives finite length at most $\nu_\ell + a_s$ with $a_s \ge 0$, $\sum_s a_s \le \ell^{-2}$. The conductor-root tie gives the tame bound.

Lean: `GlobalTheory.PtSel.sel_at` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SameCharEqOne`, lines 27–64)
Lean: `GlobalTheory.JointSix.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.PsiAtPsiAt`, lines 23–37)

At odd $\ell \nmid Q_S q d$, the cyclotomic residue image is full and tr is odd. At q, d the image contains the squares and $(\ell-1)/2 \nmid tr$. These give $\nu_\ell = 0$. For $\ell \mid Q_S$, a unit of depth $v_\ell(Q_S)$ trivial on the other conductor factors gives $\nu_\ell \le v_\ell(Q_S) + v_\ell(tr)$ by (A.512); summation proves (36.27).

Lean: `UpperTheory.Mu.sum_nu_le` (`Development.AutomorphicArithmetic.CyclotomicBounds.RootsOfUnity.NuRecord`, lines 53–62)

The Z-identity bounds the contribution at i by the sum of dimensions of its good isotypic spaces. Localization uniqueness makes their type sets disjoint in a fibre. Their direct sum lies in M_j, whose theta image is $Z_j(\mu_w)$ and whose theta map is injective by the nonzero scalars. An empty fibre contributes zero.

Lean: `GlobalTheory.MultiplicityBound.sum_mult_le` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.GoodPointMultiplicityLeRankProof`, lines 46–64)
Lean: `Formalism.MainNotions.J.theta_injective_of_scalars` (`Formalism.SharpRestriction.AdmissibleParameterJoins.NandDIVb`, lines 38–43)

□

Proof of 36.28

Proof. Let m_i be the least m with $Z^m N_i \subset P_i$. It exists and is at most H_*. We first calculate the cost of lowering this power of Z. Suppose $bZ^{j+1}N_i \subset P_i$ for $b \in \mathbf{Z}_\ell$. For $x \in N_i$, apply $(\gamma - 1)/Z$ to $bZ^{j+1}x$. The result belongs to P_i. Using $\gamma x \equiv x \bmod ZN_i$ and $\gamma Z/Z \equiv u \bmod Z$, the result equals

$$b(u^{j+1} - 1)Z^j x + bZ^{j+1}x', \qquad x' \in N_i.$$

The second term belongs to P_i. Descending from m_i to j shows that $Z^j \prod_{j<k\le m_i}(u^k - 1)$ annihilates N_i/P_i.

For $q_0 \nmid k$ the factor $u^k - 1$ is a unit. For $q_0 \mid k$ it has valuation $1 + v_\ell(k/q_0)$, by binomial expansion at u^{q_0}. Thus the valuation of the product is the sum, over $v \geq 0$, of the numbers of multiples of $q_0\ell^v$ in $(j, m_i]$. Each count is at most $(m_i - j)/(q_0\ell^v) + 1$, and at most C_h of the counts are nonzero. Consequently

$$\ell^a Z^j N_i \subset P_i \quad \text{if} \quad 0 \leq j \leq m_i, \qquad a \geq (m_i - j)\ell/q_0^2 + C_h.$$

For $j = 0$ the integer valuation of the product itself is at most $m_i\ell/q_0^2 + C_h$.

Put $t = \lceil m_i/q_0 \rceil$ and $L = C_h + 2 + t$. We claim that $Z^t Q^L N_i \subset P_i$. Write $Q = Z^{q_0} + \ell w$ with $w \in A$, using the binomial coefficients of $(1 + Z)^\ell$. A term of $Z^t Q^L$ is divisible by $\ell^{L-v} Z^j$, where $j = t + q_0 v$ and $0 \leq v \leq L$. If $j \geq m_i$ it annihilates the quotient. For $j < m_i$, use $m_i \leq q_0 t$ to obtain

$$(m_i - j)\ell/q_0^2 \leq t(1 - q_0^{-2}) - v(1 + q_0^{-1}) \leq t + 2 - v.$$

The preceding valuation bound applies with $a = L - v$. This proves the claim.

For $x \in N_{i-1}$, first apply the claim to $Q^{M_i}\varphi x \in N_i$. Multiply by $Z^{H_i + C_h + 2}$, and then use $Q^{h_i} P_i \subset A\varphi P_{i-1}$. Since $\varphi(Z) = ZQ$, this gives

$$\varphi(Z^{H_i + C_h + 2 + t} x) \in A\varphi P_{i-1}.$$

The ring A is free over $\varphi(A)$ on $1, Z, \ldots, Z^{\ell-1}$, by monic division using $(1 + Z)^\ell - 1$. Linearized Frobenius is injective, so comparison of the coordinate on 1 puts $Z^{H_i + C_h + 2 + t} x$ in P_{i-1}. We have proved

$$m_{i-1} \leq H_i + C_h + 2 + \lceil m_i/q_0 \rceil \leq H_i + C_h + 3 + m_i/q_0.$$

Iterate J times starting at c. The first J numerators are at most $2g + C_h + 3$, and the remaining term is $m_{c+J}/q_0^J \leq 2h/q_0^J$. Sum the geometric progression and apply the valuation estimate at $j = 0$. □

Proof of 36.29

Proof. Write $y = \Lambda x$. At position i choose an O-basis adapted to V_i, which is a direct summand by saturation. In each complementary coordinate divide x by the distinguished polynomial defining D_i. The remainder has integral coefficients. If it is constant after inverting ϖ, its positive-degree coefficients already vanish and its constant coefficient belongs to O. When $b_i = 0$ the residue condition is empty. We thus have $v_i \in V$ with

$$x - v_i \in (V_i)_A + D_i V_A.$$

Let $I = D_0 A + D_1 A$. The polynomials defining D_0, D_1 are relatively prime over the fraction field. Clearing the denominators in their polynomial Bézout identity shows that $I \cap O = \varpi^\alpha O$ for some $\alpha \geq 0$. Choose $f_\alpha \in D_1 A$ with $f_\alpha - \varpi^\alpha \in D_0 A$.

Subtract the two residue conditions. They give $v_0 - v_1 \in (V_0 + V_1)_A + IV_A$. We claim that its constant vector lies in $V_0 + V_1 + \varpi^\alpha V$. Take Smith coordinates for $V_0 + V_1 \subset V$. In one coordinate the assertion reads

$$c \in O, \quad c = ag + h, \quad a \in O, \quad g \in A, \quad h \in I \quad \Longrightarrow \quad c \in aO + \varpi^\alpha O.$$

For $a = 0$ this is the formula for $I \cap O$. If $a \mid c$ it holds directly. Otherwise $v_\varpi(a) > v_\varpi(c)$, and $1 - (a/c)g$ is a unit of A. Since its product with c is h, we have $c \in I \cap O$. This proves the claim in every coordinate.

Choose $u_i \in V_i$ and $v' \in V$ with $v_0 - v_1 = u_0 + u_1 + \varpi^\alpha v'$. Put $v_* = v_1 + u_1$. Modulo D_1 the vector $x - v_* - f_\alpha v'$ belongs to $(V_1)_A$. Modulo D_0 it belongs to $(V_0)_A$, since $v_* + \varpi^\alpha v' = v_0 - u_0$. Hence

$$y = \Lambda v_* + \Lambda f_\alpha v' + \nu, \qquad \nu \in \mathcal{N}(V).$$

Choose $w_*, w' \in W$ lifting $\ell^{2a_0} v_*, \ell^{2a_0} v'$, respectively. The vector $y_1 = \Lambda(w_* + f_\alpha w')$ has constant residues w_* and $w_* + \varpi^\alpha w'$ at the two positions. It belongs to $\mathcal{I}(W)$ and gives the asserted equality. The two constant vectors were lifted with the same power of ℓ. □

Proof of 36.30

Proof. Counting multiples of ℓ^j gives $v_\ell(k!) = \sum_{j\ge1}\lfloor k/\ell^j\rfloor \le k/2$. Thus $R \le 6x + 3/2$ and $D \le 6x$. Also

$$x \le \tfrac{3}{2}eh/\ell + \tfrac{1}{2}eS + 1, \qquad \log_\ell h' \le \log_\ell h + S + 1.$$

The first inequality uses $e/(\ell-1) \le 1$ in the constant term. For the second, $h + S + 1 \le \max(1,h)\ell^{S+1}$.

Write $j' = \lfloor \log_\ell h' \rfloor$. If $h' < \ell$, then $j' = 0$. Otherwise $h + S + 1 \ge \ell$, whence $3h/\ell + S \ge 2$. To see this for $S = 0, 1$, substitute $h \ge \ell - S - 1$. For $S \ge 2$ it follows from $h \ge 0$. The logarithmic inequality therefore gives, in both cases,

$$j' \le \log_\ell h + \tfrac{3}{2}S + \tfrac{3}{2}h/\ell.$$

Bounding the maximum in Y by the sum of its two entries now gives

$$\begin{aligned} Y &\le 48x + \tfrac{9}{2} + \eta + \log_\ell h' + e\big(\tfrac{3}{2}S + j'\big) \\ &\le \tfrac{147}{2}eh/\ell + (27e+1)S + (e+1)\log_\ell h + \tfrac{107}{2} + \eta \le 80B. \end{aligned}$$

The negative and positive elements of J contribute at most $L(h)$ and $L(2h)$, respectively. Hence $E' \le 2L(4h) + 2L(h) + L(2h) \le 18h$.

The quotient C/C_0 is a quotient of at most $2(ne+1)r$ copies of $O/\ell^Y O$. The submodule C_0 is a quotient of at most $ne(e+1)m$ copies of $O/\ell^{E'}O$. Add their normalized lengths, use $Y \le 80B$ in the first term and $18h \le 160(h+1)$ in the second. □

Proof of 36.31

Proof. In the Artin description of the host in Theorem 8.3, the factor at ℓ acts through its quadratic residue character. The other factors of the conductor are prime to ℓ and give unramified local extensions. Thus the ramification index is at most two. The quadratic Gauss sum in the proof of Theorem 8.3 belongs to E, has square $-\ell$, and belongs to $\mathbf{Q}_\ell(\mu_\ell)$ under the chosen local embedding. Its valuation is $1/2$ for $v_\ell(\ell) = 1$. Hence $K_0(\sqrt{-\ell})$ has degree two over K_0 and exhausts K. The polynomial $\Phi_\ell(1+Z)$ is Eisenstein over K_0, giving the asserted index. In particular restriction to the level-one cyclotomic field followed by corestriction has multiplier prime to ℓ.

Use the pointwise local compatibility on the cut in Theorem 36.26(I). Above the coefficient prime, its clause away from the two type primes gives a crystalline full realization with the level-zero Hodge weights. It applies to every x here since q, d are distinct from g_0, g_1. At a good point the semisimple decomposition of the full realization contains $\rho_i(\pi)$ and the character lines $\chi_{i,c}$, including $c = 1$.

If $0 \to U \to V \to A \to 0$ is exact with V crystalline, left exactness of crystalline periods gives

$$\dim D_{\mathrm{cris}}(V) \le \dim D_{\mathrm{cris}}(U) + \dim D_{\mathrm{cris}}(A) \le \dim U + \dim A.$$

Here dimensions of periods are over K_0 and representation dimensions are over $\mathbf{Q}_\ell$, after restriction of coefficient scalars. The end terms agree. Both constituents are therefore crystalline. Applying this along a composition series gives crystallinity of the character and rank-r semisimple constituents above. Dualizing or tensoring crystalline comparison isomorphisms gives the comparison isomorphism for the dual or tensor product. This proves the claims about V_x, W_π and their Tate duals. Apply the same dimension argument to the specified subrepresentation of the direct sum. The de Rham filtration on a subrepresentation is the intersection filtration. Its graded pieces inject into those of the sum. Dualizing negates the jumps and the Tate twist subtracts one. □

Proof of 36.33

Proof. Since $t \le \eta$, we have $0 \le c_t \le \eta$ and $t - 1 \le c_t + \eta$. The bounds on F_l therefore give

$$c_t - \eta \le m_l \le 0, \qquad t - 1 \le M_l \le c_t + \eta, \qquad \eta' \le 2\eta + 1.$$

Hence $\eta' + 4\sigma + 6 \le 2\eta + 8B + 7 < \ell$, which is (P4). Every $z \in F_l \cup F_c$ satisfies $|z - c_t| \ge 2B + t/2 + 3$; for $z \in F_c$ this follows from $|z - c_t| \ge \delta(z) \ge W_0$. An integer $y' \in [0, t-1]$ has $|y' - c_t| \le c_t$, so $|z - y'| \ge 2B + 7/2 > \sigma + 2$.

With $x = z + w$ and $y = y' + w$ this gives (P5), even for every $x \in U$. The map $z \mapsto z + w$ is injective on the set $F_l \cup F_c$, which gives the multiplicities.

Let $\varsigma = 1$. An element of F_c satisfies $z \geq c_t + W_0 > 0 \geq m_l$. Hence every $z \in F_l \cup F_c$ lies in $[m_l, M_a]$, and $z + w \in [1, h-1]$. This is (P1). Next $w = 1 - m_l \geq 1$, the inequality $w + t \leq h$ reads $t - 1 \leq M_a$, and $w + t - 1 \leq \eta'$ reads $t - 1 \leq M_l$. These give (P2) and (P3). Every element of $F_l \cup F_c \cup \{t-1\}$ is at most $c_t + W_0 + S + \eta + 1$, and $-m_l \leq \eta - c_t$. Hence $h \leq W_0 + S + 2\eta + 3$. For (P6) let $x = z + w \notin [1, \eta']$. Then $z \notin [m_l, M_l]$, so $z \in F_c$ and $\bar{x} = z + 1 - m_l$. Suppose $\bar{x} \equiv m \pmod{\ell}$ with m in the range of (P6). Then ℓ divides $N = z + 1 - m_l - m$. The bounds $m \leq M_l - m_l + 2\sigma + 4$, $M_l \leq c_t + \eta$ and $z \geq c_t + W_0$ give the lower bound below. The bounds $m \geq -2\sigma - 2$, $-m_l \leq \eta - c_t$ and $z \leq c_t + W_0 + S + \eta + 1$ give the upper bound:

$$W_0 - \eta - 2\sigma - 3 \leq N \leq W_0 + S + 2\eta + 2\sigma + 4.$$

Since $\sigma \leq \sigma_P$, the integer N lies in (36.29), a contradiction.

Let $\varsigma = -1$. An element of F_c satisfies $z \leq c_t - W_0 < c_t \leq M_l$. Hence every $z \in F_l \cup F_c$ lies in $[m_a, M_l]$, and $z + w \in [1, h-1]$, which is (P1). Next $w = 1 - m_a \geq 1$, the inequality $w + t \leq h$ reads $t - 1 \leq M_l$, and $h - \eta' = w + m_l \leq w$. These give (P2) and (P3). Every element of $F_l \cup F_c \cup \{0\}$ is at least $c_t - W_0 - S - \eta - 1$, and $M_l \leq c_t + \eta$; hence $h \leq W_0 + S + 2\eta + 3$. For (P6) let $x = z + w$ lie outside $[h - \eta', h-1] = [w + m_l, w + M_l]$. Then $z \in F_c$ and $\bar{x} = h - x = M_l + 1 - z$. If $\bar{x} \equiv m \pmod{\ell}$ with m in the range of (P6), then ℓ divides $N = M_l + 1 - z - m$, and

$$W_0 - \eta - 2\sigma - 3 \leq m_l - z - 2\sigma - 3 \leq N \leq M_l + 1 - z + 2\sigma + 2 \leq W_0 + S + 2\eta + 2\sigma + 4,$$

by $m \leq M_l - m_l + 2\sigma + 4$, $m_l \geq c_t - \eta$, $c_t - W_0 - S - \eta - 1 \leq z \leq c_t - W_0$ and $M_l \leq c_t + \eta$. Again N lies in (36.29), a contradiction. □

Proof of 36.37

Proof. The substituted series defining δ_z has zero constant coefficient and nonzero linear coefficient $\bar{z}$. The binomial identities give $\delta_z \delta_w = \delta_{zw}$, so $\delta_{z^{-1}}$ is its inverse. These identities follow coefficientwise by approximating $z, w \in \mathbf{Z}_\ell$ by nonnegative integers. Thus the displayed substitutions define an action on A_π and its fraction field.

If $y \in B_\pi^\Delta$, choose $r \geq 0$ so that $\iota(X)^r y \in A_\pi$. This product is invariant, hence belongs to $\iota(A)$. Dividing by $\iota(X)^r$ proves $\iota(B) = B_\pi^\Delta$. The identity intertwining γ and γ_π extends from A to B by taking fractions. Finally write $b = X^m a$, where $m = v_X(b)$ and $a \in A^\times$. Since φ preserves A, it carries a and a^{-1} to mutually inverse elements of A. Therefore

$$\varphi(b) = X^{\ell m} u^m \varphi(a), \qquad v_X(\varphi(b)) = \ell m.$$

Multiplication by X^a is invertible on E, so a shifted lattice is again a lattice; the kernel intersection defining K_a is a k-subspace. □

Proof of 36.38

Proof. Write $n = \dim_B E$. All constructions below use the scalar data of Definition 36.36.

Claim 1. Every $y \in B_\pi$ has a unique expression

$$y = \sum_{j=0}^{\ell-1} (1+\pi)^j \varphi_\pi(y_j).$$

The map $\psi_\pi(y) = y_0$ is k-linear, satisfies $\psi_\pi(\varphi_\pi(a)b) = a\psi_\pi(b)$, and commutes with Δ and γ_π. To see this, group Laurent coefficients modulo ℓ. This gives a unique expansion in $1, \pi, \dots, \pi^{\ell-1}$ over $\varphi_\pi(B_\pi)$. The binomial change of basis to $(1+\pi)^j$ is triangular with diagonal entries 1. The displayed semilinear identity follows by multiplying an expansion by $\varphi_\pi(a)$. For $z \in \Delta$, write $zj = r + \ell u$, where $0 \leq r < \ell$ and $u \in \mathbf{Z}_\ell$. Then

$$\delta_z((1+\pi)^j) = (1+\pi)^r \varphi_\pi((1+\pi)^u).$$

Here $r = 0$ exactly when $j = 0$, so the zeroth coordinate after substitution is $\delta_z(y_0)$. The same computation with $z = 1 + \ell$ gives $\psi_\pi\gamma_\pi = \gamma_\pi\psi_\pi$. Thus ψ_π preserves $B_\pi^\Delta = \iota(B)$, and defines ψ with all three asserted scalar operator identities.
Lean: `GlobalTheory.PsiLemmaCompare.assemble_bijective` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CyclotomicExistsIff`, lines 158–183)
Lean: `GlobalTheory.PsiShiftProof.psi_del` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.MemOfAssembleMem`, lines 333–347)
Lean: `GlobalTheory.PsiShiftProof.exists_psi` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsPullBasis`, lines 31–39)

Claim 2. For $2-\ell \le a \le 1$, one has $\psi(X^a A) = A$. For $q \in \mathbf{Z}$ and $0 \le r < \ell$, expansion of $\pi^r = ((1+\pi)-1)^r$ gives

$$\psi_\pi(\pi^{\ell q+r}) = (-1)^r \pi^q.$$

Consequently $v_\pi(\psi_\pi(y)) \ge \lfloor v_\pi(y)/\ell \rfloor$. If $x \in X^a A$, the latter bound is strictly greater than $-(\ell-1)$. The valuation of a nonzero element of $\iota(B)$ is a multiple of $\ell - 1$, so $\psi(x) \in A$. The constant term of $\psi_\pi(\iota(X))$ is the nonzero leading coefficient of $\iota(X)$; hence $\psi(X)$ is a unit of A. For $b \in A$,

$$\psi\big(\varphi(b/\psi(X))X\big) = b.$$

The argument belongs to $XA \subset X^a A$, proving surjectivity.
Lean: `GlobalTheory.PsiShiftProof.psiRing` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsPullBasis`, lines 78–97)

Claim 3. There is a unique k-linear map $\Psi : E \to E$ satisfying $\Psi(bf(x)) = \psi(b)x$. It obeys $\Psi g = g_0 \Psi$, and $\Psi(X^a f^* M_0) = M_0$ in the range of Claim 2. Choose an A-basis $b_1, \dots, b_n$ of M_0. The vectors $c_i = f(b_i)$ span E, since semilinearity expresses every $f(x)$ in their span and $Bf(E) = E$. There are n such vectors, so they form a B-basis and $P = f^* M_0 = \bigoplus_i Ac_i$. Put

$$\Psi\left(\sum_i x_i c_i\right) = \sum_i \psi(x_i) b_i.$$

Claim 1 gives the required identity on every $bf(x)$; such vectors span E, so the map is independent of the chosen basis and lattice. The identities $fg_0 = gf$ and $\psi\gamma = \gamma\psi$ prove $\Psi g = g_0 \Psi$ on these spanning vectors. Claim 2, applied to each coordinate, gives the image assertion.
Lean: `GlobalTheory.PsiShiftProof.exists_lat_basis` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsPullBasis`, lines 206–216)
Lean: `GlobalTheory.PsiShiftProof.exists_pull_basis` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsPullBasis`, lines 164–176)
Lean: `GlobalTheory.PsiShiftProof.exists_Psi` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsPullBasis`, lines 179–195)
Lean: `GlobalTheory.PsiShiftProof.pull_span` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsPullBasis`, lines 219–235)

Claim 4. If $H - I_n \in \pi^2 M_n(A_\pi)$, $1 \le j < \ell$, and $s \in \mathbf{Z}$, then

$$L_j(y) = (1+\pi)^j H \gamma_\pi(y) - y$$

is a bijection from $\pi^s A_\pi^n$ onto $\pi^{s+1} A_\pi^n$. Indeed, $\gamma_\pi(\pi) = \pi + \pi^\ell + \pi^{\ell+1}$. For every integer m, the first nonzero term of $L_j(\pi^m v)$, with $v \in k^n$, is $j\pi^{m+1}v$. All other terms have larger exponent, because $\ell \ge 5$ and $H - I_n$ has order at least two. Given $z \in \pi^{s+1} A_\pi^n$, determine the coefficient of π^s in y from the coefficient of π^{s+1} in z, and proceed successively. At every stage the new coefficient is multiplied by the same nonzero scalar $j \in k$. This recursion converges coefficientwise and proves existence and uniqueness.
Lean: `GlobalTheory.PsiShiftProof.twist_bij` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.MemOfAssembleMem`, lines 457–474)

Claim 5. The operator $x \mapsto \varphi_\pi(H)\gamma_\pi(x) - x$ maps $\pi^{\ell s} A_\pi^n \cap \ker \psi_\pi$ bijectively onto $\pi^{\ell(s+1)} A_\pi^n \cap \ker \psi_\pi$, under the hypotheses on H in Claim 4. For a tuple $y = (y_j)_{0 \le j < \ell}$, let $\mathcal{A}(y) = \sum_j (1+\pi)^j \varphi_\pi(y_j)$, componentwise. Grouping exponents and using the triangular binomial change of basis proves

$$\mathcal{A}(y) \in \pi^{\ell s} A_\pi^n \quad \Longleftrightarrow \quad y_j \in \pi^s A_\pi^n \text{ for every } j.$$

Moreover $\psi_\pi \mathcal{A}(y) = y_0$, and direct substitution gives

$$\varphi_\pi(H)\gamma_\pi(\mathcal{A}(y)) - \mathcal{A}(y) = \mathcal{A}\big((L_j(y_j))_j\big).$$

By Claim 4, the operator on the left is a bijection from $\pi^{\ell s}A_\pi^n \cap \ker \psi_\pi$ onto $\pi^{\ell(s+1)}A_\pi^n \cap \ker \psi_\pi$. It is also injective on the whole kernel, since each finite tuple of Laurent series lies in some $\pi^{\ell s}A_\pi^n$.
Lean: GlobalTheory.PsiShiftProof.op_assemble (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.MemOfAssembleMem, lines 476–483)
Lean: GlobalTheory.PsiShiftProof.core_maps (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.MemOfAssembleMem, lines 496–505)
Lean: GlobalTheory.PsiShiftProof.core_inj (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.MemOfAssembleMem, lines 507–522)
Lean: GlobalTheory.PsiShiftProof.core_surj (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.MemOfAssembleMem, lines 524–540)

Claim 6. If $\psi(x) = 0$ and $2 - \ell \le a \le 1$, then

$$x \in X^a A \quad \Longleftrightarrow \quad \iota(x) \in \pi^{\ell a} A_\pi.$$

For $a \le 0$, the thresholds $(\ell-1)a$ and ℓa select the same multiples of $\ell - 1$, since $a \ge 2 - \ell$. For $a = 1$, membership in XA gives valuation at least $\ell - 1$. The coefficient of $\pi^{\ell-1}$ must vanish, since it is the constant term of $\psi_\pi(\iota(x))$. Thus the valuation is at least ℓ. The reverse implication follows directly from $\ell > \ell - 1$.

Now assume $(g_0 - 1)M_0 \subset XM_0$, and write $g_0(b_j) = \sum_i G_{ij} b_i$. Then $G - I_n \in XM_n(A)$. In the c-basis the operator $g - 1$ is

$$T_G(x) = \varphi(G)\gamma(x) - x. \tag{A.521}$$

Also $\Psi(x) = 0$ exactly when all its c-coordinates lie in $\ker \psi$. Apply Claim 5 with $H = \iota(G)$; its entries satisfy $H - I_n \in \pi^{\ell-1}M_n(A_\pi) \subset \pi^2 M_n(A_\pi)$. The assembled operator commutes with Δ. Its unique preimage of a Δ-fixed target is therefore Δ-fixed. Descending through ι and applying Claim 6 to the source exponent a and target exponent $a + 1$ gives

$$T_G\big((X^a A \cap \ker \psi)^n\big) = (X^{a+1}A \cap \ker \psi)^n \qquad (2 - \ell \le a \le 0).$$

This is the asserted equality $(g - 1)K_a(P, \Psi) = K_{a+1}(P, \Psi)$.
Lean: GlobalTheory.PsiShiftProof.lemmaPsi (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsPullBasis, lines 239–294)

□

Proof of 36.40

Proof (outline). Claim 1. The family in (36.31) consists of A-lattices. Exactness makes $D(j)_i$ injective and $D(p)_i$ surjective. After identifying $D(U_1)_i$ with its image, the inverse image is the intersection of a B-subspace with N_i^b. This intersection is finitely generated, since it is an A-submodule of the finite A-module N_i^b. For every vector of $D(U_1)_i$, multiplying by a sufficiently large power of X puts its image in N_i^b; hence the intersection spans $D(U_1)_i$. Its image under the surjective map $D(p)_i$ is therefore finitely generated and spans $D(U)_i$.

Claim 2. In the unramified, level-one crystalline setting of the definition, the canonical residue functor has the stated compatible reduction maps, and the image $\overline{N}_i(T)$ is an A-lattice. The canonical maps are obtained from reduction of the integral étale module followed by Δ-invariants. The order of Δ is $\ell - 1$, which is a unit in O and in k, so the averaging projector makes reduction on invariants surjective. The chosen cyclotomic scalar embedding identifies the invariant scalar reduction with A. More explicitly, an invariant scalar reduces into $\iota(A)$; conversely, lift a power series coefficientwise and average its translates to lift any member of $\iota(A)$ to an invariant scalar.
Lean: GlobalTheory.RedFun.realD (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsPresTensor, lines 556–570)
Lean: GlobalTheory.Present.RedLat.red_lift (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachDeltaInvariantsReductionIsLattice, lines 232–266)

The level-one Wach construction supplies a basis $b_1, \dots, b_d$ of $N(T)$ fixed by Δ. For an invariant vector, uniqueness of its expansion in this basis forces each scalar coefficient to be invariant. Scalar reduction and lifting therefore identify $\overline{N}_i(T)$ with the A-span of the finitely many $\jmath_i(b_j)$. This proves finite generation. Here the invariant-basis assertion uses that $K/\mathbf{Q}_\ell$ is unramified, $\ell \neq 2$, and the cyclotomic ring relations at level one, all of which hold in the setting of the definition.
Lean: GlobalTheory.WachQN.invBasis_of_admitted (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachLatticeUsedPropertiesOfAdmitted, lines 53–61)

For the spanning assertion, take $x \in D_{\mathrm{et}}(T)^{\Delta}$. The étale base-change property gives an integer $m \geq 0$ and vectors $y \in N(T)$, $z \in D_{\mathrm{et}}(T)$, such that $Z^m x = y + \ell z$. Put

$$u = \prod_{\delta \in \Delta} \delta(Z) = u' Z.$$

This scalar is invariant and its reduction is nonzero. Multiply the last equality by $(u')^m$, then sum its Δ-translates. One obtains

$$|\Delta| u^m x = \mathrm{Tr}_{\Delta}((u')^m y) + \ell \, \mathrm{Tr}_{\Delta}((u')^m z).$$

The first trace is an invariant vector of $N(T)$. Applying $\jmath_i$ kills the second term. Since the reduction of $|\Delta| u^m$ is a nonzero member of A, it is invertible in B; thus $\jmath_i(x)$ belongs to the B-span of $\overline{N}_i(T)$. Surjectivity of $\jmath_i$ proves the spanning assertion.

Lean: `GlobalTheory.Present.RedLat.approx` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachDeltaInvariantsReductionIsLattice`, lines 69–99)

Lean: `GlobalTheory.Present.RedLat.exists_lat` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachDeltaInvariantsReductionIsLattice`, lines 118–210)

Claim 3. The displayed cocycle construction gives an extension and a well-defined Herr class. For $g, h \in G_K$, the two actions on (u, a) agree because $z(gh) = z(g) + gz(h)$; the same law gives $z(1) = 0$. Exactness supplies the lifts e_i, and the images of both $\varphi_i(e_{i-1}) - e_i$ and $\gamma_i(e_i) - e_i$ in $D(k)_i$ vanish. Commutation of φ_i and γ_i gives

$$(\gamma_i - 1)(\varphi_i(e_{i-1}) - e_i) = \varphi_i(\gamma_{i-1}(e_{i-1}) - e_{i-1}) - (\gamma_i(e_i) - e_i).$$

Thus the pair lies in the numerator. The same identity applied to arbitrary x_i places every displayed boundary in that numerator. Replacing e_i by $e_i + x_i$ adds exactly that boundary, proving independence of the lifts. Finally, reduction $\mathbf{Z} \to k$ sends the integer multiset to its reduced multiset with the multiplicities prescribed in the definition. The complete residue-lattice comparison is proved in `GlobalTheory.Present.RedLat.red_isLat`. □

Proof of 36.42

Proof. Put $m = h_i - w_i - a$. By (P3), $0 \leq m \leq h_i - w_i \leq \eta_i'$. The assumed divisibility says $h_i - u \equiv m \pmod{\ell}$. If $u \in U_i$ lies outside the low window, this contradicts (P6), since

$$[0, \eta_i'] \subset [-2\sigma_i - 2, \eta_i' + 2\sigma_i + 3].$$

For a jump in the low window, or in the character block, (P1)–(P3) give $1 \leq h_i - u \leq \eta_i'$. Hence $|(h_i - u) - m| \leq \eta_i' < \ell$, where the last inequality follows from (P4). An integer divisible by ℓ and of absolute value less than ℓ is zero. Thus $h_i - u = m$, which is the required equality. □

Proof of 36.43

Proof. The quotient map $T_0 \to T_0/\varpi T_0$, followed by the twist, identifies the coefficient representation with $\overline{W}$. Choose a cocycle representing each class. The exponent-one hypothesis makes its extension admissible, so the assumed model construction supplies its model lattice. An invertible equivariant change of coordinates preserves cocycles, coboundaries and linear independence. This gives the first assertion.

The passage from a cocycle to its Herr class is linear. Suppose a linear combination has zero Herr class. In its extension, subtract suitable elements of $D(\overline{W})_i$ from the chosen lifts of $\mathbf{1}_i$. The resulting lifts e_i satisfy

$$\varphi_i(e_{i-1}) = e_i, \qquad \gamma_i(e_i) = e_i.$$

The fixed-vector property gives an equivariant map $k \to E_z$ whose realization takes 1 to e_i. By k-linearity of the realization, its composite with $E_z \to k$ is multiplication by a scalar; this scalar is 1, since the image of each e_i is $\mathbf{1}_i$. Thus the extension splits and the original cocycle is a coboundary. Linear independence of the original classes forces all coefficients in the relation to be zero. □

Proof of 36.45

Proof. Choose a Frobenius element F and the continuous unramified homomorphism $\nu : G_K \to \mathbf{Z}_\ell$ normalized by $\nu(F) = 1$. Let $\ell^a X = 0$, with $a \geq 1$, and put $q = \ell^a$. The unramified character and its divided difference

$$\mu(g) = (1+q)^{\nu(g)}, \qquad w(g) = \frac{\mu(g) - 1}{q}$$

satisfy $w(g) \equiv \nu(g) \pmod{qO}$. Indeed, $\binom{z}{j} \in \mathbf{Z}_\ell$ for $z \in \mathbf{Z}_\ell$, by continuity from nonnegative integers. Thus every term $\binom{\nu(g)}{j} q^{j-1}$, for $j \geq 2$, belongs to $q\mathbf{Z}_\ell$. Every unramified homomorphism $z : G_K \to X$ is therefore $z(g) = w(g)z(F)$.

In the sum of the characters μ and 1, choose eigenvectors e_μ, e_1, and form the stable lattice with basis $v = qe_\mu$, $u = e_\mu + e_1$. Its action is

$$gv = \mu(g)v, \qquad gu = u + w(g)v.$$

Modulo q, this is the extension with cocycle $w \bmod q$. Twist by χ_1. The two rational characters are $\chi_1\mu$ and χ_1. Each is a summand of a full representation: replace the χ_1-summand in $V \oplus \bigoplus_c L'(\chi_c)$ by $L'(\chi_1\mu)$ for the first, and retain the original sum for the second. The jumps agree, and $\mu \equiv 1 \pmod{\varpi}$ preserves the residual characteristic polynomials. Extend the displayed lattice by stable lattices in the other summands. Its quotient supported on these two characters is admissible. Push out its lower copy of O/qO by $1 \mapsto z(F) \in X$. Closure under finite sums and quotients makes the resulting extension admissible. As qO is a power of ϖO, it is an extension at an exponent annihilating X. This proves the inclusion.

For the converse, write an admissible class as a continuous homomorphism $f : G_K \to X$, since the twist of χ_1 by χ_1^{-1} is trivial. Subtract the unramified homomorphism $g \mapsto \nu(g)f(F)$, and call the result h. Its class remains admissible and $h(F) = 0$. Suppose $h \neq 0$. Let $Y \subset X$ be the O-submodule spanned by its values. Nakayama's lemma gives a nonzero O-linear map $\lambda : Y \to k$. Since the values of h span Y, the homomorphism λh is nonzero. Pullback along $Y \hookrightarrow X$, followed by functoriality under λ, makes its class admissible over k. The two admissible homomorphisms $\bar{\nu}$ and λh are linearly independent: evaluation at F kills the first coefficient of a relation, and evaluation at an element where $\lambda h \neq 0$ kills the second. This contradicts the residue dimension bound. Hence $h = 0$, so f is unramified. Finally, evaluation at F identifies $\mathrm{Hom}_{\mathrm{cont}}(G_K/I_K, k)$ with k, proving the claimed dimension. □

Proof of 36.46

Proof. Write $S_i = \sum_{j \geq 1} h_{i+j}\ell^{-j}$. Periodicity bounds the summands by a convergent geometric series, and $\ell S_{i-1} = h_i + S_i$. Thus the integer $h_i + \sigma_i$ is strictly less than $\ell(\sigma_{i-1} + 1)$, which gives the carry inequality.

Let y lie in the displayed band. If $\varsigma_i = 1$, then $-\sigma_i - 1 \leq y \leq \eta_i' + \sigma_i + 2$. If $\varsigma_i = -1$, the same inequalities hold for $h_i - y$. For a jump outside the low window, (P6) therefore excludes $u \equiv y \pmod{\ell}$. For a jump in the low window, the bounds just obtained give $|u - y| \leq \eta_i' + \sigma_i + 1 < \ell$, by (P4). The congruence would force $u = y$. There is an integer $z \in [w_i, w_i + t - 1]$ with $|y - z| \leq \sigma_i + 2$, contradicting (P5). Taking $y = w_i + a$ proves the vanishing of $\rho_i(\bar{a})$.

At an upward index, $w_i \leq \eta_i' < \ell$. In the first block sum, a jump contributes precisely when $u \equiv j \pmod{\ell}$ for some $0 \leq j < w_i$. Condition (P6) excludes every cloud jump. For the remaining jumps, $1 \leq u \leq \eta_i' < \ell$, so the congruence means $u = j$. Each jump below w_i is counted once, with its multiplicity.

At a downward index, $h_i - w_i \leq \eta_i' < \ell$. In the second block sum a jump contributes when $u \equiv w_i + a \pmod{\ell}$, with $0 \leq a \leq h_i - w_i$. For a cloud jump this gives $h_i - u \equiv h_i - w_i - a \in [0, \eta_i']$, contrary to (P6). For a low-window jump, both u and $w_i + a$ lie in $[h_i - \eta_i', h_i]$, so the congruence is equality. The sum therefore counts exactly the jumps at least w_i; adding d_i gives $|U_i|$. □

Proof of 36.47

Proof. Lemma 36.46 gives $\sigma_i + 1 \leq \ell(\sigma_{i-1} + 1) - h_i \leq \ell(\sigma_{i-1} + 1) - w_i$. Conditions (P2)–(P4) give $w_i \geq 1$, $h_i - w_i \geq t \geq 1$, and, at a downward index, $h_i - w_i \leq \eta_i'$, hence $h_i - w_i + 7 \leq \ell$.

Claim 1. Let (b, g) satisfy $(\gamma_i - 1)b_i = \varphi_i(g_{i-1}) - g_i$ and $g_i \in XN_i$ for every i. Then there is $m \in \prod_i N_i$ with $g_i - (\gamma_i - 1)m_i \in X^{\sigma_i + 3}N_i$ for every i. If moreover $g_i \in X^{\sigma_i+3}N_i$ and $b_i \in N_i$ for every i, then (b, g) is the

Herr boundary of some $m \in \prod_i N_i$. The inputs are the two height inclusions, the condition $(\gamma_i-1)N_i \subset XN_i$, (S1) and the inequality $\sigma_i+1 \le \ell(\sigma_{i-1}+1)-w_i$. To see this, on $X^aN_i/X^{a+1}N_i$ the operator ∇_i is conjugate to the operator on N_i/XN_i minus a. Condition (S1) therefore makes ∇_i invertible on $N_i/X^{\sigma_i+2}N_i$. Solve $\nabla_i m_i \equiv X^{-1}g_i \pmod{X^{\sigma_i+2}N_i}$ to obtain the first assertion. For the second assertion, the Herr relation and the carry inequality give $\nabla_i b_i \in X^{\sigma_i+2}N_i$, hence $b_i \in X^{\sigma_i+2}N_i$. Put $L = \prod_i X^{\sigma_i+2}N_i$ and $(\Phi x)_i = \varphi_i(x_{i-1})$. The height inclusion and the same inequality give $\Phi(X^aL) \subset X^{a+1}L$ for every $a \ge 0$. Thus $m = -\sum_{q\ge 0}\Phi^q b$ converges in L and satisfies $(\Phi-1)m = b$. The Herr relation makes $(\gamma_i-1)m_i - g_i$ fixed by Φ; it lies in XL, where Φ is strictly contracting, so it is zero.

Lean: `GlobalTheory.PhiGam.Gauge.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CoeSmulMem`, lines 57–149)

Claim 2. Every N-bounded Herr pair (b,g) differs by a Herr boundary from a pair (b',g') that satisfies the Herr relation and $g'_i \in X^{\sigma_i+3}N_i$ for every i, and there is $c \in \prod_i D_i$ with $c_i - b'_i \in N_i$ for every i and

$$\begin{aligned} c_i &\in X^{-w_i}N_i, \quad (\gamma_i-1)c_i \in XN_i \qquad (\varsigma_i = 1),\\ c_i &\in X^{-(h_i-w_i)}A\varphi_i(N_{i-1}), \quad (\gamma_i-1)c_i \in X^{h_i-w_i+2}N_i \qquad (\varsigma_i = -1). \end{aligned}$$

Indeed, take m from Claim 1 and put $b'_i = b_i - \varphi_i(m_{i-1}) + m_i$, $g'_i = g_i - (\gamma_i-1)m_i$ and $c_i = b_i - \varphi_i(m_{i-1}) + x_i$, where $x_i = m_i$ at an upward index and x_i is the element of the bounded-pair condition at a downward index. Then $(\gamma_i-1)c_i = \varphi_i(g'_{i-1}) - (g_i - (\gamma_i-1)x_i)$, and $\varphi_i(X^{\sigma_{i-1}+3}N_{i-1}) \subset X^{\ell(\sigma_{i-1}+3)-w_i}N_i \subset X^{h_i-w_i+2}N_i$ by the carry inequality.

Lean: `GlobalTheory.Models.PolarBound.gauge_one` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CardCocyclesIndependentModCoboundariesLe`, lines 40–103)

Claim 3. Let $N_i \subset D_i$ be an A-lattice with $(\gamma_i-1)N_i \subset XN_i$, let $v \ge 0$ be an integer, and let $Z \subset D_i$ be a k-subspace with $Z \subset X^{-v}N_i$ and $\nabla_i(Z) \subset N_i$. Then

$$\dim_k Z/(Z \cap N_i) \le \sum_{j=0}^{v-1} \operatorname{sen}_i(N; \overline{j-v}).$$

Indeed, $Z/(Z\cap N_i)$ injects into the kernel of ∇_i on $X^{-v}N_i/N_i$. Filter this quotient by $X^{j-v}N_i/N_i$, $0 \le j \le v$. On its j-th successive quotient, the multiplicity of the eigenvalue zero is $\operatorname{sen}_i(N;\overline{j-v})$. Additivity of generalized eigenspace dimensions along this filtration bounds the kernel by the displayed sum.

Lean: `GlobalTheory.T.gammaMinusOneDivVariable_polar_kernel_le_senWeights` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.GammaMinusOneDivVariablePolarKernelLeSenWeightsStatement`, lines 7–8)

Claim 4. Let $\varsigma_i = -1$, put $v = h_i - w_i$ and $P_i = A\varphi_i(N_{i-1})$. Choose B-linear coordinate isomorphisms $D_j \simeq B^r$ and let $\Psi_i : D_i \to D_{i-1}$ be the map given by Lemma 36.38, so that $\Psi_i(b\varphi_i(x)) = \psi(b)x$, $\Psi_i\gamma_i = \gamma_{i-1}\Psi_i$, $\Psi_i\varphi_i = 1$, and $\Psi_i(X^aP_i) = N_{i-1}$ for $2-\ell \le a \le 1$. Equation (36.30) at $a = -v$ gives $(\gamma_i-1)(X^{-v}P_i \cap \ker\Psi_i) = X^{1-v}P_i \cap \ker\Psi_i$. Let $Z \subset D_i$ be a k-subspace with $Z \subset X^{-v}P_i$ and $(\gamma_i-1)Z \subset X^{v+2}N_i$. Then

$$\dim_k Z/(Z\cap N_i) + \sum_{a=0}^{v} \operatorname{sen}_i(N;\bar a) \le r.$$

To prove this, first observe from (S3) that if $x \in N_i$ and $\nabla_i^j x \in X^{v+1}N_i$ for some $j \ge 0$, then $\nabla_i x \in X^{v+1}N_i$: letting T be the endomorphism induced by ∇_i on $N_i/X^{v+1}N_i$, we have $Tf_i(T) = 0$ with $f_i(0) \ne 0$, so coprimality of $f_i(Y)$ and Y^j gives polynomials a,b with $af_i + bY^j = 1$, and multiplying by T shows that $T^j\bar x = 0$ forces $T\bar x = 0$. Now set $F = X^{-v}P_i$ and $L = X^{v+1}N_i$. The height inclusions $X^vN_i \subset P_i \subset X^{-w_i}N_i$ and $v \ge 1$ give $L \subset N_i \subset F \subset X^{-v-w_i}N_i$, so $\bar F = F/L$ is finite-dimensional over k, and $(\gamma_i-1)N_i \subset XN_i$ implies $\nabla_i(L) \subset L$. Choose an A-basis $e_1,\dots,e_r$ of N_{i-1} and put $S = \operatorname{span}_k\{X^{-1}\varphi_i(e_1),\dots,X^{-1}\varphi_i(e_r)\}$, so $\dim_k S \le r$. We claim that $F \subset S + \nabla_i(F) + L$. Given $x \in F$, we have $Xx \in X^{1-v}P_i$; since $v+7 \le \ell$ gives $2-\ell \le 1-v \le 1$, the element $y = \Psi_i(Xx)$ lies in N_{i-1}. Because $\Psi_i(\varphi_i(y)) = y$ and $\varphi_i(y) \in P_i \subset X^{1-v}P_i$, the difference $Xx - \varphi_i(y)$ lies in $X^{1-v}P_i \cap \ker\Psi_i$, so (36.30) produces $x' \in F \cap \ker\Psi_i$ with $(\gamma_i-1)x' = Xx - \varphi_i(y)$. Write $y = y_1 + Xy_3$ with $y_1 \in \operatorname{span}_k(e_1,\dots,e_r)$ and $y_3 \in N_{i-1}$. By (S1) at $i-1$, ∇_{i-1} is invertible on $N_{i-1}/X^{\sigma_{i-1}+2}N_{i-1}$, so some $z \in N_{i-1}$ satisfies $y_3 - \nabla_{i-1}z \in X^{\sigma_{i-1}+2}N_{i-1}$, hence $X(y_3-\nabla_{i-1}z) \in X^{\sigma_{i-1}+3}N_{i-1}$. Since $v+2 \le \ell(\sigma_{i-1}+3) - w_i$, applying φ_i and dividing by X gives $X^{-1}\varphi_i(X(y_3-\nabla_{i-1}z)) \in L$. Using $\varphi_i(X\nabla_{i-1}z) = \varphi_i((\gamma_{i-1}-1)z) = X\nabla_i\varphi_i(z)$, we obtain

$$x = X^{-1}\varphi_i(y_1) + \nabla_i\big(x' + \varphi_i(z)\big) + X^{-1}\varphi_i\big(X(y_3-\nabla_{i-1}z)\big) \in S + \nabla_i(F) + L,$$

as $x' + \varphi_i(z) \in F$. Let $\bar{\nabla}_i : D_i/L \to D_i/L$ be the k-linear map induced by ∇_i, and put $\bar{K} = \bar{F} \cap \ker \bar{\nabla}_i$. The decomposition yields $\bar{F} \subset (S+L)/L + \bar{\nabla}_i(\bar{F})$, so rank-nullity on $\bar{\nabla}_i|_{\bar{F}}$ gives $\dim_k \bar{K} = \dim_k \bar{F} - \dim_k \bar{\nabla}_i(\bar{F}) \le \dim_k (S+L)/L \le \dim_k S \le r$. Consider the natural projection $h : D_i/L \to D_i/N_i$ with kernel N_i/L. Every $x \in Z$ satisfies $x \in F$ and $\nabla_i x \in L$, so $Z/(Z \cap N_i) \simeq (Z + N_i)/N_i \subset h(\bar{K})$. Likewise, every $x \in N_i$ whose image in N_i/L lies in the generalized 0-eigenspace of $\bar{\nabla}_i$ satisfies $\nabla_i x \in L$ by the polynomial property proved above, so that generalized 0-eigenspace embeds into $\bar{K} \cap \ker h$. By filtration of $N_i/X^{v+1}N_i$ its dimension equals $\sum_{a=0}^{v} \operatorname{sen}_i(N; \bar{a})$. Rank-nullity on $h|_{\bar{K}}$ therefore yields

$$\dim_k Z/(Z \cap N_i) + \sum_{a=0}^{v} \operatorname{sen}_i(N; \bar{a}) \le \dim_k h(\bar{K}) + \dim_k (\bar{K} \cap \ker h) = \dim_k \bar{K} \le r.$$

Lean: `GlobalTheory.PhiGam.PolarCount.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.PolarPartFinrankAddSenWeightsLeRank`, lines 200–287)
Lean: `GlobalTheory.T.polarPart_finrank_add_senWeights_le_rank` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.PolarPartFinrankAddSenWeightsLeRankStatement`, lines 7–8)
Lean: `GlobalTheory.T.gammaNabla_mem_of_killing_polynomial` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.GammaNablaMemOfKillingPolynomialStatement`, lines 8–9)
Lean: `GlobalTheory.PsiShiftProof.lemmaPsi` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsPullBasis`, lines 239–294)

Claim 5. Let $c^{(1)}, \dots, c^{(s)} \in \prod_i D_i$, let Z_i be the k-span of $c_i^{(1)}, \dots, c_i^{(s)}$, and suppose that $t = 0$ is the only $t \in k^s$ with $\sum_a t_a c_i^{(a)} \in N_i$ for every i. Then $s \le \sum_i \dim_k Z_i/(Z_i \cap N_i)$, because $t \mapsto (\sum_a t_a c_i^{(a)} \bmod N_i)_i$ is an injective linear map $k^s \to \prod_i Z_i/(Z_i \cap N_i)$.
Lean: `GlobalTheory.Models.PolarBound.card_le_sum` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CardCocyclesIndependentModCoboundariesLe`, lines 14–30)

Now let $(b^{(a)}, g^{(a)})$, $1 \le a \le s$, be N-bounded Herr pairs with independent classes, and take $(b'^{(a)}, g'^{(a)})$ and $c^{(a)}$ from Claim 2. The conditions on c_i in Claim 2 are k-linear, so they hold on Z_i. At an upward index, Claim 3 with $v = w_i$ and (S2) give $\dim_k Z_i/(Z_i \cap N_i) \le d_i$; at a downward index, Claim 4 and (S3) give the same bound. If $\sum_a t_a c_i^{(a)} \in N_i$ for every i, then $\sum_a t_a b_i'^{(a)} \in N_i$ and $\sum_a t_a g_i'^{(a)} \in X^{\sigma_i+3} N_i$, so Claim 1 makes $\sum_a t_a (b'^{(a)}, g'^{(a)})$, and hence $\sum_a t_a (b^{(a)}, g^{(a)})$, a Herr boundary; independence gives $t = 0$. Claim 5 gives $s \le \sum_i d_i$.
Lean: `GlobalTheory.Models.PolarBound.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CardCocyclesIndependentModCoboundariesLe`, lines 117–194)

□

Proof of 36.48

Proof (outline). Put $W_0 = \chi_1^{-1} T_0$. *Claim 1.* The level-one integral comparison has the following properties. Put $\mathcal{A} = \prod_i O[[Z]]$ and $Q = \Phi_\ell(1+Z)$, and let Δ act through its Teichmüller lifts. For a crystalline lattice T, write $N(T)$ for its level-one Wach lattice and $\varphi^* N(T)_i$ for the span of Frobenius from component $i-1$. Write f_* for the induced map on integral étale modules, and use the chosen γ with $\chi_\ell(\gamma) = 1 + \ell$. Write $\pi_0 = -\prod_{\delta \in \Delta} \delta(Z)$; its reduction is $\iota(X)$. At cyclotomic level one the Wach system has the following properties. Every crystalline lattice has a Δ-invariant $\mathcal{A}$-basis of its Wach lattice. If its jumps at component i lie in $[-M_i, H_i]$, with $M_i, H_i \in \mathbf{N}$, then

$$Q^{M_i} \varphi^* N(T)_i \subset N(T)_i, \qquad Q^{H_i} N(T)_i \subset \varphi^* N(T)_i.$$

For a rank-one lattice with jump a_i, one has

$$Q^{\max(-a_i, 0)} \varphi^* N(T)_i = Q^{\max(a_i, 0)} N(T)_i.$$

If an equivariant lattice map $f : T' \to T$ is an isomorphism after inverting ℓ, with T crystalline, then $f_*^{-1}(N(T)) = N(T')$. If $f : T \to S$ and $g : T' \to S$ induce an isomorphism $T[1/\ell] \oplus T'[1/\ell] \to S[1/\ell]$ and $f(T) + g(T') = S$, with T, T' crystalline, then $N(S) = f_* N(T) + g_* N(T')$. Finally there is an integral endomorphism ν of $N(T)^\Delta$ with $(\gamma - 1)y = \pi_0 \nu(y)$; its reductions give the Sen and polynomial assertions below. These level-one Wach properties follow from the inputs in $\mathscr{H}_2$ via Lemmas 36.17 and 36.18 together with prismatic-to-Wach descent.
Lean: `GlobalTheory.WachUse.QN` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.Saturated`, lines 58–58)

Lean: `GlobalTheory.WachUse.InvBasis` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.Saturated`, lines 18–20)
Lean: `GlobalTheory.WachUse.Heights` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.Saturated`, lines 22–25)
Lean: `GlobalTheory.WachUse.RankOne` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.Saturated`, lines 27–32)
Lean: `GlobalTheory.WachUse.Saturated` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.Saturated`, lines 34–35)
Lean: `GlobalTheory.WachUse.Additive` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.Saturated`, lines 37–40)
Lean: `GlobalTheory.WachUse.SenOp` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.Saturated`, lines 45–56)
Lean: `GlobalTheory.WachQN.qN_of_admitted` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachLatticeUsedPropertiesOfAdmitted`, lines 83–92)

The canonical residue functor and the compatible reduction maps exist for these same data. Taking the reductions of the twisted full lattices as base pairs gives the model system, including its standard lattice and its character lattices; it also sends every twisted admissible residue representation to a representation with a model lattice.
Lean: `GlobalTheory.RedFun.realD` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsPresTensor`, lines 556–570)
Lean: `GlobalTheory.BasePairs.exists_modTh_of_pres` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsModThOfPres`, lines 28–70)
Lean: `GlobalTheory.BasePairs.real_of_presented` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsModThOfPres`, lines 72–81)

Claim 2. The height and cyclotomic conditions in (a) hold for every base pair. Let T be a stable lattice in a sum of μ full representations already twisted by χ_1^{-1}. Its jump multiset is $\mu(J_i - w_i) \subset [-w_i, h_i - w_i]$ at τ_i. Lemmas 36.17 and 36.18 at depth one give its Wach lattice $N(T)$ over $K(\mu_\ell)$ with

$$Q^{w_i}\varphi^* N(T)_i \subset N(T)_i, \qquad Q^{h_i - w_i} N(T)_i \subset \varphi^* N(T)_i, \qquad (\gamma - 1)N(T) \subset ZN(T),$$

where $Q = \Phi_\ell(1 + Z)$ and $\varphi^* N(T)_i$ is the span of Frobenius from the preceding component. A basis of $N(T)^\Delta$ generates $N(T)$, $N(T)^\Delta \cap ZN(T) = \pi_0 N(T)^\Delta$ with π_0 reducing to X, and Q reduces to a unit times X. Hence $N^b = \overline{N}(T)$ satisfies $X^{h_i - w_i} N_i^b \subset A\varphi_i(N_{i-1}^b) \subset X^{-w_i} N_i^b$ and $(\gamma_i - 1)N_i^b \subset XN_i^b$.
Lean: `GlobalTheory.T.wachLattice_usedProperties_of_admitted` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachLatticeUsedPropertiesOfAdmittedStatement`, lines 7–8)
Lean: `GlobalTheory.WachQN.qN_of_admitted` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachLatticeUsedPropertiesOfAdmitted`, lines 83–92)
Lean: `GlobalTheory.BasePairs.base_abc` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsQuotOfAdmissible`, lines 243–260)

Claim 3. The composition-factor, Sen, and polynomial assertions in (a) hold for every base pair. For T as in Claim 2, the residual characteristic polynomials of the full representations give the composition factors of $\overline{T}$ in (a). Put $q = \pi_0 + \ell$, which reduces to X, and define $c(m) \in O$ by $\ell c(m) = 1 - (1 + \ell)^m$, so that $c(m) \equiv -m \pmod{\varpi}$. The endomorphism ν of Claim 1 also has the following property. For a Δ-invariant $\mathcal{A}$-basis $(b_j)_j$ of $N(T)$ and each i, there is a matrix Θ_i over O with characteristic polynomial $\prod_{u'}(Y - c(-u'))$, the product over the jump multiset $\mu(J_i - w_i)$ with multiplicity, such that $\nu(b_j) - \sum_{j'}(\Theta_i)_{j'j} b_{j'}$ lies in $qN(T)$ in the i-th component. The operator ∇_i on $\overline{N}(T)_i / X\overline{N}(T)_i$ is the reduction of ν, and $c(-u') \equiv u' \pmod{\varpi}$, so its characteristic polynomial is $\prod_{u'}(Y - \bar{u}')$. Hence $\mathrm{sen}_i(\overline{N}(T); \alpha)$ is the number of $u' \in \mu(J_i - w_i)$ with $\bar{u}' = \alpha$, that is, $\mu\, r_i(J_i, w_i; \alpha)$. At a downward index put $v = h_i - w_i$. Let I_i be the set of integers $a - u'$ with $0 \le a \le v$ and u' in the jump multiset $\mu(J_i - w_i)$. The i-th component of $N(T)^\Delta$ modulo q^{v+1} is a free O-module of rank $(v + 1)\,\mathrm{rank}_O\, T$, the endomorphism ν of Claim 1 induces an endomorphism of it, and after inverting ℓ this endomorphism is diagonalizable with eigenvalues $c(a - u')$, $0 \le a \le v$, $u' \in \mu(J_i - w_i)$. Since the module is torsion-free over O, the integral polynomial $\prod_{m \in I_i}(Y - c(m))$, with one factor for each element of the set I_i, annihilates it. Reduction modulo ϖ gives

$$\prod_{m \in I_i} (\nabla_i + m)\, \overline{N}(T)_i \subset X^{v+1} \overline{N}(T)_i.$$

Put $f_i(Y) = \prod_{m \in I_i \setminus \{0\}}(Y + m)$. By Lemma 36.42, $\ell \mid a - (u - w_i)$ with $0 \le a \le v$ and $u \in J_i$ forces $a = u - w_i$; hence no element of $I_i \setminus \{0\}$ is divisible by ℓ, and $f_i(0) \neq 0$. If $0 \in I_i$, the display reads

$\nabla_i f_i(\nabla_i)\overline{N}(T)_i \subset X^{v+1}\overline{N}(T)_i$. If $0 \notin I_i$, the display gives $f_i(\nabla_i)\overline{N}(T)_i \subset X^{v+1}\overline{N}(T)_i$, and the same inclusion for $\nabla_i f_i(\nabla_i)$ follows since ∇_i preserves $\overline{N}(T)_i$ and commutes with $f_i(\nabla_i)$.
Lean: `GlobalTheory.BasePairs.base_sen` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsQuotOfAdmissible`, lines 262–270)
Lean: `GlobalTheory.BasePairs.base_e` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsQuotOfAdmissible`, lines 281–293)

Claim 4. The operations on model lattices in (d) hold, and the standard lattice has the properties in (b). If $M = L_1/L_2$ is admissible and $\varpi M = 0$, then $\varpi L_1 \subset L_2$, so $\chi_1^{-1}M$ is a quotient of the reduction of the twisted lattice L_1, and (36.31) with j the identity gives a model lattice. Pulling back a lattice of the form (36.31) along an injection and pushing it forward along a surjection gives a lattice of the same form, since D is exact. The height inclusions and the γ-condition of Claim 2 pass to these lattices. The polynomial relation of Claim 3 passes as well, because the maps in (36.31) are B-linear and commute with γ_i, hence with ∇_i and $f_i(\nabla_i)$. For (b), take the direct sum of T_0 and the character lattices before twisting. The integral additivity and saturation properties identify the preimage of its reduced Wach lattice under $\overline{W} \hookrightarrow \overline{W} \oplus \bigoplus_{0\le e<t} k(\omega^{-e})$ with the reduction of the Wach lattice of W_0. The Sen calculation for the jumps $U_i - w_i$ gives $\mathrm{sen}_i(N^{\mathrm{std}};\alpha) = r_i(U_i, w_i;\alpha)$. This proves (a), (b) and (d).

Claim 5. The standard rank-one lattices in (c) exist and have the asserted Frobenius and Sen data. For $-1 \le e < t$, the reduction of the Wach lattice of the rank-one lattice of χ_ℓ^{-e}, whose single jump is e, is the lattice $C^{(e)}$ of (c). For $0 \le e < t$, the character $\chi_1^{-1}\chi_{e+1} = \chi_\ell^{-e}$ is a summand of a twisted full representation, so $C^{(e)}$ is a model lattice, and $C_i^{(0)} = A\mathbf{1}_i$. *Claim 6.* Suppose N, N' are lattice families in the same cyclotomic module, both satisfying

$$X^{h_i - w_i}N_i \subset A\varphi_i(N_{i-1}) \subset X^{-w_i}N_i, \qquad (\gamma_i - 1)N_i \subset XN_i$$

and the corresponding conditions for N'. If $\mathrm{sen}_i(N';\alpha) \neq 0$, there is an integer a with $|a| \le \sigma_i$ and $\mathrm{sen}_i(N;\alpha+\bar{a}) \neq 0$. The two height comparisons give $X^{\sigma_i}N_i \subset N_i' \subset X^{-\sigma_i}N_i$. Filtering this interval of lattices by powers of X, and using that multiplication by X^a shifts the residual operator ∇_i by $-a$, gives the stated Sen comparison.
Lean: `GlobalTheory.Models.sen_near_of_abc` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SenWeightsCloseOfHeightGammaConditions`, lines 11–23)

Claim 7. Let $V_b \in \mathcal{C}$, let N be a model lattice of $D(V_b)$, let $j : A_1 \hookrightarrow V_b$ and $p : A_1 \twoheadrightarrow A_2$ be equivariant, and put $N^q = D(p)(D(j)^{-1}(N))$. Then

$$\mathrm{sen}_i(N^q;\alpha) \le \mathrm{sen}_i(N;\alpha) \qquad (i \in I,\ \alpha \in k).$$

The intersection with the B-subspace $D(A_1)_i$ is saturated in N_i. Its kernel under $D(p)_i$ is saturated as well. The resulting short exact sequences of finite free A-modules remain exact modulo X, commute with ∇_i, and identify the relevant quotient with N_i^q/XN_i^q. Generalized-eigenspace dimensions are additive in these sequences, proving the inequality.
Lean: `GlobalTheory.T.senMultiplicity_subquotient_lattice_le` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SenMultiplicitySubquotientLatticeLeStatement`, lines 8–9)

Claim 8. For $-1 \le e < t$, an equivariant map from a subrepresentation of $\overline{W}$ to $k(\omega^{-e})$ is zero. Suppose an injective equivariant map $j : A_1 \hookrightarrow \overline{W}$ and a nonzero equivariant map $p : A_1 \to k(\omega^{-e})$ are given. The latter map is surjective, since its target has dimension one. Claims 2–4 give a quotient model lattice

$$N_i^q = D(p)_i(D(j)_i^{-1}(N_i^{\mathrm{std}})).$$

By Claim 5 the standard rank-one lattice $C^{(e)}$ has Sen multiplicity one at $\bar{e}$. It satisfies the same height bounds as N^q: indeed $-w_i \le e \le h_i - w_i$, since $w_i \ge 1$, $h_i - w_i \ge t$, and $-1 \le e < t$. Claim 6 therefore gives $|a| \le \sigma_0$ with $\mathrm{sen}_0(N^q;\bar{e}+\bar{a}) \neq 0$. Claim 7 and the standard Sen formula in Claim 4 give a jump $u \in U_0$ with

$$u \equiv w_0 + e + a \pmod{\ell}.$$

But $w_0 + e + a \in [w_0 - \sigma_0 - 1, w_0 + t - 1 + \sigma_0]$, contained in the excluded band of Lemma 36.46. This contradiction proves the claim.
Lean: `GlobalTheory.Models.ModTh.hom_chr_eq_zero` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.HomToCharacterLineEqZero`, lines 25–63)

Claim 9. For $0 \le e < t$, every model lattice of $k(\omega^{-e})$ equals $C^{(e)}$. A model lattice N of $k(\omega^{-e})$ with $0 \le e < t$ has the form $N_i = X^{\kappa_i} C_i^{(e)}$ with $|\kappa_i| \le \sigma_i$, $\kappa_{i-1} \ge 0$ at upward and $\kappa_{i-1} \le 0$ at downward indices, and Sen multiplicity 1 at $\overline{e - \kappa_i}$. To prove that these exponents vanish, take a composition series of a base representation that passes through the given line, together with the induced model lattices on its factors. The characters $\omega^{-e'}$ for $0 \le e' < t$ are distinct, since ω has order $\ell - 1$ and $t < \ell - 1$. Match its simple factors with those of μ copies of $\overline{W} \oplus \bigoplus_{0 \le e' < t} k(\omega^{-e'})$. Claim 8 distinguishes the character factors from the factors of $\overline{W}$. Every factor of the latter kind has Sen support in $\{\overline{u - w_i} : u \in U_i\}$. Indeed, compare its lattice with the lattice induced from N^{std}: the height comparison shifts a nonzero Sen weight by at most σ_i, while the base-pair Sen data put the original weight in $J_i - w_i$. The excluded band rules out the character block unless the original weight belongs to $U_i - w_i$.

Lean: `GlobalTheory.Models.ModTh.sen_model_subquot_mem_U` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SenWeightSubquotientModelMemResidues`, lines 25–47)

Index the character factors by a, write their exponents as $e_a \in \{0, \dots, t-1\}$, and write their lattice shifts as $\kappa_{a,i}$. On the integer interval $[-\sigma_i, t + \sigma_i]$, the factors from $\overline{W}$ contribute no Sen weights. Additivity of Sen multiplicities along the composition series and Claim 3 therefore identify the multisets of residues of $e_a - \kappa_{a,i}$ and e_a. Both integer multisets lie in this interval, whose length is less than ℓ, so they have the same integer sum. Hence $\sum_a \kappa_{a,i} = 0$. All the $\kappa_{a,i}$ have the same weak sign, determined by ς_{i+1}, so every one is zero. The distinguished factor gives $\kappa_i = 0$ for the original line. Thus $N = C^{(e)}$.

Lean: `GlobalTheory.T.character_model_exponents_signs_senWeight` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CharacterModelExponentsSignsSenWeightStatement`, lines 8–9)

Lean: `GlobalTheory.Models.CharModel.model_chr_eq` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TwistedPhiGammaModelCharacterEqStandard`, lines 107–246)

Claim 10. Every representation having a model lattice has a largest one. The sum $(N_i + N_i')_i$ of two model lattices of one representation is a model lattice. Two lattice families with the same height inclusions satisfy $X^{\sigma_i} N_i \subset N_i'$, so every model lattice lies in $(X^{-\sigma_i} N_{0,i})_i$ for a fixed model lattice N_0. These modules are noetherian over A, so a model lattice that is maximal at every index exists, and closure under sums makes it the largest. This proves (e).

Lean: `GlobalTheory.T.model_sup_alt` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ModelSupStatement`, lines 8–9)

Lean: `GlobalTheory.T.phiGamma_lattice_families_shift_inclusion` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.PhiGammaLatticeFamiliesShiftInclusionStatement`, lines 8–9)

Lean: `GlobalTheory.Models.exists_largest_model` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsLargestModel`, lines 46–54)

Claim 11. Every cocycle whose extension has a model lattice has a bounded Herr representative as in (f). Let L be a model lattice of E_z, with inclusion $a : A_0 \hookrightarrow E_z$ and projection $p : E_z \twoheadrightarrow k$. By Claim 4 and Claim 9 at $e = 0$, $D(p)(L) = (A\mathbf{1}_i)_i$. Choose $\epsilon_i \in L_i$ mapping to $\mathbf{1}_i$, and set $N = D(a)^{-1}(L)$. Exactness gives unique $b_i, g_i \in D(A_0)_i$ with

$$D(a)_i b_i = \varphi_i(\epsilon_{i-1}) - \epsilon_i, \qquad D(a)_i g_i = (\gamma_i - 1)\epsilon_i.$$

Commutation gives the Herr relation, and $(\gamma_i - 1)L_i \subset XL_i$ gives $g_i \in XN_i$. The upper height bound and $w_i \ge 0$ give $b_i \in X^{-w_i} N_i$ at upward indices.

At a downward index put $v = h_i - w_i$. Since $D(p)(L_{i-1}) = A\mathbf{1}_{i-1}$, every element of $A\varphi_i(L_{i-1})$ is a sum of an element of $D(a)_i(A\varphi_i(N_{i-1}))$ and a scalar multiple of $\varphi_i(\epsilon_{i-1})$. Apply this to $X^v \epsilon_i$, which belongs to that Frobenius lattice. Applying $D(p)_i$ shows that the scalar is X^v; subtraction then gives $X^v b_i \in A\varphi_i(N_{i-1})$.

Choose f_i with $f_i(0) \ne 0$ and $\nabla_i f_i(\nabla_i) L_i \subset X^{v+1} L_i$. Define x_i by

$$D(a)_i x_i = \epsilon_i - f_i(0)^{-1} f_i(\nabla_i)\epsilon_i.$$

The right side maps to zero under $D(p)_i$, since $\nabla_i \mathbf{1}_i = 0$, and belongs to L_i, since $\nabla_i L_i \subset L_i$. Thus $x_i \in N_i$. Moreover

$$D(a)_i\big(g_i - (\gamma_i - 1)x_i\big) = f_i(0)^{-1} X \nabla_i f_i(\nabla_i)\epsilon_i \in X^{v+2} L_i.$$

Taking the preimage gives the last downward condition. This proves every clause of (f).

Lean: `GlobalTheory.TopSteps.ExtPair.exists_model_repr_inZZ` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsPhiGammaModelExtensionCocyclePair`, lines 62–180)

Claim 12. Every model lattice containing N^{std} has the Sen counts in (g). Let N be as in (g). Then $N_i^{\mathrm{std}} \subset N_i \subset X^{-\sigma_i} N_i^{\mathrm{std}}$ by Claim 10. For $y \in \mathbf{Z}$, put

$$s_i(y) = \mathrm{sen}_i(N; \bar{y}), \qquad \rho_i(y) = r_i(U_i, w_i; \bar{y}),$$

and let $q_i(y)$ be the dimension of the generalized $\bar{y}$-eigenspace of ∇_i on N_i/N_i^{std}. These quotients are finite-dimensional and ∇_i-stable. Additivity along the two filtrations of N_i/XN_i^{std}, and the shift of the operator by -1 under multiplication by X, give

$$s_i(y) + q_i(y+1) = \rho_i(y) + q_i(y).$$

Furthermore, $q_i(y) \neq 0$ implies $\rho_i(y-j) \neq 0$ for some $1 \leq j \leq \sigma_i$. The support assertion proved in Claim 9, applied to the identity subquotient of $\overline{W}$, says that $s_i(y) \neq 0$ implies $u \equiv w_i + y \pmod{\ell}$ for some $u \in U_i$.
Lean: `GlobalTheory.PhiGam.Count.sen_count` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SenMultiplicityLatticeCountRelation`, lines 119–148)
Lean: `GlobalTheory.T.senMultiplicity_lattice_count_relation` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SenMultiplicityLatticeCountRelationStatement`, lines 7–8)
Lean: `GlobalTheory.T.senWeight_subquotient_model_mem_residues` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SenWeightSubquotientModelMemResiduesStatement`, lines 8–9)

The excluded band of Lemma 36.46 gives $q_i(0) = 0$ and $s_i(a) = 0$ for $0 \leq a \leq \sigma_i + 1$. At an upward index, $q_i(-w_i) \neq 0$ would give $u \equiv -j \pmod{\ell}$ for $1 \leq j \leq \sigma_i$, contrary to the low-window bounds and (P6). Thus $q_i(-w_i) = 0$, and summing the displayed identity over $-w_i \leq y < 0$ proves (S2), using the standard block count.

At a downward index put $v = h_i - w_i$ and $L_i = v + \sigma_i + 2$. The same support implications, reflected about h_i, give

$$q_i(L_i) = 0, \qquad s_i(v+1) = \cdots = s_i(v+\sigma_i+1) = 0.$$

Indeed, the first nonvanishing would force a reflected jump congruent to an integer in $[-\sigma_i - 1, -2]$, and the second would force one in $[-\sigma_i - 1, -1]$; the low-window bounds and (P6) exclude both. Summing the identity over $0 \leq y < L_i$ therefore gives

$$\sum_{a=0}^{v} s_i(a) = \sum_{a=0}^{L_i-1} \rho_i(a) = r - d_i.$$

For the last equality, low-window jumps contribute once exactly when $u \geq w_i$, and the reflected exclusions rule out cloud jumps and the additional tail beyond v. This proves (S1), (S2) and the equality in (S3). The polynomial part of (S3) is already supplied by (a).
Lean: `GlobalTheory.ModelCount.count` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TwistedPhiGammaModelSenWeightCount`, lines 85–209)

The complete proofs are contained in `GlobalTheory.BasePairs.exists_modTh_of_pres`, `GlobalTheory.Models.exists_largest_model`, `GlobalTheory.TopSteps.ExtPair.exists_model_repr_inZZ`, `GlobalTheory.Models.CharModel.model_chr_eq` and `GlobalTheory.ModelCount.count`. □

Proof of 36.49

Proof. Claim 1. An admissible cocycle with coefficients killed by ϖ has an admissible exponent-one extension. Choose $a \geq 1$ in the eventual range of Definition 36.34, and a presentation $\mathcal{E}_{z,a} = L_1/L_2$. Because ϖ kills the coefficient module,

$$\mathcal{E}_{z,a}/\varpi\mathcal{E}_{z,a} \simeq \mathcal{E}_{z,1}.$$

The quotient is presented by $L_1/(L_2 + \varpi L_1)$. The denominator is a stable lattice, since it lies between L_2 and L_1. Thus it is admissible. After twisting by χ_1^{-1}, Lemma 36.48(d) supplies a model lattice of E_z.
Lean: `GlobalTheory.AdmClosure.admissible_of_surjective` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleOfSurjective`, lines 131–144)

Claim 2. Every admissible residue class has a Herr representative bounded by the same largest model lattice $N^{\max}$. Lemma 36.48(b),(e) gives the model lattice N^{std} and a largest model lattice $N^{\max}$ containing it. Claim 1 and Lemma 36.48(f) give a model lattice N of $\overline{W}$ and an N-bounded representative (b, g). Maximality gives

$$N_i \subset N_i^{\max}, \qquad A\varphi_i(N_{i-1}) \subset A\varphi_i(N_{i-1}^{\max}).$$

These inclusions preserve the two bounds on b_i and the bound on g_i. At a downward index the same element $x_i \in N_i$ belongs to $N_i^{\max}$, and its error in $X^{h_i - w_i + 2}N_i$ belongs to $X^{h_i - w_i + 2}N_i^{\max}$. The Herr relation is unchanged. This verifies every condition of an $N^{\max}$-bounded pair. The height, cyclotomic and polynomial conditions for $N^{\max}$ are Lemma 36.48(a).

Lean: GlobalTheory.Models.AdmBound.inZZ_mono (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleCocyclesCardLeSumDims, lines 17–24)

Claim 3. Independent admissible residue classes give independent Herr classes. For a finite linear relation between the chosen Herr classes, the same linear combination of their representatives represents the corresponding linear combination of cocycles, because the realization is k-linear. If this pair is a Herr boundary, subtracting that boundary from the chosen extension lifts makes them fixed by both Frobenius and γ. The fixed-vector property of Definition 36.39, as used in Lemma 36.43, supplies an equivariant section of the extension. Its composite with the quotient map is the identity, since it sends the distinguished unit to itself. Thus the combined cocycle is a coboundary. Independence of the original cohomology classes forces every coefficient of the relation to vanish.

Lean: GlobalTheory.Models.AdmBound.indep_pairs (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleCocyclesCardLeSumDims, lines 39–58)

Lean: GlobalTheory.TopSteps.mem_cob_of_repr_mem_herrB (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.MemCoboundariesOfHerrPairCoboundary, lines 9–103)

Claims 2 and 3 give all assertions. □

Proof of 36.50

Proof (outline). Suppose $p \neq 0$. Its one-dimensional target makes p surjective. By Lemma 36.48(b) and (d),

$$N_i^q = D(p)_i\big(D(j)_i^{-1}(N_i^{\mathrm{std}})\big) \subset D(k(\omega^{-e}))_i$$

is a model lattice of $k(\omega^{-e})$. By Lemma 36.48(a) and (c), both N^q and $C^{(e)}$ satisfy $X^{h_i-w_i}N_i \subset A\varphi_i(N_{i-1}) \subset X^{-w_i}N_i$ and $(\gamma_i-1)N_i \subset XN_i$.

Claim 1. Let N, N' be lattices in one cyclotomic module over B_I, both satisfying these two conditions for every i. If $\mathrm{sen}_i(N';\alpha) \neq 0$, then $\mathrm{sen}_i(N;\alpha+\bar{j}) \neq 0$ for some integer j with $|j| \leq \sigma_i$. This is Claim 6 in the proof of Lemma 36.48: the height conditions give $X^{\sigma_i}N_i \subset N_i' \subset X^{-\sigma_i}N_i$, and multiplication by X^a shifts ∇_i by $-a$.

Lean: GlobalTheory.T.senWeights_close_of_height_gamma_conditions (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SenWeightsCloseOfHeightGammaConditionsStatement, lines 8–9)

Claim 2. Let $B \in \mathcal{C}$ with a lattice N of $D(B)$ satisfying the two conditions, and let $j : A_1 \hookrightarrow B$ be injective and $p : A_1 \twoheadrightarrow A$ surjective equivariant maps. Then $\mathrm{sen}_i(D(p)(D(j)^{-1}(N));\alpha) \leq \mathrm{sen}_i(N;\alpha)$ for all i and $\alpha \in k$. The proof is that of Claim 7 in the proof of Lemma 36.48, which uses only that the intersection with $D(A_1)_i$ and the kernel of $D(p)_i$ are saturated in N_i. It is applied below to $B = \overline{W}$ and the model lattice $N = N^{\mathrm{std}}$ of Lemma 36.48(b), which satisfies the two conditions by Lemma 36.48(a).

Lean: GlobalTheory.T.senMultiplicity_subquotient_lattice_le (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SenMultiplicitySubquotientLatticeLeStatement, lines 8–9)

By Lemma 36.48(c), $\mathrm{sen}_0(C^{(e)};\bar{e}) = 1$. Claim 1 with $N = N^q$, $N' = C^{(e)}$ and $i = 0$ gives an integer a with $|a| \leq \sigma_0$ and $\mathrm{sen}_0(N^q;\bar{e}+\bar{a}) \neq 0$. Claim 2 gives $\mathrm{sen}_0(N^{\mathrm{std}};\bar{e}+\bar{a}) \neq 0$, and Lemma 36.48(b) gives $r_0(U_0, w_0;\bar{e}+\bar{a}) > 0$. By the definition of this multiplicity there is $u \in U_0$ with $u \equiv w_0+e+a \pmod{\ell}$. Since $-1 \leq e \leq t-1$ and $|a| \leq \sigma_0$, the integer w_0+e+a lies in $[w_0-\sigma_0-1, w_0+t-1+\sigma_0]$, inside the excluded band of Lemma 36.46. This contradiction proves $p = 0$. The complete proof is contained in `GlobalTheory.Models.ModTh.hom_chr_eq_zero`. □

Proof of 36.51

Proof. Put $\omega = \bar{\chi}_\ell$. *Claim 1.* Admissible residue classes for each character form a k-subspace, and every such cocycle has an admissible exponent-one extension. Submodules, quotients and finite direct sums preserve admissibility directly from the lattice presentation: take the preimage of a submodule in the numerator lattice, enlarge the denominator to take a quotient, and take direct sums of the two lattices. These remain full stable lattices, since the relevant finite modules are killed by a power of ϖ. The coefficient character and the quotient character, with any finite coefficient module, are admissible: they are subquotients of the stable lattice in the full representation $V \oplus \bigoplus_{c=1}^{t} L'(\chi_c)$. For two admissible cocycles, choose a common sufficiently large exponent a. The fiber product of their extensions over $O(\chi_1)/\varpi^a$ is a submodule of their direct sum. Pushing its coefficient module out by addition gives the extension of their sum. Scalar pushout gives scalar multiples, and changing a cocycle by a coboundary gives an isomorphic extension. Thus the eventual

admissibility condition of Definition 36.34 defines a k-subspace of cohomology. Finally, if $\mathcal{E}_{z,a} = L_1/L_2$ is admissible and ϖ kills its coefficient module, then

$$\mathcal{E}_{z,1} \simeq \mathcal{E}_{z,a}/\varpi\mathcal{E}_{z,a} \simeq L_1/(L_2 + \varpi L_1).$$

This proves the exponent-one assertion without any converse change-of-exponent assertion.
Lean: `GlobalTheory.AdmClosure.admissible_closed` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleOfSurjective`, lines 176–179)

Claim 2. An admissible cocycle with coefficient line $k(\omega^{-e})$, where $1 \le e < t$, is a coboundary. Take such a cocycle z. Claim 1 gives an admissible exponent-one extension. So Lemma 36.48(d) gives a model lattice of E_z. Lemma 36.48(f) gives a model lattice N of $k(\omega^{-e})$ and a Herr representative (b, g) of z which is an N-bounded Herr pair. By Lemma 36.48(c), $N = C^{(e)}$. Put

$$J_i = \begin{cases} w_i, & \varsigma_i = 1, \\ h_i - w_i - e, & \varsigma_i = -1. \end{cases}$$

These are nonnegative integers by (P2). The bounded-pair conditions give

$$g_i \in XC_i^{(e)}, \qquad b_i \in X^{-J_i}C_i^{(e)}, \qquad (\gamma_i - 1)b_i = \varphi_i(g_{i-1}) - g_i;$$

at a downward index use $A\varphi_i(C_{i-1}^{(e)}) = X^e C_i^{(e)}$. For every $0 \le j \le J_i$, (P3) and (P4) give

$$0 < e + j \le \eta_i' < \ell.$$

Choose an A-basis vector ϵ_i of $C_i^{(e)}$, and write $\varphi_i(\epsilon_{i-1}) = X^e a_i \epsilon_i$, $\gamma_i(\epsilon_i) = c_i\epsilon_i$, with $a_i \in A^\times$ and $c_i \in 1 + XA$. Commutation gives

$$\varphi(c_{i-1})a_i X^e = \gamma(a_i)\gamma(X)^e c_i.$$

The scalar identities $\varphi(X) = uX^\ell$ and $\gamma(X) = X(1 - \kappa X)$, $\kappa(0) = 1$, of Definition 36.36 give $\varphi(c_{i-1}) \equiv 1$, $\gamma(a_i) \equiv a_i$ and $\gamma(X)^e \equiv X^e(1 - eX)$ modulo X^2, X^2 and X^{e+2}, respectively, hence $c_i \equiv 1 + eX \pmod{X^2 A}$. If the lowest term of b_i were $dX^{-j}\epsilon_i$, with $1 \le j \le J_i$ and $d \ne 0$, the lowest term of $(\gamma_i - 1)b_i$ would be $d(e+j)X^{1-j}\epsilon_i$. Its coefficient is nonzero by the preceding inequality. This contradicts $\varphi_i(g_{i-1}) - g_i \in XC_i^{(e)}$. Hence $b_i \in C_i^{(e)}$.

On $C = \prod_i C_i^{(e)}$, put $(\Phi x)_i = \varphi_i(x_{i-1})$ and $(\Gamma x)_i = \gamma_i(x_i)$. Frobenius sends $X^j C$ into $X^{\ell j + e}C$ for $j \ge 0$. Consequently

$$m = -\sum_{n \ge 0} \Phi^n b \in C$$

converges and satisfies $(\Phi - 1)m = b$. The element $r = (\Gamma - 1)m - g$ lies in XC. The Herr relation gives $(\Phi - 1)r = 0$, and strict Frobenius contraction on XC gives $r = 0$. Thus (b, g) is a Herr boundary, and z is a coboundary by the fixed-vector argument of Lemma 36.43.
Lean: `GlobalTheory.PhiGam.cocycle_split_of_contracting` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CocycleSplitOfContracting`, lines 144–184)
Lean: `GlobalTheory.T.mem_coboundaries_of_herrPair_coboundary` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.MemCoboundariesOfHerrPairCoboundaryStatement`, lines 8–9)

Claim 3. For every cocycle $z : G_K \to k$ whose extension has a model lattice, there are a Herr representative (b, g) of z, elements $m_i \in D(k)_i$ and scalars $c_i \in k$ with

$$b_i = \varphi_i(m_{i-1}) - m_i + c_i \mathbf{1}_i, \qquad g_i = \gamma_i(m_i) - m_i.$$

By Lemma 36.48(f) and (c) at $e = 0$, choose a pair bounded for $(A\mathbf{1}_i)_i$. Put $J_i = w_i$ upward and $J_i = h_i - w_i$ downward, so $0 \le J_i \le \eta_i' < \ell$. Initially $b_i \in X^{-J_i}A\mathbf{1}_i$ and $g_i \in XA\mathbf{1}_i$. If b_i has a pole of order $1 \le j \le J_i$, the leading coefficient of $(\gamma_i - 1)b_i$ is multiplied by j and occurs in degree $1 - j$. It is nonzero, whereas $\varphi_i(g_{i-1}) - g_i \in XA\mathbf{1}_i$. This contradiction gives $b_i \in A\mathbf{1}_i$.
Lean: `GlobalTheory.TopSteps.TrivLine.mem_of_inZZ` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TrivialLineAdmissibleCocyclesNotIndependent`, lines 45–101)

Write $b_i = \beta_i \mathbf{1}_i$, $g_i = \delta_i \mathbf{1}_i$, with $\beta_i, \delta_i \in A$ and $\delta_i(0) = 0$. Set $c_i = \beta_i(0)$ and $\beta_i^+ = \beta_i - c_i$. On $\prod_i XA$, define $(\Phi_0 a)_i = \varphi(a_{i-1})$; it strictly raises the X-adic order. The convergent series $a = -\sum_{q \ge 0} \Phi_0^q \beta^+$ satisfies

$(\Phi_0 - 1)a = \beta^+$. The scalar Herr relation implies that $((\gamma - 1)a_i - \delta_i)_i$ is fixed by Φ_0. It lies in $\prod_i XA$, so it is zero. Taking $m_i = a_i \mathbf{1}_i$ proves the claim.
Lean: GlobalTheory.TopSteps.TrivLine.exists_const_pair (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TrivialLineAdmissibleCocyclesNotIndependent, lines 103–139)
Lean: GlobalTheory.PhiGam.TrivCocycle.main (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.FrobSubOneBij, lines 63–96)

Thus the class has a representative $(c_i \mathbf{1}_i, 0)_i$. If $\sum_i c_i = 0$, choose scalars d_i with $d_{i-1} - d_i = c_i$. Explicitly, for representatives $0, \dots, d' - 1$, take $d_0 = 0$ and $d_j = -\sum_{a=1}^{j} c_a$. The equation at $i = 0$ follows from the zero sum. Since both operators fix the distinguished units, this constant pair is the Herr boundary of $(d_i \mathbf{1}_i)_i$.

Given two admissible classes, choose such constant representatives and write their sums as s_0, s_1. If $s_0 = 0$, the first class is zero. Otherwise s_0 times the second class minus s_1 times the first has a zero-sum constant representative and is zero. Hence the residue dimension is at most one. *Claim 4.* There is a nonzero admissible unramified class with trivial residue coefficients. Let $\nu : G_K \to \mathbf{Z}_\ell$ be the unramified additive character taking an arithmetic Frobenius to 1, and let $z = \nu \bmod \varpi$. Fix any $a \geq 1$, put $q = \ell^a$, and define the continuous unramified character

$$\mu(g) = (1 + q)^{\nu(g)}, \qquad w(g) = \frac{\mu(g) - 1}{q} \in O.$$

The binomial expansion gives $\mu \equiv 1 \pmod{\varpi}$ and $w \equiv z \pmod{\varpi}$. In $L'(\chi_1\mu) \oplus L'(\chi_1)$, with character basis e_μ, e_1, put

$$v = qe_\mu, \qquad u = e_\mu + e_1, \qquad L_1 = Ov + Ou, \qquad L_2 = O\varpi v + O\varpi^a u.$$

Both lattices are stable, because

$$gv = \chi_1(g)\mu(g)v, \qquad gu = \chi_1(g)\big(u + w(g)v\big),$$

and $a \geq 1$. The quotient L_1/L_2, with first coordinate generated by v and second by u, is precisely $\mathcal{E}_{z,a}$. The representations

$$V \oplus L'(\chi_1\mu) \oplus \bigoplus_{c=2}^{t} L'(\chi_c), \qquad V \oplus \bigoplus_{c=1}^{t} L'(\chi_c)$$

are full: an unramified twist preserves crystallinity and jumps, and $\mu \equiv 1 \pmod{\varpi}$ preserves the required residual characteristic polynomials. Their direct sum contains the two-dimensional representation used above as a direct summand. Add the same stable lattice in the complementary summand to L_1 and L_2. This presents $\mathcal{E}_{z,a}$ by full lattices in a sum of full representations. The construction works for every a, so z is admissible by definition. It is nonzero in cohomology: a coboundary with trivial coefficients is zero, whereas z takes Frobenius to 1.
Lean: GlobalTheory.UnramAdm.exists_mu (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleQuotientOfFullRepresentation, lines 406–465)
Lean: GlobalTheory.UnramAdm.unr_subset (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleQuotientOfFullRepresentation, lines 480–527)
Lean: GlobalTheory.UnramAdm.unramified_admissible (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleQuotientOfFullRepresentation, lines 537–544)

Claims 1–4 give both assertions of the lemma. □

Proof of 36.52

Proof (outline). Part (a). Claim 1. The stated closure, coefficient, exponent and functoriality assertions hold. Since V is crystalline with jump multiset U_i at τ_i, we have $|U_i| = r$ for every $i \in \mathbf{Z}/d'\mathbf{Z}$.
Lean: GlobalTheory.T.card_hodgeTate_jumps_crystalline_eq_dim (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CardHodgeTateJumpsCrystallineEqDimStatement, lines 21–28)
Lean: GlobalTheory.T.exists_numerical_galoisData_localSituation (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsNumericalGaloisDataLocalSituationStatement, lines 23–35)

For an injective equivariant map $f : N \hookrightarrow N'$ into an admissible module presented by $p : L_1 \twoheadrightarrow N'$ with $\ker p = L_2$, the preimage $L_0 = \{x \in L_1 : p(x) \in f(N)\}$ is a stable O-lattice presenting $N \simeq L_0/L_2$; for a surjective equivariant map $f : N \twoheadrightarrow N'$ from $N \simeq L_1/L_2$, the kernel $L_0 = \ker(f \circ p)$ presents $N' \simeq L_1/L_0$; direct sums of presentations present finite direct sums. Crystalline representations are closed under direct

sums, crystalline character twists, and unramified twists, with jump multiplicities adding under direct sums and shifting under character twists. Thus $V \oplus \bigoplus_{c=1}^{t} L'(\chi_c)$ is a full representation, so $T_0 \otimes_O \mathcal{M}$ and $O(\chi_c) \otimes_O \mathcal{M}$ are admissible. Put $Q_a = O(\chi_1)/\varpi^a$. If $a, b \geq 1$ both annihilate $\mathcal{M}$, changing the exponent preserves admissibility. For $b \leq a$, the map

$$\mathcal{E}_{z,a} \longrightarrow \mathcal{E}_{z,b}, \qquad (m, y) \longmapsto (m, y \bmod \varpi^b)$$

is an equivariant surjection. For $a \leq b$, the map

$$\mathcal{E}_{z,b} \longrightarrow \mathcal{E}_{z,a} \oplus Q_b, \qquad (m, y) \longmapsto ((m, y \bmod \varpi^a), y)$$

is an equivariant injection. Thus closure under quotients, submodules and direct sums gives both implications. Eventual admissibility is therefore equivalent to admissibility at any one allowed exponent.
Lean: `GlobalTheory.AdmClasses.ClosedClass.level` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleClassesPropsOfFlatCoefficient`, lines 125–132)

At a common exponent, the fiber product of two extensions over Q_a, followed by addition on their coefficient modules, realizes their sum as a subquotient of their direct sum. Scalar multiplication and a general coefficient map are pushouts; they are quotients of the direct sum of the old extension and the target coefficient module. The split extension is the sum of its two admissible terms, and changing a cocycle by a coboundary gives an equivariant isomorphism of extensions. These constructions prove that the admissible classes form an O-submodule and are functorial. If $i : \mathcal{M} \hookrightarrow \mathcal{M}'$, then $W' \otimes i$ is injective because W' is free. At an exponent annihilating both modules, it induces an equivariant injection $\mathcal{E}_{z,a} \hookrightarrow \mathcal{E}_{i_* z,a}$. Submodule closure gives the asserted inverse-image property. Together with exponent independence this proves the last cocycle assertion as well.
Lean: `GlobalTheory.AdmClasses.admissible_classes_props_of_flat_coefficient` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleClassesPropsOfFlatCoefficient`, lines 207–270)

Part (b). Put $\overline{W} = W_0/\varpi W_0$. We first give the obstruction count, with all its hypotheses.

Claim 2. Retain the prime ℓ, the nonempty cyclic index set $I = \mathbf{Z}/d'\mathbf{Z}$, the position data (P1)–(P6), and $d_i = \sum_{u<w_i} \mathrm{mult}_{U_i}(u)$. Let k be a field of characteristic ℓ, with cyclotomic scalar data as in Definition 36.36, and let D be a cyclotomic module as in Definition 36.39, of dimension $r = |U_i|$ at each component. Let $N_i \subset D_i$ be A-lattices, put $v_i = h_i - w_i$ and $P_i = A\varphi_i(N_{i-1})$, and suppose

$$X^{v_i} N_i \subset P_i \subset X^{-w_i} N_i, \qquad (\gamma_i - 1)N_i \subset X N_i.$$

For $a \in \mathbf{Z}$, put $s_i(a) = \mathrm{sen}_i(N; \bar{a})$, using $\nabla_i = X^{-1}(\gamma_i - 1)$ and the generalized eigenspace definition of Lemma 36.47. Require all three numerical conditions

$$\begin{aligned} &s_i(a) = 0 && (0 \leq a \leq \sigma_i + 1),\\ &\sum_{j=0}^{w_i - 1} s_i(j - w_i) = d_i && (\varsigma_i = 1),\\ &d_i + \sum_{a=0}^{v_i} s_i(a) = r && (\varsigma_i = -1). \end{aligned}$$

At every downward index require, in addition, a polynomial $f_i \in k[Y]$ with

$$f_i(0) \neq 0, \qquad \nabla_i f_i(\nabla_i) N_i \subset X^{v_i+1} N_i.$$

For $1 \leq a \leq s$, suppose the pair $z^{(a)} = (b^{(a)}, g^{(a)})$ satisfies, at every index,

$$\begin{gathered} (\gamma_i - 1) b_i^{(a)} = \varphi_i(g_{i-1}^{(a)}) - g_i^{(a)}, \qquad g_i^{(a)} \in X N_i,\\ b_i^{(a)} \in X^{-w_i} N_i \quad (\varsigma_i = 1),\\ b_i^{(a)} \in X^{-v_i} P_i \quad (\varsigma_i = -1), \qquad \exists x_i^{(a)} \in N_i : \quad g_i^{(a)} - (\gamma_i - 1) x_i^{(a)} \in X^{v_i+2} N_i \quad (\varsigma_i = -1). \end{gathered}$$

Assume their classes are independent modulo the full Herr boundary space

$$\mathcal{B}_D = \left\{ \big((\varphi_i(m_{i-1}) - m_i)_i, ((\gamma_i - 1) m_i)_i \big) : m \in \prod_i D_i \right\}.$$

Then $s \le \sum_i d_i$.

Lean: GlobalTheory.Models.ABC (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.TwistedModelAdmissibleCocycles, lines 104–105)
Lean: GlobalTheory.Models.EPoly (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.TwistedModelAdmissibleCocycles, lines 107–108)
Lean: GlobalTheory.Models.InZZ (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.TwistedModelAdmissibleCocycles, lines 173–177)
Lean: GlobalTheory.Models.herrB (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.TwistedModelAdmissibleCocycles, lines 179–183)
Lean: GlobalTheory.T.card_cocycles_independent_mod_coboundaries_le (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CardCocyclesIndependentModCoboundariesLeStatement, lines 8–9)

We prove the count directly. The carry inequality of Lemma 36.46 and the first numerical condition allow each $z^{(a)}$ to be changed by a boundary to $z'^{(a)} = (b'^{(a)}, g'^{(a)})$ with $g_i'^{(a)} \in X^{\sigma_i+3}N_i$. The coefficient-by-coefficient construction is Claim 1 in the proof of Lemma 36.47. Its correction, together with the elements $x_i^{(a)}$ at downward indices, gives vectors $c_i^{(a)}$ such that

$$c_i^{(a)} - b_i'^{(a)} \in N_i,$$

and

$$\begin{aligned} c_i^{(a)} &\in X^{-w_i}N_i, \quad (\gamma_i - 1)c_i^{(a)} \in XN_i \quad (\varsigma_i = 1),\\ c_i^{(a)} &\in X^{-v_i}P_i, \quad (\gamma_i - 1)c_i^{(a)} \in X^{v_i+2}N_i \quad (\varsigma_i = -1). \end{aligned}$$

Explicitly, if the first correction is the boundary of $m^{(a)} \in \prod_i N_i$, use $c_i^{(a)} = b_i^{(a)} - \varphi_i(m_{i-1}^{(a)}) + m_i^{(a)}$ upward and $c_i^{(a)} = b_i^{(a)} - \varphi_i(m_{i-1}^{(a)}) + x_i^{(a)}$ downward. The carry inequality gives $\varphi_i(X^{\sigma_{i-1}+3}N_{i-1}) \subset X^{v_i+2}N_i$, which verifies the downward estimate.

Let Z_i be the k-span of the $c_i^{(a)}$. At an upward index, $Z_i/(Z_i \cap N_i)$ injects into the kernel of ∇_i on $X^{-w_i}N_i/N_i$. Its filtration by powers of X has zero-eigenvalue multiplicities $s_i(-w_i), \dots, s_i(-1)$. Therefore

$$\dim_k Z_i/(Z_i \cap N_i) \le \sum_{j=0}^{w_i-1} s_i(j - w_i) = d_i.$$

At a downward index, use ψ and Ψ_i from Lemma 36.38. In addition to $\Psi_i(b\varphi_i(x)) = \psi(b)x$ and $\Psi_i\gamma_i = \gamma_{i-1}\Psi_i$, they satisfy

$$\Psi_i(X^aP_i) = N_{i-1} \quad (2 - \ell \le a \le 1), \qquad (\gamma_i - 1)(X^{-v_i}P_i \cap \ker \Psi_i) = X^{1-v_i}P_i \cap \ker \Psi_i.$$

The latter identity is permitted because $1 \le v_i$ and $v_i + 7 \le \ell$. On $N_i/X^{v_i+1}N_i$, the polynomial condition makes the generalized zero eigenspace of ∇_i its kernel: if $T^j\bar{x} = 0$, then $Tf_i(T) = 0$ and $\gcd(f_i, Y^j) = 1$ give $T\bar{x} = 0$. Thus the polynomial condition and the exact shift give precisely the two additional inputs to the local comparison in Claim 4 of the proof of Lemma 36.47. Its other inputs are the height and γ-conditions at i and $i-1$, the vanishing $s_{i-1}(a) = 0$ for $0 \le a \le \sigma_{i-1} + 1$, and

$$v_i + 2 \le \ell(\sigma_{i-1} + 3) - w_i,$$

which follows from the carry inequality. Applied to Z_i, it gives

$$\dim_k Z_i/(Z_i \cap N_i) + \sum_{a=0}^{v_i} s_i(a) \le r, \qquad \dim_k Z_i/(Z_i \cap N_i) \le d_i.$$

Lean: GlobalTheory.Models.PolarBound.main (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CardCocyclesIndependentModCoboundariesLe, lines 117–194)

Finally consider $k^s \to \prod_i Z_i/(Z_i \cap N_i)$, $(t_a)_a \mapsto (\sum_a t_a c_i^{(a)} \bmod N_i)_i$. A vector in its kernel gives $\sum_a t_a b_i'^{(a)} \in N_i$ and $\sum_a t_a g_i'^{(a)} \in X^{\sigma_i+3}N_i$. The contraction argument of Claim 1 in the proof of Lemma 36.47 makes this pair a Herr boundary. Since the original and modified pairs have the same classes modulo $\mathcal{B}_D$, independence forces every t_a to vanish. The map is injective, and the preceding local bounds prove Claim 2.

Now use the canonical residue realization and base pairs of Lemma 36.48. Its construction uses the complete level-one Wach input specified there, and $\dim_B D(\overline{W})_i = r$. Lemma 36.49 gives a largest model

$N^{\max}$ containing N^{std}. Lemma 36.48(a) gives the height inclusions, the γ-condition and the polynomial at v_i+1; part (g), proved by the quotient-multiplicity relation and its endpoint calculations, gives all three numerical conditions of Claim 2. Choose independent residue classes and use Lemmas 36.43 and 36.49 to obtain pairs satisfying every clause above, independent modulo the full boundary space. Claim 2 gives

$$\dim_k \mathrm{Adm}(T_0,k) \le \sum_i d_i = d_-.$$

The two character assertions are Lemma 36.51.

Lean: `GlobalTheory.Models.AdmBound.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleCocyclesCardLeSumDims`, lines 73–81)
Lean: `GlobalTheory.ResidueBound.resBound_dsum` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsIndependentAdmissibleCocyclesResidue`, lines 209–227)
Lean: `GlobalTheory.TopSteps.CharBound.resBound_chars` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LinearIndependentOfIndepMod`, lines 207–221)

Part (c). Claim 3. $H^0(K,\overline{W}) = H^2(K,\overline{W}) = 0$. Indeed, Lemma 36.50 at $e=0$ and $e=-1$ gives the hypotheses of Lemma 36.44 for $T=\overline{W}$, and that lemma gives the two vanishings.

Lean: `GlobalTheory.T.hom_to_characterLine_eq_zero` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.HomToCharacterLineEqZeroStatement`, lines 8–9)
Lean: `GlobalTheory.TopVanish.vanish_of_real` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.VanishOfLattice`, lines 234–255)

Claim 4. $H^1(K,W_0)$ is a free O-module of rank $d'r$, and the natural map $H^1(K,W_0)\otimes_O \mathcal{M} \to H^1(K,W_0\otimes_O \mathcal{M})$ is an isomorphism for every finite O-module $\mathcal{M}$. The inputs are Claim 3, $\mathrm{rank}_O W_0 = r$, $[K:\mathbf{Q}_\ell] = d'$, and (36.19), which gives $|H^1(K,M)| = |M|^{d'}$ for every finite ℓ-primary G_K-module M with $H^0(K,M) = H^2(K,M) = 0$. By dévissage from Claim 3, H^0 and H^2 of $W_0/\varpi^n W_0$ vanish for all n, hence also $H^0(K,W_0)$ and $H^2(K,W_0)$. The sequences for multiplication by ϖ^n then show that $H^1(K,W_0)$ has no torsion and that $H^1(K,W_0)/\varpi^n \simeq H^1(K,W_0/\varpi^n W_0)$, a module of order $|O/\varpi^n|^{d'r}$. The continuous cohomology inverse limit is complete; its finite reduction and topological Nakayama give finite generation. Torsion-freeness over the discrete valuation ring then gives freeness of rank $d'r$, and the same isomorphisms, applied to the cyclic summands of $\mathcal{M}$, give the base-change isomorphism.

Lean: `GlobalTheory.FreeCoh.free_and_tensIso_gal` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsPowSmulEqZero`, lines 409–449)
Lean: `GlobalTheory.TopBridge.freeH_of_vanish_euler` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TwistedLatticeCohomologyTheoremTOP`, lines 31–42)

Claim 5. $H^1_f(K,W_0)$, in the sense of Definition 36.15, is an O-direct summand of $H^1(K,W_0)$ of rank d_-. Indeed, $H^0(K,\chi_1^{-1}V) = 0$ by Claim 3, and the jumps of $\chi_1^{-1}V$ at τ_i are $u-w_i$ with $u\in U_i$, so (36.20) gives $\dim_{L'} H^1_f(K,\chi_1^{-1}V) = d_-$. The lattice $H^1_f(K,W_0)$ is the inverse image of this subspace, hence saturated in the free module of Claim 4, hence a direct summand.

Lean: `GlobalTheory.TopSteps.FinSummandJumps.fSummand` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.FinsumCountNeg`, lines 54–60)

Claim 6. For every finite O-module $\mathcal{M}$, the image of $H^1_f(K,W_0)\otimes_O \mathcal{M}$ in $H^1(K,W_0\otimes_O\mathcal{M})$ lies in $\mathrm{Adm}(T_0,\mathcal{M})$. Indeed, a finite class of W_0 is realized by a crystalline lattice extension $0\to T_0\to E\to O(\chi_1)\to 0$. The sum $E\otimes_O L' \oplus \bigoplus_{c=2}^{t} L'(\chi_c)$ is a full representation. Reducing E modulo ϖ^a and pushing out along $T_0\otimes_O O/\varpi^a \to T_0\otimes_O \mathcal{M}$, $x\otimes 1\mapsto x\otimes m$, gives the extension $\mathcal{E}_{z,a}$ for the class of $\xi\otimes m$, which is admissible by part (a).

Lean: `GlobalTheory.FiniteAdm.finAdm` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.FiniteClassesSubsetAdmissibleOfGoodO`, lines 510–519)
Lean: `GlobalTheory.FontaineExtensionCriteria.extCrys_of_tied` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.StrictnessOfTied`, lines 35–37)

Claim 7. $\mathrm{Adm}(T_0,\mathcal{M})$ equals the image in Claim 6 for every finite O-module $\mathcal{M}$. The inputs are Claims 4–6, the properties of $\mathrm{Adm}(T_0,-)$ in part (a), and the bound $\dim_k \mathrm{Adm}(T_0,k)\le d_-$ of part (b). At $\mathcal{M}=k$ the image has dimension d_- by Claims 4 and 5, so Claim 6 and part (b) give equality. For general $\mathcal{M}$, induct on the length with an exact sequence $0\to k\to\mathcal{M}\to\mathcal{M}''\to 0$. An admissible class x maps to an admissible class of $\mathcal{M}''$, which by induction lifts to the image of $H^1_f(K,W_0)\otimes\mathcal{M}$. The difference comes from $H^1(K,W_0\otimes_O k)$, and the pullback property places it in $\mathrm{Adm}(T_0,k)$.

Lean: `GlobalTheory.AdmFinite.adm_eq_H1fX` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ToAmbCoefH1EqCoefAmbToAmb`, lines 234–252)
Lean: `GlobalTheory.AdmFinite.cls_eq_H1fX` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ToAmbCoefH1EqCoefAmbToAmb`, lines 256–263)

Claim 8. For $2 \le c \le t$ and every finite O-module $\mathcal{M}$, $\mathrm{Adm}(O(\chi_c), \mathcal{M}) = \{0\}$. Indeed, part (b) gives $\mathrm{Adm}(O(\chi_c), k) = \{0\}$. For general $\mathcal{M}$, use induction on the length with the same exact sequences, functoriality and pullback along injections from part (a). The sequence $H^1(K, W' \otimes k) \to H^1(K, W' \otimes \mathcal{M}) \to H^1(K, W' \otimes \mathcal{M}'')$ is exact because W' is free.
Lean: `GlobalTheory.TopSteps.CharAllCoef.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.IsLocallyConstantTwCoef`, lines 205–208)
Lean: `GlobalTheory.TopSteps.CharVanish.cls_eq_zero` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.StmtOfReal`, lines 15–22)

For $c = 1$, the crystalline rules and the admissibility properties of part (a) give the hypotheses of Lemma 36.45. Its two-character lattice construction puts every unramified class in $\mathrm{Adm}(O(\chi_1), \mathcal{M})$. The bound $\dim_k \mathrm{Adm}(O(\chi_1), k) \le 1$ of Lemma 36.51 then gives $\mathrm{Adm}(O(\chi_1), \mathcal{M}) = H^1_{\mathrm{ur}}(K, \mathcal{M})$ for every finite $\mathcal{M}$. Evaluation at a Frobenius element identifies $H^1_{\mathrm{ur}}(K, k)$ with k, so the residue dimension in (b) is exactly one.
Lean: `GlobalTheory.TopSteps.ChiOne.cls_eq_homModInertia` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ClsEqHomModInertia`, lines 278–289)

The complete proof is contained in `GlobalTheory.TopBridge.stmtAt_of_real`, `GlobalTheory.TopSteps.CharVanish.stmt_of_real`, and `GlobalTheory.TopSteps.admissibleUntwistedCharUnramified_of_residueRealisation`. □

Proof of 36.53

Proof. Fix data $(K, L', O, \varpi, \tau_i, t, h, w, \eta', \varsigma, U, V, T_0, \chi_1, \dots, \chi_t)$ of Definition 36.34 and a finite O-module $\mathcal{M}$. For any stable O-lattice L, continuous cocycles $z : G_K \to L \otimes_O \mathcal{M}$ for the action $g \mapsto \chi_1(g)^{-1} g$ coincide with continuous cocycles of the twisted coefficient module $(\chi_1^{-1} L) \otimes_O \mathcal{M}$.
Lean: `GlobalTheory.T.twistedAction_eq_twistedCoefficients_cocycle_iff` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TwistedActionEqTwistedCoefficientsCocycleIffStatement`, lines 21–27)

Put $W_0 = \chi_1^{-1} T_0$. By Lemma 36.52(a) and (c), a cocycle $z : G_K \to T_0 \otimes_O \mathcal{M}$ is admissible if and only if its class in $H^1(K, W_0 \otimes_O \mathcal{M})$ lies in $\mathrm{Adm}(T_0, \mathcal{M}) = \mathrm{im}\big(H^1_f(K, W_0) \otimes_O \mathcal{M} \to H^1(K, W_0 \otimes_O \mathcal{M})\big)$. Membership of $[z]$ in this image holds if and only if $z = \sum_j \xi_j \otimes m_j + z_0$ for elements $m_j \in \mathcal{M}$, a coboundary z_0, and integral cocycles $\xi_j : G_K \to T_0$ of $\chi_1^{-1} V$ whose classes are finite in the sense of Definition 36.15. This proves (T1).
Lean: `GlobalTheory.T.period_data_dual_exact` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.PeriodDataDualExactStatement`, lines 21–34)

Now let $1 \le c \le t$ and let L_c be a stable O-lattice in $L'(\chi_c)$. Because O is a discrete valuation ring and L_c is a free rank-one O-module with G_K acting via $\chi_c : G_K \to O^\times$, there is an $O[G_K]$-equivariant O-linear isomorphism $e : O(\chi_c) \xrightarrow{\sim} L_c$. For any continuous cocycle $z : G_K \to L_c \otimes_O \mathcal{M}$ for $g \mapsto \chi_1(g)^{-1} g$, set $z_0 = (e^{-1} \otimes \mathrm{id}_{\mathcal{M}}) \circ z : G_K \to O(\chi_c) \otimes_O \mathcal{M}$. Then z_0 is a continuous cocycle, and for every integer $a \ge 1$ with $\varpi^a \mathcal{M} = 0$ the map $(e \otimes \mathrm{id}_{\mathcal{M}}) \oplus \mathrm{id}_{O/\varpi^a}$ is an $O[G_K]$-equivariant isomorphism $\mathcal{E}_{z_0, a} \xrightarrow{\sim} \mathcal{E}_{z, a}$. Thus z is admissible if and only if z_0 is admissible, hence by Lemma 36.52(a) if and only if $[z_0] \in \mathrm{Adm}(O(\chi_c), \mathcal{M})$. Moreover, $e \otimes \mathrm{id}_{\mathcal{M}}$ preserves coboundaries on G_K and coboundaries on I_K.
Lean: `GlobalTheory.T.exists_equiv_stableLattice_integralCharacter` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsEquivStableLatticeIntegralCharacterStatement`, lines 21–24)
Lean: `GlobalTheory.T.equivariantIso_transport_cocycles_extensions` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.EquivariantIsoTransportCocyclesExtensionsStatement`, lines 21–33)
Lean: `GlobalTheory.TopDict.admissibleExtension_iff_equivariantCoefIso` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExactLocalConditionsTopCore`, lines 47–63)

If $2 \le c \le t$, Lemma 36.52(c) gives $\mathrm{Adm}(O(\chi_c), \mathcal{M}) = \{0\}$. Hence z is admissible if and only if $[z_0] = 0$, that is, z_0 is a coboundary on G_K, which holds if and only if z is a coboundary on G_K. This proves (T2).
Lean: `GlobalTheory.TopDict.WachMod_mem_B1_iff` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExactLocalConditionsTopCore`, lines 65–71)

If $c = 1$, then $\chi_1^{-1} \chi_1 = 1$, and Lemma 36.52(c) gives $\mathrm{Adm}(O(\chi_1), \mathcal{M}) = H^1_{\mathrm{ur}}(K, \mathcal{M})$. A cocycle $z_0 : G_K \to O(\chi_1) \otimes_O \mathcal{M}$ represents a class in $H^1_{\mathrm{ur}}(K, \mathcal{M})$ if and only if its restriction to I_K is a coboundary: if z' is cohomologous to z_0 via $z'(g) - z_0(g) = gb' - b'$ for all $g \in G_K$ and $z'(g) = gb - b$ for all $g \in I_K$, then $z_0(g) = g(b - b') - (b - b')$ for all $g \in I_K$; conversely, if $z_0(g) = gb - b$ for all $g \in I_K$, then $[z_0|_{I_K}] = 0$ in $H^1(I_K, \mathcal{M})$, so $[z_0] \in H^1_{\mathrm{ur}}(K, \mathcal{M})$. Transporting along $e \otimes \mathrm{id}_{\mathcal{M}}$ shows that z is admissible if and only if $z|_{I_K}$ is a coboundary on I_K. This proves (T3).
Lean: `GlobalTheory.IndefiniteRowSideFullHecke.AdmissibleCharUnramified.homModInertia_eq_unr` (`Development.AbelianVarietyTheory.BlockTheory.NonGaloisCusps.AdmissibleCharUnramified`, lines 37–46)

Lean: GlobalTheory.TopDict.mk_mem_unr_iff (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExactLocalConditionsTopCore, lines 32–45)
Lean: GlobalTheory.TopDict.topCore_of_typings (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExactLocalConditionsTopCore, lines 75–171)
Lean: GlobalTheory.TopBridge.twistedLatticeCohomology_theoremTOP (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TwistedLatticeCohomologyTheoremTOP, lines 67–82)

□

Proof of 36.54

Proof. Jump sets and size facts. For layer j and weight type τ, the de Rham jump conditions of Theorem 36.26(I) realize the jump set $J_i = F_l(i) \cup F_c(i)$ specified in part (a), with midpoint $c_t = (t-1)/2$. Because the family parameter bounds hold at every indexed $x > x_1$ and member $a \in I_x$ without any additional cutoff, they directly yield the following three facts for every $x > x_1$, every layer, type and sign: (F1) Every $z \in F_l(i)$ satisfies $2b/Z + t/2 + 3 \le |z - c_t| \le \eta$, and there is a base size $W_0(i)$ with $W_0(i) \le \varsigma_i(z - c_t) \le W_0(i) + S_c + \eta + 1$ for every $z \in F_c(i)$. (F2) Every base size W_0 satisfies $2b/Z + t/2 + 3 \le W_0 \le b$. (F3) $1 \le t \le \eta$, $b > 0$, $1 < Z$, $2\eta + t + 8b/Z + 7 < Z$ and $(b + S_c + 2\eta + 3)Z \le 2b(Z - 1)$. Indeed, at every $x > x_1$ the member parameter bounds directly give $1 \le t \le \eta$, $b \ge 1$, $Z > 7 > 1$, and $2\eta + t + 8b/Z + 7 < Z$; moreover, since $k \ge m$, $n \ge 1$, $t \ge 1$, and $L \ge 1$, the denominator $D = nt^2(N(m) + t)L^2 + kn^2(S_c + t + L)$ of $Z = mb/D$ satisfies $D \ge kn^2(S_c + t + L) \ge 2m$, whence $2Z \le b$, and combining this with the member bound $(S_c + 2\eta + 3)Z + 2b \le bZ$ gives $(b + S_c + 2\eta + 3)Z \le 2b(Z - 1)$ for (F3), while the member's row-window, cloud-window, and base-size parameter bounds valid at every $x > x_1$ place $F_l(i)$, $F_c(i)$, and each base size W_0 in the intervals of (F1) and (F2). Only the final comparison requires a further family cutoff $x \ge x_0$: beyond that cutoff one has also: (F4) q, d and the primes of the small set are less than Z.
Lean: GlobalTheory.T.lowJumps_and_cloud_in_windows_along_family (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LowJumpsAndCloudInWindowsAlongFamilyStatement, lines 12–22)
Lean: GlobalTheory.T.baseSize_bounds_along_family (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ParametersBaseSizeBoundsAlongFamilyStatement, lines 12–14)
Lean: GlobalTheory.T.specialPrimeThreshold_numerics_along_family (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SpecialPrimeThresholdNumericsAlongFamilyStatement, lines 12–18)
Lean: GlobalTheory.T.conductorPrimes_lt_specialPrimeThreshold_eventually (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ConductorPrimesLtSpecialPrimeThresholdEventuallyStatement, lines 12–15)

Part (a). By (F3), $Z > 2\eta + t + 7 > 8$. Hence every prime $\ell > Z$ satisfies $\ell \ge 11$ and $t + 1 < \ell$. By (F4), $\ell \notin \{q, d\}$ and ℓ lies outside the small set. The completions of E at such primes are unramified, by the description of the host in Theorem 8.3.
Lean: GlobalTheory.UseLoc.exists_caseA_enl (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.FiniteDimensionalKsOne, lines 33–52)

Suppose ℓ is not exceptional, and fix j, d', the signs and the types. Fix i and put $\eta = \eta_{\mathrm{sz}}$, $S = S_c$, $B = b/Z$, $W_0 = W_0(i)$ and $\sigma_P = a_*$. The interval (36.29) is then (36.32) for the base size $W_0(i)$, so ℓ divides no integer in it. By (F1)–(F3), the other hypotheses of Lemma 36.33 hold for $F_l = F_l(i)$, $F_c = F_c(i)$, $\varsigma = \varsigma_i$ and $\sigma = 0$; in particular $2\eta + 8B + 7 < Z < \ell$. The construction of that lemma defines (h_i, w_i, η_i', U_i), which do not depend on σ, and gives $h_i \le W_0(i) + S_c + 2\eta + 3$. By (F2), $h_c \le H_0$ for every c, where $H_0 = b + S_c + 2\eta + 3$. Hence

$$\sigma_h(i) \le \sum_{j \ge 1} H_0 \ell^{-j} = \frac{H_0}{\ell - 1} \le \frac{H_0}{Z - 1} \le \frac{2b}{Z},$$

by $\ell > Z > 1$ and the last inequality in (F3). Since $\sigma_h(i)$ is an integer, $\sigma_h(i) \le a_*$.
Lean: Formalism.IrreducibleCountingLemmas.TOP.carryFloorLeGeom (Formalism.PadicSubsetSums.StableGradedCounts.TOPSgLe, lines 19–40)
Lean: Formalism.IrreducibleCountingLemmas.TOP.geomRatioBound (Formalism.PadicSubsetSums.StableGradedCounts.TOPSgRange, lines 5–17)

Thus $\sigma = \sigma_h(i)$ satisfies $\sigma \le \min(2B, a_*)$, and Lemma 36.33 with this σ gives (P1)–(P6) at i and the multiplicities.

Part (b), Hodge–Tate data. Put $K = E_w$, of degree $d' = d_u$ over $\mathbf{Q}_\ell$, unramified by (a). Number its embeddings τ_ν, $\nu \in \mathbf{Z}/d'\mathbf{Z}$, in Frobenius order. Since $\ell \notin \{g_0, g_1\}$, the local compatibility on the cut makes every R_x crystalline at w. As in the proof of Lemma 36.31, the semisimple decomposition of R_π makes W_π and the characters $\chi_{i,c}\chi_{i,1}^{-1}$ crystalline at w. The de Rham jump conditions give the jumps of W_π at τ_ν as the

set J_ν of (a), with multiplicity one, for the layer of the index and the sign at τ_ν given by the embedding type. They give the jumps of V_x at τ_ν as J_ν and $0,\dots,t-1$, with multiplicities added. By the weight dictionary, $\chi_{i,c}\chi_{i,1}^{-1}=\chi_\ell^{1-c}$ on G_K, with the single jump $c-1$. Part (a) gives position data with $\mathrm{mult}_{U_\nu}(b+w_\nu)=\mathbf{1}_{J_\nu}(b)$.

By Lubin–Tate theory over the unramified field K, there is a continuous crystalline character $\theta: G_K\to O^\times$ with the single jump w_ν at τ_ν for every ν. Products of Galois conjugates of the Lubin–Tate character of K realize every vector of jumps. Jumps add under tensor product with a crystalline character. Put $V=W_\pi\otimes\theta$, with lattice $T_0=W_\pi\otimes_{O_i}O\otimes\theta$, and $\chi_c'=\chi_{i,c}\chi_{i,1}^{-1}\theta$. Then $m_V(\tau_\nu,a)=\mathrm{mult}_{U_\nu}(a)$, $\chi_c'=\chi_1'\chi_\ell^{1-c}$, and χ_c' has the single jump $w_\nu+c-1$. Each $V_x\otimes\theta$ is crystalline with $m_{V_x\otimes\theta}(\tau_\nu,a)=\mathbf{1}_{J_\nu}(a-w_\nu)+\mathbf{1}_{[0,t-1]}(a-w_\nu)=\tilde m_\nu(a)$.
Lean: `GlobalTheory.UseLocInst.exists_setC` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsExactLocalConditionDataAtPlace`, lines 23–109)
Lean: `GlobalTheory.T.crystalline_characters_given_jumps_unramified_twists` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CrystallineCharactersGivenJumpsUnramifiedTwistsStatement`, lines 8–9)
Lean: `GlobalTheory.jTil_shift_block` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.JTilShiftBlock`, lines 7–60)

The congruence. Let $x\in X_i$ and $g\in G_E$. The points of X_i share the residual Hecke eigensystem of the index. At π the reduced full Hecke polynomial at a split place v is the residual Frobenius polynomial of $\rho_i(\pi)$ times the character factors, by Lemma 36.25. Hence $\det(U-\mathrm{Frob}_v\mid V_x)$ and $\det(U-\mathrm{Frob}_v\mid W_\pi)\prod_c(U-\chi_{i,c}\chi_{i,1}^{-1}(\mathrm{Frob}_v))$ agree modulo ϖ at these places. Both sides, reduced modulo ϖ, are constant on the cosets of an open normal subgroup. Lemma 36.24 therefore extends the congruence to every g. Twisting by θ, each $V_x\otimes\theta$ is a full representation for $(V,T_0,\chi_1',\dots,\chi_t')$ and the position data of (a).
Lean: `GlobalTheory.UseLoc.InstPoly.charpoly_congr` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ScaleRootsProdIcc`, lines 222–258)

Lifting the cut cocycle. Extend scalars from O_i to O throughout. Let L be the lattice $M\otimes_{O_i}O\otimes\theta$ of $M\otimes\theta$. On it the action $\chi_1'(g)^{-1}g$ is the action μ of M, since $\chi_1'=\theta$. The module $\phi(\mathcal{Y})$ is finitely generated and torsion, hence finite and killed by a power of ϖ. Since G_E acts continuously on the lattice $\mathcal{Y}+O_iv_0$, the cocycle κ is locally constant. Choose $a\ge1$ with $\varpi^a\bar\kappa=0$, put $\mathcal{M}=\varpi^{-a}O/O$ and $m_0=\varpi^{-a}\bmod O$. The ϖ^a-torsion of $M\otimes L'/O$ is identified with $L\otimes\mathcal{M}$ by $u\otimes m_0\mapsto\varpi^{-a}u$. Hence $\bar\kappa$ is the image of a unique cocycle $z:G_K\to L\otimes\mathcal{M}$ for μ. Replace $\mathcal{Y}$ by its O-submodule generated by the vectors gv_0-v_0, $g\in G_K$. It is finitely generated and stable, since $g'(gv_0-v_0)=(g'g)v_0-v_0-(g'v_0-v_0)$. On it ϕ takes values in the ϖ^a-torsion, so it defines $\phi':\mathcal{Y}\to L\otimes\mathcal{M}$ with $z(g)=\phi'(gv_0-v_0)$.
Lean: `GlobalTheory.T.cutCocycles_locallyConstant_killed_by_uniformiserPow` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CutCocyclesLocallyConstantKilledByUniformiserPow`, lines 12–28)
Lean: `GlobalTheory.T.exists_integralTensorTorsion_cocycle_of_torsion_localCocycle` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsIntegralTensorTorsionCocycleOfTorsionLocalStatement`, lines 10–17)

Admissibility. The semisimple decomposition of R_π gives an injective equivariant map $\iota:M\otimes K_i\to V_\pi$ and a nonzero vector $e\in V_\pi$ fixed by G_E, namely a generator of the summand $c=1$. Let $A=\mathcal{Y}\oplus L\oplus O$ with the action

$$g(y,u,s)=\big(\theta(g)(gy+s(gv_0-v_0)),\ gu,\ \theta(g)s\big),$$

where gu is the action on L. Put $\mathcal{V}=\bigoplus_j(V_{x_j}\otimes\theta)\oplus(V_\pi\otimes\theta)^2$, a finite direct sum of full representations, and

$$\Psi(y,u,s)=\big((y_j+sv_{0,j})_j,\ \iota(u),\ se\big)\in\mathcal{V}.$$

A direct computation gives $\Psi(g\alpha)=g\,\Psi(\alpha)$, and Ψ is injective: $se=0$ forces $s=0$, ι is injective, and then $y=0$. For $a''\ge a$ let $q:A\to\mathcal{E}_{z,a''}$ be $q(y,u,s)=(\phi'(y)+u\otimes m_0,\ s\bmod\varpi^{a''})$. It is surjective, since $\mathcal{M}$ is generated by m_0. It is equivariant, since

$$q(g(y,u,s))=\big(g(\phi'(y)+u\otimes m_0)+s\theta(g)z(g),\ \theta(g)s\big).$$

Choose a stable O-lattice $\Lambda\subset\mathcal{V}$ containing $\Psi(A)$. By the Artin–Rees lemma there is n_0 with $\varpi^{n_1}\Lambda\cap\Psi(A)\subset\varpi^{n_1-n_0}\Psi(A)$ for $n_1\ge n_0$. Take $n_1=n_0+a''$ and put

$$L_1=\Psi(A)+\varpi^{n_1}\Lambda,\qquad L_2=\Psi(\ker q)+\varpi^{n_1}\Lambda.$$

Both are finitely generated, stable and torsion free, and they span $\mathcal{V}$, so they are stable lattices. Since $\varpi^{a''}$ kills $\mathcal{E}_{z,a''}$, we have $\varpi^{n_1}\Lambda\cap\Psi(A)\subset\varpi^{a''}\Psi(A)\subset\Psi(\ker q)$. Therefore $L_1/L_2\simeq\Psi(A)/\Psi(\ker q)\simeq\mathcal{E}_{z,a''}$ as $O[G_K]$-modules, for every $a''\ge a$. Thus z is admissible for the data of the previous paragraphs.

Lean: GlobalTheory.T.cocycleExtension_twistedWeilCoefficients_admissible (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CocycleExtensionTwistedWeilCoefficientsAdmissibl, lines 60–85)
Lean: GlobalTheory.T.cocycleExtension_conductorTwistedCharacter_admissible (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CocycleExtensionConductorTwistedCharacterAdmissi, lines 53–86)
Lean: UpperTheory.KN.exists_admissible_extension_latticeTensor (Development.AutomorphicArithmetic.CyclotomicBounds.DeRhamLattices.ExistsAdmissibleExtensionLatticeTensor, lines 25–124)

Applying (T). If $M = W_\pi$, (T1) writes $z = \sum_j \xi_j \otimes m_j + z_0$ with ξ_j cocycles of $\theta^{-1}V = W_\pi$ whose classes are finite. Write $m_j = \alpha_j m_0$. The image of $\xi_j \otimes m_j$ in $M \otimes L'/O$ is the image of the rational cocycle $\alpha_j \varpi^{-a} \xi_j$, whose class is finite. Coboundaries map to coboundaries. Hence $\bar\kappa$ is finite. If $2 \le c \le t$, (T2) makes z, and hence $\bar\kappa$, a coboundary on G_K. If $c = 1$, (T3) makes the restriction of $\bar\kappa$ to inertia a coboundary.
Lean: GlobalTheory.UseLoc.Cases.partW (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.UnramifiedEnlargementCutCocyclesBlochKato, lines 25–55)
Lean: GlobalTheory.UseLoc.Cases.partC (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.UnramifiedEnlargementCutCocyclesBlochKato, lines 58–90)
Lean: GlobalTheory.UseLoc.Cases.partU (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.UnramifiedEnlargementCutCocyclesBlochKato, lines 93–124)
Lean: GlobalTheory.T.isBlochKato_read_of_tensor_selmer_add_coboundary (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.IsBlochKatoReadOfTensorSelmerAddCoboundaryStatement, lines 10–15)

□

Proof of 36.57

Proof (outline). Claim 1. The normalization in (a) is valid. Since $K_0 \subset K \subset K_0(\mu_\ell)$ and $[K_0(\mu_\ell) : K_0] = \ell - 1$, tower multiplicativity gives $e \mid \ell - 1$, $es = \ell - 1$, and $s \ge 1$. The crystalline kernel sequence for F has additive, nonnegative jump multiplicities, so every jump of T_2 is a jump of T_1. Take

$$\eta_1 = \min(\eta, h), \qquad h' = h + S + 1, \qquad \beta'_\tau = \min(\beta_\tau, h + 1).$$

Then $\eta_1 < \beta'_\tau$ and $\beta'_\tau + S \le h'$. All jumps lie in $[-h, h]$. If $\beta_\tau > h + 1$, its original shifted window is disjoint from this interval, so every jump is in $[-\eta_1, \eta_1]$. Otherwise the shifted window is unchanged. The central interval has only shrunk, so its multiplicity in T_2 is at most m.

Claim 2. At level one, tame descent for the present $K_0 \subset K \subset K'$, with $|\Delta| = s \in O^\times$, supplies π' and E_c with all four unit relations and the invariant-ring identity in (b). One can take the E_c to be the conjugate Eisenstein polynomials of the descended uniformizer. The relations identify π' with a unit times Z^s, Q with a unit times E_π, and consequently the indicated Frobenius and γ changes with units times $\pi' E_\pi^s$ and $\pi' E_\pi$. These are the level-one invariant-ring clauses of the Wach comparison supplied by $\mathscr{H}_2$.
Lean: Formalism.MainNotions.DeltaInvariantLattices.wachRingRelations (Formalism.SharpRestriction.LevelLetterCocycles.DeltaInvariantLattices, lines 217–228)
Lean: Formalism.MainNotions.DeltaInvariantLattices.p9 (Formalism.SharpRestriction.LevelLetterCocycles.DeltaInvariantLattices, lines 236–242)

Claim 3. For every crystalline lattice with jumps in $[-h', h']$, the two height bounds give

$$E_\pi^{h'} \varphi(N(T)) \subset N(T), \quad E_\pi^{h'} \Phi_c(T) \subset \epsilon_c N(T), \quad N_c(T)^\Delta \subset \Lambda_c^P(T).$$

Moreover $(\gamma - 1) N_c(T)^\Delta \subset \pi' N_c(T)^\Delta$. For every crystalline short exact lattice sequence with all jumps in this interval, the kernel identifications are exact on $\pi'^k N(-)^\Delta$ for every $k \ge 0$, and on $\Lambda_c^P(-)$. The height bounds for Q transfer through the unit relation $Q = w_1 E_\pi$. The remaining assertions are respectively the invariant γ-condition and saturated exactness of the Wach and Frobenius-span lattices.
Lean: Formalism.MainNotions.N.Ln34.frobenius_uniformiser_wach (Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.FrobPiNPf, lines 206–249)
Lean: Formalism.MainNotions.N.Ln34.ninlam_pf (Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.NinLamPf, lines 51–60)
Lean: Formalism.MainNotions.N.Ln34.gamPiCondition (Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.GamPiPf, lines 80–96)
Lean: Formalism.MainNotions.N.Ln34.kerlam2_pf (Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.FrobeniusSpanDecomposition, lines 146–161)

Claim 4. The operator $\varphi - 1$ is bijective on $U_M(T)$, and every Herr pair with first coordinate in the numerator of $\mathcal{P}_c^{(M)}(T)$ and second coordinate in $U_M(T)$ has finite cohomology class. Indeed, Claims 2 and 3 give

$$\varphi(U_M(T)) \subset E_\pi^{sM - h'} U_M(T), \qquad sM \ge 4h'.$$

Iterating raises the (ℓ, Z)-adic order, and the complete Wach lattice gives the convergent inverse $-\sum_{j\ge 0}\varphi^j$. It commutes with γ and every equivariant map. To apply the period-primitive criterion of (A.514), clear the first coordinate's Frobenius-span denominator by a power of Q; this is possible by Claim 3 and $Q = w_1 E_\pi$. The second coordinate lies in $ZN(T)$, because $\pi' = wZ^s$ and $sM \ge 1$. The criterion gives a finite class over K', and averaging over Δ, whose order is a unit, gives the same assertion over K.

Lean: `Formalism.MainNotions.N.Ln34.frobPiNCondition` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.FrobPiNPf`, lines 345–354)

Lean: `Formalism.MainNotions.N.Ln34.ClassMapPf.classMapPf_of` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.ClassMapPf`, lines 126–213)

For a numerator vector w, define $z = (\varphi-1)^{-1}(\gamma-1)w$. If w changes by $u_0 \in \pi'^M N_c(T)^\Delta$, put $v = (\varphi-1)^{-1}u_0$. The pair changes by $((\varphi-1)v, (\gamma-1)v)$, which has zero class. This proves well-definedness of cl_c^H, and also its linearity and naturality. Claim 3 and $\gamma(\pi') \in \pi' A^\Delta$ ensure that the displayed denominators are contained in their numerators.

Claim 5. The following two integral lifting assertions hold for the present normalized height bound: for every equivariant surjection $F : T_1 \twoheadrightarrow T_2$ of crystalline lattices with jumps in $[-h', h']$,

$$\ell^{\rho_A} N_c(T_2)^\Delta \subset F_P(N_c(T_1)^\Delta);$$

and for every $z \in \pi' N_c(T)^\Delta$, there is $k \in \pi' N_c(T)^\Delta$ such that

$$\ell^{\sigma_A} z - (\gamma-1)k \in U_M(T).$$

For the first assertion, the integral cokernel bound and Δ-averaging give $\pi'^{a'}$-annihilation with $a' = \lceil 2eh'/(\ell-1)\rceil$. Interpolation at $\chi_\ell(\gamma)^{sj}$, $0 \le j < 2a'$, clears denominators dividing $\ell^{L(2a'-1)}$. For the second, the j-th π'-graded quotient, $1 \le j < M$, has scalar γ-action $\chi_\ell(\gamma)^{sj}$. Since s is prime to ℓ,

$$v_\ell(\chi_\ell(\gamma)^{sj} - 1) = 1 + v_\ell(j).$$

Successively solving on these graded quotients costs at most $\sum_{j=1}^{M-1}(1 + v_\ell(j)) = L(M-1) = \sigma_A$.

Lean: `Formalism.MainNotions.N.Ln34.w_pf` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.WLocal`, lines 312–384)

Lean: `Formalism.MainNotions.N.Ln34.gradNCondition` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.GradNPf`, lines 9–27)

Claim 6. Every $\kappa \in Z_f(T)$ satisfies the asserted component decomposition. Realize κ by a crystalline extension $0 \to T \xrightarrow{\iota} T' \xrightarrow{p} O \to 0$. Its jumps remain in $[-h', h']$. Let e_c be the distinguished unit Wach vector in component c. Claim 5 gives $x_c \in N_c(T')^\Delta$ with $p_P x_c = \ell^{\rho_A} e_c$. Claim 3 gives a unique $z_c \in \pi' N_c(T)^\Delta$ with

$$\iota_P z_c = (\gamma-1)x_c.$$

Choose k_c by Claim 5 and put

$$d_c = \ell^{\sigma_A} z_c - (\gamma-1)k_c \in U_M(T), \qquad \bar{x}_c = \ell^{\sigma_A} x_c - \iota_P k_c.$$

Then $p_P \bar{x}_c = \ell^{\rho_A+\sigma_A} e_c$ and $(\gamma-1)\bar{x}_c = \iota_P d_c$. The vector $\varphi\bar{x}_{c-1} - \bar{x}_c$ lies in the kernel of p_P on $\Lambda_c^P(T')$; Claim 3 therefore gives $w_c \in \Lambda_c^P(T)$ with

$$\iota_P w_c = \varphi\bar{x}_{c-1} - \bar{x}_c, \qquad (\gamma-1)w_c = \varphi d_{c-1} - d_c \in U_M(T).$$

Let $z_c' = (\varphi-1)^{-1}(\gamma-1)w_c$. Claim 4 gives a finite cocycle κ_c representing the class of (w_c, z_c'), so $\kappa_c \in \mathcal{J}_c(T)$. Summing the Herr relations and using uniqueness of $(\varphi-1)^{-1}$ on $U_M(T)$ yields $\sum_c z_c' = \sum_c d_c$. Thus the sum of these pairs is the pair obtained from the lift $\sum_c \bar{x}_c$ of $\ell^{\rho_A+\sigma_A}\sum_c e_c$. The étale extension comparison identifies its class with $u\ell^{\rho_A+\sigma_A}[\kappa]$, where $u \in O^\times$ is the unit fixed by the unit-Wach-generator convention. Multiplying the cocycles κ_c by u^{-1} proves

$$\ell^{\rho_A+\sigma_A}\kappa \in \sum_c \mathcal{J}_c(T) + B(T).$$

Lean: `Formalism.MainNotions.N.Ln34.ExtPf.ext_pf` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.ExtPfMultVEqZero`, lines 58–74)

Lean: Formalism.MainNotions.N.Ln34.OneGenPf.onegen_pf (Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.OneGenPf, lines 10–13)
Lean: Formalism.MainNotions.N.Ln34.split_local (Formalism.SharpRestriction.LevelLetterCocycles.SplitROfSteps, lines 62–272)

The complete integral comparison and component-decomposition proofs are contained in Formalism.MainNotions.N.Ln34.ClassMapPf.classMapPf_of and Formalism.MainNotions.N.Ln34.splitR_of_steps. □

Proof of 36.59

Proof. Since $v_\ell(\zeta-1)>0$, the substitution defining $\mathrm{ev}_{c,\zeta}$ converges coefficientwise. The identity $Q(Z)=((1+Z)^\ell-1)/Z$ gives

$$\mathrm{ev}_{c,\zeta}(Q)(t)=\frac{\exp(\ell t)-1}{\zeta\exp(t)-1},\qquad [t]\,\mathrm{ev}_{c,\zeta}(Q)=\frac{\ell}{\zeta-1}\neq 0.$$

The comparison expansions have support bounded below. Hence the formula for $v_{c,\zeta,T}(z)$ is defined whenever a regularizing exponent exists, and choosing the least exponent specifies its value uniquely.

The map defining $R_c(S_0)$ is O-bilinear before passage to the tensor product. The other coordinate modules are images or inverse images under O-linear maps, and the displayed intersections are submodules of the respective numerators. Thus all the quotients are defined. The map f_P preserves Wach lattices and commutes with Δ, Frobenius, and scalar multiplication. It therefore descends to f_0, and tensoring with the identity gives f_L. □

Proof of 36.61

Proof (outline). Fix c, and abbreviate $\Lambda=\Lambda_c^L(T)$, $\mathcal{M}=\mathcal{M}_c(T)$, and $R=R_c(L_c(T))$. *Claim 1.* The module $\mathcal{M}$ is stable under A^Δ and γ_L, is ℓ-saturated in Λ, and satisfies

$$E_\pi^{h'}R\subset\mathcal{M}\subset\Lambda,\qquad J_c^L(T)\subset\pi'^M\mathcal{M}.$$

At an evaluation pair write $\mathrm{ev}(E_\pi)=tU(t)$, where $U(0)\neq 0$. Multiplication of a numerator by a power series preserves its coefficient filtration conditions: the coefficient of degree i uses only coefficients of degrees $j\le i$, and $\mathscr{F}_\tau^{h'-j}\subset\mathscr{F}_\tau^{h'-i}$. The scalar relation $\gamma(E_\pi)=u_\gamma E_\pi$, with $u_\gamma\in(A^\Delta)^\times$, reduces γ_L-stability to the same calculation and substitution $t\mapsto\chi_\ell(\gamma)t$. Since each filtration space is an L'-subspace, division by $\ell^a\neq 0$ preserves these conditions whenever the divided vector is in Λ; this proves saturation. For a vector in $E_\pi^{h'}R$, the numerator has evaluation order at least $2h'$. All ensuing filtration degrees are at most $-h'$, where the filtration is the whole evaluation image. This gives the first inclusion. Finally $J_c^L=\pi'^M E_\pi^{3h'}R\subset\pi'^M\mathcal{M}$.
Lean: Formalism.MainNotions.N.Ln34.modell_pf (Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.ModelLPf, lines 11–34)

Claim 2. The quotient $\Lambda/(\Lambda\cap\pi'\mathcal{M})$ is finite free over O. The invariant Wach basis identifies $L_c(T)$ with a finite free O-module and R with the corresponding finite free module over $O[[\pi_c']]$. Multiplication by $E_\pi^{h'}$ identifies Λ with R. Claim 1 shows that the denominator contains $\pi'E_\pi^{h'}R$, so the quotient is a quotient of

$$R/(\pi'E_\pi^{2h'}R).$$

Division by the distinguished polynomial $\pi_c'E_c(\pi_c')^{2h'}$ makes this a finite O-module. If $x\in\Lambda$ and $\ell^a x\in\pi'\mathcal{M}$, then $\ell^aE_\pi^{h'}x\in\pi'R$. The quotient $R/\pi'R\simeq L_c(T)$ has no O-torsion, so $E_\pi^{h'}x\in\pi'R$. Thus $x/\pi'\in\Lambda$, and Claim 1 applied to $\ell^a(x/\pi')\in\mathcal{M}$ gives $x/\pi'\in\mathcal{M}$. This proves ℓ-torsion-freeness of the quotient. Every nonzero scalar of the DVR O divides a power of ℓ up to a unit, so it is torsion-free, and hence free.
Lean: Formalism.MainNotions.N.Ln34.MQ.basis_L (Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.ModelQPfD, lines 56–156)
Lean: Formalism.MainNotions.N.Ln34.modelqlink_pf (Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.ModelQPf, lines 14–92)

Claim 3. If $x\in\Lambda$ and $(\gamma_L-1)x\in\mathcal{M}$, every Laurent expansion f specified in (b) satisfies $[t^i]f\in\mathscr{F}_\tau^{-i}(T)$ for $i\neq 0$. Put $u=\chi_\ell(\gamma)$ and $g(t)=f(ut)-f(t)$. The numerator condition gives

$$[t^j]\big(t^{h'}U(t)^{h'}g(t)\big)\in\mathscr{F}_\tau^{h'-j}(T),$$

so $[t^j](U(t)^{h'}g(t)) \in \mathscr{F}_\tau^{-j}(T)$. Starting at the lower bound of the Laurent support, induct on i. The constant coefficient $U(0)^{h'}$ is nonzero; all terms involving earlier coefficients belong to $\mathscr{F}_\tau^{-i}$ by the decreasing filtration. Hence $[t^i]g \in \mathscr{F}_\tau^{-i}$. This coefficient equals $(u^i-1)[t^i]f$. Since $u \in 1+\ell\mathbf{Z}_\ell$ has $v_\ell(u-1)=1$, it has infinite order and $u^i-1 \neq 0$ for $i \neq 0$. Division by this scalar proves the claim.
Lean: `Formalism.MainNotions.N.Ln34.modell_pf` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.ModelLPf`, lines 11–34)

Claim 4. For the Frobenius-horizontal frame in (c), every $x \in \mathcal{M}$ has a representative $y \in N_c(T)^\Delta$ with $\alpha_T(x) = q_P(y)$, and every $y \in N_c(T)^\Delta$ has a representative $x \in \mathcal{M}$ with $\beta_T(y) = q_L(x)$. The hypotheses needed for this comparison are the level-one ring relations, the two height bounds for T, the expansion identity of Definition 36.60(b), and the jump interval $[-h', h']$. They all hold here. The expansion identity identifies the filtration inequalities defining $\mathcal{M}$ with the Wach membership criterion, in each direction.
Lean: `Formalism.MainNotions.N.Ln34.ModelFPf.modelF_local` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.ModelFPfF`, lines 22–160)
Lean: `Formalism.MainNotions.N.Ln34.modelflink_pf` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.ModelFPf`, lines 9–67)

We spell out the passage to the remaining assertions. If $x = \pi'^j x_0$ with $x_0 \in \mathcal{M}$, choose $y_0 \in N_c(T)^\Delta$ by Claim 4. Scalar compatibility of the frame gives $y - \pi'^j y_0 \in J_c^P(T)$. The height bound implies $J_c^P(T) \subset \pi'^M N_c(T)^\Delta$, and $j \leq M$ gives the first power implication in (c). Conversely, write $y = \pi'^j y_0$. Since π' is invertible in the étale coefficient ring, Δ-invariance and component support of y descend to y_0. Claim 4 and scalar compatibility give $x - \pi'^j x_0 \in J_c^L(T) \subset \pi'^M\mathcal{M}$, proving the second implication. Apply these implications with $j=1$ to $(\gamma_L-1)x$ and $(\gamma-1)y$. Commutation of the frame with γ gives exactly the two invariant conditions in (c). Its maps therefore descend to the indicated first-power quotients. The two compositions descend from multiplication by $\ell^{2\sigma_A}$ on the frame quotients, and its naturality descends as well. This proves every assertion in (c). The complete integral model comparisons are proved in `Formalism.MainNotions.N.Ln34.modell_pf`, `Formalism.MainNotions.N.Ln34.modelflink_pf`, and `Formalism.MainNotions.N.Ln34.modelqlink_pf`. □

Proof of 36.62

Proof. Write the c-component of a as $F(\pi')$, where $F(X) = \sum_{j\geq 0} a_j X^j \in O[[X]]$. Since $r^{2k} \in \ell^k O$, the following sums converge in O:

$$F(r) = \sum_{j\geq 0} a_j r^j, \qquad G_i = \sum_{j>i} a_j r^{j-i-1}.$$

The identity $F(X) = F(r) + (X-r)\sum_{i\geq 0} G_i X^i$ follows coefficient by coefficient. Repeat this division d times and substitute $X = \pi'$. This constructs the stated coefficients and remainder.

At a root r, the identity

$$\Phi_\ell(\zeta\exp(t)) = \frac{\exp(\ell t)-1}{\zeta\exp(t)-1}$$

shows that the evaluated cyclotomic polynomial has a simple zero at $t=0$. Its unit factor in $E_c(\pi')$ has nonzero constant coefficient. Factoring $E_c(X)$ by $X-r$ therefore proves that the evaluation of $\pi'-r$ has the form $tU(t)$ with $U(0)\neq 0$. Comparison of the first nonzero coefficient proves the last assertion. □

Proof of 36.63

Proof. Claim 1. The polynomial has the asserted factorization with unit leading coefficient. Both evaluation values are roots of the degree-two polynomial E_c. Divide successively by $X-r_U$ and $X-r_D$ to obtain the displayed factorization. Put $Q = \Phi_\ell(1+Z)$. Write $\pi' = wZ^s$ and $Q = w'E_\pi$, with w, w' units. Let $w_0' \in O^\times$ be the constant coefficient of w'. Choose $t_U, t_D \in O$ with $r_U^2 = \ell t_U$, $r_D^2 = \ell t_D$. The constant-coefficient identity is $\ell = w_0' a_2 r_U r_D$. Squaring and cancelling ℓ^2 gives

$$a_2\big(a_2(w_0')^2 t_U t_D\big) = 1,$$

so a_2 is a unit.

Claim 2. One has $\delta_0^2 \mid \ell$. Choose the root of unity $z \in O$ associated with r_U, so $z^\ell = 1$, $z \neq 1$, and $r_U = \pi'(z-1)$. Expanding in the exponential parameter $1 + Z = ze^t$ (so $Z = (z-1) + zt + O(t^2)$ and $\Phi_\ell(ze^t) = \frac{\ell}{z-1}t + O(t^2)$), let $u_0, u_1 \in O$ be the constant and linear t-coefficients of $w(ze^t - 1)$, and let $u_0' \in O$ be the constant term of $w'(ze^t - 1)$. Comparing linear coefficients in $Q = w'E_\pi$ gives $\frac{\ell}{z-1} = (z-1)^{s-1}\delta_0\Omega$, and multiplying by $z - 1$ gives

$$\ell = (z-1)^s \delta_0 \Omega, \qquad \Omega = -u_0' a_2 \big(u_1(z-1) + u_0 s z\big) \in O.$$

Since $2s = \ell - 1$, the product identity

$$\ell = \prod_{a=1}^{\ell-1}(1 - z^a) = (1-z)^{\ell-1} \prod_{a=1}^{\ell-1}(1 + z + \cdots + z^{a-1})$$

gives $(z-1)^{2s} = \ell t_0$ for some $t_0 \in O$. Indeed each factor in the last product is a unit: modulo the maximal ideal it is a, and $1 \leq a < \ell$. Squaring the preceding equality and cancelling the nonzero element ℓ yields $\ell = \delta_0^2 t_0 \Omega^2$, which proves the divisibility and valuation bound.

Claim 3. The asserted denominator formula is integral. For the last assertion, the numerical hypothesis gives $y \geq v$ and $2(y - v) \geq d$. The right side of the displayed formula therefore belongs to O. Substitution of $o_0 = x\ell^v c' \delta_0^d$ and $\delta_0^2 r_0 = \ell$ proves the equality.

Lean: `Formalism.MainNotions.N.Ln34.LevelExpansions.fact2` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.ExpansionLowCoeffsIntegral`, lines 252–327)

Lean: `Formalism.MainNotions.N.Ln34.LevelExpansions.caseC` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.PolyExpand`, lines 417–451)

□

Proof of 36.64

Proof (outline). Claim 1. Under the present level-one ring relations and normalized bound $h' \geq 1$, there is a frame as in Definition 36.60(b), and the reduction and lifting maps have all properties in (a). Fix a Δ-invariant A_*-basis $(b_j)_j$ of $N(T)$. The images $\bar{b}_j$ of its components at $c-1$ form an O-basis of $L_c(T)$, and the vectors $\varphi(b_j)$ form an $\mathscr{E}$-basis of $D(T)$. In component $c-1$ write $\gamma(\pi') = \pi' g(\pi')$, with $g(0) = \chi_\ell(\gamma)^s$, and let $1 + \pi' R$ be the matrix of γ whose m-th column gives the coordinates of $\gamma(b_m)$; here R has entries in $O[[\pi']]$ because $(\gamma - 1)N(T)^\Delta \subset \pi' N(T)$. By the valuation computed below, $\ell^{\sigma_A} = a_\sigma \prod_{j=1}^{M-1}(\chi_\ell(\gamma)^{sj} - 1)$ for some $a_\sigma \in O$. Solving the horizontality equations degree by degree in π', the coefficient of degree j is determined after division by $\chi_\ell(\gamma)^{sj} - 1$, and the factor ℓ^{σ_A} clears these divisions for $j < M$. This gives matrices H, H' over $O[[\pi']]$ with

$$H \equiv H' \equiv \ell^{\sigma_A} \pmod{\pi'}, \qquad (1 + \pi' R)H^\gamma \equiv H, \quad H'(1 + \pi' R) \equiv H'^\gamma \pmod{\pi'^M},$$
$$H'H \equiv HH' \equiv \ell^{2\sigma_A} \pmod{\pi'^M},$$

where H^γ is H with π' replaced by $\pi' g(\pi')$. Define $\theta_T : L_c(T) \to N(T)$ by $\theta_T(\bar{b}_j) = \sum_m H_{mj} b_m$ in component $c-1$. For $z \in N_{c-1}(T)^\Delta$ one has $(\gamma - 1)\theta_T(\bar{z}) \in \pi'^M N(T)$ and $\theta_T(\bar{z}) - \ell^{\sigma_A} z \in \pi' N(T)$. Let α_T send $\xi \otimes \bar{z}$ to the class of $\xi\varphi(\theta_T(\bar{z}))$, and let β_T send $\sum_k \xi_k \varphi(b_k)$ to the class of $\sum_j \big(\sum_k \xi_k \varphi(H'_{jk})\big) \otimes \bar{b}_j$, for $\xi, \xi_k \in \mathscr{E}$. Both maps are A^Δ-linear, α_T maps $\Lambda_c^L(T)$ into $\Lambda_c^P(T)$ and β_T maps $\Lambda_c^P(T)$ into $\Lambda_c^L(T)$, and each maps one denominator module into the other. The identity $\varphi(\pi')^M = \pi'^M E_\pi^{4h'} u$, with $u \in A^\Delta$, turns the π'^M-errors into elements of $J_c^L(T)$ and $J_c^P(T)$ after Frobenius, so the congruences for $H'H$ and HH' give $\bar{\beta}_T \bar{\alpha}_T = \bar{\alpha}_T \bar{\beta}_T = \ell^{2\sigma_A}\,\mathrm{id}$. The two horizontality congruences give the γ-compatibilities, since $\gamma(\pi') \in \pi' A^\Delta$ and $\gamma(E_\pi) \in E_\pi (A^\Delta)^\times$ make $J_c^L(T)$ and $J_c^P(T)$ stable under γ. For naturality let $f : T \to T'$ be equivariant. The vectors $f_P(\theta_T(\bar{z}))$ and $\theta_{T'}(f_0 \bar{z})$ are Δ-invariant, horizontal modulo π'^M, and congruent to $\ell^{\sigma_A} f_P(z)$ modulo π'. Their difference lies in $\pi' N(T')$ and is horizontal modulo π'^M, so the graded computation in the next paragraph, applied in component $c-1$, puts it in $\pi'^M N(T')$. This proves naturality of α. Naturality of β follows from it, the two compositions, and the implication $\ell^{2\sigma_A} x \in J_c^L(T') \Rightarrow x \in J_c^L(T')$ for $x \in \Lambda_c^L(T')$. For the expansion, π' is a unit times Z^s with $sM \geq 4h'$, and Q is a unit times E_π, so E_π vanishes at every evaluation pair. Expand $\theta_T(\bar{z})$ at $Z = \exp(t) - 1$ in component $c-1$. It is γ-invariant modulo $t^{4h'}$, γ multiplies t by $\chi_\ell(\gamma)$, and $\chi_\ell(\gamma)^i \neq 1$ for $i \geq 1$; hence its coefficients in degrees $1, \ldots, 4h' - 1$ vanish, and its

constant term is ℓ^{σ_A} times that of z. Frobenius carries this expansion to the evaluation pairs above c, and the error terms from $J_c^P(T)$ contribute to the expansion of $Q^k E_\pi^{h'} y$ only in degrees at least $k+4h'$. This gives the formula in Definition 36.60(b) for $i < k+4h'$. Thus these maps form the frame in (a).
Lean: `Formalism.MainNotions.N.Ln34.frame_pf` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.FramePf`, lines 64–68)

On $\pi'^j N_c(T)^\Delta/\pi'^{j+1}N_c(T)^\Delta$, $\gamma-1$ acts as $\chi_\ell(\gamma)^{sj}-1$. The graded modules are torsion-free over O. Consequently an invariant element of $\pi' N_c(T)^\Delta/\pi'^M N_c(T)^\Delta$ is zero. The cokernel of $\gamma-1$ on this module is killed by

$$\prod_{j=1}^{M-1}(\chi_\ell(\gamma)^{sj}-1), \qquad v_\ell\left(\prod_{j=1}^{M-1}(\chi_\ell(\gamma)^{sj}-1)\right)=\sum_{j=1}^{M-1}(1+v_\ell(j))=\sigma_A.$$

The empty product applies when $M=1$. Given a representative w of a class in $\mathcal{P}_c^{(1)}(T)$, solve for $k\in \pi' N_c(T)^\Delta$ so that

$$(\gamma-1)k \equiv \ell^{\sigma_A}(\gamma-1)w \pmod{\pi'^M N_c(T)^\Delta}.$$

Then $\ell^{\sigma_A}w-k$ is an invariant lift. The vanishing of the invariant kernel proves uniqueness after quotienting, and hence linearity and naturality. This proves (a).
Lean: `Formalism.MainNotions.N.Ln34.invLiftCondition` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.InvLiftPf`, lines 14–83)

Claim 2. For the fixed frame and normalized data of the proposition, there are a finite subset $\Omega_c^0\subset\Omega_c$, a finite free $L_r\subset L_c(T_2)$ of rank at most em, and vectors $z_1,\dots,z_k\in N_{c-1}(T_2)^\Delta$, independent of w, satisfying the spanning and upward-filtration assertions in (b). For every $w\in\mathcal{P}_c^{(1)}(T_2)$, the equality in (b) holds with d_c replaced by $d_c^0=\prod_{i\in\Omega_c^0}(1-\chi_\ell(\gamma)^i)$, and with slot coefficients and a residual vector satisfying all the stated membership conditions. The unprojected values $v_{c,\zeta,T_2}(z)$ are independent of the evaluation pair above c. They define an O-linear injection on $L_c(T_2)$, which remains injective after extending scalars to L'. Projection to the τ-component identifies their image with $D_{\mathrm{dR}}(T_2[1/\ell])_\tau$. Put $W_c=L'\otimes_O L_c(T_2)$, and let $U_\tau^a\subset W_c$ be the inverse image of $\mathscr{F}_\tau^a(T_2)$ under evaluation. The jump conditions give

$$\dim_{L'}(W_c/U_\tau^{\beta'_\tau})\le m \quad(\varsigma_\tau=1), \qquad \dim_{L'}U_\tau^{-\eta_1}\le m \quad(\varsigma_\tau=-1).$$

There are at most e embeddings above c. Decompose the residue lattice using the intersection of the upward subspaces and the sum of the downward central subspaces. This gives the generators $\bar z_j$ and a residual submodule L_r of rank at most em. Since $L_c(T_2)$ is finite free over the discrete valuation ring O, so is L_r.

The coefficient construction selects a finite subset $\Omega_c^0\subset\Omega_c$. Put $d_c^0=\prod_{i\in\Omega_c^0}(1-\chi_\ell(\gamma)^i)$. Apply $\prod_{i\in\Omega_c^0}(\gamma_L-\chi_\ell(\gamma)^i)$ to an invariant coordinate representative. On Laurent coefficients this removes the selected exponents; on $\mathcal{H}_c(T_2)$ it is multiplication by d_c^0. In the residue directions $\bar z_j$, the resulting coefficients have upward order at least $-\beta'_\tau$, and at a downward pair they are a constant plus a series of order at least $\beta'_\tau+S$. Their $\gamma-1$ differences have numerators divisible by π'. These are exactly the conditions in Definition 36.60(d), and the remaining coordinate lies in $\mathcal{I}_c^L(T_2;L_r)$.

Since $\Omega_c^0\subset\Omega_c$, one has $d_c=d_c^0 t_c$, where

$$t_c=\prod_{i\in\Omega_c\setminus\Omega_c^0}(1-\chi_\ell(\gamma)^i)\in O.$$

Apply this construction to $t_c w$. The invariant module is an O-module, and the submodule L_r and the vectors z_j were fixed before choosing w. It therefore gives admissible coefficients and the equality in (b) with d_c.
Lean: `Formalism.MainNotions.N.Ln34.u1link_pf` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.U1linkPf`, lines 16–67)
Lean: `Formalism.MainNotions.N.Ln34.FirstPropositionFrame` (`Formalism.SharpRestriction.LevelLetterCocycles.DownSlotProposition`, lines 216–231)
Lean: `Formalism.MainNotions.N.Ln34.LevelExpansions.valH_of` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.PiNSOfPi`, lines 179–235)
Lean: `Formalism.MainNotions.N.Ln34.kapPiPf_main` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.KapPiPf`, lines 217–283)

Claim 3. The finite-freeness, rank bound and injection in (c) hold for every O-submodule $S_0\subset L_c(T_2)$. Inclusion induces an injection

$$\mathcal{R}_c(T_2;S_0)\longrightarrow\mathcal{Q}_c(T_2).$$

Lemma 36.61(a) therefore gives finite generation and torsion freeness. More precisely, the image of $\Lambda_c^L(T_2;S_0)$ in $\mathcal{Q}_c(T_2)$ has a finite decreasing γ_L-stable filtration with terminal term zero. On its j-th graded quotient, γ_L acts by a scalar $\lambda_j \in O$; that quotient has rank at most $\operatorname{rank}_O S_0$, and at most $e+1$ indices have $\lambda_j = 1$. The filtration comes from successive scalar factors of $\pi' E_\pi^{2h'}$: division by each factor gives the rank bound on the graded term, and the scalar ring relations give the stated action and count. After extending scalars to L', a fixed vector on a graded term with $\lambda_j \neq 1$ is zero. Induction through the filtration therefore bounds the invariant subspace dimension by $(e+1)\operatorname{rank}_O S_0$, which bounds the rank of $\mathcal{R}_c(T_2;S_0)$. The injection into $\mathcal{H}_c(T_2)$ follows from the same inclusion and the same intersection with $\pi'\mathcal{M}_c(T_2)$.
Lean: `Formalism.MainNotions.N.Ln34.u2ring_pf` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.U2ringPfZeta0`, lines 42–73)
Lean: `Formalism.MainNotions.N.Ln34.RankBounds.u2rank_local` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.RankBounds`, lines 295–325)
Lean: `Formalism.MainNotions.N.Ln34.RankLink.rank_link_of_model` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.RankLinkProof`, lines 61–79)

Claim 4. The scalar a_c in (d) exists, and every elementary slot tensor has the asserted invariant membership. Every $i \in \Omega_c$ is nonzero and $|i| \le h'$. Counting multiples of ℓ^a in an interval of S consecutive integers gives

$$\sum_{i\in\Omega_c} (1 + v_\ell(|i|)) \le e\big(S + v_\ell(S!) + \lfloor \log_\ell h' \rfloor\big) = \kappa_A.$$

Since $v_\ell(1-\chi_\ell(\gamma)^i) = 1 + v_\ell(|i|)$, the scalar d_c is a unit times ℓ^v for an integer $v \le \kappa_A$. Multiplication by the inverse unit and by $\ell^{\kappa_A - v}$ gives a_c.

For an elementary slot tensor $f \otimes \bar z$, write $E_\pi^{h'}(\gamma f - f) = \pi' g$. At an upward pair, the expansion of g has order at least $h' - \beta'_\tau$, and $v_{c,\zeta,T_2}(z) \in \mathscr{F}_\tau^{\beta'_\tau}(T_2)$. At a downward pair the constant term in the Laurent expansion of f cancels under $\gamma - 1$, so the expansion of g has order at least $h' + \beta'_\tau + S$. Here $v_{c,\zeta,T_2}(z) \in \mathscr{F}_\tau^{-\beta'_\tau - S}(T_2)$ by the lower jump bound. The factor $\mathrm{ev}_{c,\zeta}(\pi')$ has nonzero constant coefficient. These inequalities put $E_\pi^{-h'} g \otimes \bar z$ in $\mathcal{M}_c(T_2)$, so the slot tensor belongs to $\mathcal{I}_c^L(T_2; L_c(T_2))$.

Claim 5. For the equivariant lattice surjection $F : T_1 \twoheadrightarrow T_2$ and the normalized jump bound $[-h', h']$ of the present hypotheses, every component $d \in \mathbf{Z}/n_K\mathbf{Z}$ satisfies

$$\ell^{\rho_A} N_d(T_2)^\Delta \subset F_P\big(N_d(T_1)^\Delta\big).$$

Equivalently, for every $y \in N_d(T_2)^\Delta$ there is $x \in N_d(T_1)^\Delta$ with $F_P x = \ell^{\rho_A} y$. Here ρ_A is the exponent defined in Lemma 36.57. The invariant averaging and Vandermonde-denominator argument in its proof applies to the present surjection.
Lean: `Formalism.MainNotions.N.Ln34.WKill` (`Formalism.SharpRestriction.LevelLetterCocycles.SplitROfSteps`, lines 21–22)
Lean: `Formalism.MainNotions.N.Ln34.w_pf` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.WLocal`, lines 312–384)

Apply Claim 5 with $d = c-1$ and $y = z$ to obtain $z_1 \in N_{c-1}(T_1)^\Delta$ with $F_P z_1 = \ell^{\rho_A} z$. When all signs are downward, the preceding argument puts $f \otimes \bar z_1$ in the invariant coordinate module, and naturality of the frame gives the required lift. When all signs are upward, $f \otimes \bar z \in \mathcal{M}_c(T_2)$; its image under α_{T_2} has a Wach representative. Lift that representative by Claim 5 at component c. The action of γ is trivial on the Wach lattice modulo π', so the resulting vector represents a class in $\mathcal{P}_c^{(1)}(T_1)$. Multiplying either lift by $\ell^{y_1 - \rho_A}$ gives the asserted exponent.

Claim 6. A source residue lift satisfying the invariant coordinate condition gives a slot lift at the same exponent. Precisely, suppose $a \ge 0$, $z_* \in N_{c-1}(T_1)^\Delta$, $F_P z_* - \ell^a z \in \pi' N_{c-1}(T_2)^\Delta$, and $f \otimes \bar z_*$ belongs to the invariant coordinate module. Choose $y_* \in \Lambda_c^P(T_1)$ satisfying

$$q_P(y_*) = \alpha_{T_1}(f \otimes \bar z_*).$$

The invariant clause of Lemma 36.61(c) gives $(\gamma - 1) y_* \in \pi' N_c(T_1)^\Delta$. Naturality of the frame gives $q_P(F_P y_* - \ell^a y) = 0$. The height bound $E_\pi^{h'} P_c^\varphi(T_2) \subset N_c(T_2)^\Delta$ implies

$$J_c^P(T_2) = \pi'^M E_\pi^{3h'} P_c^\varphi(T_2) \subset \pi'^M N_c(T_2)^\Delta \subset \pi' N_c(T_2)^\Delta.$$

Hence $F_P y_* - \ell^a y \in \pi' N_c(T_2)^\Delta$, which is the required slot lift.
Lean: `Formalism.MainNotions.N.Ln34.IsFrame` (`Formalism.SharpRestriction.LevelLetterCocycles.DownSlotProposition`, lines 122–151)

Lean: Formalism.MainNotions.N.Ln34.ModelF (Formalism.SharpRestriction.LevelLetterCocycles.DownSlotProposition, lines 198–211)

Claim 7. For mixed signs and $e = 2$, the slot lifting conclusion of (d) holds at exponent y_1. For mixed signs with $e = 2$, let $r_{\zeta,\tau} = [t^0]\,\mathrm{ev}_{c,\zeta}(\pi')$. The relation $\pi' = wZ^s$, with w a unit and $2s = \ell - 1$, gives $r_{\zeta,\tau} = w_c(\zeta - 1)(\zeta - 1)^s \in O$. The cyclotomic identity $(\zeta - 1)^{\ell-1} \in \ell O$ therefore gives $\ell \mid r_{\zeta,\tau}^2$. First suppose that every evaluation has the same value as some downward evaluation. Apply Lemma 36.62 to the numerator of $E_\pi^{h'}(\gamma f - f)/\pi'$. Its downward vanishing order transfers to every evaluation with that value. The lower jump bound for T_1 then puts $f \otimes \bar{z}_1$ in the invariant coordinate module, for the previously chosen lift $F_P z_1 = \ell^{\rho_A} z$. This gives a lift at exponent ρ_A.

Otherwise choose an upward evaluation (ζ_U, τ_U) whose value differs from every downward value, and choose a downward evaluation (ζ_D, τ_D). Put

$$r_U = r_{\zeta_U,\tau_U}, \quad r_D = r_{\zeta_D,\tau_D}, \quad b = \beta'_{\tau_U}, \quad B = \beta'_{\tau_D} + S, \quad \delta_0 = r_D - r_U.$$

The normalized jump inequalities give $1 \le B \le h'$ and $b \le h'$. There are two embeddings above c. Every upward embedding is τ_U, every downward embedding is τ_D, and every evaluation value is r_U or r_D. Lemma 36.63 gives $E_c(X) = a_2(X - r_U)(X - r_D)$, with $a_2 \in O^\times$ and $\delta_0^2 \mid \ell$.

Suppose first that $B + 2\eta_1 < b$. Set

$$a_0 = \epsilon_c(\pi' - r_U)^{h'-b}(\pi' - r_D)^{h'+B}.$$

The upward and downward orders put $(a_0/E_\pi^{h'}) \otimes \bar{z}$ in $\mathcal{M}_c(T_2)$. Apply α_{T_2}, choose its Wach representative, lift that representative through F_P after multiplication by ℓ^{ρ_A}, and apply β_{T_1}. Write the resulting model representative as

$$\nu = \sum_j (E_\pi^{-h'} a_j) \otimes \bar{z}_j, \qquad a_j \in A^\Delta, \quad z_j \in N_{c-1}(T_1)^\Delta.$$

For an evaluation pair, write $v_j = v_{c,\zeta,T_1}(z_j)$ and $v'_j = v_{c,\zeta,T_2}(F_P z_j)$. The model filtration conditions give $\sum_j [t^i]\,\mathrm{ev}(a_j) v_j \in \mathscr{F}_\tau^{h'-i}(T_1)$. Naturality and the composition identity of Definition 36.60(b) give, for $0 \le i \le h'$,

$$\sum_j [t^i]\,\mathrm{ev}(a_j) v'_j = \ell^{\rho_A + 2\sigma_A}[t^i]\,\mathrm{ev}(a_0) v_{c,\zeta,T_2}(z).$$

The denominator terms have order at least $4h'$, so they contribute no coefficient in this range.

Lean: Formalism.MainNotions.N.Ln34.LevelExpansions.compare (Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.TmulMkLMemInvLOfValAtMemFilC, lines 302–381)

The coefficient extraction in the next two divisions is triangular. At a root put $Y = \mathrm{ev}(\pi' - r) = tU(t)$, where $U(0) \neq 0$. If a vector series has expansion $\sum_j P_j Y^j$, then

$$[t^k] \sum_j P_j Y^j = U(0)^k P_k + \sum_{j<k} u_{kj} P_j$$

for scalars $u_{kj} \in L'$. Induction on k, and the inclusions $\mathscr{F}^{h'-j} \subset \mathscr{F}^{h'-k}$ for $j < k$, recover the filtration condition on P_k. The same induction recovers coefficient identities. Division by a power series with nonzero constant coefficient uses this recurrence as well.

Divide each a_j at r_D, using Lemma 36.62, through degree $h' - \eta_1 - 1$:

$$a_j = \sum_{k < h' - \eta_1} p_{jk}(\pi' - r_D)^k + (\pi' - r_D)^{h'-\eta_1} a'_j$$

on component c. For each such k, the downward filtration condition puts $\sum_j p_{jk} v_j$ in $\mathscr{F}_{\tau_D}^{\eta_1+1}(T_1)$. Its projection to the τ_D-component is zero. Projection to this component is injective on the evaluation image, so the entire evaluation vector is zero. The kernel of evaluation on $N_{c-1}(T_1)^\Delta$ is $\pi' N_{c-1}(T_1)^\Delta$, hence $\sum_j p_{jk} \bar{z}_j = 0$. Naturality gives zero in the target residue lattice as well. The unprojected evaluation vector is independent of the evaluation pair above c, so both zero identities also hold at the upward evaluation. Thus these polynomial terms disappear from both coefficient identities. At the upward evaluation, $\pi' - r_D$

has nonzero constant coefficient, so division by its $(h'-\eta_1)$-st power preserves the filtration conditions and the identities through degree h'.

Put $d_* = h' - b + B + \eta_1$, and divide each a'_j at r_U with remainder exponent d_*+1:

$$a'_j = \sum_{k=0}^{d_*} q_{jk}(\pi' - r_U)^k + (\pi' - r_U)^{d_*+1} a''_j.$$

Let $o_j = q_{j,d_*} \in O$, and define

$$z'_1 = \sum_j o_j z_j.$$

After the first division, the target polynomial is

$$\ell^{\rho_A+2\sigma_A} X^{h'-b}(X + r_U - r_D)^{B+\eta_1}.$$

Its coefficient of degree d_* is $\ell^{\rho_A+2\sigma_A}$. Therefore

$$F_P z'_1 - \ell^{\rho_A+2\sigma_A} z \in \pi' N_{c-1}(T_2)^{\Delta}.$$

The source coefficient belongs to $\mathscr{F}^{h'-d_*}_{\tau_U}(T_1) \subset \mathscr{F}^{\eta_1+1}_{\tau_U}(T_1)$. There are no jumps between η_1+1 and b, so it belongs to $\mathscr{F}^{b}_{\tau_U}(T_1)$. Independence of the unprojected evaluation vector gives this filtration condition at every evaluation with embedding τ_U, including those whose value of π' is r_D. Apply the source-residue construction above with $z_* = z'_1$ and $a = \rho_A + 2\sigma_A$. This constructs the invariant source slot at that exponent.

Lean: `Formalism.MainNotions.N.Ln34.LevelExpansions.ValH` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.PiNSOfPi`, lines 103–120)
Lean: `Formalism.MainNotions.N.Ln34.LevelExpansions.valH_of` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.PiNSOfPi`, lines 179–235)
Lean: `Formalism.MainNotions.N.Ln34.LevelExpansions.caseB` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.PolyExpand`, lines 352–411)

Now suppose $b \le B + 2\eta_1$. Write $f = a/E_\pi^{h'}$ on component c, and let $b_0 \in L'$ be its downward constant. Division at r_U gives $a = (\pi' - r_U)^{h'-b}A$ with A integral. At r_D, the defining congruence for f gives

$$A \equiv b_0 a_2^{h'}(\pi' - r_U)^b(\pi' - r_D)^{h'} \pmod{(\pi' - r_D)^{h'+B}}.$$

Consequently, comparison of the coefficient of degree $h'+j$, for $0 \le j < B$, gives

$$b_0 a_2^{h'} \binom{b}{j} \delta_0^{b-j} \in O.$$

If $b < B$, take $j = b$, so $b_0 \in O$ and set $y_0 = 0$. If $B \le b$, take $j = B-1$. The factorial formula for the valuation of the binomial coefficient gives $v_\ell\binom{b}{B-1} \le \lfloor \log_\ell h' \rfloor$, and hence

$$b - B + 1 + 2v_\ell\binom{b}{B-1} \le 2\eta_1 + 1 + 2\log_\ell h' \le 2y_c(\eta_1).$$

Lemma 36.63, followed by division by the unit $a_2^{h'}$, constructs $o = \ell^{y_0} b_0 \in O$ with $y_0 = y_c(\eta_1)$.

Define the corrected numerator

$$a_N = \ell^{y_0} a - o\epsilon_c E_\pi^{h'}.$$

Its upward order is at least $h' - b$, and its downward order is at least $h' + B$. The transfer assertion in Lemma 36.62 supplies these conditions at every evaluation of the respective root. Thus

$$(\ell^{y_0} f - o\epsilon_c) \otimes \bar{z} \in \mathcal{M}_c(T_2).$$

Lift this model term using its Wach representative and lift the constant slot $o\epsilon_c \otimes \bar{z}$ using z_1. Both require the factor ℓ^{ρ_A}. Their sum gives a lift at exponent $\rho_A + y_0$. The two cases are bounded by $\rho_A + \max(2\sigma_A, y_c(\eta_1)) = y_1$.

Lean: `Formalism.MainNotions.N.Ln34.LevelExpansions.mix_local` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.MixLocal`, lines 33–336)

Claim 8. There is a set $\Sigma_c \subset Z_f(T_2)$ of cardinality at most $e(e+1)m$ satisfying (e). Let $x_1, \dots, x_d \in \mathcal{I}_c^L(T_2; L_r)$ represent an O-basis of $\mathcal{I}_{\mathrm{rest}}$, so $d \le e(e+1)m$. Put $v_i = \alpha_{1,T_2}([x_i])$, choose a representative Y_i of $\mathrm{lift}_c(v_i)$, and complete it to a Herr pair (Y_i, Z_i) with $Z_i \in U_M(T_2)$. Choose a finite cocycle κ_i representing its class and put $\Sigma_c = \{\kappa_1, \dots, \kappa_d\}$.

Let κ be a generator of $\mathcal{J}_c(T_2)$, represented by a Herr pair (w, z) as in Lemma 36.57(c), and write $\bar{w}$ for the reduction of $[w]$ to $\mathcal{P}_c^{(1)}(T_2)$. Parts (b) and (c) give $X \in \mathcal{C}_{\mathrm{lift}}(T_2, c)$ and $b_i \in O$ with

$$\beta_{1,T_2}(d_c \bar{w}) = X + \sum_i b_i [x_i].$$

Apply α_{1,T_2}, and then multiply by $a_c \ell^{y_1}$. Part (d) supplies $u_1 \in \mathcal{P}_c^{(1)}(T_1)$ such that

$$\ell^{2\sigma_A + \kappa_A + y_1} \bar{w} = F_* u_1 + \sum_i a_c \ell^{y_1} b_i v_i.$$

Apply lift_c, and choose a representative W_1 of $\mathrm{lift}_c(u_1)$. By (a), with $A = \sigma_A + 2\sigma_A + \kappa_A + y_1$,

$$w' = \ell^A w - F_P W_1 - \sum_i a_c \ell^{y_1} b_i Y_i \in U_M(T_2).$$

Complete W_1 to a Herr pair (W_1, Z_1), where $Z_1 \in U_M(T_1)$, and choose a finite cocycle κ_1' representing it. The corresponding second-coordinate difference

$$z' = \ell^A z - F_P Z_1 - \sum_i a_c \ell^{y_1} b_i Z_i$$

also belongs to $U_M(T_2)$. Set $v = (\varphi - 1)^{-1} w'$. Since (w', z') is a Herr pair and $\varphi - 1$ is injective on $U_M(T_2)$, one has $z' = (\gamma - 1)v$. This pair has zero cohomology class. Naturality of the Herr class map gives

$$\ell^A \kappa - F(\kappa_1') - \sum_i a_c \ell^{y_1} b_i \kappa_i \in B(T_2).$$

Extend this relation O-linearly over the defining span of $\mathcal{J}_c(T_2)$. This proves (e).

The complete proofs are contained in `Formalism.MainNotions.N.Ln34.frame_pf` and `Formalism.MainNotions.N.Ln34.invLiftCondition` for (a), `Formalism.MainNotions.N.Ln34.u1link_pf` for (b), `Formalism.MainNotions.N.Ln34.RankBounds.u2rank_local`, `Formalism.MainNotions.N.Ln34.RankLink.rank_link_of_model` and `Formalism.MainNotions.N.Ln34.kapPiPf_main` for (c), `Formalism.MainNotions.N.Ln34.LevelExpansions.mix_local` for (d), and `Formalism.MainNotions.N.Ln34.univlink_pf` for (e). □

Proof of 36.65

Proof. Let ℓ be odd and $K_0 \subset K \subset K_0(\mu_\ell)$ as in Definition 36.56; for $\ell \in \{q, d\}$ and any place $w \mid \ell$ of E, Lemma 36.31 shows that ℓ is odd and $K = E_w$ is of this form with $e = [K : K_0] = 2$ and $s = (\ell - 1)/2$.

Fix admissible data $(T_1, T_2, F, r, S, \eta, m, h, \varsigma_\tau, \beta_\tau)$ at K. By Lemma 36.57(a), $es = \ell - 1$ and there exist normalized integers $0 \le \eta_1 \le \eta$, $1 \le h' \le h + S + 1$, and β_τ' $(\tau : K \to L')$ satisfying $\eta_1 < \beta_\tau'$ and $\beta_\tau' + S \le h'$, such that the jumps of T_1 and T_2 lie in the corresponding shifted and central intervals inside $[-h', h']$ and T_2 has at most m jumps in $[-\eta_1, \eta_1]$ at each τ. Choose (π', E) as in Lemma 36.57(b), and define M, ρ_A, σ_A and the submodules $\mathcal{J}_c(T_2) \subset Z_f(T_2)$ $(c \in \mathbf{Z}/n_K\mathbf{Z})$ as in Lemma 36.57(c), together with $\kappa_A, y_c(\eta_1), y_1$ as in Proposition 36.64.

For each $c \in \mathbf{Z}/n_K\mathbf{Z}$, Proposition 36.64(e) gives a subset $\Sigma_c \subset Z_f(T_2)$ of cardinality at most $e(e+1)m$ such that $\ell^{\sigma_A + 2\sigma_A + \kappa_A + y_1} \mathcal{J}_c(T_2) \subset I_F + O\Sigma_c$. Put $\Sigma_0 = \bigcup_{c \in \mathbf{Z}/n_K\mathbf{Z}} \Sigma_c \subset Z_f(T_2)$; then

$$|\Sigma_0| \le \sum_{c \in \mathbf{Z}/n_K\mathbf{Z}} |\Sigma_c| \le n_K e(e+1)m.$$

Set $a=\ell^{\rho_A+\sigma_A}$ and $b=\ell^{\sigma_A+2\sigma_A+\kappa_A+y_1}$. For any $\kappa\in Z_f(T_2)$, Lemma 36.57(c) writes $a\kappa=\sum_{c\in\mathbf{Z}/n_K\mathbf{Z}}x_c+\beta_0$ with $x_c\in\mathcal{J}_c(T_2)$ and $\beta_0\in B(T_2)\subset I_F$. Multiplying by b gives $bx_c\in I_F+O\Sigma_c\subset O\Sigma_0+I_F$ for every c and $b\beta_0\in I_F$, whence $ba\kappa\in O\Sigma_0+I_F$.

Lean: `Formalism.MainNotions.N.Ln34.annihilation_of_scaled_spans` (`Formalism.SharpRestriction.LevelLetterCocycles.MixedPositionHypothesis`, lines 81–101)

Comparing exponents with the formula of Lemma 36.30 gives

$$(\rho_A+\sigma_A)+\big(\sigma_A+2\sigma_A+\kappa_A+\rho_A+\max(2\sigma_A,y_c(\eta_1))\big)=2\rho_A+4\sigma_A+\kappa_A+\max\big(2\sigma_A,y_c(\eta_1)\big)=Y(\ell,e,s,h',S,\eta_1).$$

Since $\eta_1\le\eta$, we have $y_c(\eta_1)\le y_c(\eta)$ and therefore $Y(\ell,e,s,h',S,\eta_1)\le Y(\ell,e,s,h',S,\eta)$. Multiplying $ba\kappa\in O\Sigma_0+I_F$ by $\ell^{Y(\ell,e,s,h',S,\eta)-Y(\ell,e,s,h',S,\eta_1)}$ yields $\ell^{Y(\ell,e,s,h',S,\eta)}Z_f(T_2)\subset O\Sigma_0+I_F$. This establishes statement (A) at K.

Lean: `Formalism.MainNotions.N.Ln34.YA_split` (`Formalism.SharpRestriction.LevelLetterCocycles.MixedPositionHypothesis`, lines 70–74)

Lean: `Formalism.MainNotions.N.Ln34.YA_mono` (`Formalism.SharpRestriction.LevelLetterCocycles.MixedPositionHypothesis`, lines 76–79)

Lean: `Formalism.MainNotions.N.Ln34.killR_of_steps` (`Formalism.SharpRestriction.LevelLetterCocycles.MixedPositionHypothesis`, lines 105–170)

□

Proof of 36.66

Proof. Generators. Put $V_2=T_2[1/\ell]$. The quotient $Z_f(T_2)/B(T_2)$ is the integral finite cohomology $H^1_f(K,T_2)$, a submodule of $H^1(K,T_2)$. The latter is finitely generated over O, by Theorem 36.14 for the finite modules T_2/ϖ^nT_2, the Mittag–Leffler property of their cohomology and Nakayama's lemma, where ϖ is a uniformizer of O. Its torsion submodule is the image of $H^0(K,V_2/T_2)$ under the connecting map, which kills the image of $H^0(K,V_2)$. The maximal divisible submodule D of $H^0(K,V_2/T_2)$ is a direct summand with finite complement Q'. The image of $H^0(K,V_2)$ lies in D, and D modulo this image is finite and divisible, hence zero. Thus the torsion is isomorphic to Q'. Since $Q'[\varpi]\subset(V_2/T_2)[\varpi]\simeq(O/\varpi)^r$, the torsion needs at most r generators. By (36.20), the rank of $H^1_f(K,T_2)$ is $\dim_{L'}H^0(K,V_2)+\dim_{L'}D_{\mathrm{dR}}(V_2)/\mathrm{Fil}^0\le r+n_Ker$. Hence $Z_f(T_2)$ is generated modulo $B(T_2)$ by at most $(n_Ke+1)r+r\le 2(n_Ke+1)r$ elements.

Lean: `Formalism.MainNotions.N.Ln34.genA` (`Formalism.SharpRestriction.LevelLetterCocycles.LocalFormOfThreeInputs`, lines 13–19)

Lifting exponent. Put $K'=K_0(\mu_\ell)$. It contains K and equals $\mathbf{Q}_\ell(\mu_{\ell N})$ with $N=\ell^{n_K}-1$ prime to ℓ. Hence its depth in Lemma 36.19 is one, and $[K':K]=s$ is prime to ℓ. Apply the proof of the first assertion of Lemma 36.22 over K' to F, whose source jumps lie in $[-h,h]$. It scales a finite target class c to $c_\gamma P(1)\ell^{2w}c$, with $w=4h+v_\ell((4h)!)$, and lifts the result to a finite source class. The valuation of the scalar is the left side of (A.516) at depth one:

$$E'=2L(4h)+\sum_{j\in J^*}(1+v_\ell(j))+L(h),\qquad L(k)=k+v_\ell(k!).$$

Here $J^*=((J+[1,h])\setminus\{0\})\cup[1,h]$ for the union $J\subset[-h,h]$ of source jump sets, so $J^*\subset[-h,2h]\setminus\{0\}$. Restriction to K' followed by corestriction multiplies classes by s, a unit of O, and corestriction preserves finite classes. Hence $\ell^{E'}Z_f(T_2)\subset I_F$.

Lean: `Formalism.MainNotions.N.Ln34.liftA` (`Formalism.SharpRestriction.LevelLetterCocycles.LocalFormOfThreeInputs`, lines 21–26)

Lean: `Formalism.MainNotions.ElementDenominator.CongruenceModules.expLift` (`Formalism.SharpRestriction.LevelLetterCocycles.LocalTheoremOfGoodExtensions`, lines 153–155)

Lean: `Formalism.WeaklyAdmissibleFiltrations.LatHom.LiftF` (`Formalism.WeaklyAdmissibleFiltrations.Statement`, lines 265–266)

Conclusion. Let $C=Z_f(T_2)/I_F$ and let C_0 be the image of $O\Sigma_0$, with h' and Σ_0 from (A). Since $B(T_2)\subset I_F$, the module C is generated by at most $2(n_Ke+1)r$ elements. The submodule C_0 is generated by at most $n_Ke(e+1)m$ elements. Statement (A) gives $\ell^YC\subset C_0$, and the lifting exponent gives $\ell^{E'}C_0=0$. The integers satisfy $es=\ell-1$, $1\le h'\le h+S+1$ and $J^*\subset[-h,2h]\setminus\{0\}$. Lemma 36.30 with $n=n_K$ gives the bound.

Lean: `Formalism.WeaklyAdmissibleFiltrations.LatHom.col` (`Formalism.WeaklyAdmissibleFiltrations.Statement`, lines 262–263)

□

Proof of 36.68

Proof. The Hecke algebra separates the points of X_i, so its eigenspaces have direct sum. Commutation with the Hecke algebra makes every member of $\mathcal{A}$ preserve each V_x. If $u = m+v \in U_{\mathcal{S}}$, with $m \in M_i$ and $v \in V_{\mathcal{S}^c}$, then $\alpha u = \alpha m \in M_i \cap V_{\mathcal{S}} = L_{\mathcal{S}}$ for every $\alpha \in \mathcal{A}$.

Order $\mathcal{S} = \{\pi_1, \dots, \pi_q\}$, set $F_j = \bigoplus_{a \le j} V_{\pi_a}$, and put $U_j = U_{\mathcal{S}} \cap F_j$, $L_j = L_{\mathcal{S}} \cap F_j$. Let $p_j : F_j \to V_{\pi_j}$ be the projection with kernel F_{j-1}. The preceding inclusion and preservation of F_j give

$$a_{\pi_j} p_j(U_j) \subset p_j(L_j) \subset p_j(U_j).$$

The module $p_j(U_j)$ is a free O_i-lattice of rank $\dim_{K_i} V_{\pi_j}$. Consequently

$$\lambda_i\big(p_j(U_j)/p_j(L_j)\big) \le \dim_{K_i} V_{\pi_j}\, \lambda_i(O_i/a_{\pi_j}O_i).$$

The sequence

$$0 \longrightarrow U_{j-1}/L_{j-1} \longrightarrow U_j/L_j \xrightarrow{p_j} p_j(U_j)/p_j(L_j) \longrightarrow 0$$

is exact. Add its length identities for $1 \le j \le q$. The empty set gives the zero quotient. □

Proof of 36.69

Proof. For each $x \in Y \setminus Z(i)$, choose $f_x \in T_i$ with $f_x(\pi) \ne f_x(x)$. The product of $f_x - f_x(x)$, with empty product 1, vanishes on $Y \setminus Z(i)$ and is nonzero at π. Thus $\mathfrak{c}_{Y,\pi} \ne 0$. Since O_i is a discrete valuation ring, write $\mathfrak{c}_{Y,\pi} = cO_i$ with $c \ne 0$; the definition of the ideal supplies g.

Both g and β commute with the Hecke algebra, so α does also. It acts by cb on V_π. For $x \notin Y$, it vanishes because β vanishes. For $x \in Y \setminus Z_{\text{good}}(i)$, the assumption on $Y \cap Z(i)$ gives $x \notin Z(i)$, so $g(x) = 0$. Since β preserves V_x, α vanishes there too. Finally,

$$\lambda_i(O_i/cbO_i) = \lambda_i(O_i/cO_i) + \lambda_i(O_i/bO_i) \le v_{Y,\pi} + v_{g_k}(M_0).$$

□

Proof of 36.70

Proof. Put $A = H^1_f(K, L_{U'})$. Local Tate duality and the exact-annihilator assertion in Definition 36.15 give an injection

$$H^1(K, U/L_U)/H^1_f(K, U/L_U) \longrightarrow \operatorname{Hom}_O(A, L/O).$$

Coboundaries lie in Z^1_f, so $(S + Z^1_f)/Z^1_f$ injects into the ℓ^e-torsion of the target. This torsion is $\operatorname{Hom}_O(A/\ell^e A, L/O)$.

The module A is finitely generated. Let ϖ be a uniformizer of O. The exact sequence for multiplication by ϖ on $L_{U'}$ gives a surjection

$$H^0(K, L_{U'}/\varpi L_{U'}) \longrightarrow H^1(K, L_{U'})[\varpi].$$

The source has residue-field dimension at most r_U. Hence the torsion submodule of A has at most r_U generators. If $d = \operatorname{rank}_O A$, the structure theorem over O gives

$$\operatorname{length}_O(A/\ell^e A) \le e(d + r_U)\operatorname{length}_O(O/\ell O).$$

The dual of a finite-length O-module in L/O has the same length. Divide by $\operatorname{length}_O(O/\ell O)$ to obtain the assertion. □

Proof of 36.71

Proof (outline). The characters in Σ factor through a common finite quotient of I of order prime to ℓ. Averaging over this quotient shows that distinct character eigenspaces have direct sum. Thus the displayed decomposition is equivalent to the sum condition on the inertia eigenspaces. We use the prismatic and Fontaine inputs of Definition 36.15 and the local cohomological inputs of Theorem 36.14. The coefficient action on a crystalline lattice is retained in its prismatic realization: first regard the lattice as a finite free $\mathbf{Z}_\ell$-module, and then lift its scalar endomorphisms by the uniqueness of morphisms in the prismatic comparison.

Lean: `GlobalTheory.bhattScholze_coeff` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.BhattScholzeCoefficientsOfPadicIntLattices`, lines 98–164)

Lean: `GlobalTheory.AdmAllRecords.localLifting_admittedHypotheses_all_alt` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalLiftingAdmittedHypothesesAll`, lines 18–23)

Claim 1. Every ratio $\chi\psi^{-1}$, with $\chi, \psi \in \Sigma$, has Frobenius orbit length at most $N^2/4$.

Let b and c be the orbit lengths of χ and ψ. If they lie in different orbits, their ratio has period dividing $\operatorname{lcm}(b, c)$, and

$$\operatorname{lcm}(b, c) \le bc \le (b + c)^2/4 \le N^2/4.$$

If they lie in the same orbit of length b, the ratio has period dividing b. An orbit other than $\{1\}$ has length at most $N-1$, since 1 is a fixed point outside it, and $N-1 \le N^2/4$. For $\{1\}$, the ratio has period $1 \le N^2/4$. Taking the maximum proves the claim.

Lean: `GlobalTheory.BK.maxPeriod_bounds` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.MaxFrobeniusOrbitLengthDiffBoundsZmod`, lines 125–146)

Claim 2. Finite extension of the coefficient field preserves all the hypotheses of the lemma, the Frobenius orbit lengths, and the jump set. If the asserted finite-cocycle lifting inclusion holds after such an extension, it holds over the original coefficient field with the same exponent.

Let L_1/L be finite, with integer ring O_1, and replace T_j by $O_1 \otimes_O T_j$. The map on character values is injective; it preserves distinct characters, products, Frobenius orbits and the inertia decomposition. Crystalline periods and their filtrations commute with this coefficient extension. For descent, O_1/O is torsion-free over O: if $0 \ne a \in O$, $x \in O_1$, and $ax \in O$, then $x \in L \cap O_1 = O$. Hence the finite free O-module O_1 has an O-linear retraction onto O fixing 1. Extend it L-linearly to L_1. Applying this retraction to an integral lifting identity preserves cocycles, coboundaries and the finite condition, and returns the original target cocycle. This proves descent without changing the exponent.

Lean: `GlobalTheory.BaseCoef.thmT_descend` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TheoremTDescendSmallerCoefficients`, lines 21–36)

Put $a = (\ell - 1)!$ and $e = \ell^a - 1$. Then $a, e > 0$, $\ell \nmid a$, and $\ell \nmid e$. By Claim 2 we may enlarge L to contain the unramified extension of $\mathbf{Q}_\ell$ of degree a. Its Teichmüller units contain a primitive e-th root of unity. Choose a compatible root system

$$\pi_0^e = -\ell, \qquad \pi_{j+1}^\ell = \pi_j \quad (j \ge 0),$$

and write I_ℓ for the usual inertia subgroup of $G_{\mathbf{Q}_\ell}$, and define $\omega(g) = g(\pi_0)/\pi_0 \in \mu_e \subset L^\times$ for $g \in I_\ell$. For a residue $c \in \mathbf{Z}/e\mathbf{Z}$, the expression $\omega(g)^c$ is independent of its integer representative.

Lean: `GlobalTheory.TameCoeff.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsCoeffFieldContainsUnramifiedWittTameDescen`, lines 10–27)

Lean: `GlobalTheory.BKReal.PiSys` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.IsAdicCompleteOfPowLe`, lines 245–249)

The numerical hypothesis gives $40(HN^2 + N^3) < \ell$ and $N < \ell$. In particular every positive integer at most N divides a.

Claim 3. For the period datum, Frobenius and character set of the statement, with $N < \ell$, the tame-period construction supplies an auxiliary finite tame extension $K_t/\mathbf{Q}_\ell$ containing π_0 and a finite set $A \subset \mathbf{Z}/e\mathbf{Z}$ such that

$$\ell A = A, \qquad 0 \in A, \qquad |A| \le N, \qquad R(A) = r_{\mathrm{orb}},$$

where $R(A)$ is the maximum length of an orbit of a difference $c - d$, $c, d \in A$, under multiplication by ℓ. Every V crystalline over K' whose period module is the sum of its Σ-eigenspaces is crystalline over K_t, and, for every $g \in I_\ell$, its period module over K_t satisfies

$$\prod_{c \in A} \bigl(g - \omega(g)^c\bigr) D_{\mathrm{cris}, K_t}(V) = 0.$$

Here g acts on the crystalline periods through the given Galois action on $V \otimes B_{\mathrm{cris}}$.
Lean: `GlobalTheory.GoodLatt.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsTameFieldGoodLatticePotentiallyCrystalline`, lines 337–443)
Lean: `GlobalTheory.BKGal.TameType` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.IsPotential`, lines 117–121)

To obtain this set, restrict the characters to I_ℓ and write them as powers of ω. Frobenius conjugation becomes multiplication by ℓ on the exponents. The orbit lengths are at most N and divide a; the resulting exponents therefore belong to $\mathbf{Z}/e\mathbf{Z}$. The inertia decomposition for the period datum preserves the orbit lengths of their ratios. Inertia in G_{K_t} has $\omega(g) = 1$, so it fixes every crystalline period over K'. Crystalline descent gives crystallinity over K_t; applying the same eigenspace decomposition to $g \in I_\ell$ gives the displayed annihilating polynomial. In particular Claim 3 applies to both V_1 and V_2.

Claim 4. Retain a, e, L, π_j, A, and the same auxiliary field K_t chosen by the construction in Claim 3. Suppose

$$0 \in A, \quad \ell A = A, \quad |A| \le N, \quad N \ge 2, \quad 0 \in J \subset [-\eta', H], \quad |J| \le N{+}1, \quad 1 \le \eta' < H, \quad 40(HN^2{+}N^3) < \ell.$$

Let $f : T_1 \twoheadrightarrow T_2$ be an equivariant surjection of stable O-lattices. If both rational representations are crystalline over K_t, satisfy the polynomial condition of Claim 3 for A, and have all jumps in J, then

$$\ell^{7(NR(A)+\eta'(N+R(A)))} Z^1_f(T_2) \subset f_* Z^1_f(T_1) + B^1(T_2).$$

Lean: `GlobalTheory.T.lift_surjection_good_crystalline_lattices` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LiftSurjectionGoodCrystallineLatticesStatement`, lines 10–11)
Lean: `GlobalTheory.TheoremT.liftF_of_isGood` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LiftSurjectionGoodCrystallineLattices`, lines 27–107)
Lean: `GlobalTheory.IndefiniteRowSideFullHecke.ConductorExponents.Numerics` (`Development.AbelianVarietyTheory.BlockTheory.NonGaloisCusps.ConductorExponents`, lines 20–30)

The integral construction in Claim 4 first takes the stable kernel $T_0 = \ker f$ and works over the base ring $\mathfrak{S}_O = O[[X]]^a$ (where O is the ring of integers of L, $K = L$, $a = (\ell-1)!$, $e = \ell^a - 1$, Frobenius φ acts by $X \mapsto X^\ell$ together with the cyclic shift on $\mathbf{Z}/a\mathbf{Z}$, and $E(X) = X^e + \ell$, the Eisenstein polynomial of π_0 over $\mathbf{Z}_\ell$, which over O is monic with all roots of absolute value $\ell^{-1/e}$), together with the Breuil divided-power ring $S_O \subset K[[X]]^a$ of power series $\sum_{k\ge0} a_k X^k$ with $(\lfloor k/e \rfloor)!\, a_k \in O$ tending to 0 as $k \to \infty$ (so $\mathfrak{S}_O \subseteq S_O$), its j-th divided-power Eisenstein filtration ideal $\mathrm{Fil}^j S_O$ (the ℓ-adic closure in S_O of the ideal generated by $E(X)^i/i!$ for $i \ge j$), and the analytic module $D_j^{\mathrm{an}} = D_j \otimes_{\mathfrak{S}_O} R_O$, where $R_O \subset K[[X]]^a$ consists of the tuples of power series $\sum_{k\ge0} a_k X^k$ with $|a_k|\ell^{-k/e} \to 0$, that is, of functions converging on the closed disc $|X| \le \ell^{-1/e}$, which contains all roots of E. Since $\ell^{-k/e} \le |\ell|^{\lfloor k/e \rfloor} \le |(\lfloor k/e \rfloor)!|$, we have $S_O \subseteq R_O$. Realising the exact triple $0 \to T_0 \to T_1 \to T_2 \to 0$ via Bhatt–Scholze crystalline lattices produces finite free $\mathfrak{S}_O$-modules $0 \to D_0 \xrightarrow{F} D_1 \xrightarrow{G} D_2$, exact at D_0 and D_1 and with finite-length cokernel of G, equipped with semilinear Frobenius φ, tame descent action of the group $\mu_e \rtimes \mathbf{Z}/a\mathbf{Z} \simeq \mathrm{Gal}(K_t/\mathbf{Q}_\ell)$, whose inertia subgroup μ_e acts on X via ω with character set $\subseteq A$ and whose quotient $\mathbf{Z}/a\mathbf{Z}$ is generated by a Frobenius element acting on μ_e by $\zeta \mapsto \zeta^\ell$ (tame-invariant vectors are those fixed by the whole group, Frobenius elements included), logarithmic connection conn, and Frobenius height sandwich $E^H D_j \subseteq L_j$ and $E^{\eta'} L_j \subseteq D_j$ (where $L_j = \varphi^* D_j$). Write $\mathcal{M}_1 = L_1 \otimes_{\mathfrak{S}_O} S_O$ for the S_O-extension of the Frobenius image lattice L_1 and $\mathrm{Fil}^{10H} \mathcal{M}_1 = (\mathrm{Fil}^{10H} S_O) \cdot \mathcal{M}_1$ for its $10H$-th filtration submodule. The triple satisfies $G \circ F = 0$. Put $r = R(A) = r_{\mathrm{orb}}$ and $e_W = 2(N-1)r$. Under the numerical hypotheses of Claim 4, since the cokernel of G has finite length, D_1 has all jumps in J and tame characters in A, and D_2 has tame characters in A, this cokernel is killed by ℓ^{e_W} on the lattices, on the tame invariants and on the modules Y and $X^e Y$ introduced below: every element v of one of these modules for D_2 satisfies $\ell^{e_W} v = G(u)$ for some u in the corresponding module for D_1. On each D_j, define the formal monodromy series $\zeta(y) = -\sum_{k\ge1} (\ell\varphi)^k(\mathrm{conn}(y)) \in X \cdot D_j \otimes_{\mathfrak{S}_O} K[[X]]^a$ (satisfying $\zeta(y) = \ell\varphi(\zeta(y) - \mathrm{conn}(y))$) and the monodromy obstruction $\mathrm{Mon}(y) = [E \cdot \mathrm{conn}(y) - E \cdot \zeta(y)]$ in the polar quotient $D_j^{\mathrm{Pol}} = (D_j \otimes_{\mathfrak{S}_O} K[[X]]^a)/D_j^{\mathrm{an}}$; thus $\mathrm{Mon}(y) = 0$ holds precisely when $E(\mathrm{conn}(y) - \zeta(y))$ lies in D_j^{an}, meaning the φ-module extension defined by y is crystalline. The Breuil–Kisin comparison attaches a cocycle class to every element of $Y(D_j)$, the O-module of tame-invariant vectors y with $E^{\eta'} y \in D_j$. For $y \in Y(D_j)$ choose m_0 in the $W(\mathbf{C}^\flat)$-realization of D_j with $\varphi(m_0) - m_0 = y$, and a tame-invariant $\sigma \in (L_j \otimes_{\mathfrak{S}_O} S_O)[1/\ell]$ with $\mathrm{conn}(\sigma) = \zeta(y)$. Let $b_0 \in T_j \otimes_{\mathbf{Z}_\ell} B_{\mathrm{cris}}$ be the crystalline value of $\varphi(m_0)$, let σ_{cris} be the image of σ in $T_j \otimes_{\mathbf{Z}_\ell} B_{\mathrm{cris}}$, and put $b = \sigma_{\mathrm{cris}} - b_0$. The class of y is the class of the cocycle $c(g) = g(b) - b$. Changing

m_0 by an element $x \in T_j$ changes c by the coboundary of $-x$, and two choices of σ differ by a horizontal tame-invariant vector, whose image is fixed by G_{K_t} because it is horizontal and by G_{K_∞} because it is tame-invariant, hence by $G = G_{K_t} G_{K_\infty}$ (see the factorization below); hence c is determined by y modulo $B^1(T_j)$. Multiplying y by ℓ^k multiplies the class by ℓ^k, and replacing y by $y + \varphi(m) - m$ for a tame-invariant $m \in D_j$ leaves the class unchanged. Put $e_L = (N+1)\eta'$, $c_1 = \eta'(r+1)$, and $c_2 = c_1 + e_W$. Given $c \in Z^1_f(T_2)$, the period comparison represents $\ell^{e_W} c$ by a monodromy-zero element $y_2 \in Y(D_2)$; prime-power divisibility of the tame descent datum gives $\ell^{\eta'} Y(D_2) \subseteq X^e Y(D_2) + D_2^{\mathrm{inv}}$, where D_j^{inv} is the module of tame-invariant elements of D_j, and so decomposes $\ell^{\eta'} y_2 = y_2' + m_2$ with $y_2' \in X^e Y(D_2)$ and $m_2 \in D_2^{\mathrm{inv}}$, so subtracting the $(1-\varphi)$-coboundary $(1-\varphi)m_2 = m_2 - \varphi(m_2)$ shows that $y_2' + \varphi(m_2) = \ell^{\eta'} y_2 - (1-\varphi)m_2$ has zero monodromy and represents $\ell^{\eta' + e_W} c$. Write $Z(D_j) \subset D_j^{\mathrm{Pol}}$ for the O-module of tame-invariant classes of vectors v with $E^{\eta'} v \in D_j$; obstructions are measured in this integral lattice. The cokernel bound gives $y_1' \in X^e Y(D_1)$ and $m_1 \in D_1^{\mathrm{inv}}$ with $G(y_1') = \ell^{e_W} y_2'$ and $G(m_1) = \ell^{e_W} m_2$. For these lifts the tame-orbit monodromy estimate on D_1 gives $\ell^{c_1} \operatorname{Mon}(y_1' + \varphi(m_1)) \in Z(D_1)$.

Lean: `GlobalTheory.T.breuilKisinDescent_pow_smul_monodromy_add_phi_mem` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.BreuilKisinDescentPowSmulMonodromyAddPhiMemStatement`, lines 6–7)

The image of this obstruction under G vanishes, so pulling it back along F costs a further ℓ^{e_W}: there is $w_0 \in Z(D_0)$ with $F(w_0) = \ell^{c_2} \operatorname{Mon}(y_1' + \varphi(m_1))$. Solving the monodromy equation in D_0 costs $\ell^{\eta' + e_L}$: with $n = c_2 + \eta' + e_L$ there is $y_0 \in X^e Y(D_0)$ with $F(\operatorname{Mon}(y_0)) = \ell^n \operatorname{Mon}(y_1' + \varphi(m_1))$. Put $y' = \ell^n y_1' - F(y_0) \in X^e Y(D_1)$ and $m = \ell^n m_1 \in D_1^{\mathrm{inv}}$. Then $y = y' + \varphi(m) \in Y(D_1)$ has zero monodromy, and since $G \circ F = 0$, $G(y) = \ell^{k_{\exp}}(y_2' + \varphi(m_2))$ with $k_{\exp} = n + e_W = c_2 + \eta' + e_L + e_W = 4(N-1)r + \eta'(N + r + 3)$. To pass from this monodromy-zero element $y \in Y(D_1)$ back to a finite Galois cocycle $\kappa_1 \in Z^1_f(T_1)$, set $w = E(\zeta(y) - \operatorname{conn}(y))$. Because $\operatorname{Mon}(y) = 0$, we have $w \in D_1^{\mathrm{an}}$, and since $c_{\mathrm{fr}} \in S_O$ satisfies $c_{\mathrm{fr}} \varphi(E) = \ell$, we obtain $\zeta(y) = \ell\varphi(\zeta(y) - \operatorname{conn}(y)) = c_{\mathrm{fr}} \varphi(E) \varphi(\zeta(y) - \operatorname{conn}(y)) = c_{\mathrm{fr}} \varphi(w)$ identically. By the tame-orbit bound $\ell^{c_1} \zeta(\varphi(m)) \in \mathcal{M}_1$, the Breuil-ring splitting $S_O = \mathfrak{S}_O + \mathrm{Fil}^{10H} S_O$ (for $10H < \ell$, giving $\mathcal{M}_1 = L_1 + \mathrm{Fil}^{10H} \mathcal{M}_1$), the fine bound $\zeta(y') = \ell^{\ell - 14H} p + \ell^{-3H} f$ with $p \in L_1$ and $f \in \mathrm{Fil}^{10H} \mathcal{M}_1$, valid for every $y' \in X^e Y(D_1)$, and the decomposition $E^{2H} E \operatorname{conn}(y) \in L_1 + \mathrm{Fil}^{10H} \mathcal{M}_1$, valid since $E^{\eta'} y \in D_1$, there exist $P_0 \in L_1$ and $F_0 \in \mathrm{Fil}^{10H} \mathcal{M}_1$ such that $\ell^{c_1} E^{2H} w = P_0 + \ell^{-3H} F_0$. Every element of $\mathrm{Fil}^{10H} S_O$ has the form $E^{10H}\theta$ with $\theta \in R_O$, and $E^{\eta'} L_1 \subseteq D_1$ with $\eta' < H$ puts $E^{-2H} \mathrm{Fil}^{10H} \mathcal{M}_1 \subseteq E^{8H - \eta'} D_1^{\mathrm{an}} \subseteq D_1^{\mathrm{an}}$; hence $Q = \ell^{-3H} E^{-2H} F_0 \in D_1^{\mathrm{an}}$. Writing $X_0 = E^{-2H} P_0$ gives $\ell^{c_1} w = X_0 + Q$, so $X_0 = \ell^{c_1} w - Q$ lies in D_1^{an}, while $P_0 \in L_1$ and $E^{\eta'} L_1 \subseteq D_1$ give $E^{2H + \eta'} X_0 = E^{\eta'} P_0 \in D_1$. Division by E preserves integrality in R_O: if $h = \sum_k h_k X^k \in R_O$ and $Eh = u \in O[[X]]$, then comparing coefficients in $u = X^e h + \ell h$ gives $h_k = u_{k+e} - \ell h_{k+e}$, and since $\ell^i h_{k+ie} \to 0$ as $i \to \infty$, we get $h_k = \sum_{i \geq 0} (-\ell)^i u_{k + (i+1)e} \in O$. Applying this $2H + \eta'$ times to each coordinate of X_0 in an $\mathfrak{S}_O$-basis of D_1 forces $X_0 \in D_1$; consequently $\varphi(X_0) \in L_1 \subseteq L_1 \otimes_{\mathfrak{S}_O} S_O = \mathcal{M}_1$ directly without any ℓ-denominator. Meanwhile, for $F_0 \in \mathrm{Fil}^{10H} \mathcal{M}_1$ (with $10H < \ell$), Frobenius divisibility on $\mathrm{Fil}^{10H} S_O$ combined with $E^{\eta'} L_1 \subseteq D_1$ yields $\ell^{\eta'} \varphi(F_0) = \ell^{10H} v'$ for some $v' \in \mathcal{M}_1$, so $\varphi(F_0) = \ell^{3H} \cdot \ell^{2H} \cdot \ell^{5H - \eta'} v'$ (with $5H - \eta' > 0$); using $\ell^{2H} \varphi(E^{-2H}) = c_{\mathrm{fr}}^{2H}$ then gives $\varphi(Q) = c_{\mathrm{fr}}^{2H} \ell^{5H - \eta'} v' \in \mathcal{M}_1$. Therefore $\ell^{c_1} \zeta(y) = c_{\mathrm{fr}}(\varphi(X_0) + \varphi(Q)) \in \mathcal{M}_1 = L_1 \otimes_{\mathfrak{S}_O} S_O$. Let $b = \sigma_{\mathrm{cris}} - b_0$ be the potential of y constructed above, so that $\ell^{c_1} \operatorname{conn}(\sigma) = \ell^{c_1} \zeta(y) \in \mathcal{M}_1$. Every $g \in G_{\mathbf{Q}_\ell}$ factors as $g = hk$ with $h \in G_{K_t}$ and k in the subgroup G_{K_∞}, where $K_\infty = \mathbf{Q}_\ell(\pi_j^e : j \geq 0)$ is the field generated by the compatible ℓ-power roots π_j^e of $-\ell$; it does not contain π_0. Such k acts on the realization through the descent datum, so it fixes y and σ_{cris}; hence $k(m_0) - m_0$ is fixed by φ, lies in T_1, and $k(b) - b \in T_1$. For $h \in G_{K_t}$, the element $h(b) - b$ is fixed by φ and lies in Fil^0 because $\operatorname{Mon}(y) = 0$, so by Fontaine's theorem $B_{\mathrm{cris}}^{\varphi = 1} \cap \mathrm{Fil}^0 B_{\mathrm{cris}} = \mathbf{Q}_\ell$ it lies in $T_1[1/\ell]$; the bound $\ell^{c_1} \operatorname{conn}(\sigma) \in \mathcal{M}_1$, compared inside A_{cris}, gives $\ell^{c_1}(h(b) - b) \in T_1$. The cocycle identity $g(b) - b = h(k(b) - b) + (h(b) - b)$ then gives $\ell^{c_1}(g(b) - b) \in T_1$ for every g. The cocycle $g \mapsto g(b) - b$ is split by b over B_{cris}, so $\kappa_1(g) = \ell^{c_1}(g(b) - b)$ lies in $Z^1_f(T_1)$ and represents $\ell^{c_1} y$, at lifting exponent $c_{\mathrm{lift}} = c_1 = \eta'(r+1)$. Naturality of the Breuil–Kisin realisation equates $\ell^{c_{\mathrm{lift}} + k_{\exp} + \eta' + e_W} c$ with $f_* \kappa_1$ modulo an integral coboundary, and for all $N \geq 2$, $r \geq 1$, $\eta' \geq 1$ the total exponent satisfies $c_{\mathrm{lift}} + k_{\exp} + \eta' + e_W = 6(N-1)r + \eta'(N + 2r + 5) \leq 7(Nr + \eta'(N + r))$ (since $6(N-1)r < 7Nr$ and $6N + 5r \geq 17 > 5$). Multiplying by the remaining power of ℓ gives Claim 4.

The inputs to Claim 4 are now supplied by Claim 3 and the original jump hypotheses; extension of the local field does not change the jumps after passing to embeddings. Substitute $R(A) = r_{\mathrm{orb}}$. Since $C_T \geq 40 \geq 7$, multiplication by $\ell^{(C_T - 7)(N r_{\mathrm{orb}} + \eta'(N + r_{\mathrm{orb}}))}$ gives the desired inclusion over the enlarged coefficient field, and Claim 2 descends it to O. Claim 1 supplies the orbit inequality. The complete proof is contained in `GlobalTheory.TheoremTClosing.main` and `GlobalTheory.InertiaEigenvalues.theoremT_all_records`. □

Proof of 36.72

Proof. Claim 1. Every subrepresentation W of a finite direct sum of the V_x is crystalline over K', has all jumps in J, and has crystalline period module equal to the sum of its Σ-eigenspaces. The Fontaine comparison in $\mathscr{H}_2$ is exact on crystalline representations and strict for the de Rham filtration. Finite direct sums therefore preserve crystallinity and add jump multiplicities. The crystalline category is closed under subrepresentations, and the filtered exact sequence for W identifies each of its graded pieces with a subspace of the corresponding graded piece of the ambient sum. Hence no new jump occurs. On crystalline periods the inertia action on the ambient sum factors through the common finite prime-to-ℓ quotient on which the characters in Σ are defined. For each character, its averaging idempotent preserves every inertia-stable subspace. The sum of these idempotents is the identity on the ambient period module, so it is the identity on the period module of W. This proves the eigenspace assertion, including zero eigenspaces for characters which do not occur.

Lean: `GlobalTheory.StepLiftPf.sub_props` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LiftFMonoLoc`, lines 63–99)

Claim 2. The exponent furnished by Lemma 36.71 for $L_1 \twoheadrightarrow L_2$ is at most e_{fam}. By Claim 1, the two lattices satisfy that lemma with the original K', Σ, J, C_T, N, η', H; its extension and period-stability hypotheses are unchanged. The maximum ratio-orbit length is an integer, and its inequality gives

$$r_{\mathrm{orb}} \le \lfloor N^2/4 \rfloor = r', \qquad e_0 = C_T\big(N r_{\mathrm{orb}} + \eta'(N + r_{\mathrm{orb}})\big) \le e_{\mathrm{fam}}.$$

The lemma gives $\ell^{e_0} Z^1_f(L_2) \subset f_* Z^1_f(L_1) + B^1(L_2)$. The right side is an O-submodule, so multiplication by $\ell^{e_{\mathrm{fam}}-e_0}$ proves the required inclusion.

Lean: `GlobalTheory.StepLiftPf.liftFam_core` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LiftFMonoLoc`, lines 111–144)

Lean: `GlobalTheory.StepLiftPf.liftF_mono` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LiftFMonoLoc`, lines 30–35)

If the family is empty, every finite direct sum and every such W is zero, so the inclusion holds as well. □

Proof of 36.73

Proof (outline). Work over the local coefficient field at w, with integer ring O and fraction field L, and use length divided by $\mathrm{length}_O(O/\ell O)$. The coefficient identifications in the statement identify this length with λ_i. For a continuous finite-dimensional L-representation X, write $X^* = X^\vee(1)$ for its Tate dual. If $L_U \subset U$ is a stable lattice, let $L_U^* \subset U^*$ be its dual lattice for the perfect pairing with values in $O(1)$.

For representations X, U, define $\mathscr{P}(X, U; L_U)$ to be the O-span of the cocycles

$$g \longmapsto \phi(y(g)),$$

where $L_X \subset X$ is any stable lattice, $y \in Z^1_f(E_w, L_X)$, and $\phi : L_X \to U/L_U$ is an equivariant O-linear map. The finite-cocycle module in U/L_U includes the coboundaries, as in Definition 36.15.

Lean: `Formalism.MainNotions.BlochKatoDuality.spanPush` (`Formalism.SharpRestriction.LevelLetterCocycles.BlochKatoDuality`, lines 60–62)

Lean: `Formalism.WeaklyAdmissibleFiltrations.HfD` (`Formalism.WeaklyAdmissibleFiltrations.StatementTor`, lines 126–127)

Claim 1. Suppose U is de Rham. Assume either that ℓ^e lifts finite cohomology across every surjection of stable lattices in subrepresentations of finite sums of $X \oplus U$, or that X is de Rham and the analogous lifting property holds for $X^* \oplus U^*$. Then

$$\lambda_i\big((\mathscr{P}(X, U; L_U) + Z^1_f)/Z^1_f\big) \le e\big(\mathrm{rank}_O\, H^1_f(E_w, L_U^*) + \mathrm{rank}_O\, L_U\big).$$

The hypotheses include the canonical perfect Tate-dual pairings on X, X^* and U, U^*, and the chosen lattices are dual for the latter pairing.

Lean: `Formalism.MainNotions.BlochKatoDuality.Hyp` (`Formalism.SharpRestriction.LevelLetterCocycles.BlochKatoDuality`, lines 78–83)

Lean: `Formalism.MainNotions.BlochKatoDuality.LiftAll` (`Formalism.SharpRestriction.LevelLetterCocycles.BlochKatoDuality`, lines 55–58)

Lean: `Formalism.MainNotions.BlochKatoDuality.PushoutSpanLengthBound` (`Formalism.SharpRestriction.LevelLetterCocycles.BlochKatoDuality`, lines 95–102)

Lean: `Formalism.MainNotions.BlochKatoDuality.second_proposition_every_localisation` (`Formalism.SharpRestriction.LevelLetterCocycles.RamificationTwoAndConjugationAtQAndD`, lines 1405–1407)

Lean: GlobalTheory.ClosingStatement.main (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalAbsorptionLengthBoundOfBlochKato, lines 15–25)

Here one uses the identity twist on X, so no hypothesis on an auxiliary character is needed. The direct alternative is applied to the pullback lattice

$$L_\phi = \{(x,u) \in L_X \oplus U : \phi(x) = u \bmod L_U\}, \qquad 0 \longrightarrow L_U \longrightarrow L_\phi \longrightarrow L_X \longrightarrow 0.$$

Its rational representation is $X \oplus U$. The dual alternative uses the dual lattice sequence and the exact annihilator property for the integral finite conditions. In both cases the local duality assertion is applied with precisely the alternatives in Claim 1. The resulting length deduction includes the torsion in integral finite cohomology, as proved in Lemma 36.70.

Claim 2. Retain the given type-locus family and its common potentially crystalline extension. Fix a finite list $x_1, \dots, x_b \in X_s$, a vector $E \in \bigoplus_j V_{x_j}$, and a finitely generated stable O_i-submodule Y containing $(g-1)E$ for every $g \in G_E$. Let X be the local coefficient-field span of Y. Every cut cocycle from these data, with target U/L_U, belongs to $\mathscr{P}(X, U; L_U)$.

Lean: GlobalTheory.T.pushouts_mem_span_localSelmerPushouts (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.PushoutsMemSpanLocalSelmerPushoutsStatement, lines 10–11)
Lean: GlobalTheory.Pass.LocPush.main (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.PushoutsMemSpanLocalSelmerPushouts, lines 243–254)

Indeed, the local integer span of the image of Y is a stable lattice in X. The map $g \mapsto (g-1)E$ takes values in this lattice. Its rational class is finite: if $E \notin X$, it is the class of

$$0 \longrightarrow X \longrightarrow X + LE \longrightarrow L \longrightarrow 0,$$

whose middle representation is a subrepresentation of the common potentially crystalline sum; if $E \in X$, it is a coboundary. Over the common crystalline extension, Fontaine comparison makes this class finite. Restriction and corestriction descend the rational finite condition. Extension of the coefficient map from Y to its local integer span identifies the original cut cocycle with the pushout of this integral finite cocycle.

Claim 3. For every finite set $\mathcal{F}$ of cut cocycles with target U/L_U, there is a representation X, an equivariant injection

$$j : X \longrightarrow \bigoplus_{q=1}^{b} V_{x_q}|_{G_{E_w}}, \qquad x_q \in X_s,$$

and a stable complementary subspace C such that

$$\bigoplus_{q=1}^{b} V_{x_q}|_{G_{E_w}} = j(X) \oplus C, \qquad O\mathcal{F} \subset \mathscr{P}(X, U; L_U).$$

Lean: GlobalTheory.Pass.LocDual.IsSummandTw (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.IsSummandTw, lines 19–26)
Lean: GlobalTheory.T.localCocycles_mem_span_pushout_summandLattice (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalCocyclesMemSpanPushoutSummandLatticeStatement, lines 9–19)
Lean: GlobalTheory.Pass.LocDual.OneX.main (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalCocyclesMemSpanPushoutSummandLattice, lines 40–63)

To construct common data, concatenate the finite spectral lists, pair the vectors E, and take the direct sum of their modules Y. Projection onto a summand recovers each original coefficient map. For the resulting global coefficient-field span of Y, the good Galois-family condition supplies an invariant complement inside the finite sum. Local coefficient extension gives the displayed complement C. Its local integer span is a stable lattice; the finite cocycle in Claim 2 and the extended coefficient maps realize every member of $\mathcal{F}$. The complement is retained: dualizing the direct-sum decomposition embeds X^* in a finite sum of the Tate-dual family.

In the direct case, the representations X in Claim 2 and U are subrepresentations of finite sums of the given family. Every subrepresentation of a finite sum of $X \oplus U$ is consequently a subrepresentation of a finite sum of that family, so the assumed lifting property gives the first alternative of Claim 1. The common crystalline extension makes U de Rham, and also makes its Tate dual de Rham. By (36.20) and the injection $H^0(E_w, U^*) \hookrightarrow \mathrm{Fil}^0 D_{\mathrm{dR}}(U^*)$,

$$\operatorname{rank}_O H^1_f(E_w, L_U^*) \le k, \qquad \operatorname{rank}_O L_U = k,$$

since $[E_w : \mathbf{Q}_\ell] = 1$. Claim 1 therefore bounds the pushout span by $2ke$. The concatenation just described makes the fixed-data pushout sets a directed family. Every finite collection of cut cocycles occurs in one member, so the same bound holds for their whole span.
Lean: `GlobalTheory.Pass.LocDirect.liftAll_of_liftFamW` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LiftAllOfBlochKatoLiftingFamilyAtPlace`, lines 116–129)
Lean: `GlobalTheory.Pass.LocDirect.dualLat_rank` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.DualLatticeRankBound`, lines 96–109)
Lean: `GlobalTheory.T.exists_merged_module_pushouts_contain_both` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsMergedModulePushoutsContainBothStatement`, lines 8–9)

In the dual case, apply Claim 3 to a finite set $\mathcal{F}$ of cut cocycles. Both X and U are de Rham. The explicit complement for X, and the semisimple decomposition giving U as a constituent of V_π, put X^* and U^* in subrepresentations of finite sums of the dual family. The assumed lifting property gives the second alternative of Claim 1. A dual lattice for L_U has

$$\operatorname{rank}_O H^1_f(E_w, L_U^*) + \operatorname{rank}_O L_U \le 2k,$$

by the same dimension argument. Thus the quotient of $O\mathcal{F} + Z^1_f$ by Z^1_f has length at most $2ke$. Every finitely generated submodule of the quotient for $S_{w,U}$ is contained in one of these finite spans. Length is the supremum of their lengths, which proves the assertion.
Lean: `GlobalTheory.SummandDual.xdual` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.PotentialFamilySummandRestriction`, lines 204–215)
Lean: `GlobalTheory.Pass.LocDual.liftAll_of_liftFam` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LiftAllSubSumsOfLiftFamilyMatrixFamily`, lines 51–63)
Lean: `GlobalTheory.Pass.LocDualU.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CrystallinePotentialFamily`, lines 33–71)

The complete proofs are contained in `GlobalTheory.Pass.LocDirect.main` and `GlobalTheory.Pass.LocDual.main`. □

Proof of 36.74

Proof (outline). Claim 1. Under $\mathscr{H}_2$, the member has a primewise Wach system and its Fontaine comparison. For the parameters of Lemma 36.71, its lifting exponent is valid for the stated potentially crystalline families and their duals. For a rank-k target, an exponent e for finite-cohomology lifting in either family gives the local cut-cocycle bound $2ke$ over $\mathbf{Q}_\ell$. The finite classes of the nontrivial cyclotomic character blocks are coboundaries, and those of the trivial block are unramified.

At the member, the prismatic F-crystal input of $\mathscr{H}_2$ supplies a primewise Wach system with its Fontaine functor and comparison. Fix an integer $C_T \ge 40$ and a real $C' \ge 8C_T$. The four local inputs used below are derived consequences of $\mathscr{H}_2$. The local lifting input uses prismatic coefficients, Fontaine comparison and local cohomological hypotheses obtained from Bloch–Kato duality, local Tate duality and the Euler characteristic formula. The pushout input uses the rational and integral finite-condition orthogonality, the top-degree Tate pairing, local finiteness and de Rham duality. The character-block and full local Bloch–Kato inputs are derived from the same local cohomological inputs. First use Lemma 36.71 with parameters N, η', H. It gives the local lifting input for the specified extension and period datum, with the prime-to-ℓ condition on the inertia characters stated explicitly. The numerical hypotheses give

$$160H \le C_T H N^2 < C_T(HN^2 + N^3) < \ell, \qquad H < \ell/160,$$

since $C_T \ge 40$ and $N \ge 2$. For fixed $N, \eta', r_{\mathrm{orb}}$, the upper jump H enters the range condition and leaves the exponent in Lemma 36.71 unchanged. Second, by Bloch–Kato duality over $K/\mathbf{Q}_\ell$, if ℓ^e lifts finite cohomology across lattice surjections in subrepresentations of finite sums of de Rham $X \oplus U$ (or of $X^\vee(1) \oplus U^\vee(1)$), then for dual lattices $L_U \subset U$ and $L_{U'} \subset U^\vee(1)$, the O-span S of pushouts of integral finite cocycles of lattices in X along equivariant maps to U/L_U satisfies $\lambda_O((S+Z^1_f)/Z^1_f) \le e(\operatorname{rank}_O H^1_f(K, L_{U'}) + \operatorname{rank}_O L_U)$. Lemma 36.70 gives the length deduction from the absorption condition $\ell^e S \subset Z^1_f$, including the contribution from the torsion of integral finite cohomology. Third, Lemma 36.22 gives the character-block inclusions on divisible cyclotomic characters at level one. Fourth, Definition 36.15 and Theorem 36.14, including the local Euler characteristic identity (36.19), give the local Bloch–Kato finiteness and duality conditions at every place.
Lean: `GlobalTheory.T.crystallinePeriodBound_all_constant_ge_forty` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CrystallinePeriodBoundAllConstantGeFortyStatement`, lines 11–12)

Lean: GlobalTheory.T.local_absorption_length_bound_of_blochKato (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalAbsorptionLengthBoundOfBlochKatoStatement, lines 8–9)
Lean: GlobalTheory.T.theorem_tb_of_local_blochKato_duality_euler (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TheoremTbOfLocalBlochKatoDualityEulerStatement, lines 8–9)
Lean: GlobalTheory.T.blochKatoLocal_all_of_tateDuality_eulerChar (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.BlochKatoLocalAllOfTateDualityEulerCharStatement, lines 8–9)

Claim 2. Fix an index i of nonempty good spectrum with $\ell_i = g_k$. Under the norm-choice coefficient-largeness and residual-irreducibility conditions, for every realizing family (R_x), every rank-r family (ρ_x), and every $\pi \in Z(i)$ with

$$R_\pi \simeq \rho_\pi \oplus \bigoplus_{c=1}^{t} \chi_{i,c},$$

there are block-separating projector data at π with denominator valuation at most δ_X. Their chosen cyclotomic-generator element γ_0 has the following property at every $w \mid \ell_i$: if $\gamma_0 \in I_w$, then γ_0 lies in wild inertia at w. These data have the characteristic-polynomial, quotient-relation and coboundary properties used in Claim 7.

We now show how the coefficient-largeness hypothesis supplies the choice. Call i live when it admits a Hecke realization with the displayed decomposition and an absolutely irreducible residual rank-r block. The residual block of ρ_π is a residual realization: reduce the characteristic-polynomial identity of the decomposition and cancel its monic character factor from the full Hecke polynomial. Its kernel is open, since continuity passes to a rational direct summand by clearing denominators in its matrix coefficients. Residual irreducibility therefore makes i live. The rank bounds give $r \geq 2$, and matrix-algebra density gives full residual matrix span.

Lean: GlobalTheory.BlockJumpsCompare.continuous_rho (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SecondClosedOfFirstClosed, lines 94–98)
Lean: GlobalTheory.OrthogonalDesignFromSize.main (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.RowFiveOneTypePrimeOfCoefRingTakenAtLive, lines 13–41)
Lean: GlobalTheory.Pass.residualRep_spans_matrixAlgebra_of_goodPoint (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ResidualRepSpansMatrixAlgebraOfGoodPoint, lines 15–24)

At a live index, coefficient largeness supplies a reference realizing family over a finite discrete valuation ring $A/\mathbf{Z}_{\ell_i}$, an integral Hecke algebra over A, and, for each $\pi \in Z(i)$, its two complementary block projectors and a separating block operator. The reference data include a cyclotomic-generator element with the wild-inertia property above; the associated monic character-block polynomials split over O_i. Their scalar extensions give the required projector data for the reference family. This is precisely the realization clause in the norm-choice coefficient-largeness condition.

Lean: GlobalTheory.CoefDuty.Live (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.IsWildInertiaCyclotomicElement, lines 72–75)
Lean: GlobalTheory.CoefDuty.TakenWild (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.IsWildInertiaCyclotomicElement, lines 85–93)
Lean: GlobalTheory.CoefDuty.TakenAll (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.IsWildInertiaCyclotomicElement, lines 126–126)
Lean: GlobalTheory.CoefDuty.Large (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.IsWildInertiaCyclotomicElement, lines 128–128)
Lean: GlobalTheory.ConsExists.consExist (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CharacterValuesInjectiveLength, lines 325–400)

The split Frobenius density of Lemma 36.24, equality of Hecke polynomials and semisimplicity conjugate the reference family to the given family over the fraction field. Cancelling the common character summands conjugates their rank-r complements. Transport the projector data by these conjugacies. This transport preserves the chosen element γ_0, so it preserves the wild-inertia property. This proves the choice asserted in Claim 2 for every realizing family, with the same denominator bound. The first provision supplies integral characteristic-polynomial coefficients and residual eigensystem constancy; the congruence-module construction then gives the remaining properties in Claim 2. At a cut, the inherited coefficient-largeness condition supplies exactly the same argument.

Lean: GlobalTheory.ConsEveryFamilyWild.main (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsGlobalObjectsEveryFamilyWild, lines 15–28)
Lean: UpperTheory.GalS.duties_of_large_aux41 (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.AdmissibleJOfRestricted, lines 25–27)
Lean: GlobalTheory.typePrimeCongruences_forall (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LemmaOpenOfIntegrality, lines 16–17)

Lean: `GlobalTheory.ClosingLocalInputs.galoisFamilyConditions_of_sumRepresentations_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.AdmG14OfAux`, lines 13–14)

Per-point type-locus cuts. Claim 3. If Γ is a finite group, $e \in \mathbf{C}[\Gamma]$ is a central idempotent, and ζ is a primitive $|\Gamma|$-th root of unity, then every coefficient of $|\Gamma|e$ belongs to $\mathbf{Z}[\zeta]$.

For $h \in \Gamma$, put $x = h^{-1}e$. Centrality and idempotence give $x^{|\Gamma|+1} = x$. Thus the eigenvalues of left multiplication by x on $\mathbf{C}[\Gamma]$ are zero or $|\Gamma|$-th roots of unity, so its trace belongs to $\mathbf{Z}[\zeta]$. That trace is $|\Gamma|$ times the coefficient of h in e.

Lean: `GlobalTheory.CentralIdem.centralIdemCoeff` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TraceMemAdjoin`, lines 49–61)

Apply Claim 3 to $\Gamma \simeq \mathrm{GL}_r(\mathbf{F}_{g_0})^f \times \mathrm{GL}_r(\mathbf{F}_{g_1})^f$, of order $M_0 = |\,\mathrm{GL}_r(\mathbf{F}_{g_0})|^f\,|\,\mathrm{GL}_r(\mathbf{F}_{g_1})|^f$. The coefficient-root and integral-action identifications in the given type datum carry $M_0 e$ to an integral central operator. Global Artin reciprocity is used separately in the unramified character block of Claim 7.

Lean: `GlobalTheory.T.no_cut_passes_of_artin_reciprocity` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.StepsNoCutOfArtinStatement`, lines 11–12)

Claim 4. Under the local series compatibility, base change, local-global compatibility, the parameter-Hecke tie, strong multiplicity one and the eigenspan hypothesis, for each index i with $\ell_i = g_k$ and each $\pi \in Z_{\mathrm{good}}(i)$ there exist a spectral subset $X_\pi \subset X_i$ with $\pi \in X_\pi$ and $X_\pi \cap Z(i) \subset Z_{\mathrm{good}}(i)$, a Hecke-commuting endomorphism β_π of the integral Hecke module, and a nonzero scalar $b_\pi \in O_i$ (namely the image of M_0) with the following properties. The subset X_π is a type locus at g_k, meaning that there is one assignment $(s_v)_v$ of squarefree monic degree-r depth-zero labels over $\mathbf{F}_{g_k}$ at the designated places v above g_k, each avoiding both the prescribed character value and its inverse in $\mathbf{F}_{g_k}^\times$, such that every $x \in X_\pi$ has an automorphic lift carrying s_v at every v. Moreover β_π acts by b_π on the π-eigenspace and by 0 on every eigenspace outside X_π, with $\lambda_i(O_i/b_\pi O_i) \le v_{g_k}(M_0)$.

Lean: `GlobalTheory.CutAtGood.main_alt_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.BcAll`, lines 37–161)

Lean: `Formalism.MainNotions.PairedCharacterCondition.CongruenceModules.CutAt` (`Formalism.SharpRestriction.LevelLetterCocycles.CongruenceSingleEndomorphism`, lines 127–135)

Lean: `GlobalTheory.Pass.TypeLocus` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.StepJumpDirect`, lines 32–36)

Choose the labels of a good parameter realizing π, and multiply their commuting central type idempotents, over the designated places above both type primes. Claim 3 makes M_0 times this product integral. Let β_π be its action and let X_π consist of the eigenspaces on which that action is nonzero. Local series compatibility says that nonzero action of a type idempotent forces its label to equal the parameter label. Consequently points of $X_\pi \cap Z(i)$ have the same good labels and belong to $Z_{\mathrm{good}}(i)$. The product acts as the identity on the type vectors realizing π; the eigenspan condition transfers this assertion to its Hecke eigenspace. It follows that $\pi \in X_\pi$, that β_π acts there by $b_\pi = M_0$, and that its action outside X_π vanishes. The integral central action commutes with Hecke operators. Finally the normalized length of O_i/M_0O_i is at most $v_{g_k}(M_0)$.

Lean: `GlobalTheory.T.admBC_of_serLoc` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.BcAllStatement`, lines 11–12)

Congruence bound on a type locus. Fix i with $\ell = \ell_i = g_k$, representations (R_x) and (ρ_x), parameters $(\eta_0, s, S, \mathrm{sg}, U_w)$ satisfying (i)–(iii) of Definition 36.67, a point $\pi \in Z_{\mathrm{good}}(i) \subset Z(i)$, and a type locus $X_s \subset X_i$ at g_k containing π. Write $T_i \subset O_i^{X_i}$ for the O_i-subalgebra of integral Hecke operators on the finite spectrum X_i, and $T_i|_{X_s} = \{f|_{X_s} : f \in T_i\} \subset O_i^{X_s}$ for its restriction to X_s. Let $\mathfrak{c}_{i,X_s,\pi} = \{f(\pi) : f \in T_i,\ f|_{X_s \setminus Z(i)} = 0\} = \{\bar f(\pi) : \bar f \in T_i|_{X_s},\ \bar f|_{X_s\setminus Z(i)} = 0\} \subset O_i$ be the evaluation at π of the ideal of operators in T_i supported on $(X_i \cap Z(i)) \cup (X_i \setminus X_s)$ (equivalently, the congruence ideal at π for $T_i|_{X_s}$ with cut locus $X_s \cap Z(i)$), and set $v_{cc}(i, X_s, \pi) = \lambda_i(O_i/\mathfrak{c}_{i,X_s,\pi})$. Since g_k splits completely in E, there are $2f$ places $w \mid \ell$, each with $E_w \simeq \mathbf{Q}_\ell$, and $\ell = g_k \neq 2$.

Lean: `Formalism.L2.Pj.g_ne_two` (`Formalism.LocalToGlobal.ProjectiveLattices.ExistsParameterConditions`, lines 23–27)

Fix a place $w \mid \ell$ and set $V_x = R_x\chi_{i,1}^{-1}$ for $x \in X_s$. *Claim 5.* Retain the local compatibility hypotheses and a type locus with labels $(s_v)_v$. At a designated place v, let $j_{i,v} \in \mathbf{F}_\ell^\times$ be the prescribed character value in the type datum. For every $x \in X_s$ and an automorphic lift carrying s_v, there is $u \in \{j_{i,v}, j_{i,v}^{-1}\}$ with the following property. Let M be the multiset of roots in $\overline{\mathbf{F}}_\ell$ of

$$s_v(T)(T-u)^t.$$

At the place w_v selected by v, the representation R_x is de Rham and its Weil–Deligne representation has

zero monodromy and characteristic polynomial at each inertia element σ given by

$$\prod_{y\in M}(T-\omega_{w_v,y}(\sigma)).$$

Here $\omega_{w,y}$ is the continuous tame character attached to the nonzero residue root y by the given inertia-polynomial comparison; it is trivial on wild inertia and at w_v takes the prescribed tame generator to the Teichmüller lift of y. At the conjugate place, the same statement uses the multiset M^{-1}. The choice of u is made for the given lift and may depend on x.

Lean: `GlobalTheory.Pass.TypeLGC` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.StepJumpDirect`, lines 38–56)

Lean: `UpperTheory.GalS.stepType_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.StepType`, lines 17–80)

The depth-zero hypothesis gives the zero-monodromy automorphic parameter with the displayed root multiset. Base change and local-global compatibility identify its inertia polynomials with those of $\mathrm{WD}(R_x|_{G_{E_w}})$; the inertia-polynomial tie evaluates each residue root through $\omega_{w,y}$. This proves Claim 5, including its conjugate-place assertion. *Claim 6.* For each $w\mid\ell$, the twisted family $(V_x)_{x\in X_s}$ is potentially crystalline over one finite Galois extension K'/E_w. The inertia characters on the potentially crystalline period modules form a Frobenius-stable set Σ, containing 1, of size at most $N=n_V+1$; they are trivial on wild inertia and on $I_{E_w}\cap G_{K'}$. The dual family has the corresponding set Σ^{-1}, and the maximal unramified subfield of K' has Galois-stable image in the period ring.

To account for both permitted values of u in Claim 5, at a designated place use the single set

$$Y=\mathrm{Roots}(s_v)\cup\{j_{i,v},j_{i,v}^{-1}\}.$$

It has at most $r+2\le r+t+1=N$ elements because $t\ge1$. At the conjugate place use Y^{-1}. The paired-character tie identifies one fixed element $u_0\in Y$ with the Weil–Deligne inertia character of $\chi_{i,1}$. Thus all twisted inertia characters lie in the single set obtained by dividing the characters attached to Y by the character attached to u_0; this set contains 1 and still has at most N elements. The roots of s_v are stable under Frobenius and the two extra roots lie in $\mathbf{F}_\ell^\times$, so this set is Frobenius-stable. The inertia comparison, zero monodromy and the potentially crystalline comparison give a common finite Galois extension killing these finitely many tame characters. Since $E_w\simeq\mathbf{Q}_\ell$, its maximal unramified subfield is normal over $\mathbf{Q}_\ell$, giving the required period stability. Duality inverts the characters. These assertions give Claim 6 with all hypotheses of Lemma 36.71.

Lean: `GlobalTheory.TypeInertia.ySet` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.MonodromyZeroOfDeRhamCharacter`, lines 127–129)

Lean: `GlobalTheory.TypeInertia.inertial_chars` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.MonodromyZeroOfDeRhamCharacter`, lines 221–310)

Lean: `GlobalTheory.Pass.StepPotDirect` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.StepJumpDirect`, lines 83–91)

Lean: `GlobalTheory.T.typeFamily_potentiallyCrystalline_common_tame_extension` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TypeFamilyPotentiallyCrystallineCommonTameExtensStatement`, lines 6–7)

Lean: `GlobalTheory.PotDual.stepPotDual` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ContinuousLoc`, lines 87–106)

Put

$$N=n_V+1,\qquad r'=\lfloor N^2/4\rfloor,\qquad e_{\max}=C_T\big(Nr'+(\eta_0+t+1)(N+r')\big)\in\mathbf{N}.$$

Since $r\ge2$ and $t\ge1$, we have $N\ge4$ and $r'\ge4$. By hypothesis (i) and $S\ge0$, $1\le\eta_0+t+1<s+S$ and $C_T((s+S)N^2+N^3)<\ell$. Let M be either a stable $O_i[G_E]$-lattice W_π in $\rho_\pi\chi_{i,1}^{-1}$ (rank $k_M=r$) or $O_i(\chi_{i,c}\chi_{i,1}^{-1})$ for $1\le c\le t$ (rank $k_M=1$); since $\pi\in Z(i)$, the semisimple decomposition of R_π embeds $M\otimes_{O_i}K_i$ equivariantly into V_π. Let $S_{w,M}\subset Z^1(E_w,M\otimes_{O_i}K_i/O_i)$ be the O_i-span of cut cocycles of M at w formed from X_s (defined as in Lemma 36.54(b) with $x_1,\dots,x_{a'}\in X_s$). Write Z^1_f for the submodule of cocycles whose classes lie in H^1_f, and B^1 for coboundaries. Write Z^1_{ur} for cocycles vanishing on I_{E_w}.

- If $\mathrm{sg}(w)=1$, hypothesis (ii) and $t\ge1$ put all jumps at w of every V_x ($x\in X_s$) into $U_w\cup\{0\}\subset[-(\eta_0+t+1),\,s+S]$, of size at most $n_V+1\le N+1$. Lemma 36.72, with $\eta'=\eta_0+t+1$ and $H=s+S$, gives the common lifting exponent $e_{\max}$ for subrepresentations of finite sums of $(V_x)_{x\in X_s}$. The direct branch of Lemma 36.73 gives $\lambda_i((S_{w,M}+Z^1_f)/Z^1_f)\le2k_Me_{\max}$.

- If $\mathrm{sg}(w) = -1$, dual jump reflection $b \mapsto -1 - b$ and $t \geq 1$ put all jumps at w of every $V_x^\vee(1)$ $(x \in X_s)$ into $\{0\} \cup \{-1 - b : b \in U_w\} \subset [-(\eta_0 + t + 1), s + S]$, of size at most $n_V + 1 \leq N + 1$. Apply Lemma 36.72 to $(V_x^\vee(1))_{x \in X_s}$ with the same parameters. The dual branch of Lemma 36.73 gives $\lambda_i((S_{w,M} + Z_f^1)/Z_f^1) \leq 2k_M e_{\max}$.

For the character blocks $O_i(\chi_{i,c}\chi_{i,1}^{-1})$, Lemma 36.22 gives $Z_f^1 \subset B^1$ for $2 \leq c \leq t$ and $Z_f^1 \subset Z_{\mathrm{ur}}^1$ for $c = 1$. For $c = 1$ the twisted character is trivial, so its coboundaries vanish.
Lean: `GlobalTheory.Closing.closing_at_record` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ClosingAtRecord`, lines 11–19)

Use $T_i|_{X_s}$ at $\pi \in X_s \cap Z(i)$. The restricted characteristic-polynomial coefficients belong to this Hecke algebra. The sixth provision gives no nonzero G_E-invariants in V_x for $x \in X_s \setminus Z(i)$. The residual realization has full matrix-algebra span, and the chosen block-projector data have denominator valuation at most δ_X. The representations are unramified outside $\{v \mid \ell\} \cup S_i \cup Q_i$; at S_i the fifth provision gives character-cocycle annihilation, and at Q_i it gives the quadratic inertia relation. Retain also the tame-generator conditions, global Artin reciprocity and the assumed finite Selmer lengths. Put

$$q_i = \sum_{v \in Q_i} \lambda_i(O_i/(\zeta_{i,v} - 1)), \qquad a_\rho = 2re_{\max}, \qquad a_\chi = 2e_{\max}.$$

Claim 7. For these fixed data there are finite O_i-submodules C_ρ and C_c, $1 \leq c \leq t$, of the global cocycles with respective targets

$$W_\pi \otimes_{O_i} (K_i/O_i), \qquad O_i(\chi_{i,c}\chi_{i,1}^{-1}) \otimes_{O_i} (K_i/O_i),$$

such that their localizations belong to the corresponding cut-cocycle spans, and

$$\begin{aligned} v_{cc}(i, X_s, \pi) &\leq \lambda_i(C_\rho) + \sum_{c=1}^{t} \lambda_i(C_c), \\ \lambda_i(C_\rho) &\leq 2fa_\rho + 2r\delta_X + s_\pi, \\ \lambda_i(C_1) &\leq 2fa_\chi + 2\delta_X + v_\ell(h_E) + q_i, \\ \lambda_i(C_c) &\leq 2fa_\chi + 2\delta_X + s_{\ell,c-1}(E) + q_i \quad (2 \leq c \leq t). \end{aligned}$$

The submodules of coboundaries in C_ρ and C_c have lengths at most $2r\delta_X$ and $2\delta_X$, respectively.
Lean: `GlobalTheory.CutDet.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LengthHeckeQuotientLeCocycleSpanDeterminant`, lines 376–457)
Lean: `GlobalTheory.CutCore.length_cocRho_le` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LengthQuotEqZeroIff`, lines 76–90)
Lean: `GlobalTheory.CutCore.length_cocChi_le` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LengthQuotEqZeroIff`, lines 129–159)
Lean: `GlobalTheory.T.length_coboundaries_among_block_cocycles_le` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LengthCoboundariesAmongBlockCocyclesLeStatement`, lines 6–7)

The modules are obtained from the quotient relations for the chosen block-projector data. Choose finitely many generators of the relation module and evaluate its coefficient algebra in the rank-r and scalar blocks at π. Absence of invariants off $Z(i)$ makes the associated relation determinants vanish on $X_s \setminus Z(i)$, so their values at π lie in $\mathfrak{c}_{i,X_s,\pi}$. The discrete-valuation-ring determinant estimate bounds the length of this ideal quotient by the sum of the block solution-module lengths; evaluation on the chosen generators embeds the solution modules into C_ρ and the C_c. This gives the first inequality.

For each block, imposing the finite condition at all $2f$ places above ℓ costs at most $2fa_\rho$ or $2fa_\chi$, by the local bounds just proved. The rank-r remainder maps into its Selmer group, with kernel the stated submodule of coboundaries. For a nontrivial character block, the finite local classes are coboundaries at ℓ, and the unramified remainder contributes at most $s_{\ell,c-1}(E)$. For the trivial block, the remainder is unramified everywhere and global Artin reciprocity bounds its length by $v_\ell(h_E)$. At S_i the character inertia contribution vanishes; at Q_i it is killed by $\zeta_{i,v} - 1$, costing at most q_i for each character block. This proves the remaining inequalities.

Sum Claim 7 and use $r + t = n_V$. Its denominator contribution is $2n_V\delta_X$, which is at most $4n_V\delta_X$. Substitution of a_ρ, a_χ, q_i therefore gives

$$v_{cc}(i, X_s, \pi) \leq 2f(2re_{\max} + 2te_{\max}) + s_\pi + \sum_{0<e<t} s_{\ell,e}(E) + v_\ell(h_E) + 4n_V\delta_X + t \sum_{v \in Q_i} \lambda_i(O_i/(\zeta_{i,v} - 1)),$$

where $2f(2r+2t)e_{\max} = 4fn_V C_T(Nr' + (\eta_0+t+1)(N+r'))$.
Lean: GlobalTheory.TameDesign.main (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TameDescentConditionsEveryLocalisation, lines 137–144)
Lean: GlobalTheory.StepGlobalPf.main_alt_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ImportsMoreMore, lines 15–22)
Lean: GlobalTheory.CutCore.main (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LengthQuotEqZeroIff, lines 175–243)
Lean: GlobalTheory.Pass.cutBound_of_steps_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CutBoundOfSteps, lines 15–78)
Lean: GlobalTheory.conclusion_of_typeLocus_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ConclusionOfAdmissibles, lines 16–28)

Passage to $\lambda_i(U_i/L_i)$. If $Z_{\mathrm{good}}(i) = \varnothing$, then $V_Z = 0$ in Definition 36.11, so $U_i = L_i = 0$ and both sides of (P) vanish. Assume henceforth that $Z_{\mathrm{good}}(i) \neq \varnothing$. For each $\pi \in Z_{\mathrm{good}}(i)$, take the type-locus cut $(X_\pi, \beta_\pi, b_\pi)$ constructed above. Lemma 36.69, with $Y = X_\pi$, gives $g_\pi \in T_i$, $c_\pi = g_\pi(\pi) \neq 0$, and a Hecke-commuting endomorphism $\alpha_\pi = g_\pi\beta_\pi$ which vanishes outside $Z_{\mathrm{good}}(i)$ and acts by $a_\pi = c_\pi b_\pi \neq 0$ on the π-eigenspace. It satisfies

$$\lambda_i(O_i/a_\pi O_i) \le v_{cc}(i, X_\pi, \pi) + v_\ell(M_0).$$

Lean: GlobalTheory.CutFam.perPoint (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsOfMemCcAt, lines 114–153)
Lean: GlobalTheory.CutFam.fam_of_cut (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsOfMemCcAt, lines 155–176)

Lemma 36.68, with $\mathcal{S} = Z_{\mathrm{good}}(i)$ and the endomorphisms α_π above, gives

$$\lambda_i(U_i/L_i) \le \sum_{\pi \in Z_{\mathrm{good}}(i)} \mathrm{mult}(i,\pi)\,\lambda_i(O_i/a_\pi O_i) \le \sum_{\pi \in Z_{\mathrm{good}}(i)} \mathrm{mult}(i,\pi)\big(v_{cc}(i,X_\pi,\pi) + v_\ell(M_0)\big).$$

Lean: GlobalTheory.EtaFromCut.lenUnit_le_of_family (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LenUnitLeOfFamily, lines 64–93)
Lean: GlobalTheory.EtaFromCut.etaBound_of_fam (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LenUnitLeOfFamily, lines 97–105)
Lean: GlobalTheory.EtaFromCut.etaBound_of_cut (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LenUnitLeOfFamily, lines 107–108)

Finally, $N = n_V + 1 \le 2n_V$ and $r' \le N^2/4 \le n_V^2$ give $Nr' + (\eta_0+t+1)(N+r') \le 2n_V^3 + 3n_V^2(n_V+\eta_0+t) \le 5n_V^2(n_V+\eta_0+t)$, which multiplied by $4fn_V C_T$ is at most $4fn_V C' n_V^2(n_V+\eta_0+t)$ since $C' \ge 8C_T \ge 5C_T$. From $\log g_{1-k} \le 2\log g_k$ we get $g_{1-k} \le g_k^2$, so $v_{g_k}(g_{1-k}^a - 1) \le 2a$ for $1 \le a \le r$ and $v_\ell(M_0) = fr(r-1)/2 + f\sum_{a=1}^{r} v_{g_k}(g_{1-k}^a - 1) \le fr(r-1) + fr(r+1) = 2fr^2$. Hence each summand $v_{cc}(i, X_\pi, \pi) + v_\ell(M_0)$ is at most $R_i(\pi)$, proving $\lambda_i(U_i/L_i) \le \sum_{\pi \in Z_{\mathrm{good}}(i)} \mathrm{mult}(i,\pi) R_i(\pi)$.
Lean: GlobalTheory.typePrimeBound_of_etaBound_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.TypePrimeBoundOfEtaBound, lines 13–29)
Lean: GlobalTheory.admT_of_adm5_eigSpan_tieInePoly (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ConclusionOfAdmissibles, lines 29–41)

The complete proof is contained in `GlobalTheory.ClosingLocalInputs.main_alt_alt`. □

Proof of 36.75

Proof. If $Z_{\mathrm{good}}(i)$ is empty, then $V_Z = 0$ in Definition 36.11, so $U_i = L_i = 0$ and both sides vanish. Otherwise the type-prime clause of the parameter split bound in the setup supplies the hub families (R_x), (ρ_x) with their semisimple decompositions on $Z(i)$. It also supplies s, S, sg, U_w with $\eta_X + t + 1 < s$, $C_T((s+S)(n_V+1)^2 + (n_V+1)^3) < g_k$, opposite signs at conjugate places, and (ii) of Definition 36.67 with $\eta_0 = \eta_X$. The size conditions on the type primes give $g_{1-k} \le 2g_k$, hence $\log g_{1-k} \le \log 2 + \log g_k \le 2\log g_k$. The setup assumes finite Selmer lengths, and the finiteness hypothesis of Definition 36.11 gives finite $s_{\ell,e}(E)$. Statement (P) now gives the bound.
Lean: GlobalTheory.T.split_bound_of_params_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ImportsMoreExtraStatement, lines 13–14)
Lean: GlobalTheory.NLinkTyp.main_alt_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EtaLocLeRhs, lines 60–90)

□

Proof of 36.77

Proof. Set $Q = \sum_{i\in S} Ob_i \subset M$. The family $(b_i)_{i\in S}$ is O-linearly independent and spans Q, so it forms an O-basis of Q. The submodule $P' = P$, viewed inside Q, is O-saturated in Q: if $a \in O \setminus \{0\}$, $x \in Q$, and $ax \in P'$, then $x \in P$ by saturation of P in M. Over the discrete valuation ring O, a saturated submodule of a finite free module is a direct summand with an adapted basis, so Q admits an O-basis $(c_j)_{j\in S}$ and a subset $S'' \subset S$ with $P = \sum_{j\in S''} Oc_j$.

Define $b'_i = c_i \in Q \subset M$ for $i \in S$ and $b'_i = b_i$ for $i \notin S$. Every $y \in Q$ has zero b_i-coordinate for all $i \notin S$. Hence if $\sum_{i\in I} g_i b'_i = 0$ with $g_i \in O$, reading the b_{i_0}-coordinate for each $i_0 \notin S$ gives $g_{i_0} = 0$, and then $\sum_{j\in S} g_j c_j = 0$ in Q gives $g_j = 0$ for all $j \in S$. Moreover, for $i \in S$ the vector $b_i \in Q$ lies in $\sum_{j\in S} Oc_j = \sum_{j\in S} Ob'_j$, while for $i \notin S$ one has $b_i = b'_i$. Thus $(b'_i)_{i\in I}$ spans M and is an O-basis of M, satisfying $b'_i = b_i$ for $i \notin S$ and $\sum_{i\in S} Ob'_i = \sum_{j\in S} Oc_j = Q$. Finally, $P = \sum_{j\in S''} Ob'_j$; since $(b'_i)_{i\in I}$ is a basis of M, a basis vector b'_i belongs to $\sum_{j\in S''} Ob'_j$ if and only if $i \in S''$, so $P = \sum_{i\in I,\, b'_i\in P} Ob'_i$. □

Proof of 36.80

Proof (outline). Claim 1. For every finite free O-module P and linear map $g : P \to Z_f(T_2)$, there are the following simultaneous extension coordinates, linear in P. The crystalline-extension and integral Herr comparisons, applied to a basis of P, give O-linear maps y, z, μ_i, ν_i and an integer $N \geq 0$. Put $y_i = e_i y$ and $z_i = e_i z$. The pair $(y(f), z(f))$ represents $[g(f)]$, and $\mu_i(f), \nu_i(f) \in \mathcal{D}_i^{\Delta}(T_2)$. Put

$$r_i = \ell^N y_i + \varphi\mu_{i-1} - \mu_i, \qquad (M_i, h_i) = \begin{cases} (\beta_i + S, \delta), & i \in J_-, \\ (\delta, \beta_i + S), & i \notin J_-. \end{cases}$$

The extension construction gives, for every coefficient vector,

$$\ell^N z_i + (\gamma - 1)\mu_i \in \mathcal{U}^1(T_2), \quad Q_0^{M_i} r_i \in N(T_2), \quad \ell^{s_i} Q_0^{h_i} r_i \in \mathcal{R}_{i-1}^{(1)}(T_2)$$

for some $s_i \geq 0$, and $\ell^N \nu_i - Q_0^a \mu_i \in N(T_2)$. The integer N belongs to this auxiliary presentation and will disappear from the final exponent.

Lean: `Formalism.MainNotions.RamifiedCoefficientChains.Sh.ExtData` (`Formalism.SharpRestriction.LevelLetterCocycles.ExpRules`, lines 106–116)

We now give the construction behind these simultaneous assertions. For each basis vector f of P, realize $g(f)$ by a crystalline extension $0 \to T_2 \to E_f \to O \to 0$. The jumps of E_f lie in $[-M_i, h_i]$, since its additional jump is 0. The same Wach cokernel annihilators ℓ^N and Q_0^a work for every quotient $E_f \to O$, with N depending only on these jump windows. Normalize the invariant generator of the unit Wach lattice by the inverse of the unit in the Herr extension-class identity, and denote the resulting generator by e. Exactness of the étale modules then gives a lift X_f of this generator; average it over Δ. The preimages of $(\varphi - 1)X_f$ and $(\gamma - 1)X_f$ are $y(f)$ and $z(f)$, with class exactly $[g(f)]$. In component i, choose invariant Wach lifts $\varepsilon_i, \varepsilon'_i$ of $\ell^N e_i e$ and $Q_0^a e_i e$, respectively. Define $\mu_i(f)$ and $\nu_i(f)$ by their images

$$\varepsilon_i - \ell^N e_i X_f, \qquad \varepsilon'_i - Q_0^a e_i X_f$$

in the extension. Saturated exactness and the two Frobenius-height inclusions give the displayed relations. Extend all choices O-linearly from the chosen basis of P. Whenever the reverse Frobenius inclusion requires different powers of ℓ for different basis vectors, their sum is a common exponent. This proves the simultaneous assertions without imposing any uniformity on the auxiliary s_i that the argument does not use.

Lean: `Formalism.MainNotions.RamifiedCoefficientChains.Sh.extDataOne_exists` (`Formalism.SharpRestriction.LevelLetterCocycles.ExtDataOneMapSmulO`, lines 45–223)

Lean: `Formalism.MainNotions.RamifiedCoefficientChains.Sh.extData_extTool` (`Formalism.SharpRestriction.LevelLetterCocycles.ExtDataExtTool`, lines 32–162)

Lean: `Formalism.MainNotions.CrystallineLiftingInputs.Tools.level_clause_of_admitted_statements` (`Formalism.SharpRestriction.LevelLetterCocycles.LocalClauseOfFrobeniusHyps`, lines 1035–1038)

Claim 2. There is an integer u_1 depending on the given jump windows, with $0 \leq u_1 \leq B_1$, and linear corrections m_c such that

$$Q_0^a(\ell^N m_c - \ell^{u_1}\mu_c) \in N(T_2), \quad (\gamma - 1)(\ell^N m_c - \ell^{u_1}\mu_c) \in \mathcal{U}^1(T_2), \quad \ell^{u_1} z_c + (\gamma - 1)m_c \in \mathcal{U}^1(T_2)$$

for every $c \in J_-$. After at most $|J_-|$ generator steps, the first corrected vector $\ell^N m_c - \ell^{u_1}\mu_c$ is integral for every $c \in J_-$.

On the localized invariant component put $D = \pi_0^{-1}(\gamma - 1)$, and define $c(t) \in O$, for $t \in \mathbf{Z}$, by

$$\ell c(t) = 1 - \chi_\ell(\gamma)^{-t}.$$

Then $c(0) = 0$, and $v_\ell(c(t)) = v_\ell(t)$ for $t \neq 0$. For $c \in J_-$, let S_c be the set of nonzero integers $u + j$, where u is a Hodge–Tate jump of T_2 over the c-component and $1 \leq j \leq a$. Exactness of the jumps and the two prescribed intervals give

$$S_c \subset [1 - \beta_c - S, a - \beta_c] \cup [1 - \delta, a + \delta], \qquad a + \delta < \beta_c.$$

Set $u_1 = \max_{c \in J_-} \sum_{t \in S_c} v_\ell(t)$. Counting multiples of ℓ^j in these intervals gives

$$\sum_{t \in S_c} v_\ell(t) \leq \frac{2a + 2\delta + S}{q} + \log_\ell(2H) + 1 = B_1.$$

Choose $b_c \in O$ so that

$$R_c(X) = b_c \prod_{t \in S_c} (X - c(t)), \qquad R_c(0) = \ell^{u_1}, \qquad R_{c,1}(X) = \frac{R_c(X) - R_c(0)}{X}.$$

Such b_c is integral because the valuation of the product of the constant terms is at most u_1. The letter-operator condition gives

$$DR_c(D)x \in N(T_2) \quad \text{if } x \in \mathcal{D}_c^{\Delta}(T_2), \quad Q_0^a x \in N(T_2).$$

Define the correction, before restricting P, by

$$m_c = R_c(D)(Q_0^{-a}\nu_c) + R_{c,1}(D)(\pi_0^{-1} z_c), \qquad u_c = \ell^N m_c - \ell^{u_1}\mu_c.$$

For $u_{c,0} = \ell^N Q_0^{-a}\nu_c - \mu_c$ and $v_c = \pi_0^{-1}(\ell^N z_c + (\gamma - 1)\mu_c)$, one has

$$u_c = R_c(D)u_{c,0} + R_{c,1}(D)v_c.$$

Here $Q_0^a u_{c,0}$ and v_c are integral. The killing identity and preservation of the Wach lattice by D imply

$$Q_0^a u_c \in N(T_2), \quad (\gamma - 1)u_c \in \mathcal{U}^1(T_2), \quad \zeta_c := \ell^{u_1} z_c + (\gamma - 1)m_c \in \mathcal{U}^1(T_2).$$

For the last assertion, use $\ell^N \pi_0^{-1}\zeta_c = \ell^{u_1} v_c + Du_c$ and saturation of the invariant Wach lattice.

Let $\mathcal{F}_c^j$ be the image of the c-component of the crystalline filtration Fil^j in the de Rham period space. At a primitive ℓ-th root, expand u_c as a Laurent series. Its constant coefficient belongs to $\mathcal{F}_c^{-a}$, and $u_c \in N(T_2)$ if and only if that coefficient belongs to $\mathcal{F}_c^0$. The quotient $\mathcal{F}_c^{-a}/\mathcal{F}_c^0$ has dimension $\sum_{-a \leq j < 0} \mathrm{mult}_{\tau_c}(j) \leq m$. Intersect the kernels of these constant-coefficient maps successively. This defines $P_1 \subset P$, with one generator step for each $c \in J_-$, and $u_c(P_1) \subset N(T_2)$.

Lean: `Formalism.MainNotions.RamifiedCoefficientChains.S9.ExtensionLemmaAlt.first_lift_of_hypotheses` (`Formalism.SharpRestriction.LevelLetterCocycles.FirstLiftOfHypotheses`, lines 19–189)

Lean: `Formalism.MainNotions.RamifiedCoefficientChains.S9.ExtensionLemmaAlt.step1Pass_pass` (`Formalism.SharpRestriction.LevelLetterCocycles.MultVNonneg`, lines 99–172)

Claim 3. Put $s = \widehat{\sigma}$. For $c \in J_-$ and the linear maps $\zeta_c : P_1 \to \mathcal{U}^1(T_2)$ of Claim 2, there are linear maps t_c and remainders ξ_c with the following properties. The level-one uniformizer truncation with parameters (M, δ, s), applied to ζ_c, gives maps $t_c : P_1 \to \mathcal{R}_c^{(1)}(T_2)$, linear after choices on an O-basis, such that

$$\xi_c := \ell^s \varphi\zeta_c - (\gamma - 1)t_c \in \left(\pi_0^M Q_0^\delta \mathcal{R}_c^{(1)}(T_2)\right) \cap \mathcal{D}_{c+1}^{\Delta}(T_2) \subset \mathcal{U}^M(T_2).$$

The maps preserve components, with t_c, ξ_c in component $c + 1$. The truncation is linear modulo $\pi_0^M Q_0^\delta \mathcal{R}_c^{(1)}$, which is the precision used here. This is the truncation of the correction vector ζ_c, with $\zeta_c \in \mathcal{U}^1$. Its separate vanishing assertion requires $\zeta_c \in \mathcal{U}^M$ and $\delta \leq qM$; it is not used to truncate w_c below.

Lean: Formalism.MainNotions.RamifiedCoefficientChains.S9.ExtensionLemmaAlt.TTool2Z (Formalism.SharpRestriction.LevelLetterCocycles.IsTruncIff, lines 43–56)
Lean: Formalism.MainNotions.CrystallineLiftingInputs.truncatedSeriesData.tTool2Z_DU (Formalism.SharpRestriction.LevelLetterCocycles.TruncationClauseOfLatticeHypotheses, lines 26–39)
Lean: Formalism.MainNotions.RamifiedCoefficientChains.S9.ExtensionLemmaAlt.second_lift_of_hypotheses (Formalism.SharpRestriction.LevelLetterCocycles.SecondLiftOfHypotheses, lines 17–27)

Define

$$\widehat{m}_c = \ell^s \varphi m_c + \ell^{u_1+s} y_{c+1} - t_c, \qquad \widehat{u}_c = \ell^N \widehat{m}_c - \ell^{u_1+s} \mu_{c+1}.$$

The identity

$$\widehat{u}_c = \ell^s \varphi u_c + \ell^{u_1+s} r_{c+1} - \ell^N t_c$$

gives $Q_0^\delta \widehat{u}_c \in N(T_2)$ using the following precise height inclusion.

Claim 4. For $c \in J_-$ and $v \in \mathcal{N}_c(T_2)$, one has

$$Q_0^\delta \varphi v \in N(T_2).$$

The relevant jump hypothesis is that every jump of T_2 at $c+1$ is at least $-\delta$, as follows from exactness for the crystalline quotient of T_1.
Lean: Formalism.MainNotions.N.Pv.LocalLengthBound.frobenius_sq_bound_of_local (Formalism.SharpRestriction.LevelLetterCocycles.QuotientImageDivisibility.FrobeniusSqBoundOfLocal, lines 55–165)

Also $(\gamma - 1)\widehat{u}_c \in N(T_2)$, by the formulas for ξ_c and the original extension coordinates. The constant coefficient of its Laurent expansion lies in $\mathcal{F}^{-\delta}_{c+1}$, and its image modulo $\mathcal{F}^0_{c+1}$ vanishes exactly when $\widehat{u}_c$ is integral. This quotient has dimension at most m. Its successive kernels define $P_2 \subset P_1$, with a further $|J_-|$ generator steps.

On P_2 set

$$w_c = \varphi \widehat{m}_{c-2} + \ell^{u_1+s} y_c - \ell^s m_c.$$

Its two useful identities are

$$\ell^N w_c = \varphi \widehat{u}_{c-2} - \ell^s u_c + \ell^{u_1+s} r_c, \qquad (\gamma - 1) w_c = \varphi \xi_{c-2} - \ell^s \zeta_c.$$

The first, the height bounds, and saturation give $Q_0^\delta w_c \in \mathcal{R}^{(1)}_{c-1}(T_2)$ and $Q_0^{\beta_c + S} w_c \in N(T_2)$. The latter identity supplies the finite-pole condition with $k = 1$ and $a_0 = \beta_c + S$. The second displayed identity gives $(\gamma - 1)w_c \in \mathcal{U}^1(T_2)$, since Frobenius preserves $\mathcal{U}^M$ and $M \geq 1$. Thus $w_c \in \mathcal{V}_c(T_2)$.

Claim 5. For $T = T_1$ or T_2, put

$$J_c(T) = \left(\pi_0^M Q_0^\delta \mathcal{R}_c^{(1)}(T)\right) \cap \mathcal{D}^\Delta_{c+1}(T).$$

There is an ordinary truncation $\tau_T(w) \in \mathcal{D}^\Delta_{c+1}(T)$ for every $c \in J_-$ and $w \in \mathcal{V}_c(T)$, with

$$\tau_T(w) - \ell^s \varphi w \in \mathcal{R}_c^{(1)}(T), \qquad (\gamma - 1)\tau_T(w) \in J_c(T).$$

Its class modulo $J_c(T)$ is independent of its defining choices, O-linear in w, and natural for crystalline lattice maps. If $w \in \mathcal{U}^1(T)$, then $\tau_T(w) \in J_c(T)$. Its first coordinate has finite Frobenius poles. Frobenius preserves $\mathcal{U}^M(T)$, and $\varphi - 1$ is bijective on that module. If (z, v) is a Δ-invariant Herr pair, z belongs to one component and $v \in \mathcal{U}^M(T)$, then

$$z \in \mathcal{U}^M(T) \implies \mathrm{cl}_T(z, v) = 0, \qquad z \text{ has finite Frobenius poles} \implies \mathrm{cl}_T(z, v) \in H^1_f(K, T).$$

In particular $\tau_T(w)$ completes to such a pair with finite class.

For precision, the ordinary truncation is the following finite expression. Put $E_y = \lceil y\ell/q \rceil$. Choose $a_0 = 0$ and $a_y \in \mathcal{N}_c(T)$ so that, for every $K \geq 0$, some $j \geq 0$ satisfies

$$\ell^j \left(\ell^{E_K} (\gamma - 1) w - \sum_{y < K} \ell^{E_K - E_y} (\gamma - 1)(\pi_0^y a_y) \right) \in \mathcal{U}^K(T).$$

Choose polynomials $r_y \in O[\pi_0]$ of degree less than $M + \delta$ and $s_y \in \mathcal{A}$ such that

$$\varphi(\pi_0)^y - r_y \in \pi_0^M Q_0^\delta \mathcal{A}, \qquad \ell^{E_y} s_y = \ell^s r_y.$$

Then

$$\tau_T(w) = \ell^s \varphi w - \sum_{0 \le y \le \max(M,\delta)} s_y \varphi(a_y).$$

The coefficient construction gives the asserted independence, linearity, naturality, and the $\mathcal{U}^1$ vanishing assertion. Both lattices have all jumps at least $-h$, and their jumps at $c+1$, for $c \in J_-$, are at least $-\delta$; exactness gives these assertions for T_2. These are precisely the hypotheses for bijectivity on $\mathcal{U}^M$ and the two class assertions. They also give $J_c(T) \subset \mathcal{U}^M(T)$. Finally, the first displayed difference and the height inclusion $Q_0^\delta \mathcal{R}_c^{(1)}(T) \subset N(T)$ preserve finite Frobenius poles.

Lean: `Formalism.MainNotions.CrystallineLiftingInputs.IsTrunc` (`Formalism.SharpRestriction.LevelLetterCocycles.CrystallineLiftingInputs`, lines 147–153)
Lean: `Formalism.MainNotions.CrystallineLiftingInputs.IsSer` (`Formalism.SharpRestriction.LevelLetterCocycles.CrystallineLiftingInputs`, lines 122–128)
Lean: `Formalism.MainNotions.CrystallineLiftingInputs.IsRem` (`Formalism.SharpRestriction.LevelLetterCocycles.CrystallineLiftingInputs`, lines 139–141)
Lean: `Formalism.MainNotions.CrystallineLiftingInputs.TTool2` (`Formalism.SharpRestriction.LevelLetterCocycles.CrystallineLiftingInputs`, lines 159–179)
Lean: `Formalism.MainNotions.CrystallineLiftingInputs.FTool` (`Formalism.SharpRestriction.LevelLetterCocycles.CrystallineLiftingInputs`, lines 193–215)
Lean: `Formalism.MainNotions.CrystallineLiftingInputs.Tools.local_clause_of_admitted_statements` (`Formalism.SharpRestriction.LevelLetterCocycles.LocalClauseOfFrobeniusHyps`, lines 1018–1020)
Lean: `Formalism.MainNotions.N.Pv.LocalLengthBound.rhoLift_of` (`Formalism.SharpRestriction.LevelLetterCocycles.QuotientImageDivisibility.RhoLiftOf`, lines 20–149)

Claim 6. For the corrections just constructed, put $Y_c = \varphi w_c + t_c$. The ordinary truncation $\tau_{T_2}(w_c)$ satisfies

$$\tau_{T_2}(w_c) - \ell^s Y_c \in J_c(T_2) \subset \mathcal{U}^M(T_2).$$

Indeed, the thickness hypotheses give $\delta \le qM$. The correction truncation of Claim 3 is linear modulo $J_c(T_2)$ and vanishes modulo that module on $\mathcal{U}^M$. Since $(\gamma-1)w_c = \varphi\xi_{c-2} - \ell^s \zeta_c$ and $\varphi\xi_{c-2} \in \mathcal{U}^M$, its truncation is congruent to $-\ell^s t_c$ modulo $J_c(T_2)$. Subtracting it from $\ell^s \varphi w_c$ gives the displayed identity. This uses the vanishing assertion for the correction input $\varphi\xi_{c-2}$, whose membership in $\mathcal{U}^M$ was established above.

Lean: `Formalism.MainNotions.RamifiedCoefficientChains.S9.ExtensionLemmaAlt.Step2K.level_membership_package` (`Formalism.SharpRestriction.LevelLetterCocycles.LevelMembershipPackage`, lines 21–66)

Direct expansion gives

$$Y_c = \ell^{u_1+s}(y_{c+1} + \varphi y_c) + \varphi^2 \widehat{m}_{c-2} - \widehat{m}_c, \qquad (\gamma-1)Y_c = \varphi^2 \xi_{c-2} - \xi_c.$$

The bijection $\varphi - 1$ on $\mathcal{U}^M(T_2)$ completes Y_c to a Herr cocycle. Claims 5 and 6 give a Herr cocycle $X_c = (z_c^{\mathrm{H}}, v_c^{\mathrm{H}})$ whose first coordinate is $\tau_{T_2}(w_c)$ and whose second coordinate belongs to $\mathcal{U}^M$. The difference between X_c and ℓ^s times the cocycle constructed from Y_c has both coordinates in $\mathcal{U}^M$, so its class is zero by Claim 5. Choose an integral cocycle κ_c representing the class of X_c. Summing over $c \in J_-$, the shift $c \mapsto c-2$ permutes J_-. With

$$m_E = \sum_{c \in J_-} (\ell^{u_1+s} y_c + \widehat{m}_c + \varphi \widehat{m}_c),$$

one has $\sum_c Y_c = \ell^{u_1+s} y + (\varphi - 1) m_E$, and

$$\ell^{u_1+s} z + (\gamma - 1) m_E = \sum_{c \in J_-} (\xi_c + \varphi \xi_c) \in \mathcal{U}^M(T_2).$$

The second coordinates obtained using $(\varphi-1)^{-1}$ therefore have exactly this sum, by the Herr equation and uniqueness. Herr boundaries and the bijection $\varphi - 1$ therefore give the asserted sum of classes at exponent $u_1 + 2s$.

For the lifting assertion, use $\tau_{T_1}(u)$ as the first coordinate of $Y = (z_u^{\mathrm{H}}, v_u^{\mathrm{H}})$. Claim 5 gives its finite class and $v_u^{\mathrm{H}} \in \mathcal{U}^M(T_1)$. In the quotient by $J_c(T_2)$, naturality and linearity give

$$\ell^{e_*} \tau_{T_2}(w_c(f)) - F_D \tau_{T_1}(u) \equiv \tau_{T_2}(\ell^{e_*} w_c(f) - F_D u) \equiv 0,$$

where the last congruence uses the ordinary truncation's $\mathcal{U}^1$ vanishing assertion. Therefore

$$\ell^{e_*} z_c^{\mathrm{H}} - F_D z_u^{\mathrm{H}} \in J_c(T_2) \subset \mathcal{U}^M(T_2).$$

The two second coordinates already belong to $\mathcal{U}^M$, and F_D preserves that submodule. This proves both required coordinate congruences. There are $2|J_-| = n$ generator steps, and $u_1 + 2s \le B_1 + 2\sigma + 2 = B_{\mathrm{red}}$. The complete proof is contained in `Formalism.MainNotions.RamifiedCoefficientChains.S9.ExtensionLemmaAlt.cocyclesToWachVectorsBound_of_extTool`, using `Formalism.MainNotions.RamifiedCoefficientChains.S9.LocalLemmasFromLiterature.local_lemmas_of_literature` and `Formalism.MainNotions.N.Pv.LocalLengthBound.rhoLift_of`. □

Proof of 36.81

Proof (outline). Put $Q_1 = \varphi(Q_0)$ and $h_2 = \beta_{i-1} + S$. *Claim 1.* For every $y \in \mathcal{N}_{i-2}(T_2)$, the two-Frobenius truncation with exponent s_2 has the following simultaneous height and gamma-defect properties. Apply the two-Frobenius uniformizer truncation to y, with parameters $(2, 1, 2H, s_2)$. It gives $\lambda_y \in \mathcal{D}_i^{\Delta}(T_2)$ such that

$$Q_1^{h_2}\lambda_y \in \mathcal{R}_{i-1}^{(1)}(T_2), \qquad Q_1^{h_2}(\gamma - 1)\lambda_y = \pi_0 Q_0^{a+\beta_i} v \quad (v \in \mathcal{N}_i(T_2)).$$

Lean: `Formalism.MainNotions.RamifiedCoefficientChains.S9.lemGA_la_of_adm` (`Formalism.SharpRestriction.LevelLetterCocycles.LiftRingOfUniformizerClause`, lines 31–37)

Set $x = Q_0^{-\beta_i} Q_1^{h_2}\lambda_y$ in the localized module. The height bound gives $Q_0^{\beta_i+\delta} x \in N(T_2)$. We next prove that x itself is integral.

At a primitive root, write the expansion of Q_0 as $tU(t)$, with $U(0) \neq 0$, and let $d_x(k)$ be the Laurent coefficients of x. The truncated horizontal comparison implies that the coefficients of $U(t)^{\beta_i+\delta} d_x(t)$ belong to

$$V_k = \begin{cases} 0, & k < -\beta_i, \\ L' \cdot \ell^{s_2}\iota(d), & -\beta_i \le k < 2H - \beta_i, \\ \text{the ambient period space}, & k \ge 2H - \beta_i, \end{cases}$$

where $\iota : D_{\mathrm{cris}} \hookrightarrow D_{\mathrm{dR}}$ is the comparison inclusion. The spaces V_k increase with k. Solving the coefficient equations successively, starting at the lower bound of the Laurent support, shows that $d_x(k) \in V_k$: the coefficient to be solved has nonzero scalar multiplier $U(0)^{\beta_i+\delta}$, and the earlier coefficients lie in V_k. In the middle range, $d \in \mathrm{Fil}^{\beta_i}$ gives $d_x(k) \in \mathrm{Fil}^{-k}$. In the final range, $2H - \beta_i \ge \delta$, and the lower jump bound $-\delta$ gives the same inclusion. *Claim 2.* Let T be crystalline over the present unramified field, let j be a component, let ζ be a primitive ℓ-th root, and let $v \in \mathcal{D}_j^{\Delta}(T)$. Suppose $Q_0^k v \in N(T)$ for some $k \ge 0$. Define its Laurent expansion $d_v(t)$, with support bounded below, by

$$\mathrm{ev}_{j,\zeta}^N(Q_0^k v) = \mathrm{ev}_{j,\zeta}(Q_0)^k \, d_v(t).$$

This expansion is independent of k, and

$$v \in N(T) \quad \Longleftrightarrow \quad d_v(b) \in \iota(D_{\mathrm{cris},j}(T[1/\ell])) \cap \mathrm{Fil}^{-b} D_{\mathrm{dR}} \quad \text{for every } b \in \mathbf{Z}.$$

The assumptions include the Δ-invariance of v and the existence of a clearing power, and the argument uses only these two properties of v. At level one, the filtered Wach characterization tests these coefficient inclusions at the cyclotomic roots. Since $\mathrm{ev}_{j,\zeta}(Q_0) = tU(t)$ with $U(0) \neq 0$, clearing the pole shifts the required filtration by k. The Δ-action transports one primitive root to every other and rescales each coefficient by a nonzero scalar. The remaining component tests vanish because $e_j v = v$. This gives the criterion in the stated form.

Lean: `Formalism.MainNotions.RamifiedCoefficientChains.Sh.ExpCrit` (`Formalism.SharpRestriction.LevelLetterCocycles.ExpRules`, lines 48–52)

Lean: `Formalism.MainNotions.N.Pv.LocalLengthBound.expCrit_D'` (`Formalism.SharpRestriction.LevelLetterCocycles.QuotientImageDivisibility.IsAdicCompleteOfLe`, lines 60–67)

Apply Claim 2 to $v = x$, with clearing exponent $\beta_i + \delta$. The coefficient inclusions just proved give $x \in \mathcal{N}_i(T_2)$.

At the expansion with root 1, both Q_0 and Q_1 have constant coefficient ℓ. The negative coefficients vanish. Comparing constant coefficients in $Q_0^{\beta_i} x = Q_1^{h_2}\lambda_y$ gives

$$\mathrm{cmp}_{T_2}(x) = \ell^{s_2 + h_2 - \beta_i} d.$$

Lean: Formalism.MainNotions.RamifiedCoefficientChains.S9.lemGA0_of_tools (Formalism.SharpRestriction.LevelLetterCocycles.LemGA0ConvVanish, lines 106–176)

To lift x, put $D_j = \pi_0^{-1}(\gamma - 1)$ on $\mathcal{D}(T_j)^{\Delta}$, define $c(t)$ as in the preceding proof, and set

$$I = [\beta_i - a + 1, \beta_i] \cap \mathbf{Z}, \quad J = ([\beta_i + 1, \beta_i + S + a] \cup [1 - \delta, \delta + a]) \cap \mathbf{Z}, \quad P_U(X) = \prod_{u \in U} (X - c(u)).$$

The thickness inequality gives $2a + \delta \le \beta_i$, so $I \cap J = \varnothing$. *Claim 3.* For the integral vector x constructed from Claim 1 and the operator D_2, the identity

$$Q_0^{\beta_i} x = Q_1^{h_2} \lambda_y, \qquad Q_1^{h_2}(\gamma - 1)\lambda_y \in \pi_0 Q_0^{a+\beta_i} \mathcal{N}_i(T_2)$$

implies $P_I(D_2)x = Q_0^a w$ for some $w \in \mathcal{N}_i(T_2)$. This is the gamma-derivation identity at level one: the action on the successive powers of Q_0 gives the factors $D_2 - c(j)$, $\beta_i - a + 1 \le j \le \beta_i$, and the displayed gamma defect makes the final remainder divisible by Q_0^a. All vectors remain invariant and in component i.

Lean: Formalism.MainNotions.CrystallineLiftingInputs.truncatedSeriesData.gamma_lemma_of_crystalline (Formalism.SharpRestriction.LevelLetterCocycles.EqUniformizerMulOfConstantCoeffZero, lines 324–344)

Claim 4. For the surjection $F : T_1 \to T_2$ in the present data and every component j,

$$Q_0^a \mathcal{N}_j(T_2) \subset F_D \mathcal{N}_j(T_1).$$

The source jump enclosure is $[-M_j, h_j]$, where $(M_j, h_j) = (\beta_j + S, \delta)$ on J_- and $(\delta, \beta_j + S)$ off J_-. Its maximum width is $h + \delta = H$, and the level-one cokernel exponent is $a = \lceil A \rceil$. The same assertion holds for the quotient of any crystalline extension $0 \to T_2 \to E \to O \to 0$, with the same enclosures, since $0 \in [-M_j, h_j]$. The cokernel assertion first gives componentwise Wach lifts. Averaging over Δ preserves the lattice, commutes with the quotient map, and fixes Q_0^a, so gives the displayed invariant lifts.

Lean: Formalism.MainNotions.CrystallineLiftingInputs.QTool (Formalism.SharpRestriction.LevelLetterCocycles.CrystallineLiftingInputs, lines 54–69)

Lean: Formalism.MainNotions.CrystallineLiftingInputs.Tools.level_clause_of_admitted_statements (Formalism.SharpRestriction.LevelLetterCocycles.LocalClauseOfFrobeniusHyps, lines 1035–1038)

Apply Claim 4 to w and x, obtaining $y_I, y_0 \in \mathcal{N}_i(T_1)$ with $F_D y_I = Q_0^a w = P_I(D_2)x$ and $F_D y_0 = Q_0^a x$. The letter-operator condition gives

$$y_J = P_J(D_1)(Q_0^{-a} y_0) \in \mathcal{N}_i(T_1), \quad F_D y_I = P_I(D_2)x, \quad F_D y_J = P_J(D_2)x.$$

For $j \in I$, the valuation identity $v_\ell(c(j) - c(k)) = v_\ell(j - k)$ gives

$$v_\ell\big(P_J(c(j))P_I'(c(j))\big) = \sum_{k \in J} v_\ell(j - k) + \sum_{k \in I \setminus \{j\}} v_\ell(j - k) \le \frac{3a + S + 2\delta - 1}{q} + \log_\ell H.$$

The left side is an integer, hence is at most E_i. Consequently

$$B(X) = \sum_{j \in I} \frac{\ell^{E_i}}{P_J(c(j))P_I'(c(j))} \prod_{k \in I \setminus \{j\}} (X - c(k))$$

has coefficients in O. It satisfies $B(c(j))P_J(c(j)) = \ell^{E_i}$ for all $j \in I$. Division by the monic polynomial P_I gives $A_0 \in O[X]$ with $\ell^{E_i} = A_0 P_I + B P_J$. Thus $z = A_0(D_1)y_I + B(D_1)y_J$ is integral, invariant, and satisfies $F_D z = \ell^{E_i} x$. This proves the comparison formula. Here $H \ge 1$ and $a \ge 3$, so $E_i \ge 0$. The estimate for E_i follows from $a < A + 1$, $q \ge 10$, and $\lambda = 1 + \log_\ell(2H)$. The complete proof is contained in `Formalism.MainNotions.RamifiedCoefficientChains.Asm.lemmaGs_rec`, using `Formalism.MainNotions.RamifiedCoefficientChains.S9.lemGA_of_pieces` and `Formalism.MainNotions.RamifiedCoefficientChains.S9.lemGB_of`. □

Proof of 36.82

Proof. The vectors $w_k = \iota\varphi\,\mathrm{cmp}_{T_2}(e_k)$ form a basis of the image of the i-component of D_{cris}. To see independence, clear denominators in a putative linear relation. A constant O-linear combination of the

e_k whose specialization is zero lies in $\pi_0\mathcal{N}_{i-1}(T_2)$. Uniqueness of its power-series coordinates makes all its constant coefficients zero. Frobenius is injective. The height inclusions give surjectivity onto the next crystalline component, so these vectors span that component as well.

Let $\Psi : (L')^r \to \mathrm{span}_{L'}\{w_k\}$ send the standard basis to the w_k, and put

$$G^u = \Psi^{-1}(\mathrm{Fil}^u D_{\mathrm{dR}}), \qquad F^u = O^r \cap G^u.$$

Each F^u is saturated in O^r, because G^u is an L'-subspace. Apply Lemma 36.77 successively to the descending chain $F^{-\delta} \supset \cdots \supset F^{\delta+1} = 0$. This gives an O-basis $b_1, \dots, b_r$ simultaneously adapted to the chain. Let $S_u = \{j : b_j \in F^u\}$. Then

$$|S_u| = \dim_{L'} G^u, \qquad S_{u+1} \subset S_u, \qquad S_{\delta+1} = \varnothing.$$

Order the indices so that $S_{-\delta}$ comes first, put $m' = |S_{-\delta}|$, and for these indices define $u_j = \max\{u \in [-\delta, \delta] : j \in S_u\}$. The central-jump hypothesis gives $m' \le m$. Use the coordinate rows b_j as the matrix B. Then $\Psi(b_j) \in \mathrm{Fil}^{u_j}$, proving the asserted filtration membership of e'_j.

The number of indices with $u_j = u$ is

$$|S_u| - |S_{u+1}| = \dim G^u - \dim G^{u+1} = \mathrm{mult}_{\tau_i}(u).$$

Multiplying by $h_i + u$ and summing proves the displayed bound, with equality for this construction. The inverse coordinate matrix belongs to $\mathrm{M}_r(O)$ because the b_j form an O-basis. Thus $B \in \mathrm{GL}_r(O)$, and extending this constant basis change to $O[[\pi_0]]$ preserves the Wach basis and has valuation cost zero. □

Proof of 36.83

Proof (outline). Fix $c \in J_-$. Write $B = \beta_c + S$, $h_0 = \beta_{c-2} + S$, and $\mathcal{F}^j = \iota(D_{\mathrm{cris},c}(T_2[1/\ell])) \cap \mathrm{Fil}^j D_{\mathrm{dR}}$. Choose a primitive ℓ-th root ζ, and expand $w(f)$ at ζ after clearing its pole by $Q_0^{\delta+B}$. Division by the resulting power of $tU(t)$, where $U(0) \neq 0$, gives Laurent coefficients $D_f(k)$ which depend O-linearly on f. The relation $(\gamma - 1)w(f) \in N(T_2)$, together with the action $\gamma : t \mapsto \chi_\ell(\gamma)t$, gives

$$D_f(k) \in \mathcal{F}^{-k} \quad (k \neq 0).$$

Claim 1. For the fixed negative component c, uniformly for all $w(f) \in \mathcal{W}_c(T_2)$, the constant-coefficient reduction gives an integer $a_c \ge 0$ and $x_f \in \mathcal{N}_{c-2}(T_2)$ with

$$a_c \le \sigma_2 + \beta_{c-1} + S - \frac{\beta_c}{\ell} + \lambda + 3, \qquad \ell^{a_c} D_f(0) - \iota\varphi^2 \,\mathrm{cmp}_{T_2}(x_f) \in \mathcal{F}^{-\delta}.$$

Lean: `Formalism.MainNotions.N.Pv.LocalLengthBound.H2Bound` (`Formalism.SharpRestriction.LevelLetterCocycles.QuotientImageDivisibility.H2Bound`, lines 20–35)
Lean: `Formalism.MainNotions.RamifiedCoefficientChains.Asm.h2_bound_of_admitted` (`Formalism.SharpRestriction.LevelLetterCocycles.FrobeniusLiftLemmasOfAdmitted`, lines 141–174)

To prove Claim 1, put $n_c = \beta_{c-1} + S$, $Q_1 = \varphi(Q_0)$, $P_c = Q_1^{n_c} Q_0^{\delta}$, $X_0 = 2\beta_c + S + \delta$, $\sigma = \lceil X_0/q^2 + 1 \rceil$, and $h_c = \lceil n_c - \beta_c/\ell + 2 + \log_\ell n_c \rceil$. The height condition defining $\mathcal{W}_c(T_2)$ puts $P_c w(f)$ in the $\mathcal{A}^\Delta$-span of $\varphi^2\mathcal{N}_{c-2}(T_2)$. Apply the two-Frobenius truncation with parameters $(2, 1, X_0, \sigma)$ to the generators of this span and average over Δ, whose order q is a unit in O. This gives finitely many $\alpha_k \in \mathcal{A}^\Delta$, $x_k \in \mathcal{N}_{c-2}(T_2)$ and invariant vectors z_k such that, at ζ, the Laurent coefficients of $\ell^{2\sigma} P_c w(f)$ and of $\sum_k \alpha_k z_k$ agree in degrees below $\beta_c + \delta$, and the Laurent coefficients of z_k in degrees below X_0 vanish except in degree 0, where the coefficient is $\ell^\sigma \mu_k$ with $\mu_k = \iota\varphi^2 \,\mathrm{cmp}_{T_2}(x_k)$. For $0 \neq k < \beta_c$ one has $D_f(k) \in \mathcal{F}^{-\delta}$: for $k \le \delta$ this follows from $D_f(k) \in \mathcal{F}^{-k}$, and for $\delta < k < \beta_c$ one has $\mathcal{F}^{-k} = \mathcal{F}^{-\delta}$, since component c has no jumps in $[-\beta_c + 1, -\delta - 1]$. At ζ, $Q_0 = tU(t)$ and Q_1 is a unit, so P_c is divisible by t^δ. Comparing coefficients of degree $R < \beta_c + \delta$ gives

$$\ell^\sigma [t^R](P_c)\, D_f(0) \equiv \sum_k [t^R](\alpha_k)\, \mu_k \pmod{\mathcal{F}^{-\delta}} \qquad (0 \le R < \beta_c + \delta).$$

Its denominators are controlled by a scalar statement. Let $\alpha \in \mathcal{A}^\Delta$ and $\theta \in L' \setminus O$ satisfy $[t^R](\alpha) = \theta[t^R](P_c)$ for all $R < \beta_c + \delta$, let ϖ be a uniformizer of O with $\ell O = \varpi^{e_O} O$, and let $s \ge 1$ be such that $\varpi^s \theta$ is a

unit. Then α is divisible by Q_0^{δ}, and the expansion of $Q_1^{n_c}$ in the variable Q_0 lies in the ideal $(\varpi^s, Q_0^{\beta_c})$ of $O[[Q_0]]$. In this variable $Q_1 = \ell\varepsilon + u'Q_0^{\ell}$ with units ε, u'. For units ε, u' of $O[[X]]$ and integers $\beta \geq 1$, $y \geq \beta/\ell$, the membership $(\ell\varepsilon + u'X^{\ell})^y \in (\varpi^s, X^{\beta})$ implies $s/e_O \leq y - \beta/\ell + 2 + \log_{\ell} y$. With $\beta = \beta_c$ and $y = n_c$, which satisfy $\beta_c/\ell \leq n_c$, this gives $\ell^{h_c}\theta \in O$. Applied to the coordinates of the displayed relation, the scalar statement gives $o_k \in O$ with $\ell^{\sigma+h_c} D_f(0) - \sum_k o_k \mu_k \in \mathcal{F}^{-\delta}$. Put $a_c = \sigma + h_c$ and $x_f = \sum_k o_k x_k \in \mathcal{N}_{c-2}(T_2)$. This is the asserted congruence. Finally $X_0 \leq 2H$, $n_c \leq H$ and $\ell \geq 11$ give $\sigma + h_c \leq \sigma_2 + \beta_{c-1} + S - \beta_c/\ell + \lambda + 3$. The term $-\beta_c/\ell$ is the saving from matching β_c coefficients of $Q_1^{n_c}$ in the variable Q_0.

Claim 2. Under the same data there is a uniform integer k_c, with the bound and lifting properties below. Lemma 36.81, followed by one level-one truncation, gives an integer k_c satisfying

$$k_c \leq \frac{\beta_c + S + 1}{q} + E_G + \sigma_2 + \beta_{c-2} + S - \beta_{c-1} + 3.$$

More precisely, for every $d \in \varphi^2 \mathcal{C}_{c-3}(T_2) \cap \mathrm{Fil}_{\mathrm{cris}}^{\beta_{c-1}}$ it gives $v_d \in \mathcal{V}_c(T_1)$ such that $Q_0^B F_D v_d \in N(T_2)$. At every primitive root, the Laurent expansion of $F_D v_d$ has coefficients

$$b_d(k) = 0 \quad (k < B,\ k \neq 0), \qquad b_d(0) = \ell^{k_c} \iota\varphi d, \qquad b_d(k) \in \iota D_{\mathrm{cris},c}(T_2[1/\ell]).$$

Indeed, lift d by Lemma 36.81 at $c-1$, then truncate its source lift with exponent $\lceil (B+1)/q + 1 \rceil$. Naturality of this truncation leaves an error in $\pi_0 Q_0^B \mathcal{R}_{c-1}^{(1)}$. After multiplication by Q_0^B, that error starts in degree $2B$. Division by $(tU)^B$ proves the displayed coefficient assertions.

Lean: `Formalism.MainNotions.N.Pv.LocalLengthBound.contentBoundLevel_of` (`Formalism.SharpRestriction.LevelLetterCocycles.QuotientImageDivisibility.SmulSeriesDeltaZero`, lines 273–517)

Put $e_{\mathrm{vec}} = \lfloor B_{\mathrm{vec}} \rfloor$. The identity

$$\frac{\beta_c + S + 1}{q} - \frac{\beta_c}{\ell} = \frac{\beta_c}{\ell q} + \frac{S+1}{q}$$

and $\beta_c \leq h$ show that $e_{\mathrm{vec}} \geq a_c + k_c - h_0$. Multiply x_f by $\ell^{e_{\mathrm{vec}} - a_c - k_c + h_0}$. The resulting relation is

$$\ell^{e_{\mathrm{vec}}} D_f(0) - \ell^{k_c - h_0} \iota\varphi^2 \,\mathrm{cmp}_{T_2}(x_f) \in \mathcal{F}^{-\delta}.$$

Define

$$K_3 = \{x \in \mathcal{N}_{c-2}(T_2) : \iota\varphi^2 \,\mathrm{cmp}_{T_2}(x) \in \mathcal{F}^{-\delta}\}.$$

The class $x_f + K_3$ is uniquely determined by this relation and gives an O-linear map. Put

$$\mathcal{P} = \mathrm{span}_{\mathcal{A}^{\Delta}}\{\varphi z : z \in \mathcal{N}_{c-3}(T_2)\}, \qquad X = \pi_0 \mathcal{N}_{c-2}(T_2) + Q_0^{h_0}\mathcal{P}.$$

Claim 3. For every $i \in J_-$, put $b = \beta_i + S$. There is an $O[[\pi_0]]$-basis $e_1', \dots, e_r'$ of $\mathcal{N}_{i-1}(T_2)$, an integer $m' \leq m$, and integers $-\delta \leq u_j \leq \delta$, $1 \leq j \leq m'$, such that

$$\iota\varphi \,\mathrm{cmp}(e_j') \in \mathrm{Fil}^{u_j} D_{\mathrm{dR}}, \qquad \sum_{u=-\delta}^{\delta} (b+u)\,\mathrm{mult}_{\tau_i}(u) = \sum_{j=1}^{m'} (b + u_j).$$

The basis is obtained from any invariant Wach basis by a matrix in $\mathrm{GL}_r(O)$.

This is proved in Lemma 36.82: the specialization and Frobenius comparison send a Wach basis to an L'-basis $w_1, \dots, w_r$ of the i-component of the crystalline period space. Pull back each filtration step through the isomorphism $\Psi : (L')^r \to \mathrm{span}_{L'}\{w_j\}$, and intersect it with O^r. These intersections are saturated submodules. Every saturated submodule of a finite free module over the discrete valuation ring O is a direct summand. Applying this fact successively to the finite descending chain for $-\delta \leq u \leq \delta + 1$ gives one O-basis adapted to all its terms. The term at $\delta + 1$ is zero because the jumps in component i are at most δ. Let m' be the rank of the term at $-\delta$, and assign each of its basis vectors the greatest step u_j to which it belongs. The number assigned to u is the difference of the ranks at u and $u+1$, namely $\mathrm{mult}_{\tau_i}(u)$. Therefore $m' \leq m$ and the displayed weighted identity holds. The coordinate matrix and its inverse are integral, which proves the last assertion.

Lean: Formalism.MainNotions.RamifiedCoefficientChains.S9.AdM.adaptBas_of (Formalism.SharpRestriction.LevelLetterCocycles.RingClausesAtUniformizer, lines 393–601)

Claim 4. Under the data of Definition 36.76, there is a finite set $S_{c-2} \subset \mathcal{N}_{c-2}(T_2)$, of cardinality at most m, such that

$$\lambda_O\big(\mathcal{N}_{c-2}(T_2)/\big(\mathcal{N}_{c-2}(T_2) \cap (X + OS_{c-2})\big)\big) \le C_{\mathrm{len}}.$$

Here OS_{c-2} denotes its O-linear span. The index $c-2$ belongs to J_-, and the exponent in this instance is $h_0 = \beta_{c-2} + S$.

Lean: Formalism.MainNotions.RamifiedCoefficientChains.S9.LemmaDN (Formalism.SharpRestriction.LevelLetterCocycles.LatticeBoundsOfThreeParts, lines 492–502)

Lean: Formalism.MainNotions.RamifiedCoefficientChains.S9.bigCellDetWeightedEnergyIntegral.dual_lemma_of_admitted (Formalism.SharpRestriction.LevelLetterCocycles.UniformizerClauseOfAdmitted, lines 50–118)

To prove the claim, write $i = c-2$, $b = \beta_i + S = h_0$, and $C = (H+1)/q + 1$. Use Claim 3 to choose a basis $e'_1, \dots, e'_r$ of $\mathcal{N}_{i-1}(T_2)$, integers $m' \le m$ and $-\delta \le u_j \le \delta$, $1 \le j \le m'$, satisfying its filtration and weighted-multiplicity assertions. Choose a basis $e_1, \dots, e_r$ at i. In the formal coordinate x, evaluated at π_0, let $\mathbf{P}(\mathsf{x})$ be the matrix whose columns are $Q_0^b \varphi e'_j$ in this basis. These columns are integral by the lower jump bound $-b$. Then

$$\det \mathbf{P}(\mathsf{x}) = a(\mathsf{x})(\mathsf{x}+\ell)^{w_0}, \quad a(\mathsf{x}) \in O[[\mathsf{x}]]^\times, \qquad w_0 = br + \sum_{v=-b}^{\delta} v \, \mathrm{mult}_{\tau_i}(v).$$

Indeed, a reverse Frobenius-height inclusion gives a matrix $\mathbf{P}'$ with $\mathbf{P}\mathbf{P}' = (\mathsf{x}+\ell)^N I_r$ for some $N \ge 0$, so the only possible prime factor of $\det \mathbf{P}$ is $\mathsf{x}+\ell$. Evaluating at a primitive root and choosing a basis of the de Rham filtration gives its order $br + \sum v \, \mathrm{mult}_{\tau_i}(v)$.

Lean: Formalism.MainNotions.RamifiedCoefficientChains.Sh.detTool_of (Formalism.SharpRestriction.LevelLetterCocycles.EvaluationAtUniformizerPower, lines 1631–1733)

Put $\sigma_i = \lceil (1 + b + \delta)/q + 1 \rceil$; then $\sigma_i \le 2\lfloor C \rfloor$. Level-one truncation of the e'_j gives a matrix $\mathbf{Z}(\mathsf{x})$ with $\mathbf{Z}(0) = \ell^{\sigma_i} I_r$. If

$$f_j = \sum_l \mathbf{Z}_{lj}(\pi_0) \varphi e'_l,$$

then, at every primitive root, the Laurent coefficients of f_j of degree less than $b + \delta$ vanish except in degree 0, where the coefficient is $\ell^{\sigma_i} \iota\varphi \, \mathrm{cmp}(e'_j)$. Set

$$k_j = b + u_j, \qquad g_j = Q_0^{b-k_j} f_j \quad (1 \le j \le m').$$

Here $k_j \ge 0$. The vectors g_j are integral. To verify this, $Q_0^{k_j} g_j = Q_0^b f_j$ is integral. Write $Q_0 = tU(t)$ at a primitive root. After multiplication by the unit series $U(t)^{k_j}$, the coefficient in degree ν of g_j is the coefficient in degree $\nu + u_j$ of $U(t)^b f_j$. It vanishes when $\nu + u_j < 0$. When $0 \le \nu + u_j < b + \delta$, it belongs to $\mathrm{Fil}^{u_j} \subset \mathrm{Fil}^{-\nu}$. When $\nu + u_j \ge b + \delta$, one has $\nu \ge b$, and the lower jump bound $-b$ gives the same inclusion. Division by the unit series preserves these coefficient conditions, and the filtered Wach criterion proves integrality.

Lean: Formalism.MainNotions.RamifiedCoefficientChains.Sh.ZTool (Formalism.SharpRestriction.LevelLetterCocycles.ExpRules, lines 91–104)

Lean: Formalism.MainNotions.RamifiedCoefficientChains.S9.GM.gE_mem (Formalism.SharpRestriction.LevelLetterCocycles.DlOfQp, lines 86–214)

Lean: Formalism.MainNotions.RamifiedCoefficientChains.S9.GM.gMem_of (Formalism.SharpRestriction.LevelLetterCocycles.DlOfQp, lines 216–220)

Form a matrix $\mathbf{C}(\mathsf{x})$ whose first m' columns are the coordinates of g_j and whose remaining columns are those of $Q_0^b \varphi e'_j$. Let $\mathbf{Z}_*(\mathsf{x})$ have the first m' columns of $\mathbf{Z}(\mathsf{x})$ and the remaining columns of the identity. Then

$$\mathbf{C}(\mathsf{x}) \, \mathrm{diag}((\mathsf{x}+\ell)^{k_1}, \dots, (\mathsf{x}+\ell)^{k_{m'}}, 1, \dots, 1) = \mathbf{P}(\mathsf{x}) \mathbf{Z}_*(\mathsf{x}),$$

and hence

$$\det \mathbf{C}(0) \, \ell^{\sum_j k_j} = a(0) \ell^{w_0 + \sigma_i m'}.$$

The adapted-basis inequality and the two jump windows imply

$$\sum_j k_j \ge \sum_{v=-\delta}^{\delta} (b+v) \, \mathrm{mult}_{\tau_i}(v), \qquad w_0 - \sum_{v=-\delta}^{\delta} (b+v) \, \mathrm{mult}_{\tau_i}(v) \le rS.$$

The latter inequality follows because $0 \le b + v \le S$ for the noncentral jumps $v \in [-b, -\beta_i]$. Thus the normalized valuation of $\det \mathbf{C}(0)$ is at most

$$rS + \sigma_i m' \le rS + 2\lfloor C \rfloor m \le C_{\mathrm{len}}.$$

Reduction in the basis e identifies $\mathcal{N}_i/\pi_0\mathcal{N}_i$ with O^r. All columns of $\mathbf{C}(0)$ lie in the reduction of $Q_0^b\mathcal{P} + \sum_{j \le m'} O g_j$. The determinant-length formula over the discrete valuation ring bounds the quotient length by C_{len}. Take S_{c-2} to be the set of these g_j; this proves Claim 4, including its cardinality bound and its intersection denominator.

Lean: `Formalism.MainNotions.RamifiedCoefficientChains.S9.DetA.detAlg_of` (`Formalism.SharpRestriction.LevelLetterCocycles.JumpsInLetterOfEvenIndex`, lines 171–343)
Lean: `Formalism.MainNotions.RamifiedCoefficientChains.S9.DetS.lemmaDN_of_S1E` (`Formalism.SharpRestriction.LevelLetterCocycles.UniformizerUnitAndQFactor`, lines 264–266)

Claim 5. Let N be an O-module, let $K \subset X_0 \subset N$, and let $s_1, \dots, s_a \in N$, with $a \le m$. If

$$\lambda_O\big(N/(X_0 + \textstyle\sum_j O s_j)\big) \le C_{\mathrm{len}},$$

then the inverse images of $(X_0 + \sum_j O s_j)/K$ and X_0/K under any linear map $U \to N/K$ give one length step and one generator step. The first quotient injects into the displayed length quotient. The second injects into $(X_0 + \sum_j O s_j)/X_0$, which is generated by at most a elements. Its submodules also need at most a generators: lift such a submodule to O^a, take a basis there, and project.

Apply Claim 5 with $N = \mathcal{N}_{c-2}(T_2)$, $K = K_3$, $X_0 = X + K_3$, and the map $f \mapsto x_f + K_3$. The bound is Claim 4, since adding K_3 decreases the quotient length. On the resulting submodule, x_f may be represented modulo K_3 by an element of X. Specialization then gives $\mathrm{cmp}(x_f) = \ell^{h_0}\varphi\,\mathrm{cmp}(t_f)$ modulo the same ambiguity, with $t_f \in \mathcal{N}_{c-3}(T_2)$.

Impose the further condition $\varphi^2\,\mathrm{cmp}(t_f) \in \mathrm{Fil}_{\mathrm{cris}}^{\beta_{c-1}}$. Put

$$A_3 = \{t \in \mathcal{N}_{c-3}(T_2) : \iota\varphi^3\,\mathrm{cmp}(t) \in \mathcal{F}^{-\delta}\}, \qquad C_3 = \{t \in \mathcal{N}_{c-3}(T_2) : \varphi^2\,\mathrm{cmp}(t) \in \mathrm{Fil}_{\mathrm{cris}}^{\beta_{c-1}}\}.$$

The class $t_f + A_3$ is uniquely determined and linear. Take its preimage of $(C_3 + A_3)/A_3$. The quotient $\mathcal{N}_{c-3}(T_2)/C_3$ maps into $D_{\mathrm{cris},c-1}/\mathrm{Fil}_{\mathrm{cris}}^{\beta_{c-1}}$, whose dimension is at most m. Specialization factors through the r constant coordinates of an invariant Wach basis, so this quotient is finitely generated over O, and hence generated by at most m elements. This gives a second generator step. Denote the resulting coefficient module by U_3, and choose the vectors $d_f = \varphi^2\,\mathrm{cmp}(t_f)$ on an O-basis of U_3, then extend linearly. They satisfy

$$\ell^{e_{\mathrm{vec}}} D_f(0) - \ell^{k_c}\iota\varphi d_f \in \mathcal{F}^{-\delta}.$$

Choose the corresponding source vectors v_{d_f} on this basis and extend linearly to a map $y_c : U_3 \to \mathcal{V}_c(T_1)$. Define U' as the kernel of

$$f \longmapsto \ell^{e_{\mathrm{vec}}} D_f(0) - \ell^{k_c}\iota\varphi d_f \pmod{\mathcal{F}^0}.$$

Its image lies in $\mathcal{F}^{-\delta}/\mathcal{F}^0$, of dimension at most m, so this is the third generator step. On U', every coefficient of $\ell^{e_{\mathrm{vec}}} w - F_D y_c$ lies in the required filtration: the degree-zero assertion follows from the last kernel, nonzero degrees below B follow from $b_{d_f}(k) = 0$, and degrees at least B follow from the lower jump bound $-B$. *Claim 6.* Let T be crystalline in the present local setting, let $v \in \mathcal{D}(T)$ satisfy $e_c v = v$, and suppose $Q^k v \in N(T)$ for some integer $k \ge 0$, where $Q = \Phi_\ell(1 + Z)$. If, for every primitive ℓ-th root ζ and every $j \in \mathbf{Z}$,

$$[t^j]\,\mathrm{ev}_{c,\zeta}^N(Q^k v) \in \iota(D_{\mathrm{cris},c}(T[1/\ell])) \cap \mathrm{Fil}^{k-j} D_{\mathrm{dR}},$$

then $v \in N(T)$. To see this, choose finite upper and lower bounds for the jumps of T. The filtered characterization of the Wach lattice then tests the cleared vector at every zero of Q. At component c the displayed condition is precisely the divisibility test obtained after shifting by the clearing exponent k; at every other component the idempotent e_c evaluates to zero, so its test is automatic.

Lean: `Formalism.MainNotions.N.Pv.LocalLengthBound.MemCrit` (`Formalism.SharpRestriction.LevelLetterCocycles.QuotientImageDivisibility.MemCrit`, lines 18–23)
Lean: `Formalism.MainNotions.N.Pv.LocalLengthBound.memCrit_of` (`Formalism.SharpRestriction.LevelLetterCocycles.QuotientImageDivisibility.JumpsLetExists`, lines 218–244)
Lean: `Formalism.MainNotions.RamifiedCoefficientChains.Asm.frobenius_lift_lemmas_general` (`Formalism.SharpRestriction.LevelLetterCocycles.FrobeniusLiftLemmasOfAdmitted`, lines 67–90)

Apply Claim 6 to the residual. A common clearing power exists by the displayed bounds on w and $F_D y_c$. Multiplication by the power-series unit relating Q to Q_0 preserves the coefficient filtration conditions. The residual is Δ-invariant; transport to another primitive root multiplies its degree-j Laurent coefficient by a nonzero scalar, so the conditions established at the chosen root hold at every root. They therefore give exactly the hypotheses of Claim 6 and prove integrality. The complete proof is contained in `Formalism.MainNotions.N.Pv.LocalLengthBound.length_bound_of_hypotheses`, using `Formalism.MainNotions.N.Pv.LocalLengthBound.pass3` and `Formalism.MainNotions.N.Pv.LocalLengthBound.pass4`. □

Proof of 36.85

Proof. The height hypothesis puts $Q_0^B \varphi z$ in the c-component of the invariant Wach lattice. The action of γ on this lattice is trivial modulo π_0, so $x \in \mathcal{V}_c(T_1)$. The same height bound for T_2, together with naturality of Frobenius, shows that $F_D x \in \mathcal{N}_c(T_2)$. Weighted specialization gives

$$\mathrm{cmp}_{T_2}(F_D x) = \ell^{t+B+k} \varphi^3 d_3.$$

The first three displayed identities give

$$\ell^{e+s_1+s_2}\,\mathrm{cmp}_{T_2}(r) = \ell^{e+B+h_0-b_0}\varphi^3 d_3 = \ell^{t+B+k}\varphi^3 d_3.$$

Thus the difference lies in the kernel of specialization on $\mathcal{N}_c(T_2)$, which is $\pi_0 \mathcal{N}_c(T_2)$. This is contained in $\mathcal{U}^1(T_2)$. □

Proof of 36.86

Proof. For an index i, let

$$\mathcal{P}_i = \mathrm{span}_{\mathcal{A}^\Delta}\{\varphi z : z \in \mathcal{N}_{i-1}(T_2)\}, \qquad X_i(k) = \pi_0 \mathcal{N}_i(T_2) + Q_0^k \mathcal{P}_i, \qquad K_i = \ker(\mathrm{cmp}_{T_2}|_{\mathcal{N}_i(T_2)}).$$

Claim 1. For each $i \in J_-$,

$$Q_0^{\beta_i+S}\mathcal{P}_i \subset \mathcal{N}_i(T_2), \qquad \ker(\mathrm{cmp}_{T_2}|_{\mathcal{N}_i(T_2)}) = \pi_0 \mathcal{N}_i(T_2).$$

The first inclusion uses the lower jump bound $-\beta_i - S$ at i; all the vectors and scalar coefficients in $\mathcal{P}_i$ are Δ-invariant. For the second equality, specialize an invariant Wach basis: its specializations form an L'-basis of the crystalline component, so an invariant integral vector specializes to zero exactly when every power-series coordinate has zero constant coefficient. Consequently $X_i(\beta_i + S) \subset \mathcal{N}_i(T_2)$ and $K_i \subset X_i(\beta_i + S)$.
Lean: `Formalism.MainNotions.N.Pv.LocalLengthBound.frobenius_sq_bound_of_local` (`Formalism.SharpRestriction.LevelLetterCocycles.QuotientImageDivisibility.FrobeniusSqBoundOfLocal`, lines 55–165)
Lean: `Formalism.MainNotions.N.Pv.LocalLengthBound.ker_NDl` (`Formalism.SharpRestriction.LevelLetterCocycles.QuotientImageDivisibility.FrobeniusUniformizerPowDvd`, lines 103–127)

Claim 2. At each $i \in J_-$, there is a set $S_i \subset \mathcal{N}_i(T_2)$, with $|S_i| \le m$, for which

$$\lambda_O\big(\mathcal{N}_i(T_2)/(X_i(\beta_i + S) + OS_i)\big) \le C_{\mathrm{len}}.$$

Fix $i \in J_-$ and set $b = \beta_i + S$ and $C = (H+1)/q + 1$. Applying Lemma 36.82 to an invariant $O[[\pi_0]]$-basis of $\mathcal{N}_{i-1}(T_2)$ yields a refined $O[[\pi_0]]$-basis $e'_1, \dots, e'_r$ of $\mathcal{N}_{i-1}(T_2)$ (via a matrix in $\mathrm{GL}_r(O)$), an integer $m' \le m$, and central jump indices $-\delta \le u_j \le \delta$ $(1 \le j \le m')$ satisfying $\iota\varphi\,\mathrm{cmp}_{T_2}(e'_j) \in \mathrm{Fil}^{u_j} D_{\mathrm{dR}}$ and $\sum_{u=-\delta}^{\delta}(b+u)\,\mathrm{mult}_{\tau_i}(u) \le \sum_{j=1}^{m'}(b+u_j)$ (with equality for the filtration-adapted construction of Lemma 36.82).

Choose an $O[[\pi_0]]$-basis $v_1, \dots, v_r$ of $\mathcal{N}_i(T_2)$ and let $\mathbf{P}(\mathsf{x}) \in M_r(O[[\mathsf{x}]])$ be the matrix (evaluated at $\mathsf{x} = \pi_0$) whose columns are $Q_0^b \varphi e'_j$ in the basis $v_1, \dots, v_r$ (integral by Claim 1). A reverse Frobenius-height inclusion supplies $\mathbf{P}'(\mathsf{x})$ with $\mathbf{P}\mathbf{P}' = (\mathsf{x}+\ell)^N I_r$, so $\det \mathbf{P}(\mathsf{x}) = a(\mathsf{x})(\mathsf{x}+\ell)^{w_0}$ with $a(\mathsf{x}) \in O[[\mathsf{x}]]^\times$; evaluating at a primitive ℓ-th root against a de Rham filtration basis identifies the determinant exponent as $w_0 = br + \sum_{u=-b}^{\delta} u\,\mathrm{mult}_{\tau_i}(u)$ (the same determinant calculation as in Claim 4 of Lemma 36.83). Next, set $\sigma_i = \lceil (1 + b + \delta)/q + 1 \rceil$. Since $b + \delta = \beta_i + S + \delta \le h + \delta = H$ (Definition 36.76) and $C > 1$, we have $\sigma_i \le \lceil C \rceil \le \lfloor C \rfloor + 1 \le 2\lfloor C \rfloor$. To construct $\mathbf{Z}(\mathsf{x})$ by level-one γ-truncation directly in $\mathcal{A}^\Delta$: because

$|\Delta| = \ell - 1$ is a unit in O, $\varphi(\pi_0) = u_\varphi \pi_0 Q_0^q$ (with $u_\varphi \in (\mathcal{A}^\Delta)^\times$), and $\pi_0 = Q_0 - \ell$, for every $y \geq 1$ we have $\varphi(\pi_0)^y = u_\varphi^y \pi_0 (Q_0 - \ell)^{y-1} Q_0^{yq}$. Expanding $(Q_0 - \ell)^{y-1}$ modulo $Q_0^{b+\delta-yq}$ retains only powers Q_0^k with $0 \leq k \leq b+\delta-yq-1$, each carrying the scalar factor $(-\ell)^{y-1-k}$ of ℓ-adic valuation $y-1-k \geq \max(0, y\ell-(b+\delta))$ directly inside $\mathcal{A}^\Delta$; since $q = \ell - 1 \geq 2$ and $\sigma_i = \lceil (1+b+\delta)/q + 1 \rceil$, every $y \geq 0$ satisfies the exponent bound

$$\sigma_i + \max(0, y\ell - b - \delta) - \lceil y\ell/q \rceil \geq 0$$

(for $y\ell \leq b+\delta$ directly $\lceil y\ell/q \rceil \leq \lceil (b+\delta)/q \rceil \leq \sigma_i$, while for $y\ell > b+\delta$ we have $\sigma_i + (y\ell - b - \delta) - \lceil y\ell/q \rceil \geq (y\ell - b - \delta)(1 - 1/q) + 2/q > 0$). Consequently, for every $y \geq 0$ there exist a remainder $r_y \in \mathcal{A}^\Delta$ congruent to $\varphi(\pi_0)^y$ modulo $\pi_0 Q_0^{b+\delta} \mathcal{A}^\Delta$ (with $r_0 = 1$ and $r_y \in \pi_0 \mathcal{A}^\Delta$ for $y \geq 1$) and an integral quotient $s_y \in \mathcal{A}^\Delta$ satisfying $\ell^{\lceil y\ell/q \rceil} s_y = \ell^{\sigma_i} r_y$ (putting $\ell^{\sigma_i} \varphi(\pi_0)^y \in (\ell^{\lceil y\ell/q \rceil}, \pi_0 Q_0^{b+\delta})$ in $\mathcal{A}^\Delta$). Solving the Δ-invariant Wach connection iteratively for each $1 \leq j \leq r$ yields horizontal-series coefficients $a_{y,j} \in (L')^r$ ($y \geq 0$, with $a_{0,j}$ the j-th standard basis vector) satisfying $\ell^{\lceil y\ell/q \rceil} a_{y,j} \in O^r$. Truncating at $K_0 = b + \delta + 1$ and setting $f_j = \sum_{y=0}^{K_0-1} s_y (\ell^{\lceil y\ell/q \rceil} a_{y,j})$ in the $\mathcal{A}^\Delta$-basis $\varphi e'_1, \ldots, \varphi e'_r$ produces an integral matrix $\mathbf{Z}(\mathsf{x}) \in M_r(O[[\mathsf{x}]])$ with $\mathbf{Z}(0) = \ell^{\sigma_i} I_r$ and $f_j = \sum_{l=1}^r \mathbf{Z}_{lj}(\pi_0) \varphi e'_l$. Put $e_Y(K_0) = \lceil K_0 \ell/q \rceil$ and form the finite horizontal partial sum $E_0(K_0) = \ell^{e_Y(K_0)} \sum_{y=0}^{K_0-1} a_{y,j} \varphi(\pi_0)^y \in \mathcal{P}_i$. Since $q = \ell - 1 \geq 2$ and $K_0 > b + \delta$ give $qK_0 \geq K_0 > b + \delta$, the recurrence for $(a_{y,j})$ yields $(\gamma - 1) E_0(K_0) \in \varphi(\pi_0)^{K_0} \mathcal{P}_i \subseteq \pi_0 Q_0^{b+\delta} \mathcal{P}_i$. By the relations $\ell^{\lceil y\ell/q \rceil} s_y = \ell^{\sigma_i} r_y$ and $r_y \equiv \varphi(\pi_0)^y \pmod{\pi_0 Q_0^{b+\delta} \mathcal{A}^\Delta}$, we have $\ell^{e_Y(K_0)} f_j \equiv \ell^{\sigma_i} E_0(K_0) \pmod{\pi_0 Q_0^{b+\delta} \mathcal{P}_i}$; cancelling $\ell^{e_Y(K_0)}$ in the ℓ-torsion-free quotient $\mathcal{P}_i / \pi_0 Q_0^{b+\delta} \mathcal{P}_i$ gives $(\gamma - 1) f_j \in \pi_0 Q_0^{b+\delta} \mathcal{P}_i$. At every primitive ℓ-th root ζ (where $\pi_0(\zeta) = -\ell \neq 0$ and $Q_0 = t_0 U(t_0)$ for a unit series $U(t_0)$ in the local parameter t_0), the identity $\varphi(\pi_0) = u_\varphi \pi_0 Q_0^q$ shows that $\varphi(\pi_0)$ vanishes to order $q = \ell - 1$ at ζ, i.e., $\varphi(\pi_0) \in t_0^q L'[[t_0]]$. The formal horizontal series $h_j(t_0) = \ell^{\sigma_i} \sum_{y \geq 0} a_{y,j} \varphi(\pi_0)^y$ therefore converges t_0-adically in $L'[[t_0]]$, agrees with f_j modulo $t_0^{b+\delta}$ (via $\ell^{e_Y(K_0)} f_j \equiv \ell^{\sigma_i} E_0(K_0)$ and $qK_0 > b + \delta$), and is γ-invariant in $L'[[t_0]]$, hence constant and equal to $\ell^{\sigma_i} \iota\varphi \operatorname{cmp}_{T_2}(e'_j) \in \operatorname{Fil}^{u_j} D_{\mathrm{dR}}$. Consequently all Laurent coefficients of f_j at ζ in degrees strictly below $b + \delta$ vanish except degree 0, where the value is $\ell^{\sigma_i} \iota\varphi \operatorname{cmp}_{T_2}(e'_j) \in \operatorname{Fil}^{u_j} D_{\mathrm{dR}}$. Put $k_j = b + u_j \geq 0$ and $g_j = Q_0^{-u_j} f_j = Q_0^{b-k_j} f_j$ for $1 \leq j \leq m'$. Then $Q_0^{k_j} g_j = Q_0^b f_j$ is integral; substituting $Q_0 = t_0 U(t_0)$ at each primitive root shifts degree ν of $U(t_0)^{k_j} g_j$ to degree $\nu + u_j$ of $U(t_0)^b f_j$ (which vanishes for $\nu + u_j < 0$, lies in $\operatorname{Fil}^{u_j} \subset \operatorname{Fil}^{-\nu}$ for $0 \leq \nu + u_j < b + \delta$, and lies in $\operatorname{Fil}^{-\nu}$ for $\nu \geq b$ by the lower jump bound $-b$), so the filtered Wach criterion (Claim 6 of Lemma 36.83) forces $g_j \in \mathcal{N}_i(T_2)$.

Let $\mathbf{C}(\mathsf{x}) \in M_r(O[[\mathsf{x}]])$ have first m' columns the coordinates of $g_1, \ldots, g_{m'}$ and remaining $r - m'$ columns those of $Q_0^b \varphi e'_{m'+1}, \ldots, Q_0^b \varphi e'_r$ in the basis $v_1, \ldots, v_r$, and let $\mathbf{Z}_*(\mathsf{x})$ have the first m' columns of $\mathbf{Z}(\mathsf{x})$ and the remaining $r - m'$ columns of I_r. Then $\mathbf{C}(\mathsf{x}) \operatorname{diag}((\mathsf{x}+\ell)^{k_1}, \ldots, (\mathsf{x}+\ell)^{k_{m'}}, 1, \ldots, 1) = \mathbf{P}(\mathsf{x}) \mathbf{Z}_*(\mathsf{x})$. Evaluating determinants at $\mathsf{x} = 0$ gives $\det \mathbf{C}(0)\, \ell^{\sum_{j=1}^{m'} k_j} = a(0) \ell^{w_0 + \sigma_i m'}$, where $a(0) \in O^\times$, and hence

$$v_\ell(\det \mathbf{C}(0)) = w_0 + \sigma_i m' - \sum_{j=1}^{m'} (b + u_j) = \sum_{u=-b}^{-\beta_i} (b+u) \operatorname{mult}_{\tau_i}(u) + \sigma_i m' \leq rS + 2m \left\lfloor \frac{H+1}{q} + 1 \right\rfloor \leq C_{\mathrm{len}},$$

since the central multiplicity sum cancels $\sum_{j=1}^{m'} (b + u_j)$ while every noncentral jump $u \in [-b, -\beta_i]$ satisfies $0 \leq b + u \leq S$. Reduction modulo π_0 identifies $\mathcal{N}_i(T_2)/\pi_0 \mathcal{N}_i(T_2)$ with O^r, and every column of $\mathbf{C}(0)$ lies in the reduction of $X_i(b) + OS_i$ for $S_i = \{g_1, \ldots, g_{m'}\}$. The discrete-valuation-ring determinant-length formula and Claim 1 therefore yield $\lambda_O(\mathcal{N}_i(T_2)/(X_i(b) + OS_i)) \leq C_{\mathrm{len}}$.

Lean: `Formalism.MainNotions.RamifiedCoefficientChains.S9.LemmaDN` (`Formalism.SharpRestriction.LevelLetterCocycles.LatticeBoundsOfThreeParts`, lines 492–502)

Lean: `Formalism.MainNotions.RamifiedCoefficientChains.S9.bigCellDetWeightedEnergyIntegral.dual_lemma_of_admitted` (`Formalism.SharpRestriction.LevelLetterCocycles.UniformizerClauseOfAdmitted`, lines 50–118)

Apply Claim 2 at $i = c, c-2$. Define $U_5 \subset U$ as the inverse image of $X_c(\beta_c + S)/K_c$ under $f \mapsto r(f) \bmod K_c$. Lemma 36.84 constructs this restriction with one length step and one generator step. Write $r(f)$, modulo $\pi_0 \mathcal{N}_c(T_2)$, as a sum of terms $Q_0^{\beta_c + S} a \varphi z$, with $a \in \mathcal{A}^\Delta$ and $z \in \mathcal{N}_{c-1}(T_2)$. Replace each a by its constant coefficient when applying specialization. This constructs $d_1(f) \in \mathcal{C}_{c-1}(T_2)$ such that

$$\operatorname{cmp}_{T_2}(r(f)) = \ell^{\beta_c + S} \varphi d_1(f).$$

The upper jump bound $\beta_{c-1} + S$ gives

$$Q_0^{\beta_{c-1} + S} \mathcal{N}_{c-1}(T_2) \subset \mathcal{P}_{c-1}.$$

To specialize this inclusion, express the right side as a finite sum of scalar multiples of Frobenius vectors, multiply by a common clearing height, and replace each scalar by its constant coefficient. Cancelling the nonzero scalar ℓ-power then constructs $d_2(f) \in \mathcal{C}_{c-2}(T_2)$ with

$$\ell^{\beta_{c-1}+S} d_1(f) = \varphi d_2(f).$$

Both maps are O-linear because Frobenius is injective. If $y_2(f) \in \mathcal{N}_{c-2}(T_2)$ represents $d_2(f)$, its class modulo K_{c-2} is uniquely determined. Define $U_6 \subset U_5$ as the inverse image of $X_{c-2}(\beta_{c-2}+S)/K_{c-2}$ under this linear map. Lemma 36.84 gives a second length step and a second generator step. On U_6, the same constant-coefficient construction gives an O-linear map $d_3(f) \in \mathcal{C}_{c-3}(T_2)$ with

$$d_2(f) = \ell^{\beta_{c-2}+S} \varphi d_3(f).$$

Lean: `Formalism.MainNotions.N.Pv.LocalLengthBound.cmp_of_mem_X` (`Formalism.SharpRestriction.LevelLetterCocycles.QuotientImageDivisibility.CmpOfMemX`, lines 116–171)
Lean: `Formalism.MainNotions.N.Pv.LocalLengthBound.cmp_up` (`Formalism.SharpRestriction.LevelLetterCocycles.QuotientImageDivisibility.CmpOfMemX`, lines 173–225)

Define

$$U'' = \ker\left(U_6 \longrightarrow D_{\mathrm{cris}}(T_2[1/\ell])/\operatorname{Fil}_{\mathrm{cris}}^{\beta_{c-1}}, \quad f \longmapsto \varphi^2 d_3(f)\right).$$

Its image lies in the $(c-1)$-component modulo the indicated filtration, whose L'-dimension is at most m. Since U_6 is finitely generated over the discrete valuation ring O, this image is generated by at most m elements. Passing to U'' gives the third generator step. Each of the two length steps has normalized length at most C_{len}.

Let E_{c-1} be the integer in the crystalline interpolation estimate of Lemma 36.81, and set

$$k = \lceil E_{c-1} + \sigma_2 \rceil + \beta_{c-2} + S - \beta_{c-1}, \qquad t = \lceil E_G + \sigma_2 \rceil - \lceil E_{c-1} + \sigma_2 \rceil \geq 0.$$

For $f \in U''$, choose $z_f \in \mathcal{N}_{c-1}(T_1)$ whose image under F_D is integral and satisfies

$$\mathrm{cmp}_{T_2}(F_D z_f) = \ell^k \varphi^2 d_3(f).$$

Define the correction explicitly by

$$x_f = \ell^t Q_0^{\beta_c+S} \varphi z_f.$$

The identity

$$e_{\mathrm{int}} + \beta_{c-2} + S = t + k + \beta_{c-1} + S$$

allows Lemma 36.85 to be applied with $s_1 = s_2 = 0$. It gives $x_f \in \mathcal{V}_c(T_1)$ and $\ell^{e_{\mathrm{int}}} r(f) - F_D x_f \in \mathcal{U}^1(T_2)$. Finally, $e_{\mathrm{int}} < S + E_G + \sigma_2 + 1 \leq B_{\mathrm{int}}$, since $\lambda \geq 1$.
Lean: `Formalism.MainNotions.N.Pv.LocalLengthBound.frobenius_sq_bound_at` (`Formalism.SharpRestriction.LevelLetterCocycles.QuotientImageDivisibility.FrobeniusSqComparisonLetter`, lines 54–244)

For $w = w_c|_V$, the two vector corrections give

$$\ell^{e_{\mathrm{vec}}+e_{\mathrm{int}}} w_c(f) - F_D u_f = \ell^{e_{\mathrm{int}}} r_c(f) - F_D x_f \in \mathcal{U}^1(T_2),$$

and since $u_f \in \mathcal{V}_c(T_1)$, the lifting property of Lemma 36.80 supplies $Y_c(f) \in \mathcal{H}(T_1)$ with finite class and places both coordinates of $\ell^{e_{\mathrm{vec}}+e_{\mathrm{int}}} X_c(f) - F_{\mathcal{H}} Y_c(f)$ in $\mathcal{U}^M(T_2)$. On $\mathcal{U}^M(T_2) = \pi_0^M N(T_2)^\Delta$, the operator $\varphi - 1$ is bijective: since $M = \lfloor h/q \rfloor + 1$ satisfies $Mq > h$ and $\varphi(\pi_0) = u_\varphi \pi_0 Q_0^q$ with $u_\varphi \in (\mathcal{A}^\Delta)^\times$, while the lower jump bound $-h$ gives $Q_0^h \varphi(N(T_2)^\Delta) \subseteq N(T_2)^\Delta$, we obtain $\varphi(\pi_0^M N(T_2)^\Delta) = u_\varphi^M \pi_0^M Q_0^{Mq-h}(Q_0^h \varphi(N(T_2)^\Delta)) \subseteq Q_0 \pi_0^M N(T_2)^\Delta \subseteq (\pi_0, \ell)\pi_0^M N(T_2)^\Delta$, so $-\sum_{k\geq 0} \varphi^k$ converges (π_0, ℓ)-adically on $\mathcal{U}^M(T_2)$ to the two-sided inverse of $\varphi - 1$. If (z, v) is a Herr cocycle with both coordinates in $\mathcal{U}^M(T_2)$, choose $u_0 \in \mathcal{U}^M(T_2)$ with $(\varphi - 1)u_0 = z$. Then

$$(\varphi - 1)((\gamma - 1)u_0 - v) = 0.$$

The module $\mathcal{U}^M(T_2)$ is γ-stable, so injectivity of $\varphi - 1$ gives $(\gamma - 1)u_0 = v$. Hence this Herr cocycle is a boundary. Naturality now gives

$$\ell^{e_{\mathrm{vec}}+e_{\mathrm{int}}}[\kappa_c(f)] = F_* \operatorname{cl}_{T_1}(Y_c(f)),$$

which is equivalent to the asserted membership in I_F.
Lean: `Formalism.MainNotions.N.Pv.LocalLengthBound.frobenius_sq_bound_of_local` (`Formalism.SharpRestriction.LevelLetterCocycles.QuotientImageDivisibility.FrobeniusSqBoundOfLocal`, lines 55–165)
Lean: `Formalism.MainNotions.N.Pv.LocalLengthBound.rhoLift_of` (`Formalism.SharpRestriction.LevelLetterCocycles.QuotientImageDivisibility.RhoLiftOf`, lines 20–149)

□

Proof of 36.87

Proof. Choose Smith normal form for $P \subset M$. It gives a basis $(c_i)_{i\in I}$ of M, a basis $(p_j)_{j\in J_0}$ of P, an injection $\alpha : J_0 \to I$, and scalars $a_j \in O$ with

$$p_j = a_j c_{\alpha(j)}.$$

Each a_j is nonzero, since $p_j \neq 0$. Saturation gives $c_{\alpha(j)} \in P$. Conversely, expansion in the basis (p_j) expresses every element of P as an O-linear combination of these $c_{\alpha(j)}$. Take $J = \alpha(J_0)$. Both bases are bases of the same O-module, so their transition matrices are inverse matrices over O. Their determinants are units. This gives the valuation assertions. □

Proof of 36.88

Proof. Fix data of Definition 36.76, and use the modules and constants of Definition 36.78. In particular $n = d'$, $q = \ell - 1$, $a = \lceil A \rceil$, and $|J_-| = n/2$. We use Lemmas 36.80, 36.83, and 36.86, whose hypotheses are exactly these crystalline surjection and jump data under $\mathscr{H}_2$.

Claim 1. There is a finite free O-module F_0, of rank at most $(n+1)r$, and a linear map $\jmath : F_0 \to Z_f(T_2)$ surjective modulo $B(T_2)$.

Put $V_2 = T_2[1/\ell]$, choose a uniformizer ϖ of O, and let $k_O = O/\varpi O$. Write

$$a_0 = \dim_{L'} H^0(K, V_2), \qquad t_0 = \dim_{L'}\big(D_{\mathrm{dR}}(V_2)/\operatorname{Fil}^0\big), \qquad b_0 = \dim_{k_O} H^0(K, T_2/\varpi T_2).$$

Local finiteness in Theorem 36.14 makes $H^1_f(K, T_2)$ a finitely generated O-module, and the Bloch–Kato dimension identity (36.20) gives its rank $a_0 + t_0$. The exact sequence for multiplication by ϖ gives

$$0 \longrightarrow H^0(K,T_2)/\varpi H^0(K,T_2) \longrightarrow H^0(K, T_2/\varpi T_2) \longrightarrow H^1(K,T_2)[\varpi] \longrightarrow 0.$$

The first term has dimension a_0. Every torsion class belongs to $H^1_f(K, T_2)$, by the definition of the integral finite condition as the preimage of the rational finite condition. Thus the torsion subgroup of $H^1_f(K, T_2)$ needs $b_0 - a_0$ generators. The structure theorem over the discrete valuation ring shows that $H^1_f(K, T_2)$ needs at most

$$(a_0 + t_0) + (b_0 - a_0) = t_0 + b_0 \leq nr + r$$

generators. Here the crystalline comparison gives $\dim_{L'} D_{\mathrm{dR}}(V_2) = nr$, and $b_0 \leq r$. Choose integral finite cocycles representing such a set of generators and map the standard basis of F_0 to them. This proves Claim 1.

Lean: `Formalism.MainNotions.RamifiedCoefficientChains.stmt_of_stepsAdm` (`Formalism.SharpRestriction.LevelLetterCocycles.LatticeFiniteCohomologyGeneratorBound`, lines 327–331)

Claim 2. There is a descending coefficient chain starting at F_0, with at most $4n$ generator steps and $3n/2$ length steps, whose terminal module F_k satisfies

$$\ell^e \jmath(F_k) \subset I_F, \qquad e = s_{\mathrm{red}} + e_{\mathrm{vec}} + e_{\mathrm{int}},$$

where

$$\begin{aligned} s_{\mathrm{red}} &\leq \frac{2a + 2\delta + S}{q} + \lambda + 2\sigma + 2, \\ e_{\mathrm{vec}} &\leq \frac{h}{\ell q} + \frac{S+1}{q} + S + E_G + 2\sigma_2 + 6 + \lambda, \\ e_{\mathrm{int}} &= S + \lceil E_G + \sigma_2 \rceil \leq S + E_G + \sigma_2 + \lambda. \end{aligned}$$

Each exponent is a nonnegative integer, uniform in the component and coefficient vectors.

Apply Lemma 36.80 to $\jmath$. It gives a submodule P_2 reached by at most n generator steps, and maps $w_c : P_2 \to \mathcal{W}_c(T_2)$, for $c \in J_-$, such that $\ell^{s_{\mathrm{red}}} \jmath(f)$ is a signed sum of the associated integral cocycles $\kappa_c(f)$, modulo $B(T_2)$. There are no length steps in this first restriction.

Now process the $n/2$ components of J_- one at a time. Restrict every previously chosen map to the current coefficient module. Lemma 36.83 gives at most three generator steps and one length step, together with a source map y_c, for which

$$r_c = \ell^{e_{\mathrm{vec}}} w_c - F_D y_c \in \mathcal{N}_c(T_2).$$

Lemma 36.86 gives at most three additional generator steps and two length steps. Its compatibility assertion gives

$$\ell^{e_{\mathrm{vec}}+e_{\mathrm{int}}}\kappa_c(f)\in I_F$$

on the new coefficient module. These properties persist under further restriction. The common exponents of the two lemmas work for every component, so multiplying the signed sum by $\ell^{e_{\mathrm{vec}}+e_{\mathrm{int}}}$ proves Claim 2. Since $B(T_2)\subset I_F$, the sign and the coboundary do not affect this conclusion. The total numbers of steps are

$$n+(3+3)\frac{n}{2}=4n,\qquad (1+2)\frac{n}{2}=\frac{3n}{2}.$$

Every length step has normalized length at most C_{len}. If $F_{j+1}\subset F_j$ is such a step, the quotient

$$F_j/\big(F_j\cap(F_{j+1}+\jmath^{-1}(I_F))\big)$$

is a quotient of F_j/F_{j+1}, so it satisfies the length bound required in (R).
Lean: Formalism.MainNotions.RamifiedCoefficientChains.S9.lattice_bound_three_six_of (Formalism.SharpRestriction.LevelLetterCocycles.LatticeBoundsOfThreeParts, lines 414–461)

It remains to match the real witnesses in (R). Set

$$a_1=a,\qquad s_1=\frac{2a+2\delta+S}{q}+2\lambda,\qquad E_1=E_G+2\lambda=\frac{3a+S+2\delta}{q}+3\lambda,\qquad e_2=e_{\mathrm{vec}}.$$

Then $a_1\le A+1$. Since $q\ge 10$, $S\ge 0$, and $\lambda\ge 1$,

$$S-\frac{S+1}{q}+2\lambda-1\ge 0.$$

Subtracting the bound for e_{vec} in Claim 2 from the required bound for e_2 gives exactly this nonnegative quantity. Hence

$$e_2\le\frac{h}{\ell q}+E_1+2\sigma_2+2S+5+\lambda.$$

The other two estimates in Claim 2 give

$$s_{\mathrm{red}}+e_{\mathrm{int}}\le s_1+2\sigma+S+E_1+\sigma_2+1,$$

because the difference between the right side and those upper bounds is $2\lambda-1\ge 0$. Adding e_2 gives the required bound for e. Claim 1 supplies the prescribed rank of F_0, and Claim 2 supplies every chain and annihilation assertion. Thus (R) holds.
Lean: Formalism.MainNotions.RamifiedCoefficientChains.Asm.theorem_of_local_lemmas (Formalism.SharpRestriction.LevelLetterCocycles.FrobeniusLiftLemmasOfAdmitted, lines 272–290)

□

Proof of 36.89

Proof (outline). Put $h_1=\max(h,1)$. The auxiliary positive size is

$$\max\{g,S,\delta,1\}=\max(g,1)=g_*;$$

thus the height used in the lifting construction is h_1, while the height in the final colength bound remains h.
Lean: Formalism.MainNotions.WachCoordinates.gPar (Formalism.SharpRestriction.LevelLetterCocycles.WachCoordinates, lines 34–34)
Lean: Formalism.MainNotions.WachCoordinates.gPar_A (Formalism.SharpRestriction.LevelLetterCocycles.BoundedNumeratorPowerSummability, lines 278–279)
Lean: Formalism.MainNotions.WachCoordinates.hA (Formalism.SharpRestriction.LevelLetterCocycles.BoundedNumeratorPowerSummability, lines 273–273)
Lean: Formalism.MainNotions.WachCoordinates.hCor_pos (Formalism.SharpRestriction.LevelLetterCocycles.BoundedNumeratorPowerSummability, lines 281–283)
Lean: Formalism.MainNotions.WachCoordinates.hCor_neg (Formalism.SharpRestriction.LevelLetterCocycles.BoundedNumeratorPowerSummability, lines 285–287)

Claim 1. There is an absolute constant $C' \geq 1$ such that, under the size inequality with $C_2 \geq C'$, there are a nonempty cut set $\mathscr{C} \subset \mathbf{Z}/d_u\mathbf{Z}$ and backward interval lengths $v(c)$ with the following properties:

$$\mathscr{C} \neq \varnothing, \qquad J \leq v(c) \leq d_u, \qquad c - v(c) \in \mathscr{C}, \qquad c - s \notin \mathscr{C} \quad (1 \leq s < v(c)).$$

Every $c + k$, $1 \leq k \leq J$, is nonthick. Each backward interval $c, c-1, \ldots, c - v(c) + 1$ contains at most two thick positions, and $v_{\max} := \max_{c \in \mathscr{C}} v(c) \leq 3J + 1$. To obtain the cuts, use the two-case separation condition. If a and $a - k$ are thick with $1 \leq k \leq J$, the first case at a is impossible, and in the second case necessarily $k = d$. For $1 \leq k' \leq J$, the position $a - k - k'$ is not thick, since $1 \leq k + k' \leq d + J$ and $k + k' \neq d$. Thus three successive thick positions cannot have both intervening backward distances at most J. The sparse-thick cut construction gives the displayed cuts and $v_{\max} \leq 3J + 1$.

Lean: `Formalism.MainNotions.WachCoordinates.clusters_sep` (`Formalism.SharpRestriction.LevelLetterCocycles.BoundedNumeratorPowerSummability`, lines 350–361)
Lean: `Formalism.ExponentWindows.exists_cuts_exponent_bounds` (`Formalism.DistinguishedUnitIdeals.DistinguishedOrderProofs.AprimeCuts`, lines 8–23)
Lean: `Formalism.MainNotions.WachCoordinates.aprimeConfig_holds` (`Formalism.SharpRestriction.LevelLetterCocycles.BoundedNumeratorPowerSummability`, lines 380–492)

The size inequality gives $2(g + 1) \leq b_*$. In particular the thick and nonthick alternatives are disjoint. Put $d(a) = b_* + \delta + S$ at a thick position and $d(a) = g$ otherwise. The source jumps lie in $[-d(a), d(a)]$, and strictness of the crystalline filtration under the surjection gives the same enclosure for the target. Set $H_1(a) = H_2(a) = 2d(a)$. Hence

$$g_* \leq h_1, \qquad H_j(a) \leq 2h_1, \qquad H_j(c + k) \leq 2g \quad (j = 1, 2,\ c \in \mathscr{C},\ 1 \leq k \leq J).$$

Lean: `Formalism.MainNotions.WachCoordinates.JumpsSub` (`Formalism.SharpRestriction.LevelLetterCocycles.BoundedNumeratorPowerSummability`, lines 667–669)
Lean: `Formalism.MainNotions.WachCoordinates.liftA_of_thmE` (`Formalism.SharpRestriction.LevelLetterCocycles.BoundedNumeratorPowerSummability`, lines 595–647)
Lean: `Formalism.MainNotions.LocalTheoryWindows.CutsOK` (`Formalism.SharpRestriction.LevelLetterCocycles.LocalTheoryWindows`, lines 49–54)
Lean: `Formalism.MainNotions.WachCoordinates.ThickLe2` (`Formalism.SharpRestriction.LevelLetterCocycles.WachCoordinates`, lines 21–22)

For $j = 1, 2$, define

$$H_{j,\max} = \max_a H_j(a), \quad c_j = \lceil 1 + \log_\ell(H_{j,\max} + 1) \rceil, \quad s_j(c) = \sum_{v \geq 0} \frac{H_j(c + v + 1) + c_j + 3}{q_\ell^v},$$

$$\rho_{j,c} = \left\lfloor \min\{\lfloor s_j(c) \rfloor, H_{j,\max}\} \frac{\ell}{q_\ell^2} \right\rfloor + c_j.$$

The sums converge by the geometric-series estimate, and all these exponents are nonnegative. Put $c_h = \lceil 1 + \log_\ell(2h_1 + 1) \rceil$. Applying Lemma 36.28 at cyclotomic level one (and averaging over $\mathrm{Gal}(K(\mu_\ell)/K)$ of unit order q_ℓ) to F and to the unit projection of the crystalline extension realizing a target class bounds the cut remainder exponents by

$$\rho_{j,c} \leq \frac{\ell}{q_\ell^2} \left((2g + c_h + 3) \frac{q_\ell}{q_\ell - 1} + \frac{2h_1}{q_\ell^J} \frac{q_\ell}{q_\ell - 1} \right) + c_h. \tag{A.522}$$

Let ρ_* be the maximum of $\rho_{j,c}$ over cuts and both lattices, and set

$$M_* = \lfloor (h_1 + 2g_*)/q_\ell \rfloor + 1, \qquad \sigma_* = \left\lceil (3g_* + M_*/\ell^{J-1})\ell/q_\ell^2 + 1 \right\rceil, \qquad a_0 = 2\sigma_* + \rho_* + 2g_* v_{\max}.$$

Let $\mathscr{U}$ be the finite set of integer intervals

$$[a, a + t) \cap \mathbf{Z}, \qquad a \in \mathbf{Z}, \quad t \in \mathbf{N}, \qquad -2h_1 \leq a \leq 2h_1, \quad 0 \leq t \leq 3g_*, \quad a + t \leq 2h_1 + 1.$$

Define the nonnegative integer

$$\kappa_* = \max_{U_1, U_2, U_3 \in \mathscr{U}} \left(v_{\max} + \sum_{i \in (U_1 \cup U_2 \cup U_3) \setminus \{0\}} (v_{\max} + 1 + v_\ell(i)) \right).$$

The set $\mathscr{U}$ contains the empty interval, so this maximum exists.
Lean: Formalism.MainNotions.WachCoordinates.wins (Formalism.SharpRestriction.LevelLetterCocycles.WachCoordinates, lines 54–56)
Lean: Formalism.MainNotions.WachCoordinates.kapPar (Formalism.SharpRestriction.LevelLetterCocycles.WachCoordinates, lines 60–61)

Across a cut interval of length v, the cyclotomic shift-killing polynomial for the shift set $U \subset [-2h_1, 2h_1]$ (lying in three intervals of length at most $3g_*$) has valuation at 1 equal to

$$v + \sum_{i \in U \setminus \{0\}} (v + 1 + v_\ell(i)), \tag{A.523}$$

bounded uniformly by κ_* where

$$\kappa_* \le v_{\max} + 20g_*(1 + v_{\max}) + 5\{4g_*/q_\ell + \log_\ell(2h_1)\}. \tag{A.524}$$

Claim 2. For the same cuts, enclosures and exponent a_0, the opposite-sign comparisons below hold. For $0 \le s_0 < s_1 < v(c)$, write $b_0 = s_{c-s_0}$ and $b_1 = s_{c-s_1}$ when these positions are thick. Then

$$\begin{aligned} (\varsigma_{c-s_1}, \varsigma_{c-s_0}) = (-1, 1) \quad &\Longrightarrow b_0 < \ell b_1, \\ (\varsigma_{c-s_1}, \varsigma_{c-s_0}) = (1, -1) \quad &\Longrightarrow 2b_0 \le \ell b_1 \quad \text{and} \quad 2a_0 < b_1. \end{aligned}$$

For the proof, a thick position forces $h_1 = b_* + \delta + S$. Uniformly for the two enclosure functions, the cut-remainder calculation gives

$$\rho_* \le 20\{g/\ell + (b_* + \delta + S)/q_\ell^{J+1} + \log_\ell(b_* + \delta + S) + 1\}.$$

The cut construction bounds the initial exponent $2\sigma_* + \rho_* + 2g_* v_{\max}$ using this estimate and $v_{\max} \le 3J + 1$. Its numerical separation inequalities are $b_0 < \ell b_1$, $2b_0 \le \ell b_1$, and $2a_0 < b_1$. They apply to every pair of thick positions in the same backward interval, hence give the two displayed implications. If there is no such pair, the implications have no instance.
Lean: Formalism.MainNotions.WachCoordinates.aprimeConfig_holds (Formalism.SharpRestriction.LevelLetterCocycles.BoundedNumeratorPowerSummability, lines 380–492)

Claim 3. Assume the present unramified crystalline data, the source and target jump enclosures H_1, H_2, the cut properties of Claim 1, and both opposite-sign comparisons of Claim 2 for a_0. With ρ_*, σ_*, a_0 and κ_* as above, the single integer

$$e_0 = 2\rho_* + 4\sigma_* + 2a_0 + 2\kappa_*$$

has the following property: for every $\kappa \in Z_f(T_2)$, there are $\kappa_1 \in Z_f(T_1)$ and a coboundary $b \in B(T_2)$ such that $\ell^{e_0}\kappa = F\kappa_1 + b$. The hypotheses on the source are crystallinity, surjectivity of F, and the signed windows already specified; the height enclosures for the target follow by strictness. Under $\mathscr{H}_2$, the exact Fontaine and crystalline comparisons, the horizontal unit generator, Bloch–Kato duality and the local Euler characteristic supply this cut-interval lifting assertion.
Lean: Formalism.MainNotions.N.Ln03.local_theorem_of_inputs (Formalism.SharpRestriction.LevelLetterCocycles.LocalTheoremOfInputs, lines 26–85)
Lean: GlobalTheory.ThmECorA.closing (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TheoremECorollaryAClosingAllData, lines 17–31)

Fix $\kappa \in Z_f(T_2)$. Construct Frobenius-horizontal frames across each cut interval. Propagating endpoint coordinates to the preceding cut costs two factors ℓ^{σ_*} and reduces $\ell^{\rho_* + 2\sigma_*}\kappa$ modulo coboundaries to the sum of the cocycles attached to cut vectors w_c, $c \in \mathscr{C}$. Each w_c is lifted separately. Apply the target shift killer P and pass to window coordinates on the target, at cost ℓ^{σ_*}. In these coordinates every nonthick position uses the window $[-g, g]$, for either sign. Among the at most two thick positions of Claim 1, lift the downward ones at exponent $2a_0$ by Lemma 36.29 and the upward ones in their filtered submodules using the displayed comparisons. Returning to lattice coordinates on the source costs a second factor ℓ^{σ_*}. The difference between $\ell^{2\sigma_* + 2a_0}$ times the target vector and the image of this source vector is interval-divisible, and it lifts at cost ℓ^{ρ_*}. The source shift killer P' is applied last. By (A.523), the product of the values of P and P' at 1 is a unit times ℓ^k with $k \le 2\kappa_*$. After a further factor $\ell^{2\kappa_* - k}$, every $\ell^{2\sigma_* + 2a_0 + \rho_* + 2\kappa_*} w_c$ is the image of a source cut vector modulo π_0 times the Wach lattice at c, uniformly in c and κ; here π_0 is the Δ-invariant uniformizer of the level-one Wach ring. This discrepancy changes the attached cocycle only by

a coboundary, and the cocycles attached to source cut vectors lie in $Z_f(T_1)$. Adding the costs $\rho_* + 2\sigma_*$ and $2\sigma_* + 2a_0 + \rho_* + 2\kappa_*$ of the two stages, the resulting integer exponent, uniform in κ, is

$$e_{\mathrm{cl}} = 2\rho_* + 4\sigma_* + 2a_0 + 2\kappa_*, \qquad a_0 = 2\sigma_* + \rho_* + 2g_* v_{\max}.$$

Lean: `Formalism.MainNotions.WachCoordinates.WordOK` (`Formalism.SharpRestriction.LevelLetterCocycles.WachCoordinates`, lines 24–29)
Lean: `Formalism.MainNotions.WachCoordinates.initialExponentBound` (`Formalism.SharpRestriction.LevelLetterCocycles.WachCoordinates`, lines 65–67)
Lean: `Formalism.MainNotions.WachCoordinates.ePar` (`Formalism.SharpRestriction.LevelLetterCocycles.WachCoordinates`, lines 69–72)
Lean: `Formalism.MainNotions.WachCoordinates.ThmELiftAt` (`Formalism.SharpRestriction.LevelLetterCocycles.WachCoordinates`, lines 123–139)
Lean: `Formalism.MainNotions.WachCoordinates.ThmEBound` (`Formalism.SharpRestriction.LevelLetterCocycles.WachCoordinates`, lines 74–83)

Claim 4. For $\ell \geq 11$, $J \geq 1$, $1 \leq g_* \leq h_1$, and $v_{\max} \geq 1$, the displayed remainder, ceiling and killer bounds give

$$2\rho_* + 4\sigma_* + 2a_0 + 2\kappa_* \leq 800\{h_1/q_\ell^{J+1} + g_* v_{\max} + \log_\ell h_1\}. \tag{A.525}$$

In this substitution, $a_0 = 2\sigma_* + \rho_* + 2g_* v_{\max}$, so the left side is $4\rho_* + 8\sigma_* + 4g_* v_{\max} + 2\kappa_*$. The estimates use $g_* \geq 1$, $v_{\max} \geq 1$, $g_* \leq h_1$, and $\log_\ell h_1 \geq 0$; these include the case $h_1 = 1$.
Lean: `Formalism.MainNotions.WachCoordinates.thmEBound_holds` (`Formalism.SharpRestriction.LevelLetterCocycles.BoundedNumeratorPowerSummability`, lines 154–209)

Thus $e_{\mathrm{cl}} = 2\rho_* + 4\sigma_* + 2a_0 + 2\kappa_*$ satisfies $\ell^{e_{\mathrm{cl}}} Z_f(T_2) \subset I_F$ and, since $v_{\max} \leq 3J + 1$, the asserted bound. The finite-cohomology presentation argument in Claim 1 of the proof of Proposition 36.88, with $n = d_u$, uses no parity assumption and shows that the quotient $Q = Z_f(T_2)/I_F$ is generated by at most $d = (d_u + 1)r$ elements, since $Z_f(T_2)/B(T_2) \simeq H^1_f(K, T_2)$ has that generator bound and $B(T_2) \subset I_F$. The annihilation $\ell^{e_{\mathrm{cl}}} Q = 0$ makes a chosen surjection $O^d \to Q$ factor through $(O/\ell^{e_{\mathrm{cl}}} O)^d$. Consequently

$$\lambda_O(Q) \leq d e_{\mathrm{cl}}.$$

Put $R = h/q_\ell^{J+1} + (g+1)(J+1) + \log_\ell h$. Every summand of R is nonnegative, including the convention at $h = 0$. If $h \geq 1$, then $h_1 = h$, and

$$h_1/q_\ell^{J+1} + g_* v_{\max} + \log_\ell h_1 \leq 3R,$$

because $g_* v_{\max} \leq 3(g+1)(J+1)$. If $h = 0$, then no thick position exists, $g = 0$, $g_* = h_1 = 1$, and $R = J + 1$. Therefore

$$h_1/q_\ell^{J+1} + g_* v_{\max} + \log_\ell h_1 \leq 1 + 3J + 1 \leq 4(J+1) = 4R.$$

In both cases $e_{\mathrm{cl}} \leq 3200R$, so $\lambda_O(Q) \leq 3200(d_u + 1)rR$. Take $C_0 = \max(C', 3200)$. Every $C_2 \geq C_0$ gives (36.33). Enlarging C_2 strengthens the size hypothesis and increases the bound by a nonnegative multiple of R.
Lean: `Formalism.MainNotions.LocalTheoryWindows.colLeR_of_liftF` (`Formalism.SharpRestriction.LevelLetterCocycles.ArcIffLocalOfAdmissible`, lines 207–215)
Lean: `Formalism.MainNotions.WachCoordinates.corAAt_of` (`Formalism.SharpRestriction.LevelLetterCocycles.BoundedNumeratorPowerSummability`, lines 673–710)
Lean: `Formalism.MainNotions.WachCoordinates.corollary_of_Theorem_E` (`Formalism.SharpRestriction.LevelLetterCocycles.BoundedNumeratorPowerSummability`, lines 722–729)
Lean: `Formalism.MainNotions.N.Ln03.corollary_local_exists_constant` (`Formalism.SharpRestriction.LevelLetterCocycles.LocalTheoremOfInputs`, lines 112–132)

The complete proof is contained in `GlobalTheory.T.theoremE_corollaryA_closing_all_data`. □

Proof of 36.90

Proof (outline). Write $Q = Z_f(T_2)/I_F$ and $d = (d_u + 1)r$. The finite-cohomology argument in Claim 1 of the proof of Proposition 36.88 applies without a parity assumption and shows that Q is generated by at most d elements.

Claim 1. Under the unramified crystalline hypotheses of (a), put $M = \max_a M_a$, let $\mathcal{J}_0$ be the set of jumps of T_1 over all embeddings, and define

$$\mathcal{J} = \{u + j : u \in \mathcal{J}_0,\ 1 \leq j \leq M,\ u + j \neq 0\} \cup \{1, \dots, M\}, \qquad a_M = \left\lfloor \frac{M}{q_\ell} \right\rfloor + 1,$$

$$e_W = \left\lfloor \frac{H'\ell}{q_\ell^2} + 3 + \log_\ell(H'+1) \right\rfloor, \qquad e_U = \sum_{j\in\mathcal{J}} v_\ell(j) + a_M + v_\ell(a_M!).$$

Then $\ell^{2e_W+e_U} Z_f(T_2) \subset I_F$. Let $T_0 = \ker F$, let π_0 be the Δ-invariant uniformizer of the level-one Wach ring $\mathcal{A}^\Delta$, put $Q = \pi_0 + \ell$, and let $u = \chi_\ell(\gamma)$ for the chosen topological generator γ. Then $v_\ell(u-1) = 1$, $\gamma(\pi_0) \in \pi_0(u^{q_\ell} + \pi_0\mathcal{A}^\Delta)$ and $\varphi(\pi_0) \in \pi_0 Q^{q_\ell}\mathcal{A}^\Delta$. The level-one Wach comparison in $\mathscr{H}_2$, applied with the component width H', gives two lifts with exponent e_W each. First, $\ell^{e_W} N(T_2)^\Delta$ lies in the image of $N(T_1)^\Delta$. Second, for the crystalline extension $0 \to T_2 \to E \to O \to 0$ of a finite class κ, ℓ^{e_W} times the invariant generator of $N(O)$ is the image of a vector of $N(E)$. The extension comparison then represents $\ell^{e_W}\kappa$ by a Δ-invariant Wach cocycle (y,z) of T_2 with $Q^M y \in N(T_2)$ and $z \in \pi_0 N(T_2)^\Delta$, and the first lift carries $\ell^{e_W}(y,z)$ to a pair on T_1 whose cocycle defect lies on T_0. This defect is removed on T_0 with the operator $\delta = \pi_0^{-1}(\gamma - 1)$, which preserves $N(T_0)^\Delta$. Put $c(j) = (1-u^{-j})/\ell \in \mathbf{Z}_\ell$. Then $c(0) = 0$ and $v_\ell(c(j)) = v_\ell(j)$ for $j \neq 0$. Every jump of T_0 is a jump of T_1, and the crystalline eigenvalue decomposition shows that $\delta\prod_{j\in\mathcal{J}}(\delta - c(j))$ sends every invariant x with $Q^M x \in N(T_0)$ into $N(T_0)^\Delta$. The shift 0 contributes only the factor δ, and the constant term $\prod_{j\in\mathcal{J}}(-c(j))$ divides $\ell^{\sum_{j\in\mathcal{J}} v_\ell(j)}$. Put $a' = a_M - 1$ and $u_1 = u^{q_\ell}$. On $\pi_0^i N(T_0)^\Delta/\pi_0^{i+1}N(T_0)^\Delta$ with $1 \le i \le a'$, the operator γ acts by u_1^i, and $\prod_{i=1}^{a'}(1-u_1^i)$ divides $\ell^{a'+v_\ell(a'!)}$, hence divides $\ell^{a_M+v_\ell(a_M!)}$. Since $M+1 \le q_\ell a_M$, the map $1-\varphi$ sends $\pi_0^{a_M}N(T_0)^\Delta$ onto itself. These three facts lift the corrected cocycle to T_1 at the further cost ℓ^{e_U}, so the total exponent is $2e_W + e_U$.

Lean: `GlobalTheory.localConstants_from_admitted_local_Galois_statements_alt` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalConstantsFromAdmittedLocalGaloisStatements`, lines 11–14)

Lean: `Formalism.MainNotions.LiftingTheory.ePA` (`Formalism.SharpRestriction.LevelLetterCocycles.LiftingTheory`, lines 27–27)

Since $M \le H'$ and every jump lies in $[-H', H']$, $\mathcal{J} \subset [-H', 2H'] \setminus \{0\}$. Thus

$$\sum_{j\in\mathcal{J}} v_\ell(j) \le v_\ell(H'!) + v_\ell((2H')!) \le \frac{3H'}{q_\ell}.$$

The Wach, Herr, and crystalline lifting inputs in Claim 1 require only that ℓ be an odd prime and $K/\mathbf{Q}_\ell$ unramified, so they hold for every prime $\ell \ge 5$. Also $a_M \le H'/q_\ell + 1$ and $v_\ell(a_M!) \le a_M/q_\ell \le H'/q_\ell^2 + 1/q_\ell$. Combining these estimates with $2e_W \le 2H'\ell/q_\ell^2 + 6 + 2\log_\ell(H'+1)$ yields

$$2e_W + e_U \le \left(\frac{2\ell+1}{q_\ell^2} + \frac{4}{q_\ell}\right) H' + 7 + \frac{1}{q_\ell} + 2\log_\ell(H'+1) \le 23\{2 + H'/\ell + \log_\ell(H'+1)\}.$$

Indeed, for every prime $\ell \ge 5$ we have $q_\ell = \ell - 1 \ge 4$, so $7 + 1/q_\ell \le 29/4 < 46$, while $\ell((2\ell+1)/q_\ell^2 + 4/q_\ell) = \ell(6\ell-3)/(\ell-1)^2$ is strictly decreasing for $\ell \ge 5$ and attains its maximum $135/16 < 23$ at $\ell = 5$. These estimates include $H' = 0$, with $0! = 1$. Taking $e' = 2e_W + e_U$ proves (a).

Lean: `Formalism.MainNotions.LiftingTheory.arith_sizeA` (`Formalism.SharpRestriction.LevelLetterCocycles.FrobeniusPowUniformizerFactorisation`, lines 1799–1840)

Now assume (b), fix any integer $e' \ge 0$ with $\ell^{e'} Z_f(T_2) \subset I_F$, and put

$$V_0 = \frac{h}{\ell^2} + \frac{w_* + \delta}{\ell} + S + 1 + \log_\ell(2h).$$

Here $h \ge 1$, $U_0 \ge 1$, and $U_0 \le V_0$.

Claim 2. If a thick position exists and $b_*/4 < 40U_0$, there is an integer $e_0 \ge 0$ such that

$$\ell^{e_0} Z_f(T_2) \subset I_F, \qquad e_0 \le \frac{71}{10}(h/\ell + 1) + 2\log_\ell(2h) \le 160U_0.$$

For this estimate use the signed component boxes in (b): their negative heights are at most h, and their widths are at most $h + \delta$. Counting the shifted valuations in Claim 1 on these signed ranges gives the first inequality. The assumptions $20\delta, 20S \le b_*$ give $h = b_* + \delta + S \le 11b_*/10$, and $b_* < 160U_0$, together with $\ell \ge 11$, gives the second.

Lean: `Formalism.MainNotions.N.Ln17.exists_liftF_le_sg_of_thick` (`Formalism.SharpRestriction.LevelLetterCocycles.RepresentationCongruences.ExistsLiftFLeSgOfThick`, lines 9–89)

Lean: `Formalism.MainNotions.N.FrobeniusPowerSeries.lift_le_ePA` (`Formalism.SharpRestriction.LevelLetterCocycles.RepresentationCongruences.LiftLeEPA`, lines 9–15)

In this case the quotient of $(O/\ell^{e_0}O)^d$ onto Q gives $\lambda_O(Q) \le 160dU_0 \le 320d(V_0+\delta I)+320d_u m e'$.

It remains to treat the case in which there is no thick position or $40U_0 \le b_*/4$.

Claim 3. Under all the hypotheses of (b) and this last alternative, there are an integer $E \ge 0$ and a finite set $\mathcal{Z} \subset Z_f(T_2)$ such that

$$E \le 55U_0 + 13S + \delta I, \qquad |\mathcal{Z}| \le 4d_u m, \qquad \ell^E Z_f(T_2) \subset I_F + \sum_{z\in\mathcal{Z}} Oz.$$

The choices of E and $\mathcal{Z}$ are independent of the subsequently chosen annihilation exponent e'.

Lean: `Formalism.MainNotions.N.Ln17.sum_alt` (`Formalism.SharpRestriction.LevelLetterCocycles.RepresentationCongruences.LogbHBNonneg`, lines 386–406)

We now give the construction and its exponent accounting. Cut at each c for which $c+1$ is nonthick. Since a nonthick position exists and no three consecutive positions are thick, the backward lengths belong to $\{1,2,3\}$, and their sum is d_u. If no thick position exists, every length is one. Set

$$M_0' = \lfloor 2h/q_\ell \rfloor + 1, \qquad \sigma_0 = \left\lceil (2(w_*+S)+M_0')\ell/q_\ell^2 + 1 \right\rceil.$$

The Frobenius remainder construction gives one integer r_0, valid for the map F and for the unit projection of every crystalline extension representing a finite target class, with $r_0 \le 3U_0$ and $\sigma_0 \le 3U_0$.

Lean: `Formalism.MainNotions.N.FrobeniusPowerSeries.arc_setup_theorem` (`Formalism.SharpRestriction.LevelLetterCocycles.RepresentationCongruences.ArcSetupTheorem`, lines 22–38)

At a cut of length v, split off the central filtration directions in a Frobenius-horizontal basis. The exceptional finite classes form a set $\mathcal{Z}_c$ of cardinality at most $mr_c(v+1)$, where r_c is the number of thick letters in that interval. Since $r_c \le v \le 3$, this is at most $4mv$. Each complementary horizontal coordinate has a source lift with an integer exponent bounded as follows:

v	1	2	3
e_1	$\le 6U_0+1$	$\le 15U_0+1+\delta I$	$\le 34U_0+1+\delta I$.

For length two, let c be the thick letter and $c-1$ the nonthick one, write $b_0 = s_c$ and $w' = s_{c-1}$, and let a_{c-1} be the least integer at least k/q_ℓ, where k is the least exponent with Z^k killing the cokernel of F on the Wach modules at $c-1$. Here π_0 is a unit times Z^{q_ℓ}, so $\pi_0^{a_{c-1}}$ kills this cokernel because $q_\ell a_{c-1} \ge k$. Each coordinate exponent is $r_0+\sigma_0 \le 6U_0$ plus a window exponent d_c. If $\varsigma_c = 1$ and $\ell(w'+S+\delta)+\delta < b_0$, or $\varsigma_c = -1$ and $a_{c-1}+\delta+1 \le w'$, the coordinate lifts with exponent at most $12U_0+1$, respectively $9U_0+1$. If $\varsigma_c = 1$ and $b_0 \le \ell(w'+S+\delta)+\delta$, then $d_c \le \max\{0, b_0/\ell - (w'+S) + 2 + \log_\ell b_0\} + 1$. When $b_0 < \ell(w'+S)$ this is at most $3+\log_\ell b_0$. When $\ell(w'+S) \le b_0$, the window $\ell(w'+S) \le b_0 \le \ell(w'+S+\delta)+\delta$, together with $b_* \le b_0 \le b_*+\delta$, $w' \le w_*$ and $20\delta, 20S, 20w_* \le b_*$, gives (36.35) at $a = c-1$ for every constant at least 2; thus $I = 1$ and $d_c \le \delta + 3U_0$. If $\varsigma_c = -1$ and $w' \le a_{c-1}+\delta$, then one can choose $d_c \in \mathbf{N}$ with $d_c \le \lceil \max\{0, w' - (b_0+S)/\ell + 2 + \log_\ell w'\} \rceil$ and with $d_c = 0$ when $\ell w' < b_0 + S$. Otherwise $a_{c-1} \le (b_0+S+\delta)/\ell + 2U_0$ gives $d_c \le \delta + 5U_0$, and if (36.35) fails at $a = c-1$, then $\delta < 4U_0$. In all four cases the exponent is at most $15U_0+1+\delta I$. For length three with $\varsigma_{c-2} = -1$, the bound is $22U_0+1+\delta I$ if $s_{c-1} < \ell(s_{c-2}+S+\delta)$ and $13U_0+1$ otherwise. With $\varsigma_{c-2} = 1$, it is $34U_0+1+\delta I$ if $s_{c-2} < a_{c-2}+\delta+2\sigma_0+2$ and $18U_0+1$ otherwise, with a_{c-2} defined as above at $c-2$. So the displayed common bound applies. The constant in the window comparison can be any real number at least 2; take it to be 320, making its indicator precisely I from the statement.

Lean: `Formalism.MainNotions.N.FrobeniusPowerSeries.arc_length_one_two_theorem` (`Formalism.SharpRestriction.LevelLetterCocycles.RepresentationCongruences.ArcPropositionAlternative`, lines 34–69)
Lean: `Formalism.MainNotions.N.FrobeniusPowerSeries.arc_length_three_theorem` (`Formalism.SharpRestriction.LevelLetterCocycles.RepresentationCongruences.ArcPropositionAlternative`, lines 100–131)
Lean: `Formalism.MainNotions.N.FrobeniusPowerSeries.exists_liftsAt_of_arc_len_three_of_up` (`Formalism.SharpRestriction.LevelLetterCocycles.RepresentationCongruences.ArcPropositionAlternative`, lines 71–98)

Since $U_0 \ge 1$, every coordinate exponent is at most $E_1 = \lfloor 40U_0+\delta I \rfloor$, and $\sigma_0 \le E_1$. For the union of the at most three half-open shift intervals of width S, the shift killer at cut c has valuation $\kappa_c \le 13S + 3\log_\ell(2h) + 3$. Take $\kappa = \max_c \kappa_c$, and assemble the cut classes using the common exponent

$$E = r_0 + 3\sigma_0 + \kappa + E_1.$$

The remainder and ceiling bounds and $\log_\ell(2h)+1 \le U_0$ give $E \le 55U_0 + 13S + \delta I$. The union of the finite exceptional sets has at most $\sum_c 4mv(c) = 4d_u m$ elements. Choose finite cocycle representatives for that union. The Herr assembly gives the last inclusion in Claim 3, with no choice of e'.

Lean: Formalism.MainNotions.N.Ln17.sumAltW (Formalism.SharpRestriction.LevelLetterCocycles.RepresentationCongruences.LogbHBNonneg, lines 100–214)
Lean: GlobalTheory.TheoremBAdm.admissible_ranks_of_CMDatum (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalLiftingTheoremBHypothesesAdmitted, lines 26–46)

To finish, let $R \subset Q$ be generated by the images of $\mathcal{Z}$. Claim 3 makes Q/R a quotient of $(O/\ell^E O)^d$. The assumed annihilation by $\ell^{e'}$ makes R a quotient of $(O/\ell^{e'}O)^{|\mathcal{Z}|}$. Additivity of normalized length gives

$$\lambda_O(Q) \le dE + |\mathcal{Z}|e' \le d(55U_0 + 13S + \delta I) + 4d_u m e'.$$

Since $U_0 \le V_0$, $S \le V_0$, and all terms are nonnegative,

$$55U_0 + 13S + \delta I \le 68V_0 + \delta I \le 320(V_0 + \delta I).$$

This is (36.36) for the arbitrary exponent e' fixed above.
Lean: Formalism.MainNotions.N.Ln17.colLeR_of_exponent (Formalism.SharpRestriction.LevelLetterCocycles.RepresentationCongruences.LenQuotSpanNeTopDvr, lines 57–158)

The complete local comparison and short-interval construction are contained in Formalism.MainNotions.LiftingTheory.frobenius_lifting, Formalism.MainNotions.N.Ln17.sum_alt, and Formalism.MainNotions.N.Ln17.frobenius_statement_of_fontaine. □

Proof of 36.91

Proof. The parameter conditions give $f, n, t, n_V, L \ge 1$, $m, r \le t$, and $0 \le \delta_{\mathrm{pr}} \le 2t$. Since $d_u \ge 1$ and the sum of the local degrees over all places above ℓ is $2f$,

$$|\mathcal{C}| \le 2f, \qquad \sum_{u \in \mathcal{C}} d_u \le 2f, \qquad \sum_{w \in \mathcal{R}} (d_w + 2) \le 6f.$$

The three summands of D_0 are consequently bounded by

$$6B_c, \qquad 8n_V t(2f+1) \le 24B_c, \qquad 2cft^2 \le 2B_c,$$

respectively. Their sum is at most $32B_c$.

For the additional column terms, the degree contribution is at most $2fm\delta_{\mathrm{pr}} \le 4B_c$, the constant projector contribution is at most $2\delta_{\mathrm{pr}}|\mathcal{C}| \le 8B_c$, and the two invariant contributions together are at most

$$|\mathcal{C}|\, 2crnn_V L \le 4cfrnn_V L \le 4B_c.$$

Thus $D_{\mathrm{sp}} - D_0 \le 16B_c$. Finally $48 \le 60$ and $L \ge 1$, so both bounds are at most $60cfnt^2 n_V L^2$. Taking $c = 120$ proves the last assertion. □

Proof of 36.92

Proof (outline). The common local word. Set $C = \max(100, 2C_2)$.

Claim 1. Uniformly for the members of the fixed family, once the scale x is sufficiently large,

$$100 \le w''(n), \qquad 100(\eta + S_c + t + 2) \le w''(n) \le b',$$
$$b_T + b'/x \le \tfrac{3}{2}b' \quad (T \in TT), \qquad C(L_w + 1)b' \le b.$$

Lean: GlobalTheory.HTW.Roomy (Development.AbelianVarietyTheory.Definitions.GaloisGeometry.Alternates, lines 143–148)
Lean: GlobalTheory.HTW.roomy_of_large (Development.AbelianVarietyTheory.Proofs.SatakeParameters.PageLargeScaleBoundsOfLargeScale, lines 12–28)

For the proof, let $\varepsilon > 0$ be the fixed scale exponent and let A_w bound the family's progression-window coefficient, so that $L_w \le A_w \log x$. The thin-size bounds give the first line after a uniform cutoff. The top-size bound gives the first inequality in the second line for $x \ge 4$. After another uniform cutoff the scale inequality

$$C(\max(A_w, 0)\log x + 1)b' \le b$$

holds. Since $b' \geq 0$ and $x \geq 4$, it implies the last inequality in Claim 1. Take the maximum of these cutoffs and the inert-prime cutoff. Thus $\ell \geq 11$, and exclusion of the ramified host primes makes $E_u/\mathbf{Q}_\ell$ unramified.
Lean: `GlobalTheory.T.thinSize_dominates_at_large_scale` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.ThinSizeDominatesAtLargeScaleStatement`, lines 8–9)
Lean: `GlobalTheory.T.topSize_add_le_three_halves_scale` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.TopSizeAddLeThreeHalvesScaleStatement`, lines 8–9)
Lean: `GlobalTheory.T.scale_log_mul_le_size_ratio` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.ScaleLogMulLeSizeRatioStatement`, lines 8–9)

Theorem 36.26(I) supplies the full-family continuity, the good point, its semisimple decomposition into $\rho_i(\pi)$ and the character blocks, the continuity of $\rho_i(\pi)$, and crystallinity of the normalized full family at these places. Its de Rham jump conditions give the common embedding-sign condition. The period and Fontaine comparisons follow from $\mathscr{H}_2$. These are the point and comparison inputs used next.

Claim 2. For every good point, place and finite splitting coefficient extension, under Claim 1 and the point and comparison inputs just specified, there is one cyclic word with the following simultaneous properties. Its letter at a consists of s_a, its sign, and the two enclosure heights M_a, h_a. The same word works for every stable source lattice in a subrepresentation of a finite sum of twisted dual full representations and for every stable Tate-dual target lattice. The source and target are crystalline with jumps in the displayed signed windows. The comparison also bounds the sum of the central multiplicities over the embeddings above each component by m. Since the field is unramified, precisely one embedding lies over each component, so this is the required bound at τ_a. The word-size bounds give (36.37); at a nonthick letter they give $|s_a - y| \leq T_0$ with $w''(n) \leq y \leq w_B$.
Lean: `GlobalTheory.HTW.RecS.WordBase` (`Development.AbelianVarietyTheory.Definitions.GaloisGeometry.ThickLetterWordLatticeSurjectionHypotheses`, lines 30–43)
Lean: `GlobalTheory.Survey.ClusteredLetterwiseBoxes.base_of` (`Development.AbelianVarietyTheory.Counterexamples.HeckeCharacterExamples.ClusteredLetterwiseBoxes`, lines 100–114)
Lean: `GlobalTheory.HTW.Rec.recS_wordBase_natural` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.WordConditionsCommonNaturalRepresentation`, lines 24–48)
Lean: `GlobalTheory.HTW.Rec.recWord_wordBox` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.TwistedDualSizeWordBoxesAndSizes`, lines 15–34)

For the simultaneous assertion, the coefficient comparison first fixes a common sign and the labelled jumps. Enumerate the embeddings by the inverse Frobenius orbit and reconstruct the signed size and the enclosure at each letter. Apply the same componentwise jump statements to every stable source lattice and to every stable target lattice. The target's central multiplicity is the sum over the embeddings above that component, so in the unramified case it is the single sum at τ_a. No choice of lattice surjection enters the construction of the word. We now give the scalar dictionary in this construction. The integer t is odd, so $T_0 = (t+1)/2$ is an integer. Let $e_j(\tau)$ be the labelled auxiliary weight of the full family, and set

$$w_{\max} = \max\big(\{0\} \cup \{\max(0, \lfloor e_j(\tau) \rfloor) : \tau \text{ is nonthick},\ 0 \leq j < m\}\big).$$

The reconstructed word uses exactly

$$b_* = \lceil b \rceil - T_0, \qquad w_* = w_{\max} + T_0, \qquad S = \lfloor S_c \rfloor, \qquad \delta = \lfloor \eta \rfloor + t + 2.$$

Claim 1 implies $b \geq t + 2$, so the first subtraction is nonnegative and agrees with subtraction in $\mathbf{N}$. At the auxiliary label j fixed by i, the signed size is

$$s_a = \lfloor e_j(\tau_a) \rfloor - \varsigma_a T_0.$$

The common comparison sign is chosen so that $\varsigma_a = -1$ when the embedding sign agrees with it and $\varsigma_a = 1$ otherwise. The word-size estimate gives $s_a > \delta$; thus conversion of the signed integer to $\mathbf{N}$ changes nothing. These formulas identify the rounded parameters and the letter sizes before a lattice or a surjection is chosen.
Lean: `GlobalTheory.HTW.bHat` (`Development.AbelianVarietyTheory.Definitions.GaloisGeometry.Alternates`, lines 117–117)
Lean: `GlobalTheory.HTW.wMax` (`Development.AbelianVarietyTheory.Definitions.GaloisGeometry.Alternates`, lines 123–124)
Lean: `GlobalTheory.HTW.wHat` (`Development.AbelianVarietyTheory.Definitions.GaloisGeometry.Alternates`, lines 126–126)
Lean: `GlobalTheory.HTW.sHat` (`Development.AbelianVarietyTheory.Definitions.GaloisGeometry.Alternates`, lines 121–121)
Lean: `GlobalTheory.HTW.etaHat` (`Development.AbelianVarietyTheory.Definitions.GaloisGeometry.Alternates`, lines 119–119)
Lean: `GlobalTheory.HTW.Rec.recWord` (`Development.AbelianVarietyTheory.Definitions.GaloisGeometry.NotDvdCondN`, lines 87–98)
Lean: `GlobalTheory.HTW.Rec.recWord_letter_eq_pageWord` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.TwistedDualSizeWordLetterEqPageWord`, lines 10–29)

At a nonthick embedding the scalar window gives $w''(n) \le \lfloor e_j(\tau_a) \rfloor \le w_{\max} \le w_B$. Taking $y = \lfloor e_j(\tau_a) \rfloor$ gives $|s_a - y| = T_0$, and $s_a \le w_* \le w_P$. Also $S \le S_c$ and $\delta \le \eta_3$. The rounded thick-height bound gives $b_* + \delta + S \le H_B$ directly, and the component boxes are bounded by $2H_B$.
Lean: `GlobalTheory.HTW.bHat_add_etaHat_add_sHat_le_hPage` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.RoundedSizesSumLePageHeight`, lines 10–36)
Lean: `GlobalTheory.HTW.pageWord_Mi_add_hi_le` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.TwistedDualPageWordMiAddHiLeTwoHeight`, lines 23–32)
Lean: `GlobalTheory.HTW.bHat_window` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.RoundedSizeBoundWindow`, lines 9–42)
Lean: `GlobalTheory.HTW.Rec.wMax_le_wB` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.MaxSizeAtNonThickPlaceLeWindowEnds`, lines 54–70)
Lean: `GlobalTheory.HTW.pageWord_not_thick_size` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.TwistedDualOrbitPageWordNonThickSizeBounds`, lines 26–33)
Lean: `GlobalTheory.HTW.Rec.recWord_small_pos` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.OneLeScOfSizeData`, lines 36–52)

The two enclosure heights are exactly the two maxima displayed in part (a). The bounds for them and for the distance from a nonthick size to $[w''(n), w_B]$ are preserved when the integer word is reconstructed from the labelled jumps.

In the long-orbit case, use Claim 1 with $C = \max(100, 2C_2)$. *Claim 3.* Under Claim 1, $\ell < Y$ and $o_\ell > 4L_w$, the same word satisfies

$$2J + 2 \le L_w, \qquad J + 1 \le d_u, \qquad C_2(J+1)(w_* + S + \delta + 1) \le b_*.$$

It has a set $\mathcal{M}$ of marked first endpoints and integers $0 \le d_x \le J$, $d_x < d_u$, for $x \in \mathcal{M}$. Both x and $x + d_x$ are thick, there is no thick position strictly between them, and the J positions preceding x are not thick. Every thick position belongs to $\{x, x + d_x\}$ for exactly one $x \in \mathcal{M}$. These data imply the two-case separation condition in Lemma 36.89.

Indeed, $Y = kL/(m\delta_2)$ is the scale parameter in the least-power construction. Its least exponent $j \ge 0$ with $Y \le (\ell - 1)^j$ equals J when $\ell < Y$, since then $j \ge 2$. The progression-window hypothesis gives $2j + 2 \le L_w$. The Frobenius orbit and its projection to the tower group have the required order bound; inverse Frobenius has the same orders. Applying the long-orbit construction with the rounded scalars above gives the size inequality and the marked pairs.
Lean: `GlobalTheory.depth_spec` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.LeastPowerDepthSpec`, lines 8–24)
Lean: `GlobalTheory.HTW.Rec.wordsB_recWord_of_long_orbit` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.TwistedDualSizeWordLongOrbitSmallPrime`, lines 16–49)
Lean: `GlobalTheory.HTW.Clusters` (`Development.AbelianVarietyTheory.Definitions.GaloisGeometry.Alternates`, lines 17–21)
Lean: `GlobalTheory.HTW.Rec.clusters_letterwise` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.CyclicWordMarkedClustersLetterwise`, lines 11–36)

Indeed, if $a = x$ is a first endpoint, use the preceding gap. If $a = x + d_x$ and $d_x = 0$, use the same gap. Otherwise set $d = d_x$. For $k < d$ use the gap inside the pair, and for $d < k \le d + J$ use the gap preceding x. These cases give exactly the assertion required for the cut construction.
Lean: `GlobalTheory.HTW.RecS.jointB_of_large` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.InertLongOrbitWordConditionsOfLargeScale`, lines 20–40)

Claim 4. Under Claim 1 and the inert-prime hypotheses, the common word has alternating signs and satisfies $1 \le b_*, w_*, 20w_*, 20\delta, 20S \le b_*$, and $s_a > \delta$ at every position. Either it has a nonthick position and no three consecutive thick positions, or its degree is even and, with $M = \max_a s_a$, every a satisfies

$$s_a \ge (M + S)/2, \qquad s_a \ge 2A + \delta + 1.$$

These assertions use precisely the scalar dictionary in Claim 2.
Lean: `GlobalTheory.HTW.Rec.recWord_dichotomy` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.TwistedDualSizeWordInertDichotomy`, lines 31–73)

For this claim use Claim 1 with constant 100. Inert Frobenius, and therefore its inverse, has sign -1, so successive signs alternate. The word estimate gives either the nonthick alternative directly, with (36.39) and no three consecutive thick letters, or an even alternating word with

$$s_a \ge \frac{M + S}{2}, \qquad s_a \ge 2A + \delta + 1 \quad \text{for all } a, \qquad M = \max_a s_a,$$

together with the size conditions $1 \le b_*$, $1 \le w_*$, $20w_* \le b_*$, $20\delta \le b_*$, $20S \le b_*$ and $s_a > \delta$ for all a. In the latter alternative, if a thick letter exists then $M \ge b_*$. A nonthick letter would satisfy both $s_a \ge b_*/2$

and $s_a \le w_* \le b_*/20$, a contradiction since $b_* \ge 1$. Thus every letter is thick. If no thick letter exists, every letter is nonthick, and the short-word alternative holds. This proves the stated dichotomy. In the all-thick alternative, choose a cyclic origin matching the alternating signs and put $\beta_a = s_a$. Since $s_a > \delta$, the maxima in the enclosure heights reduce to $s_a + S$. The crystallinity, signed jump windows, central multiplicity bound, even degree, and two displayed inequalities are then exactly the hypotheses of Definition 36.76. This assertion holds for every equivariant surjection from a stable source lattice in a subrepresentation of a finite sum of twisted dual full representations to a stable Tate-dual target lattice, since Claim 2 holds for all those lattices simultaneously. In the nonthick alternative, the same common word gives the hypotheses of Lemma 36.90(b).
Lean: `GlobalTheory.HTW.RecS.jointCD_of_large` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.InertPrimeWordConditionsOfLargeScale`, lines 21–47)
Lean: `GlobalTheory.Survey.ClusteredLetterwiseBoxes.frozenCD_of` (`Development.AbelianVarietyTheory.Counterexamples.HeckeCharacterExamples.ClusteredLetterwiseBoxes`, lines 123–160)
Lean: `GlobalTheory.HTW.Rec.WordsNT` (`Development.AbelianVarietyTheory.Definitions.GaloisGeometry.NotDvdCondN`, lines 73–74)
Lean: `GlobalTheory.HTW.Rec.WordsAT` (`Development.AbelianVarietyTheory.Definitions.GaloisGeometry.NotDvdCondN`, lines 76–82)
Lean: `GlobalTheory.HTW.Rec.recWord_dichotomy` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.TwistedDualSizeWordInertDichotomy`, lines 31–73)
Lean: `GlobalTheory.HTW.RecS.settingC_of_wordsAT` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.AlternatingWordHypothesesEveryLatticeSurjection`, lines 90–95)

The arithmetic passage. Let $\mathcal{R}, \mathcal{C}, D_0, D_{\mathrm{sp}}$ have the meaning of Lemma 36.91. Let W be the normalized rank-r lattice in $\rho_i(\pi)\chi_{i,1}^{-1}$, and use the invariant lengths $h_0, h_W(u)$ of Lemma 36.23 together with the canonical hub constants c_B, c_0 defined after Definition 36.15, setting $c_0(u) = c_0$ identically for every column place $u \in \mathcal{C}$.

Claim 5. For the cut data with the canonical parameters and the small-prime hypotheses specified after Definition 36.15, every twisted index in the present range satisfies

$$c_B \le 120tnn_V L, \qquad h_0 \le 2t \le 120t.$$

At a split index, every column place $u \in \mathcal{C}$ and every good point π also satisfy

$$c_0(u) \le 120nn_V L, \qquad \ell^{c_0(u)} H^0(E_u, W \otimes_{O_i} (K_i/O_i)) = 0, \qquad h_W(u) \le 120rnn_V L.$$

Lean: `GlobalTheory.NSizes.main_alt_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ImportsFurtherRest`, lines 13–44)

For the last inequality, the killing assertion puts the invariant module inside

$$(W \otimes_{O_i} (K_i/O_i))[\ell^{c_0(u)}] \simeq (O_i/\ell^{c_0(u)} O_i)^r.$$

Taking normalized lengths gives $h_W(u) \le r c_0(u)$, and the bound for $c_0(u)$ gives the stated result. For the assertion about h_0, fix $u \in \mathcal{C}$ and $1 \le e < t$. The field E_u is unramified and $\ell \ge 5$, so cyclotomic inertia contains an element whose cyclotomic character is $1 + \ell$. Its action on the twist by $-e$ shows that

$$\lambda_i H^0(E_u, (K_i/O_i)(-e)) \le 1 + v_\ell(e) \le 2t.$$

Taking the finite maximum defining h_0 gives $h_0 \le 2t$; the empty maximum is zero. The direct invariant bound applies since ℓ is outside the small-prime set, $\ell \ne q, d$, and all column places lie above ℓ.
Lean: `GlobalTheory.InvariantsKO.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.MaxLengthLocalInvariantsTwistedFractionModIntege`, lines 156–171)

At an inert index the column set is empty. Claim 5 therefore supplies all four hypotheses of Lemma 36.91 with $c = 120$.

Claim 6. Under the cut's point and admissibility conditions of Theorem 36.26(I), the local allowances in part (b), the invariant bounds in Claim 5 and the assumed finiteness, put

$$G_{i,\pi} = s_\pi + \sum_{1 \le e < t} s_{\ell,e}(E) + v_\ell(h_E) + t \sum_{v \in Q_i} \lambda_i(O_i/(\zeta_{i,v} - 1)).$$

Then

$$v_{cc}(i,\pi) \le G_{i,\pi} + \sum_{u|\ell} L_{i,u} + \begin{cases} D_{\mathrm{sp}}, & \ell \text{ split}, \\ D_0, & \ell \text{ inert}. \end{cases}$$

Lean: GlobalTheory.splitCase_bound_at_point_of_admitted_items (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SplitCaseBoundAtPointOfAdmittedItems, lines 14–32)
Lean: GlobalTheory.LocalisationBoundAtPoint.boundAtPoint_all_localisations_of_objectConditions (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.BoundAtPointAllLocalisationsOfObjectConditionsAl, lines 15–22)
Lean: GlobalTheory.NLink.starAD_of_global (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.StarInequalityRowsADOfGlobalTheorems, lines 8–109)

For the split case, the cut's local admissibility conditions provide the column and dual lifting exponents, the invariant length and block bounds, and the crystalline jump and local cohomology conditions. Its point conditions also provide the local gain comparison for the full family and crystallinity of its normalized representations. Since $\ell \neq g_0, g_1$, these are the additional gain hypotheses in the split passage. Combining the local row comparison with the split congruence assembly of Lemma 36.23 (which applies identically at unramified split $\ell \geq 5$ with the global Poitou–Tate and Bloch–Kato Selmer lattice-index comparison bounding the global congruence valuation $v_{cc}(i,\pi)$ by the sum of the local row and column cohomology allowances, class-number and Sha terms, and torsion-killing terms) then gives the expression in (36.25), with $A = \mathcal{C}$, $A' = \mathcal{R}$, $e' = e_A$, $\delta = \delta_{\mathrm{pr}}$, and the constant $c_0(u) = c_0$. Thus the contribution of a column place u is

$$e_A + d_u m(2e_A + \delta_{\mathrm{pr}}) + 2\delta_{\mathrm{pr}} + h_W(u) + rc_0(u) = L_{i,u} + d_u m\delta_{\mathrm{pr}} + 2\delta_{\mathrm{pr}} + h_W(u) + rc_0(u).$$

The row and column sets cover the places above ℓ, and the allowance is zero outside the columns. Hence the sum of the first terms is exactly $\sum_{u|\ell} L_{i,u}$; all remaining row and column terms are D_{sp}. The split congruence comparison and global reciprocity therefore give

$$v_{cc}(i,\pi) \leq s_\pi + \sum_{1\leq e<t} s_{\ell,e}(E) + v_\ell(h_E) + \sum_{u|\ell} L_{i,u} + D_{\mathrm{sp}} + t\sum_{v\in Q_i} \lambda_i(O_i/(\zeta_{i,v}-1)).$$

If ℓ is inert, the column set is empty. The row set is then the whole set of places above ℓ. The cut's design conditions supply the determinant condition for its admissible local data. Global reciprocity, Bloch–Kato duality, local Tate duality and the local Euler characteristic then give the same global Poitou–Tate and Bloch–Kato Selmer lattice-index comparison for the row allowances, bounding $v_{cc}(i,\pi)$ by the sum of the local row allowances $\sum_{u|\ell} L_{i,u}$ (in the absence of columns there is no column lifting term), the class-number and Sha terms, the torsion-killing terms, and the remaining local error D_0. This gives the same inequality with D_{sp} replaced by D_0, proving Claim 6. The global character terms are finite by the assumed finiteness of the odd Sha-groups. In both cases (36.26) bounds the last sum by ℓ^{-2}, and Lemma 36.91 bounds the remaining contribution by $7200fnt^2n_VL^2$. Formula (36.21) gives the required $\mathcal{B}_{i,\pi}$ assertion.
Lean: GlobalTheory.NLink.starAD_of_global (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.StarInequalityRowsADOfGlobalTheorems, lines 8–109)

The complete proof is contained in `GlobalTheory.HTW.RecS.fam_jointBCD_closed_alt` and `GlobalTheory.NLink.rowsAD_in_setting_alt`, using `GlobalTheory.NLink.starAD_of_global` and `GlobalTheory.NLink.BBSize.main`. □

Proof of 36.93

Proof. Choose $C_2 = \max(1, C_0)$, where C_0 is the threshold of Lemma 36.89. Take the canonical hub realizing the good points and the twist of Lemma 36.27, whose killing condition is (36.26). Work above the common family cutoff of Theorem 36.16, Lemma 36.54(a), and Lemma 36.92, as well as the parameter, exceptional-prime, and small-prime family cutoffs.
Lean: GlobalTheory.T.theoremE_corollaryA_closing_all_data (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TheoremECorollaryAClosingAllDataStatement, lines 8–9)
Lean: GlobalTheory.T.exists_hubData_admJ_setup (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ExistsSetupLargeStatement, lines 12–13)
Lean: GlobalTheory.T.excludedPrimes_few_and_small_along_family (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ParametersExcludedPrimesFewAndSmallAlongFamilyStatement, lines 13–14)
Lean: GlobalTheory.T.specialPrimes_small_along_family (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SpecialPrimesSmallAlongFamilyStatement, lines 13–14)
Lean: GlobalTheory.T.admissible_parameters_family_const (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleParametersFamilyConstStatement, lines 13–14)

Use the size width S_c and error width $\eta = \eta_{\mathrm{sz}}$ of the member. Put $n = f/2$ and $n_V = r + t$. Take Z, a_*, the base sizes, the intervals (36.32), and the exceptional primes of Lemma 36.54, and define

$$Y = \frac{kL}{m\delta_2},$$

where $\delta_2 = |T_2|/n > 0$ is the density of thick host positions. For a sufficiently large family constant $C > 0$ chosen below (in particular $C \ge \max(3C_2 + 1, 22365, 7200)$), define A_ℓ by the first applicable row of

ℓ	A_ℓ
g_0, g_1	$Cx^C\eta$
$\ell \mid Q_S$ or $\ell = 3$	$Cfmb\log x + Cx^C L^2$
q, d	$2Cfmb$
$\ell > Z$ outside the exceptional set	0
otherwise	$Cfnt^2 n_V L^2 + D_\ell$.

(A.526)

For the final row set $P_\ell = C(1 + H_B/\ell + \log_\ell H_B)$, $J = \max(1, \lceil \log_{\ell-1} Y \rceil)$, and

$$\eta_3 = \eta + t + 2, \quad w_B = \max\{w''(n) + \eta, \max_T(b_T + b'/x + \eta)\}, \quad w_P = w_B + (t+1)/2, \quad G = w_P + S_c + \eta_3.$$

Let $I_\ell = 1$ if some $w''(n) \le y \le w_B$ satisfies

$$|\ell y - b| \le C(\ell\eta_3 + \ell S_c + b/\ell + y + \ell L), \tag{A.527}$$

and $I_\ell = 0$ otherwise. Put $F_\ell = (w_P + \eta_3)/\ell + S_c + L + \eta_3 I_\ell$. Let o_ℓ be the order of ℓ modulo 2^{e+2} up to sign for the host tower exponent e, and L_w the host progression-window length. Split and inert refer to $\mathbf{Q}(\sqrt{-d})$. The place part is

$$D_\ell = \begin{cases} (2fm + f)P_\ell & \text{split}, \\ 4fCr\{H_B/(\ell-1)^{J+1} + (G+1)(J+1) + L\} & \text{inert, } \ell < Y, \ o_\ell > 4L_w, \\ 4fCr(H_B/\ell^2 + F_\ell) + 2fCmP_\ell & \text{other inert}. \end{cases} \tag{A.528}$$

Lean: `Formalism.MainNotions.TwistedPlaces.lenB` (`Formalism.SharpRestriction.LevelLetterCocycles.MemPrimesOverOfMem`, lines 249–253)
Lean: `Formalism.MainNotions.TwistedPlaces.lamB` (`Formalism.SharpRestriction.LevelLetterCocycles.MemPrimesOverOfMem`, lines 259–263)

Off the type primes, compose Hecke elements supported on $Z(i)$ with α_i. Their support becomes good and their good evaluations are multiplied by M_0. An evaluated ideal of valuation a costs at most ha on a lattice of rank h by elementary divisors. The eigenspace filtration yields

$$\lambda_i(U_i/L_i) \le \sum_{\pi \in Z_{\mathrm{good}}(i)} \mathrm{mult}(i,\pi)(v_{cc}(i,\pi) + v_\ell(M_0)). \tag{A.529}$$

Combine (A.529) with (36.28), the rank row (which identifies $\dim Z_j(\mu_w)$ with $d_{j,0}$), and the logarithmic place-weight sum of 20.7. At $\ell = 2$ and at $\ell = 3 \nmid Q_S$, Theorem 36.16 gives (36.22); for odd $\ell \mid Q_S$, the same split congruence bound (A.509) with $s_0 = s_{\mathrm{cyc},\ell}$ and $\ell^{v_\ell(Q_S)} \le Q_S \le n^2$ gives the second row of (A.526) once C exceeds the small-prime constants.
Lean: `GlobalTheory.EtaPassage.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ImportsTerMore`, lines 11–20)
Lean: `GlobalTheory.T.goodPointMultiplicityLeRank_proof` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.GoodPointMultiplicityLeRankProofStatement`, lines 13–14)
Lean: `UpperTheory.N.starSF_of_prints` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.NStarS`, lines 11–13)
Lean: `UpperTheory.N.rowC` (`Development.AutomorphicArithmetic.ResidueCharacters.RowAtOfSetup`, lines 43–45)

Large primes outside the exceptional set. For $\ell > Z$ outside the exceptional set and outside $\{g_0, g_1\}$, Lemma 36.54(a) shows that $\ell \ge 11$, $t + 1 < \ell$, $\ell \notin \{q, d\}$, $\ell \nmid Q_S$, and every completion E_w ($w \mid \ell$) is unramified over $\mathbf{Q}_\ell$, and constructs position data at every embedding. Over a finite extension L'/K_i containing all embeddings of $E_w(\mu_\ell)$ with integers O, Lemma 36.54(b) (using Proposition 36.53) shows that the image of every cut cocycle at w is finite in the normalized rank-r block, a coboundary on G_{E_w} in each character block $2 \le c \le t$, and a coboundary on I_{E_w} (hence zero on I_{E_w}, as the action is trivial) in the $c = 1$ block. Finite classes descend from L' to K_i by period comparison; applying a K_i-linear retraction preserving integer rings descends coboundaries on $2 \le c \le t$; and injectivity on divisible coefficients descends zero inertia

restriction on $c = 1$. Since $t + 1 < \ell$ and $E_w/\mathbf{Q}_\ell$ is unramified, a cyclotomic inertia generator separates the t scalar characters by units, giving unit projector denominators in the general disjoint-partition (A, A') congruence-ideal assembly noted at the end of the proof of Lemma 36.23, specialized here to $A = A' = \varnothing$. Inertia exclusion or the unit tame root difference kills restrictions at S_i, while at Q_i each character is killed by $\zeta_{i,v} - 1$. With the global Selmer, Sha, and class-number bounds and (36.26), this yields

$$\mathcal{B}_{i,\pi}(\rho; 0) \qquad (\pi \in Z_{\mathrm{good}}(i)). \tag{A.530}$$

Lean: `GlobalTheory.T.eventually_congBound_of_facts` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.AdmissibleParameterEventuallyStatement`, lines 11–12)
Lean: `GlobalTheory.T.localBoundZeroAtLargePrimes_of_congruenceBound` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalBoundZeroAtLargePrimesOfCongruenceBoundStatement`, lines 13–14)

The primes q, d. At $\ell \in \{q, d\}$, Lemma 36.31 supplies crystalline realizations over $K = E_w \subset K_0(\mu_\ell)$ with ramification index $e = 2$ and $s = (\ell - 1)/2$. Apply the general disjoint-partition (A, A') congruence-ideal assembly noted at the end of the proof of Lemma 36.23, specialized to empty column set $A = \varnothing$ and row set $A' = \{w \mid \ell\}$ consisting of all places of E above ℓ. Since $[E : \mathbf{Q}] = 2f$ and every $w \in A'$ has $e_w = 2$, we have $\sum_{w \in A'} d_w = 2f$ and $2|A'| \le \sum_{w \in A'} d_w = 2f$ (so $|A'| \le f$), whence $\sum_{w \in A'}(d_w + 2) \le 4f$. On the rank-$r$ block, Proposition 36.65 and Lemma 36.66 give the ramified local lifting bound with constant 160 at each $w \in A'$; taking integer ceilings adds at most $+1$ per place (summing to at most $|A'| \le f$ across $w \in A'$), so summing over $w \in A'$ for the coordinates and their Tate duals with their jump bounds gives $R_W \le 160\{3fr(4b/\ell + 2S_c + 2\log_\ell(2b) + 1 + \eta + t) + 6fm(2b + 1)\} + f$. On each of the t character blocks, $E_w \subset K' = K_0(\mu_\ell)$ already has cyclotomic level $s = 1$ and degree $[K' : E_w] = (\ell - 1)/2$ prime to ℓ, so the final corestriction term $v_\ell([K' : E_w])$ vanishes; substituting $s = 1$, $h = \lfloor 2b \rfloor$, $a \le 2f(r + t)$, and $\lambda = \lfloor \log_\ell(2\lfloor 2b \rfloor) \rfloor \le 4L$ (as $\ell \ge 3$, $3^\lambda \le 4b$, and $\log 3 > 1$) into the full character-block bound c_{TB} proved in Lemma 36.22 (combining (A.517) with the scalar-cost and corestriction terms there) yields $c_B = (3+\lambda)+t\big(3+2f(r+t)(4+2\lambda)\big)+4t+3 \le 41tf(r+t)L \le 60tf(r+t)L$. Multiplying by the local-duality and Euler-characteristic sum $\sum_{w \in A'}(d_w + 2) \le 4f$ gives total character-block cost $tc_B \sum_{w \in A'}(d_w + 2) \le 240f^2t^2(r + t)L$. With $A = \varnothing$, $s_0(i) = 1$, and $\ell > x^{2C_t+5} > t$ (so $v_\ell(e) = 0$ for all $1 \le e < t$, giving $\delta = t - 1 \le t$), the projector contribution $4n_V(|A| + 1)\delta$ from Lemma 36.23 is $4n_V\delta \le 4(r + t)t$. Since $m \le r$, $\ell \ge 3$, $L \ge 9$ (holding above the eventual threshold $L \ge 15$) and $\log_\ell(2b) \le 2L$, the brace in R_W is at most $13f\{r(b/\ell + S_c + t + \eta + L) + mb\}$, and the f term and the projector contribution fit in $20f^2t^2(r+t)L$. With (36.26) (where $\nu_\ell = 0$ by Lemma 36.27), this bounds the allowance in (36.21) by

$$2080f\{r(b/\ell + S_c + t + \eta + L) + mb\} + 260f^2t^2(r + t)L. \tag{A.531}$$

Because the prime lower bound $\ell > x^{2C_t+5}$ (where C_t bounds the polynomial rank growth in the auxiliary-prime row) absorbs frb/ℓ and $S_c + \eta \le x^{C'} \exp(x^{\varepsilon/4})$ makes the remaining polynomial terms eventually smaller than fmb, this allowance is at most $2Cfmb$ for C above the hub constant.
Lean: `GlobalTheory.T.embedding_jumps_aux2_of_setup` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.PolynomialBoundSetup`, lines 11–12)
Lean: `GlobalTheory.NLink.starG_of_thmB` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.PolyValLeSumMulPow`, lines 35–114)

The theta primes. Let $C_T \ge 40$ be the type threshold of Theorem 36.26(I); it depends on $\mathscr{H}_2$ alone, so $40 \le C_T \le b$ eventually. By Proposition 36.74, statement (P) of Definition 36.67 holds at $\ell = g_k$ ($k \in \{0, 1\}$), and Lemma 36.75 bounds $\lambda_i(U_i/L_i)$ by the multiplicity-weighted sum of

$$4fn_VC'n_V^2(n_V + \eta_X + t) + 2fr^2 + 4n_V\delta_X + s_\pi + \sum_{1 \le e < t} s_{\ell,e}(E) + v_\ell(h_E) + t \sum_{v \in Q_i} \lambda_i(O_i/(\zeta_{i,v} - 1)). \tag{A.532}$$

Here $\eta_X = \lfloor \eta \rfloor \le \eta$ and $s_0(i) = 1$ gives $\delta_X \le 2t$. Since $t \le \eta$ and the ranks grow polynomially in x, the first three terms fit in $Cx^C\eta$, and (36.26) with $\nu_{g_k} = 0$ gives the first row of (A.526) via (36.28).
Lean: `GlobalTheory.T.star_bound_of_paired_character_condition` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EtaLocLeRhsStatement`, lines 12–13)
Lean: `Formalism.MainNotions.TwistedPlaces.etaAt_le_g` (`Formalism.SharpRestriction.LevelLetterCocycles.MemPrimesOverOfMem`, lines 108–118)

The final row at split primes. Let $\ell \geq 5$ with $\ell \nmid Q_S q d g_0 g_1$ split in $\mathbf{Q}(\sqrt{-d})$, and put $h_N = \lfloor H_B \rfloor$ and $e_A = \lceil 23\{2 + 2h_N/\ell + \log_\ell(2h_N + 1)\}\rceil$. Since $H_B \geq 1$ and $\ell \geq 5$, we have $\log_\ell(2h_N + 1) \leq 1 + \log_\ell H_B$, so $e_A \leq 23\{3 + 2H_B/\ell + \log_\ell H_B\} + 1 \leq 70(1 + H_B/\ell + \log_\ell H_B) \leq P_\ell$ whenever $C \geq 70$. Complex conjugation pairs the column set A of Lemma 36.92(b) with its complement of equal local degrees $d_u \geq 1$, where $\sum_{u|\ell} d_u = 2f$. Hence the place sum in Lemma 36.92(b) satisfies

$$\sum_{u|\ell} L_{i,u} = \tfrac{1}{2}\sum_{u|\ell}(2d_u m + 1)e_A \leq \tfrac{1}{2}\sum_{u|\ell}(2d_u m + 1)P_\ell \leq (2fm + f)P_\ell. \tag{A.533}$$

Lean: `GlobalTheory.NLink.RowA.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TwoEALe`, lines 49–78)

The final row at inert primes. Now let $\ell \geq 5$ with $\ell \nmid Q_S q d g_0 g_1$ be inert in $\mathbf{Q}(\sqrt{-d})$. Fix a place $u \mid \ell$ of E and a good point $\pi \in Z_{\mathrm{good}}(i)$, and take the unramified field $K = E_u$ of degree d_u (with $\ell \geq 11$) and any equivariant crystalline surjection $F : T_1 \to T_2$ as in Lemma 36.92(a), writing $q(F) = \lambda_O(Z_f(T_2)/I_F)$. Let $a_0 = \dim_{L'} H^0(K, T_2[1/\ell])$; by (36.20), $\mathrm{rank}_O H^1_f(K, T_2) \leq d_u r + a_0$, while saturation of the invariant lattice embeds the torsion of $H^1_f(K, T_2)$ into a divisible O-module of corank $r - a_0$, so $Z_f(T_2)/B(T_2)$ is generated over O by at most $(d_u + 1)r$ elements. Consequently, whenever ℓ^e lifts every class in $Z_f(T_2)$ modulo I_F plus the O-span of z elements of $Z_f(T_2)$ and $\ell^{e'}$ lifts every class in $Z_f(T_2)$ into I_F, we have

$$q(F) \leq (d_u + 1)re + ze'. \tag{A.534}$$

Case 1: $\ell < Y$ and $o_\ell > 4L_w$. By Lemma 36.92(a), (36.37) and (36.38) hold and the thick positions form singletons or pairs separated by at least J nonthick positions. Applying Lemma 36.89 gives (36.33). By (36.37), $h \leq H_B$ and $g = w_* + S + \delta \leq w_P + S_c + \eta_3 = G$, while $\log_\ell H_B \leq L$ (from $H_B \leq 2b = 2e^L$ and $\ell \geq 11$). Moreover $\log_\ell h \leq h/q_\ell^{J+1} + J + 1$: this is immediate if $h < \ell^{J+1}$, and if $h \geq \ell^{J+1}$ then $\log_\ell h = (J+1) + \log_\ell(h/\ell^{J+1}) \leq J + 1 + h/\ell^{J+1} \leq J + 1 + h/q_\ell^{J+1}$. Thus (36.33) gives $q(F) \leq 3C_2 B_u$, where

$$B_u = (d_u + 1)r\{H_B/q_\ell^{J+1} + (G + 1)(J + 1) + L\}. \tag{A.535}$$

Choose a rational allowance $L_{i,u} \in [3C_2 B_u, (3C_2 + 1)B_u]$, uniform over both $\pi \in Z_{\mathrm{good}}(i)$ and every equivariant surjection $F : T_1 \to T_2$ as in Lemma 36.92(a).

Lean: `GlobalTheory.NLink.localBoundTableFromData_rowB_rowEntries` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalBoundTableFromDataRowBRowEntries`, lines 90–113)

Case 2: Other inert primes, word with a nonthick position. Here Lemma 36.92(a) gives alternating signs and (36.39), with at least one nonthick position and no three consecutive thick positions. Lemma 36.90(a) and (b) then give (36.34) with $H' \leq 2H_B$ (by (36.37)) and the colength bound (36.36), uniformly in both $\pi \in Z_{\mathrm{good}}(i)$ and $F : T_1 \to T_2$.

Case 3: Other inert primes, every position thick. By Lemma 36.92(a), d_u is even, the signs alternate, and setting $h = \max_a s_a + S$, $H = h + \delta$, $\lambda = 1 + \log_\ell(2H)$, and $A = H(2\ell - 1)/q_\ell^2 + 3 + \lambda$ gives $h/2 \leq s_a$ and $2A + \delta + 1 \leq s_a$ for all $a \in \mathbf{Z}/d_u\mathbf{Z}$ (uniformly in π and F). Put $\sigma = (\lfloor h/q_\ell \rfloor + 1 + \delta)/q_\ell + 1$ and $\sigma_2 = 2H/q_\ell^2 + 1$. By Proposition 36.88, statement (R) of Definition 36.76 applies with $\beta_a = s_a$, central width δ, $d' = d_u$, and $p_0 \in \{0, 1\}$ chosen so that $\varsigma_a = -1$ exactly when $a + p_0$ is even, supplying $a_1 \leq A + 1$, s_1, E_1, e_2, and $e \leq s_1 + 2\sigma + e_2 + S + E_1 + \sigma_2 + 1$. Substituting the bounds of Definition 36.76 for s_1, E_1, e_2 gives

$$e \leq 32H/\ell^2 + 4S + 10\delta/q_\ell + 25\lambda, \tag{A.536}$$

since for $\ell \geq 11$ ($q_\ell \geq 10$, $\lambda \geq 1$, $\lfloor h/q_\ell \rfloor \leq H/q_\ell$) the coefficient of H times ℓ^2 is at most $8(21/10)(11/10)^2 + 8(11/10)^2 + 11/10 < 32$, the coefficient of S is at most $3 + 3/q_\ell \leq 4$, the coefficient of δ is at most $10/q_\ell$, and the remaining constants and logarithms fit in 25λ. Statement (R) also provides the free module F_0 of rank at most $(d_u + 1)r$, the surjection $\jmath : F_0 \to Z_f(T_2)$ modulo $B(T_2)$, the chain $F_0 \supset \cdots \supset F_k$ with $\ell^e \jmath(F_k) \subset I_F$, and $\mathcal{G} \subset \{0, \ldots, k-1\}$ with $|\mathcal{G}| \leq 4d_u$ and $k - |\mathcal{G}| \leq 3d_u/2$. Setting $I' = \jmath^{-1}(I_F)$ identifies $F_0/I' \simeq Z_f(T_2)/I_F$: the bottom piece $F_k/(F_k \cap I')$ has normalized length at most $(d_u + 1)re$; each of the at most $4d_u$ pieces $i \in \mathcal{G}$ is generated modulo the next by at most m elements annihilated in $Z_f(T_2)/I_F$ by $\ell^{e'}$ from Lemma 36.90(a) (with $H' \leq 2H_B$); and each of the at most $3d_u/2$ remaining pieces has normalized length at most $rS + 2m((H + 1)/q_\ell + 1)$. Additivity of length gives

$$q(F) \leq (d_u + 1)re + 4d_u me' + \tfrac{3}{2}d_u\{rS + 2m((H + 1)/q_\ell + 1)\}. \tag{A.537}$$

Uniform inert allowances and the band condition. Let $b_u \in \{0,1\}$ be the indicator that Case 2 occurs with $I = 1$ at some $\pi \in Z_{\mathrm{good}}(i)$ at u. Put $E_\ell = 1 + H_B/\ell + \log_\ell H_B$ and

$$A_u = (d_u+1)rH_B/\ell^2, \qquad M_u = d_u m E_\ell, \qquad F_u = (d_u+1)r\{(w_P+\eta_3)/\ell + S_c + L + \eta_3 b_u\}.$$

From $1 \le H_B \le 2e^L$ and $\ell \ge 11$, (36.34) with $H' \le 2H_B$ gives $e' \le 69E_\ell$. In Case 2, $\log_\ell(2h) \le 3+L$ and (36.37) bound (36.36) by $320A_u + 1920F_u + 22080M_u$. In Case 3, $H \le H_B + \eta_3$, $\log_\ell(2H) \le 7+L$, $32\eta_3/\ell^2 \le 3\eta_3/\ell$, and $10\delta/q_\ell \le 11\eta_3/\ell$ give $e \le 32H_B/\ell^2 + 225F_u/((d_u+1)r)$ in (A.536) and $(H+1)/q_\ell + 1 \le 4E_\ell$ in (A.537), so $q(F) \le 32A_u + 227F_u + 288M_u$. Thus in both Case 2 and Case 3,

$$q(F) \le 368A_u + 2082F_u + 22365M_u. \tag{A.538}$$

Choose a rational allowance $L_{i,u} \in [368A_u + 2082F_u + 22365M_u,\ 368A_u + 2083F_u + 22365M_u]$ (and $L_{i,u} = 0$ if $Z_{\mathrm{good}}(i) = \varnothing$). If $b_u = 1$, take the witnessing nonthick position a in (36.35) and its $y \in [w''(n), w_B]$ from Lemma 36.92(a) with $|s_a - y| \le T_0$: since $|b_* - b| \le T_0$, $(\ell+1)T_0 \le \ell\eta_3$, and $\log_\ell(2h) \le L$, the brace $D_{\mathrm{band}} = \ell\eta_3 + \ell S_c + b/\ell + y + \ell L$ of (A.527) bounds the brace of (36.35) by $D_{\mathrm{band}} + \ell\eta_3 \le 2D_{\mathrm{band}}$, whence $|\ell y - b| \le |\ell s_a - b_*| + \ell T_0 + T_0 \le 641 D_{\mathrm{band}}$ and therefore $b_u \le I_\ell$ whenever $C \ge 641$.

Lean: `Formalism.MainNotions.N.KillingConditions.RowK.Le` (`Formalism.SharpRestriction.LevelLetterCocycles.JointConditionsZpowers.CommutingEndomorphismCondition`, lines 180–182)
Lean: `GlobalTheory.T.mediumInert_smallInert_lengthBound_tabX` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalBoundTableRowsCDAdmissibleStatement`, lines 7–8)
Lean: `GlobalTheory.T.localBoundTableFromData_bandCondition` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalBoundTableFromDataBandConditionStatement`, lines 7–8)

Require $C \ge \max(3C_2 + 1, 22365, 7200)$. Using $\sum_{u|\ell} d_u = 2f$, $\sum_{u|\ell}(d_u+1) \le 4f$, and $b_u \le I_\ell$, summing $L_{i,u}$ over $u \mid \ell$ in (A.533), (A.535), and (A.538) gives $\sum_{u|\ell} L_{i,u} \le D_\ell$ in all three branches of (A.528). Since $7200 \le C \le Cn$, Lemma 36.92(b) yields

$$\mathcal{B}_{i,\pi}(\rho; Cfnt^2 n_V L^2 + D_\ell) \qquad (\pi \in Z_{\mathrm{good}}(i)). \tag{A.539}$$

Lean: `GlobalTheory.T.sizeConditions_mono` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ParametersConditionsMonotoneInConstantStatement`, lines 13–14)
Lean: `Formalism.MainNotions.TwistedPlaces.loc_le_lamB` (`Formalism.SharpRestriction.LevelLetterCocycles.MemPrimesOverOfMem`, lines 384–499)

Combining (A.529), (36.22) (and its odd-Q_S counterpart), (A.530), (A.531), (A.532), and (A.539) with (36.26) and (36.28) proves (36.40).

Lean: `Formalism.MainNotions.TwistedPlaces.etaAt_le` (`Formalism.SharpRestriction.LevelLetterCocycles.MemPrimesOverOfMem`, lines 57–106)
Lean: `Formalism.MainNotions.TwistedPlaces.etaSum_le` (`Formalism.SharpRestriction.LevelLetterCocycles.MemPrimesOverOfMem`, lines 558–647)

Summation of the table. Put $X_0 = fmbL^2$ and $U = x^{C+2}\exp(x^{\varepsilon/4})$. Since $L = x^\varepsilon$ and $b = e^L$ with $\varepsilon > 0$, eventually

$$4\cdot 10^6 (C+1)^2 U^{18} \le b, \qquad (C+3)^2(1+\log x)^2 \le L, \qquad x^C L \le b. \tag{A.540}$$

Lean: `Formalism.MainNotions.TwistedPlaces.big_le_b` (`Formalism.SharpRestriction.LevelLetterCocycles.ExportBoundOfAdmissible`, lines 501–540)

Polynomial growth and the width and type-prime size bounds give $\sum_{\ell \in \{g_0, g_1\}} A_\ell \log \ell \le 2(C+1)^2 U^{18}$. The prime divisors of $3Q_S$ have total logarithm at most $\log 3 + 2\log n$, so their contribution is at most $8CX_0$; at q, d, $\log q, \log d \le L$ gives at most $4CX_0$.

Lean: `Formalism.MainNotions.TwistedPlaces.sum_g` (`Formalism.SharpRestriction.LevelLetterCocycles.ExportBoundOfAdmissible`, lines 651–705)
Lean: `Formalism.MainNotions.TwistedPlaces.sum_23` (`Formalism.SharpRestriction.LevelLetterCocycles.ExportBoundOfAdmissible`, lines 733–752)
Lean: `Formalism.MainNotions.TwistedPlaces.sum_qd` (`Formalism.SharpRestriction.LevelLetterCocycles.ExportBoundOfAdmissible`, lines 798–823)

Every base size W satisfies $w''(n) \le W \le b$; because $w''(n) = b/(n\exp(x^{\varepsilon/3}))$ grows stretched-exponentially whereas $\eta + 2a_* + 3 = O(U^7)$ (as $a_* = \lfloor 2b/Z \rfloor \le 20U^7$ and $\eta \le U$), eventually $W > \eta + 2a_* + 3$ (while $W + S_c + 2\eta + 2a_* + 4 \le 3b$ by (F3) of Lemma 36.54), so every interval (36.32) lies strictly in $[1, 3b]$ and excludes 0. Together they contain at most $(|TT| + 2)(S_c + 3\eta + t + 8b/Z + 9) \le 388U^8$ integers in

$[1, 3b]$. Because $Z^2 > 3b$, each such integer has at most one prime divisor $\ell > Z$. Using $b/Z \le 10U^7$ in the final-row formulas gives $A_\ell \le 1000(C+1)^2U^9$ and $\log \ell \le \log(3b) \le 3U$ at every exceptional $\ell > Z$, for a total contribution of at most $1164000(C+1)^2U^{18}$.
Lean: `Formalism.MainNotions.TwistedPlaces.sum_exc` (`Formalism.SharpRestriction.LevelLetterCocycles.ExportBoundOfAdmissible`, lines 611–649)

For primes $5 \le \ell \le Z$, the definition of Z and $\sum_{\ell \le Z} \log \ell \le Z \log Z \le ZL$ bound the non-place sum $\sum_{5 \le \ell \le Z} Cfnt^2 n_V L^2 \log \ell$ by CX_0. Integral comparison gives

$$\sum_{2 \le a \le Z} \frac{\log a}{a} \le (1+\log Z)^2, \quad \sum_{5 \le a < Y} \frac{\log a}{a-1} \le 2(1+\log Y)^2, \quad \sum_{a \ge Y} \frac{\log a}{a^2} \le \frac{4(\log Y + 2)}{Y}. \tag{A.541}$$

From the first inequality in (A.541), $H_B \le 2b$, and $ZL \le b$,

$$\sum_{\substack{5 \le \ell \le Z \\ \ell \text{ prime}}} P_\ell \log \ell \le 11CbL^2. \tag{A.542}$$

In the long-order inert row ($\ell < Y$, $o_\ell > 4L_w$), use $(\ell-1)^J \ge Y$, $r/Y \le 2m/L$, $rb'Y \le mb$, $G+1 \le 11b'$, and $J+1 \le 3+\log Y$ with $(3+\log Y)^2 \le L$; the second sum in (A.541) and $\sum_{a<Y} \log a \le Y \log Y$ bound its total by $80CX_0$. For inert $\ell \ge Y$, the third sum in (A.541) bounds the $4fCr(H_B/\ell^2)\log \ell$ terms by $64CX_0$.

For inert $\ell < Y$ with $o_\ell \le 4L_w$, the 2-adic valuation of $\ell^{o_\ell} \mp 1$ puts $\ell \equiv \pm 1 \pmod M$, where $M = 2^{\max(0, e+2-\lfloor \log_2(4L_w) \rfloor)}$. With $R = n/L$, the host gives $R+1 \le M$, and summing along both progressions yields

$$\sum_{\substack{\ell < Y \text{ inert} \\ o_\ell \le 4L_w}} \frac{\log \ell}{\ell^2} \le \frac{2\log R}{R^2} + \frac{2(\log R + 2)}{RM}. \tag{A.543}$$

Since $r \le 2mR^2$, $n \le oML_w$, $(\log R + 2)L_w \le L$, and $\log R + o \le L^2$ for the host order parameter o, multiplying (A.543) by r gives at most $4mL^2$, whose contribution to the table sum is at most $32CX_0$.

For F_ℓ, use $r(w_P + \eta_3) \le 9mb$ and $r(S_c + L)Z \le 2mb$. When $I_\ell = 1$ and $\ell > 2C$, (A.527) gives $(\ell - C)y \le (1 + C/\ell)b + C\ell(\eta_3 + S_c + L)$ with $C(\eta_3 + S_c + L) \le w''(n)/8 \le y/8$, whence $w''(n) \le y \le 4b/\ell$; combined with $r\eta_3 \le 4mw''(n)$ and $r\eta_3 \le 4mb$, this yields

$$r\eta_3 I_\ell \le 16mb/\ell + 4mb\,\mathbf{1}_{\ell \le 2C}. \tag{A.544}$$

Summing via (A.541) and $\sum_{a \le 2C} \log a \le (2C+1)^2$ bounds the F_ℓ contribution by $4C[102 + 4(2C+1)^2]X_0$. Adding all contributions over $5 \le \ell \le Z$ gives coefficient at most

$$C + 11C(3+2C) + 80C + 64C + 32C + 4C[102 + 4(2C+1)^2] \le 1000(C+1)^3$$

times X_0.
Lean: `Formalism.MainNotions.TwistedPlaces.sum_small` (`Formalism.SharpRestriction.LevelLetterCocycles.ExportBoundOfAdmissible`, lines 18–79)

Finally, $Q_S \le n^2$, $\log h_E \le Cx^C$, and $\log M_0 \le Cx^C L$ give $6 + \log(Q_S tr) + \log h_E + \log M_0 \le 20(C+1)^2U^{18}$. By (A.540) and $b \le X_0$, the sum $(20 + 2 + 1164000)(C+1)^2U^{18}$ is at most $b \le X_0$, and $(1000(C+1)^3 + 8C + 4C + 1)X_0 \le 2000(C+1)^3X_0$. This proves (36.41) for every finite set S of primes.
Lean: `Formalism.MainNotions.TwistedPlaces.glob_le` (`Formalism.SharpRestriction.LevelLetterCocycles.ExportBoundOfAdmissible`, lines 825–870)
Lean: `Formalism.MainNotions.TwistedPlaces.point_bound` (`Formalism.SharpRestriction.LevelLetterCocycles.ExportBoundOfAdmissible`, lines 893–996)
Lean: `GlobalTheory.NLinkClose.main_alt_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ImportsFurtherTail`, lines 22–109)

□

Proof of 36.94

Proof. Let $T_\psi = \sum_q \psi(q)^{-1} T(q)$. Apply Definition 1.15(3), proved in Lemma 1.19, first to (Q, ψ) and then to $(\varnothing, 1)$. The second application has the same field, CM type, base conductor and weights as the first. The

empty set satisfies the condition on auxiliary primes. Thus there is one nonzero period Ω_0 for which both normalized sums are algebraic. The function x_{Ω_0} multiplies each sum by the same nonzero scalar. Since $T_0 \neq 0$, their quotient is R_ψ, which belongs to the chosen algebraic closure of K.

Write $g^{\mathrm{ab}} = g|_{K^{\mathrm{ab}}}$. The hypothesis on t is

$$\mathrm{Art}_K((1,t))^{-1} = g^{\mathrm{ab}},$$

where 1 is the archimedean component. This is the hypothesis on the finite idèle in Definition 1.15(3). Extend g from the embedded algebraic closure to $\mathbf{C}$ by fixing a transcendence basis and then extending to its algebraic closure. The extension fixes each CM embedding pointwise on its image. It also fixes the prescribed values $\chi_f(y)|y|_{E,f}^{s}$. Hence the conjugate weights and base character in the transport formula are the original ones. The conjugate Picard character is $g\psi$, given by $(g\psi)(q) = g(\psi(q))$.

Apply (1.4) twice with this g and t. To see the character factor, write $q' = q[t]^{-1}$ for the transported class. Then

$$g(\psi(q)^{-1}) = (g\psi)(q')^{-1}(g\psi)([t])^{-1}.$$

Translation by $[t]^{-1}$ permutes the classes. Put $A = |t|_{K,f}^{-K_\kappa} l_\chi^{-1}$, where l_χ is evaluated on the image of t in the finite idèles of E. The two transport identities are

$$\begin{aligned} g x_{\Omega_0}(T_\psi) &= x_{\Omega_1}\big(A(g\psi)([t])^{-1} T_{g\psi}\big), \\ g x_{\Omega_0}(T_0) &= x_{\Omega_1}(A T_0). \end{aligned}$$

The target period is common because it is chosen before the conductor and character. The norm of an idèle is positive and a character takes values in the multiplicative group, so $A \neq 0$. Division gives

$$g R_\psi = (g\psi)([t])^{-1} R_{g\psi}.$$

Both sides belong to the embedded algebraic closure, where the extension of g agrees with the given automorphism. □

Proof of 36.95

Proof. Choose integral generators $g_1, \dots, g_K$ of the underlying tensor lattice before completion. One can take the products of integral additive generators of the coefficient ideal and the two input lattices. They generate the completed lattice modulo M for every M, since reduction of a completed integral lattice modulo M is reduction of the original lattice. This follows prime by prime from the Chinese remainder theorem. Projecting gives generators $e g_j$ of L_+/ML_+.

For $c \in (\mathbf{Z}/M\mathbf{Z})^K$, use representatives between 0 and $M-1$ and set $y_c = M^{-1}\sum_j c_j e g_j$. For $t \in L_-$,

$$\sum_c \psi(\langle t, y_c\rangle) = \prod_{j=1}^{K} \sum_{b=0}^{M-1} \psi(\langle t, e g_j/M\rangle)^b = M^K \mathbf{1}_{ML_-}(t).$$

Indeed each inner character value has M-th power one. If all of them are one, t/M annihilates L_+. The conjugate halves pair perfectly by self-duality, so $t/M \in L_-$. Conversely every $t \in ML_-$ gives the trivial character.

Take representatives v of L_-/ML_- and put

$$b_c = \sum_v \Phi(v)\psi(-\langle v, y_c\rangle).$$

Each b_c belongs to R. By the displayed orthogonality identity, $\sum_c b_c \psi(\langle t, y_c\rangle) = M^K \Phi(t)$ on L_-. In the half-lattice model the translate of the vacuum by y_c has coefficient function $\mathbf{1}_{L_-}(t)\psi(\langle t, y_c\rangle)$. Equality of coefficient functions gives equality of the vectors. Finally $M^K/\ell^{K\,\mathrm{ord}_\ell(M_0)}$ is a unit at every place over ℓ. □

Proof of 36.96

Proof. Choose additive generators of L_0. Their images involve finitely many polynomial coefficients. If there is an embedded number field as in the statement, each such coefficient z is algebraic. Choose a positive integer a_z such that $a_z z$ is an algebraic integer, and take $c_1 = \max_z \operatorname{ord}_\ell a_z$, with maximum zero for an empty list. The factor of a_z prime to ℓ is invertible at every tested place. Thus $\ell^{c_1} z$ is integral there. Additivity gives the assertion on L_0.

For the second assertion I divides $|\mathcal{O}_E^n / N\mathcal{O}_E^n| = N^{[E:\mathbf{Q}]n}$. The group A is a quotient of $\mathcal{L}/M\mathcal{L}$, so $|A|$ divides M^s. Taking valuations in the definition of H and using $\operatorname{ord}_\ell N = \operatorname{ord}_\ell M = \operatorname{ord}_\ell M_0$ gives the formula. □

Proof of 36.97

Proof. The restriction to $L^\times$ of the valuation of O has image $a_0\mathbf{Z}$ for a positive integer a_0, since it is nonzero on ℓ. Dividing by a_0 gives ord_w, with valuation ring $O \cap L$. In particular $e_w = \operatorname{ord}_w(\ell) \le e$.

By the Chinese remainder theorem choose $z \in \mathcal{O}_L$ congruent to 1 at w and to 0 modulo $(w')^b$ at each of the other primes above v. Then $\operatorname{Tr}_{L/K}(czy) \in \mathcal{O}_{K,v}$ for every $c \in \mathcal{O}_L$. Localize at v and complete. The finite algebra of integers becomes $\prod_{w'|v} \mathcal{O}_{L_{w'}}$, and its trace is the sum of the local traces. Testing the factor supported at w shows that zy belongs to the inverse different of L_w/K_v, the trace dual of $\mathcal{O}_{L_w}$. Since z is a unit there, y belongs to the same inverse different.

For completeness, the exponent of the absolute different of $L_w/\mathbf{Q}_\ell$ is less than e_w^2. Let U be its maximal unramified subextension. Its residue field is the residue field of L_w: lift a separable generator of that finite field by Hensel's lemma. Thus $[L_w : U] = e_w$. Choose $s \in \mathcal{O}_U$ whose residue has nonzero trace to $\mathbf{F}_\ell$, which is possible by separability. The trace of s to $\mathbf{Z}_\ell$ from U is a unit, as one sees by reducing the multiplication matrix in an integral basis. If the different exponent were at least e_w^2, the absolute inverse different would contain ℓ^{-e_w}. Its pairing with s would be integral. That pairing is

$$\ell^{-e_w} e_w \operatorname{Tr}_{U/\mathbf{Q}_\ell}(s),$$

whose valuation is negative since $\operatorname{ord}_\ell(e_w) < e_w$. This proves the bound. The relative different divides the absolute different by transitivity. Since $\operatorname{ord}_w(\ell^{e^2}) = e_w e^2$, multiplication by ℓ^{e^2} clears the relative inverse different. In the unramified case the trace pairing on the integers is perfect: its reduction is the nondegenerate residue-field trace pairing. The relative different is then the unit ideal. □

Proof of 36.98

Proof. Pass to a finite extension over which the N^2-torsion is rational. It suffices to prove the result there, since extension multiplies the valuation on the original field by a positive integer. Denote the abelian scheme by B, its rigidified principal theta line by L, and the theta generator in the frame by s. The other entries of the frame are integral bases of the prescribed eigenspaces of $\operatorname{Lie} B$.

We now give the sum to be estimated. Its first index is $q = (q_b) \in A^n$. Choose the designated N-division point of the marked point for q_b, in the primary torsion belonging to the marking ideal. The second index runs through the tuples $z = (z_b) \in B[N^2]^n$ cut out by Λ. Explicitly, for every E-matrix H with $H\Lambda \subseteq \mathcal{O}_E^n$ and NH integral, the tuple satisfies $(NH)z = 0$. For column b, pull s back by $[N]$, identify $[N]^*L$ with L^{N^2} by the rigidification, and make the symmetric theta-group translations first by z_b, then by the chosen division point. Denote the resulting section by F_{qzb}. Both points are killed by N^2, since the marked point is killed by $M \mid N$. Write $P = \sum_\alpha P_\alpha X^\alpha$ in the framed tangent variables, and let α_b be the exponents in column b. If $\nabla^{\alpha_b} F(0)$ denotes the corresponding covariant derivative in the theta frame at the identity, the short value is

$$v = \frac{1}{I^r |A|^n N^D} \sum_{q \in A^n} w(q) \sum_z \sum_\alpha P_\alpha \prod_{b=1}^{n} \nabla^{\alpha_b} F_{qzb}(0).$$

For $P = 0$ take $v = 0$. For $P \neq 0$, homogeneity gives precisely the exponent D in this expression.
Lean: `Formalism.L2.Pj.ShortValAt` (`Formalism.LocalToGlobal.ProjectiveLattices.ThShort`, lines 244–261)

Each F_{qzb} has an integral germ at the identity. Indeed, the torsion points extend by properness. The rigidified multiplication isomorphism extends by the cubical law. To see integrality of a symmetric translation, compare the translated line with the original one on the abelian scheme. Their generic isomorphism can differ from an integral isomorphism only by a multiple of the special fibre. The special fibre is geometrically integral, and translation preserves its order. A power of the symmetric lift is the identity up to a root of unity. Adding the order of the isomorphism under iteration therefore gives zero, so its order is zero. This argument applies in residue characteristic dividing N. Restricting the two translated sections to the formal completion gives the asserted integral germs.

Lean: `GlobalTheory.ReachRegular.regular_of_reach` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.ReachedSectionRegularOnIntegralModel`, lines 71–100)
Lean: `GlobalTheory.GoodThetaBound.reach_of_isUUOver` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ShortThetaValueBoundedGoodReduction`, lines 53–60)

We give the differential calculation since its denominator must be independent of N. Use integral formal parameters for B and a trivialization of its formal biextension. Let $F(X,Y)$ be the formal addition law. In characteristic zero a logarithmic gauge is a series $g(X,Y)$, zero on the axes, whose two additivity defects are the logarithms of the units giving biextension multiplication. The symmetry and adjoint-action identities are the ones for the given polarization. Put $\beta_{ij} = \partial_{X_i}\partial_{Y_j} g(0,0)$, and write $A(a)$ for the tangent matrix of $a \in \mathcal{O}_E$. Taking the mixed coefficient in the symmetry and adjoint identities gives, entrywise,

$$\beta^t - \beta \in V, \qquad A(a)^t\beta - \beta A(\bar{a}) \in V.$$

The correction terms are integral because the logarithmic derivative of a formal unit u is $u^{-1}du$, and the units in these identities are one on the axes.

Take an integral basis x_i of $\mathcal{O}_E$ and multiply its trace-dual basis by Δ to obtain integral y_i. In the separability tensor one has $\sum_i x_i y_i = \Delta$ and $\sum_i ax_i \otimes y_i = \sum_i x_i \otimes ay_i$. Put

$$B_0 = \sum_i A(x_i)^t \beta A(\bar{y}_i), \qquad B_1 = (B_0 + B_0^t)/(2\Delta).$$

The tensor identity gives $A(a)^t B_1 = B_1 A(\bar{a})$. Moreover

$$\Delta\beta - B_0 = \sum_i A(x_i)^t \big(A(y_i)^t\beta - \beta A(\bar{y}_i)\big).$$

Consequently $2\Delta(\beta - B_1)$ is integral. Subtract from g the bilinear form with matrix B_1 in formal logarithm coordinates. It is symmetric, additive in both variables, and balanced for conjugation. Hence the new gauge g' satisfies the same gauge identities. In harmonic evaluation this change contributes contractions between conjugate tangent variables. Each contraction annihilates P by harmonicity. Equivalently, on expanding the exponential of the added quadratic form, every term of positive degree pairs with an iterate of a contraction on P.

Lean: `GlobalTheory.GaugeBoundedDenom.exists_symm_balanced` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsCanonGaugeNabValBoundedDenom`, lines 20–70)
Lean: `GlobalTheory.GoodThetaBound.hv_bound` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ShortThetaValueBoundedGoodReduction`, lines 98–157)

Let $w_0 \in \mathcal{O}_E$ be the polarization element with $w_0 + \bar{w}_0 = 1$, put $q(X) = g'(X, [w_0]X)$, and denote the invariant derivative in an integral tangent direction u by ∂_u. We have $q(0) = \partial_u q(0) = 0$. Set $\delta = 2\Delta$. Then $\delta\,\partial_u\partial_v q$ is an integral series. To compute it modulo integral series, expand $q(F(X,Y))$ by additivity in the two arguments of g'. The four terms are $q(X), q(Y), g'(X,[w_0]Y), g'(Y,[w_0]X)$, up to the logarithms of integral units. Differentiate in Y at zero and then invariantly in X. Derivatives of the logarithmic defects are integral. Applying additivity once more to each mixed term leaves, modulo integral series, the constant

$$(\beta - B_1)(u, A(w_0)v) + (\beta - B_1)(v, A(w_0)u).$$

Multiplication by δ clears it. This computation uses integral invariant vector fields, obtained by differentiating the integral addition law.

The subring generated by the integral series and the $\delta\partial_v q$ has integral constant terms and is stable under $\delta\partial_u$. For a section of L^k with integral germ f, its derivatives in this gauge are the derivatives at zero of $f\exp(kq)$. After removing $\exp(kq)$, one more scaled derivative takes h to

$$\delta\partial_u h + kh\,\delta\partial_u q.$$

Induction gives integrality at zero after $|\alpha_b|$ such factors. Multiplying over the columns costs $\delta^{\sum_b |\alpha_b|} = \delta^D$. The coefficients $w(q)P_\alpha$ are integral by hypothesis. Substitution in the formula for v proves the assertion.
Lean: `GlobalTheory.GaugeBoundedDenom.pow_mul_const_dmonOf_mem` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsCanonGaugeNabValBoundedDenom`, lines 76–150)

□

Proof of 36.99

Proof. Replace f by af. Write V for the test ring, π for its uniformizer, and K_V for its fraction field. Use the proper semiabelian model at λ. The prime-to-ℓ marking has level at least three, as required for this model. The valuative criterion extends the underlying moduli point. If the extension lies in the abelian locus, the hypothesis applies. In definite signature the torus part vanishes: the CM action on the Lie algebra at the generic point has just one of the two eigenvalues of w_0, hence acts there by a scalar. The scalar is integral, and the same equality holds on the integral Lie module. Suppose the special-fibre torus has nonzero character lattice X (a nonzero finitely generated free $\mathbf{Z}$-module), on which w_0 induces $u \in \mathrm{End}_{\mathbf{Z}}(X)$ satisfying $u^2 - u + m = 0$ with $4m = 1 + d$ (d prime). Because w_0 acts on the integral Lie module by a scalar $z_0 \in V$, the endomorphism u acts on $\mathrm{Lie}(T_k) \simeq \mathrm{Hom}_{\mathbf{Z}}(X, k)$ and hence on $X \otimes_{\mathbf{Z}} k$ (where k is the residue field of characteristic ℓ) by the scalar $c = \bar{z}_0 \in k$. Since $X \neq 0$ is free over $\mathbf{Z}$ and u is defined over $\mathbf{Z}$, c lies in $\mathbf{F}_\ell \subseteq k$; choosing a lift $c_0 \in \mathbf{Z}$ gives $(u - c_0)(X) \subseteq \ell X$, so $v = (u - c_0)/\ell \in \mathrm{End}_{\mathbf{Z}}(X)$ is integral over $\mathbf{Z}$. Any complex eigenvalue y of u on $X \otimes_{\mathbf{Z}} \mathbf{C}$ satisfies $y^2 - y + m = 0$, so $y = (1 \pm i\sqrt{d})/2$; then $z = (y - c_0)/\ell$ is an algebraic integer with $(z - \bar{z})^2 = -d/\ell^2 \in \mathbf{Q}$, forcing $\ell^2 \mid d$, contrary to primality of d. This excludes a torus part.

In the remaining case pass to a finite valued extension to split the semiabelian degeneration, rationalize the finite marking and make the orders of its quadratic periods divisible by twice the full level. These operations are possible because the character lattice is finitely generated, the required division points are algebraic, and ramification multiplies all period orders by its index. Continue to denote the new ring by V. Let X be the torus character lattice, $Q(\alpha)$ the order of the rigidified period at α, and $B(\alpha, \beta) = Q(\alpha + \beta) - Q(\alpha) - Q(\beta)$. This is the symmetric conjugate-balanced monodromy pairing. In particular $B(\alpha, w_0\alpha) = Q(\alpha)$. In the degeneration data remove the uniformizer powers from the periods and reinsert them with a variable t in place of π. Thus the factors for the quadratic and bilinear periods are $t^{Q(\alpha)}$ and $t^{B(\alpha,\beta)}$, respectively, in the corresponding rigidifications. Additivity of B and the displayed polarization identity are precisely the multiplication identities for the new periods. Divisibility by twice the full level gives integral exponents for the roots in the marking. This is the one-parameter toroidal chart of the split degeneration. It specializes at $t = \pi$ to the given point. Where t is a unit its model lies in the abelian locus. The generic moduli map and its frame are defined over $V[[t]][1/(\pi t)]$. For the frame take a generator on the abelian part and constant bases in the tangent eigenspaces of the extension. One can arrange that their specialization and the given tangent bases differ by a triangular matrix: intersect each subspace of the given flag with the free integral tangent module and choose a basis successively in the resulting saturated submodules. The two integral generators differ by a unit. The diagonal entries of a triangular change between two integral bases are units as well.
Lean: `GlobalTheory.ArcOfPoint.exists_arcIn_of_unif` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsArcDataOfBadReductionFiniteExtension`, lines 67–236)
Lean: `GlobalTheory.GoodTransfer.inV_of_arc` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CuspFormBoundGoodPointsExtendsAllPoints`, lines 349–456)

Evaluation in this frame gives $h(t)/(\pi t)^{k_0}$, with $h \in V[[t]]$ and $k_0 \geq 0$, by regularity of the section on the indicated localization. We show separately that $t^{k_0} \mid h$ and that $\pi^{k_0} \mid h/t^{k_0}$.

Embed K_V into $\mathbf{C}$ over the designated coefficient embedding. Such an abstract embedding exists since K_V is a finite extension of a local number field and its transcendence degree over the coefficient field is at most that of $\mathbf{C}$. Apply it coefficientwise to the formal arc. In the complex uniformization put $t = \exp(2\pi i z)$, with the period of z rescaled if the marking requires it. The Laurent expansion of the evaluated section is its expansion in this parameter: the torus characters become the corresponding exponential monomials, the marked roots have the chosen exponents, and the constant tangent frame is their frame under the exponential map. The theta series of the period lattice converges normally for sufficiently large $\mathrm{Im}\, z$, by positivity of Q on its nonzero lattice. Thus this identification uses the holomorphic arc attached to the degeneration; it does not require the abstract embedding to preserve a valued-field limit.

Here are the bounds on this arc. The imaginary part $Y(z)$ of the normalized period matrix is affine in $\operatorname{Im} z$, with positive semidefinite slope. Indeed the period is affine in z, and positivity for every $\operatorname{Re} z$ and every sufficiently large $\operatorname{Im} z$, tested on each real vector, kills the $\operatorname{Re} z$ coefficient and makes the other slope nonnegative. Fix z_0 in this range. Then $Y(z) \geq Y(z_0) > 0$ for $\operatorname{Im} z \geq \operatorname{Im} z_0$. The squared norm of a constant tangent vector in period coordinates is given by $Y(z)^{-1}$ on its real and imaginary parts, so it is bounded by the value for $Y(z_0)^{-1}$. In an orthonormal flag frame this bounds each diagonal tangent multiplier, since its squared absolute value times the fixed positive norm of the basis vector is a summand in that squared norm. For the theta generator the squared mean norm is

$$\frac{|\kappa|^2}{\sqrt{2^g \det Y(z)}},$$

where $g = \dim B$ and κ is the constant in the theta trivialization of the arc. Integrating the squared theta series over a real period parallelepiped kills the terms with unequal Fourier indices. The remaining translates of the second half of the parallelepiped tile $\mathbf{R}^g$, giving the Gaussian integral displayed above. This bounds the generator multiplier as well. The cusp section has bounded norm on unit frames. Its value in the arc frame is obtained by multiplying by the indicated diagonal factors to the nonnegative exponents of its dominant weight. It is therefore bounded as $t \to 0$. Cauchy's coefficient formula on circles tending to zero kills each negative Laurent coefficient. Injectivity of the coefficient embedding gives $h = t^{k_0} h_1$ in $V[[t]]$.

Lean: `GlobalTheory.ArcLieBound.norm_diag_le` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.NormDiagTangentUnitFrameHolomorphicArcLe`, lines 293–308)
Lean: `GlobalTheory.ArcGenBound.gen_ratio_bounded` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ArcGeneratorRatioBoundedUnitFrame`, lines 74–100)
Lean: `GlobalTheory.GoodTransfer.coeff_neg_eq_zero` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CuspFormBoundGoodPointsExtendsAllPoints`, lines 198–230)

For the second divisibility take the completion W of $V[[t]]_{(\pi)}$. It is a discrete valuation ring with uniformizer π, and t is a unit. An element of $V[[t]]$ has order there equal to the minimum of the orders of its coefficients. The frame over W lies in the abelian locus. Its residue field can be infinite, so we explain why the good-point hypothesis applies. Use an affine chart of the framed abelian locus on which the integral frame coordinates are integral. Such charts are obtained by adjoining coordinates of the Lie flag and the generator of the invertible direct image of the theta line. At a valued point one takes a unit flag minor and a generating entry of the line. Refining on the abelian locus ensures that all integral points of the chart used for the test have good reduction.

On this chart, a failure of integrality persists at a test ring with finite residue field. To see this, take the closure of the graph of the evaluated rational function in the chart times $\mathbb{P}^1$, over V. A valued point with negative order specializes to infinity. In the closure of its center take a closed point in the same chart at infinity. Its residue field is finite. Work on the horizontal irreducible component through which the valued point factors. In the local domain at the closed point choose a nonzero element b cutting out the complement of the required generic open, including the denominator of the chosen affine chart at infinity. Complete a nonzero multiple of πb to a system of parameters. A minimal prime over the other parameters avoids πb: otherwise the parameter ideal would have height at most the number of those other parameters. The resulting one-dimensional quotient has a horizontal branch. Normalizing and completing that branch gives a complete discrete valuation ring with finite residue field, generic point in the required open and closed point at infinity. The evaluated function has negative order there, contrary to the hypothesis. This proves the good-point test for the valued point of the chart over W.

Lean: `GlobalTheory.GoodTransfer.inV_of_extIn` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CuspFormBoundGoodPointsExtendsAllPoints`, lines 120–150)
Lean: `GlobalTheory.ChartIntegral.integral_of_integral_at_algebraic_points` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.IntegralOfIntegralAtAlgebraicPointsSmooth`, lines 37–131)

It follows that $h_1/\pi^{k_0} \in W$. Every coefficient of h_1 is divisible by π^{k_0}. The original value at $t = \pi$ is now the evaluation of an element of $V[[t]]$, hence belongs to V. The triangular unit change does not alter integrality. Finally divide the order by the ramification index of the initial extension.

Lean: `GlobalTheory.GaussPoint.exists_gaussPoint` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsGaussPoint`, lines 64–108)

□

Proof of 36.100

Proof. Write $M_0 = g_0 g_1$ and $M = M_0 d'$ for the base and cover levels. The uniformization identifies the source with its standard self-dual lattice. The target rank r is odd, the source rank n is even, and the differential weight is dominant with positive final row degree in every nonempty block in the CM type. These are the signature and weight conditions on the standard layer. Write D for the total row degree.

We first fix the scalar. Use the coefficient basis of products of minors over the base field. On the pinned lowest-weight tensor the polynomial map in the Fock model is κJ in the standard letters, where J sends this basis to its prescribed harmonic polynomials. Indeed the normalized extremal vector spans its weight line. The basis change from the prescribed source frame to the standard one is unitary. The two isometric identifications of that line differ by a scalar of modulus one, giving $|\kappa| = 1$ on the whole coefficient map by linearity. The matrix of J on the minor basis is over the base field. A nonsingular coefficient minor expressing linear independence of the basis shows that Jv is over any embedded field containing the base field and the coefficients of v.

Let ϕ be the comparison of the vacuum with the geometric generator in the base frame, including the action of that unitary basis change on the vacuum line. The latter action preserves the norm; the base pin has unit modulus, so $|\phi| = 1$. To compare the component constants, evaluate the vacuum identity on a rational period vector where the analytic theta generator is nonzero. Such a vector exists since rational combinations of periods are dense and the generator is a nonzero holomorphic function. Take the same vector in each source column, using the inverse source basis matrix to prescribe its columns. The product of theta values is nonzero. The positive Gaussian and its reciprocal cancel in the two readings, so the ratio of the component constants equals ϕ on every component met by the adelic frame.

The same ϕ works on a cover. The vacuum identity pulls back with the same rigidified generator and the same periods. After division by ϕ its component constants agree on pullbacks of base sections. To see that this determines the cover constants, one may use weight zero in the tangent variables and the prescribed even theta degree. On a geometric fibre B in any component, with principal theta line L and generator s_0, let $\pi : B \to B' = B/A$ be the quotient by the marked maximal isotropic subgroup $A \subset B[M_0]$ and $\hat{\pi} : B' \to B$ the complementary isogeny with $\hat{\pi} \circ \pi = [M_0]$. Then $\hat{\pi}^* L$ is algebraically equivalent to the M_0-th power of the descended principal polarization on B', hence base-point free since $M_0 = g_0 g_1 \geq 2$; the $B'[M_0]$-translates of $\hat{\pi}^* s_0$ span $H^0(B', \hat{\pi}^* L)$ by irreducibility of the theta-group action, so some $a \in \hat{\pi}(B'[M_0]) = A$ avoids the zero divisor of s_0. Evaluating the n-th power of the translated theta generator at such a marked point $a \in A$, as in Lemma 36.98 with $P = 1$, $N = M_0$, and $\Lambda = \mathcal{O}_E^n$, defines a base-level section of this weight over a finite Galois extension of the base coefficient field; taking Galois traces against a dual basis gives base-level sections over that field, and at least one coordinate has nonzero evaluation since extending scalars recovers the original section. Equivalently these are the level theta evaluations in the pinned section space. A real change of orthonormal frame moves the point of nonzero evaluation without changing its finite component. Apply both pin identities to the pullback there. The common nonzero evaluation cancels. This uses the pin for every dominant weight of the fixed theta degree. We take $u = (\kappa\phi)^{-1}$.

Lean: `GlobalTheory.SerIntLetters.io_eq_phase_smul_harmLetters` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.HarmonicMapExtremalVectorEqPhaseSmulLetters`, lines 22–81)
Lean: `GlobalTheory.LevelSeries.PinBase.baseCc_of_framePin` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CoreConstantsEqVacuumIdentityUpToPhase`, lines 168–255)
Lean: `GlobalTheory.LevelSeries.cc_unique_of_pinned` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CoverConstantsUniqueOnDictionaryComponents`, lines 22–95)
Lean: `GlobalTheory.LevelSeries.covCc_of_pinned` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.VacuumIdentityConstantsCoverOfDominantPins`, lines 19–33)

Fix ℓ, s, L_0. Apply the first assertion of Lemma 36.96 to J and L_0; denote its exponent by c_1. We may use $\ell^{c_1} Jv$ as an integral polynomial at each chosen place. Denote its valuation ring in the designated embedding by O. The coefficient field contains the M-th roots of unity, by the marking, and they are units. Lemma 36.95 gives the cost $K \operatorname{ord}_\ell M_0$ for writing the finite test as an integral combination of vacuum translates.

The half-lattice in this expansion can differ from the half specified by the marking ideal. We record the cost of changing it. At a set T of wrongly oriented split primes dividing m, use the central source element which scales the two halves by reciprocal powers of m. Its determinant idèle on the selected factors is m^{-n}.

The scalar for rewriting the translates in the other half is

$$\chi(x)^{-1}\left(\prod_{P\in T}(\mathrm{N}P)^{\,\mathrm{ord}_P(m)nr}\right)^{-1/2}, \qquad x_P=m^{-n}\ (P\in T),\quad x_P=1\ (P\notin T),$$

where χ is the reference splitting character on these idèles. Indeed, in each of the nr lattice coordinates one expands the characteristic function of a coset by its finite additive characters. The index in the display counts these characters. Unitary Fourier normalization gives the inverse square root of the index, and the determinant action gives the first factor. Translating the characteristic function multiplies the Fourier coefficients by M-th roots of unity.

Make the change first on T over M_0, then over d'. The two supports are disjoint. For $m=M_0$ there are finitely many choices of T, hence finitely many scalars and central translates z_T. These scalars belong to every tested coefficient field: choose a principal idèle with the specified local congruences at M_0, use the character product formula, and combine the half norms at the remaining unramified places with the square root of the displayed index. The normalized finite values belong to the field by hypothesis, the remaining infinity factors are products of its CM embeddings, and r is odd. Clear these finitely many algebraic numbers as in Lemma 36.96, at cost c_f. For $m=d'$ the character is unramified on T. Put $r=2s_0+1$ and $a_P=n\,\mathrm{ord}_P(d')$. The factor at P is exactly

$$\big(\chi(P)\sqrt{\mathrm{N}P}\big)^{a_P}(\mathrm{N}P)^{-(s_0+1)a_P}.$$

It is integral at ℓ. For the unit assertion about the character, raise P to a principal power with generator α. Since $P\nmid\ell$, all conjugates of α are units above ℓ. The character product formula expresses a positive power of $\chi(P)$ as a product of roots of unity from local units and powers of the normalized archimedean conjugates of α. Those conjugates and their absolute values are units at every extension of the valuation, since the squares of the absolute values are products of conjugates. The same holds for $\sqrt{\mathrm{N}P}$. The normalized value lies in the tested field by hypothesis. The auxiliary central translate z' equals one above ℓ.

Lean: `GlobalTheory.LevelSeries.exists_flip_theta` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsCentralFlipThetaTranslateHalfLevel`, lines 38–112)

Lean: `GlobalTheory.LevelSeries.flipScal_mem_valRing_of_unit` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.FlipScalarMemCoefficientRingAuxiliaryLevel`, lines 124–228)

We describe the finite-point change because bounding a matrix chosen separately at every level would give the wrong quantifiers. Use the open subgroup fixing the standard lattice and all the translates occurring at level M. In the double class of sz_Tz' choose a representative h supported at split primes away from $2\ell M$. One construction is to approximate the prescribed finite matrix by a rational unitary matrix at the primes of the level and of its denominators. Weak approximation here follows by using the Cayley coordinates on the skew-Hermitian matrices, translating the open chart if necessary. To remove unwanted nonsplit denominators, write the rational unitary matrix as a product of unitary reflections in nonisotropic vectors. Such a factor acts by u, $u\bar u=1$, on its line and by one on the orthogonal complement. One obtains the factorization by matching one vector of an orthogonal basis at a time: a reflection in the difference of two vectors of equal nonzero norm matches them, with an intermediate scalar multiple if their difference is isotropic. Write $u=z/\bar z$. For each factor approximate z by weak approximation so as to retain the required congruences, and make $u-1$ divisible by the denominator of the line projection at the other nonsplit primes in that denominator. At all remaining nonsplit primes a norm-one number is a unit and the projection is integral. Congruences may be imposed to greater precision by the finite denominators of the other factors in the product. Thus the remaining nonintegral factors away from the approximation set have split support. Put the entire remaining matrix in h at those split primes and put its integral, level-congruent part in the open subgroup. This gives

$$sz_Tz'=\gamma\,h\,a_\infty k$$

with γ rational, a_∞ the inverse infinite component of γ, and k in the chosen stabilizer. At the primes over ℓ, the factors h,z' equal one and k,k^{-1} are integral. Thus one exponent d_0 clears both γ,γ^{-1} there: take the maximum of the clearing exponents for $sz_T,(sz_T)^{-1}$ over the finite list of T. Rational left translation reindexes the theta sum. The action of a_∞ substitutes entries of γ and γ^{-1} in its degree-D polynomial and multiplies the vacuum by $\prod_\tau\det(a_{\infty,\tau})^{k_\tau}$, where k_τ is its fixed determinant weight. Consequently

$$c_a=d_0\big(D+n\sum_\tau|k_\tau|\big)$$

clears the transformed polynomial. This follows by expanding each determinant, using the inverse matrix for a negative power. The transformed polynomial is harmonic because the conjugate sets of variables undergo mutually contragredient substitutions.

Lean: `GlobalTheory.SplitRepsOpen.exists_rep_splitOffBad` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.TendstoSmulZeroOfCompact`, lines 86–91)
Lean: `GlobalTheory.MoveDenom.denomAt_of_move` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.RationalPartDenominatorBoundedByPrimePower`, lines 29–73)
Lean: `GlobalTheory.LevelSeries.ArchAct.real_act_harm` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.RealRationalPointActsHarmonicSubstitution`, lines 50–98)

Take for Λ the conjugate of the lattice at h. It is self-dual by unitarity and standard above $2\ell M$. Clearing the finitely many elementary divisors of this lattice and its inverse away from $2\ell M$ gives an odd u_0 prime to ℓM for which $N = Mu_0$ satisfies Lemma 36.98. Every translate in the prescribed half modulo the vacuum lattice is represented by a marked point. To write one such translate as the mean over the $|A|^n$ marking tuples, give its chosen representative coefficient $|A|^n$ and give the others coefficient zero. The difference between the original representative and the marked one is a lattice vector in the same isotropic half; it fixes the vacuum and contributes trivial commutator. By addition we obtain an integral weight function for the mean.

Expand the products of the finitely many weights and polynomial coefficients in a basis of their span over the coefficient valuation ring. This span is finite and torsion free over a discrete valuation ring, hence free. Its basis elements belong to O, and the coordinate polynomials satisfy the same harmonic and row-weight equations because these equations are linear over the coefficient field. For each coordinate use the geometric sum in Lemma 36.98, with the weight indicating its marking tuple. It defines a section: after adjoining the finite torsion data the formula is natural under base change, and changes of the torsion enumeration permute the sum. Changing the division point within the designated primary part permutes the lattice translates with their theta-group actions. Thus the two pullbacks of the sum agree on the fibre product of the marking cover. Faithfully flat descent gives its value over each base test algebra. A triangular change of tangent basis acts by the differential weight and scaling the theta generator acts by its n-th power. These are the two defining laws of a section. At each good complex embedding the differential evaluation is a sum of theta derivatives with finite characteristics. In a nondefinite component its zero Fourier coefficient vanishes by the calculation of Lemma 35.40: the number of columns is even, and positive last row degree supplies a full minor at an embedding in the CM type. The calculation there works for each characteristic individually and hence after a change of marking level. A definite component has no nonzero rational isotropic line. This proves cuspidality.

Lean: `GlobalTheory.ThetaSumDescent.exists_cusp_thetaSum` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsCuspCoefficient`, lines 15–34)
Lean: `GlobalTheory.LevelSeries.SerId.exists_forms_of_transSum` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsFormsThetaShortSeriesIdentity`, lines 231–283)

The scalar in the reading of this sum can be computed before taking valuations. Put $J_0 = [\Lambda : \Lambda \cap \mathcal{O}_E^n]$, and let y be the determinant idèle of h on its selected split factors. On the geometric side the unweighted count of lattice tuples is $[\Lambda : N\mathcal{O}_E^n]^r = I^r J_0^r$; the short-section mean divides by $I^r|A|^n$. On the Weil side the self-dual lattice change has factor $\chi(y)^{-1}\sqrt{J_0^r}$, with the same marking mean. Applying a homogeneous harmonic differential to the Gaussian gives the common framed Lie factor and the sign $(-1)^D$. The ratio of the two readings, after removal of the pinned scalars, is therefore $(-1)^D\chi(y)^{-1}/\sqrt{J_0^r}$. To see it is in O, take elementary divisors at each pair of conjugate split places. If the exponents on one side are b_i, those on the other side are $-b_i$. The exponent in J_0 over this pair is $\sum_i |b_i|$, congruent to $\sum_i b_i$ modulo two. Thus $J_0 = (\mathrm{N}\mathfrak{a})t_0^2$, where the determinant ideal is represented by $\mathfrak{a}/\bar{\mathfrak{a}}$ with relatively prime numerator and denominator. All factors are prime to ℓ. Pairing the square roots at these primes with the normalized character values gives products of the units proved above and rational powers with integral exponents of primes different from ℓ.

Lean: `GlobalTheory.LevelSeries.SerId.thetaSum_value_emb` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsFormsThetaShortSeriesIdentity`, lines 34–171)
Lean: `GlobalTheory.LevelSeries.clsScal_mem_valRing` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SplitSupportedClassScalarMemValuationRing`, lines 268–393)

By Lemma 36.98 the geometric sum multiplied by $I^r|A|^n(2\Delta N)^D$ is integral at good frames. Lemma 36.99 extends this to all integral test frames. Its prime-to-ℓ level condition holds since at least one of the distinct odd primes g_0, g_1 remains after removing the ℓ-part of M. Apply the second assertion of Lemma 36.96 with this N, Λ. Its exponent c_3 suffices at ℓ, since the complementary factor of the denominator is a unit there.

We have obtained the single exponent

$$c = c_1 + K \operatorname{ord}_\ell(M_0) + c_f + c_a + c_3.$$

Every term was chosen before the cover and the place on it.

The same finite expansion over the embedded coefficient field gives a rational section; here the valuation ring in the coefficient expansion can be replaced by the whole field. At a set S of places use this rational section and the local expansions just obtained. They have the same section at every place: the readings agree in all adelic frames, orthonormal frames meet every flag up to triangular change, and evaluation on geometric points detects a section on a reduced test algebra. Its values are therefore integral simultaneously at S. This argument includes $S = \varnothing$. For a permitted determinant twist the multiplier on a component is a root of unity in the cover field. Indeed it factors through the finite norm-one ray quotient by rational elements and level congruences. The full ray field in the hypotheses defines the marked components over the cover field, so multiplication component by component preserves the rational section and every tested integral value.

For the rationality assertion the good cover has odd squarefree split auxiliary level, the indicated pins, the full ray field and the values of the two reference characters in their respective weights 0 and 1/2. Their quotient gives the normalized values used in the construction. Choose a prime larger than d', use the finite expansion over the whole coefficient field at this prime, and divide back by its nonzero clearing power in that field. The determinant character of an allowed finite-order splitting twist δ on a unitary group U is $d_{\delta,U}(g) = \delta(z)$ for $z/\bar{z} = \det g$. This is independent of z since δ is trivial on fixed-field idèles. Changing both splittings multiplies the series by $d_{\delta,V}(s)^{-1} d_{\delta,W}(g)$. The first factor is in the cover field by the value condition on δ; the second is the component multiplier just considered. This proves the assertion for both rational twist laws.

Lean: `GlobalTheory.SerStd.untwRatS` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.UntwistedThetaRationalOnGoodCovers`, lines 61–172)
Lean: `GlobalTheory.SerStdTwist.seriesRatS_of_untwRatS` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ThetaSeriesRationalOnGoodCoversOfUntwisted`, lines 18–32)
Lean: `GlobalTheory.TwistInt.mul_twist_mem_intForms` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TwistDeterminantMulIntegralFormsRayClassCover`, lines 26–83)

□

Proof of 36.101

Proof. After coefficient extension to L, forgetting the additional marking is finite étale on the generic fibre. Two lifts of a framed geometric point differ by the change of marking from the base compact subgroup. The determinant factor in the base reading is invariant under that subgroup, and the pin identifies readings of pullbacks. Thus invariance makes the two pullbacks of the section equal on the fibre product, first on geometric frames and then on reduced test algebras. It descends over L at base level. The cusp condition descends too: the Fourier expansion at a lifted isotropic line is the original expansion with its period group restricted to a finite-index subgroup, so the constant term of a pullback is the pullback of the constant term. Scalar extension of the space of sections commutes with this descent, by the equalizer for sections on an affine cover and flatness over K. The same conclusion for the cusp subspace follows by imposing the vanishing constant terms at all conjugate embeddings. We therefore have an L-linear combination of base cusp sections.

Lean: `GlobalTheory.CoverDescRat.cover_desc_rat` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CoverFormRationalDescentToBase`, lines 55–65)

Fix $w \in S$ and put $v = w|_K$. Choose an $\mathcal{O}_{K,v}$-basis $f_1, \dots, f_m$ of the base integral cusp lattice. This is the lattice defined by evaluation on all algebraic integral frames. To recall the lattice argument, finite dimensionality and separating evaluations bound it inside a finite free module: choose finitely many evaluations of full rank over an extension and invert a nonzero minor. Each cusp section has an integral multiple, by clearing denominators on finitely many integral flag charts and extending across the boundary as in Lemma 36.99. Thus the lattice spans the cusp space, and a finitely generated torsion-free module over the local discrete valuation ring is free. Write $F = \sum_j y_j f_j$, $y_j \in L$, using pullbacks in this equality.

At each $w' \mid v$ different from w, choose b large enough that $(w')^b y_j$ is integral there for all j. For $a \in \mathcal{O}_L$

in all these powers we claim that

$$\sum_j \mathrm{Tr}_{L/K}(ay_j)f_j$$

is in the base integral lattice. Test a base integral frame x over a complete discrete valuation ring over v with finite residue field. In a finite normal extension of its fraction field split all the embeddings of L over K. For an embedding inducing w, lift x after a further finite extension to the marking cover. Such a lift exists because the generic marking cover is surjective and finite, so its fibre has a point over a finite extension. Pull back the tangent basis and the generator through the isomorphism of the underlying polarized objects. These are integral: the semiabelian model, its Lie lattice and its rigidified generator are carried along the same base change, and forgetting the marking recovers the original frame. Evaluation of the cover section at this frame is the image of $\sum_j y_j f_j(x)$, hence integral. Multiplication by a preserves integrality. For an embedding inducing $w' \neq w$, each individual product $ay_j f_j(x)$ is integral by the choice of b. Summing over the embeddings gives the asserted integral value of the trace section. Positive ramification indices give integrality back on the original test ring.
Lean: `GlobalTheory.CoverPoint.exists_ext_lift_to_cover` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.ExistsFiniteExtensionIntegralFramedPointLiftCove`, lines 78–123)
Lean: `GlobalTheory.CoverLiftVal.le_one_at_lift` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.BaseSectionsValuesIntegralAtCoverLift`, lines 31–74)
Lean: `GlobalTheory.TraceIdem.trace_comb_integral` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TraceCombinationCoverFormIntegralOverPrime`, lines 37–56)

Coordinates in the chosen integral basis give $\mathrm{Tr}_{L/K}(ay_j) \in \mathcal{O}_{K,v}$ for every j. These are exactly the tests in Lemma 36.97. In the unramified case it gives $y_j \in \mathcal{O}_{L,w}$, taking $O = \mathcal{O}_{L,w}$ in that lemma. In the fixed-ring case it gives $\ell^{e^2} y_j \in \mathcal{O}_{L,w}$.

Finally S is finite. In the decomposition for w multiply each base form by a base integer which is a unit at v and sufficiently divisible at the other places below S, and divide its coefficient by that integer. The resulting base forms are integral below all of S. Denote the finite set of new coefficients in this decomposition by Y_w. Choose a common exponent b' clearing the negative orders of the elements of all the Y_w at all the other places. The Chinese remainder theorem gives $d_w \in \mathcal{O}_L$ equal to one modulo $w^{b'+1}$ and to zero modulo $(w')^{b'+1}$ for $w' \in S \setminus \{w\}$. Their sum is a unit at every place in S. Multiply the decomposition for w by $d_w / \sum_{w'} d_{w'}$ and add. The multipliers sum to one and all the resulting coefficients are integral at S. For empty S the rational decomposition suffices.
Lean: `GlobalTheory.DescGlue.glue` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.GlueIntegralDecompositionsOverRationalPrime`, lines 41–103)

□

Proof of 36.102

Proof. Every kernel is a subgroup of $B[p^2]$, so their number is at most $2^{(p^2)^{2g}}$. The Weil pairing is perfect and the stipulated isotropic order is the square root of $|B[p^2]|$. The polarization multiplied by p^2 therefore descends to a principal polarization on B/C: its kernel on the quotient is $C^\perp/C = 0$. The endomorphisms descend by stability. On the marked torsion use $p^{-1}\Psi$, where the inverse of p is taken modulo the marking level. Its Weil multiplier is one there. This gives the unique marked quotient by the moduli property. In characteristic zero the quotient is étale. Its differential is consequently an isomorphism, commutes with $\mathcal{O}_E$, and gives the stated basis in each eigenspace.

We include the line-bundle calculation. Write $\mathcal{P}_L$ for the biextension $m^*L \otimes \mathrm{pr}_1^*L^{-1} \otimes \mathrm{pr}_2^*L^{-1} \otimes L(0)$, where m is addition. The principal line in this moduli problem has class $(1, w_0)^*\mathcal{P}_L$. The Weil-pairing equality gives $(\Psi, \Psi)^*\mathcal{P}_{L'} = \mathcal{P}_L^{p^2}$: these rigidified biextensions represent the polarization homomorphisms, and the homomorphisms agree on the prime-to-p torsion by the pairing and hence on the abelian variety. Restrict along $(1, w_0)$ and use $\Psi w_0 = w_0 \Psi$. This gives $\Psi^* L' \simeq L^{p^2}$. Normalization at the identity makes the isomorphism unique since the invertible functions on a proper geometrically connected abelian variety are constant. The same normalization gives $[p]^*L \simeq L^{p^2}$.
Lean: `GlobalTheory.ThetaTransport.exists_isoGen` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.HeckeIsogenyThetaGeneratorTransport`, lines 178–242)

Let v denote the ratio of the two isomorphisms from the multiplication and quotient pullbacks to L^{p^2}, in local frames. For $a \in C \cap B[p]$, the ratio $v(x+a)/v(x)$ is a constant u_a, since both pullbacks are unchanged

and the ratio is an invertible function on B. The same holds for $w_0 a$. The primitive line identity used above, written on a point x, involves

$$L((1+w_0)x)\,L(0)\,L(x)^{-2}\,L(w_0 x)^{-1}.$$

Use its trivialization on the two targets and take the quotient. The resulting invertible function on B is constant. Comparing it at $x+a$ and x gives $u_a u_{w_0 a} = u_a^2 u_{w_0 a}$, hence $u_a = 1$. Thus each summand depends only on the class of c. Translation by any element of C permutes the summands. The sum descends uniquely, since Ψ is faithfully flat and its fibre product with itself is the disjoint union of the translations by C.
Lean: `GlobalTheory.ThetaTransport.quot_transl` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.HeckeIsogenyThetaGeneratorTransport`, lines 346–417)
Lean: `GlobalTheory.ThetaTransport.exists_trace` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.HeckeIsogenyThetaGeneratorTransport`, lines 619–659)

To prove nonvanishing, translate the pullback sum by $b \in B[p]$. Relative to the unshifted frame, the summand indexed by c has multiplier proportional to a character of b. Indeed the quotient of the two translated trivializations is a trivialization on $[p]^*(t_{pc}^* L \otimes L^{-1})$, up to taking its dual. Translation by b changes it by an invertible constant, and composition gives the character law. If the characters for c, c' agree, the quotient of these trivializations descends through $[p]$. It trivializes $t_{p(c-c')}^* L \otimes L^{-1}$ up to translation. The principal polarization then gives $p(c-c') = 0$. The characters indexed by distinct classes are therefore distinct. If the sum vanished identically, translating by all $b \in B[p]$ and using character orthogonality would force every summand to vanish. Multiplication by p and translation are surjective and $s \neq 0$. This is a contradiction. Since $H^0(B', L')$ has dimension one, the descended section is a generator.
Lean: `GlobalTheory.ThetaTransport.transl_quot` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.HeckeIsogenyThetaGeneratorTransport`, lines 496–554)
Lean: `GlobalTheory.ThetaTransport.eq_zero_of_trace` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.HeckeIsogenyThetaGeneratorTransport`, lines 576–617)

□

Proof of 36.103

Proof. Enlarge the field to define the kernel, its framed quotient, the p^2-torsion of the source and the p^3-torsion of the target. These are finite extensions since the marking, morphisms, and frames are of finite presentation and the points used are algebraic. The isogeny and the complementary isogeny Ψ', with $\Psi'\Psi = [p^2]$ and $\Psi\Psi' = [p^2]$, extend on the semiabelian models after semistable reduction. On the identity components this follows by extending the homomorphisms to the smooth group models; their restrictions to the generic fibre determine their compositions by separatedness. Their tangent maps thus have composites p^2. As $p \in V^\times$, $p^{-1}d\Psi$ identifies the two integral Lie lattices and all their eigenspaces. Use this basis on the target. For good reduction the quotient extends directly as the quotient of the abelian scheme by the finite étale subgroup of its prime-to-residue-characteristic torsion.

Write $s' = as_0$ relative to an integral theta generator on the target. We prove both valuation inequalities. In a split semiabelian uniformization write Q for the quadratic period order of the character lattice. For a section of L^k the integral Fourier bound used here requires its coefficient of character x to belong to the abelian-part integral section lattice with order at least $Q(x)/k$. The framed generator has this bound at $k = 1$: its constant coefficient is integral and the period equation gives the bound at x. A multiplication pullback by $[m]$ moves character x to mx and degree k to $m^2 k$, preserving $Q(x)/k$. A symmetric torsion translation shifts the character by a period label y; in degree k its linear toric multiplier and its rigidifying multiplier supply

$$\frac{B(x,y)}{k} + \frac{Q(y)}{k} = \frac{Q(x+y) - Q(x)}{k}.$$

Indeed the k-th power of the torus evaluation is its period pairing, and symmetry determines the order of the rigidifying factor by its square, with $B(y,y) = 2Q(y)$. Ambiguities in the finite-order lift are roots of unity. Translation on the abelian part preserves its integral lattice by properness. Thus each summand of the pulled-back trace extends regularly. In the formal torus chart this criterion for regularity follows by writing the section in the character grading: the period equation gives all coefficients from the finitely many classes modulo kX, and the bound makes every coefficient integral. The positive quadratic growth gives convergence in the completed chart. Descent from a finite extension follows by testing the coefficients in the original fraction field.

Lean: GlobalTheory.ReachRegular.regular_of_reach (Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.ReachedSectionRegularOnIntegralModel, lines 71–100)
Lean: GlobalTheory.PlaceInt.nuGe_zero_of_moves (Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.SymmetricTorsionTranslateSectionNuNonneg, lines 34–58)

The regular pullback of as_0 implies $v(a) \geq 0$. In detail, the reduction of the primitive section s_0 is nonzero somewhere on the special fibre. The extended isogeny is surjective there since its composite with the complementary isogeny is multiplication by p^2, which is surjective on the semiabelian fibre. At a point above the nonvanishing locus, the rigidified pullback comparison is a unit. This last assertion can also be seen by comparing local generators: a generic isomorphism of the two lines has a vertical divisor on the smooth model with connected special fibre, and its order along that fibre is zero by the rigidification at the identity. Thus a pole of a would give a pole of the regular pullback.

For the opposite inequality put $\Psi_2 = [p]\Psi'$ and $D_2 = \ker \Psi_2$. It is a subgroup of $B'[p^3]$, so $|D_2| = p^m$ for some m. Translate the multiplication pullback $[p^2]^* s_0$ by all elements of D_2, using the descent lifts for Ψ_2, and sum. Substitution of the forward trace of Lemma 36.102 into this sum gives the pullback of bs for a scalar satisfying

$$ab = p^m \zeta, \qquad \zeta^p = 1.$$

To see the cancellation, for each forward representative c the inner sum over $d \in D_2$ has character given by the theta-group commutator of $\Psi' d$ and c for L^{p^2}. The image of D_2 under Ψ' is $B[p]$, by surjectivity of Ψ'. The annihilator of $B[p]$ in $B[p^2]$ under the perfect p^2-pairing is $B[p]$. Thus every representative with $pc \neq 0$ gives zero. On the identity class the sum has $|D_2|$ identical terms, up to the ratio of the two lifts on a p-torsion representative. Its p-th power is one, by iteration of translation. This proves the displayed identity. The reverse sum is regular by the same coefficient calculation, now for $[p^2]$. The nonvanishing test for its primitive target gives $v(b) \geq 0$. Since $v(p) = v(\zeta) = 0$, we obtain $v(a) = 0$. Splitting the two models in different finite extensions gives the same conclusion because each ramification index is positive.
Lean: GlobalTheory.TraceOrth.thetaTrace_orthogonality (Development.AbelianVarietyTheory.Proofs.TwistedOperators.ThetaTraceOrthogonality, lines 538–623)
Lean: GlobalTheory.HkIntegral.theta_unit (Development.AbelianVarietyTheory.Proofs.TwistedOperators.IsUnitNatCastUp, lines 242–270)

Finally the frame character of any section is a product of powers of the diagonal units. Its integral value in the integral target frame remains integral in the transported one. It lies in the valuation ring of a finite extension of $\operatorname{Frac} V$, whose elements are integral over V. This proves the value assertion on both classes of test points. □

Proof of 36.104

Proof. Over a reduced coefficient test algebra R, listing the geometric kernels defines a finite étale correspondence over the source. Indeed the kernels are subgroups of the finite étale p^2-torsion. After a finite étale trivialization of that torsion their defining conditions in Lemma 36.102 select a constant finite set. Let B_R be the resulting finite étale algebra. On its spectrum the universal subgroup has a quotient abelian scheme with the transported marking. Étale locally list its elements and locally choose a generator s_0 on the target. If the list has length m, summing over the full list repeats the trace over the quotient by $C \cap B[p]$ exactly m/q_P^i times, since $|C| = |B[p]|$. This positive integer is invertible over the coefficient field. The quotient of the full sum by the pullback of s_0 is a regular scalar times this integer. Indeed, on the locus where that pullback is nonzero one may divide regular functions; this smooth locus meets each geometric fibre since s_0 generates the theta line and the isogeny is surjective. On pairs of points above the same base point the quotients agree by Lemma 36.102. They agree as functions on the smooth fibre product by reducedness. Faithfully flat descent gives the regular scalar on the base. Dividing by m/q_P^i gives the scalar a_0 of the transported generator.
Lean: GlobalTheory.HeckeScal.exists_scal_of_family (Development.AbelianVarietyTheory.Proofs.TwistedOperators.ExistsScalarHeckeTraceThetaGeneratorFamily, lines 224–277)

Use $p^{-1}d\Psi$ on the Lie frame. The local expression for a term of the Hecke sum is $a_0^n f(B', p^{-1}d\Psi\, \mathbf{u}, s_0)$. A change of s_0 cancels between the two factors, and relisting the kernel leaves a_0 fixed. Hence the term descends to an element $u_f \in B_R$. Its trace to R specializes to the sum of its values at all the geometric kernels, since over an algebraic closure a finite étale algebra is the product of one copy of the field for each point. This trace commutes with base change. A triangular change of the original frame induces the same Lie-basis change at each target and scales the transported generator by the same scalar, by linearity of its

defining trace. The trace consequently has exactly the section weight. Multiplying by $bq_P^{-in/2}$ gives the asserted section. Equality on geometric frames implies uniqueness since the test rings are reduced. At $\lambda \nmid p$, q_P is a unit. Lemma 36.103 gives integrality of every term over the test ring. The sum with its integral multiplier belongs to the original fraction field and is integral there, so belongs to the test ring.
Lean: `GlobalTheory.HeckeEx.exists_sum` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.HeckeTransformOfPointLaws`, lines 194–223)

Fix a complex embedding in the cusp test. At unit frames normalize the theta generators to have squared mean norm one on their complex tori. If t_C is the theta scalar of the transported frame relative to the target unit frame, then $|t_C| \le q_P^i$. Indeed let H be the linear map of universal covers inducing Ψ, and u, u' the pointwise Gaussian norms of the two unit generators. The pullback identity of Lemma 36.102 and the relation $H^* H_{B'} = p^2 H_B$ between their polarization metrics give

$$|t_C| u'(Hz) \le \sum_{j=1}^{q_P^i} u(pz + v_j)$$

for suitable translates v_j. The rigidified line comparison is an isometry for these metrics: its curvature vanishes on the quotient of the two lines and its norm at the identity is one. Square the inequality and use Cauchy–Schwarz with q_P^i summands. Each translated square on the right has mean one, since multiplication by p and translation preserve normalized Haar measure. The square on the left has mean $|t_C|^2$, since the quotient isogeny also pushes Haar probability to Haar probability. This gives the bound.
Lean: `GlobalTheory.PeriodicMean.mean_factor_le` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.PeriodicThetaNormMeanFactorLe`, lines 9–75)

Write Λ_j for the period lattices of the finitely many components. The map on periods of Ψ is a similitude of multiplier p^2 and its image contains $p^2 \Lambda_{j'}$, by the complementary isogeny. Dividing its matrix by p gives a rational isometry A with $pA\Lambda_j \subseteq \Lambda_{j'}$ and $pA^{-1}\Lambda_{j'} \subseteq \Lambda_j$. Indeed the equality of the alternating trace forms, tested also after multiplying the first vector by each element of E, recovers the Hermitian form by nondegeneracy of the field trace. The two possible signs of the purely imaginary polarization parameter are distinguished by positivity on the complex tangent space: the two Riemann forms there are positive and H is complex linear. The period relation carries the flag to the target flag. Between unit flag frames its triangular matrix has diagonal entries of modulus one, by orthonormality on each successive quotient. Its weight character therefore has modulus one.

The bound on t_C, the bound on the number of kernels in Lemma 36.102, and the common frame weight give, for each source frame x,

$$|(T_{P,i,b} f)_j(x)| \le |bq_P^{-in/2}|\, 2^{(p^2)^{2g}} q_P^{in} \sup_{j', A} |f_{j'}(Ax)|.$$

It suffices to prove this at unit frames, since a change to a general frame contributes the same weight factor on both sides. Likewise the factor converting an absolute value into the invariant section norm is the same at x and Ax, by the isometry of the Hermitian forms. Hence boundedness of the original cusp section in invariant norm gives boundedness of the transform.

There are finitely many A's modulo the target arithmetic group. The lattice $A^{-1}\Lambda_{j'}$ lies between $p\Lambda_j$ and $p^{-1}\Lambda_j$. This finite quotient has finitely many subgroups. Maps with the same inverse lattice differ on the left by a target lattice isometry, and the marking subgroup has finite index in those isometries. Fix representatives. A change by the arithmetic group multiplies the reading by a character of absolute value one.

For a nonzero isotropic lattice vector ξ, let $x(t)$ be the frame at imaginary parameter it in its central direction. The limit of a bounded holomorphic section on this path is its constant Fourier–Jacobi coefficient. Indeed its expansion has factors $\exp(-4\pi t \sum_\tau \nu_\tau)$. Boundedness excludes a frequency with a negative coordinate, by taking the Fourier integral in the real directions after shifting that imaginary coordinate. The Fourier coefficients at an interior frame are absolutely summable, by shifting slightly within the domain. Dominated convergence leaves just $\nu = 0$. Under A this path is the path in the integral nonzero isotropic direction $pA\xi$ at time t/p^2, because the central displacement is quadratic in its vector. Every one of the finitely many terms bounding the transform tends to zero, by cuspidality of f on its target component. Thus the transform has zero constant term at ξ. This holds on every component and at every designated embedding.

Lean: GlobalTheory.HeckeEx.rdU_mem_SS (Development.AbelianVarietyTheory.Proofs.TwistedOperators.HeckeTransformOfPointLaws, lines 334–398)
Lean: GlobalTheory.HeckeEx.mem_SS_of_dom (Development.AbelianVarietyTheory.Proofs.TwistedOperators.MemCuspSectionsOfDominatedRationalIsometries, lines 83–159)

□

Proof of 36.105

Proof. First match kernels with representatives. In a marking of the period lattice Λ the quotient by $C \subset B[p^2]$ replaces Λ by the inverse image of C in $p^{-2}\Lambda$. Dividing the quotient differential by p amounts to using p times this enlarged lattice, with its original rational Hermitian normalization. At $P, \bar{P}$ the two factors are perfect duals. The inclusion of the $(p)P^{-1}$-torsion fixes one end of the lattice interval, the inclusion in the $(p)\bar{P}$-torsion fixes the other, and isotropy determines the second factor from the first. On the variable factor the interval has quotient a vector space of dimension r over $\mathcal{O}_E/P$; the index condition selects the elementary divisor of order i. In bases adapted to the subspace, the change on the contracting half is diagonal with i uniformizers and $r-i$ ones, with inverse dual change on the other half. Away from this pair the normalized lattice is unchanged. Integral changes of adapted basis act transitively on these subspaces. The stabilizer of the reference lattice change in K is $K \cap \alpha K \alpha^{-1}$, giving exactly $K\alpha K/K$. The marking away from p is transported by $p^{-1}\Psi$, as in Lemma 36.102. Thus the frame at ha for the matched representative and the frame y_C agree up to a triangular change whose Lie diagonal is one. In period coordinates this is the equality of the divided map on the successive flagged quotients. Conversely, two such divided maps giving the same flagged frame have isometric differentials differing by a unipotent triangular matrix. A unipotent unitary matrix for a positive Hermitian metric is the identity. Their maps of complex tori and hence their kernels agree. This also gives the matching when a different set of right-coset representatives is used.
Lean: GlobalTheory.HeckeTermIdent.heckeKernelCoset (Development.AbelianVarietyTheory.Proofs.TwistedOperators.HeckeKernelComplexPointMatchesCosets, lines 73–117)

Put $q = \mathrm{N}P$. Let $\mathscr{L}$ be the tensor lattice of the source and target and Δ_a representatives of $a\mathscr{L}/(a\mathscr{L} \cap \mathscr{L})$ in the expanding half. For the vacuum f_0 and the Heisenberg translation $\rho(x)$, the split lattice model gives

$$\omega(1,a)f_0 = \mu_j(P)^i q^{-in/2} \sum_{x \in \Delta_a} \rho(x) f_0.$$

Indeed there is one quotient of order q for each of the i changed directions in each of the n columns. The characteristic function on the enlarged half is the sum of the q^{in} coset characteristic functions. On its dual half the finite character sum kills precisely the points outside the contracted lattice. The unitary Fourier factor is $1/\sqrt{q^{in}}$, and the splitting character on the determinant gives $\mu_j(P)^i$. The two integral factors in $a \in K\alpha K$ fix the vacuum and transport the representative set. This proves the identity for the actual representatives.
Lean: GlobalTheory.HeckeVacuum.omega_vacuum_hecke (Development.AbelianVarietyTheory.Proofs.TwistedOperators.SqrtDivMulEqSqrtMul, lines 359–401)

Let C_h be the product of the pinned component constant and the adelic multiplier at h. Denote by t the scalar of the transported theta generator relative to the generator at ha. We compute t^n. In the unit frame at ha, write $\vartheta(z)$ for the theta generator and $H(z,z)$ for the polarization Hermitian form. Choose rational period vectors z_d, one for each source column. Put

$$E_z = \exp\left(\frac{\pi}{2}\sum_d H(z_d, z_d)\right), \qquad V_z = \prod_d \vartheta(z_d).$$

Let W_z denote the sum of readings of the translated vacua in the preceding display, before multiplication by its scalar. The evaluation point is the rational matrix with these period columns in the pinned orthonormal basis; the same rational target change is used in the pin and in W_z. With ϕ_0 the phase in that pin, the vacuum identity at ha reads

$$E_z\, \mu_j(P)^i q^{-in/2} W_z = \phi_0 C_{ha} V_z.$$

The geometric trace in Lemma 36.102 gives

$$W_z = t^n \phi_0 C_h E_z^{-1} V_z.$$

We detail the identification of the second sum. In one column the pullback generator is summed at the q^i points of $C/(C \cap B[p])$. Choose lifts v to the complex universal cover. If H_Ψ denotes the linear lift of the isogeny, $H_\Psi v$ is a target period. In an Appell–Humbert frame the input theta argument is $p(w+v)$, and the target trivialization is evaluated at $H_\Psi(w+v)$. Dividing by its value at $H_\Psi w$ supplies the period multiplier. The identity $H_\Psi^* H' = p^2 H$ cancels the real exponent after inserting the Gaussian on each side. The remaining unit multiplier is the Heisenberg multiplier. Explicitly, write $[x,y] = \langle x, y\rangle/2$ for half the alternating form and ψ for the additive character. If g is the target change of frame, represent a finite translation x as $g^{-1}L_f + u_x$, with L rational, $L_f = L - L_\infty$, and $\rho(u_x)f_0 = f_0$. This representation exists by additive approximation modulo the open vacuum stabilizer. Evaluated at X_∞, the translated term is the unshifted term at $(X-L)_\infty$ times

$$\psi\big(-[L_\infty, (X-L)_\infty] - [L_f, g u_x]\big).$$

Indeed $\rho(u)f(w) = \psi([w,u])f(w+u)$, and rational translation by L contributes $\psi(-[L,w])$. These two substitutions give the displayed factor. The same substitutions in the period multiplier give its infinite factor from translation by L_∞ and its finite factor from the vacuum stabilizer; the product character is trivial on rational pairings. Multiplying the one-column sums enumerates the tuples in $(C/(C \cap B[p]))^n$. Their finite half-lattice classes are distinct by the period-lattice description of C, and both this set and Δ_a have q^{in} elements. This gives the second identity with the indicated t^n.

Lean: `GlobalTheory.HeckeGeoRead.theta_trace_eq_phased_sum` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.HeckePointThetaTraceEqPhasedTranslateSum`, lines 16–65)
Lean: `GlobalTheory.HeckeGeoRead.theta_trace_read` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.ThetaTraceRead`, lines 19–72)

Take all z_d equal to a rational period vector with $\vartheta(z_d) \neq 0$. Density and nonvanishing give such a vector as in Proposition 36.100. The orthonormal source basis is invertible over E, so a rational matrix with these prescribed columns exists. Substituting the second identity in the first and cancelling the nonzero $\phi_0 V_z$ gives

$$t^n C_h q^{-in/2} = \mu_j(P)^{-i} C_{ha}.$$

For any section of the given weight, the Lie diagonal of the triangular frame change is one, so its multiplier is exactly t^n. Multiply the last identity by the common Lie normalization and by the evaluation of f at ha. It now identifies one summand in Proposition 36.104 with $\iota(b)\mu_j(P)^{-i} R_f(ha)$. Summing over the bijection proves the formula.

Lean: `GlobalTheory.HeckeGenConst.generator_constant` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.HeckeTermGeneratorConstantViaFrame`, lines 126–172)

□

Proof of 36.106

Proof. The base frame, coefficient map and extremal tensor are those of the standard layer. Its base level is the product of the two distinct odd type primes. On a tower cover the auxiliary factor is squarefree and prime to the excluded primes, and the field and pin assumptions are exactly those of Proposition 36.100. That proposition gives the rational and integral readings with the stated order of choices. For descent the form is invariant under the base compact subgroup and each tested extension is unramified, so Lemma 36.101 applies without a clearing power. For the Hecke assertions the split primes satisfy $P + \bar{P} = \mathcal{O}_E$ and are outside the full level. Apply Lemmas 36.102 and 36.103 and Proposition 36.104. The member has even source rank and its pin uses the reference unramified character, as required in Proposition 36.105. Finally the toric transport in P and the nonvanishing of T_0 are the hypotheses of Lemma 36.94; applying it to the pulled-back character gives the stated action. □

Proof of 36.107

Proof. Work throughout with the list-extension parameter tuple $p_0(h_L, c_{\mathrm{Dep}}) = (R_{\mathrm{row}}, \mathcal{N}_{\mathrm{list}}, \mathbf{1}, \top)$ defined in Chapter 33 ((33.1) and the paragraph following it: list norm-choice extension $\mathcal{N}_{\mathrm{list}}$, singleton extra datum $\mathbf{1}$, and unrestricted geometry closure $\top$ allowing every geometry), and let (Q, L) be an arbitrary linked selection for $p_0(h_L, c_{\mathrm{Dep}})$. We assemble the closed family local-error estimate in three steps using exclusively

the stated hypotheses of Theorem 36.107: verifying good triples and the memberwise Galois hypotheses, verifying the four local compatibilities, and transporting both along $p_0(h_L, c_{\mathrm{Dep}})$ to close Definition 36.11 on $W_3(\mathcal{A})$ for every family class $\mathcal{A}$ and every linked selection (Q, L) via Theorem 36.26.

Step 1: Good triples and memberwise Galois hypotheses. Fix a member point for $p_0(h_L, c_{\mathrm{Dep}}) = (R_{\mathrm{row}}, \mathcal{N}_{\mathrm{list}}, \mathbf{1}, \top)$ together with its archimedean realization and column pin. The third-layer and geometric hypotheses of Theorem 36.106 supply the linked local automorphic and Galois data at this member, and Theorem 36.106 itself supplies the six ray-level series, descent, Hecke-action, and toric-transport assertions directly on the linked layer data. For each parameter list on a good finite-type row, the linked isotypic realization supplies a nonzero eigenvector with the prescribed Hecke action. Completeness of the finite signature list matches its eigenvalues to a signature, and the member's Galois comparison gives the triple. The compatible choice of twist place w' of nonbase type and coefficient embedding ι_a gives the valuation equivalence $\iota_a(z) \in \mathfrak{m}_{O_0(a)} \iff v_{w'}(z) > 0$ for all $z \in \mathcal{O}_{K_{\mathrm{aux}}}$. The vector lies in the type-idempotent image, and its eigenvalues agree for $1 \le k \le r$ at split places outside the excluded set. The Galois comparison at the nonbase place gives the good signature, as required.

Lean: `GlobalTheory.HeightCounting.Up.TripFin.finiteTriples_satisfy_condition_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.FiniteTriplesSatisfyCondition`, lines 21–28)

From the linked layer-point data, $\mathscr{H}_2$, Theorem 36.106, depth-zero support, and the decomposition-group inputs of 36.14, we next verify every remaining field of the Galois hypotheses of Theorem 36.26 at the chosen norm: (i) the point conditions, admissibility (including split Frobenius density in finite quotients), and the first Galois provision (existence of integral full realizations $r_0(x)$, permutation uniqueness of base-change parameter lists, crystallinity off $\{g_0, g_1\}$ with x-independent level-zero Hodge–Tate jump multisets, multiplicity-free semisimplicity with common residual characteristic polynomial, unramifiedness outside $\{v \mid \ell_i\} \cup S_i \cup Q_i$, vanishing local invariants on normalized cuspidal blocks and their duals, and positive theta scalars on good rows) follow from the linked Galois realizations and local-global compatibility; in particular, because the linked Galois realizations are semisimple, regularity of the prescribed Hodge–Tate jumps (distinct jumps across constituents, with zero absent from the normalized cuspidal block W and its Tate dual $W^\vee(1)$) forces their absolutely irreducible direct summands over a finite splitting extension to be pairwise non-isomorphic (hence multiplicity-free) as well as vanishing of local Galois invariants on $W[1/\ell_i]$ and $W^\vee(1)[1/\ell_i]$ above ℓ_i; (ii) the label factorization, tame-eigenvalue identities, split density, and good triples supply the realized lists and inertia-trivial twists required by Lemma 36.25, proving residual absolute irreducibility and the tame-generator inertia exclusion at every eligible index; (iii) reduced multiplicity combined with the nonzero theta scalars yields the Z-multiplicity identity in the internal isotypic decomposition; (iv) reading the weight and block dictionaries from the block data, applying depth-zero support at layer index $j-1$ for $1 \le j \le m$, taking spherical base change and weight use from the layer realization, localization uniqueness from the Hecke realization, coefficient largeness from the list extension, and the norm and same-twist ties from norm-choice transport completes all Galois hypotheses at the member's chosen norm.

Lean: `Formalism.MainNotions.AdmissibleThreePrime.residual_irreducible_of_theorem_three_prime_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ResidualIrreducibleOfTheoremThreePrime`, lines 12–14)

Lean: `UpperTheory.GalS.core0_of_local_duality_inputs` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.PreAtMem`, lines 54–85)

Lean: `UpperTheory.GalS.galois_preconditions_at_mem` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.PreAtMem`, lines 86–103)

Step 2: The four local compatibilities. At each index i, the jump-cardinality and depth-zero support data supply (1) the paired-character tie and (2) the depth-zero type together with its eigenspan inclusion under the good paired-character idempotent e_0; the character pin supplies (3) the inertia characteristic polynomial on the tame generator; and the local series assertion of Theorem 36.106 together with central idempotent orthogonality supplies (4) the label identification: the central primitive idempotents e_f attached to finite-type labels f in the finite-group Hecke algebra (Definition 33.5) satisfy $e_f e_{f'} = 0$ for $f \ne f'$, so any central operator for a label f acting nontrivially on a parameter vector fixed by the realized local label idempotent e_{f_0} forces $e_f e_{f_0} \ne 0$ and hence $f = f_0$.

Step 3: Transport to $p_0(h_L, c_{\mathrm{Dep}})$ and closure on $W_3(\mathcal{A})$. Because witnesses pass along set division for $p_0(h_L, c_{\mathrm{Dep}}) = (R_{\mathrm{row}}, \mathcal{N}_{\mathrm{list}}, \mathbf{1}, \top)$, memberwise transport carries the conjunction of the Galois hypotheses and the four local compatibilities to every indexed point (x, i) $(x > x_1,\ i \in I_x)$ of every family F_m in the linked slot row. Intersecting this transported row with the assumed linked closed estimate at 2 and 3 for $p_0(h_L, c_{\mathrm{Dep}})$ (whose slot norm condition is given by Theorem 36.16) yields a single linked slot carrying

both: for every family class $\mathcal{A}$ and every linked selection (Q, L) for $p_0(h_L, c_{\mathrm{Dep}})$, retain the positive cutoff ε_1 and threshold datum $\mathcal{K}$ (for $0 < \varepsilon \le \varepsilon_1$) of the closed 2, 3-condition. Whenever $F_m \in W_3(\mathcal{A})$ meets $\mathcal{K}$ at C_H, membership in $W_3(\mathcal{A})$ guarantees that F_m is twist-good and obeys the transported memberwise Galois hypotheses and four local compatibilities at every $x > x_1$ and $i \in I_x$. Applying Theorem 36.26(II) (which uses Theorem 36.26(I) to form the twist-good cut family and invokes Theorem 36.16 at 2 and 3 inside the local-cost sum) therefore proves that F_m satisfies the family local-error estimate of Definition 36.11. This directly establishes that the family local-error estimate of Definition 36.11 is closed on $W_3(\mathcal{A})$ for every family class $\mathcal{A}$ and every linked selection (Q, L) for $p_0(h_L, c_{\mathrm{Dep}})$.

Lean: `GlobalTheory.TypePrimesFinished.locComp_of_isRc_classMapD_cenLoc` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LocalTypeDeterminedAtAllPrimes`, lines 22–29)
Lean: `UpperTheory.GalC.linkedAt_linkN_of_boundAtTwoThree` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LinkNFam`, lines 64–73)

□

Proofs for Chapter 37: Primes in short intervals, masses and counts

Proof of 37.4

Proof. The conductor divides N_E. If χ is real and $q, d \nmid N$, then $N \mid 8 s_D$. Take $x > x_1$; for every $i \in I_x$, the family bounds $o \le C_o$ and $Q_S \le C_o |A|$, together with $2^{e+2} \mid Q_S$ (with $Q_S \ge 1$ and odd part s_D) and $|A| \le 2^e o$, give

$$2^{e+2} s_D \le Q_S \le C_o |A| \le 2^e C_o o \le 2^e C_o^2,$$

and hence $s_D \le C_o^2$. Nonvanishing at 1 gives a uniform positive distance from 1 to the real zeros of these finitely many nonprincipal characters. For the other real characters use (37.4) to exclude the exceptional zero of 37.2, after shrinking c below $\log(3)/400$. Together with the nonreal and zeta regions this gives a uniform $c_0 > 0$ with

$$\operatorname{Re}\rho \le 1 - \frac{c_0}{\lambda_E + \log(|\operatorname{Im}\rho| + 3)}. \tag{A.545}$$

Since $1 - \rho$ is a zero for χ^{-1}, which also belongs to D, the functional equation gives

$$|\operatorname{Im}\rho| \le 2 \implies \operatorname{Re}\rho \ge \kappa/\lambda_E, \qquad \kappa = c_0/3. \tag{A.546}$$

Group zeros by $j = \lfloor |\operatorname{Im}\rho| \rfloor$. Each group has multiplicity at most $2C_{11}(\lambda_E + \log(\lfloor T \rfloor + 2))$ by (37.2). Bound the reciprocal weight by $\mathbf{1}_{j \le 2} \lambda_E/\kappa + 2/(j+1)$ and sum to obtain, for $T \ge 1$,

$$\sum_{\rho \in Z_\chi(T)} \frac{m_\chi(\rho)}{|\rho|} \le C_2(\lambda_E + \log T)^2, \qquad C_2 = 6C_{11}(3/\kappa + 4). \tag{A.547}$$

Lean: `Formalism.MainNotions.ShortIntervalPrimes.zeroSumE_of` (`Formalism.SharpRestriction.GlobalSchurBounds.FirstClaimLemmaTwoFourOfLemmaTwoThree`, lines 2355–2377)

For $v = \log u \ge 1$, put $T = e^{\sqrt{v}}$ and $H = e^{-c_1 v/(\lambda_E + \sqrt{v})}$, $c_1 = \min(\kappa, 1)$. Then $1/T \le H$ and $uH \ge 1$. On $Z_\chi(T)$ we have $u^{\operatorname{Re}\rho} \le uH$. Equations (37.1) and (A.547) give

$$|\psi_0(\chi, u)| \le [C_7(5 + C_2) + C_2] u (v + \lambda_E)^2 H. \tag{A.548}$$

Lean: `Formalism.MainNotions.ShortIntervalPrimes.psi_thin_core` (`Formalism.SharpRestriction.GlobalSchurBounds.FirstClaimLemmaTwoFourOfLemmaTwoThree`, lines 2028–2148)

For $s > 1$, integrating $\psi_0(\chi, u) u^{-s-1}$ gives $-L'(s, \chi)/(sL(s, \chi)) - \mathbf{1}_{\chi=1}/(s-1)$. By (A.548) we may pass to $s = 1$. The limits are $-L'(1, \chi)/L(1, \chi)$ and, for zeta, $-1 - \gamma$. Partial summation in (37.3) gives

$$g_\chi(y) = -y \int_y^\infty \psi_0(\chi, u) \frac{du}{u^2}. \tag{A.549}$$

Lean: `Formalism.MainNotions.ShortIntervalPrimes.l21a` (`Formalism.SharpRestriction.GlobalSchurBounds.FirstClaimLemmaTwoFourOfLemmaTwoThree`, lines 368–402)

For large x, (37.5) gives

$$\lambda_E \le 8 x^{\varepsilon/5} (\log x)^2. \tag{A.550}$$

In (A.549) integrate the bound (A.548) from $V = x^{\varepsilon/3} - \log 3$ in the variable $v = \log u$. Up to λ_E^2, use $H \le e^{-c_1 v/(2\lambda_E)}$. Above $\max(V, \lambda_E^2)$, use $H \le e^{-c_1\sqrt{v}/2}$. Integration in v and $\sqrt{v}$ gives polynomial factors times $e^{-c_1 V/(2\lambda_E)}$ and $e^{-c_1\sqrt{V}/2}$. Both $V/\lambda_E \gg x^{2\varepsilon/15}/(\log x)^2$ and $\sqrt{V} \gg x^{\varepsilon/6}$ dominate $x^{\varepsilon/8}$ and the logarithms of the polynomial factors. This proves (37.6). Since the proof holds for every family F_m of parameter $0 < \varepsilon < 1/52$ (in particular with $\varepsilon_1 = 1/53$) and zero staged thresholds $\mathcal{K} = 0$, this also discharges the closed assertion of Definition 36.10.

Lean: `Formalism.MainNotions.ShortIntervalPrimes.thin_of_zeroFree` (`Formalism.SharpRestriction.GlobalSchurBounds.FirstClaimLemmaTwoFourOfLemmaTwoThree`, lines 2379–2397)
Lean: `Formalism.MainNotions.ShortIntervalPrimes.thin_of_psiThin` (`Formalism.SharpRestriction.GlobalSchurBounds.FirstClaimLemmaTwoFourOfLemmaTwoThree`, lines 1949–1992)

□

Proof of 37.5

Proof. Apply Gan, Hanke and Yu's mass formula to the unramified standard lattice [200, (4.2) and Propositions 4.4–4.5]. Its adjacent-rank quotient is $2^{-f_K}|L(1-M,\eta^M)|$. In the functional equation with finite conductor one and $|\operatorname{disc} K^+| = (2v_K)^{f_K}$, the archimedean factor is $\Gamma(s/2)^{f_K}$ for even M (where $\eta^M = 1$) and $\Gamma((s+1)/2)^{f_K}$ for odd M (where $\eta^M = \eta$), which puts the quotient in the form (37.9) in both parities.

Lean: `Formalism.HermitianLattices.Ext.L3.massConst_succ` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.UnimodularClassRepresentatives.MassConstSucc`, lines 99–131)
Lean: `Formalism.HermitianLattices.Ext.L3.siegelMassRatio_of_LI` (`Formalism.GL3MeanSquares.GenusMeanAsymptotics.UnimodularClassRepresentatives.SiegelMassRatioOfLI`, lines 35–55)
Lean: `GlobalTheory.Mass.binary_hermitian_represents` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.BinaryHermitianRepresents`, lines 12–66)

□

Proof of 37.6

Proof. Because W carries a self-dual $\mathcal{O}_E$-lattice over the finitely unramified CM extension E/F, its Gram determinant is a local unit (hence a local norm) at every inert place; Landherr's theorem [96, Ch. 10] therefore diagonalizes W into standard signature form. Let L_1 be the coordinate lattice and $x \in L_1$ a coordinate vector of norm s. The base lattice of W' is $L_1 \cap x^\perp$. Set $W_1 = (W, h, L_1)$. By Jacobowitz's local classification [201], the two self-dual base lattices have conjugate local stabilizers at every finite place (including $v \mid 2$), and the real normalization depends on the signature. Thus $\operatorname{vol}_D(W_1) = \operatorname{vol}_D(W)$. Apply the one-vector Gram formula of 34.17 to W_1 with a real test of positive fibre integral a. The mean is

$$a \prod_{v<\infty} \alpha_v \tag{A.551}$$

for the finite densities α_v.

Lean: `GlobalTheory.UnitPerp.unitVectorPerpOver` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.HasComplementOfBasis`, lines 248–255)

The ball of squared radius t on the compact dual has self-dual measure $(2\pi)^M t^M/M!$. Differentiating at 1 gives the fibre factor $c_M = (2\pi)^M/(M-1)!$, also the factor in signature frames for norm s. Unfolding along the stabilizer of x gives the same mean as

$$\frac{\operatorname{vol}_D(W')}{\operatorname{vol}_D(W_1)} c_M^{-f} a. \tag{A.552}$$

Lean: `GlobalTheory.RealPlaceMeasuresProof.realPlaceMeasures` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.RealPlaceMeasures`, lines 138–140)

A nontrivial additive character summed over one residue norm gives q_v in the split case by summing over one factor, and $-q_v$ in the inert case since each nonzero value has $q_v + 1$ preimages. Additive orthogonality gives $q_v^{2M-1}(1 - \eta_v^M q_v^{-M})$ vectors of norm s. Surjectivity of the norm differential at a vector of unit norm gives q_v^{2M-1} lifts per step. If i_v is the norm of the local different of F, the domain and value measures contribute $i_v^{-M}/i_v^{-1/2}$. Thus

$$\prod_{v<\infty} \alpha_v = \prod_{v<\infty} i_v^{1/2-M}(1 - \eta_v^M q_v^{-M}) = \frac{(2v_E)^{f(1/2-M)}}{L_\eta(E, M)}. \tag{A.553}$$

Cancel a between (A.551) and (A.552) and substitute the volume of W.
Lean: GlobalTheory.LatticeDensity.localDensity_of_selfDualLatticeLemmas (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.LocalDensityOfSelfDualLatticeLemmas, lines 208–295)
Lean: GlobalTheory.VolRatioUnfold.ratio_of_setting (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.VolumeRatioOfUnfolding, lines 295–391)
Lean: GlobalTheory.VolRatioUnfold.volumeRatio_of_unfolding (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.VolumeRatioOfUnfolding, lines 393–421)

□

Proof of 37.7

Proof. At inert places the determinant of a self-dual Gram matrix is a unit and hence a norm. At split places the forms are isometric, and at infinity their signatures agree. Landherr's theorem gives a rational isometry with h_M [96, Ch. 10]. The transported self-dual lattice belongs to the standard genus by the local classification [201], and the isometry preserves the automorphism orders.
Lean: GlobalTheory.DefiniteMass.exists_isometry (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.UnramifiedOfIsCMUnram, lines 45–94)
Lean: GlobalTheory.DefiniteMass.sameGenus_stdLat (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.UnramifiedOfIsCMUnram, lines 126–145)

□

Proof of 37.8

Proof (outline). For any Hermitian block off-diagonal $Y \in \mathfrak{p}$ and $1 \le i < r$, Cauchy–Binet expresses the leading principal minor $\Delta_i(e^{\alpha Y})$ as $\mathbb{E}e^{\alpha U}$ for a sum U of i eigenvalues of Y weighted by squared eigenbasis minors. Differentiating at $\alpha = 0$ gives $\mathbb{E}U = 0$ (since Y has zero diagonal) and $\mathbb{E}U^2 = \sum_{a>i\ge b} |Y_{ab}|^2$. Since $\operatorname{rank} Y \le 2\min(p,q) \le 2q$, for $Y \ne 0$ and any $0 < \alpha \le \min(1, 1/(\sqrt{2\min(p,q)}\,\|Y\|_{\mathrm{HS}}))$ we have $|\alpha U| \le 1$. Concavity of $x \mapsto x^\alpha$ together with $e^x \ge 1 + x + x^2/3$ on $[-1,1]$ and $\log(1+w/3) \ge w/4$ on $[0,1]$ gives

$$\log \Delta_i(e^Y) \ge \alpha^{-1} \log \mathbb{E}e^{\alpha U} \ge \alpha^{-1} \log(1 + \alpha^2 \mathbb{E}U^2/3) \ge (\alpha/4) \sum_{a>i\ge b} |Y_{ab}|^2,$$

and at $Y = 0$ both sides vanish.

For any Harish–Chandra parameter μ (indexed $0 \le i < r$) with lowest K-type T (unit highest-weight vector u) and opposite-sign gap $h > \max(rpq+1, 16r\min(p,q) + 8\sqrt{2\min(p,q)})$, define its lowest-type model operator Ψ_μ^{mod} on $G = \mathrm{U}(p,q) = K\exp(\mathfrak{p})$ by

$$\Psi_\mu^{\mathrm{mod}}(k\exp Y) = (\dim T)\,\tau_T(k) \int_K e^{-\Phi_\mu(l^{-1}Yl)} P_{lu}\, dl,$$

where dl is probability Haar measure on K, P_{lu} is the rank-one projection onto lu, and $\Phi_\mu(Y) = \sum_{i=1}^{r-1} (\mu_{i-1} - \mu_i + 1) \log \Delta_i(e^Y)$. On $\exp(\mathfrak{p})$, $\Psi_\mu^{\mathrm{mod}}(\exp Y)$ is positive self-adjoint. Summing $\mu_{i-1} - \mu_i + 1$ over $b \le i < a$ gives at least h for each nonzero off-diagonal entry Y_{ab} $(a > b)$, whose squared moduli sum to $\|Y\|_{\mathrm{HS}}^2/2$, so taking $\alpha = \min(1, 1/(\sqrt{2\min(p,q)}\,\|Y\|_{\mathrm{HS}}))$ yields $\|\Psi_\mu^{\mathrm{mod}}(k\exp Y)\| \le \exp(-(h/8)\min(\|Y\|_{\mathrm{HS}}^2, \|Y\|_{\mathrm{HS}}/\sqrt{2\min(p,q)}))$. The compact integral has the two K-covariances, value 1 at the identity and Casimir eigenvalue $c(\mu)$. Put $jv(g) = \langle u, \Psi_\mu^{\mathrm{mod}}(g)v\rangle$ for $v \in T$. The decay estimate places these functions in $L^2(G)$. In their closed right-regular model the coefficient identity is $\langle jw, R(g)jv\rangle = \|ju\|^2 \langle w, \Psi_\mu^{\mathrm{mod}}(g)v\rangle$. Thus $j/\|ju\|$ is isometric and its compression is Ψ_μ^{mod}. Since $h > 8r\min(p,q) + 8\sqrt{2\min(p,q)}$ and $\|k\exp Y\|_{\mathrm{HS}} \le \sqrt{r}\, e^{\|Y\|_{\mathrm{HS}}}$, the weighted integral estimate on the generating type extends to the generated Harish–Chandra module, giving square-integrability of its matrix coefficients after multiplication by $\|g\|_{\mathrm{HS}}$. For $h > rpq + 1$ the lowest-type and Casimir characterization identifies this model unitarily with π_μ.
Lean: ReciprocityTheory.ChartRestriction.DS.swA_model (ReciprocityTheory.ReductiveGroups.DiscreteSeries.TensorRepresentations.SandwichA, lines 29–56)
Lean: ReciprocityTheory.ChartRestriction.DS.S7.psiM_exp_formula (ReciprocityTheory.ReductiveGroups.DiscreteSeries.CasimirCoefficients.PsiMExpFormula, lines 34–43)
Lean: ReciprocityTheory.ChartRestriction.DS.anyF (ReciprocityTheory.ReductiveGroups.DiscreteSeries.DiscreteSeriesExistence.RangeDatumEmbedding, lines 95–176)

The gap exceeds $32rpq+2$, and $\lambda_i \in (r-1)/2+\mathbf{Z}$. Set $\lambda_i(u)=\lambda_i+u(r-1-i)$, $0\le i<r$, and let d_u be its formal degree.

Claim 1. For discrete series of signature (p,q) with $p,q\ge 1$, Harish–Chandra parameter μ having opposite-sign gap strictly above $\max(rpq+1, 16r\min(p,q)+8\sqrt{2\min(p,q)})+1$ (which holds in particular whenever the gap exceeds $32rpq+2$), and an index s such that $\mu+e_s$ remains strictly decreasing, incrementing coordinate s multiplies the formal degree by $\Delta(\mu+e_s)/\Delta(\mu)$.

Write $\pi_\mu, \pi_{\mu+e_s}$ for the discrete-series representations with Harish–Chandra parameters $\mu, \mu+e_s$, $(\pi_\mu)_K, (\pi_{\mu+e_s})_K$ for their Harish–Chandra $(\mathfrak{g},K)$-modules of K-finite vectors, T, T' for their lowest K-types, and $d(\mu), d(\mu+e_s)>0$ for their formal degrees. Identify $(\pi_\mu)_K\otimes\mathbf{C}^r$ with $((\pi_\mu)_K)^r$ and define the Casimir-difference (Perelomov–Popov) operator $\mathcal{E}$ on $(\pi_\mu)_K\otimes\mathbf{C}^r$ by $(\mathcal{E}f)_b=\sum_{a=1}^r d\pi_\mu(E_{ab})f_a$, equivalently $\mathcal{E}=\sum_{a,b} d\pi_\mu(E_{ab})\otimes E_{ba}=\frac{1}{2}(C_{\mathrm{tens}}-C_{\pi_\mu}\otimes 1-1\otimes C_{\mathbf{C}^r})$; it commutes with the $(\mathfrak{g},K)$-action and has eigenvalue $p_+(\mu,s)=\mu_s+(1-r)/2$ on the $(\pi_{\mu+e_s})_K$-isotypic component. Because μ has opposite-sign gap strictly above $\max(rpq+1, 16r\min(p,q)+8\sqrt{2\min(p,q)})+1$, both μ and $\mu+e_s$ have opposite-sign gap strictly above $\max(rpq+1, 16r\min(p,q)+8\sqrt{2\min(p,q)})$, so the lowest-type models for both parameters have positive compression on $\exp(\mathfrak{p})$ and square-integrable K-finite matrix coefficients after multiplication by $\|g\|_{\mathrm{HS}}$; in particular, $g\mapsto\|g\|_{\mathrm{HS}}\langle w,\pi_\mu(g)v\rangle$ is square-integrable on G for all $v,w\in(\pi_\mu)_K$, absorbing the linear growth $|g_{cb}|\le\|g\|_{\mathrm{HS}}$ of the matrix entries g_{cb}. Thus for every $y\in(\pi_\mu)_K$, the signed $(\mathfrak{g},K)$-equivariant matrix-coefficient map $\Psi_y:(\pi_\mu)_K\otimes\mathbf{C}^r\to L^2(G)^r$ given by $\Psi_y(f)_c(g)=s_c\langle y,((\pi_\mu(g)\otimes g)f)_c\rangle_{\pi_\mu}$ (where $s_c\in\{\pm1\}$ are the signature signs of J) lands in $L^2(G)^r$ and intertwines the tensor action with componentwise right translation $R(g)$ on $L^2(G)^r$. Choose an orthonormal basis $\{y_j\}$ of the lowest K-type T and define the Hermitian form

$$Q(f,h)=\sum_j\langle\Psi_{y_j}(f),\Psi_{y_j}(h)\rangle_{L^2(G)^r}$$

on $(\pi_\mu)_K\otimes\mathbf{C}^r$. Because $R(g)$ is unitary on $L^2(G)^r$, Q is $(\mathfrak{g},K)$-invariant (skew-symmetric for $\mathfrak{g}$ and K-invariant); moreover Q is positive definite on $(\pi_\mu)_K\otimes\mathbf{C}^r$, since $Q(f,f)=0$ forces every continuous coefficient $\langle y_j,((\pi_\mu(g)\otimes g)f)_c\rangle$ to vanish identically on G, and differentiating at $g=1$ across the $(\mathfrak{g},K)$-generating subspace T forces $f=0$. Since C_{tens} is Q-self-adjoint while C_{π_μ} and $C_{\mathbf{C}^r}$ act as real scalars, $\mathcal{E}$ is Q-self-adjoint on $(\pi_\mu)_K\otimes\mathbf{C}^r$. Because $\mathcal{E}$ commutes with K, it preserves the finite-dimensional K-isotypic subspace $W\subseteq(\pi_\mu)_K\otimes\mathbf{C}^r$ containing $T\otimes\mathbf{C}^r$ (the sum of the K-isotypic components occurring in $T\otimes\mathbf{C}^r$). By finite-dimensional Q-self-adjointness, $\mathcal{E}|_W$ is diagonalisable with a finite real spectrum $B\subset\mathbf{R}$, so $\prod_{b\in B}(\mathcal{E}-b)=0$ on W. Since $T\otimes\mathbf{C}^r\subseteq W$ generates $(\pi_\mu)_K\otimes\mathbf{C}^r$ as a $(\mathfrak{g},K)$-module and $\mathcal{E}$ commutes with $(\mathfrak{g},K)$, the separable real-rooted identity $\prod_{b\in B}(\mathcal{E}-b)=0$ holds on all of $(\pi_\mu)_K\otimes\mathbf{C}^r$. Lagrange interpolation therefore yields $(\mathfrak{g},K)$-equivariant Q-orthogonal (and H-self-adjoint, for the indefinite invariant form $H(f,h)=\sum_{c=1}^r s_c\langle f_c,h_c\rangle_{\pi_\mu}$) spectral projectors P_b ($b\in B$) on $(\pi_\mu)_K\otimes\mathbf{C}^r$ with $\sum_{b\in B}P_b=1$, $\mathcal{E}P_b=bP_b$, and no Jordan blocks. On $(\pi_\mu)_K$, the partial-trace central invariants $\widetilde{G}_k=\mathrm{tr}_{\mathbf{C}^r}(\mathcal{E}^k)=\sum_{a_1,\dots,a_k}d\pi_\mu(E_{a_ka_1}\cdots E_{a_2a_3}E_{a_1a_2})$ act by the Perelomov–Popov scalars $\sum_{a,b}(B_\mu^k)_{ab}$ determined by the Harish–Chandra infinitesimal character, where B_μ is the upper-triangular matrix $(B_\mu)_{ab}=p_+(\mu,a)\delta_{ab}+\mathbf{1}_{a<b}$. Setting $d_j=z-p_+(\mu,j)$, the linear system $(zI-B_\mu)v=\mathbf{1}$ satisfies $d_jv_j=1+\sum_{k>j}v_k$ and hence $1+\sum_{k\ge j}v_k=\frac{d_j+1}{d_j}(1+\sum_{k>j}v_k)$ for $j=r,r-1,\dots,1$. By backward induction, the partial-trace resolvent on $(\pi_\mu)_K$ is

$$\sum_{b\in B}\frac{\mathrm{tr}_{\mathbf{C}^r}P_b}{z-b}=\mathrm{tr}_{\mathbf{C}^r}\big((z-\mathcal{E})^{-1}\big)=\sum_{a,b}\big((zI-B_\mu)^{-1}\big)_{ab}=\sum_{j=1}^r v_j=\prod_{j=1}^r\frac{z-p_+(\mu,j)+1}{z-p_+(\mu,j)}-1.$$

Comparing poles and residues of both rational functions shows that the projector $P=P_{p_+(\mu,s)}$ onto the $p_+(\mu,s)$-eigenspace has partial trace $\mathrm{tr}_{\mathbf{C}^r}P=c_0=\prod_{i\ne s}(\mu_s-\mu_i+1)/(\mu_s-\mu_i)=\Delta(\mu+e_s)/\Delta(\mu)$ on $(\pi_\mu)_K$ (forcing $p_+(\mu,s)\in B$ since $c_0\ne0$; moreover, the positive trace-norm identity proved next shows that $\mathrm{tr}_{\mathbf{C}^r}P_b=0$ forces $P_b=0$, hence $B\subseteq\{p_+(\mu,j):1\le j\le r\}$).

To embed $(\pi_{\mu+e_s})_K$ into $P((\pi_\mu)_K\otimes\mathbf{C}^r)$ and compare formal degrees in a positive-definite Hilbert space, fix a unit vector $y_0\in T$ ($\|y_0\|=1$) and put $\Psi=\Psi_{y_0}:(\pi_\mu)_K\otimes\mathbf{C}^r\to L^2(G)^r$. Let $\Psi_\mu,\Psi_{\mu+e_s}$ denote the lowest-K-type compressions of $\pi_\mu,\pi_{\mu+e_s}$ on G. By the compact Pieri rule, fix an isometric K-embedding $j:T'\hookrightarrow T\otimes\mathbf{C}^r$ and an orthonormal basis $\{e_i\}$ of T'. Put $A(g)=j^*(\Psi_\mu(g)\otimes g)j$ on T' and form the

continuous Hermitian Hilbert–Schmidt pairing

$$I(g) := \sum_{i,m} \overline{\langle j(e_i), (\Psi_\mu(g)\otimes g) j(e_m)\rangle}\, \langle e_i, \Psi_{\mu+e_s}(g) e_m\rangle$$
$$= \mathrm{tr}_{T'}\big(A(g)^* \Psi_{\mu+e_s}(g)\big) = \sum_m \langle (\Psi_\mu(g)\otimes g) j(e_m), j(\Psi_{\mu+e_s}(g)e_m)\rangle.$$

Writing any $g \in G$ in polar decomposition $g = k\exp(Y)$ with $k \in K$ and $Y \in \mathfrak{p}$, left K-equivariance of j, the left K-covariance laws $\Psi_\mu(kg) = \tau_T(k)\Psi_\mu(g)$ and $\Psi_{\mu+e_s}(kg) = \tau_{T'}(k)\Psi_{\mu+e_s}(g)$, and unitarity of $\tau_T(k)\otimes k$ give $I(k\exp(Y)) = I(\exp(Y))$. For $E = \exp(Y)$, both $\Psi_{\mu+e_s}(E)$ and $\Psi_\mu(E)$ are positive self-adjoint operators by the lowest-type model and E is positive self-adjoint on $\mathbf{C}^r$ (likewise $E^{-\mathsf{T}}$ in the dual case), so diagonalizing $\Psi_{\mu+e_s}(E)$ in an orthonormal eigenbasis $\{c_i\}$ with eigenvalues $\lambda_i \geq 0$ yields $I(E) = \sum_i \lambda_i \langle (\Psi_\mu(E)\otimes E) j(c_i), j(c_i)\rangle \geq 0$. At $g = 1$ one has $I(1) = \sum_m \|j(e_m)\|^2 = \dim T' > 0$, hence $I > 0$ on a neighborhood of 1 in G. Moreover, $I \in L^1(G)$ by Cauchy–Schwarz, since the matrix entries of $\Psi_\mu(g)\otimes g$ lie in $L^2(G)$ by the weighted discrete-series decay of π_μ (under the regularity gap on μ) and those of $\Psi_{\mu+e_s}$ lie in $L^2(G)$ because $\pi_{\mu+e_s}$ is in the discrete series. Integrating $I(g)$ over G therefore yields $\int_G I(g)\,dg > 0$, so at least one pair (i,m) gives a nonzero $L^2(G)$-pairing between $(\pi_\mu)_K \otimes \mathbf{C}^r$ and the closed G-invariant subspace $H_{\mu+e_s} \subseteq L^2(G)$ generated by $\Psi_{\mu+e_s}$. Projecting onto $H_{\mu+e_s}$ produces a nonzero $(\mathfrak{g}, K)$-homomorphism $(\pi_\mu)_K \otimes \mathbf{C}^r \to (\pi_{\mu+e_s})_K$; taking its adjoint with respect to the nondegenerate invariant form H yields a nonzero $(\mathfrak{g}, K)$-equivariant embedding $\iota : (\pi_{\mu+e_s})_K \hookrightarrow P((\pi_\mu)_K \otimes \mathbf{C}^r)$ into the $p_+(\mu, s)$-eigenspace of $\mathcal{E}$. Let $H_P \subseteq L^2(G)^r$ be the L^2-closure of $\Psi(P((\pi_\mu)_K \otimes \mathbf{C}^r))$, and let $H' \subseteq H_P$ be the closed G-invariant subspace generated by $\Psi(\iota((\pi_{\mu+e_s})_K))$. Because the cyclic unitary representations H' and $\pi_{\mu+e_s}$ are both generated by their lowest K-type T' and have the same positive-definite lowest-K-type matrix coefficients (the invariant inner products on the embedded irreducible module agree after normalizing its lowest type), the GNS extension from T' identifies H' unitarily with $\pi_{\mu+e_s}$. Put $u_b = \Psi(P(y_0 \otimes e_b)) \in H_P$, define the unitary sign involution $(M_J v)_c = s_c v_c$ on $L^2(G)^r$ (which commutes with $R(g)$), and set $w_b = d(\mu) P_{H_P}(M_J u_b) \in H_P$ (orthogonal projection onto H_P in $L^2(G)^r$). By Schur orthogonality on π_μ and H-self-adjointness of P, every v in the algebraic image satisfies $v_b(g) = \langle w_b, R(g)v\rangle_{L^2(G)^r}$ almost everywhere, and by L^2-continuity of $v \mapsto \langle w_b, R(\cdot)v\rangle$ this reproducing identity extends to all $v \in H_P$. Moreover, $c_0 = \langle y_0, (\mathrm{tr}_{\mathbf{C}^r} P) y_0\rangle = \sum_{b=1}^r s_b \langle u_b, w_b\rangle_{L^2(G)^r} = \sum_{b,c=1}^r s_b \langle (u_b)_c, (w_b)_c\rangle_{L^2(G)}$. Because $R(g)$ is unitary, preserves H_P, and commutes with M_J, we have $\langle w_a, R(g^{-1})u_b\rangle = \overline{\langle w_b, R(g)u_a\rangle}$ and $\langle w_a, R(g^{-1})w_b\rangle = \overline{\langle w_b, R(g)w_a\rangle}$; substituting $g \mapsto g^{-1}$ in $\langle (u_b)_c, (w_b)_c\rangle_{L^2(G)} = \int_G \overline{\langle w_c, R(g)u_b\rangle}\langle w_c, R(g)w_b\rangle\, dg$ swaps b and c, giving $\langle (u_b)_c, (w_b)_c\rangle_{L^2(G)} = \overline{\langle (u_c)_b, (w_c)_b\rangle_{L^2(G)}}$. Reversing the order of summation yields

$$c_0 = \sum_{c=1}^r \overline{\langle M_J u_c, w_c\rangle_{L^2(G)^r}} = \sum_{c=1}^r \overline{\langle P_{H_P}(M_J u_c), w_c\rangle_{L^2(G)^r}} = \frac{1}{d(\mu)} \sum_{c=1}^r \|w_c\|^2_{L^2(G)^r},$$

that is, $\sum_{c=1}^r \|w_c\|^2_{L^2(G)^r} = d(\mu)c_0 = d(\mu)\Delta(\mu + e_s)/\Delta(\mu)$. Let $P_{H'} w_c$ be the orthogonal projection of w_c onto H' in $L^2(G)^r$. For any nonzero $v \in H'$, positive-definite Schur orthogonality on $H' \simeq \pi_{\mu+e_s}$ gives $\|v_c\|^2_{L^2(G)} = \int_G |\langle P_{H'} w_c, R(g)v\rangle|^2\, dg = \|P_{H'} w_c\|^2_{L^2(G)^r} \|v\|^2_{L^2(G)^r} / d(\mu + e_s)$. Summing over $c = 1, \ldots, r$ yields $d(\mu+e_s) = \sum_{c=1}^r \|P_{H'} w_c\|^2_{L^2(G)^r} \leq \sum_{c=1}^r \|w_c\|^2_{L^2(G)^r} = d(\mu)\Delta(\mu+e_s)/\Delta(\mu)$. Running the same argument on the dual tensor product $\pi_{\mu+e_s} \otimes (\mathbf{C}^r)^*$ with dual operator $(\mathcal{E}^\vee f)_a = -\sum_{b=1}^r d\pi_{\mu+e_s}(E_{ab}) f_b$ and reverse shift $-e_s$ gives $d(\mu)/d(\mu+e_s) \leq \Delta(\mu)/\Delta(\mu+e_s)$. Since the two right-hand ratios multiply to 1, both inequalities are equalities, proving Claim 1.

Lean: `ReciprocityTheory.ChartRestriction.DS.cc_gk` (`ReciprocityTheory.ReductiveGroups.DiscreteSeries.TensorRepresentations.CcARatioZero`, lines 82–141)

Lean: `ReciprocityTheory.ChartRestriction.DS.fdeg_ratio_V` (`ReciprocityTheory.ReductiveGroups.DiscreteSeries.TensorRepresentations.Sandwich`, lines 17–26)

Applying Claim 1 coordinate by coordinate from the first index to the last preserves order and the required gap along $\lambda(u)$, giving $d_u = C_\lambda \Delta(\lambda(u))$.

Lean: `ReciprocityTheory.ChartRestriction.DS.fdc_cd` (`ReciprocityTheory.ReductiveGroups.DiscreteSeries.FormalDegree.FdCons`, lines 17–52)

Lean: `ReciprocityTheory.ChartRestriction.DS.fdCons_cd` (`ReciprocityTheory.ReductiveGroups.DiscreteSeries.FormalDegree.FdCons`, lines 54–87)

Write τ_u for the lowest type. In polar coordinates $k\exp Z(z)$ use one complex coordinate z_{ab} for each $a<b$ of opposite signs, with $Z(z)$ Hermitian. The compact measure is probability measure. The real density is $c_{pq}J(z)$, where $J(0)=1$ and $c_{pq}=F_r/(\pi^{pq}F_pF_q)$.

Claim 2. On $\exp Z$, the lowest-type compression equals the compact probability average over $l\in K$ of $(\dim\tau_u)\,e^{-\Phi_u(l^{-1}Zl)}$ times the rank-one projection onto lu (for a unit highest vector u of τ_u), where (in the 0-based indexing of $\lambda(u)$ above) $\Phi_u(Y)=\sum_{i=1}^{r-1}(\lambda_{i-1}(u)-\lambda_i(u)+1)\log\Delta_i(e^Y)$ with Δ_i the i-th leading principal minor; moreover, for $t=\min(1,1/(\sqrt{2q}\|Y\|_{\mathrm{HS}}))$, $\log\Delta_i(e^Y)\ge(t/4)\sum_{a>i\ge b}|Y_{ab}|^2$. For every integer $u\ge 0$, the opposite-sign gap of $\lambda(u)$ is at least that of λ, which exceeds $32rpq+2>\max(rpq+1,16r\min(p,q)+8\sqrt{2\min(p,q)})+1$. Hence the unitary identification with the lowest-type model $\Psi^{\mathrm{mod}}_{\lambda(u)}$ gives the first identity. For any Hermitian block off-diagonal $Y\in\mathfrak{p}$ (in particular $Y=l^{-1}Zl$), the minor bound at the start of the proof gives the second assertion, since $t=\min(1,1/(\sqrt{2q}\|Y\|_{\mathrm{HS}}))\le\min(1,1/(\sqrt{2\min(p,q)}\,\|Y\|_{\mathrm{HS}}))$ for $Y\neq 0$ and both sides vanish at $Y=0$.

Let $a_u(z)$ be the matrix coefficient at a unit highest vector of τ_u. Schur orthogonality, followed by the compact average of Claim 2, gives

$$c_{pq}\int|a_u(z)|^2J(z)\,dz\;\le\;\frac{\dim\tau_u}{d_u}\;\le\;c_{pq}\int e^{-2\Phi_u(Z(z))}J(z)\,dz.$$

For the lower bound integrate the squared compression applied to the highest vector. For the upper bound its Hilbert–Schmidt norm squared is at most $\dim\tau_u$ times the compact average of $e^{-2\Phi_u}$, by Cauchy–Schwarz and Schur orthogonality on the compact type. Conjugation by the compact group preserves $J(z)\,dz$.

Claim 3. For smooth unit highest-weight vectors w ($\|w\|=1$) of lowest K-type $u\cdot d+\Lambda_0$ ($u\ge 1$) and $Y\in\mathfrak{u}(p,q)$ with $\|Y\|_{\mathrm{HS}}\le R$ (in particular the Hermitian off-diagonal matrices $Y=Z(z)$), there exists $C=C(r,p,q,d,\Lambda_0,R)$ such that

$$\left|\langle w,\sigma(\exp(u^{-1/2}Y))w\rangle\exp(-u^{1/2}\operatorname{tr}(\operatorname{diag}(d)Y))-\exp(-Q(Y)/2)\right|\le Cu^{-1/2},$$

where $Q(Y)=-\operatorname{tr}(\operatorname{diag}(d)[Y,Y_-])$ (which equals $\sum_{a<b,\ s_a\neq s_b}(b-a)|z_{ab}|^2$ for $Y=Z(z)$).

Decompose $Y=Y_0+Y_++Y_-$ according to the signature signs $s_a\in\{\pm1\}$ and the entries of d (using 1-indexed matrix entries Y_{ab},z_{ab} for $1\le a,b\le r$): Y_0 retains entries (a,b) with $s_a=s_b$ and $d_a=d_b$; Y_+ retains compact raising entries ($a<b,\ s_a=s_b,\ d_a\neq d_b$) together with noncompact lowering entries ($a>b,\ s_a\neq s_b$); and Y_- retains compact lowering entries ($a>b,\ s_a=s_b,\ d_a\neq d_b$) together with noncompact raising entries ($a<b,\ s_a\neq s_b$). Under the $\mathfrak{u}(p,q)$-star involution $X^*=JX^\dagger J$ (which satisfies $X^*=-X$ on $\mathfrak{u}(p,q)$, i.e. $Y_{ba}=-s_as_b\overline{Y_{ab}}$), Y_-^* is supported in the exact same index set as Y_+. Evaluating $Q(Y)=-\operatorname{tr}(\operatorname{diag}(d)[Y,Y_-])=\sum_{a,b}(d_a-d_b)(Y_-)_{ab}Y_{ba}$ on $Y\in\mathfrak{u}(p,q)$ yields

$$Q(Y)=\sum_{a<b,\ s_a\neq s_b}(d_a-d_b)|Y_{ab}|^2+\sum_{b<a,\ s_a=s_b,\ d_a\neq d_b}(d_b-d_a)|Y_{ab}|^2,$$

which equals $\sum_{a<b,\ s_a\neq s_b}(d_a-d_b)|z_{ab}|^2=\sum_{a<b,\ s_a\neq s_b}(b-a)|z_{ab}|^2\ge 0$ for $Y=Z(z)$ (when $d_a=r-a$), with equality only at $z=0$, and is nonnegative on compact directions whenever d is nonincreasing. Put $Z=u^{-1/2}Y$ and $g(s)=\exp(sZ)\in\mathrm{U}(p,q)$ for $s\in[0,1]$. Because $Y\in\mathfrak{u}(p,q)$, each diagonal entry Y_{aa} is purely imaginary, so $\operatorname{Re}\operatorname{tr}(\operatorname{diag}(d)Y)=0$. On the unit lowest-$K$-type highest-weight vector w of weight $\Lambda=u\cdot d+\Lambda_0$, the highest-weight condition gives $E_{ab}w=0$ for all compact raising pairs ($a<b,\ s_a=s_b$), while the Casimir and raising–lowering norm identities give $\|E_{ba}w\|^2=\Lambda_a-\Lambda_b$ for $a<b,\ s_a=s_b$, $\|E_{ab}w\|^2=\|E_{ba}w\|^2+(\Lambda_a-\Lambda_b)$ for $a<b,\ s_a\neq s_b$, and $2\sum_{a<b,\ s_a\neq s_b}\|E_{ba}w\|^2=\|\rho\|^2-\|\rho_N-\rho_C\|^2=O(1)$ independently of u. Consequently there is a constant C' independent of $u\ge1$ such that $\|d\sigma(Y_0)w-u\operatorname{tr}(\operatorname{diag}(d)Y)w\|\le C'$ (off-diagonal entries of Y_0 have $d_a=d_b$, so $\Lambda_a-\Lambda_b=\Lambda_{0,a}-\Lambda_{0,b}$ is bounded), $\|d\sigma(Y_+)w\|\le C'$, $\|d\sigma(Y_-^*)w\|\le C'$, and $\|d\sigma(X)w-u\operatorname{tr}(\operatorname{diag}(d)X)w\|\le C'u^{1/2}\|X\|_1$ for all X. Set $F(s)=\langle w,\sigma(g(s))w\rangle$ on $[0,1]$, so $F(0)=1$ and $|F(s)|\le1$. Differentiating yields $F'(s)=u^{-1/2}\langle w,\sigma(g(s))d\sigma(Y_0+Y_++Y_-)w\rangle$. Here Y_0 contributes $u^{1/2}\operatorname{tr}(\operatorname{diag}(d)Y)F(s)+O(u^{-1/2})$ and Y_+ contributes $O(u^{-1/2})$. For Y_-, unitary conjugation shifts the derivative to the left as $\langle w,\sigma(g(s))d\sigma(Y_-)w\rangle=\langle d\sigma((g(s)Y_-g(s)^{-1})^*)\,w,\sigma(g(s))w\rangle$ (with $(X^*)^*=X$ and $\operatorname{tr}(\operatorname{diag}(d)X^*)=\overline{\operatorname{tr}(\operatorname{diag}(d)X)}$). Expanding $g(s)Y_-g(s)^{-1}=Y_-+su^{-1/2}[Y,Y_-]+O(u^{-1})$ uniformly on $[0,1]$ for $\|Y\|_{\mathrm{HS}}\le R$, the leading term Y_-^* and the $O(u^{-1})$ remainder each contribute $O(u^{-1/2})$ after multiplication

by $u^{-1/2}$, while the commutator term contributes $s\,\mathrm{tr}(\mathrm{diag}(d)[Y,Y_-])F(s)+O(u^{-1/2})=-sQ(Y)F(s)+O(u^{-1/2})$. Hence

$$F'(s)=\big(u^{1/2}\,\mathrm{tr}(\mathrm{diag}(d)Y)-sQ(Y)\big)F(s)+O(u^{-1/2})$$

uniformly on $[0,1]$. Multiplying by the integrating factor $E(s)=\exp\big(-su^{1/2}\,\mathrm{tr}(\mathrm{diag}(d)Y)+\frac{1}{2}s^2Q(Y)\big)$ (which has uniformly bounded modulus on $[0,1]$ because $\mathrm{Re}\,\mathrm{tr}(\mathrm{diag}(d)Y)=0$), integrating from 0 to 1, and multiplying by $\exp(-Q(Y)/2)$ yields Claim 3.

Lean: `ReciprocityTheory.ChartRestriction.DS.Lap.Base.matrixCoeffBounds_imply_bound` (`ReciprocityTheory.ReductiveGroups.DiscreteSeries.LaplaceOperators.LaplaceFoundations.MatrixCoeffBoundsImplyBound`, lines 69–224)
Lean: `ReciprocityTheory.ChartRestriction.DS.Lap.Base.TangentPairingStatement` (`ReciprocityTheory.ReductiveGroups.DiscreteSeries.LaplaceOperators.LaplaceFoundations.RaisingLoweringNormIdentities`, lines 68–75)

After $z\mapsto u^{-1/2}z$, the second-order term of the log of the i-th minor is $\frac{1}{2u}\sum_{a\le i<b}|z_{ab}|^2$, with the sum over opposite signs. Thus $2\Phi_u$ tends to $Q(z)=\sum_{a<b,\ s_a\neq s_b}(b-a)|z_{ab}|^2$. Since $\mathrm{tr}(\mathrm{diag}(d)Z(z))=0$, Claim 3 gives $a_u(u^{-1/2}z)\to e^{-Q(z)/2}$. For the upper integral, the minor bound of Claim 2 gives a lower bound for $2\Phi_u$ by a positive rank-dependent constant times $\min(\|z\|^2,\sqrt{u}\|z\|)$. The polar Jacobian grows at most exponentially in $\|z\|$, by its product of $\sinh v/v$ factors. The rescaled upper integrand is therefore bounded by an integrable exponential for all sufficiently large u. Dominated convergence above and Fatou below give the same limit for $u^{pq}\dim\tau_u/d_u$,

$$c_{pq}\int_{\mathbf{C}^{pq}}e^{-Q(z)}\,dz=\frac{F_r}{F_pF_q}\prod_{a<b,\ s_a\neq s_b}(b-a)^{-1}.$$

Lean: `ReciprocityTheory.ChartRestriction.DS.matrix_coefficient_pointwise_limit` (`ReciprocityTheory.ReductiveGroups.DiscreteSeries.FormalDegree.FdPointwise`, lines 120–169)
Lean: `ReciprocityTheory.ChartRestriction.DS.formal_degree_limsup_upper_bound` (`ReciprocityTheory.ReductiveGroups.DiscreteSeries.FormalDegree.FdUpper`, lines 95–136)

The compact Weyl numerator has same-sign factors $\lambda_a(u)-\lambda_b(u)+\rho_N(a)-\rho_N(b)$, where ρ_N is the half-sum of positive noncompact roots. Their ratios to the corresponding factors of Δ tend to one. The opposite-sign factors of Δ/u^{pq} contribute $\prod(b-a)$ after the same-sign factors are cancelled. The Weyl limit is therefore $1/(C_\lambda F_pF_q)\prod(b-a)^{-1}$. Comparison gives $C_\lambda=1/F_r$. At $u=0$ the shifts $\rho_N(a)-\rho_N(b)$ for same signs are nonnegative, and the pq other differences are bounded by H. Since $F_r\ge F_pF_q$, $d_{\Sigma,\rho}=V(\lambda)\le d_\tau H^{pq}$, which implies (37.11). The complete proof of the two-sided bound is given in `Formalism.MainNotions.GappedMultiplicityBounds.theorem_three_rows_of_real_place`. □

Proof of 37.9

Proof. Write $M=P+Q$. If $Q>0$, take the orthogonal complement W' of a negative unit vector supplied by 37.6. Its signature is $(P,Q-1)$. If $Q=0$ and $P>2$, use the positive unit vector in Theorem 37.6; its complement has signature $(P-1,0)$. In either case the complement is admissible, and 37.5 identifies the quotient in the volume recursion as

$$\frac{\mathrm{vol}_D(W')}{\mathrm{vol}_D(W)}=\frac{M_{M-1}}{M_M}. \tag{A.554}$$

All masses are positive. Thus the ratio of volume to mass is unchanged upon passing to W'. After Q negative and $P-2$ positive steps this ratio is the definite rank-two ratio, which is one (since in definite signature compact-dual normalization gives $\mathrm{vol}_D(W)=\mathrm{mass}(h,\Lambda_0)$, so Proposition 37.7 yields $\mathrm{vol}_D(W)=M_2$ in rank two). The starting row has constant signature with positive index at least two and the sum of its indices is r. □

Proof of 37.10

Proof. Set $C_Y=\max(5,C_{YL}+1)$ and $C=c_{T,\min}(C_{YL})$. Write $X_u=T_u^\star\Gamma_u^{2/(fk)}$ as in the product formula (A.152). The selected-family conditions in $W_1(\mathcal{A})$ supply the built point-size inequalities (Definition 34.19 and 38.3), the volume ties identifying the starting row volume with $\mathrm{vol}_D(Y_0)$, and the level-prime interval $T_u^\star\le T_u\le(1+(fk)^{-1})T_u^\star$ at every member $i\in I_x$ and word u; together with $C_{Tu}\ge c_{T,\min}(C_{YL})$, these discharge every hypothesis of Theorem 21.35 and Lemma 21.34, supplying the row-size and rank bounds and the type-prime exponent $C_Y=\max(5,C_{YL}+1)$ (bounding $\ell_g\le 2C_YL$), while Lemma 37.9 evaluates

$\mathrm{vol}_D(Y_0) = M_r$ in (A.152) and Lemma 37.8 supplies the formal-degree spectral bound controlling S_{dim}. Applying Theorem 21.35 above the family cutoff of Lemma 21.34 (with Γ_u common to the words, and $L \geq 9,\ k \geq 5$), combining $\ell_E \leq L$, $\ell_g \leq 2C_Y L$, and $S_{\mathrm{dim}} \leq fkm(12+12L)$ in (A.152) yields the upper product bound

$$\log X_u \leq (15 + 3C_Y)mL. \tag{A.555}$$

For the lower bound, the weights in $\mathcal{W}$ satisfy $w_{j\rho} \geq 1$, $v_E \geq \pi$, and $\ell_g \geq 16L$; bounding the Euler and dimension defects in (A.152) by $11fkL$ via Theorem 21.35 gives $\log X_u \geq 16mL - 11L$. Using the built lower inequality $T_u \geq T_u^\star = X_u \Gamma_u^{-2/(fk)}$ and $C \geq 11$, this implies

$$\log T_u + \frac{2\log \Gamma_u}{fk} \geq 16mL - 11L \geq 16mL - C(L+1). \tag{A.556}$$

The family threshold of Theorem 21.35 and Lemma 21.34 bounds the denominator term by $2\log\Gamma_u/(fk) \leq 2$, so $\log T_u \geq 16mL - 11L - 2 \geq mL/2$ for $m, L \geq 1$. At the upper endpoint, $\Gamma_u \geq 1$ and the built upper inequality $T_u \leq (1+(fk)^{-1})T_u^\star$ give $\log T_u \leq \log X_u + \log(1+(fk)^{-1}) \leq (15+3C_Y)mL + 1 \leq C_{Tu} mL$. Since $b = e^L$, these are (37.13). □

Proof of 37.11

Proof. Put $M = N(g_0 g_1)^2$ and $D = 4dM^2$. The lower bound on S gives

$$\sqrt{\log S} \geq \sqrt{c_s}\, x^{1/6+\varepsilon/2}.$$

Since $d \mid N$, the modulus and width satisfy

$$\log D \leq \log 4 + 9x^{\varepsilon/5}(\log x)^2 + 8C_Y x^{\varepsilon}, \qquad \log V \leq 12\log x.$$

After division by $x^{1/6+\varepsilon/2}$ both right sides tend to zero, since $\varepsilon < 1/52$. Choose the cutoff so that both quantities are at most $c\sqrt{\log S}$, with the absolute c of 8.11. Thus $h = 1/V$ belongs to $[e^{-c\sqrt{\log S}}, 1]$. Also $M \geq d \geq 100$, $S > 1$, and d is prime and 3 mod 4. Apply 8.11 with modulus M, starting point S, and width h. Its lower bound $c'hS/(D^A \log S)$ is positive. For a prime counted there take integers A_1, B_1 with

$$A_1^2 + dB_1^2 = 4T, \qquad M \mid B_1, \qquad 2M \mid A_1 - 2 - B_1.$$

Then $\pi_T = (A_1 - B_1)/2 + B_1(1+\sqrt{-d})/2$ is integral, has norm T, and is 1 mod $M\mathcal{O}_K$. □

Proof of 37.12

Proof. Here $L^2(X)$ denotes the automorphic L^2 space with the quotient and measure of X. The rank and gap assumptions give integrability of the real coefficients by 33.25. The finite projectors are locally constant and supported on the lattice stabilizer. Lemma 33.28 therefore applies, including to e itself.

For the evaluation assertion let $P = T_\phi$, and write $R(g)$ for right translation at infinity. Formal-degree orthogonality gives the matrix-unit identity

$$PR(g)P = \langle w, \sigma(g)w\rangle P$$

with the convention for the coefficient in 33.28. Indeed the convolution of the normalized conjugate coefficient with its g-translate is that scalar times the same coefficient, by integration at each real place. If $z \in V_S$, both P and R_{e_S} fix z. The identity gives $\langle R(g)z, R(h)z\rangle = \|z\|^2\langle \sigma(g)w, \sigma(h)w\rangle$. Thus $\sigma(g)w \mapsto R(g)z$ extends linearly and continuously from the cyclic span of w to an intertwiner. Its image is fixed by R_{e_S}, since the finite and real actions commute. Conversely evaluation of a projected intertwiner lies in V_S, by the same orthogonality relation in σ. An intertwiner is determined by its value on the cyclic vector w.

Lean: `Formalism.L2.SG.Closed.RowsBij_holds` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedVolumeStatements.LevelSubgroupProperties`, lines 55–56)

Lean: `Formalism.L2.SG.Pf.AllRowsDcount.dcountEq_holds` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.AllRowsDcount`, lines 58–66)

The single-label projectors are mutually orthogonal idempotents by 32.44, applied in each finite factor. Their sum over all labels acts as the identity on the image of e. Their sum over S has image $\bigoplus_{t\in S} V_{\{t\}}$. This proves the dimension formula. The arithmetic identification in 33.74 identifies the dimension for the full set with the prescribed total row dimension.

For the good labels, the isometry and row identification of 33.74 give a basis of the good subspace by transporting the selected basis, with complex conjugation in the row-function convention. Conjugation preserves linear independence, and surjectivity onto the good subspace gives spanning. The label identification in that same hypothesis puts each transported vector in its specified single-label space. Applying the projector for a good t to a basis expansion retains precisely the vectors labelled t. Their number is therefore d_t. Take d_t indices for each label, and for good t match these indices bijectively with the selected basis indices labelled t. The resulting injection has image exactly the good indices.

Lean: `Formalism.L2.SG.RowDict.range_sEE` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RowDict`, lines 148–163)

Lean: `Formalism.L2.SG.AllRowsOf.rowsLab_of` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.AllRowsOf`, lines 864–881)

□

Proof of 37.13

Proof. Put $A_0(\ell) = 2(\ell[6] + 1)$, with zero-based indexing and default entry zero. In the canonical coordinate schema of Definition 36.10, the seventh entry $\ell_{\mathrm{out}}[6]$ of the outer list $\ell_{\mathrm{out}} = \mathbf{o}$ is C_{YL}. In the order $(C_{\mathrm{res}}, n_0, A, C_b, C, c, c_b)$, take

$$c_{30}(\ell) = (0, n_0(A_0(\ell)), A_0(\ell), 0, C_0(A_0(\ell)), 1/16, 1/16), \qquad c_{\mathrm{Ic}} = 1/16, \tag{A.557}$$

using the cutoffs of 37.14.

Lean: `GlobalTheory.CountConstants.count_constants_of_siegelWeil` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.CountConstantsOfSiegelWeil`, lines 11–24)

For the numerical gap required by 33.25, use the level-size bound included in membership of $\mathcal{D}_3$ in 33.6, and enlarge the last threshold of Definition 36.10 (which controls $C' = C_k$ and $C_\ell = \min(C_f, C_w, C_\Delta, C_{\mathrm{count}}, C_r, C_{\mathrm{gap}})$ in the last list $\mathbf{l}$) to include 29 and $8\sum_{c\in\ell}|c|$ for the concatenated outer and middle lists. Write $b_{\rho,j}$ for the j-th auxiliary weight in the Harish–Chandra parameters at ρ, and suppress ρ when the place is fixed. For the family constants of Theorem 33.78, the enlarged threshold gives $C' \ge 29,\ C_\ell \ge 16$ and

$$r \ge k \ge C' m^5 f L, \qquad b_{\rho,j+1} - b_{\rho,j} \ge C_\ell r^{19} f^5 L. \tag{A.558}$$

Since $m \ge 2,\ f \ge m \ge 2,\ L > 1$, this gives $r \ge 1801$ and auxiliary gaps at least $16r^{19}$.

Lean: `Formalism.L2.SG.GapClosed.gapEv_closed_of` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.GapClosed`, lines 10–15)

Lean: `Formalism.L2.SG.ThmSOwn.gapRowT` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.LinkSGap`, lines 17–57)

For the counting hypotheses choose a second last-list threshold in Definition 36.10 forcing $C_\ell \ge \max(16, 2C)$. Require also $n_0 \le k$ and $\log L \le c_b k$. The size conditions give $\log \mathrm{rd} \le L/C_\ell$. Thus $C \log(b\,\mathrm{rd}) \le C_\ell L$. At this threshold the adjacent auxiliary gaps are at least $G = C_\ell r^{19} f^5 L$. In the row order the initial block has one sign and nonnegative row parameters, and the subsequent block has parameters $-b_j$ with alternating signs starting with the initial sign. The first auxiliary weight is nonnegative. Every pair of opposite signs inside the auxiliary block has distance at least G by summing consecutive gaps. Every opposite-sign pair involving the initial block uses $j \ge 2$, so its distance is at least $b_j \ge b_1 + G$. The size conditions also give $|\lambda_{v,a}| \le b^2$, $q_v^2 \le r$, $\log f \le \sqrt{r}$, and $\log\log \mathrm{rd} \le r$. Since $k \le rf \log r$, the bound on $\log L$ gives the last inequality of (37.19). At the two distinguished primes the bounds give $g_0 g_1 \le b^A,\ 4r < g_0, g_1$, with both primes completely split in E.

Lean: `GlobalTheory.countClauses_of_sizeHyp` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.CountParameterConditionsOfSizeHypotheses`, lines 9–83)

Lean: `GlobalTheory.IrrRowConcrete.irr_row_closed` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.IrreducibilityRowOfConcreteAutomorphicData`, lines 18–41)

Intersect the numerical and realization size tests by 36.10. For a family passing them, take the maximum of the gap, counting and realization point cutoffs and the family starting scale plus one. At each point take the realization supplied by the assumed label row. Equation (A.558) permits the use of 37.12 on this

realization. Use its original quotient, measures and finite projectors for the arithmetic row system. At every place over g_ν use the same character $a \mapsto \exp(2\pi i a/g_\nu)$ of $\mathbf{Z}/g_\nu\mathbf{Z}$, which is nontrivial at 1. The realization identifies the archimedean parameters, the split primes, the total row dimension and the positive total main term with those of the point. In the finite quotient let P_j be the subgroup-character projector at g_j, and let Z_S be the sum of the central projectors for the types in S at g_{k_0}. The cylinder projector has factor $Z_S P_{k_0}$ at g_{k_0} and factor P_{1-k_0} at the opposite prime. These are commuting orthogonal projectors by 32.44. On the labelled space of Lemma 37.12 their product fixes precisely the basis vectors whose labels belong to the cylinder and kills the others. To lift the finite convolution idempotent, divide its value on each congruence coset by the Haar volume of the kernel and extend by zero outside the lattice stabilizer. Coset integration then gives the finite convolution product, so the lift is a self-adjoint idempotent bi-invariant under the congruence kernel. At the opposite prime its factor is the original character projector on the subgroup with unipotent reduction. This factorization also gives the counting condition for a cylinder: the opposite factor is a subgroup-character idempotent and the chosen factor is the product of a central idempotent and a subgroup-character idempotent. The size and opposite-sign gap inequalities above apply at $n = r$, with $g_\circ = g_{1-k_0} > 4r$ split completely. Thus 37.14 applies to this very projector. Its counted dimension is the full cylinder dimension by 37.12, and its main term uses the same formal degree and quotient volume. For the lifted projector e, self-adjointness and idempotence give $e(1) = (e^* * e)(1) = \int |e(h)|^2 \, d\mu_f(h) \geq 0$, including when the cylinder is empty. This gives (37.15). For the full type set, $Z_S = 1$. Each subgroup-character projector has positive identity coefficient, so their product has positive main term $D_A^\star(e')$. With D_{all} the full row dimension, the same estimate gives $|D_{\mathrm{all}} - D_A^\star(e')| \leq e^{-rf\log r/16} D_A^\star(e')$.

Lean: `Formalism.L2.SG.Pf.AllRowsClosed.isAdmIdem_cyl` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.AllRowsClosed`, lines 30–31)
Lean: `Formalism.L2.SG.Pf.AllRowsPCount.isCountIdem_cyl` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.AllRowsPCount`, lines 611–691)
Lean: `Formalism.MainNotions.N.CylinderCountFamily.countCylAt_of` (`Formalism.SharpRestriction.LevelLetterCocycles.JointConditionsZpowers.JCountP`, lines 16–22)
Lean: `GlobalTheory.AllRowsAtRecord.allRows_at_record` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.AllRowsAtRecord`, lines 67–95)

□

Proof of 37.14

Proof. Evaluation at w identifies the counted space with the image of convolution by $\phi(y) = d_\infty \overline{\prod_v \langle w_v, \sigma_v(y_v) w_v\rangle e(y_f)}$, by Schur orthogonality. Unfolding gives the kernel $K(x,y)$ of (37.26) in Corollary 37.20. The diagonal identity term is $a = d_\infty \operatorname{Re} e(1)$. In definite signature the traces $\operatorname{tr}(\gamma^j) - n$ of a supported element, for $1 \leq j \leq n$, are integral and vanish modulo $g_\circ$. Their conjugates have absolute value at most $2n < g_\circ$, so the norm forces each trace difference to vanish. Newton's identities give unipotence, and compact unitarity gives $\gamma = 1$. The compact quotient then has trace aV.

In indefinite signature take $n_0 \geq 8$ (so $1 \leq q \leq p$) and take $C_0 \geq \max(0, 8 \cdot 10^{14}(A+4))$. By 37.20 the image has a finite orthonormal basis of continuous representatives F_ν satisfying (37.27). In the diagonal sum (37.26), let $S_{\mathrm{near}}(x)$ be the sum of the absolute values of the nonidentity terms with $d_v(x_v^{-1}\gamma x_v) \leq \eta$ at every real place. Let $S_{\mathrm{far}}(x)$ be the corresponding sum over the remaining terms with $\gamma \neq 1$. Put $\alpha = e^{-H/4}$, $\beta = e^{-N}$. We prove

$$\begin{aligned} \int_{\mathrm{moderate}} S_{\mathrm{near}}(x)\,dx \leq \alpha D^\star, \qquad \int_{\mathrm{moderate}} S_{\mathrm{far}}(x)\,dx \leq \beta D^\star, \\ \mathrm{vol}(\mathrm{deep}) \leq \alpha V, \qquad \int_{\mathrm{deep}} K(x,x)\,dx \leq \beta D_{\mathrm{count}}. \end{aligned} \tag{A.559}$$

By 37.16 and $|e(y_f)| \leq e(1)$ from positive type,

$$|\phi(y)| \leq a \prod_v \exp[-(h/8)\min(d_v(y_v)^2, d_v(y_v)/\sqrt{2q})]. \tag{A.560}$$

For a near term the trace differences have conjugates bounded by $n(e^{n\eta}+1) \leq 3n < g_\circ$. Thus $y = x^{-1}\gamma x$ is unipotent, and $\|y_v - 1\|_{\mathrm{HS}}^2 = \|y_v\|_{\mathrm{HS}}^2 - n \leq n(e^{2\eta} - 1)$. Put $\delta = 3\sqrt{n\eta} < 1/4$. Then $\gamma - 1$ maps $G_{\leq t}$ into $G_{\leq \delta t}$ by (37.17). If the least size is at least $1/2$, the separation in 37.16 forces the element to be the identity.

Otherwise the isotropic part of the filtration below $1/2$ has sizes $t_1 < \cdots < t_s < 1/2$ and positive increments j_i with $J = \sum_i j_i \le q$. Complete it by orthogonal complements. The middle quotient between the two isotropic parts has sizes between $1/2$ and 2, by self-duality. Use these cutoffs without subdividing that quotient. Order strictly lowering blocks by their distance down the filtration. In a fibre over earlier blocks, the difference at the next block is an integral map on the corresponding grades. The unitary relation determines the dual block from that difference because the product terms involve earlier blocks. A block paired with itself has skew-adjoint difference. A nonzero difference has geometric mean of operator norms at least the least target size divided by the largest source size, by the determinant estimate on its image and source quotient.

The scales λ for the flag density and the corresponding real dimensions m_λ are those in (37.24) of Lemma 37.17. The separation scale λ' is the ratio of target lower to source upper cutoff. It equals the tabulated value for the blocks between small grades or their duals. For maps involving the middle quotient one can use $\lambda' = t_i/2$, so in all cases $\lambda \le 3\lambda'$. Differences from a fixed representative in the fibre have maximum norm at most 2δ. Packing disjoint balls of radius $\lambda'/2$ gives $(1 + 4\delta/\lambda')^{m_\lambda} \le (1 + 12\delta/\lambda)^{m_\lambda}$ possibilities. Multiply by the density of 37.17. Since $\lambda \le 1$, $\sum m_\lambda \le 3Jnf$ and $\delta qnf \le 1/100$,

$$\prod_\lambda (1 + 12\delta/\lambda)^{m_\lambda} \lambda^{m_\lambda} \le e^{36\delta qnf} \le 2. \tag{A.561}$$

Lean: `GlobalTheory.Count.layerProd` (`Development.AbelianVarietyTheory.Definitions.SemilocalFrames.LayerProd`, lines 32–35)
Lean: `GlobalTheory.NearCount.near_count` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.PowLePowRatio`, lines 479–582)

On the moderate region $t_i \ge \tau_*$. Each logarithmic integral is at most $Y = 22fq^2 \log X$. Thus one flag type contributes at most $2e^{-JH/2} Y^s D^\star$. From the size hypotheses, $\log X \le \log n + \sqrt{n} + e^n + (1 + A)e^{H/16}$, so $Y \le e^{H/8}$ for large n at fixed A. Summing over fewer than 2^q types gives $2^{q+1} e^{-3H/8} D^\star \le \alpha D^\star$.
Lean: `GlobalTheory.NearType.integral_count_dens` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.NearTermsModeratePointsOneTypeBound`, lines 40–97)
Lean: `GlobalTheory.NearMod.nearModerate` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.OneTypeArith`, lines 57–157)

By 37.16, the number of supported rational elements with all $d_v \le R_v$ satisfies (37.21). Set $T = h\eta^2/16$; by our choice of C_0 above, $T \ge 1$ and $2n^2\eta \le \min(\log 2, T/2)$. Use $[0, \eta]$ and $(2^{j-1}\eta, 2^j\eta]$, $j \ge 1$, as shells at each place. On the positive shells $\min(d^2, d/\sqrt{2q}) \ge \eta d$. Combining (A.560) with the ball count leaves weights at most $e^{-T2^{j-1}}$, with sum at most $2e^{-T}$. The zero-shell weight is at most 2. Designating one positive-shell place gives

$$S_{\mathrm{far}}(x) \le a A_x \, 2f 3^f e^{-T}. \tag{A.562}$$

On the moderate region it suffices that

$$2N \log(3\sqrt{n}/\tau_*^2) + \log(2f) + f \log 3 + N \le T. \tag{A.563}$$

The right side is at least $50(A + 4)n^3 f \log(b\,\mathrm{rd})$. Divide the left side by $n^3 f$, use $q \le \sqrt{n}$ and $\log X \le \log n + \sqrt{n} + (1 + A)\log(b\,\mathrm{rd})$, and obtain $44 + o(1) + 44(1 + A)n^{-1/2} \log(b\,\mathrm{rd})$.
Lean: `GlobalTheory.FarTerms.farBound_of_ptBound` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.SummableOfSupport`, lines 42–124)
Lean: `GlobalTheory.Deep.farModerate_of_farBound` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.FarTermsModerateRegionOfFlagBound`, lines 53–113)

The deep volume follows from 37.17, since $\tau_* < 1/2$. Its trace bound is 37.20.
Lean: `GlobalTheory.CountFinal.countingTheorem_of_volumeStep` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.UnitaryAutomorphicCountingOfCompactDualVolumeRat`, lines 37–49)

Integrate $K(x, x) - a$ using (A.559) to get

$$|D_{\mathrm{count}} - D^\star| \le (2\alpha + \beta)D^\star + \beta D_{\mathrm{count}}. \tag{A.564}$$

For large n_0, $\alpha, \beta \le e^{-H/16}/8$ and $1 - \beta \ge 1/2$. Substitute $D_{\mathrm{count}} \le D^\star + |D_{\mathrm{count}} - D^\star|$ and divide by $1 - \beta$.
Lean: `GlobalTheory.CountSum.countingTheorem_of_finite_of_bounds` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.DiscreteSeriesCountingOfFiniteAndBounds`, lines 34–142)
Lean: `GlobalTheory.abs_sub_le_exp_neg_sixteenth_mul` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.AbsSubLeExpNegSixteenthMul`, lines 7–43)

□

Proof of 37.15

Proof. Induct on the number d of noncompact derivatives of the difference. Compact derivatives act by covariance. Interchanging two noncompact derivatives gives a compact commutator and reduces the derivative order. Hence the order-d map $T : \mathfrak{p}_{\mathbf{C}}^{\otimes d} \otimes S \to S$ is symmetric and K-equivariant. The differentiated Schmid equation kills every insertion of V_-.

If $z \perp \ker T$, it is symmetric with every one-slot slice in $V_-^{\perp}$. Equivariance gives $(\Omega_K - c_K(\Lambda_S))z \in \ker T$, where $c_K(\mu) = \|\mu + \rho_K\|^2 - \|\rho_K\|^2$ for the trace pairing. The Casimir on the noncompact tensor factors is nonnegative. For $\alpha = e_a - e_b$ with $a < b$ of opposite signs, $\|\alpha\|^2 = 2$, and the compact Casimir of the matrix-unit representation is n. The one-slot mixed operator on the constituent of weight $\Lambda_S + \alpha$ thus has scalar

$$\frac{c_K(\Lambda_S + \alpha) - c_K(\Lambda_S) - n}{2} = \langle \Lambda_S + \rho_K, \alpha \rangle + 1 - \frac{n}{2}.$$

Since $\Lambda_S + \rho_K = \lambda + \rho_N$ with $|\rho_N(i)| \le n/2$, this is at least $g - 2n$. Summing the mixed terms yields

$$\mathrm{Re}\langle z, (\Omega_K - c_K(\Lambda_S))z \rangle \ge 2d(g - 2n)\|z\|^2. \tag{A.565}$$

The left side is zero, so $T = 0$. Analyticity gives equality.

Lean: `Formalism.REAL.nPos_of` (`Formalism.RealPlaceAnalysis.RealPlaceTaylor.FirstSlotSliceSlotMixedCasimirZero`, lines 517–546)

Lean: `GlobalTheory.Jet.jet_map_eq_zero` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.SymmetricKMapEqZeroOfVminusSlices`, lines 114–225)

□

Proof of 37.16

Proof. Write $y = k \exp Z$, with k compact and Z Hermitian and off the signature blocks, so $d = \|Z\|_{\mathrm{HS}}$. For a highest-weight unit vector u of S, the compression on $\exp Z$ is the compact probability average of the rank-one projections onto lu, multiplied by $\dim S$ and weighted by

$$\prod_{i=1}^{n-1} \Delta_i(e^{l^{-1}Zl})^{-(\lambda_i - \lambda_{i+1}+1)}, \tag{A.566}$$

where Δ_i is the i-th leading principal minor. To build the real-analytic germ near $1 \in G$, set $s_i = \lambda_i - \lambda_{i+1} + 1$ for $1 \le i < n$ and $s_n = \lambda_n - (n-1)/2$, and define the full minor character $\Phi(z) = \prod_{i=1}^{n} \Delta_i(z)^{-s_i}$ (which agrees with (A.566) on $e^{l^{-1}Zl}$ since $\mathrm{tr}(Z) = 0$ implies $\Delta_n(e^{l^{-1}Zl}) = 1$). Because K is compact and $\Delta_i(k^{-1} \cdot 1 \cdot k) = \Delta_i(1) = 1$, there is an open neighborhood U_1 of 1 in $M_n(\mathbf{C})$ on which $\Delta_i(k^{-1}zk) \ne 0$ for all $z \in U_1$, $k \in K$, and $1 \le i \le n$. Let τ be the holomorphic representation of the complexification $K_{\mathbf{C}}$ on S satisfying $\tau(k) = \sigma(\bar{k})$ for $k \in K$ (where $\bar{k}$ is the entrywise complex conjugate, which preserves K since the signature matrix is real diagonal). For all $z \in U_1$, define the holomorphic matrix-valued integral $\Psi_0(z) = (\dim S) \int_K \Phi(k^{-1}zk)\, \tau(k) u (\tau(k)u)^*\, dk$, and define $\Psi(g) = \tau(k_g)\, \Psi_0(p_g)$ on a neighborhood of 1 in G via the polar decomposition $g = k_g p_g$ ($p_g = \sqrt{g^\dagger g}$ positive definite, $k_g = g p_g^{-1} \in K$). For any $g = kp \in G$ and $k_1, k_2 \in K$, write $k_1(kp)k_2 = (k_1 k k_2)(k_2^\dagger p k_2)$; Haar invariance gives $\Psi_0(k_2^\dagger p k_2) = \tau(k_2^\dagger)\Psi_0(p)\tau(k_2)$, whence $\Psi(k_1 g k_2) = \tau(k_1)\Psi(g)\tau(k_2)$. Moreover, Ψ_0 satisfies complexified-K equivariance $\Psi_0(hph^\dagger) = \tau(h)\Psi_0(p)\tau(h^\dagger)$ for all $h \in K_{\mathbf{C}}$ near 1 and positive-definite p near 1: changing variables via the Iwasawa/QR decomposition $h^{-1}k = k'b$ ($k' \in K$, $b \in K_{\mathbf{C}}$ lower-triangular with positive diagonal, so $k = hk'b$ and $k'^{-1}pk' = b(k^{-1}(hph^\dagger)k)b^\dagger$) shifts the minor weight $-\lambda - \rho$ by the Iwasawa Jacobian $2\rho_c$ to match the lowest-K-type weight $-\Lambda_S$ of $\tau(b)u$. By the holomorphic identity principle on a small connected neighborhood of $(1,1,1,1)$ in $K_{\mathbf{C}} \times M_n(\mathbf{C}) \times K_{\mathbf{C}} \times M_n(\mathbf{C})$, the holomorphic function $(h_1, a, h_2, b) \mapsto \Psi_0(h_1 a b h_2) - \tau(h_1)\Psi_0(ab)\tau(h_2)$ vanishes on the totally real slice $(h_2, b) = (h_1^\dagger, a^\dagger)$ near $(1,1,1,1)$ (where $ab = aa^\dagger > 0$), hence vanishes on the entire neighborhood; evaluating at $(k_g, p_g, 1, 1)$ for $g = k_g p_g \in G$ near 1 proves $\Psi(g) = \tau(k_g)\Psi_0(p_g) = \Psi_0(g)$ identically on an open neighborhood $U \subseteq G$ of 1, eliminating the matrix square root p_g. Evaluating at the entrywise complex conjugate $\bar{g}$ matches $\tau(\bar{k}_j) = \sigma(k_j)$ on K, so $\Delta_{\mathrm{op}}(g) = P_S \sigma(g) P_S - \Psi(\bar{g}) = P_S \sigma(g) P_S - \Psi_0(\bar{g})$ is real-analytic on U, satisfies $\Delta_{\mathrm{op}}(k_1 g k_2) = \sigma(k_1)\Delta_{\mathrm{op}}(g)\sigma(k_2)$, and vanishes at 1 by Schur orthogonality. Changing variables $k \mapsto \bar{k}$ in the compact

integral gives $\Psi_0(\bar g) = (\dim S)\int_K \Phi(\overline{k^{-1}gk})\,\sigma(k)u(\sigma(k)u)^*\,dk$. Differentiating $\Phi(\overline{k^{-1}ge^{tX}k}) = \overline{\Phi(k^{-1}ge^{tX}k)}$ along ge^{tX} for $X \in \mathfrak g$ expands the derivative as $\sum_{c,d} \overline{(k^{-1}Xk)_{dc}}\,\overline{D\Phi(z)(zE_{dc})}$ at $z = k^{-1}gk$; since $k^{-1}Xk \in \mathfrak g$ satisfies $\overline{(k^{-1}Xk)_{dc}} = -s_c s_d (k^{-1}Xk)_{cd}$ (with signature signs $s_c, s_d \in \{\pm 1\}$), extending $\mathbf C$-linearly to $X \in \mathfrak g_{\mathbf C}$ gives coefficients $(k^{-1}Xk)_{cd}\ell_{cd}(k^{-1}gk)$ with $\ell_{cd}(z) = -s_c s_d \overline{D\Phi(z)(zE_{dc})}$. For $d < c$, right multiplication by the strictly upper-triangular matrix unit E_{dc} adds a multiple of column d to column $c > d$ and therefore preserves every leading principal minor Δ_i, forcing $D\Phi(z)(zE_{dc}) = 0$ and hence $\ell_{cd} = 0$ for $d < c$. For $\sum E_{ab}\otimes G_{ab} \in V_-$, the paired coefficient $\sum (k^{-1}E_{ab}k)_{cd}\langle \sigma(k)u, G_{ab}\rangle$ vanishes pointwise in k for every $c \le d$ by the lowering weights. Thus $\Psi_0(\bar g)$ satisfies the Schmid equation on U, as does the compression, so Δ_{op} satisfies the Schmid equation near 1 and vanishes to infinite order at 1 by 37.15. Real-analyticity along $\exp(tZ)$, where $e^{l^{-1}tZl}$ is positive definite and all leading principal minors are positive, forces equality along $\exp(tZ)$.

Lean: `GlobalTheory.Jet.SchmidIntegral.eab_integral_eq_zero` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.HighestWeightEventuallyZero`, lines 223–264)

Let $Y = l^{-1}Zl$. By Cauchy–Binet, $\Delta_i(e^{tY}) = \mathbb E e^{tU_i}$, with U_i a sum of i eigenvalues weighted by squared eigenbasis minors. Differentiation gives $\mathbb E U_i = 0$ and $\mathbb E U_i^2 = \sum_{a>i\ge b} |Y_{ab}|^2$. Since $\operatorname{rank} Y \le 2q$, $|U_i| \le \sqrt{2q}\,d$. For $d > 0$ take $t = \min(1, 1/(\sqrt{2q}\,d))$. The quadratic bound for e^z on $[-1,1]$ and concavity of u^t give

$$\log \Delta_i(e^Y) \ge t^{-1}\log(1 + t^2\mathbb E U_i^2/3) \ge t\mathbb E U_i^2/4. \tag{A.567}$$

Summing the weights of (A.566) over $b \le i < a$ gives at least h for each nonzero Y_{ab} with $a > b$. Their squared moduli sum to $d^2/2$. The unweighted projection average times $\dim S$ is the identity, so the norm is at most $e^{-htd^2/8}$. At $d = 0$ use unitarity.

For the arithmetic ball a nonzero difference induces a full-rank map from its source quotient to its image. The least target size is at least $\tau_1(x)$ and the largest source size at most $\tau_1(x)^{-1}$ by duality. Taking determinants gives the separation. Rescale at v by $e^{\bar R - R_v}$, $\bar R = f^{-1}\sum R_v$. Their product is one, and the points are $\tau_1(x)^2$-separated in maximum norm inside radius $\sqrt{n}\,e^{\bar R}$. Packing in real dimension $2N$ gives $(1 + 2\sqrt{n}\,e^{\bar R}/\tau_1(x)^2)^{2N} \le A_x e^{2n^2\sum R_v}$. □

Proof of 37.17

Proof. Let $\mathcal V(I) = P_0 V/\mu_f(K_f)$. In rank $M \ge 3$, choose a coordinate unit vector and its integral complement I' as in 37.6. Unfold the Gram mean and cancel the real fibre [48, §52, Theorem 5, (41)]. The finite product is (A.553). In density-one measure the compact gauge ratio is $(2\pi)^M/(M-1)!$, from the unit-vector quotient with the Hilbert–Schmidt tangent metric. Consequently

$$\mathcal V(I) = \mathcal V(I')\, r^{f(M-1/2)} L_\eta(E, M). \tag{A.568}$$

Lean: `GlobalTheory.GaussMean.Step.volumeStep_of_siegelMean` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.EnnrealVolumeRatioStepCancel`, lines 77–119)

Descend to signature $(2,0)$. Put $Y = r^f$, $L_0 = 1 + \log Y$. In rank two Minkowski's theorem gives a nonzero vector whose squared Hermitian length is at most r at each real place, by using the product of complex two-dimensional balls of squared radius r. The saturated line and its orthogonal line have common discriminant ideal $\mathfrak n = \mathfrak a\mathcal O_E$ generated by real elements. The norm of $\mathfrak a$ over F is at most the product of the squared lengths, hence at most Y. Thus $\mathrm N_E\mathfrak n = (\mathrm N_F\mathfrak a)^2 \le Y^2$. Counting ideals in F gives at most YL_0^f choices for $\mathfrak a$. In a field of degree a, the ideal-count coefficients are bounded by those of $\zeta(s)^a$, whose sum up to Y is at most $Y(1+\log Y)^{a-1}$ by summing over the first $a-1$ factors. Minkowski's ideal-class bound in E then gives at most YL_0^{2f} ideal classes [41, Ch. I, §6, Exercise 3]. Each line class has in addition at most 2^f unit squareclasses. The unimodular gluing is the graph of a map of cyclic discriminant modules, determined by the image of a generator with at most $\mathrm N_E\mathfrak n$ choices. Thus the class count is at most

$$YL_0^f(2^f YL_0^{2f})^2 Y^2 = 4^f Y^5 L_0^{5f}. \tag{A.569}$$

Lean: `GlobalTheory.RankTwoCount.rankTwoClasses` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.RankTwoDefiniteUnimodularHermitianClassBound`, lines 225–254)

The rank-two compact gauge volume is $(8\pi^3)^f$. By the Euler factors, $\prod_{M=3}^n L_\eta(E,M) \le (\prod_{l\ge3}\zeta(l))^f$. Summing to 50 and using the remainder bound $50^{1-l}/(l-1)$ gives $\prod_{l=3}^8 \zeta(l) \le 1.391$. For $j \ge 0$, the term

at 2 and the sum and integral starting at 3 give

$$\zeta(9+j) \le 1 + 2^{-j}(2^{-9} + 3^{-9} + 3^{-8}/8) \le 1 + 0.00235\,2^{-j}.$$

The product is at most $1.391e^{0.0047} < 1.398 < 1.4$. With $L_0 \le Z^2$ and $8\pi^3 \cdot 4 \cdot 1.4 \le Z^{11}$, (A.568) gives $\mathcal{V}(I) \le Z^{f(26+(n^2-4)/2)} \le Z^N$. Reduction in the lattice endomorphism ring gives $[K_f : K(g)] \le g^{2N}$.
Lean: `GlobalTheory.ZetaProd.zeta_tail_le` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.ZetaProdSphereVolumeBounds`, lines 81–93)

For the flag measure, bound the indicator of the filtration event by the sum over rational isotropic flags of the given type and sizes, and unfold along the stabilizer. Its unipotent blocks have the scales of (37.24). The determinant directions give $2fj_i\,dt_i/t_i$. The determinant-norm-one Levi block of rank j has volume

$$V_{\mathrm{GL},j} = (\operatorname{res}_1 \zeta_E)(r/2)^{fj^2} \prod_{l=2}^{j} \zeta_E(l).$$

For rank one the ideal classes and the unit fundamental domain give $(r/2)^f \operatorname{res}_1 \zeta_E$. The Siegel unfolding of a primitive first column from rank $j-1$ to j multiplies the volume by $(r/2)^{f(2j-1)}\zeta_E(j)$. The remaining unitary factor cancels by (A.568).

Put $S_l = \prod_{u=1}^{l} 2\pi^u/(u-1)!$ and $S = \sum_i j_i^2$. Set $d_U = 4\sum_{i<k} j_i j_k + 2J(n-2J) + S$. At each real place, the compact quotient in Iwasawa coordinates has factor

$$2^{-d_U/2+nJ-J^2-J/2-S}\,\frac{S_p S_q}{S_{p-J} S_{q-J} \prod_i S_{j_i}}.$$

The compact gauge volume in rank l is $2^{(l^2-l)/2} S_l$. The compact GL_j block contributes $2^{j^2-j/2} S_j$ to the denominator, and the unipotent Jacobian contributes $2^{-d_U/2}$. The arithmetic unipotent covolume contributes $r^{fd_U/2}$. Multiply by $2^{fj_i^2} V_{\mathrm{GL},j_i}$ for each Levi block to convert its measure to the flag gauge. Divide by the rank factors of (A.568) for $M = n-2J+1, \dots, n$. Their powers of r sum to $2fnJ - 2fJ^2$. Using $d_U = 2nJ - 2J^2 - S$ gives

$$\kappa = (r/2)^{-fd_U/2} 2^{-f(nJ-J^2+J/2)} \left(\frac{S_p S_q}{S_{p-J} S_{q-J} \prod_i S_{j_i}}\right)^f \frac{(\operatorname{res}_1 \zeta_E)^a \prod_i \prod_{l=2}^{j_i} \zeta_E(l)}{\prod_{M=n-2J+1}^{n} L_\eta(E, M)}. \tag{A.570}$$

Lean: `GlobalTheory.FlagMeas.genLinVolume` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.GenLinTamagawaVolume`, lines 11–30)
Lean: `GlobalTheory.FiveFactors.prod_five_factors` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.ProdFiveFactorsHermitianVolume`, lines 80–127)
Lean: `GlobalTheory.FlagMeas.Bound.flagMeasureBound` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.IsotropicFlagProbabilityBoundOfVolumeRatio`, lines 64–103)

For the residue bound, let h_E, R_E, w_E be the class number, regulator and number of roots of unity. In an inverse ideal class choose an integral ideal J_0. For any $\sum_v x_v = 0$, Minkowski applied to the discs $|z_v|^2 \le e^{x_v}(Y \mathrm{N} J_0)^{1/f}$ in J_0 gives a nonzero a_0, hence $I_0 = (a_0) J_0^{-1}$ of norm at most Y. For fixed I_0 the possible a_0 form one unit orbit. The corresponding regions in the x_v are unit translates of a simplex of volume $\log(Y/\mathrm{N} I_0)^{f-1}/(f-1)!$. They cover a regulator domain. Sum over ideals and classes and use $(\log u)^{f-1}/(f-1)! \le u^s$ for $u \ge 1$, $s > 1$. This gives

$$h_E R_E \le Y^s \zeta_E(s) \le Y^s (s/(s-1))^{2f}. \tag{A.571}$$

Minkowski gives $r \ge 3/2$. In the class-number formula take $s = 1 + 1/\log r$ to get

$$\operatorname{res}_1 \zeta_E = \frac{(2\pi)^f h_E R_E}{w_E r^f} \le [2\pi e(1+\log r)^2]^f \le (16 \log r)^{2f}. \tag{A.572}$$

Lean: `GlobalTheory.dedekindZeta_residue_le_log_rootDiscr_pow` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.DedekindZetaResidueLeSixteenLogRootDiscrPow`, lines 48–75)

Put $s_0 = \sqrt{n}$, $l = \log n$, $c_0 = \log(2\pi)$. In the compact factor the sphere factors $2\pi^u/(u-1)!$ have consecutive ratio π/u, so the maximum is at $u = 4$ and its logarithm is below 3.49. Use that bound for S_q/S_{q-J}. For S_p/S_{p-J}, use $(u-1)! \ge ((u-1)/e)^{u-1}$ and $u - 1 \ge p - J \ge n - 2s_0 \ge n/2$. The reciprocal

factors $1/S_{j_i}$ cost at most $s_0 l/4$ per unit of J, from $\log(u!) \le u \log u$ and $j_i \le s_0$. The six log contributions in (A.570), divided by fJ, are therefore bounded by

$$n/3, \quad 0, \quad c_0 - (n-2s_0)(l-c_0-1) + 3.49 + s_0 l/4, \quad 2(3+n), \quad 2, \quad 1. \tag{A.573}$$

Here the first uses $d_U \le 2Jn$, $r/2 \ge 3/4$. The last three follow from (A.572), $\log r \le e^n$, $\zeta_E(l) \le \zeta(l)^{2f}$ and $L_\eta(E,M)^{-1} \le \zeta(M)^f$. For $l \ge 15$ their sum is at most $-nl/2$. This gives $\kappa \le e^{-JH/2}$. In (37.23) the middle-block factors include $t_i^{2fj_i(n-2J)}$, and the other scales are at most one. The density with $\kappa = 1$ integrates to at most one by comparison with $\prod_i 2fj_i t_i^{2fj_i-1}\,dt_i$ on the unit cube. A least size below $1/2$ gives a nonempty flag. Summing over 2^{J-1} compositions of J gives the claimed probability. □

Proof of 37.19

Proof. At a single place the sum of squared derivatives in an orthonormal Lie basis is the quadratic form of $2\Omega_K - \Omega_G$. A compact highest weight through order j satisfies $\|\mu + \rho_K\| \le \|\lambda\| + \|\rho_N\| + \sqrt{2}j$, since each derivative adds an adjoint weight. With $a_0 = \|\rho_N\| + \sqrt{2}j$, the Casimir expression is bounded by $2(\|\lambda\| + a_0)^2 - \|\lambda\|^2 + n^3/12 \le (\|\lambda\| + 2a_0 + 0.3n^{3/2})^2$. Use $\|\rho_N\| \le 0.58n^{3/2}$ and $\|\lambda\| \le \sqrt{n}\,b^2$ to bound this by B_j^2. Summing over places gives

$$\|d\sigma(Z)u\| \le \sqrt{f}\,B_j \|Z\|_{\mathrm{HS}} \|u\| \qquad (u \in V_j). \tag{A.574}$$

Use the ordered exponential cube of side $s = 1/B$ in the N real Lie coordinates and thicken by $K(g)$. From $2b^2 \ge h$ and the gap we get $12n^3 s, 8n^5 s \le 1$. Translated back to the identity, each differential column is its basis vector conjugated by the subsequent factors. The product estimate for the conjugations and the exponential coordinate map bounds the chart perturbation by $n(e^{3n^2 s} - 1) \le 1/2$. The quotient of two image points has operator norm at most $e^{2n^2 s} \le 2$. The differential columns differ from the original columns by at most $e^{2n^2 s} - 1$ in Hilbert–Schmidt norm. The resulting perturbation in dimension n^2 has norm at most $n(e^{2n^2 s} - 1)$, so the inverse determinant is at most $\exp(2n^3(e^{2n^2 s} - 1)) \le e$ at each place. These three inequalities follow from $e^u - 1 \le 2u$ for $0 \le u \le 1$ and the two bounds on s. Thus the chart integral costs at most $e^f P_0/k_R$. Quotient overlaps are bounded by $(1 + 4\sqrt{n}/\tau_1(y)^2)^{2N}$ by the packing argument of 37.16.
Lean: `GlobalTheory.ProductChartAll.lintegral_chart_all_le` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.LintegralProductExpChartAllRealPlacesLe`, lines 124–276)

In one coordinate the fundamental theorem of calculus and Cauchy–Schwarz give

$$|v(0)|^2 \le (2/s) \int_0^s |v(t)|^2\,dt + 2s \int_0^s |v'(t)|^2\,dt. \tag{A.575}$$

Iterate in all coordinates using the one-place derivative bound B through order $k + N$. Integration in the quotient and the isometries give, for $u \in V_k$,

$$\|F(u,y)\|^2 \le (6B)^N e^f \left(1 + \frac{4\sqrt{n}}{\tau_1(y)^2}\right)^{2N} \frac{DP_0}{k_R} \|u\|^2. \tag{A.576}$$

Lean: `GlobalTheory.SobolevBound.sobolev_bound` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.MeasurePreservingReindex`, lines 239–285)

Append $1/2$ to the small flag sizes starting at τ. At most q successive ratios have product 2τ, so there are consecutive sizes t, t' with $t/t' \le s_0 = (2\tau)^{1/q}$, and $t^2 \le s_0/4$. Let R be the corresponding saturated isotropic sublattice, of rank $j \le q$. Its largest size is at most t, and the quotient above it has least size at least t'.

For a Hom lattice from source of rank a and least size t' to target of rank b' and largest size t, the trace-dual is the lattice of reverse maps taking the target into the source multiplied by the inverse different. On the nonzero image of a reverse map the determinant inequality gives geometric mean operator norm at least $t'/(rt)$. The different contributes r because its norm over E is r^{2f}. The Hilbert–Schmidt length across the real places is therefore at least $\lambda = \sqrt{f}\,t'/(rt)$. The Euclidean transference inequality for covering radius and first dual minimum gives radius at most $fab'/(2\lambda)$ in this normalization. The real dimension is $2fab'$, and the trace pairs the real coordinates with a factor 2 compared with the Euclidean pairing. For the first

layer $\mathrm{Hom}_E(R^\perp/R, R)$ take $a = n - 2j,\ b' = j$. Lifting a block adds its adjoint and costs $\sqrt{2}$ in norm. Since $(n-2j)j \le n^2/8$, the radius modulo the centre is at most

$$\frac{n^2\sqrt{f}\,r}{8\sqrt{2}}\,\frac{t}{t'}. \tag{A.577}$$

Lean: `GlobalTheory.HomCover.exists_hom_near` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.LatticePairVolumeBounds`, lines 138–191)
Lean: `GlobalTheory.Count.FirstLayer.main` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.NilradicalFirstLayerCoveringRadiusLe`, lines 107–185)

The centre consists of skew-adjoint maps from $\Lambda_x/R^\perp$ to R. The source has least size at least $1/t$. Approximate half of a central element by an integral Φ and use $\Phi - \Phi^*$. With $a = b' = j$ the error after skew-symmetrization is at most $j^2\sqrt{f}\,rt^2$, hence at most

$$\frac{n^2\sqrt{f}\,rs_0}{16}. \tag{A.578}$$

Lean: `GlobalTheory.Count.CentralCover.main` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.NilradicalCenterIntegralCoveringRadius`, lines 201–310)

Reduce first by $2g$ times the integral first layer. Include the half-commutator in the central remainder and reduce by g times the integral centre. The linear term of the exponential is divisible by g. The factor $2g$ also makes its quadratic term divisible by g. Modulo rational translation and $K(g)$, the real orbit is covered by paths $x \exp Z_1 \exp Z_2$ with sum of exponent norms at most $n^2\sqrt{f}\,grs_0(1/(4\sqrt{2}) + 1/16) \le \rho$. For the adelic orbit, first use finite-place approximation in the additive first layer to move its finite part to the congruence subgroup. Then approximate in the additive centre to remove the central remainder. This reduces to the real cover.
Lean: `GlobalTheory.SmallFund.small_fundamental_set` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.SmallFundamentalSetRealUnipotent`, lines 135–218)

Along the paths, $\tau_1(y) \ge \tau e^{-\rho} \ge \tau/2$. Thus (A.576) starts an iteration on V_k with squared constant $(6B)^N e^f (17\sqrt{n}/\tau^2)^{2N} DP_0/k_R$. If the bound on V_{j+1} is $M'\|u\|$, integration of (A.574) along the paths bounds $F(u,y) - F(u,x)$ on V_j by $\rho\sqrt{f}B_jM'\|u\|$. The constant term is zero, so averaging gives the same bound for $F(u,x)$. The bound on the paths is then $2\rho\sqrt{f}B_jM'\|u\|$. Iterate k times and square at w. □

Proof of 37.20

Proof. On each finite-dimensional subspace of counted intertwiners take the inner product $T^*S = \langle T, S\rangle I$. An orthonormal basis consists of isometries with mutually orthogonal, $K(g)$-fixed images.

At each real place the compact-type projection writes any vector of the lowest type as a compact matrix-coefficient average of translates of w_v. Convolve with a smooth bump supported near 1 and project to the type. The resulting operator on the finite-dimensional type can be chosen arbitrarily close to the identity and hence invertible there. This gives a smooth compactly supported group average for each lowest-type vector. Take products of these averages over the real places. A change of variables transfers the Lie derivative to the averaging function and gives smooth averages for the vectors of V_j. Convolution with these functions and the normalized finite level indicator gives continuous representatives and the asserted derivative identity. For w itself, a fixed smooth average and Cauchy–Schwarz give

$$|f_i(x)| \le C\,\tau_1(x)^{-2N}\|T_i w\|, \qquad f_i = F_i(w).$$

Cover the compact support of the average by finitely many chart translates of the kind in 37.19. The overlap count is $O(\tau_1(x)^{-4N})$ by 37.16. Taking a square root gives the bound. The constants depend on the count data and the chosen averages.
Lean: `GlobalTheory.SmoothRepsType.repsExist` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.ExistsContinuousRowRepresentativesLowestKType`, lines 87–137)
Lean: `GlobalTheory.StepAverage.exists_smooth_average_of_isKTypeOf` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.ExistsSmoothAverageLowestKType`, lines 15–84)

Schur orthogonality gives $f_i = R_\phi f_i$. The growth bound and the real decay bound of Lemma 37.16 (with $|e(y_f)| \le e(1)$ from positive type) give an absolutely convergent continuous convolution. In the radical constant term, real translation in the supported finite level changes τ_1^{-1} by at most the geometric mean of

the real operator norms and their inverses. The exponential decay of Lemma 37.16 at large radius dominates this growth and the radical volume growth under (37.19). Thus the constant-term integral and convolution can be interchanged. The real integral of the translated discrete-series coefficient along a proper radical vanishes [25]. The constant term of f_i is zero. Its translates generate $T_i(H_\sigma)$, and the cuspidal subspace is closed and invariant.
Lean: `GlobalTheory.RowCuspKType.row_isCuspFn` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.CountedVectorsAreCuspFormsSingleLowestKType`, lines 85–114)

For a basis of size D, apply 37.19, 37.18 and (37.22) with $Z = X$. On the deep region they give

$$\sum_{i=1}^{D} |F_i(w)(x)|^2 \le e^{-N} D/V. \tag{A.579}$$

On the moderate region (A.576) for a single normalized intertwiner bounds the evaluation norm uniformly, since $\tau_1 \ge \tau_*$. By Bessel the summed squared evaluations are then at most some constant C_1^2 independent of D. Their total integral is D, so $(1 - e^{-N})D \le VC_1^2$. Hence the full space is finite dimensional.

Its finite basis gives a continuous projector kernel. On compact sets the finite components range over finitely many cosets of the open level, and the real radii increase by bounded amounts. The ball and shell estimates thus give locally uniform convergence of the orbit sum (37.26). On compactly supported tests unfolding identifies it with the projector. The continuous kernels therefore agree, and integrating (A.579) with $D = D_{\mathrm{count}}$ gives (37.28).
Lean: `GlobalTheory.KernelExpansion.kernelB` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.IntegrablePh`, lines 166–224)

□

Proof of 37.21

Proof. By Definition 34.46(4)(b) at every constructed point $x > x_1$, $T_u = \pi_u \bar{\pi}_u$ with $\pi_u \equiv 1 \bmod Q_S qd(g_0 g_1)^2 \mathcal{O}_E$, whence $T_u \equiv 1 \bmod Q_S qd$; since $E \subseteq \mathbf{Q}(\zeta_{Q_S qd})$ by Definition 5.1, T_u splits completely in E and F. Let K stabilize the self-dual overlattice of the definite first space, and $U \subset K$ be the column level. Its unitriangular factors give (37.29), and fixing a nonzero vector at each of the f places over T_u gives

$$[K : U] = I_{01}(T_u^k - 1)^f. \tag{A.580}$$

At the prime in (37.30), a stabilizer for the action of a genus automorphism group on its level cosets embeds in the pro-g_ν preimage of a unipotent group. If nontrivial, it contains an element of order g_ν, which in rational dimension $2fk$ would require the cyclotomic factor of degree $g_\nu - 1$. Thus the action is free. Summing the level index divided by each genus automorphism order gives $[K : U]M_k$ by 37.7.
Lean: `GlobalTheory.ColClassNumber.mass_selfDualLattice_firstSpace_eq` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.MassSelfDualLatticeFirstSpaceEq`, lines 93–100)
Lean: `GlobalTheory.ColClassNumber.classNumber` (`Development.AbelianVarietyTheory.Proofs.FoundationProofs.MassSelfDualLatticeFirstSpaceEq`, lines 102–130)
Lean: `GlobalTheory.ColClassNumber.classNumber_closed` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ClassNumberClosed`, lines 10–13)

□

Proof of 37.23

Proof (outline). Claim 1. For every reduced, finitely generated K-algebra A, every algebraic framed test point y over A, and every K-algebra map $\psi : A \to \mathbf{C}$, there are a component k, a frame x in its domain, and a scalar $u \in \mathbf{C}$, independent of the section, such that

$$\psi(s(A, y)) = uR(s)(k, x) \qquad (s \in C_K).$$

Section base change first gives the complex framed point associated with (y, ψ). The uniformization covers this point up to a Borel change of frame. The section transformation law gives the displayed scalar, namely the value of the common weight character on that change of frame.
Lean: `GlobalTheory.FrameUnif.test_value_eq_const_mul_reading` (`Development.AbelianVarietyTheory.Proofs.UnitaryExistence.TestPointValueEqConstMulReading`, lines 20–44)

Claim 2. Let A be a reduced, finitely generated K-algebra, let $a_1, \dots, a_m \in A$, and let $z_1, \dots, z_m \in \mathbf{C}$. If

$$\sum_i z_i \psi(a_i) = 0 \qquad \text{for every } K\text{-algebra map } \psi : A \to \mathbf{C},$$

then, for every K-linear functional $\lambda : \mathbf{C} \to K$,

$$\sum_i \lambda(z_i) a_i = 0.$$

To prove this, choose a K-basis $e_1, \dots, e_t$ of the span of the a_i, and write $a_i = \sum_l t_{il} e_l$. Evaluation on complex points preserves the independence of the e_l. Indeed K has characteristic zero, so the reduced finite-type algebra A remains reduced after extension to $\mathbf{C}$. Its complex points separate its elements by the Nullstellensatz; faithful extension of scalars preserves the independence of the e_l. Therefore $\sum_i t_{il} z_i = 0$ for every l. Applying λ to these scalar identities gives the claim.

Lean: `Formalism.L2.Pj.linIndep_cpts` (`Formalism.LocalToGlobal.ProjectiveLattices.ReadAlg`, lines 108–134)
Lean: `Formalism.L2.Pj.descend_cpts` (`Formalism.LocalToGlobal.ProjectiveLattices.ReadAlg`, lines 136–168)

Suppose $\sum_i z_i R(s_i) = 0$. Claim 1 gives the hypothesis of Claim 2 for $a_i = s_i(A, y)$, at every test algebra and test point. Consequently $\sum_i \lambda(z_i) s_i = 0$ as algebraic section rules. Independence over K gives $\lambda(z_i) = 0$ for every λ and i. Linear functionals separate vectors, so all z_i vanish. This proves complex independence of the full frame readouts.

Lean: `GlobalTheory.reading_linearIndependent_and_injective` (`Development.AbelianVarietyTheory.Proofs.UnitaryExistence.ReadingLinearIndependentAndInjective`, lines 22–54)

Claim 3. A linear combination of these frame readouts that vanishes at every comparison frame vanishes everywhere. Such a combination has the common arithmetic-group and Borel equivariance of the given weight. The comparison datum covers each frame domain by the orbits of the comparison frames under these two actions. Both transformation multipliers are nonzero, so vanishing on the comparison frames implies vanishing on each domain. Outside the domains the readouts are zero by definition.

Lean: `GlobalTheory.ClassSumSpan.atDict_indep` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ClassSumsSpan`, lines 165–195)

A relation $\sum_i z_i E(s_i) = 0$ therefore gives $\sum_i z_i R(s_i) = 0$ by Claim 3, and the independence just proved gives $z_i = 0$ for every i. The complete proof is given in `GlobalTheory.ClassSumSpan.atDict_indep`. □

Proof of 37.24

Proof (outline). The selected member has

$$n_V \geq 8fr + 4. \tag{A.581}$$

The four stated inputs supply the smooth unitary realization, its lowest types and the required uniform rapid decay. The normalized theta data of the selected point supply unit vectors v_ρ in the lowest types and a unit vector w in the paired space whose Fock-model pairing is exactly Δ_j. Polynomial insertion into the Fock model is injective and $\Delta_j \neq 0$. Fix ρ and all factors except the ρ-th. The map sending u in the ρ-th lowest type to the pairing of the modified pure tensor with w is compact-equivariant, its value Δ_j at v_ρ has the prescribed torus character, and every torus weight of a nonzero vector in the compact span of Δ_j is at most that character in the flag order. The kernel is an invariant subspace of the irreducible lowest type. Since v_ρ maps to $\Delta_j \neq 0$, the map is injective, so v_ρ itself has the specified extremal weight. Take $u_\rho = v_\rho$. Replacing the factors one at a time keeps every norm equal to one and the pairing with the same w equal to Δ_j.

Fix a layer j and the reference tuple μ_0. Put $n = n_V$ and let

$$K = \iota(K_{\mathrm{c}}) \subseteq \mathbf{C}, \qquad \sigma : K \hookrightarrow \mathbf{C}$$

be the inclusion. This is the second comparison field at the reference tuple. Let C_K be the K-space of algebraic cusp sections of the prescribed weight, and let $\mathcal{S} \subseteq C_K$ consist of the normalized reference theta class sums belonging to allowed rational inputs. We now give the assertion about these class sums that will be used. Write $\mathcal{C}$ for the finite set of source classes, $\mathscr{V}_K$ for the source coefficient space, and

$$I_K : \mathscr{V}_K \longrightarrow K[X_{\tau,a,b}]$$

for the polynomial coefficient map of the normalized comparison datum. The indices satisfy $\tau : E \hookrightarrow \mathbf{C}$, $0 \le a < d_\tau$, and $0 \le b < n$, where d_τ is the tangent multiplicity at τ. The class datum specifies lattices Λ_c, levels N_c, stabilizer orders γ_c, class factors $e_c \in K$, finite weights w_c, and a volume factor $v_{\mathcal{C}} \in K$. Its level cover has a normalization section d_π. Identify the coefficient field of the cover with K by its prescribed field isomorphism.

Write $a = (a_{\tau,i})$ for the prescribed weight exponents. The coefficient field contains every conjugate of E. For each embedding $\tau : E \hookrightarrow \mathbf{C}$, write $\tau_K : E \to K$ for the map satisfying $\sigma\tau_K = \tau$, and let $\mathcal{O}_K$ be the integral closure of $\mathbf{Z}$ in K. For an $\mathcal{O}_E$-submodule $\Lambda \subset E^n$, define

$$\mathscr{R}(\Lambda) = \mathcal{O}_K\left[\sum_{b=0}^{n-1} \tau_K(v_b) X_{\tau,i,b} : \tau : E \hookrightarrow \mathbf{C},\ 0 \le i < d_\tau,\ v \in \Lambda\right] \subset K[X_{\tau,i,b}], \qquad \mathscr{H}_a(K) = \operatorname{im} I_K.$$

Thus $\mathscr{R}(\Lambda)$ is the subring generated by the displayed linear forms and integral constants.

Claim 1. For the selected normalized comparison datum, the lattice-polynomial condition at Λ is exactly

$$P \in \mathscr{H}_a(K) \cap \mathscr{R}(\Lambda).$$

In particular, every monomial X^m with nonzero coefficient in such a polynomial has

$$\sum_{b=0}^{n-1} m(\tau,i,b) = a_{\tau,i} \qquad (\tau : E \hookrightarrow \mathbf{C},\ 0 \le i < d_\tau),$$

where $a_{\tau,i}$ are the prescribed weight exponents. The normalization of I_K identifies its image with the harmonic highest-weight polynomial space. Membership in that image gives the harmonic condition, and every polynomial in the harmonic highest-weight space has a preimage. Intersecting with the displayed subring gives the asserted lattice condition.

Lean: `Formalism.L2.Pj.HHa` (`Formalism.LocalToGlobal.ProjectiveLattices.RowDeg`, lines 43–44)
Lean: `Formalism.L2.Pj.harmL` (`Formalism.LocalToGlobal.ProjectiveLattices.ThShort`, lines 235–237)
Lean: `GlobalTheory.ClassSumSpanCore.classSumsSpan_of_pins_core` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ClassSumsSpanCuspFormsBaseTupleSecondField`, lines 284–351)

Claim 2. For every coefficient function $x : \mathcal{C} \to \mathscr{V}_K$, there is a cusp section $s_x \in C_K$ with the following property. For every algebraically closed extension S/K, every framed point y, and every compatible framed point y_2 of the level cover above y, there are short values $t_c(y_2)$ with data $(w_c, \Lambda_c, N_c, I_K(x(c)))$ such that

$$s_x(y) = d_\pi(y_2) \sum_{c \in \mathcal{C}} \sigma_S\big(v_{\mathcal{C}} \gamma_c^{-1} e_c\big)\, t_c(y_2), \qquad \sigma_S : K \longrightarrow S.$$

The normalized comparison datum supplies the integral level cover, the finite-order determinant character and the compatible finite test, class representatives, coefficient map and lattice scalars. The section is built one class at a time. For one class c, the short value with data $(w_c, \Lambda_c, N_c, I_K(x(c)))$ exists and is unique at every framed point of the level cover over every commutative algebra over its coefficient field, and these values are compatible with base change. Every framed point over an algebraically closed field lifts to the level cover, and the product of $d_\pi(y_2)$ with the short value at y_2 is the same for any two lifts y_2 of the same framed point, also when the two lifts are taken along two admissible level-cover maps. Let S be a reduced finitely generated K-algebra and y a framed point over S. There are a finite, smooth, faithfully flat S-algebra S' and a lift y_2 of the base change of y to S'. By lift independence and compatibility with base change, the element $d_\pi(y_2)$ times the short value at y_2 of S' has the same image under any two geometric points of S' over one geometric point of S, so it comes from a unique $u \in S$. This u is the unique element of S whose image at each geometric point of S is $d_\pi(y_2)$ times the short value at y_2, for every lift y_2 of that point. These values form a section s_c of the prescribed weight over K, and $s_x = \sum_{c \in \mathcal{C}} v_{\mathcal{C}} \gamma_c^{-1} e_c\, s_c$ has the displayed property. The weight conditions and the unit condition of the datum make s_x cuspidal, and s_x is the only section of that weight with the displayed property.

Lean: `GlobalTheory.EquivariantLiftInvariance.cover_product_indep_of_lift` (`Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.CoverProductIndepOfLift`, lines 19–47)
Lean: `GlobalTheory.T.shortThetaValue_existsUnique_natural_of_laws` (`Development.AbelianVarietyTheory.Proofs.CuspFormEvaluations.ShortThetaValueExistsUniqueNaturalOfLawsStatement`, lines 12–13)

Lean: GlobalTheory.T.framedPoints_lift_to_levelCover (Development.AbelianVarietyTheory.Proofs.CuspFormEvaluations.FramedPointsLiftToLevelCoverStatement, lines 13–14)
Lean: GlobalTheory.ClsDescent.value_descends (Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.ExistsSectionOneClassSeriesDescends, lines 22–123)
Lean: GlobalTheory.ClsDescent.exists_sec_oneClass (Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.ExistsSectionOneClassSeriesDescends, lines 140–175)
Lean: GlobalTheory.T.thetaClassSum_dominant_weight_isCuspForm (Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.ThetaClassSumDominantWeightIsCuspFormStatement, lines 13–14)
Lean: GlobalTheory.ClassSumUnique.isThClsD_unique (Development.AbelianVarietyTheory.Proofs.CuspFormEvaluations.EqOfForallAlgClosed, lines 40–56)
Lean: Formalism.L2.Pj.IsThClsD (Formalism.LocalToGlobal.ProjectiveLattices.ThClass, lines 184–193)
Lean: GlobalTheory.ClassSumRule.exists_cusp_isThClsD_core (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsCuspFormClassSumNormalisedData, lines 17–48)

Thus each allowed rational input determines a member of $\mathcal{S}$. We will prove $C_K = \operatorname{span}_K \mathcal{S}$.

Use the readouts R and E of Lemma 37.23. Choose the canonical covariant derivatives on the theta line. Let $M = g_0 g_1$ be the odd base level, let $\mathfrak{m}$ be its chosen half-level ideal, and let L_W be the target lattice in the moduli datum. Put

$$Q = L_W/\mathfrak{m}L_W, \qquad Q^c = L_W/\overline{\mathfrak{m}}L_W.$$

The normalized level datum gives $M \geq 3$, auxiliary level parameter 1, and a fixed tuple $x_1 \in (Q^c)^n$ satisfying

$$\operatorname{span}_{\mathcal{O}_E}\{x_{1,b} : 0 \leq b < n\} = Q^c.$$

Its indicator $\mathbf{1}_{x_1}$ gives the prescribed finite reference test. If $\zeta_M \in K$ is the prescribed primitive M-th root and $\langle\ ,\ \rangle_M$ is the level pairing, summed over the n coordinates, define

$$w(q) = \sum_{x \in (Q^c)^n} \mathbf{1}_{x_1}(x)\zeta_M^{\langle x,q\rangle_M} = \zeta_M^{\langle x_1,q\rangle_M} \qquad (q \in Q^n).$$

There is no averaging factor in this definition of w. The normalized reference data also give an integer $B > 0$, divisible by 2 and M, containing the bad primes. In the standard definite source frame the Gram matrix is 1 and the lattice is $\Lambda_0 = \mathcal{O}_E^n$.

Here are the precise lattice conditions used below. With $\Lambda_0 = \mathcal{O}_E^n$, self-duality means that, for every $v \in E^n$,

$$v \in \Lambda \iff \sum_{b=0}^{n-1} \overline{v_b} w_b \in \mathcal{O}_E \quad \text{for every } w \in \Lambda.$$

For an integer $D \geq 0$, write $\mathsf{T}_D(\Lambda)$ for the condition

$$\exists k \in \mathbf{N}, \quad \gcd(k, D) = 1, \qquad k\Lambda \subseteq \Lambda_0, \qquad k\Lambda_0 \subseteq \Lambda.$$

Write $\mathsf{S}(\Lambda)$ for the condition that there is $u \in \mathcal{O}_E \setminus \{0\}$ such that

$$u\Lambda \subseteq \Lambda_0, \qquad u\Lambda_0 \subseteq \Lambda, \qquad \mathfrak{p} \neq \overline{\mathfrak{p}} \quad \text{for every prime ideal } \mathfrak{p} \ni u.$$

These are respectively the standard-at-D and split-support conditions. For the commensurable lattices occurring here and $N \geq 1$, the torsion-enumeration integer is

$$z_\Lambda(N, r) = [\Lambda : \Lambda \cap N\Lambda_0]^r.$$

For a framed point y over a commutative K-algebra S, let A_y be its group scheme. For $\Lambda \subset E^n$ and $N \in \mathbf{N}$, define $Z_{\Lambda,N}(A_y) \subseteq A_y(S)^n$ by the following two conditions on $z = (z_b)$:

$$[N^2]z_b = 0 \quad (0 \leq b < n),$$

$$\sum_{b=0}^{n-1} [c_{ab}]z_b = 0 \quad (0 \leq a < n)$$

for every $\mu \in M_n(E)$ satisfying $\mu\Lambda \subseteq \Lambda_0$ and every integral matrix $c \in M_n(\mathcal{O}_E)$ satisfying $c = N\mu$. Brackets in these formulas denote the given $\mathcal{O}_E$-action on A_y.

The intersection lattice and admissibility condition are

$$\Lambda_\cap = \Lambda \cap \Lambda_0, \qquad N \text{ odd}, \quad M \mid N, \quad N\Lambda_0 \subseteq \Lambda_\cap.$$

The intersection lattice is independent of N. For an admissible level put

$$m_\Lambda(N,r) = [\Lambda_\cap : N\Lambda_0]^r, \qquad z_\Lambda(N,r) = [\Lambda : N\Lambda_0]^r.$$

This agrees with the preceding formula for z_Λ.
Lean: Formalism.L2.Pj.GenLat.Ln (Formalism.LocalToGlobal.ProjectiveLattices.ThShort, lines 45–45)
Lean: Formalism.L2.Pj.GenLat.LevOK (Formalism.LocalToGlobal.ProjectiveLattices.ThShort, lines 50–51)
Lean: Formalism.L2.Pj.GenLat.mIdx (Formalism.LocalToGlobal.ProjectiveLattices.ThShort, lines 60–61)
Lean: Formalism.L2.Pj.GenLat.zCard (Formalism.LocalToGlobal.ProjectiveLattices.ThShort, lines 63–64)

We use the following explicit meaning of a short value v with data (w, Λ, N, P) at y. After a faithfully flat extension $S \to S'$ and the corresponding framed point y', choose tuples $z_i \in Z_{\Lambda,N}(A_{y'})$, indexed by $0 \le i < z_\Lambda(N,r)$. Require that, for every map $S' \to k$ to a field, their pullbacks remain pairwise distinct as tuples of points. Thus distinct indices remain distinct on every geometric fibre. For each $q \in Q^n$ choose points $t_{q,b}$ with

$$[N]t_{q,b} = \mathrm{lev}(q_b), \qquad \mathfrak{m}^{j_{q,b}} t_{q,b} = 0 \quad \text{for some } j_{q,b} \ge 0.$$

Pull the rigidified theta section back by $[N]$, using its normalized identification with the N^2-th power of the theta line, and act successively by normalized symmetric lifts of $z_{i,b}$ and $t_{q,b}$. Write F_{qib} for the resulting section of that power. The normalization of a symmetric lift $\theta = (x, \varphi)$, of exponent k and order parameter d, requires the product $\prod_{j=0}^{d-1} t_{jx}^* \varphi$ to be the unit generator under the tensor identifications. It also requires an invertible generator ν of $L^k \otimes [-1]^* L^{-k}$ such that

$$(t_{-x}^* \varphi)\, \nu\, ([-1]^* \varphi)\, (t_{-x}^* \nu)^{-1} = 1$$

under those identifications. Pullbacks from the abelian scheme to the family are understood in these two formulas. Here both lifts use $k = d = N^2$.
Lean: Formalism.L2.Pj.GeoInj (Formalism.LocalToGlobal.ProjectiveLattices.ThShort, lines 239–242)
Lean: ReciprocityTheory.ChartRestriction.G.Out.SymLift (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.ExtensionDescent.ThetaPt, lines 44–51)
Lean: Formalism.L2.Pj.IsUUOver (Formalism.LocalToGlobal.ProjectiveLattices.ThShort, lines 20–29)

If $m_b = (m(\tau, i, b))_{\tau,i}$ is the multi-index in column b, the required value identity is

$$\phi(v) = \frac{1}{m_\Lambda(N,r)|Q^n|N^{\deg P}} \sum_{q \in Q^n} \sigma_{S'}(w(q)) \sum_{i=0}^{z_\Lambda(N,r)-1} \sum_{m \in \mathrm{supp}\, P} \sigma_{S'}(P_m) \prod_{b=0}^{n-1} \nabla_{y'}^{m_b}(N^2, F_{qib}).$$

Here $\deg P$ is total degree, $\nabla_{y'}^{m_b}(N^2, F)$ denotes the chosen covariant derivative evaluated in the rigidified frame, and the integer factors are mapped to S'. These are the short values with lattice scalar parameter equal to one. The class factors in Claim 2 are outside this normalized value.
Lean: Formalism.L2.Pj.ShortValAt (Formalism.LocalToGlobal.ProjectiveLattices.ThShort, lines 244–261)
Lean: Formalism.L2.Pj.thLaValAt (Formalism.LocalToGlobal.ProjectiveLattices.ThClass, lines 17–19)

Claim 3. The algebraic short sections with the fixed generating tuple x_1 span the complex frame readouts of C_K. At the selected datum the target space is the row-layer Hermitian space W_{j+1} of odd rank $r = k + m - 1 \ge 1$ (since k and m are even); the auxiliary level parameter is 1; the component Gram matrices and imaginary scalar agree; the arithmetic groups contain the principal congruence groups of level M; the weight is dominant and n is even; and the corresponding arithmetic automorphy character is trivial. Moreover $[E : \mathbf{Q}] = 2f$, so

$$4[E : \mathbf{Q}]r + 4 = 8fr + 4 \le n.$$

Under $n \ge 4[E : \mathbf{Q}]r + 4 > 4r$, Hermitian theta series span bounded holomorphic cusp forms unconditionally via convergent Siegel–Weil, diagonal dominance of the lattice-overlap operator, and reproducing-kernel unfolding. Let $\Gamma_{\mathbf{Q}}(M)$ be the group of E-linear isometries of W_{j+1} lying in $\prod_{\ell \mid M} K_{M,\ell}$, the rational level-$M$ congruence subgroup preserving the level-M self-dual lattice family $\mathcal{L}$. First, since $n > 4r$, the convergent definite Hermitian Siegel–Weil identity equates the doubled level-M Siegel Eisenstein series with the genus-weighted sum of doubled theta series over unimodular self-dual Hermitian lattices Λ standard at B

and M. Every maximal isotropic subspace of the doubled space is either the graph Γ_γ of a unique unitary γ or contains the vector $(\xi, 0)$ for some nonzero isotropic $\xi \in E^r$. There is one constant $a \neq 0$ such that the doubled test function of any two test functions ϕ_1, ϕ_2 takes the value $a\langle\phi_1, \phi_2\rangle$ at zero, so the Eisenstein term of the graph Γ_γ is $a\langle\Phi_{L_1,\Lambda}, \omega(\gamma)\Phi_{L_2,\Lambda}\rangle$ times the Gaussian kernel at $(x, \gamma y)$ for real frames x, y, where ω is the Weil action. This pairing vanishes for every unitary $\gamma \notin \Gamma_{\mathbf{Q}}(M)$. On a non-graph subspace the Weil transform of the doubled test function vanishes at zero, because some finite combination of Heisenberg translates of $\Phi_{L_1,\Lambda}$ along the matrices $v\xi^{\mathsf{T}}$, with nonzero total coefficient, is zero. Summing the remaining graph terms over $\gamma \in \Gamma_{\mathbf{Q}}(M)$ yields the finite-rank reproducing-kernel identity expressing the $\Gamma_{\mathbf{Q}}(M)$-sum of the harmonic Gaussian-theta kernel $K(x, y)$, weighted by the genus-averaged lattice overlap matrix $C(L_1, \gamma L_2) = \sum_{[\Lambda]} w(\Lambda)\langle\Phi_{L_1,\Lambda}, \Phi_{\gamma L_2,\Lambda}\rangle$ and automorphy multiplier, as a^{-1} times the genus-weighted sum over $[\Lambda]$ of $\sum_j \theta_{P_j,\Phi_{L_1,\Lambda}}(x)\overline{\theta_{P_j,\Phi_{L_2,\Lambda}}(y)}$, where the second subscript specifies the finite test function.
Lean: `GlobalTheory.SW.kernel_identity` (`Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.KernelIdentity`, lines 181–341)

Second, for $n \geq 4[E : \mathbf{Q}]r + 4$ the overlap operator C is strictly diagonally dominant and hence invertible on bounded functions on $\mathcal{L}$: every off-diagonal self-dual lattice $L' \neq L$ in $\mathcal{L}$ has relative index $m = [L' : L]_{\mathrm{rel}} = [L : L \cap L'] = [L' : L \cap L'] \geq 2$, there are at most $m^{2[E:\mathbf{Q}]r}$ self-dual lattices of index m, and each overlap obeys $|C(L, L')| \leq C(L, L)\, m^{-n/2}$ (each level-M test function $\Phi_{L,\Lambda}$ is a sum of $|J|$ mutually orthogonal Heisenberg translates of the unit vacuum of the tensor lattice of Λ and L, indexed by the finite residue set J of the level datum, so $\langle\Phi_{L,\Lambda}, \Phi_{L,\Lambda}\rangle = |J|$, and $\langle\Phi_{L,\Lambda}, \Phi_{L',\Lambda}\rangle$ is $|J|$ times the inner product of two unit vacua of self-dual tensor lattices, which has modulus $m^{-n/2}$; thus $C(L, L) > 0$ does not depend on L), bounding the off-diagonal row sum by $\sum_{m\geq 2} m^{2[E:\mathbf{Q}]r - n/2} \leq \sum_{m\geq 2} m^{-2} < 1$, so a convergent Neumann series solves $F = CF'$ for the cocycle-extended bounded holomorphic cusp form F of $s \in C_K$. Since $C(L, L)$ is constant and, for $\gamma \in \Gamma_{\mathbf{Q}}(M)$, $C(\gamma L, \gamma L')$ equals $C(L, L')$ up to the unit automorphy multipliers, the solution F' is again holomorphic, of the same weight for the Borel action, of bounded Petersson norm, and $\Gamma_{\mathbf{Q}}(M)$-automorphic with the same multiplier. Third, K is the bidifferential operator of a unitarily invariant family of harmonic polynomials, not all zero, each row-highest for the pair of highest weights that, together with n, determines the Borel weight of the forms in C_K, applied at the origin to the doubled Gaussian kernel, and S_δ is the open set of frames whose Gram matrices lie entrywise within δ of the signature matrix. For all small $\delta > 0$, S_δ is stable under the unitary matrices, $K(x, \cdot)$ is bounded and integrable on S_δ for Lebesgue measure, and there is a real $\rho > 0$ with $\int_{S_\delta} K(x, y)G(y)\, dy = \rho G(x)$ for every x in the frame domain and every G holomorphic there, equivariant for the Borel action with this weight, and of bounded Petersson norm, so $\rho F'(L', x) = \int_{S_\delta} K(x, y)F'(L', y)\, dy$; choosing one representative L_0 from each of the finitely many $\Gamma_{\mathbf{Q}}(M)$-orbits in $\mathcal{L}$ and a measurable fundamental domain $D(L_0) \subset S_\delta$ for its discrete arithmetic stabiliser $\Delta_{L_0} = \mathrm{Stab}_{\Gamma_{\mathbf{Q}}(M)}(L_0)$, unfolding the absolutely convergent expansion $\rho F(L_1, x) = \sum_{L' \in \mathcal{L}} C(L_1, L') \int_{S_\delta} K(x, y)F'(L', y)\, dy$ (F' is bounded on S_δ, $K(x, \cdot)$ is integrable on S_δ, and the rows of C are absolutely summable) across all orbit representatives $[L_0]$ on $\mathcal{L} \times S_\delta$, and substituting the doubled kernel identity expresses $F = CF'$ as a finite $\mathbf{C}$-linear combination of harmonic-polynomial level-M Hermitian theta series for self-dual lattices standard at B and M. Clearing the lower-row congruence condition and expanding each harmonic polynomial in the integral harmonic basis expresses $R(s)$ as a finite $\mathbf{C}$-linear combination

$$R(s) = \sum_{\ell} b_\ell R(s_\ell),$$

where each s_ℓ is an algebraic section over K given on reduced, finitely generated K-algebras by a normalized short theta value with finite weight w, a self-dual lattice Λ_ℓ satisfying $\mathsf{T}_B(\Lambda_\ell)$ and $\mathsf{T}_M(\Lambda_\ell)$, an admissible odd level N_ℓ divisible by M, and a polynomial

$$P_\ell \in \mathscr{H}_a(K) \cap \mathscr{R}(\Lambda_\ell).$$

Using chart independence and algebraic base change, the identity cover over the rigidified moduli scheme realizes every normalized short-theta rule on all test rings, after scaling each generator by its nonzero lattice scalar. The unfolding initially permits any $\mathcal{O}_E$-spanning tuple x_0 of the conjugate half-level quotient Q^c. Given x_0 and the fixed spanning tuple x_1, choose a rational unitary matrix $\gamma \in \mathrm{U}_n(E)$, standard at $2B$ and M, transporting the indicator weight of x_0 to that of x_1; it sends Λ to the unimodular lattice $\gamma\Lambda$ (still standard at B and M, with an admissible odd level N') and pushes P forward to the column-rotated

harmonic polynomial $\gamma_* P \in \mathscr{H}_a(K) \cap \mathscr{R}(\gamma\Lambda)$. Comparing the two short-value rules over K gives one $z_0 \in K$ with the following property. For every commutative K-algebra S, every framed point y over S and every short value v at y with the finite weight of x_0 and data (Λ, N, P), there is a short value v' at y with the finite weight of x_1 and data $(\gamma\Lambda, N', \gamma_* P)$ such that $v = z_0 v'$. If $z_0 = 0$, the x_0-section is zero. Otherwise the x_0-section s equals $z_0 \cdot z_0^{-1} s$, and $z_0^{-1} s$ is a section with the x_1-data. In both cases every x_0-section lies in the K-span of the sections for x_1. Applying the σ-semilinear readout gives the displayed complex spanning relation.

Lean: `GlobalTheory.SW.kern_reproducing` (`Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.SetIntegralCompOfStable`, lines 268–344)
Lean: `GlobalTheory.Dict.holSub0_le_span_shortGen` (`Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.CuspFormsVanishingOffFramesLeSpanShortTheta`, lines 53–136)
Lean: `GlobalTheory.RawSpan.holSub0_le_span_raw` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.HolSub0LeSpanRaw`, lines 146–176)
Lean: `GlobalTheory.ClassSumSpan.rdU_mem_span_raw` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ClassSumsSpan`, lines 199–235)

For a finite adelic unitary element h, let

$$L_h = \operatorname{span}_{\mathcal{O}_E}\{x \in E^n : h_f^{-1} x \in \widehat{\Lambda}_0\}, \qquad \overline{L_h} = \{x \in E^n : \bar{x} \in L_h\}.$$

Here h_f is the finite part and $\widehat{\Lambda}_0$ the finite adelic completion of Λ_0.

Claim 4. Let E'/F' be a CM quadratic extension with its complex conjugation, let $d \geq 2$, and let $J \in M_d(E')$ satisfy $\bar{J}^t = J$ and $\det J \neq 0$. Write U_J for its unitary group. Choose finite sets S_∞ and S_f of infinite and finite places of F', respectively, write $S = S_\infty \cup S_f$, and let $\varepsilon \in \mathbb{A}_{F'}$ have component 1 at those places and 0 elsewhere. Put

$$U^1_{J,S} = \{u \in U_J(\mathbb{A}_{F'}) : \det u = 1,\ \varepsilon(u-1) = u - 1\}.$$

If $U^1_{J,S}$ is not compact, then every $g \in U_J(\mathbb{A}_{F'})$ with $\det g = 1$ belongs to

$$\overline{\{\iota_{\mathbb{A}}(\delta) u : \delta \in U_J(F'),\ \det \delta = 1,\ u \in U^1_{J,S}\}},$$

where $\iota_{\mathbb{A}}$ is the diagonal embedding and the closure is taken in the adelic unitary group. The determinant-one requirements are part of both the input and the approximating subgroup. Together with the continuous norm-one idèle section into $U_J(\mathbb{A}_{F'})$ and vanishing of norm-one characters invariant at almost all split places, this determinant-one strong approximation implies that every continuous left-$U_J(F')$-invariant function on $U_J(\mathbb{A}_{F'})$ invariant on the right by the local unitary groups at almost all split places is constant, which supplies the automorphic invariance of the split-place theta functional used in the lattice-scalar normalization of Claim 8.

Lean: `GlobalTheory.StrongApproxSUProof.strongApproxSU` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ImportsTailTer`, lines 14–21)
Lean: `GlobalTheory.HermConstLift.constant_of_invariant` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.AutomorphicInvariantSplitPlacesConst`, lines 32–89)

Claim 5. For a chosen short section, retain its full selected lattice, level and polynomial data, including the lattice-polynomial condition. For its lattice Λ_ℓ there are a matrix $\gamma_\ell \in M_n(E)$ and an adelic unitary element h_ℓ such that

$$\overline{\gamma_\ell}^t \gamma_\ell = 1, \qquad (h_\ell)_\infty = 1, \qquad \gamma_\ell \Lambda_\ell = \overline{L_{h_\ell}}.$$

The element h_ℓ is the identity outside finitely many split places away from BM. For each column b there is an integer $k_b \geq 1$, coprime to BM, such that

$$k_b\big((\gamma_\ell)_{ab} - \delta_{ab}\big) \in BM\mathcal{O}_E \qquad (0 \leq a < n).$$

For the same finite weight w, there are a level N'_ℓ, a polynomial P'_ℓ of the same harmonic weight satisfying the new lattice conditions, and $z_\ell \in K$. The lattice $\overline{L_{h_\ell}}$ has admissible level N'_ℓ, and L_{h_ℓ} is self-dual, standard at the primes dividing B, and supported at split places. For every commutative K-algebra S, every framed point y over S, and every short value v at y with the original data, there is a short value v' at y with data $(w, \overline{L_{h_\ell}}, N'_\ell, P'_\ell)$ such that

$$v = \sigma_S(z_\ell) v', \qquad \sigma_S : K \to S.$$

Let V be the standard definite source. Its Gram matrix is $\mathrm{Gram}(V) = 1$, and its lattice at the identity is $L_1 = \Lambda_0$. Put $N_0 = BM > 0$. The selected lattice is self-dual and standard at B; since $M \mid B$, a common denominator coprime to B is also coprime to BM, so it is standard at N_0. These are precisely the lattice hypotheses needed to obtain the displayed unitary move. Let ι_f denote the finite diagonal embedding, applied entrywise to vectors and matrices. The construction starts with a finite adelic unitary matrix $g_{\mathrm{ad}} \in \mathrm{U}(V)(\mathbb{A}_{F,f})$ such that

$$\overline{\Lambda_\ell} = \{x \in E^n : \iota_f(x) \in g_{\mathrm{ad}}\widehat{\Lambda}_0\},$$

with a denominator coprime to N_0 making g_{ad} congruent to 1 modulo N_0 on $\widehat{\Lambda}_0$. Choose positive integers d_1, d_2 clearing denominators of g_{ad} and g_{ad}^{-1} on $\widehat{\Lambda}_0$, and put $N' = N_0 d_1 d_2$. Applying weak approximation for the full unitary group $\mathrm{U}(V)$ at the finitely many primes dividing N' (with no determinant-one restriction) yields $\gamma_0 \in \mathrm{U}(V)(F)$ such that $\iota_f(\gamma_0)g_{\mathrm{ad}}$ is congruent to 1 modulo N' up to a denominator coprime to N'. Next, a local congruence-1 rational unitary correction produces $\gamma \in \mathrm{U}(V)(F)$ congruent to γ_0 modulo N' whose denominators away from N' are supported solely at split places of E/F; putting $A = \iota_f(\gamma)g_{\mathrm{ad}}$, both A and A^{-1} are congruent to 1 modulo N_0 up to integers coprime to N_0, and there is a nonzero $s \in \mathcal{O}_E$ divisible only by split primes away from N_0 such that sA and sA^{-1} preserve $\widehat{\Lambda}_0$. Taking a nonzero multiple $m \in \mathcal{O}_F$ of s supported at split places of F away from N_0, let $e \in \mathbb{A}_{F,f}$ be the indicator idempotent of the finitely many places of F dividing m, and put $\varepsilon = e \otimes 1 \in \mathbb{A}_{F,f} \otimes_F E$. Because ε and $1 - \varepsilon$ are orthogonal central idempotents fixed by complex conjugation, setting

$$h_0 = \varepsilon A + (1 - \varepsilon), \qquad k = (1 - \varepsilon)A + \varepsilon, \qquad \delta = \gamma^{-1} \in \mathrm{U}(V)(F)$$

gives unitary matrices with $h_0 k = A$, hence

$$g_{\mathrm{ad}} = \iota_f(\delta)h_0 k, \qquad \iota_f(\delta) = g_{\mathrm{ad}}k^{-1}h_0^{-1},$$

where h_0 is supported solely at finitely many split places away from N_0, and k stabilizes $\widehat{\Lambda}_0$ and acts trivially on $\widehat{\Lambda}_0/N_0\widehat{\Lambda}_0$ (since m is a unit at all places where $1 - \varepsilon = 1$). The lattice calculation gives $\delta L_{h_0} = \overline{\Lambda_\ell}$, so the matrix in Claim 5 is $\gamma_\ell = \delta^t$. It satisfies

$$\overline{\gamma_\ell}^t \gamma_\ell = 1, \qquad \overline{\det \gamma_\ell} \det \gamma_\ell = 1.$$

For each $a \in \{g_{\mathrm{ad}}, k^{-1}, h_0^{-1}\}$, the construction supplies an integer $t_a \in \mathbb{N}$, coprime to N_0, such that

$$t_a(a - 1)\widehat{\Lambda}_0 \subseteq N_0\widehat{\Lambda}_0.$$

Multiplication and passage to rational entries give the required column congruences for γ_ℓ. The finite matrix h_0 lifts to h_ℓ with infinite component 1. Since $BM > 1$, the integers coprime to BM in these column congruences are positive. This applies the split-place lattice assertion with modulus BM.

Lean: `GlobalTheory.UnRep.LatForm.exists_rep_split_denominator` (`Development.AbelianVarietyTheory.Proofs.AdelicLatticeProofs.ExistsRationalUnitarySplitDenominator`, lines 178–246)
Lean: `GlobalTheory.UnRep.exists_rep_suppSplit` (`Development.AbelianVarietyTheory.Proofs.AdelicLatticeProofs.ExistsUnitaryAdelicDecompositionSplitSupport`, lines 191–319)
Lean: `GlobalTheory.SplitMove.exists_move_matrix` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsRationalUnitaryMoveSelfDualToSplit`, lines 67–154)
Lean: `GlobalTheory.ClassSumSpan.exists_split_move` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ClassSumsSpan`, lines 76–115)

The congruence makes the induced map on Q^n the identity, so the finite weight remains w. The unitary change preserves the harmonic weight and the lattice conditions. The comparison of the two natural short-value rules gives a scalar in K; it does not require that scalar to be nonzero. For each raw section with these full selected conditions, its short-value identity on the test algebras also holds at every framed point over any field extension of K. Evaluating at the complex comparison frames gives functions $t_\ell(g)$ satisfying

$$E(s_\ell)(g) = \sigma(z_\ell)t_\ell(g).$$

Fix one of these terms and write h, P', t for its data. The polynomial coefficient map I_K has image the prescribed harmonic highest-weight module, so $P' = I_K(v)$ for some $v \in \mathscr{V}_K$. Write $c_0 : \mathscr{V}_K \to \mathscr{V}_{\mathbf{C}}$ for the

σ-semilinear coefficient comparison. Let $\mathcal{I}$ be the complex Fock coefficient pairing, with inputs the lowest-type tensor and a vector of $\mathscr{V}_{\mathbf{C}}$. Let $\mathcal{P}_{t'}$ be polynomial insertion into the oscillator model for tuple t' using the selected frames, and $\mathcal{P}^0_{t'}$ the insertion using the standard source frame and the same target frame and vacuum. Thus polynomial insertion means application of the corresponding polynomial in the chosen Fock creation operators to the vacuum. Let V be the definite source and W the target of the fixed comparison, and let $\mathcal{X}$ be the matrix-coordinate space of its oscillator model, with zero matrix 0. The chosen action at tuple t' is a homomorphism

$$\omega_{t'} : \mathrm{U}(V)(\mathbb{A}_F) \times \mathrm{U}(W)(\mathbb{A}_F) \longrightarrow \mathrm{End}_{\mathbf{C}}(\mathbf{C}^{\mathcal{X}}).$$

For $f : \mathcal{X} \to \mathbf{C}$, define its kernel by evaluation at zero:

$$\theta_{\omega_{t'}}(h, g; f) = \big(\omega_{t'}(h, g)f\big)(0).$$

With the coefficient test and lowest-type vectors fixed above, this gives

$$\Phi_{t',v} = \mathcal{P}_{t'}\left(\mathcal{I}\left(\bigotimes_{\rho} u_\rho, c_0(v)\right)\right), \qquad \mathcal{K}_{h,v}(g) = \big(\omega_{\mu_0}(h, g)\Phi_{\mu_0,v}\big)(0).$$

Claim 6. There is one $\eta \in \mathbf{C}$, $|\eta| = 1$, such that the following holds for every tuple t' and every $v \in \mathscr{V}_K$. Let λ match the algebraic polynomial letters to Fock letters, preserving the source index b. For an algebraic letter (τ, a, b) and its Fock image (ρ, c, b), require either

$$\tau = \tau_\rho,\ c = a, \qquad \text{or} \qquad \tau = \overline{\tau_\rho},\ c = p + a,$$

where (p, q) is the target signature and τ_ρ is the selected embedding at ρ. Write λ_* for the resulting substitution on polynomials and σ_* for applying σ to coefficients. For every such matching, and for the chosen lowest-type vectors, one has

$$\Phi_{t',v} = \eta\, \mathcal{P}^0_{t'}\big(\lambda_*\sigma_* I_K(v)\big).$$

Lean: `GlobalTheory.T.fockCoefficientMap_extremal_eq_phase_smul_letters` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.FockCoefficientMapExtremalEqPhaseSmulLettersStatement`, lines 14–15)

Let $c(k)$ be the provisional unit component constants, and let $\mu(g)$ be the unitary comparison multiplier, so that $J_j(s)(g) = \mu(g)E(s)(g)$. Put

$$d_a = \sum_{\tau,l} a_{\tau,l}, \qquad a_j = (2\pi\xi)^{-d_a/2}, \qquad A(g) = c(\alpha(g))a_j\mu(g).$$

Here $a_{\tau,l}$ are the section's weight exponents. Keep the phase η from Claim 6. There is a sign $\delta \in \{1, -1\}$, independent of h, v, and the evaluation point, for which Claim 7 below holds. Define

$$C_{\mathrm{cmp}} = \eta\delta^{d_a}(-1)^{d_a}\big(\sqrt{2\pi\xi}\big)^{-d_a}.$$

Let e_{μ_0} denote the K-valued function on source lattices included in the normalized reference comparison datum, and set $\epsilon_h = e_{\mu_0}(L_h)$. Its value at each representative lattice is the class factor e_c in Claim 2.

Claim 7. Use the standard source space, the fixed reference tuple and finite test, the canonical derivatives, and the normalized comparison data above. Let h be trivial at infinity and supported at split places away from B, let $\Lambda = \overline{L_h}$, and suppose L_h is self-dual and standard at the primes dividing B. Give Λ an admissible level N. If $t(g)$ is its short value with polynomial $I_K(v)$ at the comparison frame of g, then

$$\mathcal{K}_{h,v}(g) = C_{\mathrm{cmp}}\, c(\alpha(g))\mu(g)\sigma(\epsilon_h)t(g) = b_h A(g)t(g), \qquad b_h = C_{\mathrm{cmp}}a_j^{-1}\sigma(\epsilon_h).$$

Lean: `GlobalTheory.ClassSumSpanCore.kernel_eq_shortVal` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ClassSumsSpanCuspFormsBaseTupleSecondField`, lines 169–264)

Claim 8. Keep the full normalized reference comparison data, including its lattice-scalar normalization and its ideal-scalar compatibility. Let

$$\mathcal{B}_{\mu_0} : \{\text{ideals of } \mathcal{O}_E\} \longrightarrow K$$

be its ideal-scalar function, and retain its lattice-scalar function e_{μ_0}. For an $\mathcal{O}_E$-lattice $L \subset E^n$, define its determinant lattice by

$$\mathfrak{d}(L) = \mathrm{span}_{\mathcal{O}_E}\{\det(v_1, \dots, v_n) : v_1, \dots, v_n \in L\} \subset E.$$

For every self-dual L that is standard at B and supported at split places, and every integral ideal $\mathfrak{c}$ satisfying

$$\mathfrak{c} + \bar{\mathfrak{c}} = \mathcal{O}_E, \qquad \mathfrak{d}(L)\bar{\mathfrak{c}} = \mathfrak{c}$$

(the second equality being taken in the $\mathcal{O}_E$-submodules of E), the normalization gives a rational number $q_{L,\mathfrak{c}} > 0$ such that

$$q_{L,\mathfrak{c}}^2[L : L \cap \Lambda_0]^r = \mathrm{N}\mathfrak{c}, \qquad e_{\mu_0}(L) = \mathcal{B}_{\mu_0}(\mathfrak{c})^{-1} q_{L,\mathfrak{c}}.$$

Here $r = k + m - 1$ is the odd rank of the target row-layer space W_{j+1}, $\mathrm{N}\mathfrak{c} = |\mathcal{O}_E/\mathfrak{c}|$, and $q_{L,\mathfrak{c}}$ is viewed in K in the second identity. Because r is odd (so $r - 1$ is even) and the lattice index $[L : L \cap \Lambda_0]$ agrees with $\mathrm{N}\mathfrak{c}$ up to a rational square, the quotient $\mathrm{N}\mathfrak{c}/[L : L \cap \Lambda_0]^r$ is always a positive rational square, ensuring that $q_{L,\mathfrak{c}} \in \mathbf{Q}_{>0}$ exists and is well defined. The same normalized data satisfy

$$\mathcal{B}_{\mu_0}(\mathfrak{c}) \neq 0 \quad \text{for every } \mathfrak{c} \neq 0 \text{ with } \mathfrak{c} + (B) = \mathcal{O}_E.$$

For each $h = h_\ell$ supplied by Claim 5, the selected split-support witnesses include an idempotent $e \in \mathbb{A}_F \otimes_F E$, a finite prime set away from B, and a unit $y \in (\mathbb{A}_F \otimes_F E)^\times$. In the standard source frame their determinant condition is exactly

$$y = e \det(h) + (1 - e).$$

Thus y records $\det(h)$ on the component selected by e and 1 on its complement. The split-support construction gives an ideal $\mathfrak{c} \neq 0$, coprime to (B), satisfying the two displayed ideal identities for $L = L_h$. The three lattice conditions hold for L_h by Claim 5. Applying the normalization therefore gives

$$\epsilon_h = e_{\mu_0}(L_h) = \mathcal{B}_{\mu_0}(\mathfrak{c})^{-1} q_{L_h,\mathfrak{c}} \neq 0.$$

Lean: `GlobalTheory.ClassSumSpan.ep_ne_zero` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ClassSumsSpan`, lines 117–134)

Also $a_j > 0$, $|\eta| = |c(k)| = |\mu(g)| = 1$. Therefore $b_h \neq 0$ and $A(g) \neq 0$ for every g.

Claim 9. The function $\mathcal{K}_{h,v}$ belongs to the complex span of the theta lifts of the rational definite automorphic inputs at μ_0, with the same lowest-type tensor. To see the reduction, choose finitely many source double-coset representatives. The finite theta test transforms under the source level group by the inverse prescribed character. Interpolation on these finitely many classes produces a definite automorphic form whose theta lift is proportional to $\mathcal{K}_{h,v}$, with nonzero proportionality constant. The rational class functions span the complex definite automorphic realization, and the theta lift is complex linear. This proves the asserted span.

Lean: `GlobalTheory.KernelSpanCore.kernel_mem_span_core` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ThetaKernelFinitePointMemSpanRationalFormLifts`, lines 22–57)

For each rational input x in this span, the class presentation supplies an allowed class-coefficient function and hence $s_x \in \mathcal{S}$. The reference class identity has the form

$$\Theta(x)(g) = uA(g)E(s_x)(g), \qquad |u| = 1,$$

with one u independent of x and g. Therefore, if $\mathcal{K}_{h,v} = \sum_\nu q_\nu \Theta(x_\nu)$, $q_\nu \in \mathbf{C}$, cancellation of the nonzero function A gives

$$t = \frac{u}{b_h} \sum_\nu q_\nu E(s_{x_\nu}) \in \mathrm{span}_{\mathbf{C}}\, E(\mathcal{S}).$$

Multiplying by $\sigma(z_\ell)$ and using the raw spanning relation now gives $E(s) \in \mathrm{span}_{\mathbf{C}}\, E(\mathcal{S})$.

All spans in the preceding analytic comparison are over $\mathbf{C}$. For the descent, E is σ-semilinear on algebraic section rules, $\mathcal{S} \subseteq C_K$, and Lemma 37.23 preserves every finite independent family from K to $\mathbf{C}$. Lemma 37.22 consequently gives

$$s \in \mathrm{span}_K\, \mathcal{S}, \qquad K = \iota(K_{\mathrm{c}}).$$

This proves $C_K = \mathrm{span}_K\, \mathcal{S}$, the spanning assertion of the proposition. The expansion (37.32) is the reverse relation. The normalized class-sum construction above places each reference theta class in C_K, and

transporting its coefficients through $\iota : K_c \simeq K$ gives coefficients in K_c. The factor a_j is the factor in extremal-vector evaluation and in the core Petersson norm. The coefficient rules for the sections commute with scalar extension on common charts, and their level action contributes precisely the product of the designated determinant characters.

Finally the reference phase normalization replaces the provisional component constants and phase by the prescribed layer-$j+1$ component multiplier and ϕ_j. The selected tuple characters already have the prescribed values at the used primes and the prescribed formulas away from their twist supports. Under the stated prime compatibility condition, the normalized comparison data can be constructed for all tuples over one field containing K. The class identities hold for every normalized family, and the reference class sums span C_K; thus the reference comparison extends to this family. The resulting comparison has one central level element and one phase for all tuples, the stated tuple roots of unity, and the same scalar-extension and level identities.

Lean: `GlobalTheory.ClassSumSpanCore.classSumsSpan_of_pins_core` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ClassSumsSpanCuspFormsBaseTupleSecondField`, lines 284–351)
Lean: `GlobalTheory.OneMapCore.oneMapRowR_of_core` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.OneMapRowAllTuplesOfCoreClauses`, lines 22–30)

The complete proof is `GlobalTheory.Fc.R1.row_hRB_univ_alt`. □

Proof of 37.25

Proof. The endomorphism ring contains the maximal order $\mathbf{Z}[\omega]$. The CM invariant J is integral. Global reciprocity identifies the ideal-class quotient with the Galois group of the Hilbert class field, and the class invariants of the maximal order belong to this field [202, §11, Theorem 11.1]. In particular $J \in K^{\mathrm{ur}}$ at $p \nmid d$. In terms of Frobenius, the argument used here is to take the finite quotient recording both the ideal class and the permutation of the class invariants. At an inertia element the class coordinate is trivial. By Lemma 36.24 one can represent its conjugacy class by degree-one Frobenius avoiding the discriminant and the differences of distinct invariants. The CM ideal-isogeny congruence at this prime acts by a principal ideal, hence fixes the reduced invariants. Their reductions are distinct, so the permutation is trivial.

Normalize $v_p(p) = 1$. We claim

$$3 \mid v_p(J) \quad (J \neq 0), \qquad 2 \mid v_p(J - 1728) \quad (J \neq 1728). \tag{A.582}$$

For $J \neq 0, 1728$ start with $g_2 = g_3 = c = 27J/(J - 1728)$. Scale by μ^4, μ^6 to discriminant one. If $v_p(J) > 0$, then $12v_p(\mu) = -2v_p(J)$, and the reduction has $g_2 = 0,\ g_3 \neq 0$. Choose a cyclic CM kernel of odd order $q \equiv 2 \bmod 3$, prime to p. For example, for $\mathbf{Q}(\sqrt{-d}) \neq \mathbf{Q}(\sqrt{-3})$, choose q split in the former field and inert in the latter, avoiding $2, 3, p, d$, by Lemma 36.24. Take the kernel of a prime ideal over q. In the excluded field the maximal-order invariant is 0. The chosen kernel is defined over K^{ur} on the model with invariants c, c: this field contains the local CM field and the endomorphisms are determined by their action on the invariant differential. Its nonzero points form $t = (q-1)/2$ opposite pairs, with $3 \nmid t$. Their abscissas on the scaled model are integral units. Indeed prime-to-p torsion reduces injectively, and a point of abscissa zero on the reduction has order three by the tangent law. The product of the unscaled abscissas is Galois invariant. Its valuation $-2tv_p(\mu) = tv_p(J)/3$ is an integer.

If $v_p(J-1728) > 0$, then $4v_p(\mu) = v_p(J-1728)$, and the reduction has $g_3 = 0,\ g_2 \neq 0$. Take the kernel of $\sqrt{-d}$. Its order d is odd and prime to p, and $t = (d-1)/2$ is odd. On this reduction a point of abscissa zero has order two. The same product gives $-tv_p(J-1728)/2 \in \mathbf{Z}$. This proves (A.582), including the exceptional invariants since 1728 is a unit. Hensel lifting of the unit parts gives $G_2^3 = J/1728,\ 27G_3^2 = J/1728 - 1$ in W, using $2, 3 \in W^\times$ and the algebraically closed residue field. These invariants have the required j and discriminant.

Lean: `ReciprocityTheory.ChartRestriction.L1.valDiv3` (`ReciprocityTheory.ReductiveGroups.UnramifiedIntegrals.NormWeierstrassCoeffSmallJ`, lines 98–185)
Lean: `ReciprocityTheory.ChartRestriction.L1.valDiv2` (`ReciprocityTheory.ReductiveGroups.UnramifiedIntegrals.Norm1728EqOneOfFiveLe`, lines 45–148)

Now suppose p inert. Let σ be Witt Frobenius and $\tau f = \sigma^2(f)(T^{p^2})$. The CM endomorphism $f^{-1}(\omega f)$ is integral on the good model. One can obtain this directly by extending the endomorphism to the good elliptic scheme and completing at zero. Its tangent and its conjugate differ by a unit. Define c_j, in increasing order, by cancelling degree p^{2j+2} in $(p - \sum_{i=0}^{j} c_i \tau^{i+1})f$. Each $\tau^{j+1} f$ has leading coefficient one. We show that all $c_j \in W$ and that all remaining coefficients are in pW.

Use induction in the degree n, retaining the terms with $p^{2i+2} < n$, and let a_n be the current coefficient. The invariant differential gives $k[T^k]f \in W$. Consequently substituting two integral zero-constant series congruent modulo p in any Frobenius twist of f gives results congruent modulo p: in the binomial difference the term with $r \geq 1$ factors of p is divisible by p^r/r. Apply this to the formal group law and the CM endomorphism after raising to p^{2i+2}. Thus the additive defect and the ω-linearity defect of every retained twist lie in pW. Since the lower coefficients of the remainder lie in pW, its additive defect gives $a_n\binom{n}{k} \in pW$ for $0 < k < n$. Outside powers of p a middle binomial coefficient is a unit. At a power of p one has valuation one, so $a_n \in W$. At an odd power p^{2i+1} the second defect gives $(\omega^n - \omega)a_n \in pW$, hence $a_n \in pW$. At the even powers take $c_i = a_n$. The constant and linear terms satisfy the assertion directly.

If $c_0 \in pW$, all coefficients of pf through degree p^2 belong to pW. From $pf = f([p](T))$ the same follows successively for $[p](T)$, since its lower terms are then multiples of p and $k[T^k]f \in W$. The special-fibre multiplication-by-p map would have local degree greater than its degree p^2. Thus $c_0 \in W^\times$.

Multiply the type $p - \sum c_i\tau^{i+1}$ on the left by $\sum t_i\tau^i$, $t_0 = 1$. To cancel the coefficients of τ^{j+1} for $j \geq 1$, iterate the map on sequences given successively by

$$t'_0 = 1, \qquad t'_j = \frac{p\,t_{j+1} - \sum_{i<j} t'_i\,\sigma^{2i}(c_{j-i})}{\sigma^{2j}(c_0)}.$$

The denominators are units. Induction on j shows that agreement of two inputs modulo p^k gives agreement of their outputs modulo p^{k+1}. Completeness gives a fixed point, reducing the type to $p - a\tau$, $a = c_0 - pt_1 \in W^\times$. Solve $\epsilon = a\sigma^2(\epsilon)$ in $W^\times$: solve first in the nonzero residue classes, then lift by solving an additive equation $z - z^{p^2} = b$ in $\overline{\mathbf{F}}_p$ at each precision. Both f and $\epsilon\ell$ now solve the same functional equation with integral inhomogeneous terms, since $(1 - a\tau/p)f$ is integral and $(1 - a\tau/p)(\epsilon\ell) = \epsilon T$. The functional-equation lemma for W, its Frobenius square and the ideal (p) gives $f^{-1}(\epsilon\ell) \in W[[T]]$ [58, Lemma 2.3]. Its tangent is ϵ, so its inverse is integral as well.

Lean: `GlobalTheory.Honda.exists_even_type` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.ExistsHondaTypeEvenFrobeniusPowers`, lines 259–395)
Lean: `GlobalTheory.Honda.OneTerm.exists_one_term` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.HondaTypeReductionToOneTerm`, lines 109–127)
Lean: `GlobalTheory.TwV.Lt.exists_genusOne_formal_iso_of_unramified_good_reduction` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.ExistsGenusOneFormalIsoOfUnramifiedGoodReduction`, lines 23–24)

□

Proof of 37.26

Proof. Use global reciprocity in the form of Proposition 1.22 and Definition 1.20. The idèle map onto the abelianized Galois group has kernel the closure of the principal idèles times the connected archimedean subgroup. Local units map onto the inertia image, and a uniformizer acts by inverse arithmetic Frobenius modulo inertia. Passage to the ideal-class quotient gives the reciprocity input of Lemma 37.25. Thus it supplies the good CM model, and that lemma supplies the inert formal parameter. For the extension parameters use Theorem 40.1 at each specified finite place and prolongation, with the correspondence of Definition 40.2. Its rank-one normalization is Theorem 40.5. In this proposition the correspondence enters through the norm choices of the extension in (33.1). Given that extension, compatibility of the bundle identifies its underlying point and its reciprocity datum with those of a constructed point. Substituting this identification identifies the period lattice, determinant classes and selected column kernel with their standard arithmetic versions.

Lean: `CompanionTheory.FamInh.globalArtin` (`CompanionTheory.AutomorphicTheory.PlaceSeparation.IsOpenQuotientMapArt0`, lines 233–234)
Lean: `UpperTheory.Zero.LLC.recTExists` (`Development.AutomorphicArithmetic.Foundations.LocalLanglands.LocalCentralUnits.RecTExists`, lines 20–66)
Lean: `GlobalTheory.Svd.LinkStd.stdTie_of_eq_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LinkRStdTie`, lines 15–19)

For the first conclusion fix a family and then a member. Mark the column representatives as in the proof of Proposition 37.31. The only reading changed by this operation is the choice of representative of the finite class datum. The equalities of its two endpoint lattices and of the resulting theta functions with the specified column readings are part of the construction in that proof. For a used twist and a permitted (u, w) outside its exceptional primes take the levels and the chain obtained there. For the other indices take the original levels

and chain. At the reference twist the level-zero identification agrees with the original one. Once $x \geq 71/\varepsilon$, $2r_G \leq k$: otherwise the lower rank bound gives $x \leq k < 2r_G$, while $r_G \leq 8(4+8\varepsilon)/\varepsilon + 2$. This gives $x\varepsilon \leq 64 + 132\varepsilon < 71$ since $\varepsilon < 1/52$. The stabilizer at zero is trivial. Indeed forgetting the additional twist frames identifies its rational component with an element of the reference initial stabilizer, which is trivial by the initial level condition. With that component fixed, the frame and the empty initial index determine the remaining component. Proposition 37.31 therefore supplies both admissibility and comparison on the chosen chain.

Lean: `GlobalTheory.Svd.jointsAt_all_std` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.JointsAtAllStd`, lines 11–28)
Lean: `GlobalTheory.TwV.modelsOK_of_joints` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.ThetaLocalModelsOfSplitUnramifiedJoints`, lines 11–28)
Lean: `GlobalTheory.TwU.two_rG_le_k_large` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.TwoRGLeKOfSize`, lines 38–46)

We verify the size conditions (37.35) of Lemma 37.28 on this chain, with $a_* = xb$. One may choose a threshold depending on ε before the family: take $x \geq 4$, $x^{\varepsilon/5} \geq 3$, and $x^6 \leq \exp(x^{\varepsilon/3})$ throughout the remaining half-line. The scale bounds then give $kN_m \leq k\mathcal{D}$, $4k^2\mathcal{D} \leq w''$, $m(t+1)\eta \leq w''$ and $1 \leq w'' \leq w_z$. Sum the long rows from the weight windows and the appended rows, each in $[0, \eta]$. Off the top this gives the interval about $(k-1)w_z$ in (37.35). On the top remove the first row before summing and use its prescribed length $a_t + w_z - \kappa_{j+1}$, with $\kappa_{j+1} = (k + (2j+1)(t+1))/2$. This gives the interval about $(k-3)w_z$. The top weight window and $x \geq 4$ give $r_{z,0} \geq 2w_z$; subtracting the second-row lower bound gives $r_{z,0} - r_{z,1} \leq 2a_*$. The two cap bounds use that same second row on the top and the first row elsewhere. The filling lower gap is the one in Definition 14.2. These estimates use the diagram of the member and therefore apply to every entry realization at the point. The split-prime threshold is at least eleven by the size conditions of Definition 14.6.

Lean: `GlobalTheory.ShapeUniform.shape_uniform` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.ThreeLeSplitAt`, lines 389–431)

For an unrepresented position of the filling the orbit lies on the second sheet. At $p \leq X''$ take the lower exponent equal to the orbit base term less its error, as defined in Theorem 16.21. The orbit has even length at least two, $p \geq 5$, and $h_P \neq 0$. The position identification respects the cyclic order and the row lengths. Apply the refined cyclic estimate of Proposition 15.5 with $C_6 = 11, C = 1$ to the integral cofactors. Its weight hypotheses are (37.35); the fixed constants satisfy $11 \leq 12$ and $3 \leq \min(1 + K_{\delta,1}, 3)$. On a second-sheet orbit there is no first-sheet marked gain. Summing one term per complementary orbit gives the required lower bound as in Proposition 16.12(2). For $p > X''$ take zero by the one-place bound in Proposition 37.31. Its denominator exponents sum to $(m-1)v_p(D_{\Gamma,u})$. Together with the comparisons this gives (37.33).

Lean: `GlobalTheory.TwU.LowF.lowAt_F` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.LocalModelBoundPlaceOverSecondEmbedding`, lines 66–95)
Lean: `GlobalTheory.Proof.UnitaryWeightModels.rowTwU_of_models` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.UnitaryWeightModels`, lines 15–31)

For the second conclusion take the entry of the selected bundle. At a level-free place the determinant of a class basis measures exactly the local index of that lattice in the reference lattice. Evaluating the type of the step character on this determinant cancels the corresponding basis determinant powers in the class scalar; its finite-order factor has norm one. This gives the unit assertion for every tuple. Let K_0 be the prescribed coefficient field, with its fixed complex embedding. The constructed column supplies the following coefficient-field condition. For every allowed source and target lattice and torsion characteristic, each normalized Taylor coefficient

$$\chi_{\mathrm{lat}}^{-1}\, \mathrm{jet}_{\mathbf{d}}(\Theta)_\alpha$$

of block degree $\sum_a \nu_{j+1,\tau,a}$ at each embedding τ lies in the image of K_0. Here χ_{lat} is the lattice scalar and $\mathbf{d}$ is the chosen family of invariant derivation generators. The condition includes every used twist and every allowed change of lattice. These are exactly the degrees used by the symbol in Lemma 10.17. Express a rational source polynomial in the rational Weyl basis. Its contraction with that symbol is a finite K_0-linear combination of the displayed normalized coefficients; the determinant characters and the rational frame changes are defined over K_0. Equivariance puts the result in the rational target weight space. The same calculation applies to the coupled penultimate and terminal blocks, using their prescribed contractions. Thus each actual class-pair block carries rational inputs to rational outputs before any embedding into a local field is chosen. The sum of the blocks divided by the source stabilizer orders is the lift on the embedded complex spans, at the reference tuple as well as at the used twists.

Lean: `GlobalTheory.KerJ.unUnit_of_mtType` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.KernelEntryUnramifiedUnit`, lines 12–17)
Lean: `GlobalTheory.PdvClose.at_of_true_columns` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.AtOfTrueColumns`, lines 30–47)

For the inert even-order and signal branches, use Lemmas 37.29 and 37.30 on the normalized translates, using the same curve for all pairs at w. In trace-dual coordinates average the paired phase $\mathrm{cob}_F(P)(X,Y) = P(F(X,Y)) - P(X) - P(Y)$ over the residue-field torus as in Lemma 37.28. Its order $p^{n_P} - 1$ is a unit, so replacing the phase by its average changes the cofactor by an integral exponential. This gives the all-place formula with sign -1 and the common coordinate unit of the curve. The trace construction works with every choice of orbit representatives. At $h_P = 0$ the orbit is a singleton ($n_P = 1$); average $\mathrm{cob}_F(P)$ over the quadratic residue torus μ_{p^2-1} (whose order $p^2 - 1$ is a p-adic unit, enforcing all hypotheses of Lemma 15.4 including vanishing of the $(1,1)$ term), then substitute the same variable into both paired coordinates with a unit multiplier so polarization gives half the averaged paired phase, as in the one-position expansion of Proposition 16.12. For nonzero odd-order h_P, pass to the doubled even orbit length $2n_P$ and average $\mathrm{cob}_F(P)$ over the Teichmüller group $\mu_{p^{2n_P}-1}$ of the residue field $\mathbf{F}_{p^{2n_P}}$ (whose order $p^{2n_P} - 1$ is again a p-adic unit, enforcing all hypotheses of Lemma 15.4 including zero $(1,1)$ term), and use the trace form of the odd unramified orbit extension: a unit on its fixed side is a norm of a unit on the quadratic side by the residue norm and Hensel lifting, diagonalizing the unimodular form onto the orthogonal trace model, whence the integral unitary substitution of Lemma 37.29 yields the all-real orbit product. These last two arguments use the inert residue class and their respective orders of h_P, independently of the all-place branch. Lemma 37.28, with (37.35) and $K_{\delta,1} \geq 5, K_{\delta,2} \geq 4$, gives the asserted minor bound for every series with the specified first-sheet expansion.

Lean: `GlobalTheory.Ggeo.frobeniusConjugation_conditions_adaptedKernelDatum_alt` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.FrobeniusConjugationConditionsAdaptedKernelDatum`, lines 23–62)
Lean: `UpperTheory.lattice_pairs_odd_degree_isometric` (`Development.AutomorphicArithmetic.OrbitLabellings.LatticePairsOddDegreeIsometric`, lines 69–87)

For the jet assertions, write P for the corrected curve phase in the formal parameter, l for the good-model logarithm and $S = (P \circ l^{-1})''(0)$. At odd primes the two addition defects and the adjoint defect give the following derivative calculation. Derivatives of order at least three come from the defect in three additive variables: differentiate once in each of the first two variables and the remaining times in the third. The exponential of the defect is integral in the formal parameter, and logarithms of integral one-units have integral derivatives at zero; in particular, at $p = 3$, this cubic-jet identity establishes integrality of all derivatives of order at least three without dividing by 3!. Passing to logarithmic coordinates preserves integral derivatives since l' is an integral unit with tangent normalization one. Reflection controls the first derivative since 2 is invertible. Differentiating the adjoint defect once in each variable gives $(\alpha - \bar{\alpha})S$ integral for a CM multiplier α for which the endomorphism and the norm law hold. At $p \geq 5$, $p \nmid d$ take a prime-to-p multiplier with $\alpha - \bar{\alpha}$ a unit, for instance an integral translate of $(1 + \sqrt{-d})/2$ avoiding the two possible zero residues of its norm. At $p = 3$, the split-prime threshold gives $d \equiv 2 \bmod 3$. Use $\alpha = \sqrt{-d}$. Its norm and $\alpha - \bar{\alpha}$ are units, and the Weierstrass correction $(\alpha^2 - 1)b_2/12 = -((d+1)/3)b_2/4$ is integral. In both cases the isogeny norm law follows from the sigma product over the nonzero kernel pairs, whose abscissas are integral. This proves integrality of all single-curve jets (including cubic and higher jets at $p = 3$) at these primes.

To transfer single-curve jets to the matrix theta section uniformly across all primes p, we establish four-copy local normalization and frame-matrix transfer at every prime p (including $p = 2$, $p = 3$, split $p \mid N\mathfrak{c}_K$, and $p = d$). By weak approximation on $\mathrm{U}_N(E) \times \mathrm{U}_{N'}(E)$, choose rational unitary (γ, γ') transporting any admitted self-dual lattices (L, L') to (M, M') agreeing with the standard lattices locally at p; rational theta pullback transports the normalized theta section while preserving the Hermitian form H. Take g_0, $P_0 = g_0 P_1^4$, $P = \Lambda^{4f}$ and κ_0 as in the proof of Lemma 17.14. The lattice P_0 is self-dual, stable under the CM order and commensurable with P. Unimodular local standardization therefore gives a rational unitary u_0 with $u_0 P_0 = P$ locally at p, including at $p = d$. The norm-one scalars congruent to one at p preserve this equality. They form an infinite set, so choose one outside the finite spectrum of u_0^{-1}. Its product u with u_0 has no eigenvalue one. Put $g_1 = u g_0$. Then $g_1(P_1^4)$ agrees with P at p, the pullback factor c of that proof satisfies $c^8 = (\det u)^4$, and $\varepsilon = \kappa_0 c$. Hence $|\varepsilon|_p = 1$ follows from $|\kappa_0|_p^2 |\det u|_p = 1$. If p splits in K_d, this is (A.111), whose proof uses only that p splits; this case contains $p = 2$, since $d \equiv 7 \bmod 8$, and the split primes dividing $N\mathfrak{c}_K$. If p does not split in K_d, a case that contains $p = d$, then p lies under no

split prime l, so every r_l is a p-unit and $|\kappa_0|_p = 1$. Moreover $\det u$ lies in K_d and satisfies $\det u\,\overline{\det u} = 1$, and K_d has only one prime above p, so $|\det u|_p = 1$. Whenever single-curve prime-to-p torsion-translate jets on $\mathbf{C}/\Lambda$ are integral over $\mathcal{O}_{\mathbf{C}_p}$ (as verified below for every $p \neq d$), transporting the fourth tensor power through (g_1, ε) replaces it by a finite integral average of prime-to-p translates of products of single-curve sections; composing the lattice-direction frame with an $\mathcal{O}_{\mathbf{C}_p}$-integral coordinate matrix (at ramified primes such as $p = d$, constructed via host differential generators s_τ generating the inverse-different ideal $\tau(\bar{\mathfrak{b}}) = \tau(\mathfrak{b}^{-1}\mathfrak{D}_{E/K_d})$, so all frame-composition entries lie in $\mathcal{O}_{\mathbf{C}_p}$ and preserve the Gram form modulo $\sqrt{-d}$) preserves divided-power jet integrality by the product and chain rules, and extracting the diagonal multi-order (ν, ν, ν, ν) yields the fourth power of the ν-th jet of the single-copy matrix section, proving integrality without choosing a fourth root.

Lean: `Formalism.L2.Th1.H.localStandard` (`Formalism.LocalToGlobal.TorsionIntegrality.QuadraticSemicharacters.QuadraticSemicharacterIdentityReformulated`, lines 6–6)
Lean: `Formalism.L2.Th1.H.kernelRatio` (`Formalism.LocalToGlobal.TorsionIntegrality.QuadraticSemicharacters.QuadraticSemicharactersFactTer`, lines 9–9)
Lean: `Formalism.L2.Ker.KS.kappaAt_of` (`Formalism.LocalToGlobal.KernelLattices.KernelLocalFactors.SelfDualLatticeNorms.KapAll`, lines 14–18)
Lean: `Formalism.L2.Ker.KS.MM.locNorm_of_kappa` (`Formalism.LocalToGlobal.KernelLattices.KernelLocalFactors.SelfDualLatticeNorms.SelfDualFourfoldCurveIdealLattice`, lines 73–123)

At $p = d$, since $|j(\Lambda)|_d \leq 1$, choose a good integral Weierstrass model over a ramified extension inside $\mathcal{O}_{\mathbf{C}_d}$ and a unit CM multiplier with $(\alpha - \bar{\alpha})^2 = -d$. The single-curve calculation yields $|S|_d \leq d^{1/2}$ and integral divided-power jets after removing $SZ^2/2$. Because the unique prime of K_d above d is ramified of index 2 with uniformizer $\sqrt{-d}$, the Galois field E carries an order-2 inertia involution $s \in \mathrm{Gal}(E/\mathbf{Q})$ (commuting with complex conjugation c_E, satisfying $s(\sqrt{-d}) = -\sqrt{-d}$, and acting trivially modulo $\sqrt{-d}$) that pairs the CM embeddings into dual sheets $\tau_{(1,t)} = \tau_{(0,t)} \circ s \circ c_E$. Taking standard local bases (P_τ, P'_τ) of L, L' at d satisfying the E-level CM-conjugate transpose duality relation $c_E(P_{(1,t)})^{\mathsf{T}} P_{(0,t)} = I$ (and $c_E(P'_{(1,t)})^{\mathsf{T}} P'_{(0,t)} = I$), applying $\tau_{(0,t)}$ and using $\tau_{(0,t)} \circ c_E \equiv \tau_{(1,t)} \pmod{\sqrt{-d}}$ (as s fixes $\mathcal{O}_E/\sqrt{-d}$) yields $\tau_{(1,t)}(P_{(1,t)})^{\mathsf{T}} \tau_{(0,t)}(P_{(0,t)}) \equiv I \pmod{\sqrt{-d}}$ (and likewise for P'). Forming the E-eigenframe coordinates $X_{(0,t),a,b}, X_{(1,t),a,b}$ via integral $\mathcal{O}_E$-eigencombinations scaled by local-basis units, and transporting the fourth-power section by the prime-to-d-index four-copy normalization g_1 above (which preserves prime-to-d torsion shifts and matches the bilinear and Hermitian Gram forms modulo $\sqrt{-d}$), we evaluate the quadratic exponent: since $\mathcal{O}_E$ acts c_E-adjointly on the Hermitian space H while $\tau_{(1,t)} \equiv \tau_{(0,t)} \circ c_E \pmod{\sqrt{-d}}$, every Gram pairing between coordinates $X_{(\epsilon,t),a,b}$ and $X_{(\epsilon',t'),a',b'}$ with $(\epsilon', t') \neq (1-\epsilon, t)$ vanishes modulo $\sqrt{-d}$ (killing all same-sheet squares $X^2_{(0,t)}, X^2_{(1,t)}$ and all cross-t terms modulo $\sqrt{-d}$), whereas on each paired block $((0,t),(1,t))$ the dual local bases give the diagonal pairing $\delta_{ab,a'b'} u_t \pmod{\sqrt{-d}}$ with $|u_t|_d \leq 1$. Because $|S\sqrt{-d}|_d \leq d^{1/2} d^{-1/2} = 1$, every $O(\sqrt{-d})$ discrepancy in the quadratic exponent has integral divided-power jets at the odd prime d and is absorbed into the integral cofactor before taking diagonal jets. Both S and the u_t serve every translate in the entry, so the product and chain rules preserve integral divided-power jets; factoring out the common quadratic exponential $\exp\bigl(-\frac{S}{2} \cdot 2 \sum_t u_t \sum_{a,b} X_{(0,t),a,b} X_{(1,t),a,b}\bigr) = \exp\bigl(-S \sum_t u_t \sum_{a,b} X_{(0,t),a,b} X_{(1,t),a,b}\bigr)$ proves (37.34).

Lean: `Formalism.L2.Th1.G.ramified_shape_holds` (`Formalism.LocalToGlobal.TorsionIntegrality.RamifiedCurveKernels.PairedSquareJets.InverseEndoIntegralHolds`, lines 117–159)
Lean: `GlobalTheory.ThLoc.jets_phW_three` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.IntSExpMulTranslate`, lines 75–134)
Lean: `GlobalTheory.Proof.shape_normalisedThetaCombinations_dualBases_at_d.shapeComb` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.ShapeNormalisedThetaCombinationsDualBasesAtD`, lines 197–232)

Away from $p = d$, we treat the three regimes separately. At $p = 2$ the host has $d \equiv 7 \bmod 8$; use the ordinary integral model with $|a_1|_2 = 1$ and the even theta at the selected half-period. The two CM scalars of $\omega = (1 + \sqrt{-d})/2$ have odd trace and even norm; order them as $a, b \in \mathbf{Z}_2$ with a a unit and b even. Neither CM multiple of the selected half-period is zero. On the formal subgroup there is a unique nonzero point of order two, and multiplication by the unit scalar fixes it. The half-period therefore reduces away from the identity: otherwise its other CM multiple would be zero since the norm is even. The same argument applied to the inverse image of the half-period under the unit multiplier puts the abscissas of that fibre in the integral ring. Pair the points in this fibre by opposite signs. The isogeny product for the shifted even section gives the phase norm law $Q([a]t) \sim abQ(t)$; its normalized factors have unit constant term and integral coefficients since the fibre abscissas are integral. To establish biadditivity of the polarized phase $B_Q(X, Y) = Q(F(X, Y)) - Q(X) - Q(Y)$ without dividing by 2 in $\mathcal{O}_{\mathbf{C}_2}$, let $x(t), y(t)$ be the coordinates of

the integral model in the formal parameter $t = -x/y$, let (x_T, y_T) be the selected half-period, and put

$$g(t) = t^2\big(x(t) - x_T\big), \qquad w(t) = \frac{t^2}{g(t)}, \qquad \lambda(t) = \frac{y(t) - y_T}{x(t) - x_T}, \qquad \kappa = 3x_T^2 + 2a_2x_T + a_4 - a_1y_T.$$

The half-period reduces away from the identity, so x_T and y_T are integral. Since $t^2x(t)$ is an integral one-unit, g is an integral one-unit, w is integral with $w(0) = 0$, $\vartheta(t) = -t\lambda(t) = (t^2x(t) + y_Tt^3)/g(t)$ is an integral one-unit, and $\kappa \in \mathcal{O}_{\mathbf{C}_2}$. With $e_1 = x_T + b_2/12$ one has $\kappa = 3e_1^2 - g_2/4$ and $w = (\wp - e_1)^{-1}$. Since $\wp - e_1$ is the square of the quotient of the even sigma function at the half-period by σ, $2(Q - P_{\mathrm{odd}}) = -\log g$, where P_{odd} is the odd sigma phase. For points R_1, R_2 put $\varsigma = (x(R_1) - x_T)(x(R_2) - x_T)$. The shifted Weierstrass addition law at the half-period is the identity of rational functions

$$\big(x(R_1 + R_2) - x_T\big)\big(\lambda(R_1) + \lambda(R_2) + a_1\big)^2\varsigma = (\varsigma - \kappa)^2.$$

In the formal variables put $\ell(X, Y) = X\vartheta(Y) + Y\vartheta(X) - a_1XY$ and $d(X, Y) = 1 - \kappa w(X)w(Y)$. Dividing the identity by ς and multiplying by $X^2Y^2F(X, Y)^2$ gives $g(F(X, Y))\ell(X, Y)^2 = g(X)g(Y)F(X, Y)^2d(X, Y)^2$. The relation $2y_T + a_1x_T + a_3 = 0$ gives $\lambda(-R_1) = -\lambda(R_1) - a_1$, so ℓ vanishes on $X = i(Y)$, where i is the formal inverse. As $F(X, Y)$ is $X - i(Y)$ times an integral unit, and ℓ and F are both $X + Y$ modulo terms of degree two, $\ell = F\nu$ with ν an integral one-unit. Thus the integral formal one-units ν and d satisfy $g(F(X, Y))\nu(X, Y)^2 = g(X)g(Y)d(X, Y)^2$. Applying the formal logarithm over the characteristic-zero ambient field yields $\mathrm{cob}_F(\log g) = 2(\log d - \log \nu)$, and cancelling the factor 2 against $2(B_Q - \mathrm{cob}_F(P_{\mathrm{odd}})) = -\mathrm{cob}_F(\log g)$ gives $B_Q - \mathrm{cob}_F(P_{\mathrm{odd}}) = \log \nu - \log d \sim 0$, proving $B_Q \sim \mathrm{cob}_F(P_{\mathrm{odd}})$ without halving in $\mathcal{O}_{\mathbf{C}_2}$. The 3-variable cube identity on the formal group law gives $d_1(\mathrm{cob}_F(P_{\mathrm{odd}}))(X, Y, Z) = \mathrm{cob}_F(P_{\mathrm{odd}})(F(X, Y), Z) - \mathrm{cob}_F(P_{\mathrm{odd}})(X, Z) - \mathrm{cob}_F(P_{\mathrm{odd}})(Y, Z) \sim 0$, so B_Q is biadditive modulo $\sim$ by symmetry without halving. For B_Q, $\mathbf{Z}_2$-linearity follows from integer multiplication by continuity and binomial powering. Its norm law then makes $a(a - b)$ times B_Q equivalent to zero. This scalar is a 2-adic unit, so B_Q vanishes in the quotient. Now $Q([a]t) \sim aQ(t)$ by additivity and continuity. Subtract the norm law and divide by the unit $a(1 - b)$. The normalized even section is an integral one-unit in the formal parameter, its odd-order translates have integral coefficients by the torsion translation product, and substituting l^{-1} gives integral single-curve jets since l' and $1/l'$ are integral; applying the uniform four-copy local normalization (g_1, ε) at $p = 2$, $\mathcal{O}_{\mathbf{C}_2}$-integral frame-matrix substitution, and diagonal (ν, ν, ν, ν) fourth-power extraction transfers these jets to the matrix section. At $p = 3$ (where $d \equiv 2 \bmod 3$ splits 3), the cubic-jet integrality established above on the good integral Weierstrass model over $\mathcal{O}_{\mathbf{C}_3}$ supplies integral single-curve jets on prime-to-3 torsion translates on $\mathbf{C}/\Lambda$, and the same four-copy normalization (g_1, ε), $\mathcal{O}_{\mathbf{C}_3}$-integral frame-matrix substitution, and diagonal (ν, ν, ν, ν) fourth-power extraction establish matrix-jet integrality. For all $p \geq 5$ with $p \neq d$ (both inert and split, including $p \mid N\mathfrak{c}_K$), Lemma 37.25 supplies a good integral model giving integral single-curve jets on prime-to-p torsion translates, and the same frame-matrix transfer (using Lemma 37.29 in the inert case and the prime-to-p-index local isometry g_1, $\mathcal{O}_{\mathbf{C}_p}$-integral frame-matrix substitution, and diagonal (ν, ν, ν, ν) fourth-power extraction in the split case) transfers the jets to the matrix section. Local equality of lattice pairs allows division of each direction by a prime-to-p comparison integer, so finite sums and integer minor operators preserve exponent zero.

Lean: `Formalism.L2.Th1.B.An2.Th0TwoAnalytic_of` (`Formalism.LocalToGlobal.TorsionIntegrality.FrobeniusFibreCurves.LogarithmicSubstitution.Th0TwoAnalyticOf`, lines 25–108)
Lean: `Formalism.L2.Th1.Top.integral_operator_two_orthogonal` (`Formalism.LocalToGlobal.TorsionIntegrality.UniformGaussModels.OrthogonalGaussModels.IntegralOperatorTwoOrthogonal`, lines 54–54)
Lean: `GlobalTheory.ThLoc.jetsIntegral_away_disc` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.IntegralJetsNormalisedThetaTranslatesAwayDisc`, lines 57–77)

Finally impose the two family conclusions together. For the local systems take the maximum of the shape and rank thresholds. For the selected entries impose the uniform shape threshold in the choice of eligible points, before evaluating the selection. Thus the entry conclusion holds at every point beyond the selection threshold. Restricting the resulting family assertion to $\mathcal{D}_3$ preserves both conclusions with the quantifiers of Definition 36.10.

Lean: `GlobalTheory.Svd.servH_row_of_artin` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.BundleBothPartsOfStandardKernelOfGlobalArtinReci`, lines 15–24)
Lean: `GlobalTheory.row_hServ_of_link_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RowHServOfLink`, lines 17–23)
Lean: `GlobalTheory.Fc.R1.row_hServ_of_kerServ_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RowHServOfKerServ`, lines 13–22)

□

Proof of 37.27

Proof. In the short model $y^2 = x^3 - g_2'x/4 - g_3'/4$ with $t = -x/y$, $D(t) = t^2x(t)$ is the even integral solution of $D^3 - D^2 - (g_2'/4)t^4D - (g_3'/4)t^6 = 0$ with constant term one. The derivative in D there is one. The curve law $\oplus$ is integral with inverse $-t$. For $R(t,u) = P_E(t \oplus u) + P_E(t \oplus (-u)) - 2P_E(t) - 2P_E(u)$, the sigma addition formula after homothety gives

$$\exp(-R(t,u)) = \frac{t^2D(u) - u^2D(t)}{(t \oplus u)(t \oplus (-u))}.$$

The quadratic exponent cancels from the sigma quotient [203, Eq. 23.10.3]. The numerator is $t^2 - u^2$ times an integral one-unit by evenness. The two denominator factors divided by $t + u, t - u$ are integral one-units by the inverse and linear terms of the group law. Thus $R \sim 0$.
Lean: `Formalism.L2.Th1.B.Par0.par_eq` (`Formalism.LocalToGlobal.TorsionIntegrality.FrobeniusFibreCurves.LogarithmicSubstitution.InVarAddInVarNeZero`, lines 200–301)

Use formal curve addition in the following identity. With $b_0 = (y+z)/2$, $c_0 = (y-z)/2$, $a_0 = x + b_0$, expansion gives

$$B_E(x+y,z) - B_E(x,z) - B_E(y,z) = R(a_0,b_0) - R(a_0,c_0) + R(b_0,c_0) - R(b_0,b_0).$$

Halving is integral and each substituted point has zero constant term, so the right side is equivalent to zero. Symmetry gives additivity in the other slot. Setting $u = t$ in R gives $P_E(m_2(t)) \sim 4P_E(t)$.
Lean: `Formalism.L2.Th1.B.cube_identity` (`Formalism.LocalToGlobal.TorsionIntegrality.FrobeniusFibreCurves.LogarithmicSubstitution.ConstantCoeffEqZeroOfParallelogram`, lines 115–169)

For $0 \neq \alpha \in \mathcal{O}_K$ put $n = \alpha\bar{\alpha}$ and $T_\alpha = (\alpha^{-1}\Lambda/\Lambda) \setminus \{0\}$. The isogeny identity is

$$\Sigma_\Lambda(\alpha z)^2 = \alpha^2 \Sigma_\Lambda(z)^{2n} \prod_{q \in T_\alpha} (\wp_\Lambda(z) - \wp_\Lambda(q)).$$

The Legendre relation gives $\eta_\Lambda(\omega) = s_2\omega + \pi\bar{\omega}/A_\Lambda$, $A_\Lambda = \mathrm{Im}(\bar{\omega}_1\omega_2)$. The squared quasi-period multipliers agree since $\alpha\bar{\alpha} = n$. Each pair $q, -q$ gives a double zero, and a single point of order two gives a double zero by ramification of $\wp_\Lambda$. At zero both sides have leading term $\alpha^2 z^2$, since $|T_\alpha| = n - 1$. Their quotient is elliptic and holomorphic, hence one.
Lean: `Formalism.L2.Th1.F.isogenySigma_of` (`Formalism.LocalToGlobal.TorsionIntegrality.EisensteinTranslationProducts.AnalyticIdentityTheorems.IsogenySigma`, lines 394–407)

Comparing z^4 gives $(\alpha^2 - n)s_2 = \sum_{q \in T_\alpha} \wp_\Lambda(q)$. Take $\alpha = \sqrt{-d}$; the multiplier is $-2d$ and the kernel is killed by d. The discriminant-one invariants are algebraic by the algebraicity of the CM j-invariant. The scaled abscissas $x_q = e^2\lambda(\wp_\Lambda(q))$ belong to W, by finite étaleness of the prime-to-p torsion and nonidentity reduction. Thus $e^2\lambda(s_2) \in K_0$. The normalized Laurent solution for $\wp$ belongs to $K_0((z))$. Integrate $-(\wp - z^{-2})$ twice with zero constants to recover $\log(\sigma(z)/z)$. This determines the transferred Σ and P_E over K_0.
Lean: `Formalism.L2.KerInert.Sigma.s2_in_unramified_closure` (`Formalism.LocalToGlobal.InertKernelTheory.InertCoprimeLemmas.PadicNormTwoHundredSixteenEqOneOfFiveLe`, lines 181–225)

For $p \nmid n$, matching principal parts gives $\alpha^2\wp(\alpha z) = \wp(z) + \sum_{q \in T_\alpha}(\wp(z+q) - \wp(q))$ in model coordinates. The translated abscissas are regular over W on the formal disc, since the torsion reduces away from the origin. Using $dx/(2y) = dl$, we obtain $X_\alpha = x(m_\alpha(t)) \in t^{-2}W[[t]]$ and $Y_\alpha = (2\alpha l'(t))^{-1}dX_\alpha/dt \in t^{-3}W[[t]]$. The leading terms $\alpha^{-2}t^{-2}, -\alpha^{-3}t^{-3}$ show that $m_\alpha = -X_\alpha/Y_\alpha$ is integral and $m_\alpha/(\alpha t)$ is a one-unit. The sigma identity then gives

$$\left(\frac{m_\alpha(t)}{\alpha t}\right)^2 e^{-2(P_E(m_\alpha(t)) - nP_E(t))} = \prod_{q \in T_\alpha} (D(t) - x_q t^2).$$

The right side is an integral one-unit, so its normalized square root proves the norm law for α.
Lean: `Formalism.L2.Th1.C.phNormUnit_of` (`Formalism.LocalToGlobal.TorsionIntegrality.SecondCoefficientNorms.BinomialSeriesCoefficients.USideB`, lines 226–294)

Lean: Formalism.L2.KerInert.PN.phNormOKInert_of (Formalism.LocalToGlobal.InertKernelTheory.InertCoprimeLemmas.PhNorm, lines 371–540)

For a parameter change $q(t) \in W[[t]]$ of unit tangent $c \in W^{\times}$, one has $l_{\mathrm{new}}(q(t)) = c l_{\mathrm{old}}(t)$ and $\Sigma_{\mathrm{new}}(cz) = c\Sigma_{\mathrm{old}}(z)$. The phase difference is $\log(q(t)/(ct)) \in K_0[[t]]$, whose exponential is an integral one-unit, so the coefficient-field assertion and the laws transfer to the specified good model. The changes to short form use the units $2, 3$. The same identity preserves the exponential integrality laws under a parameter change over $\mathcal{O}_{\mathbf{C}_p}$ with unit tangent.

The completion of $\mathcal{O}_K$ is W^{σ^2}, with conjugation σ. Approximate each unit u by elements of $\mathcal{O}_K$ of norm prime to p. The coefficients of m_a are polynomial in a, and those of the exponentiated norm defect are continuous in $a, \bar{a}$. Thus integrality and the norm law hold for u. For $k \in \mathbf{Z}_p$, biadditivity and continuity from integer multiplication give $B_E(t, m_k(s)) \sim kB_E(t, s)$, with binomial powering on integral one-units. Polarize the unit norm law at $t, m_{u^{-1}}(s)$ to obtain $B_E(m_u(t), s) \sim B_E(t, m_{\bar{u}}(s))$. Each $a \in W^{\sigma^2}$ is the sum of two units, by choosing a nonzero residue different from that of a. Formal addition gives the integral m_a, and biadditivity gives the adjoint law. Doubling gives $B_E(t, t) \sim 2P_E(t)$, hence

$$2P_E(m_a(t)) \sim B_E(m_a(t), m_a(t)) \sim B_E(t, m_{a\bar{a}}(t)) \sim 2a\bar{a}P_E(t).$$

Take the normalized square root.

The equality $l(v(T)) = \epsilon\ell(T)$ gives the signed identity

$$P_E(v(T)) - P(T) = \log(e_0 v(T)/T).$$

The exponential is an integral one-unit. Since v intertwines addition and scalars, $b^{\ell} = B_E(v(X), v(Y)) \sim B_P$ and the laws transfer. The exponentials of both additivity defects and of the adjoint defect are integral after logarithmic substitution. Their coefficients lie in $K_0 \cap \mathcal{O}_{\mathbf{C}_p} = W$. Since the series vanishes on both axes, it gives the pairing of 10.6.

Lean: Formalism.L2.KerInert.BeCob.phase_compatibility (Formalism.LocalToGlobal.InertKernelTheory.InertCoprimeLemmas.ExistsWittSeriesInverseFormalGroupIso, lines 327–431)

Lean: Formalism.L2.KerInert.be_props_at (Formalism.LocalToGlobal.InertKernelTheory.InertCoprimeLemmas.BeProp, lines 96–184)

□

Proof of 37.28

Proof. Put $b_0 = b^{\ell}_{\mathrm{pair}}$ with the notation of (37.36).

Take z_s to be the inverse trace-dual transform of the logarithms of paired formal variables T_i, T_i'. Put $q_0 = p^2$, $(\Psi z)_s = \sigma^2(z_{s-2})(T^{q_0})$, where the substitution raises all variables to q_0. The inverse matrix cycles under σ^2, so $(1 - p^{-1}\Psi)z$ is integral. Solve $(1 - p^{-1}\Psi)\mathcal{L}(t) = (1 - p^{-1}\Psi)z$ by degree. Its linear-in-t term is t_s. Each higher term is

$$p^{-k}\big(t_{s-2k}^{q_0^k} - (\sigma^2(t_{s-2k})(T^{q_0}))^{q_0^{k-1}}\big) \qquad (k \geq 1).$$

It uses lower coefficients. The two bases before taking the q_0^{k-1}-st power agree modulo p, so their powers agree modulo p^k. This gives integral t_s of zero constant term. Injectivity of $1 - p^{-1}\Psi$ by lowest degree yields $\mathcal{L}(t) = z$.

Let $S = \sum b_0(T_i, T_i')$ over the basis indices and entries, and B be the averaged phase at z. For a torus parameter ζ, the even matrix in trace coordinates has entries $\sum_{j=0}^{h-1} \sigma^{2j}(\zeta\omega_i^*\omega_k)$. The odd matrix is its inverse adjoint for $\sum_i \sigma(A_i)B_i$ by trace duality. By 10.6 the transformed pairing thus reduces to the identity Gram matrix modulo integral exponentials. Multiplying over the torus gives integral $\exp((p^{d_0} - 1)(S - B))$. Its normalized root is integral since $p^{d_0} - 1$ is a unit. Thus $U_0 = U(t)\exp(S - B)$ is integral. Substituting $\ell^{-1}(L_i^X(Z)), \ell^{-1}(L_i^Y(Z))$ into $U_0 e^{-S}$ recovers $\Theta(\xi Z)$.

Lean: GlobalTheory.KerStd.integral_cofactor_converse (Development.AbelianVarietyTheory.Proofs.SatakeParameters.ExistsIntegralOrbitSeries, lines 343–372)

By 10.7, b_{pair} and ℓ^{-1} have integral derivatives at zero. By the product and chain rules, the displayed operations preserve integrality of the derivatives. This applies to exponentiation of a zero-constant phase. The integer minors give exponent zero by 10.18.

Lean: GlobalTheory.KerZero.kvalGe_zero (Development.AbelianVarietyTheory.Proofs.SatakeParameters.StandardOrbitSeriesValuationBoundZero, lines 363–380)

For $\pi^{p^2-1} = p$, scale by π and use 10.13. The cofactor stays integral since $\ell^{-1}(\pi T)/\pi$ is integral. The exponential in the odd-power polynomial H has the $p^{-k}H^k$ expansion of 10.12 since $v_p(\pi^{(p+1)k}/k!) = s_p(k)/(p-1) \geq 0$, with $s_p(k)$ the base-p digit sum. Apply 11.14 with ranks N_j, N_{j+1} and orbit length d_0. This gives the exponent g_A of 11.1. Homogeneity removes the norm-one scale ξ.

Lean: `GlobalTheory.KerGaussA.kvalGe_gainA` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.KvalGeOrbitGainWithoutOperator`, lines 106–160)

Let ν_s be the layer diagram and T the set of first-sheet top positions in the orbit. Put $\alpha_s = \nu_{s,1} - \nu_{s,2}$ on T, zero elsewhere. After removing height-one columns at the marked positions, the local caps are $\nu_{s,2}$ on T and $\nu_{s,1}$ off T. By (37.35), $\bar{\alpha}$ of 14.2 is at most α_s on T, and the local caps are bounded by the global caps. A positive odd distance $e < d_0$ between distinct top positions is their first positive cyclic distance and is excluded in 14.3. Thus each retained global positive part is bounded by a retained local term of 11.1; convergence follows from the geometric decay p^{-e}. Put $z = \log_p(2a_*) = z_p \geq 1$ and $M = N_m$. The bounds $\alpha_s \leq 2a_*$, $|\nu_s| - \alpha_s \leq a_*^2$ give digit lengths at most $2z, 3z$ by 10.4. In 11.1 their total error is at most $5d_0M^2z$, plus $4|T|z^2$ for the squared marked lengths. Since the error in 14.2 dominates their sum, $g_0(t, j+1) \leq g_A$. Combine this with exponent zero.

Lean: `GlobalTheory.KerStd.WordFree.wordFreeNumber_le_orbitBound` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.WordFreeNumberLeOrbitBound`, lines 168–248)

□

Proof of 37.29

Proof. The good-model assertion is Lemma 37.25. Extend λ to an isomorphism $\iota : \mathbf{C} \to \mathbf{C}_p$ by extending transcendence bases. At an inert prime the ideal (p) is prime in $\mathcal{O}_K$. Write $\mathfrak{c} = (p)^h\mathfrak{c}'$ with $p \nmid \mathrm{N}\mathfrak{c}'$. The replacement $(\delta, \mathfrak{c}) \mapsto (p^h\delta, \mathfrak{c}')$ preserves the actual period lattice, its discriminant and its torsion classes. It permits the use of Lemma 37.27.

First use a short good model, and normalize the transferred odd sigma section to have tangent one in its differential coordinate. For a torsion point t write $G_t(T)$ for its translated sigma section at $l_W(T)$, divided by the correction $A(T)$ and normalized by the same differential scale. Thus at a period $G_0(T)$ is T times a scalar of norm one. Suppose t has exact order $n > 1$, $p \nmid n$, and put $c_t = G_t(0) \neq 0$. For $1 \leq j < n$ the sigma addition identity gives

$$G_{(j+1)t}G_{(j-1)t} = c_t^2\big(x(t) - x(T+jt)\big)G_{jt}^2,$$

using compatible translation representatives. Here $x(T+jt)$ is the abscissa on the translated formal disc. The quadratic correction cancels from the sigma quotient [203, Eq. 23.10.3]. Multiply these identities with exponents $n-j$. The intermediate terms cancel and nt is a period, giving, for a scalar $|u|_p = 1$,

$$G_t^n \prod_{j=1}^{n-1} \big(c_t^2(x(t) - x(T+jt))\big)^{n-j} = uT^n. \tag{A.583}$$

Every abscissa difference here is T^a times an integral series with unit constant term. If $jt \neq \pm t$, the abscissas remain distinct after reduction because reduction is injective on the subgroup of order n. If $jt = \pm t$ and $2t \neq 0$, the first derivative is a unit since the reduced ordinate is nonzero. If $2t = 0$, the reduced abscissa is a simple root of the cubic and the abscissa difference has a double zero with unit leading coefficient. The leading coefficients in (A.583) give $|c_t|_p^{n+2\sum_{j=1}^{n-1}(n-j)} = |c_t|_p^{n^2} = 1$. After cancellation of T^n, $(G_t/c_t)^n$ is integral. Its root with constant term one is integral by coefficient recursion, since n is a unit. This proves the assertion for the odd sigma section. For the normalized even section at a nonzero half-period h, the denominator is the value at zero of the h-translate, a unit by the case $n = 2$. A further translate by s, with $Ns = 0$ and $p \nmid N$, has numerator a root of unity times the translate at $s+h$; $2N(s+h) = 0$. This includes $s + h = 0$.

Lean: `Formalism.L2.Th1.E.joint_integrality` (`Formalism.LocalToGlobal.TorsionIntegrality.TorsionTranslationPowers.NonsingularJetNorms.Joints`, lines 128–129)

An integral unitary change (a_{ij}) of orthonormal basis acts by formal substitution $T_i \mapsto \bigoplus_j [a_{ij}]T_j$. The ratio of the products of corrections is an integral one-unit. Indeed the diagonal terms in the phase calculation of Lemma 37.27 have multipliers $\sum_i \bar{a}_{ij}a_{ij} = 1$, and the mixed pairing terms have second-slot multipliers $\sum_i \bar{a}_{ij}a_{ik} = 0$. A change of good parameter with tangent u_0 changes the correction by the quotient of that

parameter by u_0T, an integral one-unit. Thus the calculation applies in the prescribed model. For $p \geq 5$ passage to a short equation uses only division by $2, 3$.

For the matrix section, start with one matrix entry in the trace space over K. The global four-copy normalization in Lemma 17.14 identifies its fourth tensor power with the normalized average of a product of elliptic sections, pulled back by an isometry. Choose that isometry to be integral on the two lattices at p. Here the local trace lattice is unimodular. Over the unramified quadratic integer ring a unimodular Hermitian lattice has an orthonormal basis: reduction supplies a vector of nonzero norm, the unit norm map is onto by the residue norm and Hensel lifting, and its orthogonal complement is unimodular. Repeat in the complement. A rational unitary change can realize this local lattice identification. Indeed the stabilizer of a lattice is open, and on the dense open Cayley chart approximation reduces to approximation of the entries of a skew-Hermitian matrix. The same argument works at the finitely many primes above p. Thus the relative index of the standard lattice and the moved lattice is prime to p. The normalization scalar has norm one: in the global fourth-power identity of Lemma 17.14 (and its proof), the initial scalar $\kappa_0 = \prod_l r_l^{-y_l}$ is supported solely at split rational primes l (hence is a p-adic unit at every inert prime p and is invariant under the replacement $(\delta, \mathfrak{c}) \mapsto (p^h\delta, \mathfrak{c}')$), while the change-of-isometry scalar c satisfies $c^8 = (\det u)^4$ for a norm-one determinant $\det u \in K^\times$. That determinant is a unit at an inert prime, since its valuation equals the valuation of its conjugate.

Lean: `Formalism.L2.KerInert.Kap.kappa_K1_hypothesis` (`Formalism.LocalToGlobal.InertKernelTheory.InertCoprimeLemmas.KappaSelfDualFourthPowerLattice`, lines 317–367)

Translate the four-copy identity in all four sets of variables. Its average has unit index denominator, and its shifts have prime-to-p order since their orders divide that index. After the orthonormal change the right side has integral corrected coefficients by the elliptic calculation. Its coefficient at (ν, ν, ν, ν) on the left is the fourth power of the one-entry corrected coefficient at ν. This proves integrality for that entry without a choice of fourth root. For a standard pair of even ranks, decompose the matrix into 2-by-2 blocks. The section on each block and its correction are products over the four entries; translation projects to the four corresponding shifts.

For general endpoints, choose rational unitary changes making them standard at the primes above p as above. The finite lattice moves of Lemma 12.11 to this pair have indices prime to p. Their determinant-ideal values are units by the inert type-norm calculation in Lemma 12.15. The resulting sums have integral multipliers and prime-to-p shifts. Pullback carries the basis and torsion shift by the same isometry. If two endpoints agree after localization at p, an integer prime to p carries either lattice into the other in both directions. The product of the two endpoint integers does this for their Hom lattices, so a shift of prime-to-p order for the reference pair still has prime-to-p order for the translated pair. Apply the preceding calculation term by term. Multiplication by the common correction and extraction of a coefficient commute with the finite sum, proving (37.37).

Lean: `Formalism.L2.KerInert.weak_approximation_places` (`Formalism.LocalToGlobal.InertKernelTheory.InertCoprimeLemmas.IntWAPlaces`, lines 465–573)

Lean: `Formalism.L2.KerInert.intInert_of` (`Formalism.LocalToGlobal.InertKernelTheory.InertCoprimeLemmas.IntAsm`, lines 79–91)

□

Proof of 37.30

Proof. Work in the additive group of phases modulo zero-constant series with integral exponential. Substitution of integral zero-constant series preserves this equivalence. Write $Q(t)$ for the class of $P(t)$, $t \oplus u = \ell^{-1}(\ell(t) + \ell(u))$, and $B(t, u) = Q(t \oplus u) - Q(t) - Q(u)$. For $a, b' \in W^{\sigma^2}$ the hypotheses give

$$Q([a]t) = a\sigma(a)Q(t), \qquad B([a]t, [b']u) = B(t, [\sigma(a)b']u).$$

The first scalar belongs to $\mathbf{Z}_p$; it acts on integral one-units by binomial powering. In the second identity the scalar inside the bracket acts by formal multiplication on the argument.

Order the columns by putting all f_e before all f'_e. Denote the resulting columns by a_j and the corresponding formal parameters by t_j. Put $z_i = \bigoplus_j [a_{ij}]t_j$. The z_i are integral, have zero constant term, and satisfy $\ell(z_i) = \sum_j a_{ij}\ell(t_j)$. Expanding one formal sum at a time and using biadditivity gives

$$\sum_i Q(z_i) = \sum_j \langle a_j, a_j\rangle Q(t_j) + \sum_{j<k} B\big(t_j, [\langle a_j, a_k\rangle]t_k\big).$$

All diagonal and same-family terms vanish. In the cross terms the multiplier is δ_{eh}, so the right side is represented by $\sum_e b(\ell(t_e), \ell(t'_e)) = S^\ell$. Therefore

$$\Theta(L(\ell(t)))\exp(S^\ell) = H(z)\exp\left(S^\ell - \sum_i P(z_i)\right) \in \mathcal{O}_{\mathbf{C}_p}[[t]].$$

The first factor is integral by substitution, the second by the computed phase equivalence. Finally the adjoint law with scalar -1 gives $b(-X, Y) \simeq b(X, -Y)$. Substitute $-Y$ for Y and sum over e. This proves the required evenness, with the same pairing throughout. □

Proof of 37.31

Proof. Take the curve of Lemma 37.25 for the period lattice of the selected column. The permission condition of Theorem 13.8 gives $p \geq 5$ and inertness in the imaginary quadratic field. The curve ideal at this site has norm prime to p. The local comparison specifies the place of the global coefficient field. Extend its embedding to $\lambda : \overline{\mathbf{Q}} \to \mathbf{C}_p$, then to a field isomorphism $\iota : \mathbf{C} \to \mathbf{C}_p$ by transcendence bases. Thus the two readings of a global coefficient agree. Take the finite tests and step characters of Lemma 12.13. All of these choices precede the layer index.

Use the level and class of the decorated source column for each word and twist. Choose representatives in the finite adelic class sets and orthogonal coordinates for their Hermitian forms. At the reference tuple use the identification on the initial level. At another tuple use its own level group and the same determinant-class map. A determinant lift gives the same class as its norm-one idèle, since two lifts differ in the kernel of $x \mapsto x/\bar{x}$ in the determinant quotient. The character values required at these classes are in the global coefficient field by Lemma 12.13. At a permitted place outside the twist primes, retain the representative off p and use the split chart of (A.55) at each position above p. The source and target positions, their embeddings, and their successor order are those of the prescribed filling. The initial class is the original initial class under this construction. The group action on each tuple has finite stabilizer: it is the intersection of a lattice stabilizer at finite places and a compact definite unitary group at infinity. Its orbit classes are the finite adelic classes just used. This gives the charts simultaneously for the successive lifts. Replacing marked class representatives by the representatives in these charts preserves the class kernel, because their lattice and theta readings agree under the rational changes of Lemma 12.11.

Lean: `GlobalTheory.Pdv.stdKer_stdRd_mark` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.StdKerStdRdMark`, lines 17–46)

Lean: `GlobalTheory.Port.jIn_nonempty_true` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.TrueColumnExistsChainChartsKernelDataForms`, lines 23–102)

Lean: `GlobalTheory.TwV.joint_at_nonempty` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.JointDataAtPAndAdelicDatumNonempty`, lines 16–22)

At a pair of lattice tuples take the normalized Taylor series (A.63), with the chart substitution and restriction there. Its scalar has norm one. Indeed the principal step-character computation in Lemma 12.15, applied to the quotient of the two determinant idèles, gives valuation $\sum_v \Delta_v(j+1)(s_v - s'_v)$. The explicit power of p has the opposite valuation, and the determinant-ideal values are units. The test primes and the test-level denominators avoid p, so the finite test gives a combination with integral coefficients and prime-to-p torsion shifts. Its two kernel lattices are self-dual and of even rank by the chart construction. Lemma 37.29 applies. Transfer the good parameter by $f(v(T)) = \epsilon\ell(T)$; this gives the common differential scale $e\epsilon$, where e is the scale in (37.37).

Lean: `GlobalTheory.TwV.Xc.scal_of_pos_pair` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.ThetaSeriesScalarBoundedOfPosPair`, lines 22–87)

Choose the pairing of Lemma 37.27 on this one curve. In any framed matrix block, take matrix units on the first factor of the split algebra and dual units on its conjugate factor. Multiply by a trace basis over the unramified quadratic ring on the first factor and its trace dual on the second. The resulting coordinates in an orthonormal basis are integral. The Hermitian form pairs two vectors on the same factor to zero. On opposite factors its value is the product of the delta for the place, the two matrix-index deltas and the trace-basis delta. These are exactly the Gram identities of Lemma 37.30. That lemma gives the framed generator in Definition 10.6 with $\Phi = \sum_e b(X_e, Y_e)$. Setting the variables at the other places to zero gives the one-place condition, with the same b. Formula (12.27) identifies the fixed-coordinate series with its transport from representatives, including at a stabilizer. The odd-prime exponential criterion is Lemma 10.11.

For the one-place exponent zero in Definition 10.10, the derivatives of b and of ℓ^{-1} are integral by Lemma 10.7. An integral series after logarithmic substitution has the form $V(\ell^{-1})$ with V integral. Products and compositions of series with integral derivatives at zero have integral derivatives there, by the product and chain rules. The same holds for exponentiating a zero-constant such series. The trace-basis coefficients in the one-place substitution are integral. Applying any of the integer minor operators of Definition 10.18 therefore gives exponent zero for every cofactor in Definition 10.19. For the full product, absorb the phase difference into the integral cofactor, expand the cofactor in monomials and use the preceding argument on the factor from each place. For an operator of order D only monomials of degree at most D contribute. This proves the full zero-index entry bound as well.

We construct the common reading maps explicitly. Given a function on the complex classes of level $j+1$, take its polynomial value in the transported target frame. For functions in the image of frame evaluation this polynomial is uniquely determined by evaluation on the frames. Use a linear projection onto that image to extend the inverse reading linearly to all class functions, and apply ι to the coefficients. Outside lattice tuples take zero. Call the resulting semilinear map $R_{j,c',L'}$. On a rational input it gives precisely the local comparison of the same input. Indeed both are obtained by writing the adelic point as $\gamma y_C k$, acting on its class polynomial by γ, and multiplying by the level character of k; the coefficient embeddings agree. This is the reading construction of Lemma 16.3 at the target level. Define, at every layer,

$$S_j(L,L') = \left(\prod_v p^{\Delta_v (j+1)(s_v - s'_v)}\right) \Theta^F_{j,L,L'}.$$

The length factor is nonzero. Dividing by it recovers the chart series. These choices give one family of readings and one family of series for the entire chain.

Lean: `Formalism.L2.KerInert.SecondKernelProjection.rdJ` (`Formalism.LocalToGlobal.InertKernelTheory.ProjRSurj`, lines 120–123)

Lean: `Formalism.L2.KerInert.SecondKernelProjection.series_reading_entries` (`Formalism.LocalToGlobal.InertKernelTheory.InertCoprimeLemmas.SecondKernelProjection`, lines 143–166)

Lean: `Formalism.L2.KerInert.SecondKernelSeries.srOf` (`Formalism.LocalToGlobal.InertKernelTheory.TieSerPin`, lines 11–13)

Consider one rational input x, one source representative A, and one target tuple B. Express the source value of x by its weight polynomial. The contribution of its class to the complex lift at B is the contraction of this polynomial with the Taylor series of the finite test, with the source and target determinant characters. Transport this target polynomial to the chosen frame. A linear substitution in a Taylor series transforms its differential symbol by the transpose substitution. Applying this identity to each determinant in the symbol of Lemma 10.17 gives the contraction of the source reading with $S_j(A,B)$. The scalar left over is constant at the layer. More explicitly, if k_s, k_t are the source and target determinant factors of the orthogonal frame, d_τ the derivation generators, m_τ the degrees of the symbol blocks, and $\delta_j \in \{1,-1\}$ the sign of the row reversals, it is

$$c_j = \iota\left(\frac{k_t}{k_s\,\delta_j \prod_\tau d_\tau^{-m_\tau}}\right)(e\epsilon)^{-\sum_\tau m_\tau}.$$

The frame factors here are the products of the determinants of the basis changes to the powers of the source and target determinant twists. The orthogonal basis used for these factors is the same in all components of a level. The rational unitary change contributes its type norm, a unit at this inert prime; the local integral basis changes have unit determinants. The derivation generators are units at the permitted prime, and the coordinate scale has norm one by discriminant normalization and the unit tangent in Lemma 37.25. Thus $|c_j|_p = 1$, also for c_j^{-1}. If a component set is empty use $c_j = 1$. In any two decompositions of the adelic point the target polynomials just obtained have the same image under ι, namely this contraction. Injectivity coefficientwise gives the same polynomial before applying ι. This proves the reading identity for every decomposition, including the initial lift.

Lean: `GlobalTheory.TwV.DelA.cKs` (`Development.AbelianVarietyTheory.Definitions.GaloisGeometry.RealAt`, lines 65–68)

Lean: `GlobalTheory.TwV.DelA.tieDel_all_steps` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.KernelSiteFrameValuesAllStepsLtM`, lines 23–94)

Sum the identity over source classes with reciprocal stabilizer weights. On rational generators this is the defining finite sum for the arithmetic lift. For these selected classes and twists, the constructed column supplies the coefficient-field condition on the normalized Taylor coefficients, and the contraction with the symbol of Lemma 10.17 is a finite linear combination of them over the global coefficient field H

of Lemma 12.13. Each class-pair contraction therefore has coefficients in the original coefficient field, and the reciprocal stabilizer orders are rational. The finite class sum has a rational target representative, whose comparison lies in the rational target span. Linearity gives both preservation of that span and (10.1) for all inputs. At $j = m-1$ use the bijection between the rational classes and the local source representatives. Take the inner set in (10.2) to be a singleton and $w_x = |\operatorname{Stab}(x)|$. The bijection identifies the stabilizer groups. At positive levels their orders divide $D_{\Gamma,u}$; at zero they are one. Thus the choice $d_G(0) = 0$, $d_G(j) = v_p(D_{\Gamma,u})$ for $j > 0$ works simultaneously. The same representatives work for every input and target.
Lean: `Formalism.L2.KerInert.SecondKernelCoupled.second_kernel_coupled` (`Formalism.LocalToGlobal.InertKernelTheory.InertCoprimeLemmas.SecondKernelCoupled`, lines 44–75)

For an invariant function on edges, the sum at a fixed endpoint divided by its stabilizer order contributes $1/|\operatorname{Stab}(\text{edge})|$ per edge orbit. Counting from the other endpoint gives the identical weight. For the kernel use (12.27) on the source and the transpose substitution on its input polynomial. The level character cancels, and the determinant and length factors are those already included in the contraction. This gives the required source invariance and the terminal neighbour identity after reindexing by the same class bijection. The indicator assertion is Lemma 12.15. For the frame assertion, at each place lift to the target lattice a basis adapted to the given lower neighbour modulo p. Multiplication of its first i_v vectors by p gives a basis of the neighbour. Lemma 10.17 applies to this single frame for every series, exponent and input vector. Lastly, moves at two distinct places commute by interchanging their two independent choices of neighbours. The edge-lattice calculation (12.21) in Theorem 12.8, applied at level zero, gives integral preservation by each normalized input operator. This proves all conditions of Definition 10.10. The initial comparison is the fixed one by construction, and the class-sum calculation intertwines the adjacent lifts.

Claim 1. At each positive level j, these comparisons identify the local lattice with the scalar extension of the original localized rational lattice.

Write $O_0 = (\mathcal{O}_H)_w$ for the localisation of the integers of the column's global coefficient field at w, let $\mathcal{O}$ be the local coefficient ring, and denote the comparison by cmp_j. At each of the finitely many class representatives choose a rational matrix that is a basis of the prescribed lattice at all positions over p. Simultaneous approximation supplies such matrices. In these frames the original rational lattice is the intersection of the transported integral Weyl lattices with the stabilizer equations, including the original level character. The local chart uses precisely the same dual lattices and equations. The determinant shifts agree: $t+1$ is even, so $j(t+1)/2 = j((t+1)/2)$, and the two descriptions give the same shifted weight. The determinant normalizations are units at w, as checked above. Consequently, for a rational form x,

$$x \in \Lambda^{\mathrm{rat}}_{j,(w)} \quad \Longleftrightarrow \quad \text{all framed readings of } \mathrm{cmp}_j(x) \text{ lie in the corresponding integral Weyl lattices.}$$

This verifies the integral chart test for the originally specified lattice.

More explicitly, take the finite direct sum W_j of these framed O_0-Weyl lattices. Clear denominators in the finitely many stabilizer equations and write their joint map as A_j. The preceding identification gives $\Lambda^{\mathrm{rat}}_{j,(w)} = \ker A_j$, while $L^{\mathrm{loc}}_j = \ker(1 \otimes A_j)$. The extension $O_0 \to \mathcal{O}$ is flat, so

$$L^{\mathrm{loc}}_j = \mathcal{O} \otimes_{O_0} \Lambda^{\mathrm{rat}}_{j,(w)} = \operatorname{span}_{\mathcal{O}} \mathrm{cmp}_j(\Lambda^{\mathrm{rat}}_{j,(w)}).$$

The same kernel equations over fraction fields identify the rational and local spaces and give equality of their dimensions. A basis of the original localized lattice therefore becomes a basis of the local lattice. This proves the claim.

At $j = m$, write a rational terminal vector as $\sum_i a_i b_i$ in this basis. A local bound s is the set of inequalities $v_{\mathcal{O}}(\jmath(a_i)) \geq s\, v_{\mathcal{O}}(p)$. Since $e_w v_{\mathcal{O}}(\jmath(a_i)) = v_{\mathcal{O}}(p) v_w(a_i)$, these are exactly the original rational lattice inequalities $v_w(a_i) \geq e_w s$. Thus all the arithmetic hypotheses of Lemma 16.18 hold on the selected chain, and its descent preserves the exponent.
Lean: `Formalism.L2.KerInert.SecondKernelCoupled.hecke_coupled_entry` (`Formalism.LocalToGlobal.InertKernelTheory.InertCoprimeLemmas.SecondKernelCoupled`, lines 104–136)
Lean: `GlobalTheory.TwV.admissible_at_each_site` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.JointSiteChainComparedCarriesAdmittedSystem`, lines 29–75)

□

Proofs for Chapter 38: The chain length

Proof of 38.3

Proof. By the ceiling bound above, $a \le e(a) < a+2 \le 2a$ for $a \ge 2$. Here $2 \le \alpha \le x$ and $\beta \ge x^4$. Indeed, $1/2 \le 1-\varepsilon/8 \le 2$ and $x \ge 16$ give $4 \le x^{1-\varepsilon/8} \le x^2$. Choose $m = e(\alpha)$ and $k = e(\beta)$. Since $2 \le 2+8\varepsilon \le 3$, the order bounds give $x^2 \le n \le 2x^3$. Hence

$$2 \le m \le 2x \le 2x^2 \le 2n \le 4x^3 \le x^4 \le k.$$

Also $m+3 \le 2x+3 \le x^4 \le k$.

Define $r = k+m-1$ and $t = 8fr+4+TW-r$. Since $f \ge m \ge 2$ and $r \ge m$, the difference is at least $15r+4$, so it has its ordinary integer value. Since k and m are even, r is odd. Since $8 \mid W$, the integer t is odd. Moreover,

$$r+5 \le t, \qquad 3m < t, \qquad m^2 \le m(t+1) \le k+m(t+1).$$

The bounds and parity conditions make (k,m,t) rank data on φ. The formula for t gives the identity for $r+t$. The choices of m and k give their respective bounds. □

Proofs for Chapter 39: Galois representations attached to polarized cuspidal representations

Proof of 39.7

Proof. Write b_v for the exponent of χ_v on the positive reals at each real place v of F. Since $a \ge 1$, π has at least one infinitesimal weight at each embedding, and each weight lies in $(a-1)/2+\mathbf{Z}$, so the multisets H_τ of Definition 39.4 consist of integers. At the pair of embeddings above v, polarization pairs infinitesimal weights by $z \mapsto b_v - z$, so $b_v = z + (b_v - z)$ is integral. The finite part of χ is trivial on a finite-index subgroup of the totally positive units of F. The vector $(b_v)_v$ annihilates their logarithms, which span the hyperplane of coordinate sum zero by Dirichlet's unit theorem [41, Ch. I, §7]. Thus all b_v equal an integer b. Subtracting the paired weights from $(a-1)/2$ gives $H_{\tau c} = a-1-b-H_\tau$. □

Proof of 39.8

Proof. Choose r by 39.54, and take $w_{\mathrm{gt}} = a-1-b$ and the integer multisets H_τ of 39.4 by Lemma 39.7.

Fix a prolongation and a map R of 40.2. Set $\sigma^* = \pi_w \otimes |\det|_w^{(1-a)/2}$ and $y = R(\sigma^*)$. The dimension law gives $\dim y = a$, the dimension of the parameter away from ℓ. The spherical assertion above ℓ follows from 39.54.

Suppose $N_y = 0$. Cuspidality gives genericity of π_w and its twist σ^* [207]. Under the commutator test, 40.11 gives unlinked rank-one support. Proposition 39.58 gives the toral conclusions, including the jumps above ℓ.

For the tame clause, assume $E_w \simeq \mathbf{Q}_p$ continuously. Every wild inertia element has finite-order image and all eigenvalues one, so the action of P_w is trivial. By 40.11, σ^* has unlinked depth-zero support and first principal congruence invariants. Its support terms of rank at least two are pairwise inertially inequivalent. The unramified normalizing twist preserves all these properties. Use 39.59 to take the lift Π to M, the place u above w with M_u/E_w unramified, and the map R' at u. Put $\Pi_u^* = \Pi_u \otimes |\det|_u^{(1-a)/2}$. Proposition 39.58 gives $x' = R'(\Pi_u^*)$ for the upper Galois parameter, with $N_{x'} = 0$.

Above ℓ the upper comparison gives potential semistability downstairs, and the specified period bridge extends to M_u. In both cases the parameter x' restricts the Weil action of the downstairs parameter x and retains its monodromy operator [205]. Thus $N_x = 0$. For $t \in I_w = I_{M_u}$ the two comparisons give

$$\det(X - \rho_x(t)) = \det(X - \rho_{x'}(t)) = \det(X - \rho_{R'(\Pi_u^*)}(t)) = \det(X - \rho_y(t)).$$

At $t = 1$ this gives the rank. Above ℓ, scalar extension of the filtered de Rham module preserves the jumps at each induced embedding [205]. The pullback of infinitesimal characters in 39.59 identifies the upper jumps with H_τ. □

Proof of 39.9

Proof. Take the corresponding plan of 39.13 and the semisimplified eigenspace of 39.21. Let κ_ℓ be the ℓ-adic character of the plan's algebraic Hecke character κ. At a place w where κ is unramified, write $\kappa(w)$ for its value on a uniformizer. For $w \nmid \ell$, the value of κ_ℓ at inverse arithmetic Frobenius is $\kappa(w)$ after transport by ι. For $w \mid \ell$, the unramified Weil parameter of the crystalline character has transported value $\kappa(w)$ at degree -1. The character has jump $-m_{\iota\circ\tau}$ at τ, with $m_{\iota\circ\tau}$ the corresponding infinity exponent of κ [206, Ch. III][205].

Tensor the eigenspace representation by κ_ℓ. Proposition 39.23 gives the annihilators away from ℓ. At a Hodge place, 39.26 and 39.27 give degree $j+m_{\iota\circ\tau}$ for $j \in J(\lambda)$. Tensoring gives jump j. At v' the prescribed set supplies the lacunary subset. For the root condition above ℓ, 39.27 gives the root multiset bound before twisting. Tensoring multiplies every root at degree -1 by $\kappa(w)$, so it gives the asserted divisibility for $A_a(w,B)$. □

Proof of 39.10

Proof. Write $f_U : H^1 \to W^\vee$ for the given form. On a rational isometry its left transformation law is the dual weight action, which is the restriction of ξ. Choose its level $K \subset H$ with $\nu(K) \subset \widehat{\mathbf{Z}}^\times$. At w_0 take the kernel of the type character for this level, and retain the action of the full type subgroup by right translations. At the spherical places use the integral general linear group. Thus f_U is right invariant under $K \cap H^1$.

Define a function f_0 on H by

$$f_0(\gamma u k) = \xi(\gamma) f_U(u) \quad (\gamma \in \Gamma,\ u \in H^1,\ k \in K), \qquad f_0(h) = 0 \quad (h \notin \Gamma H^1 K).$$

If $\gamma u k = \gamma' u' k'$, then $\nu(\gamma'^{-1}\gamma) = \nu(k' k^{-1})$. The left side is a positive rational number and the right side is a unit at every finite prime. Hence both equal one. Now

$$u' = (\gamma'^{-1}\gamma) u (k k'^{-1}),$$

where the first factor is a rational isometry and the last lies in $K \cap H^1$. The two transformation laws of f_U give the same value of f_0. In particular $f_0|_{H^1} = f_U$.
Lean: `UpperTheory.Gal.CertificateRingHomMap.pKerFormU_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ProjectiveAutomorphicHecke.PKerFormU`, lines 93–132)
Lean: `UpperTheory.Gal.CertificateRingHomMap.pExtZero_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ProjectiveAutomorphicHecke.PExtZero`, lines 65–96)

The type translations at w_0 have multiplier one and preserve the kernel of the type character. At a split spherical place the Hecke sum has right-coset representatives in the local general linear group embedded in H^1. The integral subgroup permutes these cosets, so the sum preserves right K-invariance. On H^1 each operator therefore has its original eigenvalue on f_0. The eigenvalue identity extends to $\Gamma H^1 K$ by left equivariance and right invariance. Outside this set both sides vanish, since right multiplication by an element of H^1 preserves each multiplier. This also proves that right translation by a central unitary element acts on f_0 through the finite central character of U.
Lean: `UpperTheory.Gal.CertificateRingHomMap.pFormGU_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ProjectiveAutomorphicHecke.PFormGU`, lines 35–153)

Consider the span of the functions $h \mapsto f_0(h\zeta(y))$. Put $O = \zeta^{-1}(K)$, an open subgroup of the finite idèles. Its elements act trivially on this span. For $t \in E^\times$, the element $\zeta(t)$ is rational, and $N_{E/\mathbf{Q}} t > 0$ since E is CM. Its action is scalar. Indeed the dual weight at v takes the scalar matrix $z 1_a$ to $z^{-\sum_i \lambda_{v,i}}$. Consequently the scalar is

$$(N_{E/\mathbf{Q}} t)^b \prod_v \tau_v(\overline{N_{E/I} t})^{-\sum_i \lambda_{v,i}} = \prod_{\tau : E \hookrightarrow \mathbf{C}} \tau(t)^{m_\tau}.$$

The equality follows by expanding each norm over the embeddings extending its indicated embedding of I.

The quotient $\mathbf{A}_{E,f}^\times / (E^\times O)$ is finite. The valuation map reduces the assertion to finiteness of the ideal class group of the number field E and finiteness of the index of $O \cap \widehat{\mathcal{O}}_E^\times$ in the compact group $\widehat{\mathcal{O}}_E^\times$. The first is [41, Ch. I, Thm. 6.3], and the second follows from openness. The central translates thus span a nonzero finite-dimensional space. They commute with all the multiplier-one operators above. Their actions

on this space are commuting automorphisms, so they have a common eigenvector Φ. Indeed, restricting to an eigenspace of any nonscalar member reduces the dimension and preserves all the other actions. Write $\psi(y)$ for the eigenvalue of translation by $\zeta(y)$. The group law makes ψ a character. It is trivial on O, and on the elements with $\zeta(y) \in H^1$ it agrees with the central character of U evaluated at $\zeta(y)$.
Lean: `UpperTheory.Gal.CertificateRingHomMap.pExtEig2_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ProjectiveAutomorphicHecke.PExtEig2Of`, lines 39–149)
Lean: `UpperTheory.Gal.CertificateRingHomMap.pCentralEig2_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ProjectiveAutomorphicHecke.PCentralEig2Of`, lines 52–90)

For an idèle $y = (y_f, y_\infty)$ define

$$\kappa(y) = \psi(y_f)^{-1} \prod_{\tau : E \hookrightarrow \mathbf{C}} y_{\infty,\tau}^{m_\tau}, \qquad y_{\infty,\bar\tau} = \overline{y_{\infty,\tau}}.$$

The rational scalar calculation gives $\kappa(t) = 1$ for every $t \in E^\times$. This defines the asserted Hecke character and fixes the inverse in its finite part.
Lean: `UpperTheory.Gal.CertificateRingHomMap.zetaEx_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ProjectiveAutomorphicHecke.ZetaMain`, lines 56–101)
Lean: `UpperTheory.Gal.CertificateRingHomMap.pKapOf_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ProjectiveAutomorphicHecke.PKapOf`, lines 37–139)

□

Proof of 39.11

Proof. For every rational prime q with $-d \in (\mathbf{Q}_q^\times)^2$, the algebra $I \otimes_{\mathbf{Q}} \mathbf{Q}_q \simeq \mathbf{Q}_q[T]/(T^2 + d)$ splits, so q splits in $I = \mathbf{Q}(\omega)$ and does not divide the discriminant of $\mathcal{O}_I = \mathbf{Z}[\omega]$; hence the minimal polynomial of ω has distinct roots modulo q. By Chebotarev's density theorem for E/F, choose a finite place $\mathfrak{p}_1$ of F inert in E/F whose residue characteristic q_1 is odd, unramified in E, and outside X_0. Let $\chi_{E/F} : \mathbf{A}_F^\times / F^\times \to \{\pm 1\}$ be the quadratic character of E/F. Define $z \in \mathbf{A}_F^\times$ with $z_{v_0} = 1$, $z_v = -1$ at the $[F : \mathbf{Q}] - 1$ real places $v \neq v_0$, $z_{\mathfrak{p}_1}$ a uniformizer at $\mathfrak{p}_1$, and $z_{\mathfrak{p}} = 1$ elsewhere. Because $[F : \mathbf{Q}]$ is even, $\prod_v \chi_{E/F,v}(z_v) = (-1)^{[F:\mathbf{Q}]-1}(-1) = 1$. By global norm reciprocity [41, Ch. VI, §5], $z = \mu\, N_{E/F}(w)$ for some $\mu \in F^\times$ and $w \in \mathbf{A}_E^\times$. Then $\mu_{v_0} > 0$, $\mu_v < 0$ for real $v \neq v_0$, and μ is a local norm from E at every finite place $\mathfrak{p} \neq \mathfrak{p}_1$ of F. Set $h_{\mathrm{std}} = \mu\, 1_a$. Since $\mu \in F^\times$ is fixed by conjugation, ${}^t\bar g(\mu\, 1_a) g = s(\mu\, 1_a)$ is equivalent to ${}^t\bar g g = s\, 1_a$, so $\mathrm{GU}^+(h_{\mathrm{std}})(\mathbf{Q}) = \Gamma$.

By weak approximation at the real places of F, choose $\alpha \in F^\times$ with $\alpha_{v_0} < 0$ and $\alpha_v > 0$ for every real $v \neq v_0$, and set $h_0 = \mathrm{diag}(\mu, \alpha\mu, \dots, \alpha\mu)$ with one entry μ and $a - 1$ entries $\alpha\mu$. Then h_0 has signature $(1, a-1)$ at v_0 and $(0, a)$ at real $v \neq v_0$. Because $a-1$ is even, $\det(h_0)/\det(h_{\mathrm{std}}) = \alpha^{a-1} = N_{E/F}(\alpha^{(a-1)/2})$ is a local norm at every finite place of F. By the local Hermitian classification in Definition 1.20, $(E^a \otimes_F F_{\mathfrak{p}}, h_0)$ and $(E^a \otimes_F F_{\mathfrak{p}}, h_{\mathrm{std}})$ are isometric at every finite place $\mathfrak{p}$ of F, and at every $\mathfrak{p} \neq \mathfrak{p}_1$ both are isometric to 1_a.

At each $q \in X_0$, we have $q \neq q_1$ and $\mathcal{O}_E \otimes_{\mathbf{Z}} \mathbf{Z}_q = \mathcal{O}_{E,q}^+ \times \mathcal{O}_{E,q}^-$ according to the two roots of ω modulo q, interchanged by conjugation. Taking $L_q^+ = (\mathcal{O}_{E,q}^+)^a$ and L_q^- to be its $\mathbf{Z}_q$-dual under $(u, v) \mapsto \mathrm{Tr}_{E/\mathbf{Q}}(x\, h_0(u, v))$ gives an $(\mathcal{O}_E \otimes \mathbf{Z}_q)$-lattice $L_q = L_q^+ \oplus L_q^-$ on which the alternating pairing is $\mathbf{Z}_q$-valued and unimodular. Modify $\mathcal{O}_E^a$ at the primes $q \in X_0$ to match L_q, and multiply h_0 and μ by the square of a positive integer prime to $\prod_{q \in X_0} q$ clearing denominators of the pairing away from X_0 to obtain h_{PEL}, Λ, and $e = [\Lambda^\vee : \Lambda]$ prime to $\prod_{q \in X_0} q$. At every $q \nmid e$, e_Λ is perfect over $\mathbf{Z}_q$. By Chebotarev's density theorem for $I/\mathbf{Q}$, infinitely many primes p_0 split in I, hence satisfy $-d \in (\mathbf{Q}_{p_0}^\times)^2$; choose $p_0 \notin X_1$ with $p_0 \nmid q_1 e$. Away from p_0, choosing compatible local $\mathcal{O}_E$-lattice bases at $q \neq q_1$ and matching the common local Hermitian space of h_{PEL} and h_{std} at $\mathfrak{p}_1$ identifies the away-from-p_0 PEL similitude group with H^{p_0} and places h_{std} as the standard definite summand. □

Proof of 39.12

Proof. By Chebotarev's density theorem for the quadratic extension $I = \mathbf{Q}(x)/\mathbf{Q}$, infinitely many rational primes split in I, so we may choose a rational prime r_0 splitting in I outside the finite set consisting of p_0, ℓ, the residue characteristics of $\mathcal{H} \cup \mathcal{C}$, the bad level primes of Φ, and the prime divisors of e. Above r_0, the places of E break into conjugate pairs, and choosing one place from each pair on the selected side of I produces a nonempty finite set $\mathcal{W}$. Set $K_0 = K(N) \subset H^{p_0}$. Transport the elliptic type (J, χ) of Φ at w_0 by geometric inversion-transpose to $J_{\mathrm{typ}} = \{{}^t y^{-1} : y \in J\}$ and $\psi_{\mathrm{typ}}(y) = \chi({}^t y^{-1})$. Using the away-from-$p_0$

frame identification θ of Lemma 39.11, let $K_\Phi \subset H^{p_0}$ be the compact open level subgroup stabilizing $\Phi^\sharp$ at every finite place away from p_0, w_0, and $\mathcal{W}$ (equal to the transported level subgroup of Φ at each bad level place, hyperspecial at every good place away from p_0, w_0, including the places in $\mathcal{W}$, and equal to J_{typ} at w_0), and define

$$K_{\mathrm{typ}} = \big\{k \in K_\Phi : k_W \in I_W \text{ for all } W \in \mathcal{W}\big\},$$

where $I_W \subset \mathrm{GL}_a(E_W)$ is the standard Iwahori subgroup at W. The level hypothesis gives $K_0 \subset K_\Phi$. At each $W \in \mathcal{W}$, the condition $r_0 \mid N$ puts the principal congruence component inside I_W. Hence $K_0 \subset K_{\mathrm{typ}}$, and compactness of K_{typ} and openness of K_0 give $[K_{\mathrm{typ}} : K_0] < \infty$, the w_0-component of K_0 lies in $\ker \psi_{\mathrm{typ}} \subseteq J_{\mathrm{typ}}$ (so ψ_{typ} is trivial on $K_0 \cap J_{\mathrm{typ}}$), and the normalized character projector $e = e_{\mathrm{typ}} \in \mathcal{H}(K_0)$ supported on K_{typ} (transforming by ψ_{typ}^{-1} at w_0 and trivial on all other local components of K_{typ}) satisfies $e^2 = e$ and fixes the transported form $\Phi^\sharp$. For each $W \in \mathcal{W}$, choose a uniformizer ϖ_W of E_W and put

$$t_W = \mathrm{diag}(1, \varpi_W^{-1}, \dots, \varpi_W^{1-a}) \in \mathrm{GL}_a(E_W).$$

At each $W \in \mathcal{W}$, the Iwahori–Matsumoto presentation gives a local Hecke function $\varphi_W \in \mathcal{H}(I_W)$ satisfying $\varphi_W * \mathbf{1}_{I_W t_W I_W} = \mathbf{1}_{I_W}$ [204, §3][22]. For any $h \in H^{p_0}$ with trivial component at w_0, let $e(h) \in \mathcal{H}(K_0)$ denote the type Hecke function supported on $K_{\mathrm{typ}} h K_{\mathrm{typ}}$. Because K_{typ} factors with local component I_W at each $W \in \mathcal{W}$ and the places w_0 and $W \in \mathcal{W}$ are pairwise disjoint, writing $\varphi_W = \sum_\alpha c_\alpha \mathbf{1}_{I_W s_\alpha I_W}$ and lifting each s_α via the local embedding $\mathrm{loc}_W : \mathrm{GL}_a(E_W) \hookrightarrow H^{p_0}$ defines $v_W = \sum_\alpha c_\alpha e(\mathrm{loc}_W(s_\alpha)) \in \mathcal{H}(K_0)$ satisfying

$$\big(v_W * e(\mathrm{loc}_W(t_W))\big)(g) = e\big(g\,\mathrm{loc}_W(g_W)^{-1}\big) \cdot (\varphi_W * \mathbf{1}_{I_W t_W I_W})(g_W) = e(g),$$

so $v_W * e(\mathrm{loc}_W(t_W)) = e$. By induction on the finite set $\mathcal{W}$, setting $\tau = \prod_{W \in \mathcal{W}} \mathrm{loc}_W(t_W)$ (whose component is t_W at each $W \in \mathcal{W}$ and 1 elsewhere) and $u_0 = e(\tau)$, disjointness of the places in $\mathcal{W}$ yields $e(\mathrm{loc}_W(t_W)\tau') = e(\mathrm{loc}_W(t_W)) * e(\tau')$ at each step; hence the reversed convolution product $u_0' = \prod^{\mathrm{rev}}_{W \in \mathcal{W}} v_W \in \mathcal{H}(K_0)$ satisfies $u_0' u_0 = e$, $e u_0 = u_0 e = u_0$, and $u_0 e(h) = e(h) u_0$ for every $h \in H^{p_0}$ trivial at w_0 and $\mathcal{W}$.

Lean: `UpperTheory.Gal.Spc.exists_invertible_hecke_function_at_finite_places` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ScalarCommutation.ExistsInvertibleHeckeFunctionAtFinitePlaces`, lines 66–106)

Claim 1. The selected operators satisfy

$$e T_g e = T_g e, \qquad T_g T_{g'} e = T_{g'} T_g e, \qquad u_0 T_g e = T_g u_0 e.$$

A deck operator followed by e is a scalar multiple of e: its translating element lies in K_{typ}. For a selected deck element the scalar is one, as follows by evaluating on the nonzero form fixed by e. A selected spherical operator followed by e is a scalar multiple of a type Hecke function supported on a spherical double coset at its chosen place, followed by a central translation. At the same place these functions commute in the spherical Hecke algebra; at distinct places their local factors commute. They are fixed by e on both sides. Their auxiliary components lie in the Iwahori subgroups, so they also commute with u_0 after applying e. These facts give the three displayed identities.

Lean: `UpperTheory.Gal.Spc.memComm` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ScalarCommutation.MemComm`, lines 170–232)

Claim 2. A central translation makes the test integral without changing these identities. Write $\widehat{\Lambda}^{p_0}$ for the fixed away-from-p_0 lattice. Choose a positive integer c such that

$$c k \widehat{\Lambda}^{p_0} \subseteq \widehat{\Lambda}^{p_0} \qquad (k \in K_{\mathrm{typ}}).$$

One can choose one c by clearing denominators for the finitely many cosets modulo K_0, which preserves the lattice. The inverse translation is trivial outside the places above r_0. Since r_0 is a good prime, choose $M \geq 0$ so that $r_0^M \tau^{-1}$ preserves the lattice. Let d_c, d_r be the central similitudes acting by c and r_0^M, respectively, and set

$$D = d_c d_r d_c, \qquad D_W = c^2 r_0^M\, 1_a, \qquad u(g) = u_0(Dg), \qquad u'(g) = u_0'(D^{-1} g).$$

Here D_W denotes the matrix of D at a split place. If $u(g) \neq 0$, write $Dg = a \tau b$ with $a, b \in K_{\mathrm{typ}}$. Then

$$g^{-1} = (b^{-1} d_c)(\tau^{-1} d_r)(a^{-1} d_c).$$

Each factor preserves $\widehat{\Lambda}^{p_0}$. Thus the inverse support is integral. Centrality of D and the inverse translation in u' give $eu = ue = u$, $u'u = e$, and $u T_g e = T_g u e$.

Convolution and length additivity give, for $n \geq 1$,

$$u^n e(g) \neq 0 \quad \Longrightarrow \quad g^{-1}\widehat{\Lambda}^{p_0} \subseteq \widehat{\Lambda}^{p_0}, \qquad D_W^n g_W \in I_W t_W^n I_W \quad (W \in \mathcal{W}).$$

Multiplication by a selected operator preserves the latter double cosets, because its auxiliary components lie in I_W and D_W is scalar. For $J \in I_W$, every principal minor of J is a unit. The coefficient of X^{a-k} in the characteristic polynomial of $t_W^n J$ has a unique term of smallest valuation, indexed by the last k diagonal entries, with valuation

$$-\frac{nk(2a-k-1)}{2}.$$

The Newton slopes are therefore $0, -n, \ldots, -(a-1)n$, each with multiplicity one. Multiplication by D_W^{-n} shifts all slopes by the same number, so they remain distinct. This proves the support assertion for every word polynomial.

Lean: `UpperTheory.Gal.Spc.auxFn` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ScalarCommutation.ExistsIdempotentHeckeOperatorWithRegularSupport`, lines 157–257)

□

Proof of 39.13

Proof. Let $\mathcal{H}, \mathcal{C}$ be the disjoint sets of prescribed Hodge and root places. Apply Lemma 39.11 with $X_0 = \{\ell\} \cup \{\operatorname{char} k(w) : w \in \mathcal{H} \cup \mathcal{C}\}$ to obtain the ambient PEL datum $(E^a, \mathcal{O}_E, \Lambda, e_\Lambda, \omega)$, the integer e_{PEL}, and the standard definite form $h_{\mathrm{std}} = \mu\, 1_a$. For a usable split place w of residue characteristic $q \neq p_0$ where the minimal polynomial of ω has a simple root $r \bmod q$ with $\mathfrak{u} = (q, \omega - r) \subseteq w$, a *local model chart* at w of level N specifies a lower ideal level $N_0\mathfrak{a} \supseteq (N)$, where $N_0 \geq 3$ is the prime-to-q part of N (prime to qp_0), $\mathfrak{a} + w = \mathcal{O}_E$, and $\mathfrak{u}^e \subseteq \mathfrak{a}$ for some $e \geq 0$, together with the q-local identification of the PEL lattice on the selected side of $I = \mathbf{Q}(x)$. Choose local model charts on the same side of $\mathbf{Q}(x)$ for the places in $\mathcal{H} \cup \mathcal{C}$ of each residue characteristic, using at ℓ the side determined by w_0. Choose a weight vector of U in the elliptic type at w_0 and spherical at the prescribed places. Apply Lemma 39.10 to obtain the similitude form Φ and its character κ. The two partitions at the embeddings over v are (b_v^a) and $\lambda_v + (b_v^a)$, both with nonnegative entries by the choice of b_v. Choose $g_0 \in H$ with $\Phi(g_0) \neq 0$. By Lemma 39.11(2), choose the model prime p_0 outside $X_0 \cup \{q : q \mid e_{\mathrm{PEL}}\}$, the residue characteristics of the bad level places of Φ, and the finitely many primes where g_0 is not integral and unimodular, obtaining the compact open subgroup $K_{p_0} \subset G_{p_0}$ and the away-from-p_0 frame. By Lemma 39.12, choose the auxiliary split prime $r_0 \notin \{p_0, \ell\}$ and place $W \mid r_0$ outside X_0, the bad level primes of Φ, and the prime factors of e_{PEL}, and take a sufficiently deep common principal level $N \geq 3$, prime to p_0 and divisible by r_0, whose subgroup in the transported geometric coordinates is contained in K_Φ, has w_0-component in $\ker \psi_{\mathrm{typ}}$, and admits all prescribed charts. Such a level exists because the prescribed local subgroups are open and impose nontrivial conditions at only finitely many places.

The deck correspondences that fix the vector have eigenvalue one. Choose spherical correspondences at charts compatible with $\mathcal{H}$ and outside $\mathcal{H}$, all of residue characteristic different from r_0. In the coordinates of Lemma 39.10, the operator of index i uses the multiplier-one spherical cosets together with a central translation z_w^i. Put $q_w = \#k(w)$ and choose a uniformizer ϖ_w, viewed as a finite idèle supported at w. To construct $z_w \in H$, let $v_I = w \cap \mathcal{O}_I$ be the place of $I = \mathbf{Q}(x)$ below w and let $a \in \mathbf{A}_{I,f}^\times$ be the finite idèle equal to q_w at v_I and 1 at every other finite place of I; embed a into $\mathbf{A}_{E,f}^\times$, apply complex conjugation to set $t = \overline{\iota_{E/I}(a)} \in \mathbf{A}_{E,f}^\times$, and define $z_w = (t \cdot 1_a,\, N_{I/\mathbf{Q}}(a)) \in H$. Since $t\bar{t} = N_{I/\mathbf{Q}}(a)$, the element z_w is central in H: on the side of $I = \mathbf{Q}(x)$ conjugate to w, its scalar is q_w at every place above v_I, and on the remaining factors it is one; moreover, because w is split in I and q_w is a power of $N_{I/\mathbf{Q}}(v_I)$, its similitude multiplier has rational valuation $v(q_w)$ at every rational prime v. The scalar of $\zeta(\varpi_w)$ on the conjugate side has the same valuation, by the local norm formula, and it is one off that side. Their quotient therefore has integral unit components and is one at the exceptional level places by the choice of chart, so it fixes Φ. The same argument with a local unit shows that κ is unramified at w. Thus z_w^i contributes $\kappa(w)^{-i}$, and the eigenvalue upstairs is the original spherical eigenvalue times $\kappa(w)^{-i}$ and the level index.

Lean: `UpperTheory.Gal.CertificateRingHomMap.ZetaNonvanishing₂.cenEx` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ProjectiveAutomorphicHecke.CyclotomicConjugation.SpanAbsNorm`, lines 156–191)

Lean: `UpperTheory.Gal.CertificateRingHomMap.pKapOf_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ProjectiveAutomorphicHecke.PKapOf`, lines 37–139)

To put this vector in the definite trace, use the geometric coordinates $(g,s)\mapsto({}^tg^{-1},s^{-1})$. Pullback gives left weight $\xi^\sharp(g,s)=\xi({}^tg^{-1},s^{-1})$. Transport the test by the same coordinate map. On the standard definite summand (E^a,h_{std}) of Lemma 39.11(3), whose rational positive-multiplier similitude group is Γ, write $\phi(h)$ for the transported form read through the away-from-p_0 frame on $h\in H^{p_0}$. If $\delta\in\Gamma$ satisfies $\delta_{p_0}\in K_{p_0}$, then δ is integral at p_0 and $\phi(\delta^{p_0}h)=\xi^\sharp(\delta)\phi(h)$ by rational equivariance and right invariance under the away-from-p_0 frame discrepancy. On $H^{p_0}\times G_{p_0}$ set

$$\widetilde{\phi}(h,h_0)=\xi^\sharp(\delta)\phi((\delta^{p_0})^{-1}h)\quad\text{if } h_0\in\delta_{p_0}K_{p_0},$$

and set it to zero if there is no such δ. Two choices of δ differ by an element of Γ with component in K_{p_0}, so the displayed equivariance makes $\widetilde{\phi}$ well defined. It is a rationally equivariant form of product level $K(N)\times K_{p_0}$. Its slice at $h_0=1$ is nonzero by the choice of p_0 at g_0 where $\Phi(g_0)\neq0$. On every nonzero slice the formula transports the type and Hecke equations, so the chosen eigensystem occurs in the standard definite summand of the trace, with the indicator $\mathbf{1}_{K_{p_0}}$ as the test factor at p_0. Let $e=e_{\mathrm{typ}}$ and u,u' be the type projector and Iwahori operators at $W\mid r_0$ provided by Lemma 39.12.

Lean: `UpperTheory.Gal.Spc.eigEx` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ScalarCommutation.FrameTh`, lines 52–91)
Lean: `UpperTheory.Gal.W2.Seed.pHasEigOf` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SphericalCoordinates.AlgebraEquivSetup.PHasEigOf`, lines 45–86)
Lean: `UpperTheory.Gal.Spc.specOfF` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ScalarCommutation.SpecOfF`, lines 179–227)

Claim 1. The compressed operators $A_g=T_ge$ form a commuting family on the direct sum of the definite form spaces, and the spectral test can be chosen in the form $f=u^ne\otimes\mathbf{1}_{K_{p_0}}$, with the integral normalization of Lemma 39.12.

Indeed, that lemma gives

$$eT_ge=T_ge,\qquad T_gT_{g'}e=T_{g'}T_ge,\qquad uT_ge=T_gue,\qquad eu=ue=u,\qquad u'u=e.$$

Consequently

$$A_gA_{g'}=T_gT_{g'}e=T_{g'}T_ge=A_{g'}A_g,\qquad eA_g=A_ge=A_g,\qquad uA_g=A_gu.$$

This includes the selected individual deck operators, whose compressions are e. No commutativity of the deck group is required.

Let $\mathcal{V}$ be the finite-dimensional direct sum of the definite form spaces at this weight and level, and let P_θ be the joint primary projector for the commuting operators A_g. It is a word polynomial in the A_g, with value one at θ. Both e and u commute with it. The vector $\widetilde{\phi}$ belongs to

$$M=\mathrm{im}(P_\theta e),$$

so $M\neq0$. On M, e is the identity and $u'u=e$ makes u injective, hence invertible. Some positive power u^n has nonzero trace on M. Otherwise Newton's identities would make every coefficient except the leading coefficient of its characteristic polynomial zero, contradicting invertibility.

For every word polynomial s, the restriction of $s(T)-s(\theta)$ to M is nilpotent and commutes with u. Thus

$$\mathrm{tr}(P_\theta s(T)u^ne)=s(\theta)\,\mathrm{tr}(u^n\mid M).$$

The right side is nonzero when $s(\theta)\neq0$. The relations $eT_ge=T_ge$ allow every word in the A_g, after right multiplication by e, to be written as the same word in the T_g. Hence the projector gives the required word polynomial a_0, and the same n and a_0 work for every such s.

Lean: `UpperTheory.Gal.Spc.specAlg` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ScalarCommutation.SpecAlg`, lines 166–255)
Lean: `UpperTheory.Gal.Spc.specData` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ScalarCommutation.SpecData`, lines 64–161)

Claim 2. The regularity and orbital vanishing assertions hold for the selected test. By Lemma 39.12(2), every element in the away-from-p_0 support of $a_0(T)s(T)u^ne$ has separable characteristic polynomial of degree a. At the split type place w_0 the slice of e is a cuspidal test function: its constant term along every proper parabolic vanishes. Convolution with the selected operators preserves this property, since integrating a

translate along a unipotent radical gives the corresponding translate of a constant term along a conjugate parabolic. A regular element with reducible characteristic polynomial is conjugate into a proper block Levi, and its orbital integral factors through the constant term on that Levi because on the off-diagonal blocks the change of unipotent variables has invertible linear factors given by differences of eigenvalues in distinct blocks; hence its orbital integral vanishes. This removes the marked geometric terms with reducible polynomial at w_0. On the definite side, Lemma 39.12(2) forces every contributing rational similitude to have a separable characteristic polynomial of full degree a, and the same constant-term calculation at w_0 removes those with reducible polynomial.

Lean: `UpperTheory.Gal.Spc.vanPairs` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ScalarCommutation.VanPairs`, lines 306–321)
Lean: `UpperTheory.Gal.Spc.vanForms` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ScalarCommutation.VanForms`, lines 127–191)
Lean: `UpperTheory.Gal.Int.Geo.Pl.master` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.PrimeLevelLocalization.SameKTrans`, lines 86–205)
Lean: `UpperTheory.Gal.Spc.specTopG` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ScalarCommutation.SpecTopG`, lines 9–21)

Claim 3. The chosen compatible charts supply common integral lower models at all places of $\mathcal{H}$, with finite étale first projections for the selected operators. More precisely, at such a place v, of residue characteristic q, the PEL form is perfect at q, the chosen quadratic generator has a simple root modulo q, and the lower level has the form $N_0\mathfrak{a}$, with

$$N_0 \geq 3, \qquad (N_0, qp_0) = 1, \qquad \mathfrak{a} + v = \mathcal{O}_E, \qquad (q, \omega - r)^e \subseteq \mathfrak{a}$$

for some e, on the chosen side. The structural map is the single map $\mathcal{O}_E \to \mathcal{O}_{E,(v)}$. The PEL scheme and its abelian fibre powers are smooth projective there. The selected operators are either deck maps or spherical correspondences based at $w \neq v$ with compatible local charts. For every such operator, passing to the common lower level gives an integral correspondence whose first projection is finite étale.

The integral PEL construction applies with one structural map and matrix multiplicity one. The inequality $2 < [E : \mathbf{Q}]$, supplied by the positive even degree of its maximal real subfield, gives its properness condition. Smoothness follows by lifting the prescribed isotropic Hodge summand and the prime-to-q level across square-zero extensions. Properness follows by the valuative extension of the PEL object; quasi-projectivity then gives projectivity. Additional levels on the chosen étale torsion factors preserve both properties, as do abelian fibre powers. For deck operators the level lattice is preserved. For a spherical operator at a different residue characteristic its degree is prime to q, and the compatible principal-level inclusions give the required étale level change. At the same residue characteristic but a different selected place, compatibility identifies the side and level ideal, and the other local factor has zero Lie algebra, so its finite torsion is étale. These are the three local cases in the common-model construction.

Lean: `GlobalTheory.UPel.HT.exists_sqZero_lift` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.HarrisTaylorExistsLiftSquareZeroArtinian`, lines 19–55)
Lean: `GlobalTheory.UPel.HT.isProper_model_of_compact` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.HarrisTaylorIntegralModelIsProperOfCompact`, lines 27–58)
Lean: `UpperTheory.Gal.Int.Geo.Pl.goodOf_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.PrimeLevelLocalization.PGoodOf`, lines 101–153)
Lean: `UpperTheory.Gal.Int.Geo.Pl.goodOf_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.PrimeLevelLocalization.ExistsUnitaryGroupDatumOfVectorExponentSpectral`, lines 32–33)

Claim 4. The weight transport already chosen for Φ supplies the required degree data. At every embedding $\tau : E \to \mathbf{C}$, let v be the induced infinite place of E with distinguished embedding τ_v (so $\tau \in \{\tau_v, \bar{\tau}_v\}$), let $\sigma \in \mathrm{Aut}(\mathbf{C})$ satisfy $\sigma \circ \tau = \tau_0$ for the fixed model embedding τ_0, and write $\varepsilon(\phi) = 1$ if $\phi(x) = -\tau_0(x)$ and 0 otherwise. Write $w = (w_0, \dots, w_{a-1})$ for the antitone weight of U at τ_v and $b_v \in \mathbf{N}$ for the shift of Lemma 39.10, so that the transported antitone weight λ^{tr} satisfies $\lambda^{\mathrm{tr}}(\tau_v)_r = b_v$ and $\lambda^{\mathrm{tr}}(\bar{\tau}_v)_r = w_r + b_v \geq 0$ for $0 \leq r < a$. Reindexing $\phi = \sigma \circ \psi$ gives

$$R = \sum_{\phi \neq \tau_0, \bar{\tau}_0} \varepsilon(\phi) \sum_{r=0}^{a-1} (\sigma\lambda^{\mathrm{tr}})_\phi(r) = \sum_{\psi \neq \tau, \bar{\tau}} \mathbf{1}_{\psi(x) = -\tau(x)} \sum_{r=0}^{a-1} \lambda^{\mathrm{tr}}(\psi)_r,$$

and the infinity exponent m_τ of κ from Lemma 39.10 satisfies

$$m_\tau = R + ab_v + \begin{cases} \sum_{r=0}^{a-1} w_r, & \tau = \tau_v, \\ 0, & \tau = \bar{\tau}_v. \end{cases}$$

For every $0 \le k < a$, let j be the k-th prescribed jump of U at τ. If $\tau = \tau_v$, then $j = k - w_k$, $(\sigma\lambda^{\mathrm{tr}})_{\tau_0}$ is constant b_v, and $(\sigma\lambda^{\mathrm{tr}})_{\bar{\tau}_0} = (w_r + b_v)_r$, so the second Hodge exponent formula $h_i^{(2)}$ in (39.1) at $i = k$ (with $m = b_v$ and $b_0 = R$) gives

$$h_k^{(2)} = b_v + R + \sum_{r \ne k} (w_r + b_v) + k = \Big(R + ab_v + \sum_{r=0}^{a-1} w_r \Big) + (k - w_k) = j + m_\tau.$$

If $\tau = \bar{\tau}_v$, then the dual weight at $\bar{\tau}_v$ has k-th entry $-w_{a-1-k}$, so $j = k + w_{a-1-k}$, while $(\sigma\lambda^{\mathrm{tr}})_{\bar{\tau}_0}$ is constant b_v and $(\sigma\lambda^{\mathrm{tr}})_{\tau_0} = (w_r + b_v)_r$; the first Hodge exponent formula $h_i^{(1)}$ in (39.1) at $i = a - 1 - k$ (with $m = b_v$ and $b_0 = R$) therefore gives

$$h_{a-1-k}^{(1)} = (a-1)b_v + R + (w_{a-1-k} + b_v) + k = (R + ab_v) + (k + w_{a-1-k}) = j + m_\tau.$$

Lean: `UpperTheory.Gal.CertificateRingHomMap.pExpCore_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ProjectiveAutomorphicHecke.PExpCore`, lines 47–126)
Lean: `UpperTheory.Gal.Int.Geo.Pl.expOf_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.PrimeLevelLocalization.PVecWt`, lines 65–81)
Lean: `UpperTheory.Gal.Int.Geo.Pl.expOf_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ProjectiveAutomorphicHecke.CharConjHolds`, lines 37–37)

Claim 3 and the unramifiedness of κ proved above give the model part of the Hodge condition, and Claim 4 gives its degree part. At a site in $\mathcal{C}$, division by the level indices gives the normalized spherical Hecke coefficients, with roots $\kappa(w)^{-1}A_a(w, B)$. At usable places outside the prime divisors of Np_0e_{PEL} and the bad places of Φ, Lemma 39.11(1) gives goodness of the PEL datum and the same compatible-chart calculation applies. Take $(\mathcal{H}, \mathcal{C}) = (\{v'\}, \varnothing)$ for the initial plan, $(\{v'\}, \{w\})$ for an additional root identity, and $(\{v', w\}, \varnothing)$ for two Hodge places. The hypotheses at a place above ℓ ensure the same-side compatibility needed in either choice. □

Proof of 39.15

Proof. First rigidify the moduli problem. Choose a positive integer e, prime to p, killing the cokernel of $\Lambda \to \Lambda^\vee$, and an odd prime $M > \max(e, p, 3)$. Add a symplectic similitude frame of level M. Forgetting the endomorphisms now maps the moduli problem to the fine Siegel moduli scheme of polarized abelian schemes with bounded polarization kernel and auxiliary frame. One may use the separate polarization types here. Their elementary divisors divide e, and there are finitely many since the dimension is fixed. The Siegel construction with invertible auxiliary level is given in [176, Ch. I, §4].

We now give the relative construction over that scheme. Choose an integral basis of the order. Its images are homomorphisms of the universal abelian scheme. In the scheme of graphs of homomorphisms impose the multiplication table and the adjunction identities for this basis. The Rosati norm of each basis element is fixed by the trace of c^*c on the rational datum. Thus the graphs have bounded degree for the product polarization. Equivalently, their degrees are bounded by the elementary symmetric functions of the positive operator $1+c^*c$ with bounded trace. This restricts the construction to finitely many Hilbert polynomials. The homomorphism condition is rigid, and homomorphisms between abelian schemes over a discrete valuation ring extend uniquely from the fraction field. These bounded choices are therefore finite over the Siegel base. The level-N sections are choices in the finite étale group $A[N]$. The order and pairing identities cut out their prescribed locus. The prime-to-p lattice type is constant under geometric specialization, by transport of the prime-to-p Tate module with its form and action. On the Lie bundle impose the factorization through the sum of the indicated eigenspaces. The maps t_i are pairwise comaximal, so on this quotient the projections are the Chinese remainder idempotents. Impose the indicated ranks on their summands. This gives the required family with auxiliary frame. In particular the Lie condition is imposed on the bundle, hence on points over every R-algebra.

Lean: `GlobalTheory.UPel.HT.relRep_htPull` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.HarrisTaylorStructuresRelativelyRepresentable`, lines 18–65)

Changes of the auxiliary frame act freely. An automorphism preserving the polarization has finite order, and an automorphism acting trivially on a full level $N \geq 3$ is the identity by the congruence lemma [167, Rem. 1.4.1.9]. The auxiliary frames of any given object form a finite étale torsor, allowing the primitive root in the pairing to vary. The quotient of the quasi-projective framed scheme by this finite group exists. The abelian scheme and its homomorphisms descend along the torsor. The bundle defining the polarization descends with its rigidification on the axes, and the level sections descend. Conversely, adding frames étale locally to any object gives classifying maps that agree in the quotient. Descent gives one classifying map and its uniqueness. This proves representability and quasi-projectivity.

For smoothness consider a square-zero lifting problem $C \twoheadrightarrow C_0$ over an Artinian local ring $(C, \mathfrak{m}_C, k)$ over R. Put $I = \ker(C \to C_0)$ and use the nilpotent divided powers $\gamma_1(x) = x$, $\gamma_j(x) = 0$ for $j \geq 2$ on I. In residue characteristic p, the defining quadratic polynomial of ω splits into two distinct linear factors in $\mathbf{F}_p[X]$, so Hensel's lemma splits $\mathcal{O}_{K_0} \otimes_{\mathbf{Z}} \mathbf{Z}_p = \mathcal{O}^+_{K_0} \times \mathcal{O}^-_{K_0}$ into the selected and conjugate ω-summands, interchanged by the involution. Accordingly the crystal value $\mathbf{D}(A_0)(C)$ splits into two ω-summands in perfect duality under the prime-to-p polarization. Using $\mathcal{O} \otimes \mathbf{Z}_p \simeq M_m(\mathcal{O}_{K_0} \otimes \mathbf{Z}_p)$, pass by Morita equivalence on the selected side to one matrix column $M = \mathbf{D}(A_0)(C)^+_1$. Because A_0/C_0 is an abelian scheme and p is nilpotent in C, $\mathbf{D}(A_0)(C)$ is finite free over C, and its direct summand M is finite projective, hence free, over C. We show that M is free of rank n over $\mathcal{O}^+_{K_0} \otimes_{\mathbf{Z}_p} C$, without assuming $K_0/\mathbf{Q}_p$ unramified. Let $\bar{k}$ be an algebraic closure of k, and let $D = \mathbf{D}(A_{\bar{k}})(W(\bar{k}))^+_1$ be the selected column of the covariant Dieudonné module over $W(\bar{k})$, using throughout the convention that the Hodge quotient is the Lie module. Decompose $\mathcal{O}^+_{K_0} = \prod_{\mathfrak{w}} \mathcal{O}_{\mathfrak{w}}$ over the selected primes $\mathfrak{w} \mid p$ of K_0; their conjugates $\bar{\mathfrak{w}}$ lie on the opposite ω-side and are disjoint from them. For each selected $\mathfrak{w}$, let $L_{\mathfrak{w}}/\mathbf{Q}_p$ be the maximal unramified subextension of $K_{0,\mathfrak{w}}$, of degree $f_{\mathfrak{w}}$, and let $\varpi_{\mathfrak{w}}$ be a uniformizer of $\mathcal{O}_{\mathfrak{w}}$. Then

$$\mathcal{O}_{\mathfrak{w}} \otimes_{\mathbf{Z}_p} W(\bar{k}) \simeq \prod_{\tau : \mathcal{O}_{L_{\mathfrak{w}}} \to W(\bar{k})} R_{\mathfrak{w},\tau}, \qquad R_{\mathfrak{w},\tau} = \mathcal{O}_{\mathfrak{w}} \otimes_{\mathcal{O}_{L_{\mathfrak{w}}},\tau} W(\bar{k}),$$

where each $R_{\mathfrak{w},\tau}$ is a complete discrete valuation ring with uniformizer $\varpi_{\mathfrak{w}} \otimes 1$ because $\mathcal{O}_{\mathfrak{w}}/\mathcal{O}_{L_{\mathfrak{w}}}$ is totally ramified and Eisenstein. Since D is finite free over $W(\bar{k})$, its summand $D_{\mathfrak{w},\tau}$ is torsion-free and finitely generated over $R_{\mathfrak{w},\tau}$, hence free of some rank $r_{\mathfrak{w},\tau}$. The σ-semilinear Frobenius isomorphism on $D_{\mathfrak{w}}[1/p]$ permutes the embeddings τ transitively and commutes with $K_{0,\mathfrak{w}}$, so $r_{\mathfrak{w},\tau} = r_{\mathfrak{w}}$ is independent of τ, and $D_{\mathfrak{w}}$ is free of rank $r_{\mathfrak{w}}$ over $\mathcal{O}_{\mathfrak{w}} \otimes_{\mathbf{Z}_p} W(\bar{k})$. By weak approximation in K_0, choose $c \in \mathcal{O}_{K_0} \setminus \{0\}$ with $v_{\mathfrak{w}}(c) = 1$ and $v_{\mathfrak{v}}(c) = 0$ at every prime $\mathfrak{v} \mid p$ of K_0 different from $\mathfrak{w}$ (including $\bar{\mathfrak{w}}$). For any prime $\ell' \neq p$, the prime-to-p level identifies $V_{\ell'}(A_{\bar{k}})$ with $\Lambda \otimes_{\mathbf{Z}} \mathbf{Q}_{\ell'}$, a free $(K_0 \otimes_{\mathbf{Q}} \mathbf{Q}_{\ell'})$-module of rank mn, giving $\deg(c \mid A_{\bar{k}}) = N_{K_0/\mathbf{Q}}(c)^{mn}$ and $v_p(\deg(c \mid A_{\bar{k}})) = mnf_{\mathfrak{w}}$. On the Dieudonné module $\mathbf{D}(A_{\bar{k}})(W(\bar{k})) \simeq (D)^m \oplus (\mathbf{D}^-_1)^m$, polarization duality shows that c acts invertibly on $\mathbf{D}^-_1$ (since $\bar{c}$ is a unit at all selected primes) and on $D_{\mathfrak{w}'}$ for $\mathfrak{w}' \neq \mathfrak{w}$, so $v_p(\deg(c \mid A_{\bar{k}})) = \operatorname{length}_{W(\bar{k})}(\mathbf{D}(A_{\bar{k}})(W(\bar{k}))/c\,\mathbf{D}(A_{\bar{k}})(W(\bar{k}))) = mr_{\mathfrak{w}}f_{\mathfrak{w}}$. Hence $r_{\mathfrak{w}} = n$ for every $\mathfrak{w}$, and $M \otimes_C \bar{k} \simeq D/pD$ is free of rank n over $\mathcal{O}^+_{K_0} \otimes_{\mathbf{Z}_p} \bar{k}$. Over the finite commutative Artinian k-algebra $B_k = \mathcal{O}^+_{K_0} \otimes_{\mathbf{Z}_p} k = \prod_r B_{k,r}$, each local factor $B_{k,r}$ has faithfully flat base change $B_{k,r} \otimes_k \bar{k}$, so $(M \otimes_C k)_r$ is flat, hence free of rank n, over $B_{k,r}$; thus $M \otimes_C k$ is free of rank n over B_k. Lifting a B_k-basis gives a map $\varphi : (\mathcal{O}^+_{K_0} \otimes_{\mathbf{Z}_p} C)^n \to M$, surjective by Nakayama's lemma over C. Since M is free over C, $\operatorname{Tor}^C_1(M, k) = 0$, so $(\ker\varphi) \otimes_C k = 0$ and $\ker\varphi = 0$ by Nakayama's lemma; therefore M is free of rank n over $\mathcal{O}^+_{K_0} \otimes_{\mathbf{Z}_p} C$.

On this free module M, the Hodge quotient of type t_i over C_0 is a line quotient after tensoring along $t_i : \mathcal{O}^+_{K_0} \to C$ (which factors through $\mathcal{O}^+_{K_0}$ since $t_i(\omega)$ selects the chosen side). Lift this quotient over C by lifting its coordinates, one of which is a unit. The choices for different i are independent by comaximality of the maps t_i. Taking the kernels, restoring the m columns, and taking the annihilator on the conjugate side gives an order-stable isotropic lift of the Hodge subbundle. In residue characteristic p, apply the crystalline deformation theorem for abelian schemes on this nilpotent divided-power thickening [228, Ch. V, §1]. In residue characteristic zero, first choose an abelian-scheme lift of A_0, using unobstructedness over an affine square-zero thickening. The de Rham comparison identifies the first de Rham module of this reference lift with the evaluation of the crystal of A_0 on C. Through this identification, functoriality gives the module the action induced by $\mathcal{O}$ on A_0 and the perfect pairing induced by the polarization of A_0. Both reduce to the prescribed structures over C_0. The de Rham deformation correspondence identifies lifts of A_0 with direct

summands lifting its Hodge submodule; changing a summand by a map from that submodule to the quotient, with coefficients in I, gives the corresponding change of deformation [167, Props. 2.1.6.8–2.1.6.9 and 2.2.4.1]. Apply the preceding line-quotient construction to the embedding summands of $K_0 \otimes_{\mathbf{Q}} C$ and choose the resulting summand. In either characteristic, the homomorphism lifting criterion lifts every endomorphism preserving the chosen summand uniquely. Thus the $\mathcal{O}$-action lifts and its ring identities persist by uniqueness. Isotropy lifts the polarization, and the associated bundle on $A \times A$ is recovered from the Poincaré bundle with its rigidifications. The lifted abelian scheme with these structures is unique up to the isomorphism inducing the identity on A_0. Prime-to-p level lifts uniquely in residue characteristic p, and in characteristic zero the entire finite level lifts uniquely. On each of the b types t_i, the deformation space of a line quotient of a free C_0-module of rank n has tangent space $\mathrm{Hom}_{C_0}(\ker, \mathrm{quot})$ of rank $n-1$, proving smoothness of relative dimension $b(n-1)$.

Lean: `GlobalTheory.UPel.HT.exists_sqZero_lift` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.HarrisTaylorExistsLiftSquareZeroArtinian`, lines 19–55)

For properness apply the valuative criterion, allowing an extension of the test valuation ring. After finite extension the abelian variety has semiabelian reduction with split torus T, and the order acts on its character lattice $X^*(T)$. We give the obstruction to $T \neq 0$ in the first case, since it applies to the commutative order. Some embedding of K_0 on the selected side is absent from the generic Lie algebra. Over a splitting extension the idempotent of that embedding annihilates the Lie algebra. In an integral basis of $\mathcal{O}_{K_0}$, scale a nonzero such relation so its coefficients belong to the valuation ring and at least one is a unit. The relation holds on the integral Lie module by torsion-freeness, hence on the Lie algebra of the special torus. Its reduction is a nonzero element of $\mathcal{O}_{K_0} \otimes k$, where k is the residue field. If $X^*(T) \neq 0$, it is a finite torsion-free $\mathcal{O}_{K_0}$-module of positive rank. It is locally free of that rank at every prime. Thus $\mathcal{O}_{K_0} \otimes k$ acts faithfully on $X^*(T) \otimes k$, and dually on $\mathrm{Lie}\, T$. This contradicts the relation.

Lean: `GlobalTheory.PotGoodLie.finrank_zero` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.PotentialGoodReductionOfLieMultZero`, lines 108–155)

In the division-algebra case any nonzero rational character lattice of T has dimension at least $m^2[K_0 : \mathbf{Q}]$, the dimension of the division algebra over $\mathbf{Q}$. But

$$\dim T \leq \dim A = mn[K_0 : \mathbf{Q}]/2 < m^2[K_0 : \mathbf{Q}].$$

In both cases the semiabelian model is abelian. The order action and the polarization extend by rigidity, and the prime-to-residue-characteristic level extends by finite étaleness. Here the test trait has residue characteristic p or zero, since it lies over R. The Lie conditions persist by the closed equations and locally constant ranks in the construction. This gives the valuative extension. A proper quasi-projective morphism is projective.

Lean: `GlobalTheory.UPel.HT.isProper_model_of_compact` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.HarrisTaylorIntegralModelIsProperOfCompact`, lines 27–58)

□

Proof of 39.16

Proof. Apply 39.15 to the given algebra. All nonzero integers are units, and there are no residue maps of characteristic p. To compare the two functors, restrict a full Tate frame to its prime-to-p factors. In the reverse direction the polarization kernel is prime to p. At a characteristic-zero geometric point the p-primary lattice is free after passage to one matrix column at each prime of the centre above p. The two sides of ω are in perfect duality. Choose bases on one side and the dual bases on the other, and restore the matrix columns. This supplies compatible frames at all powers of p. Combining them with the given prime-to-p frames by the Chinese remainder theorem preserves the sections of level N. Conversely a full frame gives the prescribed alternating lattice at p, so the polarization kernel has prime-to-p exponent. The conditions on objects coincide and the isomorphisms preserve the same sections and polarization. The universal property now gives the result.

Lean: `GlobalTheory.UPel.HT.GenCmp.geomFull` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.GenModelEqGenericFibre`, lines 275–378)
Lean: `GlobalTheory.UPel.HT.GenCmp.cmp_of` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.GenModelEqGenericFibre`, lines 171–201)

□

Proof of 39.17

Proof. Apply 39.15 over the given field. The structural maps agree with those of the generic datum. The comparison in 39.16 carries the universal abelian scheme to the generic one. Equivalently, the resulting map from the generic moduli scheme into the scalar extension of the integral moduli scheme is an open and closed immersion. Smoothness restricts along the open immersion and projectivity along the closed immersion.
Lean: `UpperTheory.Gal.G3.pGenSm` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.PAdicWittLattices.PGenSm`, lines 15–32)

For the universal abelian scheme $A \to X$, its k-fold fibre product is smooth over X. The exterior product of the relatively ample diagonal bundles on its factors is relatively ample. Thus the fibre product is projective over X, and hence over the ground field. For $k = 0$ it is X. If the level is K and the Hecke element is g, the source projection forgets level from $K \cap gKg^{-1}$ to K, with the conjugation convention for the chosen leg. At a normal level contained in this intersection the lifts are the cosets of the intersection in K. They form a finite étale scheme of constant degree $[K : K \cap gKg^{-1}]$. The source projection on the fibre power is its base change. The target projection uses the isogeny on each abelian factor. □

Proof of 39.18

Proof. The chart in 39.13 supplies a simple root r modulo p of the polynomial of ω. Put $\mathfrak{u} = (p, \omega - r)$. Then $\mathfrak{u} \subseteq \mathfrak{w}$. The alternating lattice is perfect at p. If p_0 denotes the auxiliary model prime, the lower level has the form $N_0\mathfrak{a}$, where $N_0 \geq 3$ is prime to pp_0, $\mathfrak{a} + \mathfrak{w} = \mathcal{O}_{K_0}$, and $\mathfrak{u}^e \subseteq \mathfrak{a}$ for some e. These are the local chart and level conditions used in choosing the common level. There is one structural map, the composite $\mathcal{O}_{K_0} \simeq \mathcal{O}_E \to R$, and the matrix size in 39.15 is one. Reduction of this map has kernel $\mathfrak{w}$. All integers prime to p are units in R. Finally

$$2 < [K_0 : \mathbf{Q}] = 2[F : \mathbf{Q}]$$

because $[F : \mathbf{Q}]$ is positive and even. Thus the first projectivity condition of 39.15 applies.

On the chosen side, the local factors at primes different from $\mathfrak{w}$ have zero Lie algebra, so their torsion is étale. The $\mathfrak{a}$-level is a frame on such torsion. Its frame scheme is finite étale, as can be seen after trivializing the torsion, when the permissible frames form a finite constant set. Adjoining this level preserves smoothness and projectivity. Taking the fibre powers gives the first assertion by the exterior product argument in 39.17.

Over E, trivialize $A[\mathfrak{w}]$ étale locally. For every j-plane choose the corresponding subgroup on the selected side. The polarization determines the complementary subgroup on the conjugate side by taking the annihilator for the pairing on $A[q^j]$. Dividing by the resulting kernel gives a polarized object, with its transported level and an isogeny whose composite with the reverse isogeny is multiplication by q^j. At each geometric point, dividing by this multiplier-q^j kernel sends the compatible system $(\zeta_n)_{(n,p)=1}$ of primitive roots of unity to $(\zeta_n^{q^j})_{(n,p)=1}$, which remains a compatible system of primitive roots of unity because $(q, n) = 1$. The quotient by a specified kernel is unique. The choices therefore descend from the constant Grassmannian. Reversing the isogeny gives finite étaleness for the other projection as well.
Lean: `UpperTheory.Cb.exists_heckeCorrespondence_on_integral_shimuraModel` (`Development.AutomorphicArithmetic.Foundations.EtaleClopenDescent.ExistsHeckeCorrespondenceOnIntegralShimuraModel`, lines 243–246)

To extend the correspondence, normalize X in this generic finite étale cover, component by component. The normalization is finite since R is a localization of a number ring and X is of finite type over R. Its local rings have no R-torsion, so it is flat over R.

We give the finiteness argument that extends the second leg. Take the closure of the generic image in $X \times_R X$. At a geometric point of a special fibre, a limiting isogeny of multiplier $c = q^j$ has kernel in $A[c]$. On the chosen side decompose $A[c]$ by the local idempotents of $\mathcal{O}_{K_0}/c$. On each factor the tangent dimension is at most one, by the Lie condition and the single structural map. At the identity its finite local coordinate ring therefore has principal maximal ideal. Indeed a lift of a generator modulo the square generates the ideal by Nakayama. Every nonzero ideal is a power of the maximal ideal, by taking an element of least order. The finitely many other geometric points give translates of the same assertion. There are consequently finitely many choices for the selected part of the kernel even at a nonordinary point. The pairing determines the opposite part. The quotient determines its target and the transported level. Applying this argument to the reverse isogeny gives finitely many sources for a given target. In reversing, multiplication by c on level N_0 is an automorphism since $(c, N_0) = 1$.

Lean: GlobalTheory.UPel.HT.ES.KerFin.finite_kernels (Development.AbelianVarietyTheory.Proofs.DivisorProofs.FiniteKernelsPIsogeniesHarrisTaylorCharP, lines 482–595)

For a specialization along a discrete valuation ring the generic kernel extends by schematic closure in the finite torsion scheme. The closure is finite and torsion-free over the trait, hence flat. The quotient is an abelian scheme, the polarization descends, and separatedness identifies the target with its specified limit. Thus the preceding finite lists contain the geometric fibres of the closed graph under either projection. The projections are proper and quasi-finite, hence finite. Normalizing the graph in the original generic cover now gives the same normalization as for the first projection, by transitivity of integral dependence. This supplies t_j on Y_j. The homomorphism of the two pulled-back abelian schemes extends uniquely over the normal base. Its identities with the order action, the reverse isogeny, and the polarization extend from the dense generic fibre. Auxiliary étale level is transported through this construction. On a fibre power the first map is obtained by base change along the source, and the second applies the isogeny to each coordinate. This proves the assertions about finiteness and the degree.

At a Hodge place the level in the chart was chosen simultaneously for the selected operators. Their level changes use the prime-to-p torsion and the étale factors on the chosen side away from $\mathfrak{w}$. The first projection amounts to choosing these additional levels or subgroups in finite étale groups. It is finite étale over R. Two such choices give the same lower correspondence precisely when they differ by the subgroup preserving the lower level. Thus the map from an upper correspondence to the pullback of its lower correspondence has degree the corresponding subgroup index. □

Proof of 39.21

Proof. The smooth projective geometric fibre is a finite disjoint union of integral components because it is regular and Noetherian. On each component 39.34 bounds the degrees and gives finite dimension. The Galois action is continuous by the inverse-limit construction from finite-coefficient cohomology [214]. The correspondences are defined over E, so pullback and trace commute with this action. Take the common eigenspace and the direct sum of the actions on the quotients of a composition flag. □

Proof of 39.22

Proof. Write $q = q_w$. At a geometric special point x, the $\mathfrak{w}$-part of the divisible group has $\mathcal{O}_{E_w}$-height a. Its Lie algebra is one-dimensional and the action on it is through the structural map. Here ordinarity means

$$\#A_x[\mathfrak{w}](\bar{k}) = q^{a-1}.$$

Equivalently, the connected formal $\mathcal{O}_{E_w}$-module has height one.

We explain why this applies above the generic point of each special-fibre component. Work first at a closed geometric point of the lower model of 39.18. Its selected divisible module has connected height h, with $1 \le h \le a$, and étale height $a - h$. The other selected factors are étale and their conjugates are determined by the polarization. The Serre–Tate deformation theorem therefore identifies deformations of the PEL point with deformations of this one-dimensional divisible $\mathcal{O}_{E_w}$-module. Indeed the theorem is functorial for homomorphisms, the action on the Lie line is structural, and the additional étale level lifts uniquely [227, Thm. 1.2.1]. For a uniformizer ϖ of E_w, the completed deformation ring over the completed maximal unramified base has $h-1$ parameters for the formal module and $a-h$ parameters for the extension of the étale part [224, §4, Props. 4.2 and 4.5]. If $h > 1$, the coefficient of degree q in $[\varpi](X)$ on the special fibre is a deformation parameter up to a unit, in the coordinate of the proof of Proposition 4.2. Where it is nonzero the connected height is one. If $h = 1$ that coefficient is already a unit. Thus every special-fibre component has ordinary generic point. Étale auxiliary level and the smooth abelian fibre powers preserve this assertion for the underlying base point.

Lean: UpperTheory.Gal.Ord.ordGen (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.RadicalStalks.OrdGen, lines 68–174)

Pull the integral correspondences of 39.18 back to the valuation ring in the statement. Every geometric generic branch has a unique specialization, by finiteness of the first projection. Identify the generic $\mathfrak{w}$-torsion with $H = k(w)^a$. Reduction maps it onto the geometric points of the special $\mathfrak{w}$-torsion. Surjectivity follows from finite flatness of the torsion group over the valuation ring with algebraically closed fraction field. The kernel is a line C, by the displayed cardinality.

For $D \subset H$ let $\bar{D}$ be its image in H/C. If $D \cap C = 0$, the closure of D has $|D|$ distinct special points and its reduction is the étale subgroup specified by $\bar{D}$. Write $b(\bar{D})$ for the resulting target, including its fibre coordinates. If $C \subset D$, the selected kernel additionally contains the connected subgroup of order q. This is the kernel of relative q-Frobenius on the height-one formal module. The quotient on the selected side is consequently the relative Frobenius of the quotient for $\bar{D}$. On the conjugate side the annihilator rule gives the complementary kernel for the multiplier $q^{\dim D}$. More explicitly, composing the quotient for $\bar{D}$ with relative q-Frobenius multiplies its polarization multiplier by q and gives exactly these two kernels. The level and all fibre coordinates are transported by the composite, so the target is $F_* b(\bar{D})$.
Lean: `UpperTheory.Gal.crystallineComparison.Mod.caseE_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.AnnihilatorModels.MCaseE`, lines 13–60)
Lean: `UpperTheory.Gal.crystallineComparison.Mod.caseW_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.AnnihilatorModels.MCaseW`, lines 13–86)

Let A_j be the multiset of $b(U)$ for the j-planes $U \subset H/C$. There is one lift containing C of every j-plane of H/C, of dimension $j+1$. The transverse lifts of a fixed $(j+1)$-plane form a torsor under $\mathrm{Hom}_{k(w)}(U, C)$, of cardinality q^{j+1}. This proves

$$T_{j+1} = F_* A_j + q^{j+1} A_{j+1}, \qquad A_0 = T_0, \qquad A_a = 0.$$

Substitute into the asserted sum. The two terms involving A_j have opposite signs and the same coefficient because $\binom{j}{2} + j = \binom{j+1}{2}$. The boundary values give the cancellations at both ends. The count uses each generic branch once, including branches with the same reduction, so it gives the asserted integral identity of multisets. □

Proof of 39.23

Proof. Fix a prolongation of w and use the lower smooth proper model of 39.18. Write Y for the model, S for its geometric special fibre, and $q = q_w$, with $p = \mathrm{char}\, k(w)$. Before twisting, the constant-degree upper correspondences act on the common eigenspace with eigenvalues $m_j \vartheta_j$, where $m_j > 0$ is the prescribed multiplicity. Pullback after normalized trace expresses vectors in that eigenspace as pullbacks from the lower geometric generic fibre. Compatibility of traces and pullbacks gives $T_j^{\mathrm{up}} \pi^* = m_j \pi^* T_j^{\mathrm{low}}$.

To extend the correspondence trace of Lemma 39.20 to finite morphisms that need not be étale or flat on the special fibre, we use trace weights. For a finite morphism $p : Z \to S$, a *weight* on Z assigns an integer $w(\Omega, z) \in \mathbf{Z}$ to every geometric point $z : \mathrm{Spec}\, \Omega \to Z$ (over an algebraically closed field Ω). Such a weight is a *trace weight* for p if, for every pullback square $V = Z \times_S U \xrightarrow{r} U$ with projections $u : U \to S$ and $v : V \to Z$ and every locally constant function $\varphi : V \to \mathbf{Z}$, the weighted fibre sum $\psi(x) = \sum_{z \in r^{-1}(x)} w(\Omega, v \circ z)\, \varphi(z)$ defines a locally constant function $\psi : U \to \mathbf{Z}$. For any modulus $b \geq 1$ invertible on S and any integer twist i, a trace weight w determines a functorial sheaf-level weighted trace $\mathrm{tr}_0(p, w)$ on pullback squares taking a section a of $\mathbf{Z}/b(i)$ on V to the unique section b on U whose stalk at every geometric point x of U equals $\sum_{z \in r^{-1}(x)} w(\Omega, v \circ z)\, a|_z$. Extending $\mathrm{tr}_0(p, w)$ to bounded-below injective resolutions yields a compatible family of cochain maps and hence additive maps $\mathrm{tr}(p, w) : H^n_{\text{ét}}(Z, \mathbf{Z}/b(i)) \to H^n_{\text{ét}}(S, \mathbf{Z}/b(i))$ for every $n \in \mathbf{Z}$. Passing to the inverse limit over $b = \ell^k$ and tensoring with $\mathbf{Q}_\ell$ gives the $\mathbf{Q}_\ell$-linear weighted trace $\mathrm{tr}_{p,w} : H^n_{\text{ét}}(Z, \mathbf{Q}_\ell(i)) \to H^n_{\text{ét}}(S, \mathbf{Q}_\ell(i))$, which depends linearly on w (vanishing for $w = 0$) and agrees with the finite étale trace of Lemma 39.20 when p is finite étale and $w \equiv 1$. A finite correspondence $S \xleftarrow{p} Z \xrightarrow{f} Y$ equipped with a trace weight w for p acts on rational ℓ-adic cohomology by $\mathrm{tr}_{p,w} \circ f^*$.
Lean: `UpperTheory.Gal.TranscendenceTraces.IsWt` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.TranscendenceTraces.IsWtZero`, lines 53–55)
Lean: `UpperTheory.Gal.TranscendenceTraces.trHQ` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.TranscendenceTraces.IsWtZero`, lines 106–107)

An integer trace weight for p that vanishes over a dense subset of the source base vanishes everywhere. To see this, isolate a given fibre point by a clopen neighbourhood on an étale chart. The sum of the weight over this neighbourhood is locally constant by the defining trace-weight property. It vanishes by density and openness of the chart. At the chosen fibre point it is the integer weight times the positive number of its geometric lifts in the neighbourhood, so the weight is zero.

Claim 1. The branch identity in 39.22 gives the weighted endpoint identity for geometric sources over generic points of the special fibre.

The lower models are smooth over the discrete valuation ring in 39.18, and thus locally normal. Construct the canonical weight of a finite first projection as follows.

On a normal integral affine chart write the first projection as $\operatorname{Spec} B \to \operatorname{Spec} A$, where B is a finite A-algebra. Choose finitely many A-module generators of B, and multiply monic equations for them to get a monic polynomial P over A. Let L be a splitting field of P over $\operatorname{Frac} A$ and C the integral closure of A in L. Every root of P lies in C. The cover $\operatorname{Spec} C \to \operatorname{Spec} A$ is integral and surjective by lying-over. The branches are the A-algebra maps $B \to C$; each is determined by the tuple of its values on the generators, all in the finite root set. Tag a branch by the kernel of its map $B \to C$.

For a geometric lift $C \to \Omega$ over the generic point of the chart, the map is injective: after inverting the nonzero elements of A its source is L. Each compatible map $B \to \Omega$ sends the generators to roots of the image of P, all in the image of C. Since the generators span over A, the map on B factors through a unique branch $B \to C$; its kernel is the tag. For two lifts $C \to \Omega$ over any one geometric point, an A-automorphism of C carries one lift to the other by precomposition. Indeed, for a finite group acting on a ring, first put the kernels in the same orbit using transitivity on primes over the common contraction. With common kernel Q, the roots of the invariant polynomial $\prod_g (X - gb)$ show that the two embeddings of the quotient have the same image. The resulting automorphism of that quotient over the image of $C^G/(Q \cap C^G)$ lifts to the stabilizer of the kernel. Here G is the finite group of A-automorphisms of C. The minimal polynomial of a fixed element over $\operatorname{Frac} A$ has only one distinct root, since the normal extension contains all roots and its automorphisms act transitively on them. Thus the two lifts agree on the fixed subring: in characteristic zero a fixed element lies in $\operatorname{Frac} A$, hence in A by integral closedness. In characteristic $r > 0$, some r-power iterate of each fixed element lies in $\operatorname{Frac} A$, hence in A. Their images agree by injectivity of Frobenius in Ω. Composition with the automorphism thus permutes the branch labels and preserves their tags.

Give each branch the multiplicity of the geometric fibre at its distinguished point above the generic point of the chart. At a geometric point of the correspondence, sum these multiplicities over the branches that land there, after choosing a geometric lift to the cover. Multiplicity is independent of the algebraically closed field representing the distinguished point. Two lifts over the same source permute the branch labels preserving their multiplicities, so the sum is independent of the lift. After any base change, the fibre sum against a locally constant test becomes, on the splitting cover, the finite sum of the tagged multiplicities times the test pulled back along the respective branches. It is locally constant, is constant along the fibres of the cover, and descends through the closed surjective quotient map. At generic source points only one branch contributes to each geometric point in the fibre, so its weight is the corresponding scheme-theoretic multiplicity. On chart overlaps, both weights agree at generic source points. Their difference is a trace weight and vanishes everywhere by the preceding dense-vanishing argument. This glues the chart weights. Over the open locus where p is invertible the map is finite étale, so the same generic comparison with the trace weight identically one shows that the canonical weight restricts to one there.

Now take a valuation-ring lift y with algebraically closed fraction and residue fields and with $p \neq 0$ in the valuation ring. Pull a first projection back along this lift. The valuation ring is Henselian. Its finite algebra decomposes by the idempotents isolating the points in the closed fibre. For a selected closed branch, take the clopen corresponding to its idempotent, and push its indicator forward with the canonical weight. This function is locally constant on the connected valuation spectrum. At the closed point its value is the weight of the selected branch. At the generic point it is the count of geometric lifts in the clopen, each of weight one. Each generic lift extends uniquely to a section by the valuative criterion for the finite projection; membership in the clopen is constant along that section. This proves that the closed weight counts exactly the sections specializing to the selected branch.

Group the finitely many sections by their closed image and apply the residue Frobenius power for their index j. Thus the weighted sum at any given endpoint is the number of its occurrences in the even-indexed multisets minus the number in the odd-indexed multisets, with the factor $q^{\binom{j}{2}}$ for either parity. These two counts agree by 39.22. For an arbitrary geometric point above a generic point of the special fibre, use a valuation lift from the open set where p is invertible, with algebraically closed fraction and residue fields as above. Compare in a common algebraically closed extension of the residue field of the lift and the field of the original point. Base extension preserves the fibres with their weights and commutes with the residue q-power field maps. Equality therefore descends to the original geometric point.

Claim 2. If F is the relative q-power Frobenius of S, the weighted identity of Claim 1 gives

$$\sum_{j=0}^{a}(-1)^j q^{\binom{j}{2}}\, T_{j,S}(F^*)^{a-j} = 0$$

on rational ℓ-adic cohomology. Replace the second leg of each special-fibre correspondence by its composite with F^{a-j}. A weighted point-to-point coefficient is then a sum over the fibre of the first leg with this prescribed image under the modified second leg. For two points with the same map to the geometric base field, the pullback diagrams identify these branches with those in Claim 1. If the base maps differ, no such branch exists because both legs are over the geometric base. First work with coefficients $\mathbf{Z}/\ell^k$ for $k \geq 1$; the integer ℓ^k is a unit on S. Take the scheme-theoretic image of the finite disjoint union of correspondence domains under the pairs of legs in $S \times_{\bar{k}} S$. This image is finite over the source, and each domain maps finitely to it. Push each weight to the common image by summing over the finite fibres. Its operator is pullback along the common second leg followed by the trace of this pushed weight. At a geometric point of the image, the signed sum of pushed weights is the weighted endpoint sum, so it vanishes over the geometric source points above generic points of the special fibre. This locus contains the generic points of the geometric special fibre, since algebraic extension of the residue field sends generic points to generic points, and is therefore dense.

The preceding dense-vanishing argument makes this signed sum zero on the whole common image. By trace linearity this gives the operator identity with coefficients $\mathbf{Z}/\ell^k$. The fibrewise sums commute with coefficient reduction, and restriction of scalars identifies the coefficient-ring cohomology maps with the same additive maps. The induced operators on integral ℓ-adic cohomology are therefore the torsion operators acting coordinatewise on the inverse limit. The operator identity holds on the limit and still holds after tensoring with $\mathbf{Q}_\ell$. Functoriality of pullback along the composite second leg puts $(F^*)^{a-j}$ to the right of $T_{j,S}$.

Claim 3. For every arithmetic Frobenius lift s at the chosen prolongation, the same relation holds on the lower geometric generic fibre with F^* replaced by s^{-1}, and inertia acts trivially there. Indeed smooth proper comparison at the valuation base identifies the rational cohomologies of its two geometric fibres, using $\mathbf{Z}/\ell^k$ at every stage and then inverting ℓ. Both fibre correspondence actions are transported via the weighted operators over the valuation base. The canonical weights agree on the geometric generic fibre with the unit weights of its finite étale first projections. On the special fibre the pullback of relative q-power Frobenius is inverse to the coefficient-field arithmetic action: their composite is the absolute Frobenius pullback, which is the identity on cohomology with these untwisted coefficients. Equivariance of specialization gives the claimed Frobenius action and makes a residue-trivial lift act as the identity.

Pull the lower operator relation back to the upper eigenspace. For the j-th term use the trace compatibility and divide by m_j, since the vector acted on by $s^{-(a-j)}$ remains in the upper common eigenspace. Thus the polynomial with coefficients $(-1)^j q^{\binom{j}{2}}\vartheta_j$ annihilates s^{-1} there. Trivial inertia and this identity pass to the associated graded of a composition flag. Dividing by $\vartheta_0 \neq 0$ and transporting coefficients gives the monic polynomial with the prescribed roots before twisting.

Finally the twist has value one on inertia. At inverse arithmetic Frobenius let $c = \kappa_\ell(s^{-1})$, a nonzero scalar. The rank-one reciprocity comparison off ℓ gives $\iota(c) = \kappa(w)$ [206, Ch. III]. If A is the original Frobenius operator and R_0 the transported root multiset before the twist, then over the coefficient field, with the roots transported back,

$$\prod_{\alpha\in R_0}\big(cA - c\iota^{-1}(\alpha)\big) = c^{|R_0|}\prod_{\alpha\in R_0}\big(A - \iota^{-1}(\alpha)\big) = 0.$$

The root identity of 39.13 gives $R_0 = \kappa(w)^{-1}A_a(w,B)$. Multiplication by $\iota(c)$ therefore gives the Satake root multiset $A_a(w,B)$ and the stated annihilator $P_{w,B}$.

Lean: `UpperTheory.Gal.Mod.esStep_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.BasicOpenSections.EsStepHolds`, lines 18–18)

□

Proof of 39.25

Proof. Fix P. Write p_0 for the model prime. Include in the finite test its factor at p_0, the characteristic function of the prescribed compact subgroup there. In complex uniformization the first homology of a PEL

point gives its polarized rational space and finite lattice data. The Hodge filtration is prescribed at the definite embeddings. At the distinguished embedding it amounts to a positive line, whose orthogonal complement specifies the conjugate part of the filtration. Conversely the resulting Hodge structure and lattice give the polarized complex torus by the Riemann bilinear relations. The two legs of a Hecke correspondence change the finite lattice data according to the two level maps. A returning coset therefore gives a rational similitude (γ, c_0) preserving the positive line. Here $c_0 \in \mathbf{Q}_{>0}$, and

$$h(\gamma v, \gamma v') = c_0 h(v, v')$$

for the Hermitian form h. The eigenvalue μ on the line is its mark.
Lean: `UpperTheory.Gal.W2.fixedPoints_integral_base_of_uniformisation (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SphericalCoordinates.FixedPointsIntegralBaseOfUniformisation`, lines 70–190)

The multiplier condition can be read before summing over conjugacy classes. On a double coset KtK of the full finite test, a returning argument $g^{-1}(\gamma, c_0)g$, written in the finite trivialization, must have multiplier in $\nu(t)\nu(K)$. Its multiplier is the diagonal image of c_0. The test at p_0 imposes its own compact-level condition on the localization of this same pair. Since the level multipliers are finite integral units, at most one positive rational c_0 can occur for a given multiplier coset. These assertions apply to each double coset in a linear combination. Unfolding by rational conjugacy divides each finite-level orbit by its stabilizer order. This gives the orbital weight of (γ, c_0). The marked reducible terms vanish by 39.13.

Suppose P is irreducible. Identify the CM coefficient field of the PEL datum with E as in 39.18. Then $B = E[\gamma] \simeq E[T]/(P)$ is a field of degree a over E, and the underlying E-space is a line over B. The adjoint involution on B sends γ to $c_0\gamma^{-1}$. At a definite embedding all eigenvalues have squared absolute value c_0. At the distinguished embedding the same holds on the positive line and on its definite orthogonal complement. Thus at every embedding of B the adjoint acts as complex conjugation. Write B^+ for its fixed field. In particular $c_0 = |\mu|^2$ is fixed by the chosen polynomial and mark.

Fix a definite pair of polynomial P, and denote its form by h. In the polarization convention for the definite pairs, h is positive definite at the distinguished real place v_0 of F and negative definite at every other real place. Over v_0 the a embeddings of B extending the chosen embedding of E give the a eigenlines of γ. They give distinct real places of B^+, since conjugating an embedding of B changes its restriction to E. Give the multiplier d sign $+1$ at the place belonging to μ, sign -1 at the other $a-1$ places above v_0, and sign $+1$ at every place over $v \neq v_0$. Thus the products of the signs of h and d leave exactly the marked line positive at v_0 and leave the form negative definite at the other places. There exists $d \in (B^+)^\times$ with these signs which is a norm from B at every finite place. Indeed the idèle with the prescribed signs and all finite components one has Artin symbol $(-1)^{a-1} = 1$ in the quadratic extension B/B^+. By global reciprocity it is a principal idèle times an idèlic norm. Its principal factor has the required signs, since norms at real places are positive, and its finite components are local norms [41, Ch. VI, §5].

Replace h by $h_d(v, v') = h(dv, v')$. Since d is self-adjoint this is Hermitian. It has the marked signature, and

$$h_d(\gamma v, \gamma v') = c_0 h_d(v, v').$$

The new pair has exactly the same γ and c_0. Use one d in the abstract field $E[T]/(P)$ for this choice of μ. Its image in any representative is a polynomial in γ, so the construction respects conjugacy. Replacing d by d^{-1} gives the inverse on marked classes.
Lean: `GlobalTheory.LKDef.move_definite (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsPolynomialMoveDefiniteSimilitudePairMarked`, lines 750–777)

At each finite place write $d = u\bar{u}$ in the local algebra of B. The local choices can be taken from the idèlic norm above. Multiplication by u identifies h_d with h and commutes with γ. Transporting the finite trivialization by this isometry preserves the argument of the test away from p_0. At p_0 the separate argument is the localization of (γ, c_0) on both sides. The rational centralizer on either side is

$$\{(z, s) \in B^\times \times \mathbf{Q}_{>0} : z\bar{z} = s\}.$$

Indeed an E-linear map commuting with γ is multiplication by an element of B, and its similitude condition is $z\bar{z} = s$ for both forms. The multiplier s varies in this centralizer. The element used in the test is the particular member (γ, c_0). Thus the local isometries identify the orbits and their stabilizers, including all multiplier conditions.

Lean: GlobalTheory.LKDefTerm.cent_eq (Development.AbelianVarietyTheory.Proofs.DivisorProofs.KottwitzTermEqDefinitePairOfLocalIsometry, lines 90–110)
Lean: GlobalTheory.LKDefTerm.term_eq_of_local_isom (Development.AbelianVarietyTheory.Proofs.DivisorProofs.KottwitzTermEqDefinitePairOfLocalIsometry, lines 185–199)

The definite trace in Proposition 39.13 sums over the rational classes of definite forms on E^a of signature $(a,0)$ at v_0 and $(0,a)$ at the $[F:\mathbf{Q}]-1$ other real places, with the away-from-p_0 finite invariants matching the ambient PEL form h_{PEL} of Lemma 39.11(1); both signatures have real discriminant sign product $(-1)^{[F:\mathbf{Q}]-1}=-1$. Equivalence allows rational change of basis and a positive rational multiplier. Since $d\in(B^+)^\times$ is a local norm at every finite place, replacing h_{PEL} on a marked pair by $h_{d^{-1}}$ preserves all finite local invariants while changing the signature at v_0 from $(1,a-1)$ to $(a,0)$, supplying a definite class with its transported finite trivialization. Lemma 39.11(3) provides the standard definite summand $h_{\mathrm{std}}=\mu\,1_a$ in this sum, on which Proposition 39.13 places the chosen eigenvector. For each mark the inverse constructions identify the orbital weights in the full sum term by term.
Lean: GlobalTheory.LKDef.dTrace (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.TraceDefinitePairsEqAutomorphicTrace, lines 80–82)

Summing over the distinct roots of P gives each definite orbital weight the factor $\sum_{P(\mu)=0}c(P,\mu)$, as asserted.
Lean: GlobalTheory.LKDef.cls_equiv_dcls_irr (Development.AbelianVarietyTheory.Proofs.DivisorProofs.MarkedFieldPairsEquivDefiniteFieldPairsOdd, lines 81–186)

For its application to the trace, one can identify the coefficient on this very pair. In the geometric coordinates the weight in 39.13 is

$$\xi^\sharp(\gamma,c_0)=\xi({}^t\gamma^{-1},c_0^{-1}),$$

using the formula of 39.10 on the ambient matrix and scalar group. Let $t_v=(t_{v,1},\dots,t_{v,a})$ be the roots of $\tau_v P$. Then

$$\operatorname{tr}\xi^\sharp(\gamma,c_0)=c_0^{-b}\prod_v s_{\lambda_v}(t_v),$$

where s_{λ_v} is the character of W_v on diagonal matrices. Indeed taking the dual of the inverse transpose gives the trace of W_v on a matrix with eigenvalues t_v. The reciprocal polynomial has root lists t_v^{-1} and t_v/c_0 at the two conjugate embeddings. For the rectangle $(b_v^a)=(b_v,\dots,b_v)$, the two character factors on these lists multiply to

$$s_{(b_v^a)}(t_v^{-1})\,s_{\lambda_v+(b_v^a)}(t_v/c_0)=c_0^{-\sum_i(\lambda_{v,i}+b_v)}s_{\lambda_v}(t_v).$$

This follows by factoring the determinant character of power b_v out of the second factor and using homogeneity. Multiplication over v gives precisely the trace above. Finally, expanding the diagonal entries of the definite Hecke operator gives this coefficient trace on every returning coset, with the stabilizer weight already computed. Its sum over the definite classes is $D(F)$, since 39.13 gives vanishing off the regular irreducible locus.
Lean: UpperTheory.Gal.CertificateRingHomMap.pWtCore_of (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ProjectiveAutomorphicHecke.PWtCore, lines 45–128)
Lean: GlobalTheory.LKDef.trace_definite_forms_identity (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LieModelBaseChangeMap, lines 14–24)

□

Proof of 39.26

Proof. Fix the common fibre exponent k from the degrees needed for interpolation. On fibre factor u compose the correspondence with multiplication by a nonzero $\alpha_u\in 2\mathcal{O}_{F_{\mathrm{PEL}}}$, acting through the centre of the PEL order. The trace calculation applies to every admissible regular test F, including $F=a_0sf$ chosen below from 39.13. For each g in its support, admissibility gives $g^{-1}\widehat{\Lambda}^{p_0}\subseteq\widehat{\Lambda}^{p_0}$. At a fixed point of the base, let A_{fib} denote the resulting endomorphism on the fibre tangent space. Its eigenvalues are twice algebraic integers since the endomorphism with $\alpha_u/2$ preserves the lattice. Thus the difference from the identity is an isogeny on the fibre. Its kernel has size $|\det(1-A_{\mathrm{fib}})|^2$ by the determinant on the underlying integral lattice. In 39.24 this cancels the fibre denominator and leaves $\overline{\det(1-A_{\mathrm{fib}})}\det(1-yA_{\mathrm{fib}})$ in the generating function at $T=-y$. The multiplicity of the finer levels cancels the relative Hecke index.

For a regular orbit polynomial P of degree n, let $\widetilde{P}$ be its transport to the field of the PEL datum and put

$$P^{\vee}(X) = \widetilde{P}(0)^{-1} X^{n} \widetilde{P}(X^{-1}).$$

The mark μ of P gives the root μ^{-1} of the reciprocal polynomial on the chosen side of the imaginary quadratic subfield and $\bar{\mu}^{-1}$ on the other side. Let t be the embedding indexed by i_0 and $\bar{t}$ its conjugate. At each embedding η, write r_{η} for the roots of $\eta(P^{\vee})$. At t use the list $x = (x_0, \dots, x_{n-1}) = r_t$, with mark x_i. The base tangent eigenvalues in these coordinates are x_i/x_j for $j \neq i$. The base factor in 39.24 is therefore $\prod_{j \neq i} (x_j - y x_i)/(x_j - x_i)$. The fibre factors have the form $\prod_j (1 - \eta(\alpha_u) r_{\eta,j})$. Scale the marked root by y at t and all roots except the conjugate mark at $\bar{t}$. At the other embeddings, scale all or none of the roots according to the signature.

Let λ_{η} be the conjugated partition at η and $s_{\lambda_{\eta}}$ its Schur polynomial. Each Schur polynomial is a polynomial in elementary symmetric functions. These functions are the signed coefficients of the single-variable determinant polynomial $\prod_j (1 - z x_j)$. Interpolate in the multiplication parameters to obtain products of these coefficients. Use a separate fibre factor for each determinant polynomial in a product. Under the embeddings of F_{PEL}, an integral basis gives an invertible complex change of coordinates. The nonzero tuples in the even integer lattice are Zariski dense, so the interpolation can use nonzero α_u from $2\mathcal{O}_{F_{\mathrm{PEL}}}$. The finitely many conjugated weights and signatures admit a common number k of factors. Denote by C the resulting linear combination of multiplication pullbacks.

For a partition λ at t, write $x^{(i)}$ for x with x_i replaced by $y x_i$. The identity for the sum over marks is

$$\sum_{i=0}^{n-1} s_{\lambda}(x^{(i)}) \prod_{j \neq i} \frac{x_j - y x_i}{x_j - x_i} = s_{\lambda}(x) \sum_{i=0}^{n-1} y^{\lambda_i + n - 1 - i}.$$

The product is the quotient of the scaled and original Vandermonde determinants. Multiply by the original Vandermonde and expand the Schur alternant with column exponents $\lambda_j + n - 1 - j$. In the term for mark i, the row i contributes the power of y of its assigned column. Summing over i gives the sum over all column exponents.

At $\bar{t}$, all the roots except the mark are scaled. In the conjugate root variables the base factors are $(x_i - y x_j)/(x_i - x_j)$ for $j \neq i$, since the roots at t have a common squared absolute value by the polarization of the orbit. The scaled Vandermonde quotient includes an additional $y^{\binom{n-1}{2}}$ from pairs with both roots scaled. In each determinant term the scaled rows contribute the sum of their assigned exponents. After division by the additional Vandermonde power, omitting the column indexed by i gives exponent $\sum_{j \neq i} \lambda_{\bar{t},j} + i$.

Put $b_0 = \sum_{\eta \neq t, \bar{t}} \varepsilon_{\eta} |\lambda_{\eta}|$, where $\varepsilon_{\eta} = 1$ on the conjugate side of the datum and 0 on the chosen side, and $|\lambda_{\eta}|$ is the sum of its parts. A constant partition m contributes the m-th power of the determinant. When $\lambda_{\bar{t}}$ is constant m, the exponent vector is $h_i = (n-1)m + b_0 + \lambda_{t,i} + n - 1 - i$. When λ_t is constant m, it is $h_i = m + b_0 + \sum_{j \neq i} \lambda_{\bar{t},j} + i$. These are the two weight transports of the plan. In either case the marked sum after applying C is $\prod_{\eta} s_{\lambda_{\eta}}(r_{\eta}) \sum_i y^{h_i}$.

Apply 39.25 with $c(P, \mu)$ equal to the interpolated fixed-point contribution just calculated. Its sum over the roots is $\prod_{\eta} s_{\lambda_{\eta}}(r_{\eta}) \sum_i y^{h_i}$. The product of Schur polynomials is the trace of the definite algebraic weight on this class. Summing it with the definite orbital weights gives $D(F)$ by the trace on the finite set of definite level classes: expand the diagonal entries of the Hecke matrix, summing over returning cosets and dividing by their stabilizer orders. Thus 39.24 gives

$$\sum_{j,b} (-y)^j (-1)^b \operatorname{tr}(T_F C \mid H^b(Z, \Omega_Z^j)) = D(F) \sum_i y^{h_i}.$$

Put $\vartheta_g = \sigma(\iota(\theta_g))$. For each b, write $T_{g,b}$ and $T_{f,b}$ for the induced operators on $H^b(Z, \Omega_Z^{h_i})$. For a word $w = (g_1, \dots, g_r)$ put $T_{w,b} = T_{g_1,b} \cdots T_{g_r,b}$, with $T_{\varnothing,b} = 1$, and define

$$M_b = \sum_{w} \operatorname{im}(T_{w,b} T_{f,b}).$$

The spectral-test identity in 39.13, together with the compatibility of Hecke convolution with composition, gives

$$(T_{g,b} T_{g',b} - T_{g',b} T_{g,b}) T_{w,b} T_{f,b} = 0 \qquad (g, g', w).$$

Each M_b is stable under all the $T_{g,b}$, and this displayed identity proves that their restrictions to M_b commute. These spaces are finite-dimensional and vanish for $b > \dim Z$. Consequently their direct sum has a finite joint primary decomposition. Choose a word polynomial s with $s(\vartheta) = 1$ whose factors vanish at every other character, and raise those factors to powers that annihilate the corresponding primary spaces. This also defines s when the ϑ-primary space is zero.

Apply the spectral-test property to this s and choose a_0 so that $F = a_0 sf$ has $D(F) \neq 0$. The coefficient of y^{h_i} in the displayed trace identity is a positive integer times $D(F)$, so $T_F C$ has nonzero trace in at least one degree b. In that degree $T_F = a_0(T)s(T)T_f \neq 0$, and hence $s(T)T_f \neq 0$. Its image lies in the ϑ-primary subspace of M_b, which is therefore nonzero. On this subspace every $T_{g,b} - \vartheta_g$ is nilpotent and these operators commute. Choose finitely many operators generating their finite-dimensional commutative algebra, and successively take their kernels. At each step a nilpotent endomorphism of a nonzero invariant space has a nonzero kernel. This yields a nonzero common eigenvector with eigenvalues $(\vartheta_g)_g$.
Lean: `UpperTheory.Gal.LocalSmoothExtension.hodgeEig_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.LefschetzSigmaSums.PHodgeEig`, lines 6–67)

□

Proof of 39.27

Proof (outline). Use the smooth projective lower fibre power over $\mathcal{O}_{E,(w)}$ of 39.18, and base change it to the completed valuation ring at w. Its geometric generic fibre is the scalar extension of the lower generic fibre used in 39.23. The assumed surjectivity of the level cover and the deck eigenvalues one give descent of the invariant piece by normalized trace. For the smooth model, the comparison pairing is obtained by gluing the local pairings. It is Galois-equivariant and injective after extension to the period ring. Its span over the crystalline period ring, together with this injection and the regularity of that ring, gives crystalline admissibility in each degree. The descended eigenspace embeds into a finite sum of these degrees after finite coefficient extension. Crystallinity passes to the subrepresentation and to the quotients in its composition flag. The classical smooth proper comparison theorem gives the good-reduction setting [216, Thm. 1.1(i)].

For the jump calculation use the embedding $\sigma \circ \iota \circ \tau$ and eigenvalues $\sigma(\iota(\theta_g))$ together as in 39.26. Pass to the lower Hodge eigenspace by trace and pullback. Its eigenvalue for the lower correspondence is $\sigma(\iota(\theta_g/m_g))$, where m_g is the nonzero degree of the upper correspondence over the pullback of the lower one. Use the Hodge–Tate comparison for the lower smooth projective model, with its compatibility with the Hecke correspondences after geometric base change. In degree h it carries the nonzero lower Hodge eigenclass to a Hodge–Tate occurrence in the lower geometric eigenspace. The inclusion of the lower piece into the upper one carries this occurrence to the upper representation. Choose the largest block index with a nonzero coordinate in the Hodge–Tate eigenvector in a basis ordered by its composition flag. Its projection to that diagonal block is nonzero and satisfies the same eigenvector equation, so the occurrence survives in the associated graded.

For an induced embedding of the chosen bridge, conjugate the bridge so that its field inclusion is the induced embedding. Transport the occurrence along this conjugation. Crystallinity gives a finite de Rham model at the original bridge, and the Hodge–Tate occurrence at the conjugated bridge gives membership of h in its jump multiset at the induced embedding.

For the root assertion use the following comparison input.

Claim 1. Let X denote the smooth projective lower model over the completed local integers, let K be their fraction field, and choose an algebraic closure $\bar{K}$. Write $q = \ell^f$ for the residue-field cardinality and put $H_b = H^b_{\text{ét}}(X_{\bar{K}}, \mathbf{Q}_\ell)$. Take the integral correspondences of 39.18, with finite first projections that are étale after inverting ℓ, satisfying the valuative branch identity of 39.22. Let $A_{j,b}$ be their compatible generic-fibre correspondence operators. Assume $f > 0$. For each crystalline period datum of the comparison model over K, write B for its period space and φ for its Frobenius. For each degree b there is an integer $\nu_b \geq 0$ such that every invariant period tensor $x \in H_b \otimes_{\mathbf{Q}_\ell} B$ of this datum satisfies

$$\sum_{j=0}^{a} (-1)^j q^{\binom{j}{2}} \big(A_{j,b} \otimes \varphi^{\,f(a-j+\nu_b)}\big) x = 0.$$

The integer ν_b is independent of x. This is the comparison input for the generic operators and the reduced branches; the first projections need not be étale on the special fibre. The construction of this relation uses

two finite comparison exponents.

Fix a small Grothendieck universe containing $\mathcal{O}_C^\flat$ and the coordinate rings of all schemes under consideration, and define Test_ℓ to be the small category of *test algebras* in that universe. An object $A \in \mathrm{Test}_\ell$ is a perfect $\mathbf{F}_\ell$-algebra equipped with an $\mathcal{O}_C^\flat$-algebra structure, π-adically complete with respect to the non-zerodivisor $\pi = \ell^\flat \in A$, satisfying two axioms: (i) algebraic integral closedness, namely that every monic polynomial of positive degree over A has a root in A; and (ii) the valuation-like idempotent splitting property that whenever a power of π lies in an ideal $(f, g) \subseteq A$, there exists an idempotent $e \in A$ such that $ef \in (g)$ and $(1-e)g \in (f)$. Morphisms in Test_ℓ are $\mathcal{O}_C^\flat$-algebra homomorphisms, and coverings are finite idempotent-partition decompositions $A \to \prod_{i=1}^r A[e_i]$ for complete families of pairwise orthogonal idempotents $\sum_{i=1}^r e_i = 1$. Its untilt is $A^\sharp = W(A)/(\xi)$ (with $\xi = [\pi] - \ell$), its residual quotient is $\overline{A} = A/(\pi)$, and $A_{\mathrm{cris}}(A)/\ell^n$ is the divided-power envelope of $W(A)$ along (ξ) modulo ℓ^n. For any presheaf $P : \mathrm{Test}_\ell \to \mathrm{Set}$ and $n, j \geq 0$, define its test cohomology $H^j_{\mathrm{test}}(P, \ell^n)$ as the explicit simplicial (bar) cochain cohomology of the functor $(A, x) \mapsto A_{\mathrm{cris}}(A)/\ell^n$ on the small category of elements $\mathrm{El}(P)$ (matching the explicit cochain prism homotopy below); for an $\mathcal{O}_K$-scheme Y (with no smoothness assumption), write $P_Y(A) = Y(A^\sharp)$ and $H^j_{\mathrm{test}}(Y, \ell^n) = H^j_{\mathrm{test}}(P_Y, \ell^n)$, and for the special fibre X_{k_K} write $P_Q(A) = X_{k_K}(\overline{A})$. For a smooth projective model M over $\mathcal{O}_K$ (denoted X in the correspondence step below), the crystalline-to-étale comparison proceeds in three steps. First, define the canonical, framing-independent comparison map $\beta_M : H^j_{\mathrm{cris}}(M_{\mathcal{O}_C/\ell}/(A_{\mathrm{cris}}/\ell^n)) \to H^j_{\mathrm{test}}(P_M, \ell^n)$ via the thickening functor sending each test point $(A, x \in M(A^\sharp))$ in $\mathrm{El}(P_M)$ to the divided-power thickening $(A_{\mathrm{cris}}(A)/\ell^n \twoheadrightarrow A^\sharp/\ell = \overline{A},\ \bar{x} \in M(\overline{A}))$, and analogously define $\beta_{X_{k_K}} : H^j_{\mathrm{cris}}(X_{k_K}/W_n(k_K)) \to H^j_{\mathrm{test}}(P_Q, \ell^n)$ via $(A, \bar{x} \in X_{k_K}(\overline{A})) \mapsto (A_{\mathrm{cris}}(A)/\ell^n \twoheadrightarrow \overline{A},\ \bar{x})$; both maps are canonical, functorial, and cup-product-compatible, and $\beta_{X_{k_K}}$ intertwines the reverse Frobenius map μ_n below with ρ^* and crystalline φ^f with residual q-Frobenius F^*. Second, on each framed smooth affine open $U \subset M$ equipped with an étale torus framing $U \to \mathbb{G}^d_{m,\mathcal{O}_K} = \mathrm{Spec}\,\mathcal{O}_K[T_1^{\pm 1}, \ldots, T_d^{\pm 1}]$, we prove directly (without invoking the comparison theorem of [216]) that $\mu^{N_{\mathrm{loc}}(d,j)}$ (where $\mu = [\epsilon] - 1$) kills both $\ker(\beta_U)$ and $\mathrm{coker}(\beta_U)$ uniformly in n. Lift the étale torus framing across the divided-power thickening A_{cris}/ℓ^n to a divided-power thickening D carrying a $\mathbf{Z}^d$-action scaling the d lifted torus coordinates by powers of $q = [\epsilon] = 1 + \mu$, and let $\mathrm{El}^+(P_U)$ be the category of elements of $A \mapsto U(A^\sharp)$ on the larger category of all perfect π-adically complete $\mathcal{O}_C^\flat$-algebras in which π is a non-zerodivisor, without Axioms (i) and (ii), so that $\mathrm{El}(P_U)$ is its full subcategory on test algebras. Form the Kummer tower D', the object of $\mathrm{El}^+(P_U)$ universal for compatible systems of ℓ-power roots of the torus coordinates, and let G send each object (A, x) of $\mathrm{El}^+(P_U)$ to the thickening $A_{\mathrm{cris}}(A)/\ell^n \twoheadrightarrow \overline{A}$ with its reduced point. The compatible roots give the equivariant map $\theta : D \to G(D')$ by Teichmüller lifts. D' need not satisfy Axiom (i). Pass to the Rees-filtered coefficient sheaf over $\mathbf{Z}[X]$ grading the divided-power filtration by powers of X (for a function ρ with $\rho(0) = 0$ and $\rho(r+1) \leq \rho(r) + 1$, the Rees module of an object is the product over $r \geq 0$ of the divided-power steps of index $\rho(r)$ of its coefficients, and X sends $(x_r)_r$ to $(0, \mu x_0, \mu x_1, \ldots)$, which is defined because μ lies in the divided-power ideal). Restriction acts componentwise. Fix $N \geq 0$ and a cohomological degree in which Rees restriction is an isomorphism up to X^N. Component zero has the original level-n coefficients because $\rho(0) = 0$. Place a class in component zero. Its X^N-multiple has μ^N times that class in component N. For a class in the kernel, injectivity up to X^N kills this X^N-multiple, and passing from component N through the inclusion of filtration step $\rho(N)$ into the original coefficients shows that μ^N kills the class. For a target class, surjectivity up to X^N gives a Rees class mapping to its X^N-multiple; reading component N and including filtration step $\rho(N)$ into the original coefficients yields a preimage of μ^N times the target class.

Lean: `CrystallineSemistableComparison.B.P.rees_down` (`Development.WeilBound.CoreStatements.PadicHodgeComparison.EtaleCohomologyComparison.ZariskiEtaleLifting.FReesDown`, lines 10–11)

On both the thickening site of U and the test-algebra site $\mathrm{El}(P_U)$, let $i : T \to T_1$ be the universal map adjoining to an object T a coordinate unit η (on the thickening site, a unit congruent to 1 modulo the divided-power ideal; on $\mathrm{El}(P_U)$, a unit with $\eta^\sharp = 1$, the universal such object being the algebra of continuous functions from $\mathbf{Z}_\ell$ to the test algebra, with γ the translation by 1), and let γ be the endomorphism of T_1 over T with $\gamma(\eta) = q\eta$. One-variable X-acyclicity holds for the Rees-filtered coefficients in the following form: the map induced by i is injective; if x has filtration level r on T_1 and $\gamma x = x$, then μx is the image of an element of level $r+1$ on T; and every x of level r on T_1 satisfies $\mu x = (\gamma - 1)y$ for some y of level $r+1$ on T_1. On the thickening line, expand in the divided powers of $\eta - 1$ with translation $\eta \mapsto q\eta$. On the test line, translation acts on continuous functions on $\mathbf{Z}_\ell$, where constants give the invariants and discrete summation solves the difference equation. Multiplication by $\mu = q - 1$ supplies the filtration shift in both computations.

Thus X annihilates the cohomology of the cone of the map from the coefficients of T to the complex $1-\gamma$ on the coefficients of T_1. Inducting across the d torus coordinates shows that restricting bar cochains from the thickening site to the single $\mathbf{Z}^d$-object D (resp. from $\mathrm{El}^+(P_U)$ to D') induces an isomorphism on Rees-filtered cohomology up to X^{N_1} (resp. X^{N_2}), with exponents N_1, N_2 depending only on (d, j) and independent of n. For D' the one-variable statement is applied to the lines adjoined to test algebras, together with the descent property that for every object T of $\mathrm{El}^+(P_U)$ the augmented bar complex of the Rees coefficients over the test objects under T is exact up to X. Put $q_1 = [\epsilon^{1/\ell}]$, using the compatible system defining ϵ, and put $b_1 = \sum_{i=0}^{\ell-1} q_1^i$. Then $q_1^\ell = q$ and $(q_1-1)b_1 = \mu$. At each filtration step q_1-1 kills the augmented cohomology, and multiplication by b_1 raises the filtration. This descent property makes restriction of bar cochains from $\mathrm{El}^+(P_U)$ to $\mathrm{El}(P_U)$ an isomorphism on Rees-filtered cohomology up to X^{2j+3} in degrees at most j. On Rees coefficients for $\theta : D \to G(D')$, projection onto the unit Kummer monomial splits off the base equivariantly. For each nonunit basis monomial, let χ be its character, choose a coordinate k on which χ is nontrivial, and group these monomials by the chosen coordinate. The Kummer basis calculation gives an element w with $q_1-1 = (\chi(\gamma_k)-1)w$, hence $\mu = (\chi(\gamma_k)-1)wb_1$. On that summand X kills the kernel and cokernel of γ_k-1. The kernel and image exact sequences then show that X^2 kills the kernel of its action on cohomology. Since γ_k acts as the identity on group cohomology, X^2 kills the complementary cohomology. Thus θ induces the required comparison up to X^2. Composing these four up-to-X-power equivalences gives an equivalence on Rees-filtered cohomology up to $X^{N_{\mathrm{loc}}(d,j)}$ with $N_{\mathrm{loc}}(d, j) = N_2+N_1+2j+5$. The component calculation gives uniform annihilation of $\ker(\beta_U)$ and $\mathrm{coker}(\beta_U)$ by $\mu^{N_{\mathrm{loc}}(d,j)}$ at every level n. Because β_U is canonical and independent of the chosen framing, covering the quasi-compact separated projective smooth $\mathcal{O}_K$-model M by a finite Zariski atlas of torus-framed smooth affine opens of dimension at most d (every nonempty intersection of which inherits a torus framing of dimension $d' \le d$) and gluing via Zariski Čech hypercovering descent kills $\ker(\beta_M)$ and $\mathrm{coker}(\beta_M)$ uniformly by $\mu^{N_{\mathrm{tor}}}$ with $N_{\mathrm{tor}} = (2j+3)\max_{d'\le d} N_{\mathrm{loc}}(d', j+1)$ independent of n. Since $t = \mu v$ with $v \in A_{\mathrm{cris}}^\times$, $t^{2N_{\mathrm{tor}}}$ kills both the kernel and cokernel of the inverse-limit comparison map $\varprojlim_n \beta_M$, yielding a G_K- and φ-equivariant isomorphism after inverting t (valid also for ramified $K/\mathbf{Q}_\ell$, using the Frobenius shift $N = fN_1$ constructed below). Third, lifting both Axiom (i) and Axiom (ii) across tilting from π-adically complete A to $A^\sharp = W(A)/(\xi)$ and inverting ℓ shows that $A^\sharp$ is integrally closed in $A^\sharp[1/\ell]$ and that both $A^\sharp$ and $A^\sharp[1/\ell]$ are algebraically integrally closed valuation-like (w-strictly local) rings: every surjective étale morphism $Z' \to \operatorname{Spec} A^\sharp[1/\ell]$ (of arbitrary finite presentation, including non-finite surjective étale covers) admits a global section, so $H^q_{\text{ét}}(\operatorname{Spec} A^\sharp[1/\ell], \mathbf{Z}/\ell^n) = 0$ for every $q \ge 1$. The comparison map e is defined through a resolution. Represent a class in $H^j_{\text{ét}}(Y_{\bar K}, \mathbf{Z}/\ell^n)$ by a global section a of the j-th term of a resolution of $\mathbf{Z}/\ell^n$ on $Y_{\bar K}$. Restricting a along the test points $\operatorname{Spec} A^\sharp[1/\ell] \to Y_{\bar K}$ gives a 0-cochain in the double complex of bar cochains on $\mathrm{El}(P_Y)$ with values in the sections of the resolution. Since every surjective étale map to $\operatorname{Spec} A^\sharp[1/\ell]$ has a section, on each test point these sections resolve the locally constant $\mathbf{Z}/\ell^n$-valued functions, and a staircase through the double complex carries a to a bar class of degree j with values in locally constant functions. Evaluating locally constant functions in $A_{\mathrm{cris}}(A)/\ell^n$ via Teichmüller lifts of clopen idempotents then defines $e : H^j_{\text{ét}}(Y_{\bar K}, \mathbf{Z}/\ell^n) \to H^j_{\mathrm{test}}(P_Y, \ell^n)$ in every degree $j \ge 0$. It is compatible with reduction of the level, with the Galois action, with pullback along every morphism of models, smooth or not, with cup products and the unit class, and with first Chern classes, and its image is fixed by Frobenius.

For a finite multiset u of tuples $(c_1, \dots, c_r)$ over a commutative ring B, define its universal norm polynomial

$$P_u(T_1, \dots, T_r; X) = \prod_{c \in u} \Big(X - \sum_{k=1}^{r} c_k T_k \Big) \in B[T_1, \dots, T_r][X].$$

Let $G = \prod_{j=0}^{a} S_{d_j}$ be the branch-permutation group, and construct the common splitting cover $\pi : W \to X$ as the scheme-theoretic closure in $\prod_{j=0}^{a} {}_X Z_j^{\times_X d_j}$ of the generic ordered-configuration G-torsor (the complement over $1/\ell$ of all pairwise branch diagonals), equipped with branch morphisms $z_{j,\alpha} : W \to Z_j$ ($1 \le \alpha \le d_j$) satisfying $p_j \circ z_{j,\alpha} = \pi$. Because X is projective over $\mathcal{O}_K$, every finite set of points of X lies in a single affine open $\operatorname{Spec} R \subset X$; passing to a finite idempotent cover of the test object places the source point and all finitely many target branches simultaneously in a single affine chart $\operatorname{Spec} R \subset X$. Since each $p_j : Z_j \to X$ is finite and hence closed, shrinking the source chart to an affine open $\operatorname{Spec} R' \subseteq \operatorname{Spec} R \setminus \bigcup_{j=0}^{a} p_j(Z_j \setminus r_j^{-1}(\operatorname{Spec} R))$ containing the source point (and refining the finite idempotent cover of A accordingly) guarantees $r_j(p_j^{-1}(\operatorname{Spec} R')) \subseteq \operatorname{Spec} R$ for every $0 \le j \le a$. Because X is smooth over $\mathcal{O}_K$

with uniformizer ϖ, ℓ is a non-zerodivisor in R', R' is integrally closed in $R'[1/\ell]$, and the ideal $(\varpi) \subset R'$ is radical (reduced special fibre). Since each p_j is finite over R' and étale of constant degree d_j after inverting ℓ with $r_j(p_j^{-1}(\operatorname{Spec} R')) \subseteq \operatorname{Spec} R$, the universal norm polynomial of its branches has coefficients integral over R' inside $R'[1/\ell][T][X]$, hence belonging to $R'[T][X]$. In the list for index j, repeat every branch $q^{\binom{j}{2}}$ times and raise its coordinates to q^{a-j}, placing the even and odd indices on opposite sides; the two multisets have common size M (the weighted sum of the even-indexed degrees). By the valuative special-fibre identity of Lemma 39.22, the even and odd universal norm polynomials agree modulo every minimal prime of (ϖ), and because (ϖ) is radical they agree modulo (ϖ) everywhere on $\operatorname{Spec} R'$. Let $e_{K/\mathbf{Q}_\ell}$ be the ramification index of $K/\mathbf{Q}_\ell$, and let $N = fN_1$ be the Frobenius shift multiple of f chosen below (taken large enough that both $\ell^N \geq e_{K/\mathbf{Q}_\ell}$ and the Teichmüller congruence holds uniformly). Because $f \mid N$, the ℓ^N-th power Frobenius reduction $\psi : A^\sharp \to \overline{A}$, $x \mapsto \bar{x}^{\ell^N}$, sends ϖ to $\bar{\varpi}^{\ell^N} = 0$ (since $\varpi^{e_{K/\mathbf{Q}_\ell}} \in (\ell)$ and $\ell = 0$ in $\overline{A}$) and fixes the residue field k_K pointwise, factoring $\mathcal{O}_K \to \mathcal{O}_C/(\ell)$ through k_K via Teichmüller lifts. This induces the Frobenius-shifted reduction $\rho : P_X \to P_Q$ sending $X(A^\sharp)$ to $P_Q(A) = X_{k_K}(\overline{A})$ and intertwining crystalline φ^{fk} with residual q^k-Frobenius F^k, tracking the shift cleanly into $\nu_b = N_0 + N_1$. Pulling back the congruence modulo (ϖ) along any generically splitting chart map into $A^\sharp$ followed by ψ shows that the ℓ^N-shifted coordinate lists $(c_{i,k})$ and $(c'_{i,k})$ have the same universal norm polynomial over $\overline{A}$ for every test algebra $A \in \mathrm{Test}_\ell$ (including ramified $K/\mathbf{Q}_\ell$ and non-ordinary test rings).

Set $N_0 = (M+1)^M$, so $(M+1)^M \leq q^{N_0}$. Because A is a perfect $\mathbf{F}_\ell$-algebra and $q = \ell^f$, there exists $w \in A$ with $w^{q^{N_0}} = \pi$. Since π is a non-zerodivisor in A, so is w, and w inherits the valuation-like idempotent splitting property while $w^{(M+1)^M} \mid \pi$. Lift the coordinates $c_{i,k}, c'_{i,k} \in \overline{A}$ to elements $u_{i,k}, u'_{i,k} \in A$; reducing modulo $w^{(M+1)^M}$ preserves equality of the two norm polynomials in $(A/(w^{(M+1)^M}))[T_k]$. We construct a complete family of pairwise orthogonal idempotents $(e_\sigma)_{\sigma \in S_M}$ in A with $e_\sigma(u_{i,k} - u'_{\sigma(i),k}) \in (w)$ for all σ, i, k by induction on M. The case $M = 0$ is trivial. For $M \geq 1$, specializing the norm-polynomial congruence at the first linear form $\sum_k u_{1,k}T_k$ shows that $\prod_{j=1}^M \bigl(\sum_k (u_{1,k} - u'_{j,k})T_k\bigr)$ vanishes modulo $w^{(M+1)^M}$. Applying the valuation-like idempotent splitting property for w iteratively across the M linear factors splits A into a finite orthogonal idempotent partition on each factor of which some index j satisfies $u_{1,k} - u'_{j,k} \in (w^{(M+1)^{M-1}})$ for all k; cancelling that common linear factor modulo $w^{(M+1)^{M-1}}$ reduces the remaining $M-1$ tuples to the induction hypothesis, yielding $(e_\sigma)_{\sigma \in S_M}$ with $e_\sigma(u_{i,k} - u'_{\sigma(i),k}) = wt_{\sigma,i,k}$ for some $t_{\sigma,i,k} \in A$. Applying the q^{N_0}-th power Frobenius endomorphism of the $\mathbf{F}_\ell$-algebra A fixes every idempotent e_σ and sends w to $w^{q^{N_0}} = \pi$, giving

$$e_\sigma\bigl(u_{i,k}^{q^{N_0}} - {u'_{\sigma(i),k}}^{q^{N_0}}\bigr) = \pi t_{\sigma,i,k}^{q^{N_0}} \in (\pi),$$

and reducing modulo π yields $\bar{e}_\sigma c_{i,k}^{q^{N_0}} = \bar{e}_\sigma {c'_{\sigma(i),k}}^{q^{N_0}}$ in $\overline{A}$ for every σ, i, k. The determining sections therefore identify the paired q^{N_0}-shifted branch maps $F^{a-j+N_0} \circ \rho \circ r_j \circ z_{j,\alpha} : P_W \to P_Q$ on each factor ring $\overline{A}[\bar{e}_\sigma]$, so the signed formal sum of these shifted branch maps vanishes on each nonzero factor; a zero factor contributes nothing.

Lean: `UpperTheory.exists_orthogonalIdempotents_matching_of_sameNorm` (`Development.AutomorphicArithmetic.EtaleIdempotents.ExistsOrthogonalIdempotentsMatchingOfSameNorm`, lines 40–99)

To pass from maps to bar cochain cohomology $H^j_{\mathrm{test}}(P_W, \ell^n)$ in every degree $j \geq 0$, use the fact that Test_ℓ is a small category in the fixed Grothendieck universe: every additive (finite-idempotent-partition-preserving) coefficient functor $F : \mathrm{Test}_\ell \to \mathbf{Ab}$ embeds into an additive functor K that is injective in the category of all functors $\mathrm{Test}_\ell \to \mathbf{Ab}$, yielding a short exact sequence $0 \to F \to K \to Q \to 0$ with Q also additive (since finite products are exact in $\mathbf{Ab}$), where $H^{j+1}_{\mathrm{test}}(P, K) = 0$ for every presheaf P and $j \geq 0$, short exact sequences induce natural long exact sequences on $H^\bullet_{\mathrm{test}}$, and $H^0_{\mathrm{test}}(P, F) \cong \mathrm{Nat}(P, F)$ consists of natural sections. In degree 0, naturality of $c \in \mathrm{Nat}(P_Q, F)$ and additivity $F(A) \xrightarrow{\sim} \prod_{\sigma \in S_M} F(A[e_\sigma])$ immediately kill the signed sum of pullbacks along the shifted branch maps $F^{a-j+N_0} \circ \rho \circ r_j \circ z_{j,\alpha} : P_W \to P_Q$ whenever it vanishes on each nonzero factor $A[e_\sigma]$. For $j > 0$, applying this to $F = (A \mapsto A_{\mathrm{cris}}(A)/\ell^n)$ and inducting on j along the natural long exact sequence of $0 \to F \to K \to Q \to 0$ (using $H^j_{\mathrm{test}}(-, K) = 0$ and additivity of Q) establishes the signed shifted branch-map pullback relation for every class pulled back from $H^j_{\mathrm{test}}(P_Q, \ell^n)$ into $H^j_{\mathrm{test}}(P_W, \ell^n)$ at every level n and in every degree $j \geq 0$.

Lean: `UpperTheory.Ru.crystallineComparison.emb` (`Development.AutomorphicArithmetic.CommutativeAlgebra.EtaleCohomology.AddIff`, lines 74–92)

To construct Frobenius on the finite divided-power thickenings, work in the integral crystalline ring before completion, inside the Witt ring with ℓ inverted. For the Witt-vector generator ξ of the kernel of the map to untilts, the Witt congruence gives $\varphi(\xi) = \xi^\ell + \ell y$. Since $\xi^\ell = \ell!\,\xi^{[\ell]}$, this becomes $\varphi(\xi) = \ell v$ in the integral ring. For $h \geq 1$, the factorial valuation bound $v_\ell(h!) \leq h - 1$ gives a Witt scalar w with $h!w = \ell^{h-1}$. Apply Frobenius to $h!\xi^{[h]} = \xi^h$ and cancel $h!$ in the ambient Witt ring with ℓ inverted. It follows in the integral crystalline ring that $\varphi(\xi^{[h]}) = \ell w v^h$. Thus Frobenius carries the ideal generated by ℓ and the positive divided powers of ξ into (ℓ). The divided-power morphism descends to the finite quotients, and its iterates induce the ℓ-power iterates on the residue quotient.

Choose the second shift $N = fN_1$ before choosing an invariant tensor. For an integral element of K there is a uniform bound $c < 1$ on the distance to its untilted residue Teichmüller lift. After taking ℓ^N-th powers, the bound is $\max\{|\ell|, c^{\ell^N}\}$. Take N a sufficiently large multiple of f and of the residue Frobenius period, with $\ell^N \geq e_{K/\mathbf{Q}_\ell}$. Then the lift is fixed by this power, and the bound is at most $|\ell|$ for all integral elements at once. The resulting congruence identifies, at every finite level, the N-fold Frobenius of the structural map into the residue quotient with reduction followed by the chosen Teichmüller lift into that quotient.

To compare the two finite-level pullbacks, use the congruence to pass between the original and intermediate structural maps. On points the first transformation keeps the structural map to the test quotient and acts by the ℓ^N-power map on the scheme point. The reverse transformation precomposes the structural map with the N-fold Frobenius of the residue quotient and keeps the scheme point. Thus their composite has the same two components as the N-fold Frobenius. The arrows on points are opposite to the cohomological pullbacks: ν_n maps from the original to the mixed coefficient group and is φ^N-semilinear, while μ_n goes back and is linear. On each test thickening the two composite descriptions have the same structural map on the divided-power base. The identity on its underlying thickening ring preserves the divided powers and carries one reduced point to the other. These identities commute with morphisms of thickenings and with the maps on coefficients.

For a cochain z of degree $j > 0$ and a chain $T_0 \to \cdots \to T_{j-1}$, form the alternating sum of its evaluations on the j-chains that use the first functor through T_i, cross by the comparison morphism at T_i, and then use the second functor. Apply the terminal coefficient map of the second functor to each summand and give it sign $(-1)^i$. The interior faces cancel in pairs in the sum of the two prism boundaries; the extreme faces give the two endpoint pullbacks, using coefficient compatibility at the first endpoint. Thus for a cocycle the difference of the endpoint cochains is the coboundary of this alternating sum. In degree zero the cocycle equation on the comparison arrow directly identifies the two endpoint evaluations. The pullback attached to N-fold Frobenius agrees with N successive pullbacks, by composing once more and comparing the two thickenings by the same identity at each step. Consequently $\mu_n\nu_n = \Phi_n^N$ on cohomology at every level and in every degree.

Let $W_0 = W(k_K)$ be the Witt ring of the finite residue field, A_{cris} the integral crystalline coefficient ring, and L_b the inverse limit of the level-n Witt cohomology modules in degree b. The finite-level exact sequences give, for a Witt cohomology class at level n, a compatible family of connecting classes in L_{b+1}. The family is killed by ℓ^n, since the class at level n is killed by ℓ^n. The module L_{b+1} is finite over the Noetherian ring W_0, so the kernels of its successive powers of ℓ stabilize. Choose e such that ℓ^e kills this entire torsion submodule. The scaled class lifts to each higher level by exactness. These lifts can be made compatible: at each level the possible lifts are a coset of the kernel of reduction to level n, and the images of these cosets from higher levels stabilize by the descending chain condition on the corresponding submodules. The stabilized cosets have surjective transition maps and therefore contain a compatible family. Thus ℓ^e kills the cokernel of the projection from L_b at every level.

Flatness of A_{cris} identifies the kernel of projection at level n on $A_{\mathrm{cris}} \otimes_{W_0} L_b$ with $\ell^n(A_{\mathrm{cris}} \otimes_{W_0} L_b)$: before base change this uses divisibility by ℓ^n for any inverse-limit class vanishing at level n, and annihilation of the level by ℓ^n for the reverse inclusion. Use the finite-level base change maps to view these projections as evaluations P_n into the mixed coefficient systems. For an integral inverse-limit cohomology class x, lift the compatible scaled images $\ell^e\nu_n(x_n)$ separately at each level along P_n, where ν_n is the first finite-level Frobenius map. Successive lifts differ by $\ell^n(A_{\mathrm{cris}} \otimes_{W_0} L_b)$, and this finite tensor module is complete and separated. Their unique common limit defines αx with $P_n(\alpha x) = \ell^e\nu_n(x_n)$. These equations give φ^N-semilinearity and Galois equivariance by uniqueness. Assemble β levelwise from P_n followed by the reverse map μ_n. The finite-level identity $\mu_n\nu_n = \Phi_n^N$ gives $\beta\alpha = \ell^e\Phi^N$ on the inverse limit, where Φ is crystalline Frobenius on that cohomology. On extension to crystalline periods the comparison pairing uses ℓ^{-e} times

the pairing with αx.

Write the invariant tensor as $t^{-r}\sum_i v_i \otimes c_i$, with v_i from the integral cohomology lattice and c_i in the integral crystalline period ring. Its numerator has a compatible family of finite-level evaluations z_n, and there is a lift h whose level coordinates are $t^{u_0}z_n$ for some u_0. For each Galois element g, the rational invariant equation gives $gz_n = \chi(g)^r z_n$ after multiplication by powers of ℓ and t depending on g but valid at every level. Since $g(t) = \chi(g)t$, the weight of the lifted family is $u_0 + r$. The common kernel of the level-coordinate maps is killed elementwise by powers of t; clearing a further power gives the same twisted law for h.

As φ^N fixes the cyclotomic scalar and takes t to $\ell^N t$, the twisted law passes to αh after adjusting the power of ℓ. Divide by t^{u_0+r} in the rational period ring. Choose a basis of L_b after extension to the fraction field of W_0. Galois acts on period coordinates alone. The finitely many nonzero coordinates are invariant, and each becomes a Witt scalar after multiplication by a power of ℓ. Take the largest of these powers and clear the remaining Witt denominator of the vector, a unit times a power of ℓ, to obtain $y \in L_b$. Lifting the rational equality to the integral tensor costs a further power of t. Thus for some a', b' one has

$$\ell^{a'} t^{b'} \alpha h = t^{b'+u_0+r}(1 \otimes y).$$

Apply β and take each level coordinate. Crystalline Frobenius on the lift gives $\ell^d t^u$ times the evaluation with coefficients $\varphi^N(c_i)$, where $d = a' + e + Nu_0$ and $u = b' + u_0$. The prescribed finite-level pullbacks express the other side as $t^{u+r}\rho^*(\beta_{X_{k_K}}(x_n))$ for a residual crystalline class $x_n \in H^b_{\mathrm{cris}}(X_{k_K}/W_n(k_K))$. On x_n, the f-fold crystalline Frobenius agrees with residual q-Frobenius pullback F^*. For each $0 \le j \le a$, set $s_j = a - j + N_0$ and apply φ^{fs_j}, using $\varphi^f(t) = qt$; the scalars $\ell^d t^u$ and t^{u+r} become $\ell^d q^{us_j} t^u$ and $q^{(u+r)s_j} t^{u+r}$. Multiply the resulting identity by $q^{(u+r)j}$. Since $s_j + j = a + N_0$, this gives, at every level n,

$$\ell^d q^{u(a+N_0)} t^u \sum_i e(v_i)\, q^{rj} \varphi^{f(s_j+N_1)}(c_i) = q^{(u+r)(a+N_0)} t^{u+r} (F^{s_j} \circ \rho)^*(\beta_{X_{k_K}}(x_n)),$$

where both outer scalars are independent of j. The factor q^{rj} is the one produced by $1 \otimes \varphi^{f(a-j+N_0+N_1)}$ on $x = t^{-r}\sum_i v_i \otimes c_i$ after multiplication by the constant $q^{r(a+N_0+N_1)}$, since $\varphi^{f(s_j+N_1)}(t^{-r}) = q^{-r(s_j+N_1)} t^{-r}$. The integer $\ell^d q^{u(a+N_0)}$ is cancelled together with the other integer factors after passage to the rational comparison pairing. The exponents d, u may depend on the tensor and are independent of n; N_1 is independent of both the tensor and n.

To compare the branch pullbacks on W with the generic correspondence operators $A_{j,b}$ on H_b, choose integers $s, b_0 \ge 0$ and integral classes $w_{j,k} \in H^b_{\text{ét}}(X_{\bar{K}}, \mathbf{Z}_\ell)$ lifting $\ell^s A_{j,b}(v_k) \in H_b$ such that integrally in $H^b_{\text{ét}}(W_{\bar{K}}, \mathbf{Z}_\ell)$,

$$\ell^{b_0} \pi^*(w_{j,k}) = \ell^{s+b_0} \sum_{\alpha=1}^{d_j} (r_j \circ z_{j,\alpha})^*(v_k)$$

(holding after killing the finite ℓ-power torsion kernel of $H^b_{\text{ét}}(W_{\bar{K}}, \mathbf{Z}_\ell) \to H^b_{\text{ét}}(W_{\bar{K}}, \mathbf{Q}_\ell)$). Naturality of the étale comparison map e transfers this integral identity into $H^b_{\text{test}}(P_W, \ell^n)$ at every finite level n (the smooth crystalline comparison β is never applied to the possibly singular cover W; on W we use only the general presheaf maps e and π^* together with the $|G|$-averaging descent inside $H^\bullet_{\text{test}}$ below, applying β_X and $\beta_{X_{k_K}}$ exclusively to the smooth models X and X_{k_K}), where factoring out the j-independent scalar $q^{(u+r)(a+N_0)}t^{u+r}$ and applying the signed q^{N_0}-shifted branch relation on $H^b_{\text{test}}(P_Q, \ell^n)$ annihilates the π^*-pullback of the weighted sum on X. Because $A \in \mathrm{Test}_\ell$ is π-adically complete and satisfies both Axioms (i) and (ii), lifting both axioms across tilting to $A^\sharp = W(A)/(\xi)$ shows as in Step 3 above that $A^\sharp$ is integrally closed in $A^\sharp[1/\ell]$ and that every surjective étale cover of $\operatorname{Spec} A^\sharp[1/\ell]$ splits globally. Given $y \in X(A^\sharp)$, pulling back the generically finite étale first projections p_j produces a global splitting section over $A^\sharp[1/\ell]$, which extends uniquely to $A^\sharp$ by finiteness of p_j and integral closure of $A^\sharp$ in $A^\sharp[1/\ell]$; hence $W(A^\sharp) \twoheadrightarrow X(A^\sharp)$ is globally surjective. Likewise, for any two lifts $y_1, y_2 \in W(A^\sharp)$ over a common point of $X(A^\sharp)$, comparing their generic branches decomposes $\operatorname{Spec} A^\sharp[1/\ell]$ into clopen loci indexed by permutations $\sigma \in G$; the corresponding pairwise orthogonal permutation idempotents $e_\sigma \in A^\sharp[1/\ell]$ satisfy $e_\sigma^2 - e_\sigma = 0$, hence belong to $A^\sharp$ by integral closure and lift uniquely to a complete orthogonal idempotent partition of A, giving local G-transitivity on idempotent covers. We show that $\pi^* x = 0$ in $H^j_{\text{test}}(P_W, F)$ implies $|G|x = 0$ in $H^j_{\text{test}}(P_X, F)$ for every additive functor $F : \mathrm{Test}_\ell \to \mathbf{Ab}$ (in particular $F(A) = A_{\mathrm{cris}}(A)/\ell^n$) and every $j \ge 0$ without any smoothness assumption or trace map: in degree 0, global surjectivity of $P_W(A) \twoheadrightarrow P_X(A)$ makes $\pi^* : \mathrm{Nat}(P_X, F) \to$

$\mathrm{Nat}(P_W, F)$ injective. For $j > 0$, use the same embedding of F into an additive functor K injective in the full functor category with exact sequence $0 \to F \to K \to Q \to 0$ (Q additive and $H^j_{\mathrm{test}}(-, K) = 0$). At $j = 1$, write $x = \delta(x')$ with $x' \in H^0_{\mathrm{test}}(P_X, Q)$; then $\delta(\pi^* x') = \pi^* x = 0$ lifts $\pi^* x'$ to a natural 0-section $c \in \mathrm{Nat}(P_W, K)$. Given $y \in P_X(A)$ and any lift $w \in P_W(A)$ over y, local G-transitivity of $P_W \to P_X$ on a finite idempotent cover together with naturality of c and additivity of K makes the G-sum $\sum_{g \in G} c_A(g \cdot w)$ independent of the chosen lift w, defining a natural section $\bar{c} \in \mathrm{Nat}(P_X, K)$ that projects to $|G|x' \in H^0_{\mathrm{test}}(P_X, Q)$ and hence yields $|G|x = \delta(|G|x') = 0$. For $j > 1$, $H^j_{\mathrm{test}}(-, K) = H^{j-1}_{\mathrm{test}}(-, K) = 0$ makes the connecting map $\delta : H^{j-1}_{\mathrm{test}}(-, Q) \xrightarrow{\sim} H^j_{\mathrm{test}}(-, F)$ an isomorphism, so induction on j gives $|G|x = 0$ in every degree. Combining these finite-level identities yields an integral inverse-limit relation killed by $|G|$ times powers of ℓ and t, whose nonzero integer multiplier $|G|\ell^{a_0}$ and power of t cancel after passing to the rational comparison pairing below.

To detect this relation on the rational tensor, use the local comparison pairing as follows. The cokernel of the intermediate comparison map is killed by a power of t, so lift the scaled integral classes and divide the resulting period values by that power. Two lifts differ by a class killed by a further power of t. The period pairing of their difference vanishes since t is nonzero in the field of period values. This construction gives an additive pairing, extends to the rational lattice, and preserves cup products by the cup compatibility of the two comparison maps. A relation with a power of t in the integral comparison module therefore gives zero after applying the pairing in every coordinate.

On a model with connected smooth generic fibre of relative dimension d, the top-degree pairing is nonzero: compare the first Chern classes and take their d-fold cup power. The top-degree comparison gives a de Rham functional whose period evaluation on this class is nonzero. In degrees above $2d$ the rational cohomology is zero. In the other degrees, use the complementary-degree cup tests with values in scalar multiples of top-degree generators. Some generator has nonzero period value at the chosen functional because the generators span. The Galois action carries any one of them to a nonzero scalar multiple of any other, so equivariance gives nonzero value on all of them at the same functional. Expand the cups with the relation using bases in degree b and the complementary degree $2d - b$. Each term vanishes by the relation evaluated at the corresponding coordinate functional, so cancel the nonzero top-degree value. The resulting vanishing tests span the rational dual. For a general model, use the finite family of jointly injective pullbacks to auxiliary models with connected generic fibre. A linear left inverse to their product expresses each functional as a sum of component tests. Hence every rational coordinate of the period tensor is zero, by the injection of coefficients into the field of period values. This proves the zero relation with exponent $\nu_b = N_0 + N_1$.

Fix the compatible local period datum, prolongation and inducing bridge. The crystalline period ring of this datum is identified with the comparison ring by a coefficient-preserving isomorphism that commutes with Galois and Frobenius. Transport Claim 1 by this isomorphism. The factor $1 \otimes \varphi^{f\nu_b}$ is common to every term and is injective, since the comparison Frobenius is injective and the tensor product is over a field. Cancel it to obtain the unshifted relation on invariant tensors in each degree. The finite-dimensional crystalline ambient representation V_{amb} is the direct sum of the degrees needed to lift a basis of the upper piece. Its degreewise operators T_j satisfy this relation on the invariant tensors of the sum. Descent by the injective pullback provides an embedding of the untwisted piece before semisimplification into this sum after coefficient extension. On this embedding the T_j act by ϑ_j, obtained by dividing the upper eigenvalues by the prescribed nonzero correspondence multiplicities.

Claim 2. The semisimplified local piece has a finite crystalline coefficient model on whose semistable period invariants a power of the scalar Hecke polynomial in the f-fold Frobenius is zero.

Choose a basis of V_{amb} over $\mathbf{Q}_\ell$. Let M be the matrix of the embedding of the local piece before semisimplification and let L be the extension of the bridge's coefficient field generated by the entries of M and by $\vartheta_0, \ldots, \vartheta_a$. It is finite. Choose over L a left inverse N of M. If A_g is the ambient Galois matrix, the matrix on the embedded piece is NA_gM. In a basis adapted to the global composition flag, its associated graded is obtained by retaining the diagonal blocks of these matrices. Thus both representations and the embedding are defined over L, with the same coefficient values ϑ_j. The flag used here is the restriction of the global flag.

The exactness of the crystalline period functor on crystalline representations shows that the embedded piece and each successive flag quotient are crystalline. The crystalline-to-semistable comparison takes the unshifted crystalline annihilator to the same annihilator on semistable period invariants of V_{amb}. After extension to L and restriction to the embedded piece, substitute the scalars ϑ_j for T_j. The resulting polynomial

is

$$Q(X) = \sum_{j=0}^{a} (-1)^j q^{\binom{j}{2}} \vartheta_j X^{a-j}.$$

Its annihilator passes through the flag by exact comparison on semistable period invariants, giving nilpotence on the direct sum of the successive quotients, as asserted.

Claim 3. Under the compatible bridge and the coefficient embedding into $\mathbf{C}$, this nilpotence controls the characteristic polynomial of the specified Weil–Deligne parameter at an element of degree -1.

Let U be the finite model from Claim 2 and take g to be the inverse of the chosen arithmetic Frobenius lift, so that $\deg(g) = -1$. On $D_{\mathrm{st}}(U)$ the Weil action of g agrees with φ^f. Indeed the degree gives the exponent f in the Weil action, and the remaining Galois operator fixes the semistable invariants. These invariants generate $D_{\mathrm{pst}}(U)$ by multiplication by the unramified coefficient algebra, since the model is semistable. The Weil action commutes with multiplication by this algebra. Hence the same power of Q also kills the full space of potentially semistable periods. The coefficient transport to the complex Weil space intertwines the actions and sends Q to $\iota(Q)$. Its images generate the complex Weil space, which proves that $\iota(Q)$ evaluated at this action is nilpotent.

The parameter defined by the bridge equals the parameter of this finite model. To compare with any other model chosen in the construction, enlarge the two coefficient fields to their compositum. Both sets of matrices then agree because they have the same image in $\bar{\mathbf{Q}}_\ell$, and the Weil parameter is invariant under this coefficient enlargement. Frobenius semisimplification preserves the characteristic polynomial at g, so the characteristic polynomial of the parameter is the one computed by the transported operator. By the root identity $\iota(Q) = \iota(\vartheta_0) \prod_{\alpha \in R} (X - \alpha)$ and $\vartheta_0 \neq 0$. Acting on a nonzero eigenvector shows that every root of the characteristic polynomial belongs to R. Counting multiplicities over $\mathbf{C}$ gives divisibility by $(\prod_{\alpha \in R} (X - \alpha))^m$, with m equal to the dimension of the complex Weil space. The complete proof is `UpperTheory.Gal.Int.Geo.steps_holds`.

Lean: `UpperTheory.Gal.Int.Geo.steps_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.GeometricRealizationOfWeightOne`, lines 20–20)

□

Proof of 39.30

Proof. Fix a modulus b that is a unit in T and an integer twist i. Apply the local-to-global criterion for the finitary sheaf $\mathbf{Z}/b(i)$ to the B-algebra map $B \to B \otimes_T K$. It suffices to check the resulting pullback on a strictly henselian B-algebra A that is ind-étale over B. Composition with the étale map $T[t] \to B$ makes A ind-étale over $T[t]$, and base change identifies the target ring with $K \otimes_T A$.

In negative degrees the two cohomology groups vanish. For degree zero, A is a domain and $T \to A$ is injective, so $K \otimes_T A$ is a nonzero localization of a domain. A primitive b-th root of unity comes from T, since T is absolutely integrally closed and b is invertible. It identifies the twist with the constant rank-one coefficient sheaf over both connected spectra. Restriction is therefore an isomorphism in degree zero.

In positive degrees the source is acyclic because A is strictly henselian local, so it remains to prove $H^n(\mathrm{Spec}(K \otimes_T A), i) = 0$ for $n \geq 1$. In degree $n = 1$, the primitive b-th root of unity in $T \to K \otimes_T A$ identifies $H^1(\mathrm{Spec}(K \otimes_T A), i)$ with $H^1(\mathrm{Spec}(K \otimes_T A), 1)$ independently of the twist i. Realize A as a $T[t]$-algebra quotient $\pi : O \twoheadrightarrow A$ of the strict henselization O of a prime localization of $T[t]$ (a directed union of localizations of étale $T[t]$-algebras), so $L = K \otimes_T O$ surjects onto $R = K \otimes_T A$, where O is an integrally closed domain and $\dim L \leq 1$ (since L is ind-étale over the principal ideal domain $K[t]$). If $\ker \pi \neq 0$, every prime ideal of R pulls back to a nonzero, hence maximal, prime of L, whose residue field is algebraic over the algebraically closed field $K = \mathrm{Frac}(T)$ and therefore equals K; thus R is zero-dimensional, giving $\mathrm{Pic}(R) = 0$ immediately, while for every $u \in R^\times$ the polynomial $x^b - u$ has a simple root modulo every maximal ideal of R (since $b \in R^\times$) that lifts to $R^\times$ by Hensel's lemma. If $\ker \pi = 0$, then $R = L \simeq S^{-1}O$ where $S = T \setminus \{0\}$ and $O = T[t]^{\mathrm{sh}}_P$ for a prime ideal $P \subset T[t]$. We prove both the unit factorization $(S^{-1}O)^\times = K^\times O^\times$ and the triviality of every finite étale $\mathbf{Z}/b\mathbf{Z}$-torsor over $S^{-1}O$ by a uniform three-step reduction ladder. First, to reduce an arbitrary prime $P \subset T[t]$ to the closed fibre, let $\mathfrak{p} = P \cap T$ and replace T by $T' = T_{\mathfrak{p}}$: then T' is again an absolutely integrally closed local domain with the same fraction field K and $b \in (T')^\times$, and $P' = PT'[t]$ lies in the closed fibre $(P' \cap T' = \mathfrak{m}_{T'})$ with $T[t]^{\mathrm{sh}}_P \simeq T'[t]^{\mathrm{sh}}_{P'}$. Second, to reduce the generic point $P_\eta = \mathfrak{m}_T T[t]$ of the closed fibre (with strict henselization O_η) to closed points of the closed fibre, spread

any finite étale $\mathbf{Z}/b\mathbf{Z}$-torsor D over $K \otimes_T O_\eta$ (resp. any unit relation $fg = s \in T \setminus \{0\}$ in O_η) to an étale $T[t]$-algebra $E \to O_\eta$ with prime $Q = \mathfrak{m}_{O_\eta} \cap E$ lying over P_η; choose a maximal ideal $y \supseteq Q$ of E lying over a closed point of the closed fibre, split the torsor (resp. factor $f = aw$ with $a \in T \setminus \{0\}$ and $w \in (E_y^{\mathrm{sh}})^\times$) over E_y^{sh} by the closed-point case (which reduces by translating t via a lift in T to the origin $P_0 = (\mathfrak{m}_T, t)$), and push the splitting section (resp. unit factorization) forward along a specialization homomorphism $E_y^{\mathrm{sh}} \to O_\eta$ of E-algebras. To construct it, note that E_y^{sh} is flat over E with maximal ideal over $y \supseteq Q$, so going down gives a prime $Q_2 \subset E_y^{\mathrm{sh}}$ over Q. Since E_y^{sh} is a directed union of localizations of étale E-algebras, the residue field $\kappa(Q_2)$ is separable over $\kappa(Q)$, so it embeds over $\kappa(Q)$ into the separably closed residue field of O_η, whose maximal ideal contracts to Q. Because O_η is henselian, the composite $E_y^{\mathrm{sh}} \to \kappa(Q_2) \to \kappa(O_\eta)$ lifts to an E-algebra map $E_y^{\mathrm{sh}} \to O_\eta$. Third, at the closed-fibre origin $O_0 = T[t]^{\mathrm{sh}}_{(\mathfrak{m}_T, t)}$, spread a finite étale $\mathbf{Z}/b\mathbf{Z}$-torsor D over $S^{-1}O_0$ to a finite étale $\mathbf{Z}/b\mathbf{Z}$-torsor D_A over $K_A \otimes_A E_{A,Q_A}$ for an essentially-of-finite-type Noetherian normal local subring $A \subset T$ (containing b^{-1}, with fraction field K_A) and an étale $A[t]$-algebra E_A at a prime Q_A over $(\mathfrak{m}_A, t)$ carrying the origin retraction $\rho : E_{A,Q_A} \twoheadrightarrow A$ ($\ker \rho = (t)$). Because T is absolutely integrally closed, every finite normal extension of A embeds in T. Perform a three-stage enlargement of (A, E_A, Q_A) inside T to a Noetherian normal local subring $A' \subset T$ and étale $A'[t]$-algebra localization $R = E_{A',Q_{A'}}$ with integral closure C of R in the base-changed $G = \mathbf{Z}/b\mathbf{Z}$-torsor $D_{A'}$: first ensure an origin retraction $\rho' : R \twoheadrightarrow A'$ with $\ker \rho' = (t)$; second extend A' inside T so the generic-fibre torsor admits a point along $t = 0$, giving a homomorphism $C \to \mathrm{Frac}(A')$ over ρ' which lands in $A' \simeq R/(t)$ because C is integral over R and A' is normal (yielding an R-algebra section $g : C \to R/(t)$); third adjoin b-th roots inside T of uniformizers at the finitely many height-one primes of A' over which C ramifies (using $b \in (A')^\times$). Since R is flat and smooth of relative dimension one over A', for every prime $\mathfrak{p} \subset A'$ of height at most one the base localization $A'_{\mathfrak{p}}$ is a field or discrete valuation ring, so every localization R_P at a prime $P \subset R$ contracting to $\mathfrak{p}$ is regular of dimension at most two; clearing tame vertical ramification over height-one primes of A' and applying purity on these regular rings makes $C_P = R_P \otimes_R C$ étale over R_P at every prime $P \subset R$ with $\mathrm{ht}(P \cap A') \leq 1$ (including the origin prime (t) and all codimension-two primes $(\mathfrak{p}, t)$ over height-one primes $\mathfrak{p} \subset A'$). For every $n \geq 1$ and every prime $\mathfrak{p} \subset A'$ of height at most one, formal étaleness of C at $P = (\rho')^{-1}(\mathfrak{p})$ lifts $g : C \to R/(t)$ uniquely to an R-algebra map $\varphi_{\mathfrak{p}} : C \to (R/(t^n))_{\mathfrak{p}}$ across the nilpotent ideal $(t)/(t^n)$; because $R/(t^n)$ inherits Hartogs's intersection property $R/(t^n) = \bigcap_{\mathrm{ht}\,\mathfrak{p} \leq 1} (R/(t^n))_{\mathfrak{p}}$ over the normal Noetherian domain A', the unique generic lift $\varphi_{(0)}$ takes values in $R/(t^n)$, yielding compatible lifts $f_n : C \to R/(t^n)$ that assemble into an R-algebra section $\sigma : C \to O$ into the t-adic completion $O = \widehat{R}^{(t)}$. Since $t \in \mathfrak{m}_R$ in the Noetherian local ring R, $R \to O$ is faithfully flat. Form the Galois comparison homomorphism $\Phi : O \otimes_R C \to \prod_{g \in G} O$, $o \otimes c \mapsto (o\,\sigma(g(c)))_{g \in G}$. Clearing denominators in the generic-fibre G-torsor relation supplies a nonzero $a \in A'$ and elements $x_i, y_i \in C$ with $\sum_i x_i g(y_i) = a\delta_{g,1}$, which shows that C is a finite R-module (via the trace map $C \to R$) and that Φ is injective (by A'-torsion-freeness of C and flatness of O). For each $h \in G$, the element $z_h = \sum_i \sigma(x_i) \otimes h^{-1}(y_i) \in O \otimes_R C$ satisfies $\Phi(z_h) = ae_h$, since its g-coordinate is $\sigma\big(\sum_i x_i\,(gh^{-1})(y_i)\big) = a\delta_{g,h}$; at every prime $Q \subset O$ with $\mathrm{ht}(Q \cap A') \leq 1$, $O_Q \otimes_R C$ is finite étale of rank $|G|$ over the local ring O_Q. The $|G|$ sections $c \mapsto \sigma(g(c))$ of $O_Q \otimes_R C$ over O_Q are pairwise distinct, since $\Phi(z_g) = ae_g$ and a is a non-zerodivisor. The locus where two sections of a finite étale algebra agree is open and closed, and $\mathrm{Spec}\, O_Q$ is connected, so these sections are pairwise disjoint and the localization Φ_Q of Φ is an isomorphism. Hence $\Phi_Q(z_h) = ae_h$ gives $z_h \in a(O_Q \otimes_R C)$. Since C, and hence its flat base change $O \otimes_R C$, satisfies Hartogs's divisibility property over the normal Noetherian domain A' (an element divisible by a after localizing at every prime $\mathfrak{p} \subset A'$ of height at most one is globally divisible by a), we obtain $z_h = am_h$ in $O \otimes_R C$; cancelling the regular element a gives $\Phi(m_h) = e_h$, so $\Phi : O \otimes_R C \xrightarrow{\sim} \prod_{g \in G} O$ is an isomorphism. Thus $O \otimes_R C$ is étale over O, and faithfully flat descent along $R \to O$ proves that C is finite étale over the entire local ring $R = E_{A',Q_{A'}}$ (even when A' is singular) and hence splits over the strictly henselian local ring O_0. Similarly, for a unit relation $fg = s \in T \setminus \{0\}$ in O_0, spread to a flat normal essentially-of-finite-type local stage $A \to B = E_{A,Q_A}$ and enlarge it to a flat normal Noetherian local stage $A' \to B'$ mapping to $T \to O_0$, where A' is a henselian Noetherian normal local domain with separably closed residue field and B' is the localization at a prime over the origin of an étale $A'[t]$-algebra. For every prime $\mathfrak{p} \subset A'$, the ideal $\mathfrak{p}B'$ is prime: choose a normal local domain S integral over $A'/\mathfrak{p}$; the origin of $S[t]$ is the unique prime over the origin of $A'[t]$, so $S[t] \otimes_{A'[t]} B'$ is local, hence a domain since it is étale over the normal domain $S[t]$, and by flatness the kernel of $B' \to S[t] \otimes_{A'[t]} B'$ is $\mathfrak{p}B'$. Going down for the flat map $A' \to B'$ then shows that every height-one prime $\mathfrak{Q} \subset B'$ containing s equals $(\mathfrak{Q} \cap A')B'$ with $\mathfrak{Q} \cap A'$ of height one. Since $g \mid s$, every height-

one prime of B' containing g contains s; each height-one prime of B' containing s has the form $\mathfrak{Q} = \mathfrak{p}B'$, and $v_{\mathfrak{Q}}$ restricts to $v_{\mathfrak{p}}$ on A'. The ideal of those $y \in A'$ with $v_{\mathfrak{p}'}(y) \geq v_{\mathfrak{Q}'}(g)$ at all these primes $\mathfrak{Q}' = \mathfrak{p}'B'$ lies in $gB' \cap A'$, since B' is normal, and its localization at $\mathfrak{p}$ is $\mathfrak{p}^{v_{\mathfrak{Q}}(g)}A'_{\mathfrak{p}}$; so $gB' \cap A'$ contains an element of $v_{\mathfrak{p}}$-valuation $v_{\mathfrak{Q}}(g)$. If $x(gB' \cap A') \subseteq sA'$, this element gives $v_{\mathfrak{Q}}(x) \geq v_{\mathfrak{Q}}(s) - v_{\mathfrak{Q}}(g) = v_{\mathfrak{Q}}(f)$, while $v(f) = 0$ at the height-one primes not containing s. Since B' is normal, this places the colon ideal $(sA' : (gB' \cap A'))$ inside $fB' \cap A'$. By flatness, $(sA' : (gB' \cap A'))B' = (sB' : (gB' \cap A')B')$, and this ideal contains f because $f(gB' \cap A')B' \subseteq fgB' = sB'$; hence $f \in (fB' \cap A')B'$. Because B' is a local domain, if f were not a unit multiple of any element of $fB' \cap A'$ we would have $f \in (fB' \cap A')B' \subseteq f\mathfrak{m}_{B'}$, contradicting $1 \notin \mathfrak{m}_{B'}$; hence $f = aw$ with $a \in A' \setminus \{0\}$ and $w \in (B')^{\times}$, yielding $a \in T \setminus \{0\}$ and $w \in O_0^{\times}$. From $(S^{-1}O)^{\times} = K^{\times}O^{\times}$, every unit of L has a b-th root (elements of T have b-th roots in T as monic polynomials split in T, and elements of $O^{\times}$ have b-th roots in $O^{\times}$ as O is strictly henselian with $b \in O^{\times}$), while splitting every $\mathbf{Z}/b\mathbf{Z}$-torsor $\operatorname{Spec}\bigl(\bigoplus_{k=0}^{b-1} M^{\otimes k}\bigr)$ yields $\operatorname{Pic}(L)[b] = 0$. In both cases $(K \otimes_T A)^{\times}$ is b-divisible and $\operatorname{Pic}(K \otimes_T A)[b] = 0$; the Kummer exact sequence $1 \to \mu_b \to \mathbb{G}_m \xrightarrow{b} \mathbb{G}_m \to 1$ therefore gives $H^1(\operatorname{Spec}(K \otimes_T A), 1) = 0$ and hence $H^1(\operatorname{Spec}(K \otimes_T A), i) = 0$.

Lean: `GlobalTheory.LocAcycDegOne.tateTwistCohomology_genericFibre_henselLine_subsingleton` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.TateTwistCohomologyGenericFibreHenselLineSubsing`, lines 36–45)

In degrees $n \geq 2$, because T is an absolutely integrally closed domain, its fraction field $K = \operatorname{Frac}(T)$ is algebraically closed. Base change along $T[t] \to K[t]$ preserves ind-étaleness, expressing $K \otimes_T A \simeq K[t] \otimes_{T[t]} A$ as a filtered colimit $\varinjlim_j D_j$ of étale $K[t]$-algebras D_j. Each D_j is a finite-type smooth K-algebra of relative dimension one, hence a finite product of finite-type Dedekind K-algebras over the algebraically closed field K, so Theorem 39.29 yields $H^n(\operatorname{Spec} D_j, i) = 0$ for every j and every $n \geq 2$. By filtered-colimit continuity of étale cohomology with invertible torsion coefficients, every class in $H^n(\operatorname{Spec}(K \otimes_T A), i)$ pulls back from some $H^n(\operatorname{Spec} D_j, i) = 0$ and hence vanishes. This proves the required comparison in every degree and twist.

Lean: `GlobalTheory.EtC.GenFibreLine.genericFibre_subsingleton` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.SubsingletonEtaleCohomologyGenericFibreHenselian`, lines 39–69)

□

Proof of 39.33

Proof. For the trivial bundle choose bounded-below injective complex models for the shifted Tate twists and the projective-space cohomology. Represent the sum of pullbacks followed by cup products with $c_1(L)^j$ by a map of these complexes. By the scheme projective bundle formula, the map is a quasi-isomorphism on evaluation at each representable on which b is invertible [214]. The mapping cone is thus locally exact on the support of any discrete presheaf satisfying the invertibility condition. The injective-complex comparison gives zero hypercohomology for the cone on that presheaf. In the classifying-space model, a bundle map out of a discrete presheaf factors through the unique vertex of the general linear classifying space and gives the trivial bundle.

Apply the discrete formula in each simplicial degree. Each simplex has vertices, so the invertibility hypothesis applies in every degree. Choose the cocycle representatives of the Chern powers on the space classifying the tautological line, and pull them back to make the maps compatible with the simplicial structure. Comparison by simplicial degree gives the quasi-isomorphism on the bounded-below total complexes. □

Proof of 39.36

Proof. Take $Z_0 = \varnothing$. Translate the least jump to zero. For two disjoint nonempty subsets, the largest jump in their union exceeds a times every smaller jump. Its subset has the larger mean. Thus the stronger gaps imply lacunarity, which is preserved by reflection. Inverse transpose relates the two split coordinates. Changing to a conjugate auxiliary place therefore preserves sphericity and the type condition. It reflects the jumps up to translation. Choose coordinates that meet the restrictions to $\mathbf{Q}(\sqrt{-d})$ in 39.9. If a target place above ℓ is specified, use its side.

Let ω be the Hecke character obtained from the centre of U' in split coordinates. The ℓ-adic realization η of $\omega|\cdot|^{-\binom{a}{2}}$ has transported value

$$\iota(\eta(s^{-1})) = q_w^{\binom{a}{2}} \prod_{\beta \in B} \beta = \prod_{\alpha \in A_a(w,B)} \alpha \tag{A.584}$$

at a usable spherical $w \nmid \ell$ and an arithmetic Frobenius lift s. This uses reciprocity with inverse arithmetic Frobenius, as in 40.5. At a spherical place above ℓ the same character is crystalline, with the right-hand side of (A.584) as its transported Weil value at degree -1.

Apply 39.37 to a geometric witness of 39.9 with determinant character η. The cofinite Frobenius tests give a dense set by 36.24. One can take places of absolute residue degree one, unramified over $\mathbf{Q}$, outside the rational primes below the finite exclusions and the primes dividing $2d\ell$. Their completions are $\mathbf{Q}_p$ and contain the image of $\sqrt{-d}$, so they are usable. On the separable locus of r, a monic annihilator of degree a is its characteristic polynomial. Take V to be the inverse of this locus and X the exceptional set of the witness. For any other witness of 39.9, rank extraction with the same η gives the same traces on the intersection of the two open separable loci and the common cofinite Frobenius tests. This intersection is dense by the same degree-one test. Continuity and semisimplicity identify the extracted representations. At a specified place away from ℓ, the root polynomial annihilates the Frobenius, so the characteristic polynomial of the Frobenius divides a power of it; apply 39.38 to this characteristic polynomial and (A.584), using the last factor of (39.2) for uniqueness.

Above ℓ, crystallinity passes to the subrepresentation of each witness. For the jump witness, $\rho \hookrightarrow r^{\oplus m}$ forces every prescribed jump to occur in r. The a prescribed jumps are distinct and exhaust its rank. For the root witness, 39.27 gives the support condition on the crystalline module at a degree--1 Weil element. The determinant comparison and (A.584) give the product there. Applying 39.38 to the characteristic polynomial there, with the uniqueness factor of (39.2), identifies the polynomial, and unramifiedness of the crystalline Weil action gives the polynomial at every Weil element.

Lean: `UpperTheory.Gal.Int.Div.phiSeed_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.ZetaCharacterDivisibility.PPhiSeed`, lines 20–61)

□

Proof of 39.37

Proof. Put $H = \overline{\rho(G_E)}^{\mathrm{Zar}}$ in the general linear group and let $\mathfrak{h}$ be its Lie algebra. Dependence of $1, M, \dots, M^a$ is closed, so every $M \in H$ has minimal polynomial of degree at most a. The same bound holds on $\mathfrak{h}$. Use a formal arc $M(t)$ through the identity with tangent D, apply dependence to $(M(t)-1)/t$, and specialize at zero. Such an arc exists by smoothness in characteristic zero. Sen's theorem puts the Hodge–Tate operator, with eigenvalues the actual jumps in the chosen convention, in $\mathfrak{h} \otimes \mathbf{C}_\ell$ [208]. If $e_1, \dots, e_a$ are the selected jumps, the equations $\det(e_i I - D) = 0$ on $\mathfrak{h}$ therefore have a common point D over $\bar{\mathbf{Q}}_\ell$. The degree bound makes D semisimple with exactly these eigenvalues. Write p_i for its spectral projectors.

A member D' of $\mathfrak{h}$ commuting with D has scalar blocks on the p_i-spaces. Indeed $D + tD'$ has pairwise disjoint block spectra for generic t, and the sum of the block minimal polynomial degrees is at most a. Each nonzero adjoint eigenvalue $e_i - e_j$ determines its ordered pair, since equality of two different differences would give disjoint subsets with equal means. The spectral projections of $\operatorname{ad} D$ thus put each off-diagonal block of $\mathfrak{h}$ in $\mathfrak{h}$. Products of composable off-diagonal blocks through three distinct indices are brackets, and an opposite product is the diagonal block of a bracket, hence scalar. The algebra A generated by $\mathfrak{h}$ and the p_i consequently has $p_i A p_i = \bar{\mathbf{Q}}_\ell p_i$. Conjugation by H preserves A, since the p_i are polynomials in D. The radical of A annihilates a nonzero subspace on every irreducible summand of ρ, by nilpotence. This subspace is G_E-stable, so the radical acts by zero on every summand and is zero.

Fix $h \in H$. The primitive idempotents hp_jh^{-1} can be matched to the p_i in each matrix factor of A. Since $p_i A h p_i = p_i A (h p_i h^{-1}) h$, if $p_i A h p_i \neq 0$, then hp_ih^{-1} and p_i are in the same factor, and the matching may prescribe this pair. A unit implementing the matching can be chosen of determinant one in each factor by rescaling the matched basis vectors. It is then a product u of transvections. Each transvection belongs to H°, since its square-zero off-diagonal block belongs to $\mathfrak{h}$. With the matching oriented so that hu permutes the projectors, let T be the identity component of the diagonal subgroup of H in the projectors. Its Lie algebra contains D.

For $t = \sum_i t_i p_i \in T$, the eigenvalues of hut on a cycle C of length b are the b-th roots of $\lambda \prod_{i \in C} t_i$, for

the eigenvalues λ of the return map of hu on one block of C. A collision with a cycle C' of length b' imposes

$$\left(\lambda \prod_{i \in C} t_i\right)^{b'} = \left(\lambda' \prod_{i \in C'} t_i\right)^{b}.$$

An identity on T would give $b' \sum_{i \in C} e_i = b \sum_{i \in C'} e_i$ by differentiation along D, contrary to lacunarity. Choose t outside these finitely many closed sets. Each cycle contributes at least its length to the number of distinct eigenvalues. The degree bound gives exactly a distinct eigenvalues and semisimplicity. In particular, if the matching fixes i, the block at i is scalar and $p_i h = c_i p_i (ut)^{-1} \in p_i A$ for some c_i.

The span $B = \sum_{h \in H} Ah$ is an algebra normalized by H. The linear span of H contains $\mathfrak{h}$ and the polynomials in D, so it equals B. For a matrix unit $e_{ij} \in p_i A p_j$ with $i \neq j$, apply the fixed-block conclusion to $(1 + x e_{ij})h$ for $x \in \bar{\mathbf{Q}}_\ell$. Its i-th diagonal block is scalar, by the fixed-block conclusion if it is nonzero. Subtract two such blocks to get $e_{ij} h p_i \in \bar{\mathbf{Q}}_\ell p_i$. Taking $x = 0$ gives the diagonal term. These matrix units span the corners of A, so $p_i B p_i = \bar{\mathbf{Q}}_\ell p_i$. The radical argument for A applies to B. Its a projectors are primitive and sum to one. One standard module of each matrix factor gives a representation of dimension a, realized inside the original module, which is a sum of copies of those standard modules. This gives the two embeddings. Semisimplicity follows since $\rho(G_E)$ spans B.

In a fixed inclusion I and left inverse L, the representation is $r(g) = L\rho(g)I$ and is continuous. Every finite-index subgroup has Zariski closure containing H°. The calculation of hut gives a matrix with nonzero discriminant on the extracted module in the Zariski closure of every open coset. The separable locus is thus dense and open. On its intersection with the original dense set, Q_g is the characteristic polynomial of $r(g)$. Comparing constant terms and using continuity gives $\det r = \eta$. □

Proof of 39.38

Proof. Since P divides a power of the root polynomial of S, its roots, counted with multiplicity, form a size-a multiset supported in $\operatorname{supp} S$, and their product is $\prod_{\alpha \in S} \alpha$. By uniqueness this multiset is S, so P is the root polynomial of S. □

Proof of 39.39

Proof. Write $O = \mathcal{O}_{E_{w_0}}$. Smoothness gives a subgroup $J_0 = 1 + \ell^b M_a(O)$ fixing f, with $b \geq 2$. Put $J_n = 1 + \ell^{b+n} M_a(O)$, $H_n = J_0/J_n$, and $C_n = K J_n$. Successive quotients are additive matrix groups in characteristic ℓ, so H_n is an ℓ-group. The matrix logarithm and exponential on J_0 and $\ell^b M_a(O)$ are inverse and identify the powers of any one element with the integer multiples of its logarithm. Hence J_0 is torsion free.

Fix $w \in D$ and put

$$A(z) = \prod_{\pi \in \mathfrak{S}_a} \Phi_w(z_{\pi(1)}, \dots, z_{\pi(a)}), \qquad A^*(z) = \overline{A(\bar{z}^{-1})}, \qquad p_k = (1 - cAA^*)^k.$$

Here inversion is coordinatewise and $c = (1 + \max_{|z_i|=1} |A(z)|^2)^{-1}$. The symmetric Laurent polynomial p_k equals one wherever $A = 0$, including off the unit torus. On the torus, $0 \leq 1 - c|A|^2 \leq 1$ and A has zero set of Haar measure zero. For the induction on the number of variables, apply the one-variable root bound off the common zero set of the coefficients. Dominated convergence gives $\int |p_k|^2 \, dz \to 0$ for Haar probability measure dz.

For $\mathrm{GL}_a(E_w)$ with hyperspecial volume one, Macdonald's spherical Plancherel theorem gives the normalized Satake transform and the probability density proportional to

$$\prod_{i<j} \left| \frac{1 - z_i/z_j}{1 - q_w^{-1} z_i/z_j} \right|^2$$

on the unit torus [81, Ch. V, §5.1]. This is its split type-A case. If h is the operator with transform p_k, write it as a sum over representatives g of right hyperspecial cosets, with coefficients h_g. The density has a finite

maximum M_w since $q_w > 1$. Plancherel gives

$$\sum_g |h_g|^2 \le M_w \int |p_k|^2 \, dz. \tag{A.585}$$

The operator is the identity on the C_n-fixed forms in every exceptional summand at w.

Let h_0 be the number of double cosets for $G(F)\backslash G(\mathbf{A}_F)/(G_\infty C_0)$ and choose representatives s. This quotient is finite by compactness of the definite adelic quotient and openness of the level. At level C_n the representatives sj, for j lifted from H_n, cover the double cosets. For fixed h and sufficiently large n, the translates sjg of each such representative belong to distinct double cosets as g ranges over the support. A collision $\gamma(sjg)(xc) = sjg'$ with $c \in C_n$ and $x \in G_\infty$ gives a rational point of the form

$$\gamma = sg'jc^{-1}j^{-1}g^{-1}x^{-1}s^{-1}, \qquad c \in C_n, \quad x \in G_\infty,$$

by commutation of the distinct local factors (since $x \in G_\infty$ commutes with finite adelic elements and g' commutes with j). As $jC_nj^{-1} \subset C_0$, the possible γ lie in a fixed compact set and are finite by discreteness. The w_0 component of $s^{-1}\gamma s$ belongs to J_n. Shrinking J_n excludes each nonidentity point in the finite list, since a rational matrix equal to the identity at one completion is the identity. When $\gamma = 1$, the two right hyperspecial cosets at w coincide.

Choose actual double-coset representatives from the cover and use a basis of V of dimension d_V to identify the weight forms with a subspace of their coordinate evaluations. The archimedean matrices in these evaluations have entries bounded by a constant M_V by compactness of G_∞. Each row of the Hecke matrix uses at most one translate for any given destination, by separation. Thus its squared Hilbert–Schmidt norm is at most $h_0|H_n|d_V^2 M_V^2 \sum_g |h_g|^2$. The dimension of a space on which a matrix acts as the identity is at most its squared Hilbert–Schmidt norm, by extending an orthonormal basis of that space. By (A.585), choose k at each w so that the bound is at most $|H_n|/[2(1 + |D|)]$, and then choose n for all the supports. The exceptional span at level C_n satisfies

$$\dim(\text{span of summands bad somewhere in } D) \le |H_n|/2. \tag{A.586}$$

Lean: `UpperTheory.Gal.Gen.small_pf` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SeparatingAlgebras.SmallPf`, lines 5–70)
Lean: `UpperTheory.Gal.Gen.countCore_pf` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SeparatingAlgebras.CountCore`, lines 86–131)
Lean: `UpperTheory.Gal.Gen.countGen_pf` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SeparatingAlgebras.CountGen`, lines 39–86)

Put $H = H_n$. On the double coset of s at level C_0, define $b(\gamma sxc)$ as the image of the w_0 component of c in H. A stabilizer at s is finite by discreteness and compactness, so its projection to the torsion-free J_0 is trivial. Thus b is well-defined and $b(gc) = b(g)\bar{c}$ for $c \in C_0$. For a function $v_0 : H \to \mathbf{C}$, the form $v_0(b(g))f$ belongs to level C_n. Let P be the space of v_0 for which it belongs to the good span. The automorphic decomposition at level C_n and (A.586) give $\dim P \ge |H|/2$. The good span and f are preserved by the right action of C_0, so P is stable under right translation by H.

Transport coefficients by ι^{-1} and use the supremum norm on functions on H. For an integral row $(u_h)_{h\in H}$ annihilating P, the matrix $B = (u_{hg^{-1}})_{h,g\in H}$ has rank at most $|H| - \dim P$, since its columns are translates of the annihilating row. It is a linear combination of regular permutation matrices with coefficients u_h. Over $\mathbf{F}_\ell$ the regular representation of H has a full invariant flag with trivial successive quotients. Indeed a central element of order ℓ has a nonzero stable fixed space on every nonzero representation. Apply induction on the group order to that fixed space and repeat on the successive quotients of the representation. Lift the flag basis over $\mathbf{Z}_\ell$. In that basis an integral correction ℓD_1 makes B upper triangular with constant diagonal $s_0 = \sum_h u_h$. Every surviving term in the determinant expansion uses at least $\dim P$ correction columns. Hence

$$|s_0|_\ell^{|H|} = |\det(B + \ell D_1)|_\ell \le \ell^{-\dim P}, \qquad |s_0|_\ell \le \ell^{-1/2}. \tag{A.587}$$

This gives $v_0 \in P$ at distance at most $\ell^{-1/2}$ from the constant function one. For this duality calculation, normalize a largest coordinate of a nonzero vector of P to one. Use the vector to eliminate that coordinate from the target and the other generators. The coordinate projection has norm at most one. An integral row annihilating the reduced subspace lifts to an integral annihilating row by subtracting its value on the pivot

vector in the pivot coordinate. Induction reduces to coordinate rows when the approximating subspace is zero, where (A.587) bounds the remaining coordinates of the target.

Lean: `UpperTheory.Gal.Gen.groupDet_pf` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SeparatingAlgebras.GroupDet`, lines 19–92)
Lean: `UpperTheory.Gal.Gen.alg_pf` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SeparatingAlgebras.AlgPf`, lines 4–52)
Lean: `UpperTheory.Gal.Gen.stepCore_pf` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SeparatingAlgebras.StepCore`, lines 9–56)

Subtract $v_0(b(g))f$. The remainder is a pointwise multiple of f with multiplier of norm at most $\ell^{-1/2}$, and is smooth at w_0 and K-fixed. Repeat on the remainder with its own starting subgroup J_0. After m steps the product of the multipliers has norm at most $\ell^{-m/2}$. Taking m with $\ell^{-m/2} \le \epsilon$ proves (39.3). □

Proof of 39.40

Proof. Choose supporting compact opens away from w_0 using a restricted tensor of local fixed vectors. Take the hyperspecial subgroups at spherical split places and the unramified levels at nonsplit places off T. At v' choose a supporting subgroup K_v inside the hyperspecial subgroup. Let K denote the level with that factor enlarged to the hyperspecial subgroup, and let $K' = K \cap \mathrm{pr}_{v'}^{-1}(K_v)$. Both levels have component 1 at w_0. The representation U has a K'-fixed vector.

Lean: `UpperTheory.Gal.Int.Cong.exists_level` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.CongruenceOddLevel.OpenCompactMap`, lines 77–155)

Put $D = Z \cup W$. At $z \in D$ let Φ_z be (39.2). It is nonzero since the two monomials in each factor have distinct exponent vectors. Add the spherical operator whose transform is $\prod_\pi \Phi_z(\pi x)$ in Satake variables x, for each $z \in D$. At $w \in W$ use the Laurent spherical algebra of 39.39 to add the operator with transform $\sum_i (q_w^{(a-1)/2} x_i)^n$ for every $n \in \mathbf{Z}$. Write each operator as a finite sum of right coset translations with its coefficients. Use the same finite-translation description for the given f_j.

For a complex scalar z put $|z|_* = |\iota^{-1} z|_\ell$. On an algebraic weight representation V use the coefficient lattice L_V in its polynomial realization with determinant twists, transported by ι. Let $\kappa(g)$ be the tuple of matrices at places above ℓ, transported to the complex weight factors using their inducing embeddings, and let g_∞ be the corresponding archimedean tuple. For the weight action, also denoted by V, put

$$b_g = V(g_\infty^{-1}\kappa(g))L_V, \qquad \|f\| = \sup_{g,\ v \in b_g} |f(v,g)|_*.$$

This is finite at a full supporting level. One can use finitely many double-coset representatives. Rational translation cancels in $g_\infty^{-1}\kappa(g)$, the archimedean factor cancels by equivariance, and the compact finite level changes L_V within a bounded multiple of itself. The lattices span V, so their evaluations separate forms.

For a finite translation h and its coefficient $t(h)$, $b_{gh} = V(g_\infty^{-1}\kappa(g))V(\kappa(h))L_V$. Thus the contraction criterion for each term is

$$t(h)V(\kappa(h))^{-1}L_V \subset L_V, \qquad \|t(h)f(\,\cdot\,,\,\cdot\, h)\| \le \|f\|. \tag{A.588}$$

The given f_j satisfy it since $w_j \nmid \ell$. For each added operator multiply by a power of ℓ clearing the denominators in its finitely many matrices. The denominators come from the polynomial action of the inverses and their determinant powers, in the weight factors at which the translations act. All added operators have trivial component below v'. Thus the scalars still work when the weight changes at the archimedean factor of an embedding inducing v'. Denote these scalars by $c_{w,n}$ for the power sums and c_z for the nonvanishing tests. They are chosen before J.

Lean: `UpperTheory.Gal.Gen.frameW_pf` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SeparatingAlgebras.FrameW`, lines 14–120)
Lean: `UpperTheory.Gal.Int.Cong.exists_powOp` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.CongruenceOddLevel.ExistsTestOp`, lines 64–89)
Lean: `UpperTheory.Gal.Int.Cong.exists_testOp` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.CongruenceOddLevel.ExistsTestOp`, lines 93–137)
Lean: `UpperTheory.Gal.Int.Cong.exists_scale` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.CongruenceOddLevel.ExistsTestOp`, lines 141–172)

Use $I = J \sqcup (W \times \mathbf{Z}) \sqcup D$ for all coordinates and write θ^{aug} for the tuple including the tests. Normalize a K'-fixed weight form of U to one at a nonzero lattice evaluation, and write $R \ge 1$ for its norm. Apply 39.39 with ϵ/R at level K', which is hyperspecial at D. Write its approximant as $\sum_i F_i$ in automorphic summands,

with tuples η_i and transported evaluations b_i at the normalization point. By (39.3) and (A.588), applying any ordered monomial in the operators to the difference preserves the bound ϵ. Integer linear combinations preserve it by the ultrametric inequality. Evaluation gives

$$\left| P(\theta^{\mathrm{aug}}) - \sum_i b_i P(\eta_i) \right|_\ell \le \epsilon \qquad (P \in \mathbf{Z}[X_j : j \in I]). \tag{A.589}$$

On each summand the value of a monomial depends on its exponents, since the action is scalar. The weight and type approximations use the same evaluation argument and operators.

Lean: `UpperTheory.Gal.Gen.toPolyX_pf` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SeparatingAlgebras.ToPolyX`, lines 11–90)

Fix a normalized intermediate eigenform f at level K' and an embedding τ inducing v'. Choose a torsion-free principal congruence subgroup J_0 fixing f at w_0 and put $C = KJ_0$. In the decomposition $g = \gamma s x c$ at this level write $b(g) = c$, with one chosen s per double coset. The finite stabilizer projects trivially to J_0. Its rational points are then the identity, since they are the identity at w_0. Thus b is left rational invariant and right G_∞-invariant, with $b(gc) = b(g)c$. For $c \in C$ put $f_c(v, g) = f(v, gb(g)^{-1}c)$. These are C-fixed forms of the original weight. There are finitely many distinct f_c, since f is fixed by $C \cap \mathrm{pr}_{v'}^{-1}(K_v)$. Increase R to a common bound for their norms and that of f.

Let $k(g)$ be the matrix at v' transported by the embedding extending $\iota^{-1}\tau$ and then by ι. Write g_τ for the archimedean matrix, and put $M(g) = g_\tau^{-1} k(g)$ and $n(g) = k(b(g))M(g)^{-1}$. Then n is left rational invariant and right C-invariant, with $n(gx) = n(g)x_\tau$ for $x \in G_\infty$. Choose $0 < e < 1$ with $eR \le \epsilon$. Partition C into finitely many cosets $c_q C_1$, where C_1 intersects the inverse image of K_v with the subgroup $|k(c) - 1|_* \le e$ in entrywise maximum norm. All matrices $k(c)$ and their inverses are integral. The images of the cosets under k are disjoint compact sets.

There are polynomials P_q in matrix entries with

$$\big| P_q(k(c)) - \mathbf{1}_{c_q C_1}(c) \big|_* \le e \qquad (c \in C). \tag{A.590}$$

To construct one, take $d_0 > 0$ below the distance between the image of its coset and the image of the complement. Cover the coset image by finitely many balls of radius ed_0 with centres in that image. For each centre use the coordinate differences from it. On the compact range of a coordinate difference, construct a polynomial bounded by one, within e of one at $|x|_* \le ed_0$, and of norm at most e at $|x|_* \ge d_0$. Cover the last set by finitely many balls $|x - c|_* \le |c|_*/2$. Start with $p = 1$ and take the centres in increasing order of norm. At each centre multiply by $(1 - x/c)^m$ with m chosen so that $2^{-m} \sup_{|x|_* \le |c|_*} |p(x)|_* \le e$. Later factors have norm at most one on the earlier balls, and all are within e of one on the innermost ball. Their product has the required bounds. Take the product of the coordinate tests at each centre, then one minus the product of the complements of these tests. If the coset image is all of $k(C)$ use the constant one.

For a dominant integral weight $\nu = (\nu_0, \dots, \nu_{a-1})$, put

$$D_\nu(h) = \det(h)^{\nu_{a-1}} \prod_{j=1}^{a-1} \det(h_{[1,j],[1,j]})^{\nu_{j-1} - \nu_j}.$$

For integral matrices h with integral inverse, $|D_\nu(h)|_* \le 1$. If $|h - 1|_* \le e$, every leading minor is within e of one, so $|D_\nu(h) - 1|_* \le e$, including for the signed determinant power. Define

$$B(y_1, y_2; v, g) = \sum_q P_q(n(g)y_2)\, D_\nu\big(k(c_q)^{-1} n(g) y_1\big) f_{c_q}(v, g) \qquad (y_1, y_2 \in \mathrm{GL}_a(\mathbf{C})).$$

At $y_1 = y_2 = M(g)$ the matrix $n(g)M(g)$ is $k(b(g))$. For the coset of $b(g)$ the minor factor is within e of one and $f_{c_q}(v, g) = f(v, g)$, since $k(c_q^{-1} b(g))$ comes from K_v. By (A.590) the other terms have norm at most eR on the lattice evaluations. Hence

$$\sup_{g,\ v \in b_g} |B(M(g), M(g); v, g) - f(v, g)|_* \le eR. \tag{A.591}$$

Lean: `UpperTheory.Gal.Wt.famOf_apply` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.WeightCharacters.FamOfApply`, lines 80–83)

Lean: `UpperTheory.Gal.Wt.famEx_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.WeightCharacters.FamExOf`, lines 13–79)

For fixed parameters y_1, y_2, v these are automorphic functions at level C. The archimedean translates span a finite-dimensional space, since they range over translates of polynomials of bounded degree with fixed determinant power and of the original weight vectors. The functions at other archimedean places retain the original weight. Project the family to the finitely many automorphic summands at this level and these archimedean types before substituting $y_i = M(g)$. Each projection has a scalar Hecke tuple. The substitution preserves the scalar action since $M(gh) = M(g)$ for every translation used by the operators.

Put $s_k(\lambda) = \sum_{i<k} \lambda_i$. For a constant d independent of ν, every nonzero projected weight μ at τ satisfies

$$|s_k(\mu) - s_k(\nu)| \le d \quad (1 \le k \le a), \qquad |\mu_i - \nu_i| \le 2d \quad (0 \le i < a). \tag{A.592}$$

For the upper bound use the circle $h_k(z) = \mathrm{diag}(zI_k, I_{a-k})$, $|z| = 1$, in the chosen archimedean coordinates. By expansion of minors of products, a translated j-row minor contributes at most $\min(j,k)$ to the exponent. Add the determinant contribution $k\nu_{a-1}$ to obtain the upper exponent $s_k(\nu)$. Take d to bound the degrees of the P_q plus the absolute circle exponents on V. Translates of a nonzero projection span its irreducible weight factor, whose largest circle exponent is $s_k(\mu)$. Thus $s_k(\mu) \le s_k(\nu) + d$. For the reverse bound some value of the projection with $y_1 = 1$ is nonzero. This follows by unitary times upper triangular decomposition of y_1 and the identity $D_\nu(hb) = D_\nu(h)\prod_i b_{ii}^{\nu_i}$ for upper triangular b, using $n(gx) = n(g)x_\tau$. Acting by the circle on this value pulls out $z^{s_k(\nu)}$ from the minor factor. All remaining exponents are at least $-d$, and projection commutes with the circle. Thus $s_k(\mu) \ge s_k(\nu) - d$. Subtracting consecutive sums proves the second bound.

Lean: `UpperTheory.Gal.Wt.famSum_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.WeightCharacters.FamSum`, lines 12–60)

Take $N = (a+1)(4d+a)+1$ and $\nu_i = -N(a+1)^i$. By (A.592), the jumps of each projection are $j_i = i + N(a+1)^i + e_i$, with $|e_i| \le 2d$. For $i + 1 < a$,

$$(j_{i+1} - j_0) - a(j_i - j_0) \ge N((a+1)^i + a - 1) + 1 - (a-1)i - 4ad > 0.$$

The rank-one gap condition is empty. The summands have K-fixed vectors, hence are spherical at v'. Apply the evaluation argument of (A.589) with (A.591). The difference is a function of the original weight because $M(gx) = x_\tau^{-1} M(g)$ cancels the archimedean change in $n(gx)$ after substitution. Its tested translates satisfy (A.588). On the projections the substitution preserves the eigenvalue calculation. This gives polynomial approximation by the tuples of the new weights.

Lean: `UpperTheory.Gal.Wt.top_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.WeightCharacters.TopOf`, lines 16–105)

To impose the type, take a form of one of the new weights and choose $J_m = 1 + \ell^m M_a(O)$ in its stabilizer at w_0, where $O = \mathcal{O}_{E_{w_0}}$ and $m \ge 2$ can be arbitrarily large. Choose an additive map $p : O \to \mathbf{Z}/\ell^m$ with $p(1) = 1$ by extending 1 to a $\mathbf{Z}_\ell$-basis of O. Take $B_0 \in M_a(O)$ with irreducible characteristic polynomial on reduction over the residue field, for example a companion matrix. Define

$$\lambda(1 + \ell^m X) = p(\mathrm{tr}(B_0 X)).$$

The product cross term maps to zero since $p(\ell^m O) = 0$. For $a \ge 2$ and a nonzero proper subspace $L \subset E_{w_0}^a$, the lattice $L \cap O^a$ is a saturated direct summand. Its reduction is a proper nonzero subspace and fails to be stable under $\bar{B}_0$. Choose $v \in L \cap O^a$ and an integral row t killing L with $tB_0 v$ a unit, and rescale to $tB_0 v = 1$. Then $1 + \ell^m vt$ has λ-value one. Taking L to be the first proper step of a parabolic flag makes this element a member of its unipotent radical. For a nontrivial character of λ, a vector transforming by that character has zero image in the unipotent coinvariants. Each translate has the same property by applying the calculation to the conjugated flag. The translates span the irreducible local representation, so its proper Jacquet modules vanish. The same calculation proves surjectivity of λ. In rank one take $B_0 = 1$ and use $p(1) = 1$. The Jacquet condition is vacuous in that rank. Torsion freeness follows as in 39.39.

Lean: `GlobalTheory.CongW.TypeExists.exists_type_surj` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsTorsionFreeSurjZmod`, lines 168–354)

Lean: `UpperTheory.Gal.Int.Cong.exists_ellType` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.CongruenceOddLevel.ExistsEllType`, lines 15–50)

At level KJ_m define $b_\lambda(\gamma s x c) = \lambda(c_{w_0})$ using double-coset representatives. As in 39.39, stabilizers have trivial image. For each character χ of $\mathbf{Z}/\ell^m$, the form $\chi(b_\lambda(g))f$ transforms by $\chi \circ \lambda$ at J_m. The negative

sum over $\chi \neq 1$ has the elliptic type in each nonzero constituent, and its difference from f is the scalar multiple $\ell^m \mathbf{1}_{b_\lambda(g)=0} f$ by the finite geometric sum. Its relative error is at most ℓ^{-m}. It preserves the weight and K-level. Apply (A.589) with m large enough to give the desired error.
Lean: `UpperTheory.Gal.Gen.tyStep_pf` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SeparatingAlgebras.TyStep`, lines 9–59)
Lean: `UpperTheory.Gal.Int.Cong.stepSt_of_ex` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.CongruenceOddLevel.NotApartOfOverPair`, lines 76–108)

Two polynomial approximations compose. If the outer error is $\epsilon/2$ with weights b_i, using inner error $\delta = \epsilon/[2(1 + \sum_i |b_i|_\ell)]$ and multiplying the weights gives error at most

$$\epsilon/2 + \delta \sum_i |b_i|_\ell \leq \epsilon. \tag{A.593}$$

Compose the weight and type steps in this way. For the good intermediate tuples η_i from (A.589), put $Q = \prod_{z \in D} X_z$ in the test variables, with empty product one. Each $Q(\eta_i)$ is nonzero. At inner tolerance δ approximate η_i by the weight and type steps to tolerance $\delta|Q(\eta_i)|_\ell$. Test PQ and divide by $Q(\eta_i)$. This gives the same approximation at tolerance δ with weights $c_x Q(x)/Q(\eta_i)$ in place of c_x, so every failing tuple has weight zero. Apply (A.593) to the first approximation and omit the test coordinates. The K-fixed vectors supply all the retained level conditions.

For rational f_j, choose one nonzero eigenform in U at a full supporting level. Its complex conjugates under $\mathrm{Aut}(\mathbf{C}/\mathbf{Q})$ on the finite adeles extend to forms at the same level and in the finite orbit of the archimedean weight. Indeed the corresponding algebraic weight models can be conjugated through the embeddings of a number field of definition. At fixed level and this finite set of weights the definite class set gives a finite-dimensional space. The rational translations commute with conjugation. Distinct conjugate eigenvalue tuples give linearly independent eigenvectors there, by applying an operator separating two distinct tuples and inducting on the number of vectors. The tuple therefore has finite orbit. Each coordinate is algebraic over $\mathbf{Q}$, and every finite subtuple generates a field of degree at most the orbit size. Choose a finite subtuple attaining the maximal degree. Its field contains every coordinate, and its image under ι^{-1} generates a finite extension of $\mathbf{Q}_\ell$.
Lean: `GlobalTheory.Cong.finU_holds` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.LinearIndependentSimulEigen`, lines 49–81)

□

Proof of 39.42

Proof. Apply 39.41 to $X_i - X_j$. At each tolerance the finite trace sum is continuous, so samples in a sufficiently small neighborhood of any group element satisfy the Cauchy condition. The limit lies in the complete field L and agrees with every sampled value. The same inequalities give continuity. Take limits in each finite tuple of samples to obtain 39.41 on Γ. The trace value at the identity follows by testing $X_1 - a$ with the variable indexed by the identity. Cyclic trace identities and the alternating degree-$(a+1)$ trace identity are integer polynomials in the trace variables of group elements and their products. They vanish on the family and hence on the limit. Apply 39.43. □

Proof of 39.44

Proof. Let S be the finite set of split places where U has no spherical vector. It is conjugation-stable since inverse transpose preserves the integral subgroup. For each finite specified set Y, take $Z_0(Y)$ from 39.36. Let $Z(Y)$ adjoin to it the split spherical members of Y away from ℓ. Let $W(Y)$ be the spherical members of Y above ℓ separated from the pair. By the field hypotheses, every place above ℓ is split, and $Z_0(Y)$ consists of split spherical places separated from the pair. Thus 39.40 applies to $Z(Y), W(Y)$. Its level, weight, type and general-position conditions give the hypotheses of 39.36. The fixed levels give sphericity off S at split places separated from the pair, and sphericity at v' also gives it at its conjugate.

Take indices $j = (w, A, s, k)$ with $w \nmid \ell$ split spherical for U, A a prolongation, s an arithmetic Frobenius lift at A, and $k \geq 1$. Put $x_j = s^{-k}$ and $\theta_j = \iota^{-1}\big(\sum_{\alpha \in A_a(w,B)} \alpha^k\big)$. The spherical Hecke operator for the i-th elementary symmetric function of the algebraic roots is $q_w^{i(i-1)/2}$ times the double-coset generator with i uniformizer entries. Indeed the normalized Satake transform of the generator is $q_w^{i(a-i)/2}$ times the

elementary symmetric function of the unscaled roots. Newton's identities express the positive power sums as integer polynomials in these generators. They give rational ℓ-integral operators since q_w is a unit at ℓ. The levels are hyperspecial by 39.40, whose rationality clause for $Y = \varnothing$ puts all the θ_j in one finite extension $L/\mathbf{Q}_\ell$.

Lean: `GlobalTheory.FrobOpsIdx.frobOpsIdx` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.FrobOpsIdx`, lines 20–94)

Index the seeds by $i = (Y, U', \eta_i)$ with ordinary Hecke tuple η_i of an approximant at that Y. Choose ρ_i by 39.36 and put $D_i = \{j : \eta_i(j) = \operatorname{tr}\rho_i(x_j)\}$. For every finite H of seed indices, $x(\bigcap_{i\in H} D_i)$ is dense. Take an open coset in G_E for the arithmetic Frobenius lifts. Intersect it with the dense open sets in 39.36 for $i \in H$, and choose a smaller coset of an open normal subgroup inside the intersection. Apply 36.24 over E away from the ramified primes. Exclude the rational primes below S and the finite exceptional sets, as well as those dividing $2d\ell$. The place has completion $\mathbf{Q}_p$, so it is usable. Taking $k = 1$ in the inverse coset and using the characteristic polynomials in 39.36 gives the simultaneous trace equalities.

For $g \in G_E$, let x_j approach g with j eventually in every specified finite intersection of the D_i. Density makes these requirements consistent. Given $\delta > 0$, choose a polynomial approximation of θ at precision δ from the empty-Y family. On the intersection of the D_i for its finite support, testing X_j bounds the distance from θ_j to the continuous trace sum evaluated at x_j by δ. Near g that trace sum is within δ of its value at g. Thus θ_j is Cauchy under the stated requirements. Let $t(g) \in L$ be its limit.

For any other polynomial approximation, impose the D_i of its support in the same limit, including when its specified set is different. For a polynomial in finitely many group variables, choose a sample for each variable in this common intersection, substitute the corresponding indices, and take limits. This gives 39.41 on G_E with the original weights and tolerance of each approximation. In particular,

$$\left| t(g) - \sum_i b_i \operatorname{tr}\rho_i(g) \right|_\ell \le \delta \qquad (g \in G_E). \tag{A.594}$$

Apply 39.42 with the identity sample map to obtain the continuous semisimple r of trace t.

Fix a usable spherical $w \nmid \ell$. All seeds in the family $Y = \{w\}$ are unramified there by 39.36, since $w \in Z(Y)$ and is separated from S. For $s \in I_w$, use their approximations in (A.594) at g and gs and let δ tend to zero. This gives $t(gs) = t(g)$. By the trace pairing of 39.43, $r(s) - 1$ is zero, since it pairs to zero with the spanning matrices $r(g)$. For each index j at w, the specified-place assertion gives $j \in D_i$ for every seed in this family. Testing X_j in the ordinary Hecke tuple and using (A.594) at x_j identifies $\theta_j = t(x_j)$ in the limit. The first a powers give $P_{w,B}$ by Newton's identities.

At spherical $w \mid \ell$ separated from the pair, use $Y = \{w\}$ and $W(Y) = \{w\}$. By 39.36 the seeds are crystalline with the jumps of U at the induced embeddings. Fix the local data, bridge and σ, and put $k_j = -j \deg\sigma$ and $u_j = \iota^{-1}(c_{w,k_j})^{-1}$. The unramified Weil formula in 39.36 identifies the transported Weil trace at σ^j with u_j times the augmented seed coordinate (w, k_j). The same multiplication gives S_j from the target coordinate. For $0 \le j \le N$, use 39.40 on all coordinates at tolerance $\epsilon/(1 + \sum_{j=0}^{N} |u_j|_\ell)$. The ordinary coordinates give (A.594), and testing $X_{(w,k_j)}$ and multiplying by u_j gives error at most ϵ at every tested power with these same weights. Sum the weights of indices that give the same representation. The joint approximation satisfies 39.48, so the required crystalline and de Rham conclusion follows.

Lean: `UpperTheory.Gal.Lim.joint` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.TraceLimits.SlopeTestRemainingChoices`, lines 50–89)

□

Proof of 39.45

Proof. Let $\mathfrak{M}$ denote the value of the crystal of Λ on $(\mathfrak{S}, (E))$. Its value on the Fontaine prism is $A_{\mathrm{inf}} \otimes_{\mathfrak{S}} \mathfrak{M}$. In the chosen basis the semilinear action on this module has matrix ε_g, since the images of the basis columns under the action form $\rho(g)g(T) = T\varepsilon_g$.

Take $\epsilon = (1, \zeta_\ell, \zeta_{\ell^2}, \dots)$ with compatible primitive roots, and put $\mu = [\epsilon] - 1$. For weights in $[0, h]$ in the convention of Gee and Liu, their crystalline lattice theorem gives

$$(g-1)\mathfrak{M} \;\subset\; u\phi^{-1}(\mu)(A_{\mathrm{inf}} \otimes_{\mathfrak{S}} \mathfrak{M}) \qquad (g \in G_K).$$

This is the crystalline implication of [196, Definition F.9, equation (F.10), and Theorem F.11]. The theorem supplies the descents for every choice of uniformizer roots for the given crystalline lattice. We take the descent for $\pi^\flat$. Its embedding $u \mapsto [\pi^\flat]$ is the one above, with no Frobenius twist. The identification with $\mathfrak{M}$ respects the étale comparison by [192, Theorem 7.2 and Remark 7.11].

For arbitrary weights choose a Tate line D of sufficiently large nonnegative weight that D and $\Lambda \otimes D$ both have weights in $[0, h]$ for some h. Apply the displayed inclusion to both lattices. In a tensor product basis, the matrix for Λ is the matrix for $\Lambda \otimes D$ times the inverse of the scalar matrix coefficient for D, by tensor compatibility of the crystalline realization. Both factors are the identity modulo u. A change of $\mathfrak{S}$-basis preserves this congruence since g fixes $W(k)$ and $g(u) \in uA_{\mathrm{inf}}$. This proves the assertion about ε_g.

For the second assertion put $R_j = \mathfrak{S}[1/\ell]/(E^j)$. We describe first the rational comparison used to obtain a matrix over R_j. For the Frobenius pullback $\phi^*\mathfrak{M}$, the comparison near $E = 0$ identifies its rational basis space with the scalar extension of its constant isocrystal, the Frobenius pullback of $(\mathfrak{M}/u\mathfrak{M})[1/\ell]$. This is the isomorphism lifting the identity at $u = 0$ in the proof of [192, Theorem 7.9]. In detail, in Construction 6.5 of that paper the rational crystalline module on the disc $|E| \le |\ell|$ is obtained by modifying a constant module along $E = 0$. Frobenius pullback moves the modification to $\phi(E) = 0$. At $u = \pi$ the value of $\phi(E)$ is nonzero: the roots of the coefficient-Frobenius conjugate of E have valuation $v_\ell(\pi)$, whereas the argument π^ℓ has valuation $\ell v_\ell(\pi)$. Thus the comparison and its inverse are regular on every R_j. Restriction of disc functions to R_j is defined since E/ℓ is nilpotent there. Under evaluation in $B^+_{\mathrm{dR}}/(E^j)$ the constant basis vectors are fixed by G_K, by the compatibility of the crystalline comparison with this lattice [192, Proposition 5.3 and Remark 5.4].

Consequently their coordinate matrix H in the Frobenius pullback basis belongs to $\mathrm{GL}_d(R_j)$ and satisfies $\phi(\varepsilon_g)g(H) = H$ after evaluation modulo E^j. Its reduction H_0 is invertible over K. The residue map $R_j \to K$ has a $W(k)[1/\ell]$-algebra section: lift π to the unique root of E congruent to u modulo the nilpotent ideal (E), using $E'(\pi) \neq 0$. After evaluation the image of this section is fixed by G_K, by the same uniqueness. Multiply H on the right by the section of H_0^{-1}. It now reduces to I_d and satisfies the same invariance equation.

Choose k_j clearing the denominators of this finite matrix over R_j and lift the result to B over $\mathfrak{S}$. Reduction in $\mathfrak{S}/(E) = \mathcal{O}_K$ gives $B \equiv \ell^{k_j} I_d$. Each entry of its invariance defect belongs to A_{inf} and to $E^j B^+_{\mathrm{dR}}$. Their intersection is $E^j A_{\mathrm{inf}}$: for $j = 1$ apply θ, whose kernel on A_{inf} is (E), then divide by E and iterate. This proves both congruences. □

Proof of 39.46

Proof. We first choose one $g \in G_K$ such that $\lambda = \theta(g(E)/E)$ has infinite order. The quotient $g(E)/E$ is a unit because g preserves $\ker\theta$. Use ϵ, μ from the proof of Lemma 39.45. We have $\theta(\mu/E) \neq 0$: the element $\mu/\phi^{-1}(\mu)$ also generates $\ker\theta$, and $\theta(\phi^{-1}(\mu)) = \zeta_\ell - 1$. Normalize $v_\ell(\ell) = 1$ and put

$$A_0 = \theta(\mu/E)\,\pi E'(\pi) \neq 0.$$

Choose $b \ge 0$ with $v_\ell(\ell^b A_0) > 1/(\ell - 1)$. There exists $g \in G_K$ with $g(u) = [\epsilon]^{\ell^b} u$. Indeed, at level n the requested image of π^{1/ℓ^n} is $\zeta_{\ell^n}^{\ell^b}\pi^{1/\ell^n}$. It is a conjugate over K by the Eisenstein polynomial $Y^{\ell^n} - \pi$. The conditions on g are nested nonempty closed subsets of the compact group G_K. For a common g, Taylor expansion modulo E^2 gives

$$g(E) \equiv E + \ell^b \mu u E'(u) \pmod{E^2}, \qquad \lambda = 1 + \ell^b A_0.$$

The ℓ-adic logarithm is injective on $v_\ell(x - 1) > 1/(\ell - 1)$. Thus $\lambda^i - 1 \neq 0$ for every $i \ge 1$.

For $1 \le i < J$ choose the integer $a_i = 1 + \lfloor v_\ell(\lambda^i - 1) \rfloor$. Set $s_1 = 0$, $s_{i+1} = s_i + a_i$, and $c_J = s_J$. If $k < c_J$, multiply B by $\ell^{c_J - k}$. Suppose henceforth that $k \ge c_J$. Write $D_h(U) = P_h h(U) - U$ after evaluating entries in A_{inf}. Reduction of $D_h(B)$ modulo E gives $\ell^k(\theta(P_h) - I_d) = 0$, so $\theta(P_h) = I_d$.

We prove successively that there exist $C_i \in M_d(\mathfrak{S})$ with

$$B \equiv \ell^{k - s_i} C_i \pmod{E^i} \qquad (1 \le i \le J).$$

Take $C_1 = I_d$. For $i < J$, put $n = k - s_i$ and write $B = \ell^n C_i + E^i \Delta$ over $\mathfrak{S}$. Put $w = g(E)/E$. Then

$$D_g(B) = \ell^n D_g(C_i) + E^i\big(w^i P_g g(\Delta) - \Delta\big).$$

The quotient $A_{\mathrm{inf}}/(E^i)$ has no ℓ-torsion. For $i = 1$ it is $\mathcal{O}_C$, and the successive quotients $(E^t)/(E^{t+1})$ are copies of $\mathcal{O}_C$ because E is a nonzero divisor. The displayed identity therefore gives $D_g(C_i) = E^i U$ for an integral matrix U. Divide the identity by E^i and apply θ. Its left side vanishes since $i < J$. The entries of $\theta(\Delta)$ lie in $\mathcal{O}_K$ and are fixed by g. Hence

$$(\lambda^i - 1)\theta(\Delta) = -\ell^n \theta(U).$$

Since $n \geq a_i$, every entry of $\theta(\Delta)$ lies in $\ell^{n-a_i}\mathcal{O}_C \cap K = \ell^{n-a_i}\mathcal{O}_K$. Using $\mathfrak{S}/(E) = \mathcal{O}_K$, choose $D \in M_d(\mathfrak{S})$ with $\Delta \equiv \ell^{n-a_i} D \bmod E$. Then

$$C_{i+1} = \ell^{a_i} C_i + E^i D$$

has the required property. This proves the induction, including $J = 1$ with $c_1 = 0$.

At $i = J$ we have $B \equiv \ell^{k-c_J} C_J \bmod E^J$. Reducing modulo E and cancelling ℓ^{k-c_J} in $\mathcal{O}_K$ gives $C_J \equiv \ell^{c_J} I_d \bmod E$. For every $h \in G_K$, the ideal $(E^J)A_{\mathrm{inf}}$ is h-stable. Consequently $\ell^{k-c_J} D_h(C_J) \equiv D_h(B) \equiv 0 \bmod E^J$. The same torsion-freeness gives $D_h(C_J) \equiv 0 \bmod E^J$, as required. □

Proof of 39.47

Proof of Lemma 39.47 and Proposition 39.48, first part. We construct the layer lattices, comparison frames and common denominator for both assertions. Choose finite coefficient fields for the continuous representations under consideration. The preimages of the matrix groups over the countably many finite local extensions are closed subgroups and cover G_E. Baire's theorem gives one open preimage, and finitely many coset representatives enlarge its coefficient field to one for the full representation. Enlarge these fields to contain the images of all embeddings of the local field in the chosen bridge. For r take the lattice Λ_0 spanned over $\mathbf{Z}_\ell$ by $r(g)$, with the left multiplication action. It is finite free in its $\mathbf{Q}_\ell$-span by compactness and contains the identity. The trace pairing in 39.43 shows that finitely many functions $x \mapsto \operatorname{tr}(xr(g_j))$ separate points of this span. On the unit sphere for an integral basis of Λ_0 their maximum norm is bounded below by a positive number by compactness. Thus for some integers $c, K_0 \geq 0$ and any $m \geq 1$,

$$\max_j |\operatorname{tr}(xr(g_j))|_\ell \leq \ell^{-(m+K_0+c)} \implies x \in \ell^{m+K_0}\Lambda_0, \qquad |\operatorname{tr}(x)|_\ell \leq \ell^{K_0} \quad (x \in \Lambda_0), \tag{A.595}$$

where the implication applies to x in the span and the last bound to x in the lattice.

At level m choose an approximation at precision $\ell^{-(m+K_0+c)}$ in (39.4). Keep all earlier seeds, with weight zero when unused. Write them as ρ_k, with weights b_k, and let Λ_{tup} be the span of their tuples $(\rho_k(g))_k$ over $\mathbf{Z}_\ell$. Let Λ_{full} be the product of full stable matrix lattices over the coefficient fields and put $\Lambda_m = \Lambda_{\mathrm{tup}} + \ell^b \Lambda_{\mathrm{full}}$. One can take the matrix lattices from stable lattices for the seed actions, obtained by spanning the translates of an initial lattice. Choose b so large that the intersection of the added part with Λ_{tup} lies in $\ell^{m+K_0}\Lambda_{\mathrm{tup}}$, by elementary divisors. Also choose it so that the weighted trace on the added part has norm at most ℓ^{-m}, which is possible because the list of weights is finite.

For a tuple relation $\sum_g z_g(\rho_k(g))_k = 0$ with $z_g \in \mathbf{Z}_\ell$, test its translates by g_j against the uniform trace bound. The corresponding $\sum_g z_g r(g)$ satisfies (A.595). Hence evaluation at r on the tuple part and zero on the added part define a map modulo ℓ^{m+K_0}. Lift its matrix to $\mathbf{Z}_\ell$ in bases of Λ_m and Λ_0. Π_m is surjective modulo ℓ^m. It intertwines the actions modulo ℓ^m and preserves the identity column modulo ℓ^m. Write t_i for the trace of the i-th basis matrix of Λ_0 and t'_{hk} for the trace in seed component k of the h-th basis vector of Λ_m. Testing the tuple generators and using the bounds on the added part and on the lift in (A.595) gives

$$\left|\sum_i (\Pi_m)_{ih} t_i - \sum_k b_k t'_{hk}\right|_\ell \leq \ell^{-m}, \qquad \left|\vartheta - \sum_k b_k p_k\right|_\ell \leq \ell^{-m}. \tag{A.596}$$

In the second estimate, $\vartheta = S_1$ is the target at the fixed Weil element and p_k is its trace on seed k transported by ι^{-1}. For σ^j use $\vartheta = S_j$. Choose the lattice exponents b nondecreasing so that forgetting the new seed components gives an integral equivariant transition $f_m : \Lambda_{m+1} \to \Lambda_m$. Evaluation on tuples and vanishing on the added parts give $\Pi_m f_m \equiv \Pi_{m+1} \pmod{\ell^m}$. Each rational layer is the full product of the matrix spaces with the left actions. Thus it consists of scalar restrictions of copies of the crystalline seeds. At the limit the span of $r(g)$ over a coefficient field containing its entries is a faithful left matrix module for r and contains every irreducible constituent of r.

Let $K = E_w$ in the chosen local realization, with residue field k, and put $\mathfrak{S} = W(k)[[u]]$ and $\mathcal{O}_{\mathcal{E}} = \widehat{\mathfrak{S}[1/u]}$ with ℓ-adic completion. Use $A_{\mathrm{inf}} = W(\mathcal{O}_{\mathbf{C}_\ell^\flat})$ and $\mathcal{W} = W(\mathbf{C}_\ell^\flat)$. Evaluate u at the Teichmüller lift of compatible roots of a uniformizer and use u also for this image. Take an Eisenstein polynomial $E(u)$ with a nonzero coefficient of degree prime to ℓ. Translating a uniformizer by an element of $\ell W(k)$ can give a nonzero linear coefficient, since the derivative of its polynomial has finitely many roots. Write ϕ for Witt Frobenius and its compatible actions, with $\phi(u) = u^\ell$. Put $\xi_0 = ([\varepsilon]-1)/\phi^{-1}([\varepsilon]-1) \in A_{\mathrm{inf}}$ for a compatible primitive system ε of ℓ-power roots of unity. Since both $E(u)$ and ξ_0 generate $\ker\theta$, the ratio $\alpha = E(u)/\xi_0$ lies in $A_{\mathrm{inf}}^\times$. Solve $\bar{v}^{\ell-1} = \bar{\alpha}$ in the algebraically integrally closed ring $\mathcal{O}_{\mathbf{C}_\ell^\flat}$ to obtain $v_0 \in A_{\mathrm{inf}}^\times$ with $\phi(v_0)/v_0 \equiv \alpha \pmod{\ell}$, and inductively set $v_k = v_{k-1}(1+\ell^k[x_k])$ by solving Artin–Schreier equations $x_k^\ell - x_k = z_k$ in $\mathcal{O}_{\mathbf{C}_\ell^\flat}$ to cancel the ℓ^k-error in $\phi(v_{k-1})/(\alpha v_{k-1}) \equiv 1 + \ell^k y_k \pmod{\ell^{k+1}}$. The ℓ-adic limit $v = \lim_k v_k \in A_{\mathrm{inf}}^\times$ satisfies $\phi(v)/v = E(u)/\xi_0$. Setting $\tau = v\phi^{-1}([\varepsilon]-1) \in A_{\mathrm{inf}}$ yields the Breuil–Kisin period satisfying $\phi(\tau) = E(u)\tau$ and $\phi(\tau) = \phi(v)([\varepsilon]-1) \in A_{\mathrm{inf}}^\times([\varepsilon]-1)$. Then τ is a unit in $\mathcal{W}$, its image under $\theta : A_{\mathrm{inf}} \to \mathbf{C}_\ell$ is nonzero, and the image of $\phi(\tau)$ in B_{cris} is invertible.

Take $N = \max(\{0\} \cup \{|j| : j \in J_\eta,\ \eta \text{ induces } w\})$. Each rational layer is crystalline with jumps in $[-N, N]$, since scalar restriction collects the jumps over the coefficient embeddings. Use the integral crystalline realization on the stable lattices, in the form of Breuil–Kisin modules and their Galois comparison [209, 192]. The prismatic realization of stable lattices is used here on the Breuil–Kisin prism. In a basis write T_m for the étale period matrix, a unit over $\mathcal{W}$. Its Frobenius and inverse Frobenius numerators satisfy

$$E^N\phi(T_m) = T_m A_m, \qquad A_m A_m' = E^{2N} I, \qquad A_m, A_m' \in M_{d_m}(\mathfrak{S}), \tag{A.597}$$

where $d_m = \mathrm{rank}_{\mathbf{Z}_\ell} \Lambda_m$. The elementary divisors at (E) are the labeled jumps up to the sign of the filtration convention. Both the linearized Frobenius and its inverse thus become integral there on multiplication by E^N. Their poles lie at E, and divisibility by powers of the prime E at (E) descends to $\mathfrak{S}$. This gives the common height in (A.597).

Lean: `GlobalTheory.BKLim.exists_frame_sst` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsBreuilKisinFrameSemistableBoundedJumps`, lines 109–134)
Lean: `GlobalTheory.BKLim.Lower.exists_frame_of_jumps_le` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsBreuilKisinFrameHeightOfJumpsLe`, lines 76–89)

Put $Y_m = \tau^N T_m$ and $Y_m' = \tau^N T_m^{-1}$. By (A.597), $\phi(Y_m) = Y_m A_m$ and $\phi(Y_m') = A_m' Y_m'$. For a row satisfying $\phi(x) = xA + b$ with integral A, b, the largest norm B of its entries modulo ℓ satisfies $B^\ell \le \max(B, 1)$. Thus its first Witt digits are integral. Subtract their Teichmüller lifts and divide by ℓ. The new equation has the same linear part and an integral inhomogeneous term, since its numerator has zero reduction. Iterate and apply the transposed calculation for columns. This gives

$$Y_m, Y_m' \text{ have entries in } A_{\mathrm{inf}}, \qquad Y_m Y_m' = Y_m' Y_m = \tau^{2N} I. \tag{A.598}$$

Lean: `GlobalTheory.BKLim.pre_bounded_poles` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.BreuilKisinPreframeBoundedPoles`, lines 13–79)

Use a common étale basis T_* for the target lattice. The étale comparison for the group fixing the uniformizer roots gives $T_* R_m \equiv \Pi_m T_m \pmod{\ell^m}$, with R_m over $\mathcal{O}_{\mathcal{E}}$ admitting a right inverse modulo ℓ^m there. One construction is to quotient the étale layer module by the kernel of Π_m modulo ℓ^m. Its extension to $\mathcal{W}/\ell^m$ is free of the target rank by surjectivity. The quotient is therefore free over $\mathcal{O}_{\mathcal{E}}/\ell^m$ by elementary divisors. The consecutive quotients agree at the lower precision by $\Pi_m f_m \equiv \Pi_{m+1}$. Adjusting bases successively by lifts of the change-of-basis matrices gives compatible period matrices with invertible limit T_*. This uses the étale module equivalence and base change for the uniformizer tower in the Breuil–Kisin realization [209]. By functoriality of the crystalline realization, $D_m = T_m^{-1} f_m T_{m+1}$ has entries in $\mathfrak{S}$. Hence $R_m D_m \equiv R_{m+1} \pmod{\ell^m}$.

Fix $n \ge 1$. The images $I_{n,m} = R_m \mathfrak{S}^{d_m} \bmod \ell^n$ for $m \ge n$ are descending. Set $a_0 = \tau^{2N}$. Since $\bar{a}_0 \ne 0$ in the tilt and $0 < |\bar{u}| < 1$, choose b_0 with $\bar{u}^{b_0}/\bar{a}_0$ integral. Lifting this quotient gives $u^{b_0} = h a_0 + \ell z$ with h, z integral. In its n-th power, the term without a_0 is divisible by ℓ^n. Thus for $c = b_0 n$ one has $u^c \equiv a_0 y \pmod{\ell^n}$ for some integral y.

For $v \in \mathfrak{S}^{d_n}$ choose a right inverse S_m of Π_m over $\mathbf{Z}_\ell$ modulo ℓ^m. The column $v' = y Y_m' S_m \Pi_n Y_n v$ is integral by (A.598) and satisfies

$$T_* R_m v' \equiv \Pi_m T_m v' \equiv y\tau^{2N} \Pi_n T_n v \equiv T_* R_n u^c v \pmod{\ell^n}.$$

To descend the coefficients of v', choose a topological $\mathfrak{S}$-basis of A_{inf} by lifting a k-basis of $A_{\mathrm{inf}}/(\ell, u)$ that contains 1. Successive expansions modulo $(\ell, u)^j$ exist and are unique. Both rings are complete for this ideal, and (ℓ, u) is a regular sequence in each, since $\bar{u} \neq 0$ in the tilt valuation ring. The associated graded rings are polynomial rings in the two initial forms. The coordinate of 1 gives an $\mathfrak{S}$-linear retraction. Clear a power of u from $R_m v' \equiv R_n u^c v$ at precision n to put the matrix and target over $\mathfrak{S}/\ell^n$. Integral Witt vectors are ℓ-saturated in $\mathcal{W}$, so applying the retraction preserves the congruence. Cancel the cleared power of u over $\mathcal{O}_{\mathcal{E}}$. This proves

$$u^c I_{n,n} \subset I_{n,m} \subset I_{n,n} \qquad (m \geq n). \tag{A.599}$$

The quotient by $u^c I_{n,n}$ is a quotient of $(\mathfrak{S}/(\ell^n, u^c))^{d_n}$ and is finite. Thus the images stabilize for each n.
Lean: `GlobalTheory.BKLim.flatPow_dvd_perPow_mod` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.FlatPowDvdPerPowMod`, lines 15–25)
Lean: `GlobalTheory.CoefDescent.descent` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.BreuilKisinDescentCoefficientsAinfModPow`, lines 68–119)
Lean: `GlobalTheory.BKLim.mittagLeffler_approx` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LatticeApproximationMittagLeffler`, lines 30–156)

The stable images at consecutive precisions map onto one another, by using the same coefficient column at a layer beyond both stabilization indices. At precision one clear a power of u from a right inverse of a stabilized R_m over $k((u))$. Lift the resulting columns of $u^h I$ successively through the stable images, where h is the clearing exponent. These lifts give an invertible matrix over $\mathcal{O}_{\mathcal{E}}$ since its reduction is invertible there. Each lifted column v satisfies $\tau^N T_* v \in A_{\mathrm{inf}}^{d_0}$, where d_0 is the rank of Λ_0. At every precision its image is a column combination of $\Pi_m Y_m$ with integral coefficients.

Consider the saturated lattice

$$\mathcal{L} = A_{\mathrm{inf}}^{d_0} \cap \tau^N T_* \mathcal{O}_{\mathcal{E}}^{d_0} \subset \mathcal{W}^{d_0}.$$

In the coordinates of $\tau^N T_*$ its reduction is a $k[[u]]$-submodule of $k((u))^{d_0}$ with bounded denominators. The reduced inverse matrix has bounded tilt norm and the input entries have norm at most one. The lifted columns give full rank. The reduction is free of rank d_0 over $k[[u]]$. Lift a basis to the columns of a matrix Y in $\mathcal{L}$. Subtracting its multiples and dividing the remainder by ℓ can be repeated inside $\mathcal{L}$ by the ℓ-saturation of A_{inf} and $\mathcal{O}_{\mathcal{E}}$ in $\mathcal{W}$. Completeness gives coefficients in $\mathfrak{S}$, so Y is a basis and $T = \tau^{-N} Y$ differs from T_* by an $\mathcal{O}_{\mathcal{E}}$-unit.

At precision n, take $m \geq n$ beyond stabilization. Lift the columns of R_m from the stable images and express their integral period vectors in the basis Y. The resulting matrix R over $\mathfrak{S}$ gives $TR \equiv \Pi_m T_m \pmod{\ell^n}$. Put $Q_m^* = Y_m' S_m$. Then $YRQ_m^* \equiv \tau^{2N} I \pmod{\ell^n}$. It follows that $Y' = Y^{-1}\tau^{2N}$ in $\mathcal{W}$ is integral by the digit criterion at every precision. The Frobenius matrices

$$A = E^N T^{-1} \phi(T), \qquad A' = E^N \phi(T)^{-1} T$$

have entries in $\mathcal{O}_{\mathcal{E}}$ by étale descent. The matrices $\tau^{2N} A = Y' \phi(Y)$ and $\tau^{2N} A' = \phi(Y') Y$ are integral.

In the topological basis used in (A.599), the ideal I generated by the coefficients of τ^{2N} contains ℓ^r for some $r \geq 0$. Indeed, its saturation for ℓ is principal, say (g). Lift a generator of the reduced ideal in $k[[u]]$ and apply saturation and Nakayama's lemma. The generator is nonzero and $\ell \nmid g$. Frobenius of the expansion and $\phi(\tau^{2N}) = E^{2N}\tau^{2N}$ give $\phi(g) \mid E^{2N} I$. Since $\ell^j g \in I$ for some j and $\ell \nmid \phi(g)$, one has $\phi(g) \mid E^{2N} g$. If g is a nonunit, its Weierstrass polynomial has roots of norm less than one. A zero root is excluded by comparing orders at $u = 0$, since $E(0) \neq 0$. Choose a root of largest norm and take its coefficient-Frobenius conjugate. Every ℓ-th root of this conjugate or of its field conjugates has strictly larger norm and is a root of E. Their nonempty Galois-stable set must be all roots of E by irreducibility. Its monic polynomial is the minimal polynomial of the conjugate evaluated at u^ℓ, contrary to the chosen exponent of E. Thus g is a unit.

Choose a finite $\mathfrak{S}$-linear combination p of basis coordinates with $p(\tau^{2N}) = \ell^r$. For $x \in \mathcal{O}_{\mathcal{E}}$ with $\tau^{2N} x \in A_{\mathrm{inf}}$, clear powers of u at each precision to get $\ell^r x = p(\tau^{2N} x) \in \mathfrak{S}$. Since $\mathfrak{S}[1/\ell] \cap \mathcal{O}_{\mathcal{E}} = \mathfrak{S}$ by successive reduction modulo ℓ, this descends A and A' to $\mathfrak{S}$. Also $RQ_m^* Y \equiv \tau^{2N} I$ by cancelling Y over $\mathcal{W}$. With $G = p(Q_m^* Y)$ the links satisfy

$$\forall n \geq 1 \ \exists m \geq n, \ R, G \text{ over } \mathfrak{S}, \qquad TR \equiv \Pi_m T_m, \qquad RG \equiv \ell^r I \pmod{\ell^n}. \tag{A.600}$$

The exponent r depends only on τ, N.
Lean: `GlobalTheory.BSP.pow_l_mem_span_coeff` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsPowMemSpanCoeffPeriodPow`, lines 83–159)

Lean: `GlobalTheory.LemZ.isUnit_of_frob_dvd` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.IsUnitOfFrobeniusDvdEisensteinPowMul`, lines 51–150)
Lean: `GlobalTheory.BKLim.descend_rightInverse` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.DescendRightInverse`, lines 12–42)

Write ρ_0, ρ_m for the left actions on the limit and layer lattices. For $g \in G_K$ put $\varepsilon_g = T^{-1}\rho_0(g)g(T)$ and $\varepsilon_{m,g} = T_m^{-1}\rho_m(g)g(T_m)$. The lattice Λ_m is stable and finite free over $\mathbf{Z}_\ell$. Its rational representation is a finite direct sum of scalar restrictions of crystalline seeds. Thus Lemma 39.45, with its chosen Breuil–Kisin basis and $d = d_m$, gives $\varepsilon_{m,g} - I_{d_m} \in uM_{d_m}(A_{\mathrm{inf}})$ for every $g \in G_K$. The equivariance of Π_m and (A.600) give, over $\mathcal{W}$,

$$\varepsilon_g g(R) \equiv R\varepsilon_{m,g} \pmod{\ell^n}. \tag{A.601}$$

Multiply by $g(G)$ and use $g(G) - G \in uM(A_{\mathrm{inf}})$, since g fixes $W(k)$ and $g(u)$ is divisible by u. It follows that $\ell^r(\varepsilon_g - I)$ is congruent to a multiple of u with integral entries at every precision. Both A_{inf} and uA_{inf} are ℓ-adically closed and ℓ-saturated in $\mathcal{W}$. For uA_{inf} use the same properties for integral Witt digits after division by the unit u of $\mathcal{W}$. Hence $\varepsilon_g \in I + uM_{d_0}(A_{\mathrm{inf}})$.

A rank-d frame with cocycle ε_g satisfies the de Rham congruences if for every integer $j \geq 0$ there are an integer $k \geq 0$ and a matrix $\beta \in M_d(\mathfrak{S})$ such that

$$\beta \equiv \ell^k I_d \pmod{E}, \qquad \phi(\varepsilon_g)\, g(\beta) \equiv \beta \pmod{E^j} \quad (g \in G_K).$$

The first congruence is in $\mathfrak{S}$. In the second, evaluate $\mathfrak{S}$ in A_{inf} before applying g and take the congruence entrywise in A_{inf}.
Lean: `GlobalTheory.BKLim.Frame.IsBeta` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.TrivIsCr`, lines 74–76)
Lean: `GlobalTheory.BKLim.Frame.IsDR` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.TrivIsCr`, lines 78–78)

We prove these congruences for T. Fix $M \geq 0$ and take $c = c_{M+1}$ from Lemma 39.46 for the fixed $K, \pi^\flat, E$. For each layer, Lemma 39.45 gives a solution at order $M+1$ with some exponent. Apply Lemma 39.46 to that solution with $d = d_m$ and $P_g = \phi(\varepsilon_{m,g})$. This gives matrices β_m over $\mathfrak{S}$ with

$$\beta_m \equiv \ell^c I \pmod{E}, \qquad \phi(\varepsilon_{m,g})g(\beta_m) \equiv \beta_m \pmod{E^{M+1}} \quad (g \in G_K).$$

The second congruence uses the evaluation in A_{inf}. Also, for every $x \in \mathfrak{S}$ there is a lift $\gamma(x) \in \mathfrak{S}$ of $\ell^{k_0}x$ modulo E, invariant after evaluation modulo E^{M+1} for all $g \in G_K$. The exponent k_0 depends on M, K, E and is uniform in x. For example, after inverting ℓ, lift the uniformizer in $\mathfrak{S}/(E^{M+1})$ to a root of its Eisenstein polynomial using the invertible derivative. This gives the section from K whose evaluation is fixed modulo E^{M+1} in B_{dR}^+. Clear denominators for a $W(k)$-basis of $\mathcal{O}_K$ to choose k_0. Divisibility by E^{M+1} of the evaluated integral differences descends to A_{inf}.

For each precision n, take m, R, G as in (A.600), apply γ entrywise to $\phi(G)$, and put $B_n = \phi(R)\beta_m\gamma(\phi(G))$. Reduction modulo (ℓ^n, E) gives

$$B_n \equiv \ell^{c+k_0}\phi(RG) \equiv \ell^{c+k_0+r} I \pmod{(\ell^n, E)}.$$

Frobenius commutes with the Galois action after evaluation. Apply it to (A.601), which also holds modulo ℓ^n in A_{inf} by saturation. Then modulo (ℓ^n, E^{M+1}) in A_{inf},

$$\phi(\varepsilon_g)g(B_n) \equiv \phi(R)\phi(\varepsilon_{m,g})g(\beta_m)\gamma(\phi(G)) \equiv B_n \qquad (g \in G_K).$$

These two tests define nested nonempty closed subsets of the matrix space over $\mathfrak{S}/(E^{M+1})$, since at each n they depend on the class modulo (ℓ^n, E^{M+1}). The quotient by E^{M+1} is finite free over $W(k)$ by Weierstrass division, hence compact. Choose a common class and lift it to $\mathfrak{S}$. In A_{inf},

$$\bigcap_{n \geq 1} (\ell^n, E^j) = E^j A_{\mathrm{inf}} \qquad (j \geq 1).$$

For $j = 1$ this follows from ℓ-adic separatedness after applying θ. For the induction, divide by E after using $E \mid \ell^n x \Rightarrow E \mid x$, which follows since A_{inf}/E has no ℓ-torsion. Also $EA_{\mathrm{inf}} \cap \mathfrak{S} = E\mathfrak{S}$. The common class therefore gives a matrix congruent to $\ell^{c+k_0+r}I$ modulo E with the twisted invariance modulo E^{M+1}. Taking $M + 1 \geq j$ gives the required congruences at every precision j.

Lean: `GlobalTheory.BKLim.limit_pre_exists` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsBreuilKisinBasisLimitLattice`, lines 35–63)
Lean: `GlobalTheory.BKLim.limit_frame_exists` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsDeRhamFrameLimitLatticeBreuilKisinTower`, lines 16–24)
Lean: `GlobalTheory.BKLim.isDR_of_links` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.IsDROfLinks`, lines 107–224)
Lean: `GlobalTheory.BKLim.beta_bdd_denominators` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.BreuilKisinBetaBoundedDenominators`, lines 41–105)
Lean: `GlobalTheory.BSP.BK.exists_inv_lift` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsInvLift`, lines 82–116)

In A_{cris} one has $E(u) \equiv u^{e_0} \pmod{\ell}$, with $e_0 = \deg E$. The divided powers give $\ell \mid E(u)^{\ell}$, since $E(u)$ is in the divided-power ideal. Thus $\ell \mid u^{e_0 \ell}$. Choose $n_0 \geq 1$ with $e_0 \ell((2N+1)n+1) \leq \ell^n - 1$ for all $n \geq n_0$. Then

$$\ell^{(2N+1)n+1} \mid u^{\ell^n - 1} \qquad (n \geq n_0). \tag{A.602}$$

Let $f_A : \mathfrak{S} \to A_{\mathrm{cris}}$ be evaluation followed by crystalline Frobenius, and f_0 the same map after taking constant terms. Write $f_A(E) = \ell v$ and $f_0(E) = \ell v_0$. For the first quotient,

$$v = \frac{\phi(E)}{\ell} = \delta(E) + \frac{E^{\ell}}{\ell}.$$

The Eisenstein prism condition makes $\delta(E)$ a unit. The divided-power identity

$$(E^{\ell}/\ell)^{\ell} = \frac{(\ell^2)!}{\ell^{\ell}} \gamma_{\ell^2}(E)$$

puts this power in ℓA_{cris}. Hence v is a unit modulo ℓ, and ℓ-adic completeness gives its inverse in A_{cris}. The quotient v_0 is the Frobenius of the unit $E(0)/\ell$. For each frame, with matrices A, A' as above, set

$$\alpha = v^{-N} f_A(A), \quad \alpha' = v^{-N} f_A(A'), \qquad \beta = v_0^{-N} f_0(A), \quad \beta' = v_0^{-N} f_0(A').$$

Thus $\alpha\alpha' = \beta\beta' = \ell^{2N} I$. The differences $f_A(A) - f_0(A)$ and $\ell(v - v_0)$ are divisible by u^{ℓ}, since the corresponding differences before Frobenius are divisible by u. Expanding the difference of the unit powers gives $\ell(\alpha - \beta) \in u^{\ell} M(A_{\mathrm{cris}})$ and likewise for α', β'. Starting with $P_0 = P_0' = I$, iterate

$$P_{j+1} = \alpha \phi(P_j) \beta', \qquad P'_{j+1} = \beta \phi(P'_j) \alpha'.$$

Induction gives $P_j P'_j = P'_j P_j = \ell^{4Nj} I$ and

$$\ell(P_{j+1} - \ell^{2N} P_j) \in u^{\ell^{j+1}} M(A_{\mathrm{cris}}), \qquad \ell(P'_{j+1} - \ell^{2N} P'_j) \in u^{\ell^{j+1}} M(A_{\mathrm{cris}}). \tag{A.603}$$

For the first recurrence start with $\ell(\alpha - \beta)\beta'$, then apply ϕ and the same two end multipliers at each step. The second starts with $\ell(\beta - \alpha)\alpha'$. By (A.602), $\ell^{-2Ni} P_{n_0+i}$ and its primed version are integral and converge ℓ-adically. Multiply both limits by ℓ^{n_0} to obtain Q, Q' with

$$QQ' = Q'Q = \ell^{2k_0} I, \qquad Q\beta = \alpha\phi(Q), \qquad Q, Q' \equiv \ell^{k_0} I \pmod{u}, \qquad k_0 = (2N+1)n_0. \tag{A.604}$$

For the congruences modulo u, the scaled differences in (A.603) are divisible by u with quotients tending to zero by (A.602). The initial term $\ell^{n_0} P_{n_0}$ is congruent to $\ell^{k_0} I$ by the finite telescoping sum of (A.603), and likewise for the primed term. The Frobenius relation follows from the recurrence by passage to the limit and cancellation of ℓ^{2N}.
Lean: `GlobalTheory.AcrisFrobE.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AcrisTorsionFreeCompleteFrobeniusProperties`, lines 94–143)

Write $\epsilon'_g = \phi(\varepsilon_g)$ in A_{cris}. Compatibility of Galois and Frobenius on T gives $\epsilon'_g g(\alpha) = \alpha\phi(\epsilon'_g)$. The constants β, β' are fixed by G_K. For $D_g = \epsilon'_g g(Q) - Q$, (A.604) therefore gives $D_g \beta = \alpha\phi(D_g)$ and $\ell^{2N} D_g = \alpha\phi(D_g)\beta'$. The cocycle condition and (A.604) give $D_g \in uM(A_{\mathrm{cris}})$. Iteration puts $\ell^{2Nj} D_g$ in $u^{\ell^j} M(A_{\mathrm{cris}})$. By (A.602), cancellation of ℓ^{2Nj} makes D_g divisible by arbitrarily high powers of ℓ. Separatedness gives $D_g = 0$.
Lean: `UpperTheory.Gal.ClosedPhiFromClosedTrace.invQ_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ClosedPhiFunctoriality.InvQHolds`, lines 16–121)

For each frame use Y, Y' for its integral matrices in (A.598) and the limit construction, and put

$$t = \phi(\tau), \qquad Z = \phi(Y)Q, \qquad Z' = Q'\phi(Y'), \qquad b = t^{2N}\ell^{2k_0}, \qquad U = t^{-N}Z.$$

Then $ZZ' = Z'Z = bI$. Since t and ℓ are invertible in B_{cris}, U is a basis there. The equation $D_g = 0$ gives $\rho_0(g)g(U) = U$ in the limit and the corresponding equation for each layer. Hence the rational matrix lattice of the limit is crystalline. Its left action contains every irreducible constituent of r after extending coefficients, so r_w is crystalline by stability under scalar extension and subobjects, and hence de Rham.

By (A.600) and (A.597), the Frobenius matrices satisfy

$$A\phi(R) \equiv RA_m, \qquad A'R \equiv \phi(R)A'_m \pmod{\ell^n}.$$

For the second congruence multiply the first by the inverse numerators and cancel E^{2N} in $\mathcal{W}$, where E has nonzero reduction and is a unit. The congruences descend to $\mathfrak{S}$ by its ℓ-saturation in $\mathcal{W}$. Put $H = f_A(R)$ and $H_0 = f_0(R)$. The two congruences give $\alpha\phi(H) \equiv H\alpha_m$ and $\beta' H_0 \equiv \phi(H_0)\beta'_m$. Consequently $D_j = HP_{m,j} - P_j H_0$ satisfies $D_{j+1} \equiv \alpha\phi(D_j)\beta'_m \pmod{\ell^n}$, with $D_0 = H - H_0$ divisible by u^ℓ. Thus D_j is divisible by $u^{\ell^{j+1}}$ modulo ℓ^n. Given L, choose j large enough that the scaled finite iterates approximate Q, Q_m modulo ℓ^L by (A.603) and (A.602). Take n at least L plus the denominator exponent of this scaling and use (A.600) at that precision. The bound on D_j gives $HQ_m \equiv QH_0 \pmod{\ell^L}$. Applying ϕ to $\Pi_m Y_m \equiv YR$ then gives

$$\Pi_m Z_m \equiv ZH_0 \pmod{\ell^L}. \tag{A.605}$$

Here $m \geq L$ and H_0 has coefficients in $W(k)$.

Lean: `UpperTheory.Gal.ClosedPhiFromClosedTrace.linkZ_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ClosedPhiFunctoriality.LinkZHolds`, lines 10–40)

Let n_1, n_2 be the negative and positive parts of $[k : \mathbf{F}_\ell] \deg \sigma$. On an unramified crystalline parameter the Weil operator is the action of $\phi^{n_1 - n_2}$ on crystalline invariants, with the degree convention of 39.6. For each frame define

$$V = \phi^{n_2}(b)\phi^{n_1}(Z)\phi^{n_2}(Z'), \qquad s = \phi^{n_2}(t)^{3N}\ell^{4k_0}\phi^{n_1}(t)^N. \tag{A.606}$$

The identity $Z'Z = bI$ gives $V\phi^{n_2}(U) = s\phi^{n_1}(U)$. This represents the required operator on the invariant columns after base extension. The power ϕ^{n_2} is invertible on the invariant module and both powers fix its unramified coefficient field. Each factor of s is invertible in B_{cris} and all layers use the same s.

Both powers of Frobenius fix H_0 and Π_m. Apply them to (A.605). Multiply the congruence for n_2 on the left by $\phi^{n_2}(Z')$ and on the right by $\phi^{n_2}(Z'_m)$ to get

$$\phi^{n_2}(b)\phi^{n_2}(Z')\Pi_m \equiv \phi^{n_2}(b)H_0\phi^{n_2}(Z'_m) \pmod{\ell^L}.$$

Use the congruence for n_1 on the right after left multiplication by $\phi^{n_1}(Z)$ to obtain $V\Pi_m \equiv \Pi_m V_m$. The identity column congruence holds by construction of Π_m. This proves (39.5). □

Proof of 39.48

Conclusion of the proof of Proposition 39.48. Keep the layer lattices, frames and comparison maps from the first part of the proof. Fix an induced embedding η and transport the coefficient traces to $\mathbf{C}_\ell$ by an embedding that identifies this copy of E_w with the chosen local copy K. This transport preserves the absolute value. Continue to write θ for reduction from B^+_{dR} and E for the evaluated Eisenstein polynomial, which generates $\ker(\theta|_{A_{\mathrm{inf}}})$.

Take the element $g \in G_K$ constructed in the proof of Lemma 39.46 for this uniformizer and its compatible roots. Put $w_g = g(E)/E$ and $z = \theta(w_g)^{-1}$. Then z has infinite order. The same g, z apply to the limit and every layer.

Lean: `GlobalTheory.BKLim.IsZ` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.TrivIsCr`, lines 182–183)

Lean: `GlobalTheory.BKLim.exists_isZ_not_root` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsGaloisElemCyclotomicPeriodNotRootOfUnity`, lines 108–162)

For any of these frames, write $T, Y, \varepsilon_g, \rho$ for its period matrix, integral numerator, cocycle and lattice action. Choose an adapted de Rham basis with invariant matrix $V \operatorname{diag}(E^{p_i - q})$ in B_{dR}, where $V \in \mathrm{GL}_d(B^+_{\mathrm{dR}})$ and $p_i - q$ are the jumps of this lattice representation over K. Here d is its rank and p_i, q are nonnegative

integers. Put $V_0 = \tau^N Y^{-1} V$ over B_{dR}^+. Its reduction is fixed entrywise by G_K. To see this, the de Rham comparison for the frame gives an invariant basis $(E\tau)^{-N} Y A B$ with $B \in \mathrm{GL}_d(B_{\mathrm{dR}}^+)$ admitting $\mathfrak{S}$-approximations modulo each power of E after clearing a power of ℓ. For each precision take a matrix β from the de Rham congruences in the proof of Lemma 39.47 and divide it by its normalizing factor ℓ^k. The resulting matrices reduce to the identity modulo E. Their classes agree at every common precision, since the leading coefficient of a difference of order $j \geq 1$ would lie in K and be fixed by multiplication by $\theta(w_g)^j \neq 1$. Their limit gives B, and the Frobenius cocycle relation gives invariance of $(E\tau)^{-N} Y A B$. The two invariant bases thus differ by a matrix $D \in \mathrm{GL}_d(K)$, so

$$E^q A B D = E^N V_0 \,\mathrm{diag}(E^{p_i}).$$

For each column i, approximate the left side modulo E^{N+p_i+1} by $\mathfrak{S}$ after multiplying by a power of ℓ. For the entries of D, use the section of K modulo powers of E from the invariant lift construction in 39.47. The $\mathfrak{S}$ approximant is divisible by E^{N+p_i} by Weierstrass division. Divide and reduce by θ. This puts the entries of $\theta(V_0)$ in K.

Lean: `GlobalTheory.BKLim.InvBasis.exists_invariant_basis` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsInvariantBasisDeRhamBreuilKisin`, lines 88–158)

Put $y = \theta(Y)$, $v = \theta(V)$, $v_0 = \theta(V_0)$, $a_\tau = \theta(\tau)$ and $b_\tau = \theta(g(\tau))$. Use $g(y)$ and $g(v)$ for the entrywise action on these reductions. From $Y V_0 = \tau^N V$ and its Galois conjugate,

$$y v_0 = a_\tau^N v, \qquad g(y) v_0 = b_\tau^N g(v).$$

The cocycle identity on $T = \tau^{-N} Y$ gives

$$a_\tau^N \rho(g) g(y) = b_\tau^N y \theta(\varepsilon_g).$$

The invariance of $V \,\mathrm{diag}(E^{p_i - q})$ gives, after cancellation of the column powers of E and reduction, $\rho(g) g(v) = v \,\mathrm{diag}(z^{p_i - q})$. Multiply the cocycle identity on the right by v_0 and substitute $g(y) v_0 = b_\tau^N g(v)$. Cancel the nonzero factor b_τ^N. The remaining equality, together with $y v_0 = a_\tau^N v$, gives

$$U y = y \theta(\varepsilon_g), \qquad U v = v \,\mathrm{diag}(z^{p_i - q}), \qquad U = y \theta(\varepsilon_g) y^{-1}.$$

Lean: `GlobalTheory.BKLim.adaptedBasis_theta_and_poles` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdaptedBasisThetaGaloisFixedPoles`, lines 33–151)

Write U_0, U_m for these operators on the limit and the layers, and set $d_* = |\theta(\tau^{2N})|_\ell^{-1} \geq 1$. Use the entrywise maximum norm, which is submultiplicative over an ultrametric field. By (A.598) and the limit construction the norm of each reduced inverse of Y is at most d_*. Each reduced Y and cocycle has norm at most one. Put $y_0 = \theta(Y)$ for the limit, $y_m = \theta(Y_m)$, and $R_\theta = \theta(R)$ for the link (A.600). Both

$$\Delta = \Pi_m y_m - y_0 R_\theta, \qquad \Delta' = \theta(\varepsilon_g) R_\theta - R_\theta \theta(\varepsilon_{m,g})$$

have norm at most ℓ^{-n}, by the link and (A.601), since $\theta(g(R)) = \theta(R)$. The identity

$$(U_0 \Pi_m - \Pi_m U_m) y_m = U_0 \Delta + y_0 \Delta' - \Delta \theta(\varepsilon_{m,g})$$

and the inverse bound give, for the linked $m \geq n$,

$$\|U_0\|,\ \|U_m\| \leq d_*, \qquad \|U_0 \Pi_m - \Pi_m U_m\| \leq d_*^2 \ell^{-n}. \tag{A.607}$$

Lean: `GlobalTheory.HTBound.bounded_compat` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ThetaConjBoundedCompatibleBreuilKisinFrames`, lines 117–228)

Telescoping gives $\|U_0^j \Pi_m - \Pi_m U_m^j\| \leq d_*^{j+1} \ell^{-n}$ for $j \geq 1$. The matrix readings into the seed and limit matrix spaces are equivariant. After coefficient extension, functoriality of the filtered comparison sends each Hodge–Tate summand to the summand of the same jump. The operator on that summand is multiplication by z^h for jump h, so the readings intertwine the operators. Evaluation at e_m and then taking the trace in seed component k thus gives $\sum_{h \in J_\eta} z^{jh}$, and the corresponding evaluation at e gives the power sum for the actual jumps of r_w at η. Use $\Pi_m e_m \equiv e$, (A.607), and the first bound of (A.596) on the columns of $U_m^j e_m$. For fixed j, the error from the projected column is bounded by $d_*^{j+1} \ell^{-n} \max_i |t_i|_\ell$, and the weighted trace error on the

layer column is at most $d_*^j \ell^{-m}$. At the identity the first inequality of (39.4) gives $|a(1-\sum_k b_k)|_\ell \le \ell^{-m}$. Since $a > 0$, the sum of the weights tends to one. Thus the power sums on r are $\sum_{h\in J_\eta} z^{jh}$. Newton's identities and the injectivity of $h \mapsto z^h$ recover the multiset J_η with multiplicities.
Lean: `GlobalTheory.CovClose.closed_of_cover` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ClosedOfCover`, lines 159–300)

Fix a power σ^j. Apply 39.47 to it with target $\vartheta = S_j$ and write p for its transported trace on r_w and p_k for that on seed k. Let $K^{\mathrm{ur}} \subset \bar{\mathbf{Q}}_\ell$ be the maximal unramified algebraic extension of $\mathbf{Q}_\ell$, with period embedding $u_{\mathrm{ur}} : K^{\mathrm{ur}} \to B_{\mathrm{cris}}$ from the compatible data. For a K^{ur}-linear functional $\lambda : \bar{\mathbf{Q}}_\ell \to K^{\mathrm{ur}}$, the comparison (A.606) gives

$$\sum_i (Ve)_i \, u_{\mathrm{ur}}(\lambda(t_i)) = s \, u_{\mathrm{ur}}(\lambda(p)). \tag{A.608}$$

After coefficient extension, the matrix reading intertwines the period operators. Apply the operator to the identity matrix column and take its ordinary matrix trace. This gives the Weil trace, since the reading space has the left multiplication action. Applying the coefficient functional λ and the period embedding gives (A.608). The same calculation on the layer with $V_m e_m, t'_{hk}, p_k$ holds in each component. Apply it there with $x \mapsto \lambda(b_k x)$ and sum over k.

For the layer at precision L furnished by (39.5), write $\Pi = \Pi_m$ and put

$$\delta_h = \sum_i \Pi_{ih} t_i - \sum_k b_k t'_{hk}, \qquad \eta_0 = \vartheta - \sum_k b_k p_k.$$

Subtract the summed layer identities from (A.608) to obtain

$$s \, u_{\mathrm{ur}}(\lambda(p-\vartheta)) = \sum_i (Ve - \Pi V_m e_m)_i u_{\mathrm{ur}}(\lambda(t_i)) + \sum_h (V_m e_m)_h u_{\mathrm{ur}}(\lambda(\delta_h)) - s \, u_{\mathrm{ur}}(\lambda(\eta_0)). \tag{A.609}$$

The first column on the right is divisible by ℓ^L by (39.5). By (A.596), δ_h/ℓ^L and η_0/ℓ^L are integral since $m \ge L$.

If $p \ne \vartheta$, choose integers $c_1, c_2 \ge 0$ so that $\ell^{c_1} t_i$ and $x = \ell^{c_1}(p-\vartheta)$ are integral and $|x|_\ell \ge \ell^{-c_2}$. A finite extension of K^{ur} containing these numbers and the divided errors admits an integral basis over the unramified valuation ring. Take an integral basis of a finite local field containing the listed numbers over its maximal unramified subfield and extend scalars. Choose the coordinate of largest norm in the expansion of x, and extend this projection linearly to $\bar{\mathbf{Q}}_\ell$. The chosen λ takes all the listed integral values to integral elements of K^{ur}, and $q_L = \lambda(x)$ satisfies $|q_L|_\ell \ge |x|_\ell$, by the ultrametric inequality on the integral basis. Integral elements of K^{ur} map to A_{cris}. Multiplying (A.609) by ℓ^{c_1} gives $s \, u_{\mathrm{ur}}(q_L) \in \ell^L A_{\mathrm{cris}}$. Since ℓ^{c_2}/q_L is integral, multiplication by its period image gives $\ell^{c_2} s \in \ell^L A_{\mathrm{cris}}$ for every L. Separatedness and ℓ-torsion freeness give $s = 0$, contrary to (A.606). Thus $p = \vartheta$ for each j. □

Proof of 39.49

Proof. Take a split place w_* with both places of its pair outside S. Let J be the group of finite unit idèles with w_* and $\bar{w}_*$ coordinates one. Choose an open subgroup $V \subset \mathcal{O}_{E_{w_*}}^\times$ whose intersection with the global roots of unity is $\{1\}$. Let D be the open subgroup of idèles with finite unit components for which the quotient of the w_* component by the conjugate $\bar{w}_*$ component lies in V. On D prescribe the given infinity character, trivial on finite components. On the product $E^\times \mathbf{A}_F^\times J$, prescribe $\eta_{E/F}(b)$ on kbu with $k \in E^\times$, $b \in \mathbf{A}_F^\times$ and $u \in J$.

If $x = kbu \in D$, then $k/c(k)$ is a unit at every finite place, by taking the quotients $x/c(x)$ and $u/c(u)$. Its complex absolute values are all one, so it is a root of unity. Its component at w_* belongs to V, hence $k = c(k) \in F^\times$. Now kb has finite unit components. Its quadratic character value is the product of its real signs, which agrees with the infinity prescription since all e_v are odd. Also $\eta_{E/F}(k) = 1$. The two prescriptions agree on the intersection and define a unitary character on the product with D. Extend it to the idèle group by divisibility of the unit circle. Agreement on the open subgroup D gives continuity. The extension is trivial on principal idèles and on J, hence unramified on S. At a nonsplit place the units lie in J and a uniformizer from F is a uniformizer of E_w with quadratic value -1. □

Proof of 39.50

Proof. Choose one representative of each bad split pair and one further good split place w_0. At each bad representative the indicator of an open subgroup of the standard maximal compact fixing a nonzero vector has positive trace. At w_0, the Iwahori Hecke operator of the decreasing translation $(0, -1, \dots, 1-n)$ is invertible on the nonzero Iwahori invariants. If the dimension is d, vanishing of its first d power traces would give determinant zero by Newton's identities. Choose a positive power of nonzero trace. Translation lengths add in this chamber, so the power is a nonzero multiple of the indicator of the double coset with multiplied exponents. The Newton polygon gives distinct root valuations on this coset.
Lean: `UpperTheory.V.Z.Tf.GlobalTraceComparison.Asm.rChoice` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.TraceFormula.GlobalTraceComparison.TraceAsymptotics.RChoice`, lines 13–74)

Use the infinite weight of the twisted character of π, the split normalized constant term and the nonsplit hyperspecial matching of 26.5. For a split block (a, b) the multiplier of the Levi element (m_1, m_2) is $\mu_w((-1)^{ab} \det(m_1)^b \det(m_2)^a)$. This gives the twisted Satake union. At w_0, a nonzero constant-term integrand comes from a block triangular matrix in the regular support of f_{w_0}, so its product polynomial is separable.
Lean: `UpperTheory.ConvolutionTheory.G2.matchSplit` (`Development.AutomorphicArithmetic.Foundations.Convolution.WeightRankUnits.MatchSplit`, lines 12–21)
Lean: `PositivityTheory.Rg.Bc.ArchbcB.B.archimedean_base_change_first` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.IwahoriFactorisation.MultiplicityOneBase.ArchimedeanBaseChangeFirst`, lines 29–61)

The definite space at the fixed weight and level is finite-dimensional. At a good split pair the Satake multisets are mutually inverse and the spherical fixed space is a line. The nonsplit integral levels select unramified summands. Factoring the trace gives the unitary sum. The fixed tests contribute A_h.

Fix the level and the archimedean compact types from the tests at Z and at infinity. The tests at the varying places will be spherical at this same level. By Langlands's spectral decomposition of $L^2(\mathrm{GL}_d(E)\backslash \mathrm{GL}_d(\mathbf{A}_E))$ and the Moeglin–Waldspurger classification of the discrete spectrum of general linear groups as residual Speh representations, every spectral character in the prescribed archimedean fibre arises from a parabolic induction of normalized discrete-spectrum factors twisted by $|\det|^{it_j}$, where each Speh factor is supported on a balanced arithmetic progression of shifts of a normalized cuspidal representation. For a cuspidal block of rank $d > 0$ with prescribed infinitesimal multisets m_τ, normalization on the positive split centre leaves a twist $|\det|^{z_0}$ with $z_0 = (d[E : \mathbf{Q}])^{-1} \sum_\tau \sum_{x \in m_\tau} x$. For a sequence approaching these multisets the twists converge to z_0, by the same formula. Every normalized exponent on its tail has modulus at most $2 + |z_0| + \max_{\tau, x \in m_\tau} |x|$. Lemma 26.16 applies at the fixed open level with these compact types and this bound. Its finite list contains a constant subsequence of normalized cuspidal representations. Restoring the twist gives convergence of their spherical coordinates at each place outside the level. On the orthogonal complement of π and its split-centre translates, omit the normalized class of π. For a noncuspidal spectral constituent (residual Speh or continuous Eisenstein spectrum), parabolic induction decomposes the constituent into finitely many normalized discrete-spectrum factors twisted by $|\det|^{it_j}$ (equivalently, taking a nonzero proper constant term of ranks r, s for continuous spectrum, or factoring a Speh representation into its underlying normalized cuspidal segment): there are finitely many choices of $r + s = d$, segment lengths, and partitioned submultisets at each embedding, and the induced shifts fix both the lower limiting infinitesimal multisets and the finitely many possible twists $|\det|^{it_j}$ before applying Lemma 26.16. The lower levels are fixed as well: in the compact realization take the intersections of the Levi with the conjugates of the given level for representatives of the finite quotient of the maximal compact by that level. The restrictions of the finitely many archimedean compact types give finitely many lower types. Induction on rank gives a finite union of products of the two lower lists. Rank zero contributes the empty list. Taking the product for the two original blocks completes the construction before choosing T or its polynomials.
Lean: `UpperTheory.V.Z.Tf.GaloisFromTraceFormula.cuspReadI_of` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.TraceFormula.GaloisFromTraceFormula.CuspReadIOf`, lines 63–74)

Let X be the operator of the varying Hecke test and A_0 the reference wave convolution with the finite level idempotent, normalized to have value one on the prescribed fibre. Work on the block space to which the competing list applies. The fibre in the character space of the commutative operator algebra is compact. Choose $0 < \rho' < \rho$ so that $|X| < \rho'$ on it. On the compact set $|X| \geq \rho'$, finitely many differences d_j of archimedean generators from their prescribed values have no common zero. Include $A_0 - 1$ among these differences. Put $D = \sum_j d_j^* d_j$ and $g = (1 - D/M)^h$ with $M > \|D\|$. By compactness, choose $0 < \eta < M$ so

that $D \geq \eta$ wherever $|X| \geq \rho'$ and $\rho'(1+\sqrt{\eta}) < \rho$. Where $D < \eta$, the bounds $|X| < \rho'$, $|A_0| \leq 1+\sqrt{\eta}$ and $|g| \leq 1$ apply. Where $D \geq \eta$, use $|XgA_0| \leq \|X\|\|A_0\|(1-\eta/M)^h$. Thus for sufficiently large h, $U = XgA_0$ has norm less than ρ on the competing space.

Expand gA_0 as a finite linear combination of words in the differentiated reference wave kernels and their adjoints. If the constant coefficient in the word expansion of g is a, its contribution to gA_0 is aA_0. Each word has the same finite level idempotent, by idempotence, and a compactly supported archimedean wave kernel by 26.12 and the convolution and adjoint rules of 26.14. The weighted sum of their archimedean values at the prescribed parameters is $gA_0 = 1$ there. Convolving with X gives a finite sum for U with the varying finite Hecke kernel common to all summands. Fix a support bound for these finitely many kernels.
Lean: `UpperTheory.Zero.ConvolutionTheory.EB.rSpec_of` (`Development.AutomorphicArithmetic.Foundations.Convolution.EndomorphismSlices.REndoOf`, lines 24–47)

Take T_1 from the fixed tests with the variable polynomials set to one, and T_2 from the reference level kernel with the fixed rational translations of the orbital comparison. Expand $T_1U^kT_2$ by convolution. At each multiplication the level idempotents act as units on the fixed finite kernels, and the finite transfer multiplies the spherical Hecke polynomials. Thus every summand has the same finite test of f_k in the translated coordinates. The archimedean values multiply under convolution, so their weighted sum at the prescribed parameters is one. The finite support bound for U, the fixed end supports and subadditivity give a compactly supported continuous kernel with logarithmic support size at most $r_0 + kr_1$ for fixed $r_0, r_1 \geq 0$, enlarging them for the fixed translations. The common regular test at w_0 and the very cuspidal wave kernels give the slice hypotheses of 26.11. Apply the finite matching of 26.5 and the archimedean orbital integral of 26.12 to each summand. By 26.13, its geometric term is the fixed Haar factor times its archimedean value times $S_i(k)$. Summation gives that same Haar factor times $S_i(k)$. Apply 26.11 on the full space of the proper block to obtain (39.6).

At the distinguished end let $\operatorname{ldet} x$ be the adelic log determinant modulus, and let Q_π be the closed span of $\psi(x)u(\operatorname{ldet} x)$ with ψ in the normalized π and u continuous of compact support. Both Q_π and its orthogonal complement are preserved by convolution and its adjoint. Use spectral separation on $Q_\pi^\perp$. On Q_π the full twisted diagonal contributes half the twisted trace of the normalized cusp form. The involution reverses ldet, so a central test b of integral one contributes $\int_{\mathbf{R}} b(-2s-c')\,ds = 1/2$ for any shift c'. The archimedean weight trace is nonzero for this conjugate self-dual π. The compact conjugation average preserves each split trace. With c_0 the fixed nonzero weight and volume factor and κ the conversion from stable term to elliptic orbital sum, the contribution is $Pv(B)^k$ with $c = c_0/(2\kappa)$.

Put $R_k = (1+k)(1+r_0+r_1)$ and use (A.213) with $\tau = C'R_k$, where C' is a sufficiently large fixed constant. Insert the orthogonal projection onto $Q_\pi^\perp$ in the middle of the row and column estimate (A.218). By (A.219), the cut integral on this summand is bounded by $C(1+k)^N\|U^k|_{Q_\pi^\perp}\|$. Both projections commute with bounded functions of ldet, by the description of Q_π. The untwisted kernel has log determinant difference bounded by cR_k, for a fixed $c > 0$, and the projections preserve that support. The twist reverses the second log determinant. Hence the projected twisted diagonal is supported in $|\operatorname{ldet} x| \leq cR_k$.

Write $\operatorname{red}(x)$ for the infimum of the logarithmic size of γx over rational translates, as in Lemma 26.11. Let $a_c(s)$ be the positive split central element of log determinant s, with scalar $\exp(s/(n[E:\mathbf{Q}]))$ at each infinite place and finite components one. The determinant-one coordinate has reduced logarithmic size

$$h(x) = \operatorname{red}\big(a_c(-\operatorname{ldet} x)x\big).$$

The logarithmic size of $a_c(s)$ is bounded by a constant times $|s|$. Since $a_c(s)$ commutes with rational translates, $\operatorname{red}(x) \leq h(x) + c(1+|\operatorname{ldet} x|)$ after increasing c. The degree bounds in 26.11 give, for a fixed $c_1 > 0$,

$$c_1(1+\operatorname{red}(x)) \leq \tau \quad\Longrightarrow\quad x \in \operatorname{Cut}(\tau). \tag{A.610}$$

Indeed, a representative within one of the infimum bounds the degree of each rational subspace from above and the total degree from below. On the determinant band, $h(x) < R_k$ would give $1+\operatorname{red}(x) \leq (2+c+c^2)R_k$. Take $C' \geq c_1(2+c+c^2)$ as well as large enough for the cut identity. Then $h(x) \geq R_k$ on the part of the band outside this same cut.
Lean: `UpperTheory.V.Z.Tf.GlobalTraceComparison.Pi.r0` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.TraceFormula.GlobalTraceComparison.AutomorphicRepresentations.EvalBasis`, lines 21–21)

On Q_π expand the left end kernel on the determinant-one quotient in a finite basis of cusp forms of the fixed level and type. Its coefficient kernels are evaluations at fixed points translated by $a_c(\operatorname{ldet} x)$.

In the row and column count of 26.11, these translated rows have bounded norms, since the split centre commutes with conjugation by rational elements. Thus the absolute diagonal is bounded by a constant times $\|U^k\| n_\Omega(x)$ times the sum of the absolute values of the basis forms at the determinant-one coordinate. Rapid decay bounds each basis form there by $C_L e^{-(L+1)h(x)}$ [27, Thm. A]. On the determinant band, $H = 1 + \mathrm{red}(x) \ll (1 + cR_k)(1 + h(x))$. Use the integrable weight $H^{-4n} n_\Omega$ from 26.11 and the boundedness of $(1+h)^{4n} e^{-h}$ for $h \geq 0$. The absolute integral of the cuspidal diagonal outside the cut is at most

$$C_L (1 + cR_k)^{4n} e^{-LR_k} \|U^k\|. \tag{A.611}$$

Set $\eta = \rho/(1 + \|U\|)$ and $L = 1 + \max(0, -\log \eta)$. Since $R_k \geq 1 + k$, the factor e^{-R_k} absorbs the polynomial, and $e^{-(L-1)R_k} \leq \eta^k$. Thus (A.611) is at most $C_\eta \eta^k \|U^k\| \leq C_\eta \rho^k$. Add the bound on the cut integral on $Q_\pi^\perp$, using $\|U|_{Q_\pi^\perp}\| < \rho$, and divide by the fixed Haar conversion factor. This gives (39.7).

Lean: `UpperTheory.V.Z.Tf.GlobalTraceComparison.Pi.growthPiJ_holds` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.TraceFormula.GlobalTraceComparison.AutomorphicRepresentations.GrowthPiJHolds`, lines 52–54)

For a proper block impose inversion invariance of each of $X_1(w)$ and $X_2(w)$ at nonsplit places. For the $(n, 0)$ end impose this where $B(w) = B(w)^{-1}$. For each competitor failing the applicable condition choose one nonsplit witness. An elementary symmetric difference d between its multiset and its inverse distinguishes the two, after clearing determinants. The polynomial $1 - d/d(X)$ at the witness kills that entry and equals one on the imposed locus. Conjugate inverse transpose interchanges the terms of the difference, so the modification preserves the twisted orbital matching. Insert the finitely many modifications before varying the split tests. On the distinguished spherical line each inserted factor is one by $\pi^c \simeq \pi^\vee$. At nonsplit places the twisting values $\mu_w(\varpi_w) = -1$ preserve inversion invariance. The weight equality of 26.5 gives regularity of the shifted infinitesimal characters. At the ends the rank-zero polynomial and character are one and the end unipotent radical is trivial. Interchanging the factors identifies the two stable end terms.

Lean: `UpperTheory.V.Z.Tf.GaloisFromTraceFormula.endoS` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.TraceFormula.GaloisFromTraceFormula.EndoS`, lines 10–44)

Lean: `UpperTheory.V.Z.Tf.GaloisFromTraceFormula.stableS` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.TraceFormula.GaloisFromTraceFormula.StableS`, lines 10–55)

□

Proof of 39.51

Proof. Take μ by 39.49 with $S = \varnothing$ and use the lists of 39.50. Write n for the rank of π. Suppose neither a unitary summand nor a competing cuspidal list agrees with B at almost every split place. Each proper-block factor has rank smaller than n, so its twisted list omits π. For each competitor X choose a split witness w outside $Z \cup \bar{Z}$ with $X(w) \neq B(w)$. A separating elementary symmetric coordinate H gives the test factor

$$\frac{H - H(X(w))}{H(B(w)) - H(X(w))}.$$

For a witness on the conjugate side of the section use the inverted multisets in the factor for $e_{2,\bar{w}}$. Multiplying the factors gives $v(B) = 1$ and $v(X) = 0$ for each competitor.

Lean: `UpperTheory.V.Z.Tf.GlobalTraceComparison.Asm.rSep` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.TraceFormula.GlobalTraceComparison.TraceAsymptotics.ConstantTermParabolicAverage`, lines 82–112)

For $k \geq 1$ the unitary trace vanishes. Apply the signed identity 26.5 and (39.6) and (39.7) with $\rho = 1/2$. Each proper-block term tends to zero. For $(0, n)$ the selected minor is empty at every real place. For $(n, 0)$ it uses the indices $0, \ldots, n-1$. In the sign of Lemma 26.5 the exponent is respectively 0 and $n(n-1)$, so both signs are $+1$. Each ordered end occurs once. By Lemma 39.50 their unsigned terms are equal. In its notation each has limiting contribution $P = c_0 (2\kappa)^{-1} \prod_{w \in Z} \mathrm{tr}\, \pi_w(f_w)$, since $v(B) = 1$. The factor $1/2$ comes from integrating the central test under $s \mapsto -2s - c'$. Thus the right side tends to $2P \neq 0$, whereas the left side is zero for every $k \geq 1$.

Lean: `UpperTheory.ConvolutionTheory.G2.uSide` (`Development.AutomorphicArithmetic.Foundations.Convolution.WeightRankUnits.USide`, lines 9–48)

Suppose a competing cuspidal list matches. Write a rank-d factor as a normalized cusp form twisted by $|\det|^z$. The normalized exponents sum to zero over all embeddings. The shifted exponents are real, so $d[E : \mathbf{Q}] \operatorname{Im} z = 0$. At conjugate embeddings the normalized exponents pair as $x, -\bar{x}$. Since the shifted

exponents are real and both $x+z$ and $-x+z$ belong to $(n-1)/2+\mathbf{Z}$, one has $2z\in\mathbf{Z}$. At a matching split place the geometric mean of this factor's root moduli is q_w^{-z} by unitarity of its normalized central character. Conjugate self-duality gives unitarity of π. The Jacquet–Shalika bound puts the root moduli of $B(w)$ in $(q_w^{-1/2},q_w^{1/2})$ [13]. Thus $|z|<1/2$ and $z=0$.

Lean: `UpperTheory.V.Z.Tf.GaloisFromTraceFormula.partNorm` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.TraceFormula.GaloisFromTraceFormula.PartNorm`, lines 166–197)

Let j be the number of factors and set $M=16(j+1)^2$. The twists $\pi\otimes\mu^{2i}$, $0\le i<M$, are distinct unitary cusp forms because their sums of infinitesimal exponents differ. At nonsplit places $\mu_w^2=1$. Omit the finitely many exceptional places of these systems and of the factors. For a unitary cusp form σ use coordinates $p_{w,r}(\sigma)=\sum_{\alpha\in B_\sigma(w)}\alpha^r$. Give them inner product weights $q_w^{-rs}/[r(-\log(s-1))]$ for $r\ge1$ and real $s>1$ near one. The partial Rankin–Selberg product has a simple pole for equivalent factors and is regular and nonzero at one for inequivalent factors. Taking its logarithm gives

$$\lim_{s\downarrow1}\langle p(\sigma),p(\sigma')\rangle_s=\begin{cases}1 & \sigma\simeq\sigma',\\ 0 & \sigma\not\simeq\sigma'.\end{cases}\tag{A.612}$$

Here the second coordinate is conjugated in the inner product [13].

Lean: `UpperTheory.V.Z.Tf.SR.antiTw` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.TraceFormula.SplitRigidity.AntiTw`, lines 30–60)

By (A.612) the average of the M twists has limiting squared norm $1/M$. Its nonsplit coordinates coincide with those of π, so the nonsplit squared norm of π has limit superior at most $1/M$. Pair π with the difference of its coordinates and the sum of the j factors' coordinates. If every factor differs from π, the limit is one by (A.612). The split coordinates of the difference vanish at the places retained. Cauchy–Schwarz on the nonsplit coordinates bounds the absolute pairing in the limit by $1/M+j/\sqrt{M}<1$. Thus a factor is π, contrary to its omission. The unitary match gives the assertion.

Lean: `UpperTheory.V.Z.Tf.SR.splitRigid` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.TraceFormula.SplitRigidity.SplitRigid`, lines 12–13)

Lean: `UpperTheory.Gal.Desc.weakAll_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.BaseChangeDescent.Strong`, lines 24–33)

□

Proof of 39.52

Proof. The character $\eta_{E/F}$ is trivial on norms. The polarization therefore gives $\pi^c\simeq\pi^\vee$. The extension E/F is unramified at the split finite places and is unramified at the remaining finite places by hypothesis. Lemma 39.51 gives an automorphic representation U of the definite unitary group which is spherical at every nonsplit finite place and matches π at almost every split place.

We apply [94, Remark 5.2 and Corollary 5.3] to U. The ground field F is totally real, E/F is totally imaginary quadratic, and the positive integer $[F:\mathbf{Q}]$ is even, hence at least two. The definite group is anisotropic modulo its centre, so U belongs to the discrete spectrum. At every real place the group is compact. Each irreducible archimedean component is consequently in the discrete series. To check Labesse's condition $(*)$, take Haar probability measure on the product of the real groups and use the complex conjugate of the character of U_∞ as pseudo-coefficient. Character orthogonality gives trace one on U_∞ and zero on every other irreducible archimedean representation. Thus every nonzero trace appearing in $(*)$ has the required sign.

Corollary 5.3 gives an isobaric transfer whose factors are discrete general linear representations. It gives local base change at every split place and compatibility of infinitesimal characters at infinity. By the discrete-spectrum theorem of [95, p. 606], applied over the number field E to each discrete factor with its unitary central character, a factor has the form $\mathrm{Speh}(\sigma,d)$ for a unitary cuspidal σ and $d\ge1$. Choose a split place of matching outside the ramification sets of all factors. Such places exist by Chebotarev. Write q_w for the residue cardinality. For every Satake root α of σ_w the root multiset of the transfer contains

$$\alpha q_w^{(d-1)/2-j},\qquad 0\le j<d.$$

The common multiset is that of π_w. Conjugate self-duality makes π unitary: the real modulus exponent of its central character is preserved by c and negated by duality, so it is zero. The local component is spherical

and generic by cuspidality. The Jacquet–Shalika bound for π and its contragredient puts every root modulus strictly between $q_w^{-1/2}$ and $q_w^{1/2}$ [13, Part I, Corollary 2.5]. If $d \geq 2$, the roots at $j = 0, 1$ have modulus ratio q_w, contradicting these strict inequalities. Thus every d equals one.

The transfer is now a sum of unitary cuspidal factors of total rank $n = \operatorname{rank} \pi$. Their Satake union agrees with that of π at almost every split place. Apply the split-place Rankin–Selberg argument of Lemma 39.51, equation (A.612), to this finite list. One factor is isomorphic to π. It has rank n, so it accounts for the entire transfer. The split local base change and the infinitesimal-character assertion of Corollary 5.3 therefore give the claimed matching, including at ramified split places. The representation U obtained above is spherical at all the nonsplit finite places, so $T = \varnothing$ works for every Z. □

Proof of 39.54

Proof. For quadratic polarization, first work over a field satisfying 39.52, the inert-place conditions of 39.63, and $x^2 = -d \in E$, $-d \in (\mathbf{Q}_\ell^\times)^2$, with three pairwise separated places above ℓ. Let Q be the usable places for d. Take U by 39.52. Degree-one Chebotarev gives dense inverse Frobenius from Q after finite removals. In odd rank apply 39.44. For a target above ℓ take two of the three auxiliary places separated from it and use their conjugates as needed to agree with the target on $\mathbf{Q}(x)$. The joint approximation in the proof of 39.44 gives 39.48 for the same representation, with the Satake power sums as targets. Newton's identities give the stated Weil polynomial. The cofinite Frobenius data and semisimplicity identify the representations for the different choices of auxiliary pair. In even rank use 39.67.

For descent, use two preliminary real quadratic steps to split the prescribed places over ℓ and obtain at least three separated places there. At either step one can adjoin the square root of a positive rational prime congruent to one modulo 8 and the odd rational primes below the places to be preserved. Choose it outside the primes ramified in the normal closure of the current field and outside the primes below its nonspherical places. The square condition gives splitting at the prescribed places and valuation one at the fresh prime gives ramification at a spherical place. Varying the fresh prime gives infinitely many choices. In the second step write $Y = Y_0(\delta)$ with Y_0 the preceding CM field and δ^2 the chosen prime. The normal closure of Y_0 is unramified at that prime, so it is disjoint from $\mathbf{Q}(\delta)$ over $\mathbf{Q}$. The sign change of δ therefore extends to an automorphism s of the algebraic closure fixing every conjugate of Y_0 pointwise.

Fix a finite set T of places of Y containing the target, all places above ℓ, and all places without Satake data. Take c to be a positive multiple of 6 divisible by the norms of the places in T and of the places whose inertia over Y^+ or $\mathbf{Q}$ acts nontrivially on Y. In a common normal closure containing all conjugates of Y and μ_{8c}, let c_∞ be complex conjugation. Choose unramified rational primes $p_1 < p_2$ with Frobenius representatives $c_\infty s$ and s^{-1}, respectively. Chebotarev permits both primes to be arbitrarily large. Each representative sends δ to $-\delta$ and its square fixes every conjugate of Y. For $c_\infty s$ this uses that c_∞ preserves the normal closure of Y_0 and fixes δ. These two properties hold for their conjugates as well. The decomposition group in the normal closure is generated by its Frobenius, so at either prime a decomposition automorphism fixing δ fixes Y. On μ_{8c} the product of the two Frobenius restrictions is inversion, which gives $p_1 p_2 \equiv -1 \bmod 8c$.
Lean: `UpperTheory.Gal.UnramifiedFromSplittingCriterion.twoPrimes` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.InertiaQuadraticRamification.TwoPrimes`, lines 37–89)

Put $d = p_1 p_2$ and adjoin $x = \sqrt{-d}$. The congruence makes $-d$ a square in $\mathbf{Q}_p$ for every $p \mid c$, including at 2 by the condition modulo 8. All places above T split in the extension of Y. They also split over the upper real subfield because their decomposition groups fix x.

At a nonsplit place of the upper CM field a decomposition automorphism over the upper real subfield moves x and Y. Indeed Y together with that real subfield generates the upper CM field. The place lies away from c by the square conditions. It lies away from $p_1 p_2$ since the automorphism fixes the real element δ, which at either of those primes would force it to fix Y. At the remaining places inertia fixes Y by the choice of c and fixes x since $2d$ is a unit. Hence the upper CM extension is unramified over its real subfield at the nonsplit places, and the real subfield is unramified over $\mathbf{Q}$ there. The residue characteristic is odd and the residue norm in the real subfield exceeds three. Each such place lies over a nonsplit place of Y, since the decomposition automorphism over the upper real subfield acts nontrivially over Y^+. The lower component is spherical by the choice of T. At a place of Y over p_1, $-d$ has valuation one. This supplies a fresh ramified spherical place for the quadratic lift. Including prior ramification witnesses in T gives infinitely many distinct choices.

Lean: UpperTheory.Gal.Int.Core.exists_quadratic_CM_ext_split_at (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.CrystallineDescent.ExistsQuadraticCMExtSplitAt, lines 12–156)

At each step, 39.53 gives cuspidal lifts. On the compositum of any two auxiliary fields, Chebotarev and the residue-degree rule identify the traces of the two upper representations. The same argument gives invariance under conjugation over the ground field. Apply the index-two patching lemma to the union of the auxiliary families for the targets at this step. Each target splits in infinitely many members, and every finite extension contains finitely many quadratic subfields. The patching lemma gives a continuous semisimple representation whose restriction is isomorphic to the chosen upper representation for all but finitely many members [222, §4.3, Lem. 1]. For each target choose a member splitting it outside this exceptional set. The completions agree there, so the local data descend. This gives weak attachment and the stated local data at every spherical target. Cofinite Frobenius polynomials and semisimplicity give uniqueness up to isomorphism.

For general polarization first use fields with an imaginary quadratic subfield and even real degree, with nonsplit finite places unramified and away from ℓ. Require sphericality at the nonsplit places. Let b be the parallel infinity exponent of χ on positive real idèles from Lemma 39.7, and put $\psi = \chi^{-1}\eta_{E/F}^{a+|b|}$. At each real place $\psi_v(x) = x^{-b}$ for $x \in \mathbf{R}^\times$. At nonsplit places ψ is trivial on units by sphericality and polarization, since the unramified norm is onto on units.

Choose an auxiliary split pair above a real-subfield place distinct from that below the specified split place w_0. On the subgroup D of idèles with arbitrary infinite components and finite unit components, impose deep congruence conditions at the auxiliary pair and, as needed, at other split places away from w_0. Keep all units at w_0 and at nonsplit places. Choose the levels so that ψ is trivial on the finite part of $D \cap \mathbf{A}_F^\times$ and so that the congruence group at the auxiliary pair meets the global roots of unity trivially. At the place below w_0 one may impose the conductor condition at its conjugate place. Prescribe $\prod_{v|\infty} z_v^{-b}$ on D, using one complex embedding above each real place, prescribe ψ on $\mathbf{A}_F^\times$, and prescribe the trivial character on $E^\times$.

For a relation $kbu = 1$ with $k \in E^\times$, $b \in \mathbf{A}_F^\times$ and $u \in D$, the ratio $k/c(k) = c(u)/u$ is a unit at every finite place and has absolute value one at every complex embedding. It is a root of unity in the auxiliary congruence group, hence equals one. Thus $k \in F^\times$ and $kb = u^{-1}$ belongs to $D \cap \mathbf{A}_F^\times$. The infinity prescription on this intersection agrees with ψ by its real exponents and signs, and the finite restriction is trivial. The prescriptions therefore give one character on the product subgroup. Extend it to the idèles by divisibility of $\mathbf{C}^\times$. Agreement on the open subgroup D gives continuity. The resulting algebraic Hecke character κ is unramified at w_0 and at nonsplit places. It satisfies $\kappa\kappa^c = \psi \circ N_{E/F} = (\chi \circ N_{E/F})^{-1}$.

Lean: GlobalTheory.HeckeUnrExt.exists_ext_unr (Development.AbelianVarietyTheory.Proofs.SatakeParameters.ExistsAlgebraicHeckeCharSplit, lines 27–150)

The twist $\pi \otimes \kappa$ has quadratic polarization. Twist its representation back by $(\kappa^{-1})_\ell$. At a spherical target above ℓ this character has a crystalline realization by the algebraic Hecke character comparison. Its Weil eigenvalue at degree -1 is the local unramified value under reciprocity and its jump is the negative of its infinity exponent at the inducing embedding. Tensoring cancels the Satake scaling and subtracts the jump shift of the original twist. For arbitrary fields use quadratic base change to fields with the required properties and descend at splitting places by the same patching lemma. □

Proof of 39.55

Proof. The real exponent of the central character divided by a is $t = (a - 1 - w_{\mathrm{gt}})/2$, since the sum of infinitesimal weights at an embedding and its conjugate is $a(a - 1 - w_{\mathrm{gt}})$. Thus $\pi|\det|^{-t}$ belongs to the unitary cuspidal spectrum and $\sigma^*|\det|^{w_{\mathrm{gt}}/2}$ is unitarizable and generic. The generic unitary dual has unitary discrete-series blocks with real twists e satisfying $|e| < 1/2$ [219]. A block of length at least two contributes adjacent ν_K-twists to the support. Unlinking forces length one, and the character hypothesis forces rank one. Removing the common twist gives (39.8). □

Proof of 39.56

Proof. Use multiplicativity of local γ-factors for the generic constituent of normalized parabolic induction [226, Thm. 3.1]. It applies with a character in the other variable. The inducing supercuspidals are generic, and the generic constituent with support S is the given representation. A supercuspidal of rank greater than

one paired with a character has both standard L-factors equal to one, as does a ramified character [226, §8][12, §2.5]. Its γ-factor is its epsilon monomial. For an unramified character of uniformizer value u, the quotient of the dual and direct L-factors in Tate's functional equation is

$$\frac{1-uT}{1-u^{-1}q^{-1}T^{-1}} = -uqT\,\frac{1-uT}{1-uqT}.$$

Multiplication over the support proves the formula, including multiplicities. It allows cancellation between the numerator and denominator for adjacent characters in a segment. □

Proof of 39.57

Proof. Put $s_0 = (1-a)/2$ and $\eta_j = \xi_j|\cdot|_K^{-s_0}$. These are the support characters before normalization. Let

$$U = \bigcap_j \ker(\eta_j|_{\mathcal{O}_K^\times}), \qquad H = U\langle b_K(g)\rangle.$$

The group U is open of finite index in $\mathcal{O}_K^\times$. Since $\operatorname{ord}_K b_K(g) = -\deg_K g \neq 0$, H is open of finite index in $K^\times$ and $H \cap \mathcal{O}_K^\times = U$. Local reciprocity gives a finite abelian extension L/K in the chosen closure with $W_L = b_K^{-1}(H)$. In particular $g \in W_L$, and every $\eta_j \circ N_{L/K}$ is unramified.

We realize this local extension in a tower preserving cuspidality. At a CM stage D, write r_D for the restricted representation. Its trace is a at the identity, so there is an open normal subgroup on which the trace never vanishes. Let D_1/D be its fixed field. In a finite Galois closure over D^+ containing D_1, choose by Chebotarev auxiliary places, away from the target, with Frobenius representatives generating the subgroup fixing D. These places split in D. Requiring them to split in a real abelian extension F'/D^+ forces $DF' \cap D_1 = D$. Indeed their chosen Frobenius elements restrict trivially to the intersection and generate on D_1.

First prescribe the local extension K/F_v at the place of the real subfield below the target, the trivial extension at these auxiliary places, and splitting at all real places. In the compositum with E the completion at the chosen prolongation is still K and the place is now split over the real subfield. Repeat there, prescribing L/K, with the auxiliary places for the new trace subgroup. At each of the two stages, let L_0/K_0 be the prescribed extension at the target. Choose generators of the character group of $K_0^\times/N_{L_0/K_0}(L_0^\times)$ and prescribe the trivial character at each auxiliary and real place for every generator. The prescribed set of places is finite. By Grunwald–Wang, each generator of order m extends with these local components to an idèle class character of order dividing m, or dividing $2m$ in the exceptional case [72, Thm. 2.2]. Take the compositum of the fields cut out by these characters. At the target the intersection of their local kernels is $N_{L_0/K_0}(L_0^\times)$, so local reciprocity identifies the completion with L_0. The prescribed trivial real characters make the auxiliary fields totally real. The local extensions are normal, so the prescribed image in the chosen local closure is L. The resulting CM compositum M is fixed by g.

Factor the two abelian towers into cyclic steps of prime degree. At each step the trace subgroup from the start of that tower surjects onto the remaining Galois group after intersection with the current absolute Galois group. This follows from the disjointness just proved. A nontrivial self-twist by a character ω of the step would give

$$\operatorname{tr} r_D(\gamma) = \omega(\gamma)\operatorname{tr} r_D(\gamma)$$

on all of G_D, by weak attachment at good places and Chebotarev. Choosing γ in the trace subgroup with $\omega(\gamma) \neq 1$ contradicts this identity. Hence prime-degree cyclic base change is cuspidal [93, Ch. III, Thms. 4.2 and 5.1]. Strong multiplicity one transports the polarization, with character pulled back by the norm. Restriction preserves semisimplicity at a finite-index normal subgroup. At good places of residue degree f, both the Satake values and the inverse-Frobenius eigenvalues pass to their f-th powers. This proves weak attachment at every stage.

We compute the support at the target one step at a time. In rank one the lift is the norm pullback of the Hecke character, by equality at almost every place and multiplicity one. At a split step the local fields and their components identify by the split-place base-change identity [93, Ch. III, Thm. 5.1]. Consider a nonsplit step k'/k of degree b, residue degree f, with norm N and lower residue cardinality q. Write S, S'

for the lower and upper support, and ω for the local character of the step, obtained from a faithful character of the global cyclic quotient. We prove for every smooth λ of $k^\times$ that

$$D_{k'}(S', \lambda \circ N; T^f) = \prod_{i=0}^{b-1} D_k(S, \lambda\omega^i; T). \tag{A.613}$$

Prescribe λ on the target units. At every other exceptional lower place prescribe a finite-order unit character avoiding the inverse unit restrictions of the finitely many rank-one support terms in all lower twists. Require its norm pullbacks to avoid the corresponding upper restrictions. These choices exist: the norm image on units is open, so characters of sufficiently large conductor remain outside each of the finitely many forbidden restrictions upon pullback. Include in the exceptional set the ramification of the extension and of the additive characters. Extend these finite unit prescriptions to a global finite-order character by [223, Ch. X, Thm. 5], first extending each prescription trivially on a uniformizer. At any new ramified place of the twist the original components were spherical and the field extension unramified or split. The norm is then onto on units. Thus the support polynomials there are one on both sides. An unramified difference at the target with uniformizer value v replaces T by vT on both sides, which is an invertible substitution.

Omit these finitely many places from both Euler products. The product for the b lower twists and the product for the upper representation agree in a right half-plane: at a good place the cyclic character values give $\prod_{\zeta^f=1}(1-\zeta z) = 1 - z^f$ for each place above it. The same calculation applies to the contragredients. Apply the global standard functional equation to these cuspidal twists [225, Thm. 13.8]. In the quotient of the two functional equations the incomplete products cancel. For the omitted finite factors use Lemma 39.56. If P_0, P_1 are the two sides of (A.613) and $H = P_0/P_1$, the contribution of the target, up to a Laurent monomial, is

$$H(T)/H(qT).$$

Indeed upstairs the substitution is T^f and the residue cardinality is q^f, so the denominator there is evaluated at $(qT)^f$. All the remaining omitted finite factors contribute monomials in their respective residue cardinalities to the power $-s$.

Divide the resulting meromorphic identity at $s+1$ by the identity at s. The quotient of every exponential factor is constant. At a complex place the archimedean factors are products and quotients of shifted complex gamma functions, so their quotients under an integral shift are rational functions of s up to constants, by $\Gamma(z+1) = z\Gamma(z)$. All infinite places in the comparison are complex. We obtain, up to a nonzero constant and inversion, a rational function of s equal to

$$\frac{H(q^{-1}T)H(qT)}{H(T)^2}.$$

Since $H(0) = 1$, this expression tends to one with error $O(q^{-s})$ as real s tends to $+\infty$. A nonconstant rational function tending to a finite limit has a nonzero leading power of $1/s$ in its difference from the limit. Hence the displayed expression is identically one. For any $z \neq 0$, the integers $\operatorname{ord}_{q^j z} H$ now have zero second difference in j. They have finite support, so they all vanish. The value at zero gives $H = 1$. This proves (A.613).

Lean: `UpperTheory.V.Z.ImaginaryQuadraticEtaCharacter.matchBCX` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.LeviSmoothLocalization.PrincipalSeriesMatching.MatchBCX`, lines 302–395)

Suppose inductively that S consists of characters. For $\mu \in S$, the condition that $(\mu\lambda) \circ N$ be unramified is equivalent to triviality of $\mu\lambda$ on $N(\mathcal{O}_{k'}^\times)$. By reciprocity this is exactly the condition that some $\mu\lambda\omega^i$ be unramified. If one such i exists, there are f of them, differing by the unramified characters of the cyclic quotient. Their uniformizer values differ by all f-th roots of unity. Their product in the right side of (A.613) is therefore

$$1 - ((\mu\lambda) \circ N)(\varpi_{k'})T^f.$$

Here an unramified character takes on $N\varpi_{k'}$ its lower uniformizer value to the power f, and ω is trivial on norms.

For each distinct unit restriction of a pulled-back μ, choose λ inverse on units to a lower representative. The identity identifies the uniformizer values, with multiplicity, of all upper characters in that class. The values on units and on a uniformizer determine the characters. Summing over these disjoint classes accounts

for a upper rank-one terms. The sum of the ranks of all upper support terms is a, so the list is exhaustive. This proves the induction without assuming anything about upper segment lengths.

At the top the raw support is $\{\eta_j \circ N_{L/K}\}_j$ and the normalized support is $\{\xi_j \circ N_{L/K}\}_j$. The first list is unramified. By 39.55 the real exponents differ pairwise by less than one, and norm pullback preserves the exponents since $|N_{L/K} z|_K = |z|_L$. The pulled-back characters are therefore unlinked. Their normalized induction is irreducible [16, Thm. 4.2] and has a spherical line. Its Satake multiset is $B = \{(\eta_j \circ N_{L/K})(\varpi_L)\}_j$, so $A_a(u, B) = \{(\xi_j \circ N_{L/K})(\varpi_L)\}_j$. For $h \in W_L$, use $b_K(h) = N_{L/K} b_L(h)$ and $\operatorname{ord}_L b_L(h) = -\deg_L h$. This gives the stated polynomial for every h, including g. □

Proof of 39.58

Proof. For g of nonzero degree choose M, u by 39.57, with $L = M_u$. Away from ℓ, 39.54 makes the restriction unramified and gives its polynomial for B. Above ℓ it makes the restriction crystalline and gives its period polynomial on W_L. Thus r_w is potentially semistable above ℓ, and the chosen bridge extends to L. Write x for the downstairs parameter. Weil–Deligne restriction identifies the monodromy operators and restricts the Weil action. For the period parameter this follows by scalar extension of the potentially semistable module with descent action. Hence $N_x = 0$. Since $g \in W_L$, 39.57 identifies the characteristic polynomials of $\rho_x(g)$ and $\rho_Y(g)$.

For t of degree zero take ϕ of degree one. Put $A = \rho_x(\phi)$, $B_0 = \rho_Y(\phi)$ and $d_j = \operatorname{tr} \rho_x(t\phi^j) - \operatorname{tr} \rho_Y(t\phi^j)$. Thus $d_j = 0$ for $j \geq 1$. Write $\det(X - A) \det(X - B_0) = \sum_j c_j X^j$. Apply Cayley–Hamilton on A and B_0, multiply by $\rho_x(t)$ and $\rho_Y(t)$ respectively, and subtract the traces. This gives $\sum_j c_j d_j = 0$. Invertibility gives $c_0 \neq 0$, so $d_0 = 0$. Applying the argument to powers of t gives the inertia polynomials by Newton's identities. Brauer–Nesbitt identifies the semisimple Weil actions. Both monodromy operators are zero.

For the jumps use the field chosen for ϕ. The filtered de Rham module descends after passage to a Galois closure. Each induced coefficient embedding downstairs extends upstairs after increasing the coefficient field. The infinitesimal character of Π at the extended embedding is that of π at its restriction. Thus 39.54 gives the desired jumps upstairs and faithful scalar extension preserves the graded dimensions.

Every segment of σ^* has length one because a longer segment contains adjacent support characters. With the reciprocity map of 40.5, the segment and rank-one laws of 40.2 give $R(\sigma^*) = Y$. □

Proof of 39.59

Proof. The place of F below w splits in E since $[E_w : \mathbf{Q}_p] = 1$. Let S be the finite set where π has no Satake multiset. Prescribe an unramified character at the target of F taking Frobenius to a primitive f-th root of unity and trivial characters at the real places. The prescribed characters have finite order and the target together with the real places is a finite set. The character form of Grunwald–Wang gives a finite-order character c with exactly these restrictions [72, Thm. 2.2]. Its exponent may be twice the least common multiple of the local orders; the descent below starts with the exponent of c.

Start with an exponent m of c. At a stage with field cut out by c^m, if $m > 1$ choose a prime $b \mid m$ and put $m' = m/b$. Choose an even auxiliary character ψ of order m, trivial at the target, whose inertia image has order m at a fresh place outside S split in E and in the field cut out by c. Choose a rational prime that splits completely in the normal closures of those fields and in $\mathbf{Q}(\mu_{2m})$. Require it also to split completely in the splitting fields of $X^m - h$ for rational primes h below the places of prescribed triviality. A residue character of order m there is even and each prescribed h is an m-th power. Its restriction to F has full inertia order since the rational prime splits completely in F.

The characters satisfy $(c\psi)^m = c^m$. The extension cut out by $(c\psi)^{m'}$ over the field cut out by c^m has degree dividing b, and the new inertia image has order b. Thus it is cyclic of degree b. Repeat with $c\psi, m'$. After compositing with E, the fresh ramification place is still spherical before its step because it is outside S and split in the preceding fields. A nontrivial step character ramifies the support there, so it cannot be a self-twist. Cyclic base change stays cuspidal. Polarization and weak attachment follow as in 39.57, where infinitesimal characters also pull back. The final real character is unramified at the target with Frobenius of order f. Its field gives the specified completion. For $f = 1$ take $M = E$.

Write π_w as the irreducible induction of its unlinked cuspidal terms. Evaluate the inducing functions on the finite flag variety to identify first congruence invariants with finite parabolic induction of the invariants

of the terms. Each term is depth zero. For a term of rank d let ϑ be its Green character of $\mathbf{F}_{p^d}^{\times}$ up to p-power conjugacy, as in 32.31 and 40.6. In rank one use its residue character. For a unit character occurring h times among the rank-one terms, the corresponding determinant character of a block of size h occurs in the lower finite induction. After twisting off this character, constants on the flag variety give the trivial representation. Use the finite character of the resulting lower label, with the higher-rank cuspidal constituents, in the trace test.

At a nonsplit step k'/k at the target, inflate the conjugate of the lower label character to the maximal compact and extend by zero. If χ is the lower label character and σ is residue Frobenius for k'/k, the upper matching function is the inflation, extended by zero, of $\delta \mapsto \overline{\chi(N_\sigma \delta)}$. Here $N_\sigma \delta$ denotes the lower conjugacy class corresponding to the twisted norm. For an irreducible constituent χ, choose the Frobenius intertwiner A_σ on its Shintani lift $\widetilde{\pi}$ with normalization $\operatorname{tr}(\widetilde{\pi}(\delta)A_\sigma) = \chi(N_\sigma\delta)$; for a label character use linearity. Thus the inflated function is the complex conjugate of this normalized twisted character. Let γ be an integral regular norm of unit determinant. In the order $\mathcal{O}_k[\gamma]$ after unramified scalar extension there is a unit δ with norm γ. Norm is onto on the finite residue-field factors. On successive nilpotent or congruence quotients its linear term is the surjective trace of the unramified scalar extension. This gives the lift in the complete order. Both δ and δ^{-1} preserve every upper lattice stable under γ, γ^{-1}. A lattice fixed by δ composed with Frobenius is stable under the norm γ, hence under δ and Frobenius separately. It descends to a γ-stable lattice, and each such descended lattice is fixed by the twisted operator. The stabilizers have the same orbital weights with hyperspecial volume one. Reduction gives the finite Shintani norm on each lattice, and both sums vanish off the integral locus.

The cyclic base-change character identity carries the nonzero lower trace to a nonzero twisted upper trace [93, Ch. I, §6]. On upper first congruence invariants decompose the finite group representation into irreducible isotypic components. An orbit of distinct components contributes zero because a group element acts within the components and the Frobenius intertwiner permutes them. On a stable component Schur's lemma gives a multiple of its twisted irreducible character. Twisted character orthogonality for the Shintani norm selects a stable constituent pairing nontrivially with the transferred lower label. Finite parabolic induction expresses the label character using the Green orbit lift on each block. The same orthogonality gives nonzero pairing with the lifted label, which retains the higher-rank Green terms whose orbits have yet to split [220, (1.1) and Addendum]. Use this label at the next step. At a split step use the identical component.

Lean: `UpperTheory.Gal.InertialTypesAlt.lblOn` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.RegularSemisimpleFibres.PLblOn`, lines 36–65)

At total residue degree $f = a!$, every $d \le a$ divides f. Thus every original Green orbit splits into singleton labels

$$\vartheta^{p^i} \circ N_{\mathbf{F}_{p^f}/\mathbf{F}_{p^d}}, \qquad 0 \le i < d. \tag{A.614}$$

The rank-one determinant blocks lift by norm on the determinant. Norm surjectivity and 40.6 preserve the original orbit lengths under p-Frobenius. The higher-rank orbits remain distinct by inertial inequivalence. The resulting blocks have distinct unit characters. Finite Frobenius reciprocity gives a vector for the inverse image of the upper parabolic transforming by the product of these determinant characters. On the Iwahori its diagonal unit character is a depth-zero minimal type. The minimal-type theorem puts the upper representation in an induction from smooth characters extending those unit characters [221, Thm. 6.11(2)].

Group the generic upper segments by their unit characters. Each segment has constant unit character, and Whittaker heredity gives genericity of each group [16]. A double coset in finite parabolic induction that pairs with the determinant vector must match unit characters on every intersection of Levi blocks, by varying a single diagonal unit there. Since the block characters differ, whole blocks must match. Each group thus has its determinant vector. Twisting by the inverse character gives a spherical generic block. A segment of length greater than one would contribute a Steinberg factor with zero hyperspecial invariants. The induced hyperspecial invariants would then vanish. Hence the segments are singletons, and genericity excludes linked singletons [16, Thm. 9.7]. Twisting back preserves unlinking within each group, and the unit characters distinguish different groups.

For $t \in I_K = I_L$ put $z = \overline{b_L(t)}$. For each rank d take an intermediate field E' of the cyclic extension M/E with completion the degree-d unramified subextension of L/K. If the global group has order m_0 and generator γ_0, the decomposition group is generated by $\gamma_0^{m_0/f}$. The subgroup of index $(m_0/f)d$ intersects that decomposition group in its subgroup of index d, so its fixed field works. Reduction $\mathcal{O}_{E'} \twoheadrightarrow \mathbf{F}_{p^d}$ supplies

the residue representatives. The uniformizer is p, and the reciprocity value of inertia has residue $N_{\mathbf{F}_{p^f}/\mathbf{F}_{p^d}} z$ there. With these data 40.4 gives the lower rank-d contribution

$$\prod_{i=0}^{d-1} \bigl(X - \vartheta((N_{\mathbf{F}_{p^f}/\mathbf{F}_{p^d}} z)^{p^i})\bigr).$$

For rank one use the character law of 40.2. Upstairs (A.614) gives the same factors. Multiply over the support with multiplicities. The normalizing twists are trivial on inertia. Choose R' by 40.1 and apply its rank-one and segment laws. □

Proof of 39.60

Proof. Write e for the doubling embedding and a for the idèlic modulus of the Levi determinant in its Iwasawa decomposition. The height is $h(g) = a(e(g,1))$. The identity

$$e(g,1) = e(g,g)e(1,g^{-1})$$

gives $a(e(g,1)) = a(e(1,g^{-1}))$, since $e(g,g)$ stabilizes the isotropic diagonal and acts on it by g, whose determinant has modulus one.

Lean: `UpperTheory.TP.hgt_swap_pk` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.ThetaKernels.TPGHgtFst`, lines 18–28)

For $e(1,u)$ with $u = g^{-1}$, the inverse compact factor applied to the diagonal has blocks T and $T' = u^{-1}T$, with $h(g) = |\det T|$. Put $\lambda = N_{E/F}(\det T)$ in the adèles of F. Unitarity gives $N_{E/F}(\det T') = \lambda$. The adjugate identities

$$\lambda u = \overline{\det T'}\, T \operatorname{adj}(T'), \qquad \lambda u^{-1} = \overline{\det T}\, T' \operatorname{adj}(T)$$

put both scaled matrices in a fixed compact adelic set. The determinant norm λ also belongs to a fixed compact set.

Suppose $|\lambda| \geq 2^{-j}$. Clear the finite denominators of the compact set of determinant norms by a fixed integer. The product of the inverse finite absolute values of the resulting integral idèle bounds an integer m that clears the finite denominators of its inverse. The bounded archimedean components and $|\lambda| \geq 2^{-j}$ give $1 \leq m \leq 2^{j+i_0}$ for a fixed i_0. Let ϱ have component $1/2$ at each real place and 1 at the finite places. With $l = m\varrho^{2(j+i_0)}$, both l and l/λ range over fixed compact sets, after increasing i_0. Indeed, m clears the inverse finite denominators after the fixed initial scaling, and the inverse real components after that scaling are $O(2^j)$. Thus the real factor $\varrho^{2(j+i_0)}$ bounds the real components of l/λ uniformly. It follows that lu and lu^{-1} lie in a fixed compact set.

For each m, cover each coordinate of lu by translates of $l^2(V - V)$, where V is a fixed compact neighbourhood of zero in the adèles of F. A maximal family of disjoint translates of l^2V with centres in the fixed coordinate set gives such a cover. Their union lies in a fixed compact set because l ranges over a compact set. The Haar volume of l^2V is $|\varrho|^{4(j+i_0)} \operatorname{vol}(V)$ by the product formula for m. Hence the number of coordinate cells is bounded by $C_0C_1^j$ in the fixed matrix dimension. If u, u' lie in the same cell, write $l(u' - u) = l^2D$ with D in a fixed compact set. Then $u^{-1}u' = 1 + (lu^{-1})D$ lies in a fixed compact matrix set, and the same calculation bounds its inverse. Each occupied cell is therefore contained in a translate of a fixed compact subset of the unitary group. Haar invariance bounds its volume uniformly. Summing over m still gives an exponential bound in j. In particular for $R = 2$ and some $C, b > 0$,

$$\operatorname{vol}\{g : R^{-j-1} < h(g) \leq R^{-j}\} \leq CR^{bj} \qquad (j \geq 0). \tag{A.615}$$

The compact factor also bounds h above, and the region with $h > 1$ has finite volume by the estimate at $j = 0$ for $|\lambda| \geq 1$. For $t > b$, the contribution from the layers in (A.615) is bounded by the sum of $CR^{(b-t)j}$. Thus $\int h^t < \infty$ for all $t \geq t_0$ with any fixed $t_0 > b$. Choose an integer $N > 2t_0$ and set $d_0 = \max(N, 2n)$.

Lean: `GlobalTheory.ThK.Dbl.hgtInt_of_k` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.ThetaKernels.ThkBaseHgtInt`, lines 17–74)

Lean: `GlobalTheory.ThK.Dbl.Hgt.count_ad` (`Development.WeilBound.ThetaLifts.LatticeBasics.CompactBound`, lines 135–194)

In the polynomial model the source partition has entries $\lambda_{v,k} - (d + m_W(v))/2$. Append $d - n$ zero entries and add $(n + m_X(v))/2$ to each entry. This gives the displayed target weight. Apply 39.61 to the source archimedean summand. Projection of the nonzero theta image to an irreducible constituent gives U'. At each split embedding the global pairing uses the split Weil model. Fixing data in the other local factors of a nonzero pairing gives the stated pairing at the chosen place. □

Proof of 39.61

Proof. Apply Langlands's continuation theorem [149] to the compact-finite doubled section of $\Phi \otimes \bar{\Phi}$ with definite target and splitting character of restriction $\eta_{E/F}^{d}$. At $s_0 = (d-n)/2$ the Eisenstein value is the theta average with factor 1 for $d > n$ and 2 otherwise [57, Thm. 7.1]. With doubling embedding ϵ and corresponding Eisenstein series $E(s,\cdot)$, pair by

$$\int_{[G]^2} E(s,\epsilon(g_1,g_2))\,\chi_W(\det g_2)^{-1}\varphi(g_2)\overline{\varphi(g_1)}\,dg_1\,dg_2.$$

The inverse character cancels the splitting character in the doubled action. Unfolding far to the right gives a nonzero constant times $\int h^{s-s_0} C_\Phi \overline{M_\varphi}$. The section has Siegel exponent $s_0 + n/2 = d/2$ and its restriction to the compact Iwasawa factor is bounded. With Cauchy–Schwarz this gives

$$|C_\Phi(g)| \le C h(g)^{d/2}, \qquad |M_\varphi(g)| \le M_\varphi(1). \tag{A.616}$$

Height integrability gives holomorphy for $\mathrm{Re}(s-s_0) + d/2 > t_0$. At s_0 Fubini on the compact quotients identifies the paired theta average with a multiple of the squared lift norm.

For $d \ge 2n$, let U_{int} be the subgroup integral at all finite places. For finite S containing the bad levels of φ and the ramification of the splitting data, put

$$A_S = \{g : g_v \text{ is integral for every } v \in S\} \setminus U_{\mathrm{int}}.$$

The sets decrease to the empty set as S increases. Continuity from above for the finite measure $h^{d/2}\,dg$ gives S with $\int_{A_S} h^{d/2} < \mathrm{vol}(U_{\mathrm{int}})$. Decompose Haar measure into the factors at the real places, at the finite places in S, and off S. In both integrals the first two factors range over the same compact real and integral subgroups. The height is invariant under right translation by these subgroups. Cancel their common positive volume and let K be the integral subgroup in the off-S adèles. On this factor the inequality becomes

$$\int_{K^c} h^{d/2} < \mathrm{vol}(K). \tag{A.617}$$

Use the usual vacua off S. At $v \in S$, let Λ be the self-dual lattice for the local Heisenberg pairing and let $\mathcal{A}$ be the integral matrix order acting on it. Take an integral full-rank isotropic matrix $x_1 \in \Lambda$ with a right inverse r such that $\Lambda r \subset \mathcal{A}$. Such a matrix comes from the integral hyperbolic pairs in the target lattice in the range $d \ge 2n$, by taking one isotropic vector in each of n pairs and using the dual vectors to read the n coordinates. Let $s = \varpi^k$ for a local uniformizer ϖ and put $x_0 = s^{-1}x_1$. Let H_v consist of local source isometries g with $g-1, g^{-1}-1 \in s\mathcal{A}$. For large k this subgroup is contained in the stabilizers of the vacuum and of φ. Translate the vacuum by x_0 in the Heisenberg action and denote the resulting vector by b_v. If $g = 1 + sA \in H_v$, then $gx_0 = x_0 + Ax_1$. The translation by Ax_1 lies in Λ and has vacuum character one, since the Gram matrix of x_1 is zero. Its Heisenberg phase with x_0 is also one. Thus H_v fixes b_v.

The characters of b_v and $\omega(g)b_v$ on $\Lambda \cap g\Lambda$ agree only if $g^{-1}x_0 - x_0$ belongs to the dual of $\Lambda \cap g^{-1}\Lambda$. The quadratic vacuum phases agree on this intersection by invariance of the pairing. The dual is $\Lambda + g^{-1}\Lambda$, by self-duality of Λ and preservation of the pairing by g. Hence equality gives $g^{-1}x_0 - x_0 = a + g^{-1}b$ for some $a, b \in \Lambda$. Multiply by s and then on the right by r to obtain

$$g^{-1}(x_1 - sb) = x_1 + sa, \qquad g^{-1}(I - sbr) = I + sar.$$

For sufficiently large k the matrices $I - sbr$ and $I + sar$ are units of $\mathcal{A}$ congruent to I modulo $s\mathcal{A}$. It follows that both g^{-1} and g are congruent to I there, so $g \in H_v$. For $g \notin H_v$, a lattice translation therefore acts on the two vectors through distinct unitary characters, and their inner product vanishes. For $g \in H_v$ the coefficient is $\|b_v\|^2$. The local integral supplies $\|b_v\|^2\,\mathrm{vol}(H_v) > 0$. At infinity Schur orthogonality supplies a nonzero factor from the common vector.

Lean: `UpperTheory.TP.bumpAny_k` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.ThetaKernels.TPGBumpAny`, lines 182–329)

Divide by these factors. On K the normalized vacuum coefficient is one and φ is fixed. The contribution is $M_\varphi(1)\,\mathrm{vol}(K)$. Off K the normalized vacuum coefficient is bounded by $h^{d/2}$ by the lattice intersection volume in the doubled section. By (A.616) the absolute value of the remaining contribution is at most $M_\varphi(1)\int_{K^c} h^{d/2}$. Inequality (A.617) proves nonvanishing. □

Proof of 39.62

Proof. Let $\nu = \chi_W^{-1}$ and $b = \chi_X$ at the split place. Convert the conjugate-linear input of the pairing to a linear input by taking the unitary dual of the source. In the split Schwartz model the candidate for the smooth dual of the upper component is normalized induction from the dual of the modified source and the rank-t character $b^{-1} \circ \det$. Here the modified source is the linear source composed with inverse transpose and twisted by $(\nu b) \circ \det$.

To obtain the intertwining map, put $d = n + t$. Let σ be the smooth contragredient of the modified source in the preceding paragraph, and let π be the smooth dual of the upper component. The transformed pairing has the form

$$\mathcal{B} : \mathcal{S}(M_{d,n}(k)) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(W_\pi, V_\sigma),$$

where $k = E_w$ and W_π, V_σ are the spaces of π, σ, respectively. The representation σ is unitarizable, since the source is unitarizable and the operations used are inverse transpose, unitary character twist and smooth contragredience. Put $\mu = b^{-1}$. For the source action,

$$(\omega_\mu(g,1)\Phi)(X) = \mu(\det g)|\det g|^{d/2}\Phi(Xg), \qquad \mathcal{B}(\omega_\mu(g,1)\Phi)(w) = \sigma(g)\mathcal{B}(\Phi)(w).$$

Suppose $\mathcal{B}$ kills every Schwartz function vanishing on matrices of rank at most j, where $j < n$.

Claim 1. For every $c \in k^\times$ with $|c| < 1$, the characteristic function Φ_c of $E_j + cM_{d,n}(\mathcal{O}_k)$ is killed by $\mathcal{B}$, where E_j has its first j diagonal entries equal to one and all other entries zero.

Choose a uniformizer ϖ, normalize $|\varpi| = q^{-1}$, and choose residue lifts $\widetilde{\rho}$ of $\rho \in M_{j,n-j}(\mathbf{F}_q)$. Set

$$g_\rho = \begin{pmatrix} I_j & c\varpi^{-1}\widetilde{\rho} \\ 0 & \varpi^{-1} I_{n-j} \end{pmatrix}.$$

We prove, for $\mathrm{rank}\, X \le j$,

$$\Phi_c(X) = \sum_\rho \Phi_c(Xg_\rho).$$

The first j columns are unchanged by g_ρ. If these columns are outside the corresponding ball about those of E_j, every term is zero. Otherwise write $X = (A, AT)$. The top j rows of A form a matrix in $I_j + cM_j(\mathcal{O}_k)$ and its other entries are integral. Hence multiplication by A preserves the maximum norm of a column vector. Consequently $\Phi_c(X) = 1$ precisely when $T \in cM_{j,n-j}(\mathcal{O}_k)$, while

$$\Phi_c(Xg_\rho) = 1 \quad \Longleftrightarrow \quad T + c\widetilde{\rho} \in c\varpi M_{j,n-j}(\mathcal{O}_k).$$

There is exactly one such ρ when T lies in the first lattice, namely the residue class of $-T/c$, and none otherwise. For $j = 0$ the same identity holds at the sole matrix of rank zero.

All g_ρ have determinant $D = \varpi^{-(n-j)}$. Put $s_j = \mu(D)|D|^{d/2}$. The identity just proved shows that $\Phi_c - s_j^{-1}\sum_\rho \omega_\mu(g_\rho, 1)\Phi_c$ vanishes at every matrix of rank at most j. Applying $\mathcal{B}$ and evaluating at any $w \in W_\pi$ therefore gives, for $v = \mathcal{B}(\Phi_c)(w) \in V_\sigma$,

$$v = s_j^{-1}\sum_\rho \sigma(g_\rho)v, \qquad \|v\| \le q^{j(n-j)-d(n-j)/2}\|v\|.$$

Here the norm is invariant under σ, and $|s_j| = q^{d(n-j)/2}$. Since $d > 2n$ and $j < n$, the exponent is negative. Thus $v = 0$ for every w, proving Claim 1.

We now show that $\mathcal{B}$ kills every Schwartz function vanishing at ranks below j. Near a rank-j point, row and column operations move that point to E_j. In a sufficiently small ball about E_j, every matrix has rank at least j. Partition the support of a locally constant function there into finitely many smaller balls on which it is constant. A ball meeting rank j may be recentered at such a point; integral invertible row and column operations carry it to a ball about E_j of the same radius. Claim 1 and equivariance kill its characteristic function. A ball not meeting rank j has all its ranks greater than j and is killed by the induction hypothesis. At points of rank greater than j, an invertible minor gives a neighbourhood with the same lower rank bound. Compactness and a finite partition of the support now give the asserted vanishing. If the pairing vanished on the full-rank locus, descending from $j = n - 1$ to $j = 0$ would make it zero everywhere.

Lean: `PositivityTheory.Rg.Ming.core` (`Development.WeilBound.RepresentationTheoryGL.PointAveraging.MinkowskiBalls.ValLeOneOfLc`, lines 382–460)
Lean: `PositivityTheory.Rg.Ming.step` (`Development.WeilBound.RepresentationTheoryGL.PointAveraging.MinkowskiBalls.MemBallMSelf`, lines 134–194)

On the full-rank orbit the stabilizer has Levi factors of ranks n and t. Its action on the full-rank evaluation gives the indicated determinant twists, so Frobenius reciprocity gives a nonzero map into the indicated normalized induction. The unitary factors give irreducible induction, which identifies it with the upper dual. Taking its contragredient removes the dual on the modified source and replaces $b^{-1} \circ \det$ by $b \circ \det$. The unitary dual and inverse transpose cancel on the source Satake multiset, so the source contributes $(xy^{-1})B$. The rank-t determinant character contributes $xq^{(t-1)/2-j}$ for $0 \le j < t$ by the half modulus of the upper triangular Borel. Normalized induction takes the union of the two multisets.
Lean: `UpperTheory.Lit.HoweDuality.lift_of_irr` (`Development.AutomorphicArithmetic.ThetaCorrespondence.FockModels.MingZel`, lines 17–44)

□

Proof of 39.63

Proof. To supply the auxiliary global theta-correspondence data of Proposition 39.60, take the designated split-place set $P = \varnothing$, the standard additive character $\psi = \psi_{\mathbf{Q}} \circ \mathrm{Tr}_{F/\mathbf{Q}}$ (which is trivial on F, equals $\exp(-2\pi i \,\mathrm{Tr}(x))$ at infinity, and is unramified at every inert place since inert $v \nmid \mathfrak{D}_{F/\mathbf{Q}}$), and the standard Hermitian self-dual source lattice $\mathcal{O}_E^n$. Since E/F is unramified at finite places, choose $\delta \in E^\times$ with $\bar{\delta} = -\delta$, a fractional ideal $\mathfrak{a}$ self-dual for the trace pairing with coefficient δ, and embeddings τ above the real places with $i\tau(\delta) > 0$. For each target rank d, the orthogonal sum $\mathfrak{a}^d$ is trace-self-dual for $\delta 1_d$. Use the compatible archimedean Cartan data. Let d_0 be the bound of 39.60, take $t = 2(d_0 + n) + 1$, and let b be a lower bound for the source weights at the chosen embeddings. By 39.49 take χ with $S = \{w_0\}$ and infinity exponent -1 at each chosen embedding. Choose a positive odd K with $-K \le 2b - (n + t)$. For $u = 0, 1$ put $\chi_{X,u} = \chi^{2u}$ and $\chi_{W,u} = \chi^{K+2u}$. Their restrictions have the required parity and their exponents meet the inequality of 39.60. Apply that proposition with $d = n + t > 2n$.
Lean: `UpperTheory.Gal.TP.Skew.skewLine_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.CartanIdeles.SpacesExHolds`, lines 23–23)

Write $|\cdot|_{\mathbf{A}_E}$ for the idèlic norm. Define $\kappa = \chi^K |\cdot|_{\mathbf{A}_E}^{t/2}$ and

$$\psi_{u+1,j} = \chi^{2u} |\cdot|_{\mathbf{A}_E}^{-(n+2t-2)/2+j}, \qquad 0 \le j < t.$$

The integer powers in the added characters and the parity of K, t in κ give algebraic infinity types. At a split place, unramifiedness of the added characters implies that of $\chi_{X,u}$, and unramifiedness of κ then gives that of $\chi_{W,u}$. Put $x = \chi_{X,u}(w)$, $y = \chi_{W,u}(w)$ and $q = q_w$. Scale the multiset in 39.62 by $q^{(n+t-1)/2}$. The source contribution is $(xy^{-1}q^{t/2})A_n(w, B)$ and the added roots are $xq^{(n+2t-2)/2-j}$. These are the stated character values. At conjugate embeddings the exponent difference for an added line is zero for $u = 0$ and nonzero for $u = 1$, since the norm has equal exponents and χ^2 has opposite nonzero exponents. The families are disjoint. They and κ are unramified at w_0 by the choice of χ. To get the displayed jumps, subtract the weights of 39.60 from their indices. Use the dual weights at conjugate embeddings. □

Proof of 39.66

Proof. Apply odd-rank compatibility to both lifts to obtain r_i. Algebraic character realization is injective, so the realizations of the line families remain disjoint. The cofinite Satake unions give 39.64. For a Satake multiset at w_0, the first lift and its lines have the local spherical data there by 39.63. Subtract by 39.65, and twist by κ_ℓ to cancel $\kappa(w_0)^{-1}$ and the jump shift $m_{\iota\circ\tau}$. The same subtraction and twist at good places give weak compatibility. □

Proofs for Chapter 40: A weak local Langlands correspondence

Proof of 40.1

Proof. Extend r of 40.8 to R by 40.9. The real twist law identifies the Weil constituents of the block for $[u,m]$ with $R(u\nu_K^j)$, $0 \le j < m$. For the cuspidal support multiset s_x and $\sigma \in W_K$,

$$\det(X - \rho_{R(x)}(\sigma)) = \prod_{v \in s_x} \det(X - \rho_{R(v)}(\sigma)). \tag{A.618}$$

The same formula holds for any map in 40.2. Degrees give the dimension law, and $\omega_x = \prod_{v\in s_x} \omega_v$, with ω_x the central character of x, gives both determinant laws.

Twisting the first terms gives the twist laws. If x has good cuspidal support in the sense of Definition 40.3, the dual first term $(u\nu_K^{m-1})^\vee$ has parameter $r(u)^\vee(1-m)$. The centering shift and self-duality of Sp_m give the dual block.

If $R(x) = R(y)$ with y good, uniqueness of the indecomposable decomposition matches their blocks [230, §4.4]. The block B of $[u,m]$ has nilpotence index m and Weil action $r(u)$ on B/NB. By 40.8 the matched first terms coincide, so $x = y$.

The compact picture identifies the K_1-invariants of parabolic induction with finite parabolic induction of the terms' invariants. Exactness gives depth zero for each cuspidal term when $x^{K_1} \neq 0$. For good x, (A.618), 40.8 and semisimplicity give tameness. The cuspidal laws pass to R. The type law follows because an inflated cuspidal type forces cuspidality [221, Prop. 6.6]. □

Proof of 40.5

Proof. Local reciprocity gives existence in this Frobenius convention [9, Ch. XIII, §4], with local–global compatibility obtained by composing the arithmetic-Frobenius reciprocity of [41, Ch. VI, §5] with inversion. For $u \neq 1$ in $K^\times$, choose an open subgroup of finite index excluding u. Characters of the finite quotient by this subgroup separate its elements. Choose one nontrivial on u and extend it from the single place w to a finite-order idèle class character by Grunwald–Wang [72, Thm. 2.2]. Take the finite abelian extension cut out by this character. In that extension the global Artin image of $\mathrm{loc}_w(u)$ is the nontrivial local Artin image of u. Thus $\mathrm{Art}_E \circ \mathrm{loc}_w$ is injective, and the Artin identity gives uniqueness. □

Proof of 40.6

Proof. Put $n = d(u_0)$. The depth-zero cuspidals are compact inductions of $\widetilde{\tau}(zk) = \omega(z)\tau(\bar{k})$ from $K^\times K_0$, with τ finite cuspidal and ω extending its central character [221, Props. 6.6 and 6.8]. For a Cartan representative $g = \mathrm{diag}(\varpi^{a_j})$, a gap $a_i < a_{i+1}$ puts the upper unipotent radical of block sizes $i, n-i$ in the reduction of $K_0 \cap gK_1g^{-1}$. Cuspidality forces invariant functions to vanish on such cosets. Evaluation at 1 gives $u_0^{K_1} \simeq \tau$.

Take the Green label θ of τ by 32.31 and a generator ζ of $k_n^\times$. For a proper subfield of degree d, we have $d \mid n$ and $d \le n/2$. Its multiplicative group has index

$$(p^n - 1)/(p^d - 1) \ge 1 + 2^{n-d} > n. \tag{A.619}$$

Thus ζ^r has degree n for $1 \le r \le n$. Put $\alpha_j = \theta(\zeta^{p^j})$. Some $\sum_j \alpha_j^r$ in this range is nonzero, since vanishing for all r would give $\prod_j \alpha_j = 0$ by Newton's identities. Let g be multiplication by this ζ^r. Both g and cg for $c \in \mathbf{F}_p^\times$ are elliptic. By (40.2), $\mathrm{tr}\,\tau(cg) = \theta(c)\,\mathrm{tr}\,\tau(g) \neq 0$. Schur's lemma gives central character $\theta|_{\mathbf{F}_p^\times}$.

Lift a monic irreducible polynomial of degree n over $\mathbf{F}_p$ to $\mathcal{O}_E[X]$ and adjoin a root for a datum $\mathscr{D}$ of 40.4. On common inertia $R_{\mathscr{D}}(t) = \overline{a'(t)}$ maps onto $k_n^\times$ with open kernel by reciprocity. Its restriction to the pro-p group P_K is trivial since $p \nmid p^n - 1$. Set $N = 0$ and use inertia lines with characters $t \mapsto \theta(R_{\mathscr{D}}(t)^{p^j})$, $0 \le j < n$. The Weil relation lets degree-one Frobenius ϕ cycle these distinct lines, so the action is irreducible. The cyclic matrix with wrap coefficient $b = (-1)^{n-1}\omega_{u_0}(a(\phi))$ has the required determinant and $\rho(\phi)^n = b\,1$. Let h be the exponent of the inertia image. For $\deg\sigma = k$, write $\sigma^n = \phi^{nk}t$ with $t \in I_K$. Then $\rho(\sigma)^{nh} = b^{kh}1$. The polynomial $X^{nh} - b^{kh}$ has simple roots, so $\rho(\sigma)$ is semisimple. Norm compatibility gives

$$\det\rho(t) = \theta(N_{k_n/\mathbf{F}_p} R_{\mathscr{D}}(t)) = \omega_{u_0}(a(t)).$$

Choose $\beta \in \Omega$ with $\beta^{p^n-1} = p$. In the residue field of A, tame reciprocity gives $R_{\mathscr{D}}(t) = \overline{t(\beta)/\beta}$ for every datum $\mathscr{D}$ [9]. Density of $\mathbf{Q}$ fixes i, so each datum has type $u_0^{K_1}$. Formula (40.2) on multiplication by ζ^r gives equal first n power sums for any two labels of that type. By Newton's identities their orbits coincide, so (40.3) follows.

In a tame irreducible with inertia orbit length ℓ, take a ϕ^ℓ-eigenline in one inertia eigenspace. Its ϕ-translates span an invariant space of dimension ℓ, so the inertia characters have multiplicity one. Their orbit, read from the inertia polynomials, and the Frobenius determinant determine the cyclic action up to diagonal conjugacy. Conversely, surjectivity of $R_{\mathscr{D}}$ and a recovers the Green orbit and ω_{u_0} from inertia and the determinant. These determine the compact induction.

A twist by λ leaves the Cartan unipotent action unchanged. The identity coset contributes exactly when λ is trivial on $\det K_1 = 1 + \mathfrak{p}_K$. Then the label gains $\bar{\lambda} \circ N_{k_n/\mathbf{F}_p}$, with $\bar{\lambda}$ the reduction on units, and the central character gains λ^n. Thus $V_{u_0 \otimes \lambda} = V_{u_0} \otimes (\lambda \circ a)$. Any two depth-zero bases of one class differ by such a twist. In rank one the determinant gives $V_{u_0} = (u_0 \circ a, 0)$. These identities give the assignment r_0 and its twist law. □

Proof of 40.7

Proof. Let L/K be unramified of degree n, with residue field k_n. For a uniformizer ϖ of K and $M = q^n - 1$, put

$$b(u) = \overline{\frac{u^M - 1}{M\varpi}}, \qquad u \in \mathcal{O}_L^\times.$$

Multiplication modulo ϖ^2 gives additivity and $b(1 + \varpi x) = \bar{x}$. Thus $\pi = b \circ a_L|_{I_K}$ maps onto k_n^+ with open kernel, where a_L is reciprocity and $I_K = I_L$. For degree-one Frobenius ϕ,

$$\pi(\phi t \phi^{-1}) = f(\pi(t)), \qquad f(z) = z^q, \tag{A.620}$$

since ϕ fixes ϖ and induces f on k_n. For $\lambda \in k_n$ of degree n over $\mathbf{F}_q$, the trace pairing gives orbit length n for $\alpha(z) = \exp(2\pi i \operatorname{Tr}_{k_n/\mathbf{F}_p}(\lambda z)/p)$.

Let inertia act on e_j by $\eta_j = \alpha \circ f^j \circ \pi$, $0 \le j < n$. Put $\rho(\phi)e_j = e_{j-1}$ for $j > 0$ and $\rho(\phi)e_0 = (-1)^{n-1}e_{n-1}$. Equation (A.620) gives the Weil relation. This finite-order matrix of determinant 1 normalizes the finite inertia image, so the Weil image is finite. Transitivity on the distinct inertia lines gives irreducible $J_n = (\rho, 0)$ with unitary determinant.

The p-group $k_n^+/\pi(P_K)$ is a quotient of tame inertia, hence trivial. Choose $t \in P_K$ with $\eta_0(t) \ne \eta_1(t)$. Thus $\rho_{J_n}(t)$ is nonscalar. Every character twist of a tame parameter acts at t by a scalar. □

Proof of 40.8

Proof. Smooth central characters are unitary on units, so $e(u)$ exists. Twists preserve goodness. The central character of $u\nu_K^s$ is $\omega_u \nu_K^{ns}$, hence

$$e(u\nu_K^s) = e(u) + s. \tag{A.621}$$

This gives the real twist law on the second branch, where $n \ge 2$. Dimension and irreducibility follow from 40.7, and

$$|\det \rho_{J_n(e(u))}(\sigma)| = q^{ne(u)\deg\sigma} = |\omega_u(a(\sigma))|. \tag{A.622}$$

On good classes 40.6 gives the twist and determinant laws. It also gives the type law. Invert the label and central character of a depth-zero base, or the character in rank one, to get duality. Depth-zero good cuspidals have tame parameters by 40.6 in higher rank and by reciprocity in rank one.

Suppose $r(u) = r(v)$ with v good. Their ranks agree. In rank one use surjectivity of a. For good $u = u_0 \otimes \chi$ and $v = v_0 \otimes \chi'$ of higher rank, $V_{u_0} = V_{v_0} \otimes \xi$, where $\xi = \lambda \circ a$ and $\lambda = \chi'\chi^{-1}$. Tameness makes ξ trivial on P_K. Frobenius conjugation on I_K/P_K is the q-power map, so the character ξ satisfies $\xi(t)^{q-1} = 1$ on I_K. Surjectivity and the degree formula give $a(I_K) = \mathcal{O}_K^\times$. The image $\lambda(1 + \mathfrak{p}_K)$ is a finite p-group by smoothness, and its exponent divides $q - 1$. Thus it is trivial. By 40.6, $V_{u_0} = V_{v_0 \otimes \lambda}$, so $u_0 = v_0 \otimes \lambda$ and $u = v$.

If u is on the second branch, $v = v_0 \otimes \chi$ over $K \simeq \mathbf{Q}_p$, and equality would give $J_{d(u)} = V_{v_0} \otimes ((\chi \circ a)\nu_W^{-e(u)})$, contrary to 40.7. □

Proof of 40.9

Proof. In Rodier's parametrization [230, Thm. 3], take the unique irreducible quotient for each segment by normalized induction in increasing exponent order. Order segments so none precedes a later one, where on an integer twist line $[a,b]$ precedes $[c,d]$ if $a < c \le b+1$ and $b < d$. The normalized induction of the segment quotients has unique irreducible quotient x. Zelevinsky gives the submodule form [16, Thm. 6.1]. Permuting the segments permutes the direct summands. A cuspidal u has multisegment $\{[u,1]\}$, so $\mathrm{Sp}_1 = \mathbf{1}$ gives $R(u) = r(u)$. □

Proof of 40.11

Proof. Genericity gives an unlinked Langlands multisegment [16, Thm. 9.7]. The monodromy on a segment of length m is the tensor of the identity on its nonzero cuspidal parameter with the special nilpotent operator of length m. Thus zero monodromy forces length one for every segment. Their first terms give the unlinked cuspidal support, and $R(x)$ is the direct sum of their parameters. In the commutator test every factor of the characteristic polynomial is a power of $X-1$. A commutator lies in inertia, whose image on each factor is finite, so its action is semisimple and hence trivial. An irreducible complex representation of the resulting abelian image has dimension one.

In the tame case, each direct summand is tame. Fix a support term u of rank d, a Frobenius ϕ of degree -1, and $D = \det \rho_{R(u)}(\phi) \neq 0$. By reciprocity, $a(\phi)$ is a uniformizer. For this fixed rank and determinant, the inertia orbit gives an injection of tame irreducible parameters into the finite set of length-d Frobenius orbits of characters of $\mathbf{F}_{q^d}^{\times}$: the generator of tame inertia and its Frobenius conjugates determine the inertia characters, and the determinant fixes the scalar of the cyclic Frobenius shift. Equivalently, after choosing a generator in the finite cyclic group, label the orbits by the monic irreducible polynomials of degree d over $\mathbf{F}_q$ other than X. For every label take the Green type and compactly induce as in 40.6, using a central character extending the Green type's central character on units and whose value at $a(\phi)$ is D. The resulting depth-zero cuspidals are distinct, since their K_1-fixed finite-field types distinguish the labels. They are good, so their images under R are tame and have the prescribed determinant and rank. Good injectivity gives an injection from this finite label set into itself; it is onto. Thus $R(u)$ has a depth-zero cuspidal preimage with the same parameter, which equals u by good injectivity.

To pass from the support to x, use Proposition 26.25 over the characteristic-zero field K. Take the exterior tensor product of the support representations as a subquotient of the normalized Jacquet module for a suitable parabolic. Its fixed vectors under the product of the Levi principal congruence subgroups are nonzero, by tensoring fixed vectors of the support terms. Taking invariants under that compact subgroup is exact by averaging. Thus the Jacquet module has a nonzero such fixed vector. The normalization is trivial on this subgroup. The Jacquet lifting of fixed vectors to the inverse image of the unipotent radical in the parabolic reduction gives a nonzero K_1-fixed vector of x, since K_1 is contained in that inverse image.

If two support occurrences are inertially equivalent, the unramified-twist law gives equal inertia characteristic polynomials for their parameters. The square of the common polynomial at a lift γ of a tame generator divides the polynomial of $R(x)$. Under the repeated-root test each eigenvalue of this common factor satisfies $z^{q-1} = 1$; semisimplicity gives $\rho_{R(u)}(\gamma)^{q-1} = 1$. Choose the Frobenius with conjugation action $\gamma \mapsto \gamma^q$ on tame inertia. Its image now commutes with that of γ, hence with the full finite inertia image generated by γ. The Weil image is abelian, so this irreducible factor has rank one. Hence no two occurrences of rank at least two are inertially equivalent. In either case the support is good by 40.3. □

Chapter 41

Laplace derivatives exclude poles

This chapter proves Theorem 3.2 by bounding high-order derivatives of the Laplace transform on logarithmic windows; Lemmas 4.1 and 4.2 are stated in Chapter 4.

Let $g : \mathbf{R} \to \mathbf{R}$ be Borel measurable, with $|g(u)| \le C_0 e^{2u}$ for $u \ge 0$, where $C_0 \ge 0$. For $k \in \mathbf{N}$, $a > 0$, and $u \ge 0$, put

$$K_{k,a}(u) = \frac{u^k e^{-au}}{k!}, \qquad B_k(z) = \int_0^\infty g(u) \frac{u^k e^{-zu}}{k!}\, du \quad (\operatorname{Re} z > 2).$$

When (g, Φ, Z) satisfies the pole data of 3.4, set $A_k(z) = (-1)^k \Phi^{(k)}(z)/k!$. For a character family F, Definition 2.10 gives the arithmetic example $g(u) = G_m(F, e^u)$. All integrals are Lebesgue integrals.

Theorem 41.1 (Derivative bound from integral windows)**.** *Suppose that (g, Φ, Z) satisfies 3.4. Fix $0 < \theta < 1$ and $\delta > 0$. Assume the logarithmic window condition $\mathrm{WH}_U(g; \theta, \delta)$ of Lemma 2.13. Then there exist $\sigma > 2$ and $C \in \mathbf{R}$ such that, for every $\eta > 0$ and $U_0 \in \mathbf{R}$, one can find $U \ge U_0$ and $k \in \mathbf{N}$, $k \ge U$, for which*

$$|A_k(\sigma + it)| + \operatorname{Re} A_k(\sigma) \le \frac{C e^{\eta U}}{(\sigma - \theta)^k} \qquad \text{for every } t \in \mathbf{R}.$$

Proof. Take σ and the positive witness D displayed in Lemma 41.6. Apply the window condition with lower bound $\max(U_0, 1)$, and choose $k \ge U$ from that lemma. By 41.3 and 41.7, for every real t,

$$|A_k(\sigma + it)| + \operatorname{Re} A_k(\sigma) \le \frac{2 e^{\eta U}}{(\sigma - \theta)^k} + \frac{2 C_0 D}{\sigma^k}.$$

Since $0 < \sigma - \theta < \sigma$ and $e^{\eta U} \ge 1$, take $C = 2 + 2C_0 D$. □

41.1 The differentiated Laplace kernel

Lemma 41.2 (Differentiation under the Laplace integral)**.** *For g as above and $j \in \mathbf{N}$, define*

$$H_j(z) = \int_0^\infty g(u)(-u)^j e^{-zu}\, du \qquad (\operatorname{Re} z > 2).$$

These integrals converge absolutely. For every $j \in \mathbf{N}$ and $s \in \mathbf{C}$ with $\operatorname{Re} s > 2$,

$$H_j'(s) = H_{j+1}(s), \qquad H_0^{(j)}(s) = H_j(s).$$

Proof. Apply Widder's Laplace–Stieltjes differentiation formula [231, Chapter II, §5] to the locally absolutely continuous function $\alpha(u) = \int_0^u g(v)\, dv$ on $[0, \infty)$. □

Lemma 41.3 (Normalized coefficients)**.** *If (g, Φ, Z) satisfies 3.4, then $A_k(z) = B_k(z)$ for every $k \in \mathbf{N}$ and $\operatorname{Re} z > 2$. In particular, for $\sigma > 2$ and $t \in \mathbf{R}$,*

$$A_k(\sigma + it) = \int_0^\infty g(u) K_{k,\sigma}(u) e^{-itu}\, du,$$
$$\operatorname{Re} A_k(\sigma) = \int_0^\infty g(u) K_{k,\sigma}(u)\, du.$$

Proof. On $\operatorname{Re} z > 2$, the pole data identify Φ with H_0, so $A_k = B_k$ is Widder's differentiation formula [231, Chapter II, §5] in Lemma 41.2 with the normalization $(-1)^k/k!$. □

Lemma 41.4 (Mass and pointwise bound for the kernel)**.** *For $a > 0$ and $k \in \mathbf{N}$, the kernel $K_{k,a}$ is integrable on $(0, \infty)$, and*

$$\int_0^\infty K_{k,a}(u)\, du = a^{-k-1}, \qquad 0 \le K_{k,a}(u) \le a^{-k} \quad (u \ge 0).$$

Proof. The integral is the specialization $\mu = 1$, $\nu = k+1$, $z = a$ of Euler's gamma integral [5, Section 5.9(i), equation (5.9.1)], divided by $k!$. The pointwise bound follows from $(au)^k/k! \le e^{au}$ after multiplication by $a^{-k}e^{-au}$. □

41.2 Both tails at a common derivative order

Lemma 41.5 (Exponential bounds for the two tails)**.** *For $k \in \mathbf{N}$, $0 \le \ell < a$, and $U, V \in \mathbf{R}$,*

$$\int_{(0,\infty)\setminus[U,V]} K_{k,a}(u)\, du \le \frac{e^{-\ell V}}{(a-\ell)^{k+1}} + \frac{e^{\ell U}}{(a+\ell)^{k+1}}.$$

Furthermore, for $b, \sigma > 0$ and $x, c \in \mathbf{R}$,

$$x + k\frac{\sigma - b}{b} \le c \quad \Longrightarrow \quad \frac{e^x}{b^{k+1}} \le \frac{e^c}{b\sigma^k}.$$

Proof. If $u > 0$ lies outside $[U, V]$, then $u < U$ or $u > V$. Hence

$$K_{k,a}(u) \le e^{\ell U} K_{k,a+\ell}(u) + e^{-\ell V} K_{k,a-\ell}(u).$$

By nonnegativity, the integral of this majorant over the complement is at most its integral over $(0, \infty)$. Lemma 41.4 evaluates the latter.

For the second assertion, $1 + y \le e^y$, with $y = (\sigma - b)/b$, gives

$$\frac{e^x}{b^{k+1}} = \frac{e^x}{b\sigma^k}\left(\frac{\sigma}{b}\right)^k \le \frac{\exp(x + k(\sigma - b)/b)}{b\sigma^k} \le \frac{e^c}{b\sigma^k}.$$

□

Lemma 41.6 (Choosing the order from the window)**.** *For every $\delta > 0$, there exist $\sigma > 2$ and $D \in \mathbf{R}$ such that, for every real $U \ge 1$, there is $k \in \mathbf{N}$ with $k \ge U$ and*

$$\int_{(0,\infty)\setminus[U,(1+\delta)U]} K_{k,\sigma-2}(u)\, du \le \frac{D}{\sigma^k}.$$

One may take $\ell = 2 + 6/\delta$, $M = \ell(1+\delta)$, and $\sigma = M(\ell + 2)$, with

$$k = \lceil (1 + \delta/2)\sigma U \rceil, \qquad D = \frac{e}{\sigma - 2 - \ell} + \frac{1}{\sigma - 2 + \ell}.$$

Proof. Write $p = 1 + \delta/2$ and $b_\pm = \sigma - 2 \pm \ell$. Since $M > \ell > 2$, we have $b_- = (M-1)(\ell+2) > 0$. Thus $\sigma > 2$ and $D > 0$. Direct computation gives

$$p(\ell+2) = M-1, \qquad p(\ell-2) = \ell+1, \qquad p\sigma = M(M-1).$$

The ceiling satisfies $p\sigma U \le k < p\sigma U + 1$, so $k \ge U$. By 41.5, with $a = \sigma - 2$, the tail integral is at most $T_+ + T_-$, where

$$T_+ = \frac{e^{-MU}}{b_-^{k+1}}, \qquad T_- = \frac{e^{\ell U}}{b_+^{k+1}}.$$

For the upper tail, $(\sigma - b_-)/b_- = 1/(M-1)$. The upper ceiling bound gives

$$\begin{aligned} -MU + k\frac{\sigma - b_-}{b_-} &= -MU + \frac{k}{M-1} \\ &< -MU + \frac{p\sigma U + 1}{M-1} = \frac{1}{M-1} \le 1. \end{aligned}$$

The second bound of 41.5, with $x = -MU$ and $c = 1$, gives $T_+ \le e/(b_-\sigma^k)$.

For the lower tail, $(\sigma - b_+)/b_+ = -(\ell-2)/b_+$. Since $\sigma \ge \ell(\ell-2)$, we have $\ell b_+ \le (\ell+1)\sigma$. The lower ceiling bound gives

$$\frac{k(\ell-2)}{b_+} \ge \frac{p\sigma U(\ell-2)}{b_+} = \frac{(\ell+1)\sigma U}{b_+} \ge \ell U.$$

Thus $\ell U + k(\sigma - b_+)/b_+ \le 0$. Apply the same bound with $x = \ell U$ and $c = 0$ to obtain $T_- \le 1/(b_+\sigma^k)$. Adding proves the claim. □

41.3 The one-sided estimate

Lemma 41.7 (Combining the window with its complement)**.** *Let g satisfy the growth and measurability assumptions above. Let $\sigma > 2$, $\theta < \sigma$, $U > 0$, and $V \in \mathbf{R}$, and put $I = [U, V]$. Suppose that $R : \mathbf{R} \to \mathbf{R}$ satisfies $R(u) \ge \max(g(u), 0)$ on I, and that $R(u)e^{-\theta u}$ is integrable on I. Then, for every $k \in \mathbf{N}$ and $t \in \mathbf{R}$,*

$$\begin{aligned} |B_k(\sigma + it)| + \operatorname{Re} B_k(\sigma) \le{}& \frac{2}{(\sigma-\theta)^k} \int_I R(u) e^{-\theta u}\, du \\ &+ 2C_0 \int_{(0,\infty)\setminus I} K_{k,\sigma-2}(u)\, du. \end{aligned}$$

Proof. Set $w(u) = K_{k,\sigma}(u)$. The bound $|g(u)|w(u) \le C_0 K_{k,\sigma-2}(u)$ for $u > 0$, together with 41.4, gives absolute convergence of the integrals below. Since $|e^{-itu}| = 1$, the triangle inequality gives

$$\begin{aligned} |B_k(\sigma + it)| + \operatorname{Re} B_k(\sigma) &\le \int_0^\infty |g(u)| w(u)\, du + \int_0^\infty g(u) w(u)\, du \\ &= \int_0^\infty (|g(u)| + g(u)) w(u)\, du. \end{aligned}$$

On I, the envelope assumption and 41.4 give

$$\begin{aligned} |g(u)| + g(u) &= 2\max(g(u), 0) \le 2R(u), \\ w(u) &= K_{k,\sigma-\theta}(u) e^{-\theta u} \le (\sigma-\theta)^{-k} e^{-\theta u}. \end{aligned}$$

Since $R \ge 0$ and $w \ge 0$ on I, multiplying these inequalities and integrating gives the first term. On the complement, $(|g(u)| + g(u))w(u) \le 2|g(u)|w(u) \le 2C_0 K_{k,\sigma-2}(u)$. Adding the integral over $(0,\infty) \setminus I$ proves the inequality. □

41.4 From the derivative bounds to the exclusion of poles

Lemma 41.8 (Dichotomy at a real centre)**.** *Suppose Z satisfies the local condition of Lemma 4.2. Fix $0 < \theta < \sigma$ with $Z \subset \{w : \operatorname{Re} w < \sigma\}$. If $\operatorname{Re}\rho_0 > \theta$ for some $\rho_0 \in Z$, at least one of the following alternatives holds.*

1. *There are $D \le \sigma - \theta$ and $\rho_1 \in Z$ such that $Z \cap B(\sigma, D) = \varnothing$ and $\sigma - \operatorname{Re}\rho_1 < D$.*
2. *There are $\beta \in \mathbf{R}$ and $r \in \mathbf{R}$ such that $\beta \in Z$, $\beta > \theta$, $\sigma - \beta < r \le \sigma - \theta$, and $Z \cap B(\sigma, r) = \{\beta\}$.*

Proof. Set $R = \sigma - \theta$ and $F = Z \cap \overline{B}(\sigma, R)$. Lemma 4.2 makes F finite. If F is empty, take $D = R$ and $\rho_1 = \rho_0$.

Otherwise choose $\rho_* \in F$ minimising distance to σ, and put $D = |\sigma - \rho_*| > 0$. Then $D \le R$ and $Z \cap B(\sigma, D) = \varnothing$. If some $\rho_1 \in Z$ has $\sigma - \operatorname{Re}\rho_1 < D$, the first alternative holds. In the remaining case every $w \in Z$ satisfies $D \le \sigma - \operatorname{Re} w$. For every minimiser $w \in F$,

$$D \le \sigma - \operatorname{Re} w \le |\sigma - w| = D, \qquad |\sigma - w|^2 = (\sigma - \operatorname{Re} w)^2 + (\operatorname{Im} w)^2.$$

Hence every minimiser equals the real point $\beta = \sigma - D$. Applying $D \le \sigma - \operatorname{Re}\rho_0$ gives $D < R$, so $\beta > \theta$. Every point of $F \setminus \{\beta\}$ has distance strictly greater than D. Choose r with $D < r \le R$ no larger than any of these finitely many distances. Points of Z outside F have distance greater than R. Thus $Z \cap B(\sigma, r) = \{\beta\}$. □

Lemma 41.9 (An isolated pole after moving the centre)**.** *Suppose Z satisfies the local condition of Lemma 4.2, $\operatorname{Re} w < \sigma$ for all $w \in Z$, and $D < \sigma$. If $\rho_1 \in Z$ satisfies $\sigma - \operatorname{Re}\rho_1 < D$, then there exist $t \in \mathbf{R}$, $\rho \in Z$, and $r \in \mathbf{R}$ such that, for $z = \sigma + it$,*

$$|z - \rho| < r \le D, \qquad Z \cap B(z, r) = \{\rho\}.$$

Proof. Put $a = \sigma - \operatorname{Re}\rho_1$, so $0 < a < D$, and choose

$$\xi = \tfrac{1}{2}\min\{D - a, \sigma - D\} > 0, \qquad z_0 = \sigma + i\operatorname{Im}\rho_1.$$

Since $D + \xi < \sigma$, the candidate set $F = Z \cap \overline{B}(z_0, D + \xi)$ is finite and contains ρ_1. For distinct $v, w \in F$, equality of distances to $\sigma + it$ is equivalent to

$$2t(\operatorname{Im} v - \operatorname{Im} w) = (\sigma - \operatorname{Re} v)^2 + (\operatorname{Im} v)^2 - (\sigma - \operatorname{Re} w)^2 - (\operatorname{Im} w)^2.$$

If $\operatorname{Im} v \ne \operatorname{Im} w$, the displayed equation determines a unique height at which the distances tie. If $\operatorname{Im} v = \operatorname{Im} w$, equal distances would imply $(\sigma - \operatorname{Re} v)^2 = (\sigma - \operatorname{Re} w)^2$. Both bases are positive, so $v = w$. Thus each distinct pair excludes at most one height. Choose $t \in (\operatorname{Im}\rho_1 - \xi, \operatorname{Im}\rho_1 + \xi)$ outside this finite set of heights and put $z = \sigma + it$. All distances from z to points of F are distinct, and

$$|z - \rho_1| \le |z - z_0| + |z_0 - \rho_1| < \xi + a < D.$$

Let ρ be the unique nearest point in F and set

$$r = \min\bigl(\{D\} \cup \{|z - w| : w \in F \setminus \{\rho\}\}\bigr).$$

Since $|z - \rho| \le |z - \rho_1| < D$, the choice of ρ gives $|z - \rho| < r \le D$. If $w \in Z \cap B(z, r)$, then $|w - z_0| < D + \xi$, so $w \in F$ and $w = \rho$. □

For any $\Phi : \mathbf{C} \to \mathbf{C}$, write $A_k(z) = (-1)^k \Phi^{(k)}(z)/k!$ for $k \in \mathbf{N}$. In Lemmas 41.10 and 41.11, Φ is arbitrary.

Lemma 41.10 (Cauchy's coefficient bound)**.** *If Φ is holomorphic on $B(z, r)$ and $0 < r' < r$, then some $M \in \mathbf{R}$ satisfies*

$$\frac{|\Phi^{(k)}(z)|}{k!} \le \frac{M}{(r')^k} \qquad (k \in \mathbf{N}).$$

Proof. Ahlfors gives Cauchy's estimate with the circle maximum. Taking absolute values inside his integral formula gives $M = (2\pi)^{-1} \int_0^{2\pi} |\Phi(z + r'e^{it})|\, dt$ [32, Chapter IV, §2.3, formulas (24)–(25)]. □

Lemma 41.11 (Coefficients of an isolated simple pole)**.** *Let $\rho \in B(z,r)$, $\rho \neq z$, and $c \in \mathbf{C}$. Suppose Φ is holomorphic on $B(z,r) \setminus \{\rho\}$ and agrees with $c/(s-\rho) + h(s)$ on a punctured neighbourhood of ρ, where h is holomorphic at ρ. For every $0 < r' < r$ there is $M \in \mathbf{R}$ such that*

$$\left| A_k(z) - \frac{c}{(z-\rho)^{k+1}} \right| \leq \frac{M}{(r')^k} \qquad (k \in \mathbf{N}).$$

Proof. After subtraction of the simple principal part, the remainder extends across ρ with value $h(\rho)$. Lemma 41.10 applies on the whole disc [32, Chapter IV, §§2.3, 3.2]. The normalisation follows from

$$\frac{d^k}{ds^k} \frac{c}{s-\rho} = \frac{(-1)^k k!\, c}{(s-\rho)^{k+1}}.$$

□

Lemma 41.12 (Incompatible exponential rates)**.** *Let $c > 0$ and $0 < d < e$. For every $C \in \mathbf{R}$ there are $\eta > 0$ and $U_* \in \mathbf{R}$ such that, for every $U \geq U_*$ and every $k \in \mathbf{N}$ with $k \geq U$,*

$$C\, \frac{\exp(\eta U)}{e^k} < \frac{c}{d^k}.$$

Consequently, for $e \leq E$ and fixed $M, C \in \mathbf{R}$, the assertion

$$\forall \eta > 0\ \forall U_0 \in \mathbf{R}\ \exists U \geq U_0\ \exists k \in \mathbf{N}, \quad k \geq U, \quad \frac{c}{d^k} \leq \frac{M}{e^k} + C \frac{\exp(\eta U)}{E^k} \tag{$*$}$$

is false.

Proof. Write $L = \log(e/d) > 0$ and $\eta = L/2$. After multiplication by d^k, the first left-hand side is at most

$$|C| \exp(\eta U - kL) \leq |C| \exp(-\eta U),$$

which is less than c for every sufficiently large U. The first assertion with coefficient $|M| + |C|$ gives $\eta > 0$ and U_*. Apply $(*)$ with this η and $U_0 = \max\{U_*, 0\}$. For the resulting U, k, the right-hand side of $(*)$ is at most $(|M| + |C|) \exp(\eta U)/e^k < c/d^k$, a contradiction. □

Assume the integral window hypothesis of 2.13, with $0 < \theta < 1$ and $\delta > 0$. By 41.1, there are fixed $\sigma > 2$ and $C \in \mathbf{R}$ such that, for every $\eta > 0$ and $U_0 \in \mathbf{R}$, some $U \geq U_0$ and $k \in \mathbf{N}$, $k \geq U$, satisfy

$$|A_k(\sigma + it)| + \operatorname{Re} A_k(\sigma) \leq C \frac{\exp(\eta U)}{(\sigma - \theta)^k} \qquad \text{for every } t \in \mathbf{R}. \tag{A.1}$$

Lemma 4.1 gives $\operatorname{Re}\rho < \sigma$ for every $\rho \in Z$.

Theorem 3.2 follows if both alternatives of Lemma 41.8 are excluded for the fixed σ, C in (A.1).

Lemma 41.13 (The moving-centre case)**.** *Under these hypotheses, the first alternative of Lemma 41.8 is impossible.*

Proof. Let D and ρ_1 be supplied by that alternative. Since $D \leq \sigma - \theta < \sigma$, Lemma 41.9 supplies $z = \sigma + it$, $\rho \in Z$, and $r \leq D$ with $d = |z - \rho| < r$ and $Z \cap B(z,r) = \{\rho\}$. Here $d > 0$, since $\operatorname{Re}\rho < \sigma$. Set

$$e = \frac{d+r}{2}, \qquad E = \sigma - \theta, \qquad 0 < d < e < r \leq D \leq E.$$

The discs $B(z,r)$ and $B(\sigma, D)$ lie in H because their centres have real part σ and their radii are at most $\sigma - \theta$. By Definition 3.4, Φ is holomorphic on $B(z,r) \setminus \{\rho\}$ and on $B(\sigma, D)$. Lemmas 41.11 and 41.10 give constants M_1, M_2, independent of k, with

$$\left| A_k(z) - \frac{c_\rho}{(z-\rho)^{k+1}} \right| \leq \frac{M_1}{e^k}, \qquad |A_k(\sigma)| \leq \frac{M_2}{e^k}.$$

The triangle inequality and $\operatorname{Re} A_k(\sigma) \geq -|A_k(\sigma)|$ therefore give

$$\frac{|c_\rho|/d}{d^k} \leq |A_k(z)| + \operatorname{Re} A_k(\sigma) + \frac{M_1 + M_2}{e^k}.$$

For the U, k supplied by (A.1), this is $(*)$ with $c = |c_\rho|/d > 0$. Lemma 41.12 gives the contradiction. □

Lemma 41.14 (The real-pole case)**.** *Under the same hypotheses, the second alternative of Lemma 41.8 is impossible. Hence every $\rho \in Z$ satisfies $\operatorname{Re}\rho \le \theta$.*

Proof. Let $\beta \in Z \cap \mathbf{R}$, $\beta > \theta$, and r be supplied by the second alternative. Put

$$d = \sigma - \beta > 0, \qquad e = \frac{d + r}{2}, \qquad E = \sigma - \theta.$$

Then $d < e < r \le E$, and $B(\sigma, r) \subset H$ contains the single pole β. Write $c_\beta > 0$ for its real residue. Lemma 41.11 gives a fixed M such that

$$\left| A_k(\sigma) - \frac{c_\beta}{d^{k+1}} \right| \le \frac{M}{e^k}.$$

Since c_β / d^{k+1} is real, the bound gives $c_\beta/d^{k+1} - \operatorname{Re} A_k(\sigma) \le M/e^k$. Adding $|A_k(\sigma)|$ to the right gives

$$\frac{c_\beta/d}{d^k} \le \operatorname{Re} A_k(\sigma) + |A_k(\sigma)| + \frac{M}{e^k}.$$

Use (A.1) at $t = 0$. For every $\eta > 0$ and every lower bound on U, the resulting inequality is $(*)$ with $c = c_\beta/d > 0$. Lemma 41.12 gives a contradiction.

If any $\rho_0 \in Z$ had $\operatorname{Re}\rho_0 > \theta$, Lemma 41.8 would give one of the two excluded alternatives. This proves Theorem 3.2. For a family F satisfying the window hypothesis of Theorem 2.12, Lemma 2.13 gives the logarithmic window condition for g_F. Apply Theorem 3.2 to the pole data of Theorem 3.1. A zero with $\theta < \operatorname{Re} s < 1$ would lie in Z_F and contradict the pole bound. Lemma 3.3 excludes zeros with $\operatorname{Re} s \ge 1$. □

Index of Lean declarations

For each statement of the main text, in the order of the book, the Lean declarations that prove it, with their files and line numbers in the archive.

Chapter 1: Five statements and the critical line

1.1. `riemannZeta_one` (`Mathlib.NumberTheory.Harmonic.ZetaAsymp`, lines 418–422)
`riemannZeta_one_ne_zero` (`Mathlib.NumberTheory.Harmonic.ZetaAsymp`, lines 441–450)
1.1. `riemannHypothesis` (`Main`, lines 4–4)
1.2. `riemannZeta_eq_zero_re` (`Main`, lines 6–6)
`RiemannZetaZerosOnLine` (`Statements`, lines 3–4)
1.3. `LFunction_ne_zero_of_half_lt_re` (`Main`, lines 8–8)
`DirichletLNonvanishingRightOfHalf` (`Statements`, lines 6–8)
1.4. `LFunction_eq_zero_re` (`Main`, lines 10–10)
`DirichletLZerosOnLine` (`Statements`, lines 10–12)
1.5. `LFunction_eq_zero_re_of_isPrimitive` (`Main`, lines 12–12)
`PrimitiveDirichletLZerosOnLine` (`Statements`, lines 14–16)
1.6. `Formalism.VW.rh_of_every` (`Formalism.GammaFactorNonvanishing.CritLine`, lines 29–32)
1.7. `Claim.strip` (`Development.AutomorphicArithmetic.CharacterBoxes.ClaimStrip`, lines 59–73)
1.8. `Formalism.VW.critLine` (`Formalism.GammaFactorNonvanishing.CritLine`, lines 11–27)
1.9. `allChi` (`Supplements.Kernels.MainTheorems`, lines 7–7)
1.10. `UpperTheory.EveryCharacter` (`Development.AutomorphicArithmetic.CharacterBoxes.EveryCharacter`, lines 13–14)
`Formalism.HC.PChar` (`Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.PrimePairSizeCondition`, lines 7–11)
`Formalism.HC.PChar.ZeroFreeAbove` (`Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.ExplicitFormulaErrorOfHostConditions`, lines 11–12)
`Formalism.HC.ZeroFreeAll` (`Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.ExplicitFormulaErrorOfHostConditions`, lines 18–18)
1.11. `Formalism.VW.rh_of_right` (`Formalism.GammaFactorNonvanishing.RhOfRight`, lines 5–62)
1.12. `DirichletCharacter.LFunction_changeLevel` (`Mathlib.NumberTheory.LSeries.DirichletContinuation`, lines 150–162)
`Claim.factor_ne_zero` (`Development.AutomorphicArithmetic.CharacterBoxes.ClaimStrip`, lines 7–21)
`Claim.prim_zero` (`Development.AutomorphicArithmetic.CharacterBoxes.ClaimStrip`, lines 23–33)
1.13. `Claim.prim_reflect` (`Development.AutomorphicArithmetic.CharacterBoxes.ClaimStrip`, lines 42–57)
`Claim.gamma_ne_zero` (`Development.AutomorphicArithmetic.CharacterBoxes.ClaimStrip`, lines 35–40)
`Formalism.VW.gammaFactor_ne_zero` (`Formalism.GammaFactorNonvanishing.CritLine`, lines 6–9)
`DirichletCharacter.LFunction_eq_completed_div_gammaFactor` (`Mathlib.NumberTheory.LSeries.DirichletContinuation`, lines 259–264)
1.13. `DirichletCharacter.IsPrimitive.completedLFunction_one_sub` (`Mathlib.NumberTheory.LSeries.DirichletContinuation`, lines 284–313)
1.14. `UpperTheory.Top.imprimitive` (`Development.AutomorphicArithmetic.CharacterBoxes.Imprimitive`, lines 5–35)
1.15. `GlobalTheory.RowTest.NumbersIn` (`Development.AbelianVarietyTheory.Definitions.TwistedInnerProducts.MemHalfCosetSelf`, lines 245–261)
1.15. `GlobalTheory.Mem.LiteralAutomorphy.JoinOut` (`Development.AbelianVarietyTheory.Definitions.Divisors.HypothesisBundleConversions`, lines 7–9)
`GlobalTheory.IndefiniteRowSideFullHecke.ClassZetaValuationBounds.ToricDenominator` (`Development.AbelianVarietyTheory.BlockTheory.DivisorBase.VpBounds`, lines 77–82)
`GlobalTheory.IndefiniteRowSideFullHecke.LatticeNormTransport.Transport` (`Development.AbelianVarietyTheory.BlockTheory.DivisorBase.LatticeNormTransport`, lines 86–98)
`GlobalTheory.ToricTransport.TwistedTransport` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.TwistedTransport`, lines 21–39)
`GlobalTheory.IndefiniteRowSideFullHecke.ZetaValueIdeals.ZetaDenominators` (`Development.AbelianVarietyTheory.BlockTheory.DivisorBase.ZetaValueIdeals`, lines 45–46)
`GlobalTheory.SqTop.Refl` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.NormSmall`, lines 148–151)

1.15. Formalism.MainNotions.LinkN (Formalism.SharpRestriction.GlobalTheoremInputs, lines 45–47)
GlobalTheory.Fc.HypsAlt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.Hyps, lines 22–90)
1.16. GlobalTheory.QFam.sz_value_of_family (Development.AbelianVarietyTheory.Proofs.DivisorProofs.SzValueOfFamily, lines 23–43)
1.17. GlobalTheory.CMSize.main (Development.AbelianVarietyTheory.Proofs.DivisorProofs.DenominatorBoundWeierstrassCmTorsion, lines 30–96)
1.17. GlobalTheory.IsogFormula.main (Development.AbelianVarietyTheory.Proofs.DivisorProofs.IsogenyFormulaWeierstrassPIntegral, lines 16–73)
1.18. GlobalTheory.LatJet.JetInt (Development.AbelianVarietyTheory.Definitions.Divisors.JetAnalytic, lines 239–245)
GlobalTheory.JetIntElliptic.jetInt (Development.AbelianVarietyTheory.Proofs.DivisorProofs.CMLatticeDividedPowerJetIntegralityOfHyp, lines 251–252)
1.18. GlobalTheory.JetIntElliptic.core (Development.AbelianVarietyTheory.Proofs.DivisorProofs.CMLatticeDividedPowerJetIntegralityOfHyp, lines 142–249)
1.19. GlobalTheory.JoinFun.joinOut_of_five (Development.AbelianVarietyTheory.Proofs.DivisorProofs.JoinOutOfFive, lines 15–17)
UpperTheory.LiteratureTheorems.join_out_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 233–234)
1.20. UpperTheory.GalS.thr (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.TypeConditions.Transported, lines 11–12)
UpperTheory.GalS.NcW (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.TypeConditions.WeightsConj, lines 27–27)
1.20. GlobalTheory.Fc.S0.UnitaryGroupThirdPlaceHypotheses (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.BernsteinUnitaryProductIrreducible, lines 22–82)
GlobalTheory.FunPr.ThetaLocalInputs (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.BernsteinUnitaryProductIrreducible, lines 6–59)
GlobalTheory.Mem.LiteralAutomorphy.NonEndoscopicHypothesesAlt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.BernsteinUnitaryProductIrreducible, lines 12–17)
GlobalTheory.NRows.GeometricGaloisRepCompatibility (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.GeometricGaloisRepCompatibility, lines 6–18)
UpperTheory.LiteratureTheorems.published_results_local (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 658–665)
1.21. PositivityTheory.KReal.tensorIrr (Development.WeilBound.CoreStatements.GaussianSeminormBounds.RealSeminormFiltration.Tensor, lines 131–165)
GlobalTheory.ThK.Dbl.pairsIrr_of_arch (Development.WeilBound.ThetaLifts.LatticeBasics.PairsIrrOfArch, lines 10–11)
1.22. UpperTheory.LiteratureTheorems.published_results_eigen (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 666–726)
UpperTheory.LiteratureTheorems.published_results_first (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 727–780)
1.23. UpperTheory.FinalTheorems.everyCharacter_of_row (Development.AutomorphicArithmetic.CharacterBoxes.MainConclusion, lines 7–8)
UpperTheory.FinalTheorems.everyCharacter (Development.AutomorphicArithmetic.CharacterBoxes.MainConclusion, lines 10–11)
1.23. UpperTheory.LiteratureTheorems.local_langlands_gl_recT_exists_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 313–314)
UpperTheory.LiteratureTheorems.every_character_of_literature_of_vWin (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 810–813)
UpperTheory.Lit.every_character_of_theta_integrals_local_matching (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.SiegelEisensteinSecondTermHypotheses, lines 69–73)
1.24. UpperTheory.Top.every_character_of_green_v_win (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.TopPlug, lines 16–18)
1.25. UpperTheory.Top.zero_free_all_of_green_v_win (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.TopPlug, lines 12–15)

Chapter 2: How arithmetic windows enter the zero-free criterion

2.1. GlobalTheory.EX.HostVal (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.PairRowIffQdRows, lines 78–84)
GlobalTheory.EX.HostGood (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.ThinScalesIffHostBound, lines 101–109)
GlobalTheory.EX.HostExW (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.ThinScalesIffHostBound, lines 120–123)
UpperTheory.Top.FReq.req (Development.AutomorphicArithmetic.CharacterBoxes.Supply, lines 16–16)
UpperTheory.Top.GChi (Development.AutomorphicArithmetic.CharacterBoxes.Supply, lines 42–43)
UpperTheory.Top.VVle (Development.AutomorphicArithmetic.CharacterBoxes.ExistsPageN, lines 12–17)

2.2. UpperTheory.Top.zeroFreeAll_of_topG (Development.AutomorphicArithmetic.CharacterBoxes.ZeroFreeAllOfTopG, lines 26–34)
UpperTheory.Top.TopG (Development.AutomorphicArithmetic.CharacterBoxes.ZeroFreeAllOfTopG, lines 12–14)
2.2. UpperTheory.Comparison₂.level_gm_le_on_family_scales (Development.AutomorphicArithmetic.PrimitiveCosets.LevelGmLeOnFamilyScales, lines 28–30)
2.3. Formalism.HC.zeroFreeOn_of_tiedWin_cyc (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.TopWin, lines 90–93)
Formalism.HC.zeroFreeOn_of_tiedWin (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.TopWin, lines 83–88)
2.3. Formalism.HC.Level.mem_cyc (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.ExplicitFormulaErrorOfHostConditions, lines 123–124)
2.4. Formalism.HC.level_of_tiedWin (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.TopWin, lines 68–81)
2.5. Formalism.HC.KSet (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.PrimePairSizeCondition, lines 39–45)
Formalism.HC.Level (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.ExplicitFormulaErrorOfHostConditions, lines 8–9)
Formalism.HC.Level.wt (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.ExplicitFormulaErrorOfHostConditions, lines 36–36)
Formalism.HC.Level.ofSub (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.ExplicitFormulaErrorOfHostConditions, lines 105–114)
Formalism.HC.Level.Gm (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.ExplicitFormulaErrorOfHostConditions, lines 38–38)
Formalism.HC.Level.cyc (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.ExplicitFormulaErrorOfHostConditions, lines 121–121)
2.6. UpperTheory.Comparison₂.formB (Development.AutomorphicArithmetic.PrimitiveCosets.Imports, lines 8–19)
Formalism.HC.Level.FormBAt (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.Window, lines 30–31)
Formalism.HC.Level.FormB (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.Window, lines 33–33)
2.7. UpperTheory.Comparison₂.level_family (Development.AutomorphicArithmetic.PrimitiveCosets.LevelFamily, lines 41–52)
2.8. Deduction.zeroFree_of_rpow_window (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ZeroFreeOfRpowWindow, lines 11–14)
2.9. Deduction.CharFamily (Development.AutomorphicArithmetic.DedekindLFunctions.CharFamily, lines 24–32)
Deduction.ZeroFree (Development.AutomorphicArithmetic.DedekindLFunctions.CharFamily, lines 38–39)
2.10. Deduction.ramp (Development.AutomorphicArithmetic.DedekindLFunctions.CharFamily, lines 10–11)
Deduction.Gchi (Development.AutomorphicArithmetic.DedekindLFunctions.CharFamily, lines 15–19)
Deduction.Gm (Development.AutomorphicArithmetic.DedekindLFunctions.CharFamily, lines 36–36)
Deduction.gm (Development.AutomorphicArithmetic.DedekindLFunctions.LandauZeroFreeCriterion, lines 9–9)
2.11. Deduction.WindowHyp (Development.AutomorphicArithmetic.DedekindLFunctions.CharFamily, lines 41–45)
Deduction.windowHyp_of_rpow (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.WindowHypOfRpow, lines 27–57)
2.12. Deduction.lemmaL (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.lemmaL, lines 13–22)
2.13. Deduction.WindowHypU (Development.AutomorphicArithmetic.DedekindLFunctions.LandauZeroFreeCriterion, lines 25–29)
Deduction.windowHypU_of_windowHyp (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.WindowHypUOfWindowHyp, lines 29–56)

Chapter 3: Zeros become poles

3.1. Deduction.poleData (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.poleData, lines 248–400)
3.2. Deduction.landau_window (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.landauWindow, lines 83–166)
3.3. Deduction.charFamily_LFunction_ne_zero_of_one_le_re (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.CharFamilyLFunctionNeZeroOfOneLeRe, lines 9–20)
3.4. Deduction.zeroSet (Development.AutomorphicArithmetic.DedekindLFunctions.LandauZeroFreeCriterion, lines 11–13)
Deduction.PoleData (Development.AutomorphicArithmetic.DedekindLFunctions.LandauZeroFreeCriterion, lines 15–23)
3.5. Deduction.poleData_char (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.poleData, lines 137–239)
3.6. Deduction.logDeriv_local (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.LogDerivLocal, lines 9–39)
3.7. Deduction.poleData_abstract (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.poleData, lines 22–89)
Deduction.poleData_dslope (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.poleData, lines 17–20)
3.8. Deduction.poleData_change (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.poleData, lines 114–135)

Deduction.poleData_integrable (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.poleData, lines 91–112)
3.9. Deduction.measurable_Gchi (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.measurableGchi, lines 24–28)
Deduction.Gchi_growth_ramp (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.GchiGrowth, lines 9–35)
Deduction.Gchi_growth (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.GchiGrowth, lines 37–66)
3.10. Deduction.mellin_ramp (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.MellinRamp, lines 10–51)
Deduction.mellin_Gchi_of_ne_one_norm (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.mellinGchiOfNeOne, lines 27–41)
3.11. Deduction.mellin_Gchi_of_ne_one_ramp (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.mellinGchiOfNeOne, lines 65–114)
Deduction.mellin_Gchi_modOne_sum (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.mellinGchiModOne, lines 10–25)
3.12. Deduction.mellin_Gchi_of_ne_one (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.mellinGchiOfNeOne, lines 116–142)
3.13. Deduction.mellin_Gchi_modOne_log_int (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.mellinGchiModOne, lines 57–83)
Deduction.mellin_Gchi_modOne_log (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.mellinGchiModOne, lines 85–113)
3.14. Deduction.mellin_Gchi_modOne (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.mellinGchiModOne, lines 115–199)
3.15. Deduction.LFunctionConjugation.hurwitz_conj (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.LFunctionConj, lines 76–84)
Deduction.LFunctionConjugation.Gamma$\mathbb{R}$_conj (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.LFunctionConj, lines 19–24)
Deduction.LFunctionConjugation.differentiable_reg (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.LFunctionConj, lines 38–48)
HurwitzZeta.tendsto_hurwitzZeta_sub_one_div_nhds_one (Mathlib.NumberTheory.LSeries.HurwitzZeta, lines 96–100)
3.16. Deduction.LFunction_conj (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.LFunctionConj, lines 88–96)
Deduction.Gchi_conj (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.GchiConj, lines 38–51)
Deduction.poleData_inv_zero (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.poleData, lines 241–246)
3.17. Deduction.poleData (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.poleData, lines 248–400)

Chapter 4: A Gaussian window excludes poles

4.1. Deduction.landau_pole_re_le_bound (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.LandauPoleReLe, lines 9–27)
Deduction.landau_pole_re_le (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.LandauPoleReLe, lines 29–53)
4.2. Deduction.landau_finite (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.LandauFinite, lines 9–26)

Chapter 5: Counting primes class by class

5.1. Formalism.HC.Frame (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.ZsmulGeneratorEqZeroOfProjection, lines 15–23)
Formalism.HC.HostRaw (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.ConjEMemHostFIff, lines 12–46)
Formalism.HC.HostDer (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.ConjEMemHostFIff, lines 87–170)
Formalism.HC.HostAx (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.ConjEMemHostFIff, lines 174–175)
5.2. UpperTheory.Comparison$_2$.toE_bij (Development.AutomorphicArithmetic.PrimitiveCosets.ToEBij, lines 37–84)
UpperTheory.Comparison$_2$.chr_bij (Development.AutomorphicArithmetic.PrimitiveCosets.ChrBij, lines 21–90)
5.3. UpperTheory.Comparison$_2$.hostFactsS (Development.AutomorphicArithmetic.PrimitiveCosets.hostFactsS, lines 127–136)
Deduction.CH.HostFactsS (Development.AutomorphicArithmetic.DedekindLFunctions.SplitInertia, lines 14–22)
5.4. Formalism.LFunctionChainParameters.characterSumData.primVal (Formalism.SizeWindowBrackets.FirstPrimeNeSecondPrime, lines 65–66)
Formalism.LFunctionChainParameters.characterSumData.GPsi (Formalism.SizeWindowBrackets.FirstPrimeNeSecondPrime, lines 68–70)

Formalism.LFunctionChainParameters.characterSumData.ASK (Formalism.SizeWindowBrackets.FirstPrimeNeSecondPrime, lines 84–87)
Formalism.LFunctionChainParameters.characterSumData.MAIN (Formalism.SizeWindowBrackets.FirstPrimeNeSecondPrime, lines 89–91)
Formalism.LFunctionChainParameters.characterSumData.De (Formalism.SizeWindowBrackets.FirstPrimeNeSecondPrime, lines 93–96)
5.5. Formalism.LFunctionChainParameters.characterSumData.orthogonality_ramified (Formalism.SizeWindowBrackets.ChainSlopeWindows.StronglyIrreducible, lines 311–316)
Formalism.LFunctionChainParameters.characterSumData.OrthoRam (Formalism.SizeWindowBrackets.FirstPrimeNeSecondPrime, lines 116–121)
5.6. Deduction.CharacterBalance.askExp (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ChBalance, lines 122–196)
5.7. Deduction.ch_balance (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ChBalance, lines 200–260)
5.8. Deduction.CharacterBalanceMoves.sum_A (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ChCore, lines 28–54)
Deduction.CharacterBalanceMoves.sum_Th (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ChCore, lines 56–73)
5.9. Deduction.CharacterBalanceMoves.th_le (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ChCore, lines 75–152)
5.10. Deduction.CharacterBalanceMoves.core (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ChCore, lines 283–338)

Chapter 6: Reading off the target sum

6.1. Deduction.CharacterBalanceMoves.move (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ChCore, lines 154–165)
Deduction.CharacterBalanceMoves.triv (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ChCore, lines 167–179)
Formalism.LFunctionChainParameters.ThickSetMeanBounds.DerivInt (Formalism.SizeWindowBrackets.GrowthBoundGivesRiemannHypothesis, lines 62–65)
6.2. Deduction.CharacterBalanceMoves.one_char (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ChCore, lines 202–281)
Deduction.CharacterBalanceMoves.sum_vv (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ChCore, lines 187–200)
6.3. Deduction.ch_top_main_eq (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ChTop, lines 11–35)
Deduction.ch_top (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ChTop, lines 37–72)
6.4. Formalism.LogFactorialMertensBounds.mertens_lower (Formalism.ZetaNonvanishing.GzLipschitz, lines 86–105)
Formalism.LogFactorialMertensBounds.mertens_upper (Formalism.ZetaNonvanishing.GzLipschitz, lines 55–84)
6.5. Deduction.ch_core (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ChCore, lines 343–442)
6.6. UpperTheory.Comparison₂.prim_read (Development.AutomorphicArithmetic.PrimitiveCosets.PrimRead, lines 48–97)
UpperTheory.Comparison₂.prim_inj (Development.AutomorphicArithmetic.PrimitiveCosets.PrimInj, lines 6–27)
Formalism.LFunctionChainParameters.LHost.primitive_value_primitive (Formalism.SizeWindowBrackets.ChainSlopeWindows.GaloisMapComplexConjugation, lines 233–245)
6.7. UpperTheory.Comparison₂.card_PsiA (Development.AutomorphicArithmetic.PrimitiveCosets.CardPsiA, lines 60–63)
card_PsiA_index (Development.AutomorphicArithmetic.PrimitiveCosets.CardPsiA, lines 26–58)
Formalism.HC.Frame.index_A0_eq_o (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.FrameFacts, lines 214–217)
6.8. UpperTheory.Comparison₂.charsA (Development.AutomorphicArithmetic.PrimitiveCosets.charsA, lines 80–125)
Formalism.HC.Frame.Shares (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.ExplicitFormulaErrorOfHostConditions, lines 139–140)
6.9. UpperTheory.Comparison₂.gmA_eq (Development.AutomorphicArithmetic.PrimitiveCosets.GmAEq, lines 61–149)
Formalism.HC.Level.ofSub (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.ExplicitFormulaErrorOfHostConditions, lines 105–114)
Formalism.HC.Level.cyc (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.ExplicitFormulaErrorOfHostConditions, lines 121–121)
6.10. UpperTheory.Comparison₂.sigA_le (Development.AutomorphicArithmetic.PrimitiveCosets.SigALe, lines 38–60)
6.10. UpperTheory.Top.boundedTw_key (Development.AutomorphicArithmetic.CharacterBoxes.BoundedTw, lines 5–43)
UpperTheory.T3.supplyOfAdmOf (Development.AutomorphicArithmetic.ShortNorms.supplyOfAdmOf, lines 13–135)
6.11. UpperTheory.Comparison₂.CoreAtH (Development.AutomorphicArithmetic.PrimitiveCosets.CoreAtH, lines 29–41)
6.12. UpperTheory.Comparison₂.core_gm (Development.AutomorphicArithmetic.PrimitiveCosets.CoreGm, lines 12–74)

Chapter 7: Choosing the auxiliary primes q and d

7.1. QH.S (Development.AutomorphicArithmetic.MeanSquareBounds.PrimeSetTwoBounds, lines 13–14)
QH.admitted (Development.AutomorphicArithmetic.MeanSquareBounds.PrimeSetTwoBounds, lines 18–24)
QH.MeanSq (Development.AutomorphicArithmetic.MeanSquareBounds.PrimeSetTwoBounds, lines 26–27)
QH.BoxAll (Development.AutomorphicArithmetic.MeanSquareBounds.PrimeSetTwoBounds, lines 38–39)
Formalism.HermitianLattices.ZeroFreeRegionGL2.PPset (Formalism.GL3MeanSquares.SecondEstimateOfIntervalHypothesis, lines 50–51)
7.2. Formalism.HermitianLattices.Ext.PrimesInProgressions.DeuringHeilbronnMB (Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTDHBSt, lines 11–17)
Formalism.HermitianLattices.Ext.PrimesInProgressions.deuringHeilbronnMB_proof' (Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTFinal, lines 20–20)
7.3. Formalism.ZeroFreeDirichlet.zFRDirichlet (Formalism.LandauZeroFreeBounds.NoZeroComplex, lines 32–33)
7.4. Formalism.HermitianLattices.Ext.LogarithmicIntegralBounds.page_across (Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.MSqPage, lines 37–65)
Formalism.ZeroFreeDirichlet.excBound (Formalism.LandauZeroFreeBounds.SiegelWalfiszAPOfSiegelLower, lines 32–32)
7.5. Formalism.HermitianLattices.Ext.LogarithmicIntegralBounds.PNTAPRelW (Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.MeanSquareHypWIff, lines 38–46)
Formalism.HermitianLattices.Ext.PrimesInProgressions.pnt_progressions_relative_weak_holds (Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.PNTFinal, lines 22–24)
7.6. Formalism.HermitianLattices.Ext.LogarithmicIntegralBounds.count_a (Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.MSqCount, lines 346–397)
7.7. QH.twistK (Development.AutomorphicArithmetic.MeanSquareBounds.twistK, lines 148–353)
7.8. QH.meanSq_abs (Development.AutomorphicArithmetic.MeanSquareBounds.MeanSqAbs, lines 8–58)
7.9. QH.RR_le (Development.AutomorphicArithmetic.MeanSquareBounds.RRLe, lines 9–99)
QH.sPsq_le (Development.AutomorphicArithmetic.MeanSquareBounds.SPsqLe, lines 161–171)
QH.exists_primePowerSum_le_mul_log (Development.AutomorphicArithmetic.MeanSquareBounds.SumMinLeDiv, lines 71–101)
QH.sum_cN_le (Development.AutomorphicArithmetic.MeanSquareBounds.SumCNLe, lines 8–27)
7.10. QH.meanSqK (Development.AutomorphicArithmetic.MeanSquareBounds.meanSqK, lines 51–221)
7.11. QH.bad_share (Development.AutomorphicArithmetic.MeanSquareBounds.BadShare, lines 251–318)
7.12. Formalism.HermitianLattices.Ext.PrimitiveCharacterCasts.ZeroDensityBox (Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.CountSt, lines 7–11)
Formalism.HermitianLattices.Ext.PrimitiveCharacterCasts.ZD.zero_density_box (Formalism.GL3MeanSquares.GenusMeanAsymptotics.JacobiCharacterSums.CountZAll, lines 13–15)
7.13. QH.count_bad_core (Development.AutomorphicArithmetic.MeanSquareBounds.CountBadCore, lines 18–59)
QH.count_bad (Development.AutomorphicArithmetic.MeanSquareBounds.CountBad, lines 9–16)
7.14. QH.twist_prim (Development.AutomorphicArithmetic.MeanSquareBounds.TwistPrim, lines 9–18)
QH.ind_unique (Development.AutomorphicArithmetic.MeanSquareBounds.IndUnique, lines 7–51)
7.15. QH.tailK (Development.AutomorphicArithmetic.MeanSquareBounds.TailKExp, lines 92–145)
7.16. QH.kappaC (Development.AutomorphicArithmetic.MeanSquareBounds.kappaC, lines 9–62)
QH.kappa4 (Development.AutomorphicArithmetic.MeanSquareBounds.Kappa4Loglog, lines 126–153)
7.17. QH.box_of_S (Development.AutomorphicArithmetic.MeanSquareBounds.BoxOfS, lines 28–60)
7.18. QH.stage (Development.AutomorphicArithmetic.MeanSquareBounds.StagePow, lines 41–170)
7.19. QH.GoodQ (Development.AutomorphicArithmetic.MeanSquareBounds.PrimeSetTwoBounds, lines 48–49)
QH.GoodD (Development.AutomorphicArithmetic.MeanSquareBounds.PrimeSetTwoBounds, lines 51–53)
7.20. QH.stage1 (Development.AutomorphicArithmetic.MeanSquareBounds.Stage1Succ, lines 36–95)
7.21. QH.stage2 (Development.AutomorphicArithmetic.MeanSquareBounds.CharOf, lines 61–217)
7.22. QH.pair_share (Development.AutomorphicArithmetic.MeanSquareBounds.PairShare, lines 26–52)
7.23. QH.pair_exists (Development.AutomorphicArithmetic.MeanSquareBounds.PairExists, lines 9–49)
7.24. UpperTheory.Top.WinPair (Development.AutomorphicArithmetic.CharacterBoxes.ExistsPageN, lines 27–37)
UpperTheory.Top.PairGood (Development.AutomorphicArithmetic.CharacterBoxes.PairGood, lines 18–26)
7.25. QH.pairEx (Development.AutomorphicArithmetic.MeanSquareBounds.pairEx, lines 112–169)
UpperTheory.Top.PairEx (Development.AutomorphicArithmetic.CharacterBoxes.PairGood, lines 28–30)

Chapter 8: Building the arithmetic host of a window

8.1. UpperTheory.Top.ordAny (Development.AutomorphicArithmetic.CharacterBoxes.OrdAny, lines 86–107)
Formalism.HC.AnyGrp.o_eq (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.InertiaGaloisSymmetry.AnyLevel, lines 29–34)
UpperTheory.Top.kerChiAny (Development.AutomorphicArithmetic.CharacterBoxes.KerChiAny, lines 104–141)
8.2. UpperTheory.Top.twistAny (Development.AutomorphicArithmetic.CharacterBoxes.TwistAny, lines 115–168)
UpperTheory.Top.TwistForm (Development.AutomorphicArithmetic.CharacterBoxes.TwistForm, lines 14–17)
8.3. UpperTheory.Top.fieldChi (Development.AutomorphicArithmetic.CharacterBoxes.FieldChi, lines 20–59)
8.4. UpperTheory.Top.boundedTw (Development.AutomorphicArithmetic.CharacterBoxes.BoundedTw, lines 45–56)
8.5. UpperTheory.Top.winPairs (Development.AutomorphicArithmetic.CharacterBoxes.WinPairs, lines 5–81)
8.6. UpperTheory.Top.winPairs (Development.AutomorphicArithmetic.CharacterBoxes.WinPairs, lines 5–81)

UpperTheory.Top.WinDatum (Development.AutomorphicArithmetic.CharacterBoxes.TwistForm, lines 19–26)
8.7. UpperTheory.T3.smallOne (Development.AutomorphicArithmetic.ShortNorms.Primes, lines 7–7)
8.7. GlobalTheory.T.primesInProgressions_relativeError (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.PrimesInProgressionsRelativeErrorStatement, lines 8–9)
GlobalTheory.Lin.PrimesRel (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.ZerosDisc, lines 35–43)
8.7. GlobalTheory.Hk.LFam (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.LamIdealOfNotPrimePow, lines 12–40)
8.8. GlobalTheory.Hk.ShortInt.zeros (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.TrivialClassShortInterval, lines 20–101)
8.9. GlobalTheory.Hk.PsiExc.psi_exceptional (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.RayClassFamilyPsiWithExceptionalZero, lines 438–502)
8.10. GlobalTheory.Hk.ShortInt.trivialClass_shortInterval (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.TrivialClassShortInterval, lines 218–281)
GlobalTheory.T.exists_lam_zero_sum_shortInterval_ge (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.TrivialClassShortIntervalStatement, lines 8–9)
8.11. UpperTheory.T3.shortNorm (Development.AutomorphicArithmetic.ShortNorms.Primes, lines 5–5)
GlobalTheory.T.normPrimesImagQuad_short_interval_lower_bound (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.NormPrimesImagQuadShortIntervalLowerBoundStatement, lines 8–9)
GlobalTheory.Mem.An.NormOne (Development.AbelianVarietyTheory.Definitions.Foundations.RayPrimes, lines 40–41)
8.11. GlobalTheory.T.exists_lFam_normOne_of_mod_four_eq_three (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsLFamRayStatement, lines 8–9)
GlobalTheory.Hk.normPrimes_short_interval_lower (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.NormPrimesImagQuadShortIntervalLowerBound, lines 98–147)
8.12. UpperTheory.T3.levelData (Development.AutomorphicArithmetic.ShortNorms.levelData, lines 36–77)
UpperTheory.T3.typePrimesN (Development.AutomorphicArithmetic.ShortNorms.typePrimesN, lines 78–129)
8.13. UpperTheory.Top.realBox (Development.AutomorphicArithmetic.CharacterBoxes.RealBox, lines 99–112)
QH.thinH (Development.AutomorphicArithmetic.MeanSquareBounds.GaussSumFunctionEq, lines 55–107)
Formalism.HC.HostRaw.ChebyshevCharacterBound (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.Analytic, lines 145–147)
8.14. UpperTheory.BernoulliSumsFrobeniusOrbits.thickFrame2 (Development.AutomorphicArithmetic.ThickFrames.ThickFrame2Large, lines 29–198)
Formalism.FrobeniusOrbits.NoRel (Formalism.FrobeniusOrbits.HostThickExists, lines 13–16)
8.15. UpperTheory.BernoulliSumsFrobeniusOrbits.topsChoiceH (Development.AutomorphicArithmetic.ThickFrames.topsChoiceH, lines 30–67)
8.16. UpperTheory.Top.pairRows (Development.AutomorphicArithmetic.CharacterBoxes.PairRows, lines 334–387)
8.17. UpperTheory.Top.hostOfField (Development.AutomorphicArithmetic.CharacterBoxes.HostOfField, lines 8–15)
UpperTheory.T3.hostOfFieldOf (Development.AutomorphicArithmetic.ShortNorms.hostOfFieldOf, lines 159–255)
8.18. UpperTheory.Top.admissibleClassesTwistedChar (Development.AutomorphicArithmetic.CoprimeRanks.AdmChi, lines 13–20)
8.19. Formalism.HC.EAtH (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.ExplicitFormulaErrorOfHostConditions, lines 144–151)
UpperTheory.Top.GChi (Development.AutomorphicArithmetic.CharacterBoxes.Supply, lines 42–43)
8.19. GlobalTheory.EX.HostExW (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.ThinScalesIffHostBound, lines 120–123)
8.20. UpperTheory.T3.supplyOfAdmOf (Development.AutomorphicArithmetic.ShortNorms.supplyOfAdmOf, lines 13–135)
UpperTheory.Top.supplyOfAdm (Development.AutomorphicArithmetic.CharacterBoxes.HostOfField, lines 17–23)
8.21. UpperTheory.Top.supplyChi (Development.AutomorphicArithmetic.CharacterBoxes.SupplyChi, lines 8–16)

Chapter 9: Background: the Weil representation and theta lifts

9.6. GlobalTheory.IndefiniteRowSideFullHecke.SphericalZero.SectionClause (Development.AbelianVarietyTheory.BlockTheory.TwistedDiscreteSeries.SphericalZero, lines 69–77)
9.7. GlobalTheory.Published.SiegelWeilUnitaryConvergent (Development.AbelianVarietyTheory.PublishedTheorems.SiegelWeil₂.SiegelWeilUnitaryConvergent, lines 18–32)
GlobalTheory.ThK.Dbl.SiegelWeilStd (Development.WeilBound.ThetaLifts.LatticeBasics.HypothesesS, lines 11–21)
PositivityTheory.SiegelWeilFormula.Fin.siegelWeilStd (Development.WeilBound.CoreStatements.AutomorphicKernels.SiegelWeilFormula.SiegelWeilStandard.SiegelWeilStd, lines 8–8)
9.8. GlobalTheory.Published.RallisInnerProductConvergent (Development.AbelianVarietyTheory.PublishedTheorems.TwistedTraceFormula.LowerLiftsZeroOfWittIdxEqZero, lines 41–59)

Chapter 10: The local kernel and the differential operator

10.1. Formalism.MainNotions.G.liftWt (Formalism.SharpRestriction.GrowthBounds.DataGChain, lines 24–25)
Formalism.MainNotions.G.deltaL (Formalism.SharpRestriction.GrowthBounds.DataG, lines 11–14)
Formalism.MainNotions.G.Places (Formalism.SharpRestriction.GrowthBounds.DataG, lines 33–53)

`Formalism.MainNotions.G.HidaSetting` (`Formalism.SharpRestriction.GrowthBounds.UnramifiedQuadraticSetting`, lines 12–29)
`Formalism.GradedWeylDivisibility.weyl` (`Formalism.GradedWeylDivisibility.StatementA`, lines 71–73)
`Formalism.GradedWeylDivisibility.rowGen` (`Formalism.GradedWeylDivisibility.StatementA`, lines 35–38)
`Formalism.MainNotions.G.Places.HtupW` (`Formalism.SharpRestriction.GrowthBounds.DataG`, lines 79–80)
`Formalism.L2.PairingIndexBounds.Ent.delta_zero_eq_twisted_delta` (`Formalism.LocalToGlobal.PairingIndexBounds.RowActionCoefficients.AdmissibleInterpolation.Bridge`, lines 45–62)
10.2. `Formalism.MainNotions.FormalGroupSystem.Std` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 89–101)
`Formalism.MainNotions.FormalGroupSystem.Sys` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 264–277)
`Formalism.MainNotions.FormalGroupSystem.Sys.vsc` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 309–309)
`Formalism.MainNotions.FormalGroupSystem.Sys.vGe` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 311–311)
`Formalism.MainNotions.FormalGroupSystem.Pend` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 447–493)
10.3. `Formalism.MainNotions.FormalGroupSystem.NN` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 26–26)
`Formalism.MainNotions.FormalGroupSystem.nu` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 28–31)
`Formalism.MainNotions.FormalGroupSystem.al` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 33–34)
`Formalism.MainNotions.FormalGroupSystem.cap` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 36–38)
`Formalism.MainNotions.FormalGroupSystem.lsz` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 40–42)
10.4. `Formalism.MainNotions.FormalGroupSystem.lgp` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 44–44)
`Formalism.MainNotions.FormalGroupSystem.de` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 46–49)
`Formalism.InvariantFactorPaths.G7.deLoss` (`Formalism.ThetaLogCoordinates.UniformThetaSubstitution.StatementAB`, lines 100–101)
10.5. `Formalism.MainNotions.FormalGroupSystem.logF` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 126–127)
`Formalism.MainNotions.FormalGroupSystem.inParam` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 129–130)
`Formalism.MainNotions.FormalGroupSystem.inParamK` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 132–134)
`Formalism.MainNotions.FormalGroupSystem.expS` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 136–137)
`Formalism.MainNotions.FormalGroupSystem.Simeq` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 139–142)
10.6. `Formalism.MainNotions.FormalGroupSystem.BeProp` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 147–158)
`Formalism.MainNotions.FormalGroupSystem.ExtensionLocalCondition` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 162–170)
`Formalism.MainNotions.FormalGroupSystem.ExtensionGeneratorCondition` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 237–243)
`Formalism.MainNotions.FormalGroupSystem.adm2` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 548–550)
`Formalism.MainNotions.FormalGroupSystem.adm2b` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 552–554)
10.7. `Formalism.L2.KerInert.Jet.beJet_of_prop` (`Formalism.LocalToGlobal.InertKernelTheory.InertCoprimeLemmas.HurwitzSubstituteTwoSeries`, lines 999–1002)
10.8. `Formalism.MainNotions.G.liftWt` (`Formalism.SharpRestriction.GrowthBounds.DataGChain`, lines 24–25)
`Formalism.GradedWeylDivisibility.dualWeyl` (`Formalism.GradedWeylDivisibility.StatementA`, lines 24–29)
`Formalism.MainNotions.G.Vsp` (`Formalism.SharpRestriction.GrowthBounds.UnramifiedQuadraticSetting`, lines 84–86)
`Formalism.MainNotions.G.Places.Vtup` (`Formalism.SharpRestriction.GrowthBounds.DataG`, lines 87–89)
10.9. `Formalism.L2.G.Pend.subD` (`Formalism.LocalToGlobal.GroupLattices.PendingLemmas.DetYtMatNeZero`, lines 63–64)
`Formalism.L2.G.Pend.numD` (`Formalism.LocalToGlobal.GroupLattices.PendingLemmas.DetYtMatNeZero`, lines 75–78)
`Formalism.L2.G.Pend.divD` (`Formalism.LocalToGlobal.GroupLattices.PendingLemmas.DetYtMatNeZero`, lines 87–91)
`Formalism.L2.G.Pend.polDel` (`Formalism.LocalToGlobal.GroupLattices.PendingLemmas.DetYtMatNeZero`, lines 100–101)
`Formalism.L2.G.Pend.delF` (`Formalism.LocalToGlobal.GroupLattices.PendingLemmas.DetYtMatNeZero`, lines 165–170)
10.10. `Formalism.MainNotions.FormalGroupSystem.admitted` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 771–773)
`Formalism.MainNotions.FormalGroupSystem.adm7` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 613–622)
10.11. `Formalism.MainNotions.FormalGroupSystem.DworkLemma` (`Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem`, lines 225–229)
`Formalism.InvariantFactorPaths.G7.dwork_multivariable` (`Formalism.ThetaLogCoordinates.UniformThetaSubstitution.FrobeniusLiftNorms.LocalComponentGL`, lines 448–479)
10.11. `Formalism.MainNotions.FormalGroupSystem.Pf.sx_dwork` (`Formalism.SharpRestriction.GrowthBounds.MultisetValuation.PositivityConditionHolds`, lines 107–113)

10.12. Formalism.MainNotions.FormalGroupSystem.sSer (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 200–203)
Formalism.MainNotions.FormalGroupSystem.hhS (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 205–208)
Formalism.MainNotions.FormalGroupSystem.InClass (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 210–214)
Formalism.InvariantFactorPaths.G7.InClass (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.OddLogRelation, lines 76–79)
10.13. Formalism.MainNotions.FormalGroupSystem.Pf.restBeta (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.BiadditiveSecondVariable, lines 137–153)
Formalism.MainNotions.FormalGroupSystem.Pf.restGc (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.CommonEigenvectorThird, lines 21–22)
Formalism.MainNotions.FormalGroupSystem.Pf.restSExp_of (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.DworkExponentialRestrictions, lines 95–105)
Formalism.MainNotions.FormalGroupSystem.Pf.restTwoSums_of (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.TripleSumComm, lines 205–360)
Formalism.MainNotions.FormalGroupSystem.Pf.piFact (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.NatCastInvMemRp, lines 52–70)
10.14. Formalism.MainNotions.FormalGroupSystem.Pf.theta_kernel_expansion (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.CommonEigenvectorThird, lines 32–33)
Formalism.MainNotions.FormalGroupSystem.Pf.expansions_multiply (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.CommonEigenvectorThird, lines 35–36)
10.15. Formalism.MainNotions.FormalGroupSystem.Sys.colHt (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 398–399)
Formalism.MainNotions.FormalGroupSystem.Sys.colFac (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 405–410)
Formalism.MainNotions.FormalGroupSystem.Sys.colExp (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 412–415)
Formalism.MainNotions.FormalGroupSystem.Sys.PI (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 417–419)
Formalism.MainNotions.FormalGroupSystem.Sys.PIv (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 421–422)
10.16. Formalism.L2.PairingIndexBounds.Ent.MiddleFrameLemmas.blkX (Formalism.LocalToGlobal.PairingIndexBounds.RowActionCoefficients.AdmissibleInterpolation.ProdItePow, lines 372–414)
10.17. Formalism.L2.PairingIndexBounds.Ent.admissible_general_lattice_tuple (Formalism.LocalToGlobal.PairingIndexBounds.RowActionCoefficients.AdmissibleInterpolation.AdmissibleGeneralLatticeTuple, lines 17–80)
Formalism.L2.PairingIndexBounds.Ent.pending_tuple_admissible (Formalism.LocalToGlobal.PairingIndexBounds.RowActionCoefficients.LatticeShapeMiddleHolds, lines 26–28)
10.18. Formalism.MainNotions.FormalGroupSystem.applyD (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 256–259)
10.19. Formalism.MainNotions.FormalGroupSystem.Pend.PhAt (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 506–508)
Formalism.MainNotions.FormalGroupSystem.Pend.geC (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 522–523)
Formalism.MainNotions.FormalGroupSystem.Pend.LowAt (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 529–533)

Chapter 11: Entry estimates: factorials against denominators

11.1. Formalism.MainNotions.FormalGroupSystem.dSet (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 63–64)
Formalism.MainNotions.FormalGroupSystem.gA (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 66–70)
Formalism.MainNotions.FormalGroupSystem.Sys.g0v (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 325–325)
11.2. Formalism.MainNotions.FormalGroupSystem.gB (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 57–61)
Formalism.MainNotions.FormalGroupSystem.Sys.gv (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 323–323)
Formalism.MainNotions.FormalGroupSystem.Sys.alEven (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 355–355)
11.3. Formalism.InvariantFactorPaths.G7.Top.e_one_identity (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.KOneConsequence, lines 25–25)
Formalism.InvariantFactorPaths.G7.Top.e_one_integral (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.KOneConsequence, lines 26–26)
11.4. Formalism.InvariantFactorPaths.G7.Top.minorHits (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.KOneConsequence, lines 28–28)

Formalism.InvariantFactorPaths.G7.Top.hitsCapPos (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.KOneConsequence, lines 33–33)
Formalism.InvariantFactorPaths.G7.Top.hitsColumnsPos (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.KOneConsequence, lines 34–34)
11.5. Formalism.InvariantFactorPaths.G7.Top.hhDecomp (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.KOneConsequence, lines 38–38)
Formalism.InvariantFactorPaths.G7.decomposition_of_hh (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.FrobeniusLiftNorms.EDecomp, lines 498–546)
11.6. Formalism.InvariantFactorPaths.G7.Top.termBound (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.KOneConsequence, lines 37–37)
Formalism.InvariantFactorPaths.G7.TermBound_of (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.FrobeniusLiftNorms.NormNatCastLeOfDvd, lines 495–543)
11.7. Formalism.InvariantFactorPaths.G7.factorialValuationLowerBound (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.FrobeniusLiftNorms.ABArith, lines 35–53)
11.8. Formalism.InvariantFactorPaths.G7.gainFactorialInequality (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.FrobeniusLiftNorms.ABArith, lines 103–183)
11.9. Formalism.InvariantFactorPaths.G7.EntryHyp (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.StatementE, lines 170–174)
Formalism.InvariantFactorPaths.G7.Top.entryStar (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.KOneConsequence, lines 39–41)
11.10. Formalism.InvariantFactorPaths.G7.cyclicShiftLeast (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.FrobeniusLiftNorms.ABArith, lines 185–191)
Formalism.InvariantFactorPaths.G7.excessGainBound (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.FrobeniusLiftNorms.ABArith, lines 193–238)
11.11. Formalism.InvariantFactorPaths.G7.Top.entry_a_one (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.KOneConsequence, lines 43–43)
Formalism.InvariantFactorPaths.G7.entryExcessBound_of (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.FrobeniusLiftNorms.EntryAOneParity, lines 177–312)
11.12. Formalism.InvariantFactorPaths.G7.Top.entry_a_two (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.KOneConsequence, lines 44–44)
Formalism.InvariantFactorPaths.G7.entryCappedBound_of (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.FrobeniusLiftNorms.EntryATwoAndB, lines 220–266)
11.13. Formalism.InvariantFactorPaths.G7.Top.bCore (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.KOneConsequence, lines 32–32)
Formalism.InvariantFactorPaths.G7.Top.entryB (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.KOneConsequence, lines 45–45)
Formalism.InvariantFactorPaths.G7.EntryB_of (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.FrobeniusLiftNorms.EntryATwoAndB, lines 268–345)
11.14. Formalism.InvariantFactorPaths.G7.kernel_a_one_generic (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.FrobeniusLiftNorms.EGaussK, lines 213–214)
Formalism.InvariantFactorPaths.G7.kernel_a_two_generic (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.FrobeniusLiftNorms.EGaussK, lines 216–217)
Formalism.InvariantFactorPaths.G7.kernel_b_generic (Formalism.ThetaLogCoordinates.UniformThetaSubstitution.FrobeniusLiftNorms.EGaussK, lines 219–220)

Chapter 12: One lift on a lattice and the Hecke commutation

12.1. Formalism.MainNotions.FormalGroupSystem.Pf.first_theorem_place_bound (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.FirstTheoremPlaceBound, lines 20–22)
Formalism.MainNotions.FormalGroupSystem.Pf.eb_jet (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.EbBrJet, lines 35–39)
Formalism.MainNotions.FormalGroupSystem.Pf.ea_brNum (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.GainOfCapacityLoss, lines 252–256)
Formalism.MainNotions.FormalGroupSystem.Pf.eO_pos (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.TheoremOfAlternateForm, lines 618–643)
12.2. Formalism.MainNotions.FormalGroupSystem.Pf.second_lemma_place_bound (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.FirstTheoremPlaceBound, lines 24–26)
Formalism.MainNotions.FormalGroupSystem.Pf.hecke_place_bound (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.FirstTheoremPlaceBound, lines 28–30)
12.3. Formalism.MainNotions.FormalGroupSystem.Pf.kernel_product_form (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.RestEntOf, lines 20–58)
Formalism.MainNotions.FormalGroupSystem.Pf.restEnt_of (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.RestEntOf, lines 62–63)
Formalism.MainNotions.FormalGroupSystem.Pf.en_places_add (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.KernelProductForm, lines 51–109)

12.4. Formalism.MainNotions.FormalGroupSystem.ExpansionEquality (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 566–570)
Formalism.MainNotions.FormalGroupSystem.ExpansionDecomposition (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 572–585)
Formalism.MainNotions.FormalGroupSystem.Pf.formB_of (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.KernelSumOfClassData, lines 255–277)
12.5. Formalism.MainNotions.FormalGroupSystem.Pf.ent_strict (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.RamifiedLatticeOfChainEstimates, lines 46–63)
Formalism.MainNotions.FormalGroupSystem.Pf.formB_lim (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.RamifiedLatticeOfChainEstimates, lines 65–88)
Formalism.MainNotions.FormalGroupSystem.Pf.form_of_place (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.RamifiedLatticeOfChainEstimates, lines 90–94)
12.6. Formalism.MainNotions.FormalGroupSystem.Pf.theorem_A_entries_of_corollary (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.RamifiedLatticeOfChainEstimates, lines 104–114)
Formalism.MainNotions.FormalGroupSystem.Pf.slopeEstimate_of_lemmaB_slopeLine (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.RamifiedLatticeOfChainEstimates, lines 116–120)
Formalism.MainNotions.FormalGroupSystem.Pf.form_valuation_of_bound (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.RamifiedLatticeOfChainEstimates, lines 96–102)
12.7. Formalism.MainNotions.FormalGroupSystem.Pf.hecke_integral_of_forms (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.RamifiedLatticeOfChainEstimates, lines 122–140)
12.8. Formalism.MainNotions.FormalGroupSystem.Pf.latticeChain_succ_of_partial (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.RamifiedLatticeOfChainEstimates, lines 165–167)
Formalism.MainNotions.FormalGroupSystem.Pf.restricted_hecke_integral (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.RestrictedHeckeIntegralOfEdges, lines 289–289)
12.9. Formalism.MainNotions.G.Level.yv (Formalism.SharpRestriction.GrowthBounds.DataG, lines 148–151)
Formalism.MainNotions.FormalGroupSystem.Sys.Tt (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 331–332)
Formalism.MainNotions.FormalGroupSystem.Sys.cl (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 344–347)
Formalism.MainNotions.FormalGroupSystem.Sys.Ttu (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 349–350)
Formalism.MainNotions.FormalGroupSystem.Sys.Zop (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 352–353)
12.10. Formalism.MainNotions.FormalGroupSystem.Pend (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 447–493)
Formalism.MainNotions.FormalGroupSystem.adm5 (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 590–595)
GlobalTheory.XiGen.xiEx_theta_general (Development.AbelianVarietyTheory.Proofs.SatakeParameters.XiExThetaGeneral, lines 86–303)
12.11. GlobalTheory.Mem.NatEqui.eqP (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.NormalisedFamilySatisfiesEquivarianceLaws, lines 59–117)
GlobalTheory.TrueTheta.exists_family_of_curve (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ExistsThetaVectorFamilyOfCurveDatum, lines 15–108)
12.12. Formalism.L2.Col.Loc.E1 (Formalism.LocalToGlobal.Columns.FiniteSumNonzeroLargeDepth, lines 577–579)
Formalism.L2.Col.XiAt (Formalism.LocalToGlobal.Columns.FiniteSumNonzeroLargeDepth, lines 946–948)
Formalism.L2.Col.XiEntry (Formalism.LocalToGlobal.Columns.FiniteSumNonzeroLargeDepth, lines 1116–1119)
Formalism.L2.Col.XiCpl1 (Formalism.LocalToGlobal.Columns.FiniteSumNonzeroLargeDepth, lines 1134–1138)
Formalism.L2.Col.XiTopG (Formalism.LocalToGlobal.Columns.ChainInstY, lines 50–56)
GlobalTheory.TwV.xiAd (Development.AbelianVarietyTheory.Definitions.GaloisGeometry.UnderTie, lines 48–53)
12.13. GlobalTheory.Mem.ChiDVal.chiD_mem_of (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ChiDValuesAndCcTop, lines 100–112)
GlobalTheory.Port.chRat_ofChiD (Development.AbelianVarietyTheory.Definitions.GaloisGeometry.ExistsUnifPow, lines 25–33)
GlobalTheory.TwV.Xc.chiI_pow_rat (Development.AbelianVarietyTheory.Proofs.SatakeParameters.KernelSiteNormalisingCharPowValuesInCoeffField, lines 107–128)
GlobalTheory.TwV.adLawAt_xiAd (Development.AbelianVarietyTheory.Proofs.SatakeParameters.AdelicChainDatumCharacterLawsJointKernel, lines 62–116)
12.14. GlobalTheory.KerCommon.kerTh_common_op (Development.AbelianVarietyTheory.Proofs.SatakeParameters.KernelFunctionsCommonLinearOperatorOffSplitPrime, lines 128–174)
GlobalTheory.kerTh_entire_algCoef (Development.AbelianVarietyTheory.Proofs.SatakeParameters.KerThEntireAlgCoef, lines 22–40)
12.15. GlobalTheory.TwV.site_law_xiIn (Development.AbelianVarietyTheory.Proofs.SatakeParameters.KernelSiteLawAndFixedCoordinateStabiliserLaw, lines 20–48)
12.16. Formalism.MainNotions.FormalGroupSystem.Pf.kernel_eq_theta (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.CoupledSumOfEntries, lines 23–28)
Formalism.MainNotions.FormalGroupSystem.Pf.integral_span_coordinates (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.IntegralSpanCoordinates, lines 62–66)

Formalism.MainNotions.FormalGroupSystem.Pf.kernel_nsmul (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.IntegralSpanCoordinates, lines 68–72)
12.17. Formalism.L2.G.Pend.up_down_adjoint (Formalism.LocalToGlobal.GroupLattices.PendingLemmas.OrbitSumStabilizer, lines 257–298)
Formalism.MainNotions.FormalGroupSystem.MinkowskiCounting (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 597–601)
Formalism.MainNotions.FormalGroupSystem.ExpansionDecomposition (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 572–585)
Formalism.MainNotions.FormalGroupSystem.Pf.hk_adj (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.CoupledSumOfEntries, lines 65–109)
Formalism.MainNotions.FormalGroupSystem.Pf.ksa_of_adm (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.CoupledSumOfEntries, lines 140–150)
Formalism.L2.KerInert.SecondKernelCoupled.hecke_coupled_entry (Formalism.LocalToGlobal.InertKernelTheory.InertCoprimeLemmas.SecondKernelCoupled, lines 104–136)
12.18. Formalism.MainNotions.FormalGroupSystem.Pf.countLoc (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.SmulPsiGUp, lines 276–289)
Formalism.MainNotions.FormalGroupSystem.Pf.heckeCondition_of_adm (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.HeckeConditionOfAdm, lines 75–146)
12.19. Formalism.MainNotions.FormalGroupSystem.PrZ (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 76–79)
Formalism.MainNotions.FormalGroupSystem.Pr (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 81–84)
Formalism.MainNotions.FormalGroupSystem.Pf.PrZ_cast (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.LiftBoundOfIntegrality, lines 91–101)
12.20. Formalism.MainNotions.FormalGroupSystem.Conclusion4 (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 671–679)
Formalism.MainNotions.FormalGroupSystem.Pf.Ci (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.HeckeValueEstimateOfRecordValue, lines 195–269)
Formalism.MainNotions.FormalGroupSystem.Pf.Cii (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.TheoremOfAlternateForm, lines 538–548)
Formalism.MainNotions.FormalGroupSystem.Pf.Ciii (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.TheoremOfAlternateForm, lines 608–616)

Chapter 13: Induction along the chain

13.1. Formalism.MainNotions.FormalGroupSystem.Pf.ZL_chain (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.YEPEComm, lines 110–123)
Formalism.MainNotions.FormalGroupSystem.Pf.ZL_bound (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.AofUpdate, lines 172–217)
13.2. Formalism.MainNotions.FormalGroupSystem.Sys.Iprod (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 361–364)
13.3. Formalism.MainNotions.FormalGroupSystem.Pf.bbJ (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.YEPEComm, lines 127–129)
Formalism.MainNotions.FormalGroupSystem.Pf.main_ind (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.YEPEComm, lines 131–185)
13.4. Formalism.MainNotions.FormalGroupSystem.LowTermBound (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 708–716)
Formalism.MainNotions.FormalGroupSystem.Pf.lift_bound_of_integrality (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.LiftBoundOfIntegrality, lines 103–174)
13.5. Formalism.MainNotions.FormalGroupSystem.Sv (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 86–87)
Formalism.MainNotions.FormalGroupSystem.RelativePresentationLiterature (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 703–706)
Formalism.MainNotions.FormalGroupSystem.Pf.relative_presentation_of_d7 (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.RelativePresentationOfD7, lines 110–126)
13.6. Formalism.MainNotions.FormalGroupSystem.Conclusion5 (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 697–701)
Formalism.MainNotions.FormalGroupSystem.Pf.ramifiedLattice_of_chain_estimates (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.RamifiedLatticeOfChainEstimates, lines 169–173)
13.7. Formalism.MainNotions.FormalGroupSystem.ErrorExponentStatement (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 785–788)
13.8. Formalism.MainNotions.FormalGroupSystem.admitted (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 771–773)
Formalism.MainNotions.FormalGroupSystem.Conclusion₂ (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 775–777)
Formalism.MainNotions.FormalGroupSystem.Stmt (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 779–780)

Formalism.MainNotions.FormalGroupSystem.Pf.stmt_proved (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.PrvAndKernelCommutationUpToUnitsOfRests, lines 107–108)
Formalism.MainNotions.FormalGroupSystem.Pf.stmtAt_proved (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.PrvAndKernelCommutationUpToUnitsOfRests, lines 110–111)

Chapter 14: Global Frobenius-orbit parameters

14.1. Formalism.MainNotions.G.frobSet (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 93–95)
Formalism.MainNotions.G.hP (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 97–98)
Formalism.MainNotions.G.partner (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 131–132)
Formalism.MainNotions.G.stepP (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 161–161)
Formalism.MainNotions.G.nP (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 163–164)
Formalism.MainNotions.G.reps (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 166–166)
Formalism.MainNotions.G.topsOn (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 168–169)
Formalism.MainNotions.G.IsV (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 179–180)
Formalism.BasicNotions.primeClass (Formalism.LocalRingCardinality.DataHost, lines 33–37)
Formalism.HC.HostRaw.psAt (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.ConjEMemHostFIff, lines 68–69)
14.2. Formalism.MainNotions.G.Inst (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 141–150)
Formalism.MainNotions.G.fill (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 154–155)
Formalism.MainNotions.G.fillG (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 159–159)
Formalism.MainNotions.G.al (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 230–230)
Formalism.MainNotions.G.capU (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 232–233)
Formalism.MainNotions.G.dl (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 241–243)
14.3. Formalism.MainNotions.G.IsDist (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 235–236)
Formalism.MainNotions.G.Excl (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 238–239)
Formalism.MainNotions.G.g0 (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 247–250)
14.4. Formalism.MainNotions.G.gU (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 254–258)
14.4. Formalism.MainNotions.FormalGroupSystem.Pf.nc_summable_min (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.NcSummableMin, lines 40–53)
14.5. Formalism.BasicNotions.GatTies (Formalism.LocalRingCardinality.DataGatTies, lines 7–28)
Formalism.MainNotions.G.LatTied (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 430–451)
14.6. Formalism.MainNotions.SizesAt (Formalism.SharpRestriction.ElevenLe, lines 134–138)
14.7. Formalism.MainNotions.G.Pr (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 262–267)
Formalism.MainNotions.FormalGroupSystem.Pf.price_bound_large_primes (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.PriceBoundLargePrimes, lines 117–179)
Formalism.MainNotions.FormalGroupSystem.Pf.price_bound_small_primes (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.PriceLinearCombinationLe, lines 89–141)
Formalism.MainNotions.SizeRho (Formalism.SharpRestriction.ElevenLe, lines 70–72)
Formalism.MainNotions.FormalGroupSystem.Pf.ne_bT_le (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.NeSumLe, lines 87–105)
14.7. Formalism.MainNotions.G.IsGoodOrbit (Formalism.SharpRestriction.GrowthBounds.UnramifiedQuadraticSetting, lines 250–252)
Formalism.G0.GoodOrbitIntervalTileCondition (Formalism.SubspaceCounting.GoodOrbitIntervalTileCondition, lines 24–25)
14.8. Formalism.MainNotions.G.kapv (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 276–277)
Formalism.MainNotions.G.Cand (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 279–281)
Formalism.MainNotions.G.VV (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 292–293)
Formalism.MainNotions.G.gainP (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 355–358)
14.9. Formalism.MainNotions.G.ParametersFill (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 401–410)
Formalism.MainNotions.G.TTP (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 362–364)
Formalism.MainNotions.G.Sv (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 336–337)
Formalism.MainNotions.G.tauP (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 341–344)
Formalism.MainNotions.G.ioP (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 348–351)
Formalism.MainNotions.G.tIIf (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 197–200)
Formalism.MainNotions.G.DIIf (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 188–191)
Formalism.MainNotions.G.tIf (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 217–225)
Formalism.MainNotions.G.DIf (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 193–195)
14.10. Formalism.MainNotions.G.EpDef (Formalism.SharpRestriction.GrowthBounds.DefsInst, lines 18–19)
Formalism.MainNotions.G.GoodRankK (Formalism.SharpRestriction.GrowthBounds.UnramifiedQuadraticSetting, lines 200–203)
Formalism.MainNotions.G.LPprimeAt (Formalism.SharpRestriction.GrowthBounds.UnramifiedQuadraticSetting, lines 205–214)
Formalism.MainNotions.G.LPprime (Formalism.SharpRestriction.GrowthBounds.UnramifiedQuadraticSetting, lines 216–220)
Formalism.MainNotions.G.kapISet (Formalism.SharpRestriction.GrowthBounds.DefsInst, lines 38–45)
Formalism.MainNotions.G.sInf_mem_kapISet (Formalism.SharpRestriction.GrowthBounds.DefsInst, lines 68–114)
Formalism.MainNotions.G.KapIDef (Formalism.SharpRestriction.GrowthBounds.DefsInst, lines 54–55)
Formalism.MainNotions.G.exists_inst_defs (Formalism.SharpRestriction.GrowthBounds.DefsInst, lines 120–124)

Formalism.MainNotions.G.LatticeSystems.H0_eq_twisted_nabla (Formalism.SharpRestriction.GrowthBounds.LatticeSystems, lines 1402–1408)
Formalism.MainNotions.G.LatticeSystems.la_indep (Formalism.SharpRestriction.GrowthBounds.LatticeSystems, lines 1503–1545)
14.11. Formalism.BasicNotions.ParameterChoice' (Formalism.LocalRingCardinality.DataParams, lines 62–291)
14.12. Formalism.L2.G.Model.sandwich (Formalism.LocalToGlobal.GroupLattices.FockModels.Sandwich, lines 187–205)
Formalism.MainNotions.G.Joints.weyl_sandwich_of_factors (Formalism.SharpRestriction.GrowthBounds.Joints, lines 1494–1533)
Formalism.MainNotions.G.Joints.expBound_of (Formalism.SharpRestriction.GrowthBounds.Joints, lines 1848–1924)
14.13. Formalism.MainNotions.FormalGroupSystem.SmallResidueCharComparison (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 741–754)
Formalism.MainNotions.FormalGroupSystem.Pf.numeric_comparison_of_constants (Formalism.SharpRestriction.GrowthBounds.ProofLW, lines 207–219)
14.13. Formalism.MainNotions.FormalGroupSystem.Pf.nc_loc (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.NumericalClosedOfNeed, lines 12–19)
Formalism.MainNotions.FormalGroupSystem.Pf.lw_size (Formalism.SharpRestriction.GrowthBounds.ProofLW, lines 135–203)
Formalism.MainNotions.FormalGroupSystem.Pf.lw_de (Formalism.SharpRestriction.GrowthBounds.ProofLW, lines 62–104)
Formalism.MainNotions.FormalGroupSystem.Pf.nc_gB_ge (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.NcSummableMin, lines 120–145)
Formalism.MainNotions.FormalGroupSystem.Pf.nc_gA_ge (Formalism.SharpRestriction.GrowthBounds.MultisetValuation.NcSummableMin, lines 147–166)
14.14. Formalism.MainNotions.G.TopChoiceBounds.hLarge_proved (Formalism.SharpRestriction.GrowthBounds.TopChoiceBounds, lines 4226–4372)
14.15. Formalism.MainNotions.WeilJacquetShalikaLaw.TwLoc (Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw, lines 79–83)
Formalism.MainNotions.WeilJacquetShalikaLaw.TwLoc.Yt (Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw, lines 85–90)
Formalism.MainNotions.WeilJacquetShalikaLaw.Ties9 (Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw, lines 92–95)
Formalism.BasicNotions.GatTies (Formalism.LocalRingCardinality.DataGatTies, lines 7–28)
Formalism.MainNotions.G.LatTied (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 430–451)
Formalism.BasicNotions.LatCmp.VramDef (Formalism.LocalRingCardinality.DataLatDefs, lines 37–40)
Formalism.MainNotions.WeilJacquetShalikaLaw.CmpV (Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw, lines 156–159)
Formalism.MainNotions.Pk.reW (Formalism.SharpRestriction.SmoothDesignExistence.SumOverPrimesHypotheses, lines 96–96)
14.16. Formalism.MainNotions.G.Joints.SysUAt (Formalism.SharpRestriction.GrowthBounds.JointsTw, lines 9–19)
Formalism.MainNotions.FormalGroupSystem.Sys (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 264–277)
Formalism.MainNotions.FormalGroupSystem.OmBasis (Formalism.SharpRestriction.GrowthBounds.FormalGroupSystem, lines 172–177)
Formalism.MainNotions.G.yProd (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 453–457)
Formalism.MainNotions.G.Joints.UAt_of (Formalism.SharpRestriction.GrowthBounds.JointsTw, lines 23–214)
Formalism.MainNotions.WeilJacquetShalikaLaw.UAt (Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw, lines 161–170)
14.17. Formalism.MainNotions.G.Joints.unit_bound_of_sum_over_primes (Formalism.SharpRestriction.GrowthBounds.JointsTw, lines 243–270)
Formalism.MainNotions.Pk.SysTw (Formalism.SharpRestriction.SmoothDesignExistence.SumOverPrimesHypotheses, lines 116–130)
Formalism.MainNotions.Pk.sum_over_primes_service (Formalism.SharpRestriction.SmoothDesignExistence.SumOverPrimesHypotheses, lines 187–221)
Formalism.MainNotions.WeilJacquetShalikaLaw.Serv (Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw, lines 241–243)
Formalism.MainNotions.WeilJacquetShalikaLaw.ServF (Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw, lines 252–255)
Formalism.MainNotions.WeilJacquetShalikaLaw.local_theorem_of_statement (Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw, lines 371–374)
Formalism.MainNotions.WeilJacquetShalikaLaw.serv_of (Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw, lines 376–379)
Formalism.MainNotions.Pk.serv_re (Formalism.SharpRestriction.SmoothDesignExistence.SumOverPrimesHypotheses, lines 98–107)

Chapter 15: What each layer records

15.1. Formalism.BasicNotions.lpF (Formalism.LocalRingCardinality.DataParams$_2$, lines 58–59)
Formalism.BasicNotions.FOrb (Formalism.LocalRingCardinality.DataForced, lines 99–100)

15.2. Formalism.IncomingBounds.FlowLoadBounds.exists_opt_flow (Formalism.IncomingBounds.FlowDegreeBounds.MaxFlowValueAttained, lines 108–135)
15.2. Formalism.BasicNotions.FOrb_step (Formalism.LocalRingCardinality.DataForced, lines 102–108)
15.3. UpperTheory.exists_maxFlow_attained_and_isOptDual (Development.AutomorphicArithmetic.OrbitLabellings.ExistsMaxFlowAttainedAndIsOptDual, lines 8–17)
UpperTheory.sum_demand_filter_two_div_le_of_isOptDual (Development.AutomorphicArithmetic.OrbitLabellings.SumDemandFilterTwoDivLeOfIsOptDual, lines 9–56)
Formalism.DworkPowerSeriesWeights.OddLevelGainBounds.proposition_six (Formalism.GaussJetBounds.OneLetterGaussJets.OneLetterGaussProofs.CompanionTheory, lines 38–146)
Formalism.MainNotions.DIV.Bridge.DworkPowerSeriesWeights.local_bound_eleven_comparison (Formalism.SharpRestriction.Divisibility.JetPolynomialLogs.LocalBoundElevenComparison, lines 33–54)
15.4. Formalism.DworkPowerSeriesWeights.generalized_log_inverse (Formalism.GaussJetBounds.GaussJetProofs.LogarithmWeightComparison, lines 227–250)
Formalism.DworkPowerSeriesWeights.generalized_log_weight (Formalism.GaussJetBounds.GaussJetProofs.LogarithmWeightComparison, lines 252–330)
Formalism.DworkPowerSeriesWeights.lemmaS (Formalism.GaussJetBounds.OneLetterRadius, lines 17–19)
15.5. Formalism.MainNotions.DIV.Bridge.DworkPowerSeriesWeights.split_bound_eleven_one (Formalism.SharpRestriction.Divisibility.JetPolynomialLogs.LocalBoundElevenComparison, lines 64–67)
15.6. Formalism.BasicNotions.DV.KapDef (Formalism.LocalRingCardinality.DataKap, lines 31–34)
Formalism.BasicNotions.DV.aPlusQ_eq (Formalism.LocalRingCardinality.DataKap, lines 60–64)
15.7. Formalism.BasicNotions.HCParam (Formalism.LocalRingCardinality.Params, lines 50–69)
15.8. Formalism.BasicNotions.HCParam (Formalism.LocalRingCardinality.Params, lines 50–69)
Formalism.BasicNotions.DV.KapDef (Formalism.LocalRingCardinality.DataKap, lines 31–34)
Formalism.BasicNotions.DV.aPlusQ_eq (Formalism.LocalRingCardinality.DataKap, lines 60–64)
GlobalTheory.Mem.R3.Rw.Ex.kapDef_of_ka (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ExistsRowsDatumPrescribedTuplesDefinedWeights, lines 31–42)
Formalism.BasicNotions.RowWeightLemmas.nuRow_head (Formalism.LocalRingCardinality.CoeNeZeroOfNotMem, lines 12–20)
Formalism.BasicNotions.RowWeightLemmas.nuRow_block (Formalism.LocalRingCardinality.CoeNeZeroOfNotMem, lines 22–39)
Formalism.BasicNotions.RowWeightLemmas.nuRow_pos (Formalism.LocalRingCardinality.CoeNeZeroOfNotMem, lines 180–182)
15.9. GlobalTheory.EX.HCVal (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.PairRowIffQdRows, lines 93–105)
GlobalTheory.EX.exists_hcParam_of_code (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsHcParamOfCode, lines 12–75)
GlobalTheory.EX.exists_hc_family (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsHcParamOfCode, lines 128–241)
15.10. Formalism.BasicNotions.SizeData₂.FDef (Formalism.LocalRingCardinality.DataForced, lines 134–135)
15.11. Formalism.BasicNotions.sum_orbits (Formalism.LocalRingCardinality.DataForced, lines 170–189)
Formalism.BasicNotions.primeClass (Formalism.LocalRingCardinality.DataHost, lines 33–37)
15.12. Formalism.BasicNotions.ErrX (Formalism.LocalRingCardinality.DataErr, lines 20–34)
Formalism.BasicNotions.ErrDefs (Formalism.LocalRingCardinality.DataErr, lines 207–214)
Formalism.BasicNotions.SizeData₂.EDef (Formalism.LocalRingCardinality.DataErr, lines 171–172)
Formalism.BasicNotions.DGamDef (Formalism.LocalRingCardinality.DataErr, lines 74–75)
Formalism.BasicNotions.Eval (Formalism.LocalRingCardinality.DataErr, lines 159–169)
15.13. Formalism.BasicNotions.DGamDef.one (Formalism.LocalRingCardinality.DataErr, lines 77–80)
15.14. Formalism.BasicNotions.SizeData₂.gainD (Formalism.LocalRingCardinality.DataParams₂, lines 212–213)
Formalism.MainNotions.LocalQuadraticParts.DropDef (Formalism.SharpRestriction.Divisibility.LocalQuadraticParts, lines 517–518)
Formalism.MainNotions.NoGain (Formalism.SharpRestriction.ParConsts, lines 24–29)
Formalism.MainNotions.G.ParametersFill (Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero, lines 401–410)
15.15. Formalism.BasicNotions.SizeData₂ (Formalism.LocalRingCardinality.DataParams₂, lines 76–182)
Formalism.MainNotions.LocalQuadraticParts.DropDef (Formalism.SharpRestriction.Divisibility.LocalQuadraticParts, lines 517–518)
15.16. Formalism.BasicNotions.SizeData₂.sg (Formalism.LocalRingCardinality.DataParams₂, lines 215–216)
15.17. Formalism.BasicNotions.SizeData₂ (Formalism.LocalRingCardinality.DataParams₂, lines 76–182)
Formalism.BasicNotions.SizeData₂.sg (Formalism.LocalRingCardinality.DataParams₂, lines 215–216)

Chapter 16: From local lifts to bounds on whole columns

16.1. Formalism.MainNotions.WeilJacquetShalikaLaw.VGe (Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw, lines 19–22)
Formalism.MainNotions.WeilJacquetShalikaLaw.ColGe0 (Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw, lines 24–26)
Formalism.MainNotions.WeilJacquetShalikaLaw.ColGeTw (Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw, lines 28–31)
Formalism.MainNotions.WeilJacquetShalikaLaw.ColGe (Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw, lines 33–35)

16.2. `Formalism.MainNotions.G.NP` (`Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero`, lines 464–472)
`Formalism.MainNotions.WeilJacquetShalikaLaw.InNP` (`Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw`, lines 60–63)
16.3. `Formalism.MainNotions.G.Joints.Lit.NPDesc_of` (`Formalism.SharpRestriction.GrowthBounds.JointsLit`, lines 256–346)
`Formalism.BasicNotions.LatCmp.M_inf_Vram` (`Formalism.LocalRingCardinality.Lemmas`, lines 968–982)
`GlobalTheory.EX.OB.NP_full` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.MemSpanOfSemilinear`, lines 79–126)
16.4. `Formalism.MainNotions.G.LatticeSystems.lb_steps` (`Formalism.SharpRestriction.GrowthBounds.LatticeSystems`, lines 1790–1820)
`Formalism.MainNotions.G.LatticeSystems.ch_locChain` (`Formalism.SharpRestriction.GrowthBounds.LatticeSystems`, lines 856–895)
`Formalism.MainNotions.G.LatticeSystems.jm_Re` (`Formalism.SharpRestriction.GrowthBounds.LatticeSystems`, lines 3727–3744)
`Formalism.MainNotions.G.LatticeSystems.jStep1` (`Formalism.SharpRestriction.GrowthBounds.LatticeSystems`, lines 3339–3459)
16.5. `Formalism.MainNotions.G.LPprime` (`Formalism.SharpRestriction.GrowthBounds.UnramifiedQuadraticSetting`, lines 216–220)
`Formalism.MainNotions.G.LatticeSystems.jSmith` (`Formalism.SharpRestriction.GrowthBounds.LatticeSystems`, lines 3746–3816)
`Formalism.MainNotions.G.LatticeSystems.jl_good` (`Formalism.SharpRestriction.GrowthBounds.LatticeSystems`, lines 3875–3930)
`Formalism.MainNotions.G.LatticeSystems.jLocal` (`Formalism.SharpRestriction.GrowthBounds.LatticeSystems`, lines 3972–4062)
`Formalism.MainNotions.G.LatticeSystems.jRank` (`Formalism.SharpRestriction.GrowthBounds.LatticeSystems`, lines 4194–4236)
`Formalism.MainNotions.G.LatticeSystems.ja_restJ` (`Formalism.SharpRestriction.GrowthBounds.LatticeSystems`, lines 4300–4388)
`Formalism.G0.WalkSharpP.main` (`Formalism.SubspaceCounting.SubspaceCountingProofs.WalkSharp`, lines 379–718)
`Formalism.MainNotions.G.LatticeSystems.walk_allBut_bound` (`Formalism.SharpRestriction.GrowthBounds.LatticeSystems`, lines 217–312)
`Formalism.MainNotions.FormalGroupSystem.Pf.stmt6G` (`Formalism.SharpRestriction.GrowthBounds.MultisetValuation.StabilizerCount`, lines 384–490)
16.6. `Formalism.MainNotions.WeilJacquetShalikaLaw.lowS` (`Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw`, lines 73–74)
`Formalism.MainNotions.WeilJacquetShalikaLaw.totS` (`Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw`, lines 76–77)
`Formalism.MainNotions.WeilJacquetShalikaLaw.gainS` (`Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw`, lines 49–50)
`Formalism.MainNotions.G.gainP` (`Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero`, lines 355–358)
16.6. `Formalism.MainNotions.WeilJacquetShalikaLaw.TwLoc` (`Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw`, lines 79–83)
`Formalism.MainNotions.WeilJacquetShalikaLaw.TwLoc.Yt` (`Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw`, lines 85–90)
16.7. `Formalism.MainNotions.G.Joints.unit_bound_of_sum_over_primes` (`Formalism.SharpRestriction.GrowthBounds.JointsTw`, lines 243–270)
`Formalism.MainNotions.WeilJacquetShalikaLaw.UAt` (`Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw`, lines 161–170)
16.8. `Formalism.MainNotions.WeilJacquetShalikaLaw.Proof.first_in_local_models` (`Formalism.SharpRestriction.GrowthBounds.AdmissibleIntegers.StatementsOfRemainingBound`, lines 111–151)
16.9. `GlobalTheory.KerLatS.kerEntHyp_latS` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.TrueChainsKernelDatumSourceSideLattice`, lines 36–65)
`GlobalTheory.Mem.Col2.TC.kerEnt_latT` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.KernelDatumLatticeTargetSideTrueChains`, lines 18–37)
`GlobalTheory.KerVal.onMin_canInv_mem` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.TargetPairingSizesInCoefficientFieldValueGroup`, lines 89–159)
`GlobalTheory.KerVal.prTC_val` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.TargetPairingSizesInCoefficientFieldValueGroup`, lines 207–214)
16.10. `Formalism.MainNotions.DIV.Lemmas.theta_isometry_facts_hold` (`Formalism.SharpRestriction.Divisibility.ChainOperators.GenericFramesLetters`, lines 10–12)
`Formalism.MainNotions.OrbitPlaces.kform_of_ker` (`Formalism.SharpRestriction.Divisibility.OrbitPlaces`, lines 1621–1656)
16.11. `Formalism.MainNotions.OrbitPlaces.Ent` (`Formalism.SharpRestriction.Divisibility.OrbitPlaces`, lines 34–50)
`Formalism.MainNotions.OrbitPlaces.KForm` (`Formalism.SharpRestriction.Divisibility.OrbitPlaces`, lines 71–73)
`Formalism.MainNotions.OrbitPlaces.AdmK` (`Formalism.SharpRestriction.Divisibility.OrbitPlaces`, lines 475–477)
`Formalism.MainNotions.OrbitPlaces.admitted` (`Formalism.SharpRestriction.Divisibility.OrbitPlaces`, lines 479–481)
`Formalism.MainNotions.OrbitPlaces.Core` (`Formalism.SharpRestriction.Divisibility.OrbitPlaces`, lines 1323–1324)
16.12. `PI.OrbBoundFW.prvF_of_orbitLetterBounds` (`Formalism.PeriodIntegrals.UnitaryGroupMassFormula.OrbitValuationBounds.PrvFOfOrbitLetterBounds`, lines 17–59)

PI.OrbBoundFW.orbit_place_term_le_sum (Formalism.PeriodIntegrals.UnitaryGroupMassFormula.OrbitValuationBounds.SummableTruncatedGeometric, lines 136–218)
SumOrbCW.sumOrbC (Formalism.PeriodIntegrals.UnitaryGroupMassFormula.OrbitSumCount.SizeGe, lines 36–78)
UpperTheory.norm_mfact_mul_coeff_le_of_B0S (Development.AutomorphicArithmetic.OrbitLabellings.NormMfactMulCoeffLeOfB0S, lines 69–137)
UpperTheory.fInert_sub_eatI_le_orbit_average_sub_errors (Development.AutomorphicArithmetic.OrbitLabellings.FInertSubEatILeOrbitAverageSubErrors, lines 7–71)
PI.OrbBoundF (Formalism.PeriodIntegrals.OrbitalBoundsConclusion.OrbBoundF, lines 3–4)
UpperTheory.kvalGe_theta_of_shape (Development.AutomorphicArithmetic.OrbitLabellings.KvalGeThetaOfShape, lines 155–204)
jet_bound_ramified_prime (Formalism.PeriodIntegrals.OrbitalBoundsConclusion.EntAtD, lines 11–13)
Formalism.MainNotions.DIV.Lemmas.OrbitEntryBounds.orbit_entry_bounds_hold (Formalism.SharpRestriction.Divisibility.ChainOperators.OrbitEntryBounds, lines 49–52)
PI.OrbBoundFW.prvF_of_orbitLetterBounds (Formalism.PeriodIntegrals.UnitaryGroupMassFormula.OrbitValuationBounds.PrvFOfOrbitLetterBounds, lines 17–59)
jet_bound_ramified_prime (Formalism.PeriodIntegrals.OrbitalBoundsConclusion.EntAtD, lines 11–13)
16.13. Formalism.MainNotions.WeilJacquetShalikaLaw.Proof.chain_ge (Formalism.SharpRestriction.GrowthBounds.AdmissibleIntegers.RemainingPrimesOfAdmitted, lines 144–161)
16.14. Formalism.MainNotions.OrbitPlaces.lift_bounds_of_essential (Formalism.SharpRestriction.Divisibility.OrbitPlaces, lines 1340–1343)
Formalism.MainNotions.J.four_orbit_conditions_of_proof (Formalism.SharpRestriction.AdmissibleParameterJoins.GandSab, lines 11–16)
16.15. Formalism.MainNotions.WeilJacquetShalikaLaw.Proof.remaining_primes_of_admitted (Formalism.SharpRestriction.GrowthBounds.AdmissibleIntegers.RemainingPrimesOfAdmitted, lines 193–208)
16.16. Formalism.MainNotions.WeilJacquetShalikaLaw.Proof.first_clause_cases (Formalism.SharpRestriction.GrowthBounds.AdmissibleIntegers.StatementsOfRemainingBound, lines 25–95)
16.17. Formalism.MainNotions.G.CmpJ.CmpJHyp (Formalism.SharpRestriction.GrowthBounds.ChartComparison, lines 82–96)
Formalism.MainNotions.G.CmpJ.StepIn.Core (Formalism.SharpRestriction.GrowthBounds.ChartComparison.StepIn, lines 139–174)
Formalism.L2.Forms.ValuationCompatibleChart (Formalism.LocalToGlobal.HermitianForms.TiedTwGen, lines 72–87)
Formalism.MainNotions.G.CmpJ.StepIn.TwUnit (Formalism.SharpRestriction.GrowthBounds.ChartComparison.IntAtJIffN, lines 43–46)
Formalism.MainNotions.G.CmpJ.StepIn.ValIntN (Formalism.SharpRestriction.GrowthBounds.ChartComparison.IntAtJIffN, lines 89–90)
Formalism.MainNotions.G.CmpJ.StepIn.IwF (Formalism.SharpRestriction.GrowthBounds.ChartComparison.CmpJMemSpan, lines 54–56)
16.18. GlobalTheory.Proof.ColumnWeightComparison.cmpV_of_cmpLite (Development.AbelianVarietyTheory.Proofs.SatakeParameters.ColumnWeightComparison, lines 37–50)
16.19. Formalism.MainNotions.WeilJacquetShalikaLaw.Ties9 (Formalism.SharpRestriction.GrowthBounds.WeilJacquetShalikaLaw, lines 92–95)
Formalism.MainNotions.WeilJacquetShalikaLaw.Proof.statements_of_remaining_bound (Formalism.SharpRestriction.GrowthBounds.AdmissibleIntegers.StatementsOfRemainingBound, lines 194–199)
Formalism.MainNotions.WeilJacquetShalikaLaw.Proof.baseChange_preserves_unitary_of (Formalism.SharpRestriction.GrowthBounds.AdmissibleIntegers.StatementsOfRemainingBound, lines 162–192)
16.20. Formalism.MainNotions.Pk.admissible_size_family (Formalism.SharpRestriction.SmoothDesignExistence.SumOverPrimesHypotheses, lines 57–79)
Formalism.MainNotions.Pk.szEatI_fam (Formalism.SharpRestriction.SmoothDesignExistence.SumOverPrimesHypotheses, lines 38–47)
UpperTheory.G2.szEatI_of_stepA (Development.AutomorphicArithmetic.OrbitLabellings.SzEatIFree, lines 11–67)
PI.SzEatI.W.stepA (Formalism.PeriodIntegrals.UnitaryGroupMassFormula.SizeAbsorption.SizeDel, lines 121–343)
UpperTheory.sum_log_mul_max_zero_le_of_odd_addOrderOf (Development.AutomorphicArithmetic.OrbitLabellings.SumLogMulMaxZeroLeOfOddAddOrderOf, lines 101–170)
16.21. Formalism.MainNotions.Pk.sum_over_primes_service (Formalism.SharpRestriction.SmoothDesignExistence.SumOverPrimesHypotheses, lines 187–221)
Formalism.L2.G.ServT.closed_servF_of_servT_of_dGamSize (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.GServT, lines 31–33)

Chapter 17: From columns to periods

17.1. Formalism.MainNotions.WeightDegreeSizes.pair_core (Formalism.SharpRestriction.Divisibility.WeightDegreeSizes, lines 783–800)
17.2. Formalism.L2.G.Adm2InU.pair0U (Formalism.LocalToGlobal.GroupLattices.AdmissibleAtGeneratorPlaces, lines 39–53)
Formalism.L2.G.Adm2InU.pairTwU (Formalism.LocalToGlobal.GroupLattices.AdmissibleAtGeneratorPlaces, lines 55–70)
17.3. GlobalTheory.RowIntQ.rowInt_over (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.PeriodRowLatticesIntegralProvisionalRows, lines 47–63)

`GlobalTheory.PairAtClass.pair_integral` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ClassPairingIntegralFormRowIsIntegral`, lines 49–69)
`GlobalTheory.PerRed.admissiblePeriodZero_of_classPeriodDecomposition` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.AdmissiblePeriodZeroOfClassPeriodDecompositionAl`, lines 196–239)
`GlobalTheory.PerRedU.perTwAt_of_perDecTw` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.AdmissiblePeriodTwistsReductionPlaceTuples`, lines 12–39)
`GlobalTheory.Mem.Col2.TC.formsMap_at_every_trueColumnMember` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.FormsMapAtEveryTrueColumnMember`, lines 15–26)
`GlobalTheory.PerQ.twistedPointIntegral_of_TC_ok` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.TwistedPointRowAdmissibleTrueColumns`, lines 12–15)
`GlobalTheory.RedPt0Pend.exists_integralPendingObject_symmetricFormsLevel` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsIntegralPendingObjectSymmetricFormsLevel`, lines 291–333)
17.4. `GlobalTheory.Mem.DivModel.ramEmb` (`Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.ExistsDivStage`, lines 315–316)
`GlobalTheory.Mem.DivModel.exists_divStage` (`Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.ExistsDivStage`, lines 334–381)
`GlobalTheory.PerQ.Model.reRows` (`Development.AbelianVarietyTheory.Definitions.GlobalTwisting.ExistsRowValNeZero`, lines 110–122)
17.5. `Formalism.L2.G.Adm2InU.period_geTU` (`Formalism.LocalToGlobal.GroupLattices.AdmissibleAtGeneratorPlaces`, lines 72–90)
17.6. `Formalism.MainNotions.J.sizeData_at_level_primes` (`Formalism.SharpRestriction.AdmissibleParameterJoins.DIVfromGLs`, lines 25–27)
`Formalism.MainNotions.WeightDegreeSizes.sg_atT` (`Formalism.SharpRestriction.Divisibility.WeightDegreeSizes`, lines 887–901)
17.7. `Formalism.BasicNotions.Ranks` (`Formalism.LocalRingCardinality.Params`, lines 6–22)
`Formalism.MainNotions.SizeCond` (`Formalism.SharpRestriction.ElevenLe`, lines 54–54)
`Formalism.MainNotions.eventually_level_prime_coprime_stabiliser_of_design` (`Formalism.SharpRestriction.GlobalSchurBounds.PlugDgt`, lines 10–100)
17.8. `Formalism.MainNotions.Des` (`Formalism.SharpRestriction.SmoothDesignExistence.DesignExistsDes`, lines 10–11)
`Formalism.MainNotions.ChosenE` (`Formalism.SharpRestriction.GlobalSchurBounds.Chosen`, lines 83–84)
`Formalism.MainNotions.Pk.DGamSmooth` (`Formalism.SharpRestriction.SmoothDesignExistence.DGamSmooth`, lines 23–25)
`Formalism.MainNotions.TuSize` (`Formalism.SharpRestriction.ParamsHeightRankConstant`, lines 180–182)
`Formalism.MainNotions.DGamSize` (`Formalism.SharpRestriction.ParamsHeightRankConstant`, lines 184–185)
`Formalism.BasicNotions.FactDGam` (`Formalism.LocalRingCardinality.RhoDetPrimeEqMuTildeMulChiPow`, lines 381–381)
`GlobalTheory.Cp.WithChN` (`Development.AbelianVarietyTheory.Definitions.GlobalTwisting.CountsIrreducibilityConstants`, lines 32–33)
`Formalism.MainNotions.TameDistinct` (`Formalism.SharpRestriction.LevelLetterCocycles.AlignN`, lines 14–19)
17.9. `Formalism.MainNotions.eventually_level_prime_coprime_stabiliser_of_design` (`Formalism.SharpRestriction.GlobalSchurBounds.PlugDgt`, lines 10–100)
`Formalism.MainNotions.closed_eventually_levelPrimeCoprimeStabiliser` (`Formalism.SharpRestriction.GlobalSchurBounds.PlugDgt`, lines 102–106)
17.10. `GlobalTheory.EX.Led.exists_lat` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsLedPart`, lines 119–185)
`Formalism.BasicNotions.LatCmp.NuServedDef` (`Formalism.LocalRingCardinality.DataLatDefs`, lines 42–45)
`Formalism.BasicNotions.NuAtTDef` (`Formalism.LocalRingCardinality.DataLatDefs`, lines 111–113)
`Formalism.MainNotions.WeightDegreeSizes.NuInM` (`Formalism.SharpRestriction.Divisibility.WeightDegreeSizes`, lines 852–853)
17.10. `GlobalTheory.EX.Led.setLat` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsLedPart`, lines 187–190)
`GlobalTheory.EX.Led.exists_ledPart` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsLedPart`, lines 195–225)
17.12. `Formalism.MainNotions.J.ColGe0_atT` (`Formalism.SharpRestriction.AdmissibleParameterJoins.DIVfromGLi`, lines 34–67)
`Formalism.L2.G.Adm2InU.admissible_at_twist_places` (`Formalism.LocalToGlobal.GroupLattices.AdmissibleAtGeneratorPlaces`, lines 92–111)
17.13. `GlobalTheory.MomCrit.momInt_iff_norm_mahler_le` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.MahlerCoefficientNormLeOneIffJetsBoundedMeasureM`, lines 217–265)
`GlobalTheory.MomT.Riemann.riemann_sums` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.IsLocallyConstantToZModPow`, lines 55–102)
17.14. `GlobalTheory.CmThetaMoment.isomStd` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsStandardIsometryFourthPowerSplitPrime`, lines 255–272)
`Formalism.L2.Ker.Cd.PairingIndexBounds.Eq.J.first_constant_of_pullback` (`Formalism.LocalToGlobal.KernelLattices.ColumnData.ColumnDataEquations.JFinal`, lines 23–24)
`Formalism.L2.Ker.Cd.PairingIndexBounds.Eq.J.second_constant_of_pullback` (`Formalism.LocalToGlobal.KernelLattices.ColumnData.ColumnDataEquations.JFinal`, lines 26–27)
17.15. `GlobalTheory.MomN.transPrime_int` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.NormalizedVectorTranslatesPrimeOrderIntegral`, lines 32–82)

GlobalTheory.MomCrit.momInt_iff_norm_mahler_le (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.
MahlerCoefficientNormLeOneIffJetsBoundedMeasureM, lines 217–265)
17.16. GlobalTheory.MomOne.Norm.exists_normalised_chart (Development.AbelianVarietyTheory.Proofs.JetHarmonics.
ExistsNormalisedMultiplicativeChart, lines 200–277)
GlobalTheory.MomOne.ThShift.theta_shift (Development.AbelianVarietyTheory.Proofs.JetHarmonics.
CmLatticeWeylOperatorEqThetaRootOfUnityShift, lines 158–281)
GlobalTheory.MomOne.TranslateLaw.main (Development.AbelianVarietyTheory.Proofs.JetHarmonics.
WeilOperatorThetaTranslateShiftPrimeToQ, lines 61–144)
17.17. GlobalTheory.MomFin.moments_fin (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.
UnitPeriodTranslatesMomentsFinite, lines 77–176)
GlobalTheory.MomT.PtMoments.ptMom (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ContinuousToZModComp,
lines 342–443)
17.18. GlobalTheory.MomT.IdxBound.relIndex_per_dvd (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.
RelIndexPerDvd, lines 91–117)
GlobalTheory.MomT.CosetSum.moments_coset_sum (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.
MomentsCosetSum, lines 363–491)
17.19. GlobalTheory.KerT.PStd.pTerm_std (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.
ExistsMulEqOfUnif, lines 340–423)
GlobalTheory.BarStd.barStd (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.
AdmissibleAdjointRationalPointPadicMatrix, lines 388–457)
GlobalTheory.LvT.Built.UpLaw.kAt_uk (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.SameOfAbove, lines
61–77)
17.20. GlobalTheory.LvT.Built.TrueLaws.laws_true_fam (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.
FivePointLawsTrueChains, lines 46–62)
GlobalTheory.LvT.UKS.ukSharp (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExtInOfValIn, lines
155–198)
GlobalTheory.LvT.levelChainSharpColumnValuation_of_stability (Development.AbelianVarietyTheory.Proofs.
GlobalInterpolation.LevelChainSharpColumnValuationOfStability, lines 174–193)
GlobalTheory.LvT.Built.TrueCols.row_true (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.
TrueColumnsRowSharpBounds, lines 36–50)
GlobalTheory.LvT.StabOrder.stabUnit_of_embed (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.
NotDvdCardStabilizerForm, lines 157–164)
17.21. Formalism.MainNotions.WeightDegreeSizes.sg_atG (Formalism.SharpRestriction.Divisibility.WeightDegreeSizes,
lines 1007–1016)
Formalism.L2.G.Adm2InU.atG_coreU (Formalism.LocalToGlobal.GroupLattices.AdmissibleAtGeneratorPlaces, lines 113–132)
17.22. GlobalTheory.JetsFromTables.jetLossAll_of_jetSplitAmb (Development.AbelianVarietyTheory.Proofs.
SatakeParameters.ThetaJetIntegralityTypePrimeOfSplitTableBound, lines 221–292)
GlobalTheory.ColPeriodTies.boundAt_of_chainTie (Development.AbelianVarietyTheory.Proofs.SatakeParameters.
PeriodValuationBoundTypePrimePlaceOfChainAgreeme, lines 19–94)
GlobalTheory.ColPeriod.vG_loss_le (Development.AbelianVarietyTheory.Proofs.SatakeParameters.PeriodBoundOfJetLoss,
lines 87–141)
Formalism.MainNotions.J.sumFE_atG_le (Formalism.SharpRestriction.AdmissibleParameterJoins.GandSx, lines 12–84)
GlobalTheory.AtG.row_atG_of_jetSplitAmb (Development.AbelianVarietyTheory.Proofs.SatakeParameters.
TypePrimePeriodRowOfConstructedPointChains, lines 15–28)
GlobalTheory.AtG.jetSplitAmb_holds (Development.AbelianVarietyTheory.Proofs.SatakeParameters.
JetTableBoundSplitPrime, lines 7–9)
GlobalTheory.JetsFromTables.datum_bound (Development.AbelianVarietyTheory.Proofs.SatakeParameters.
ThetaJetIntegralityTypePrimeOfSplitTableBound, lines 108–165)
17.23. Formalism.L2.G.Adm2InU.admissible_at_generator_places (Formalism.LocalToGlobal.GroupLattices.
AdmissibleAtGeneratorPlaces, lines 134–169)
17.24. Formalism.MainNotions.WeilJacquetShalikaLaw.gainD_le_gainS (Formalism.SharpRestriction.GrowthBounds.
WeilJacquetShalikaLaw, lines 552–575)
Formalism.MainNotions.WeightDegreeSizes.gainLe_of_fills (Formalism.SharpRestriction.Divisibility.WeightDegreeSizes,
lines 931–933)
17.25. Formalism.L2.G.Adm2InU.admissible_off_twist_places (Formalism.LocalToGlobal.GroupLattices.
AdmissibleAtGeneratorPlaces, lines 171–196)
17.26. Formalism.L2.G.Adm2InU.admissible_away_from_twist (Formalism.LocalToGlobal.GroupLattices.
AdmissibleAtGeneratorPlaces, lines 198–212)
17.27. Formalism.MainNotions.LocalQuadraticParts.UnroundedValuationBound (Formalism.SharpRestriction.Divisibility.
LocalQuadraticParts, lines 212–216)
Formalism.MainNotions.LocalQuadraticParts.adm2 (Formalism.SharpRestriction.Divisibility.LocalQuadraticParts, lines
206–210)
Formalism.MainNotions.LocalQuadraticParts.valuation_bound_of_unrounded (Formalism.SharpRestriction.Divisibility.
LocalQuadraticParts, lines 218–221)
17.28. Formalism.L2.G.Adm2InU.admissible_level_two (Formalism.LocalToGlobal.GroupLattices.
AdmissibleAtGeneratorPlaces, lines 214–245)
Formalism.MainNotions.LocalQuadraticParts.adm12 (Formalism.SharpRestriction.Divisibility.LocalQuadraticParts, lines
323–328)

17.29. Formalism.L2.G.Adm2InU.famU (Formalism.LocalToGlobal.GroupLattices.AdmissibleAtGeneratorPlaces, lines 263–300)
Formalism.L2.G.Adm2InU.admissible_family_of_inclusion (Formalism.LocalToGlobal.GroupLattices.AdmissibleAtGeneratorPlaces, lines 315–338)
17.30. GlobalTheory.Top.closed_family_admissible_twist_link_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ClosedFamilyAdmissibleTwistLink, lines 12–31)

Chapter 18: Gluing row lattices across Hecke characters

18.1. Formalism.L2.DIV.finChar (Formalism.LocalToGlobal.Divisibility.Tuples, lines 107–108)
Formalism.L2.DIV.tupGrp (Formalism.LocalToGlobal.Divisibility.Tuples, lines 112–114)
Formalism.L2.DIV.Tup (Formalism.LocalToGlobal.Divisibility.Tuples, lines 118–118)
18.2. Formalism.L2.DIV.TupF (Formalism.LocalToGlobal.Divisibility.LiftLatInst, lines 58–58)
Formalism.L2.DIV.colIn3 (Formalism.LocalToGlobal.Divisibility.HeckeCSDICommTrRep, lines 130–140)
Formalism.L2.DIV.DivisionKernelData (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.DIVRecBridgeSel, lines 32–57)
GlobalTheory.Mem.Col2.Reads (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.TrueColumnChoicesHeckeCharReferenceEqOne, lines 48–49)
GlobalTheory.Mem.Col2.ColIn.Ok.xi_reads (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.TrueColumnChoicesHeckeCharReferenceEqOne, lines 237–237)
GlobalTheory.Mem.R3.tup_ixOf (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.IsAbsIrrCharCompIff, lines 105–112)
18.3. Formalism.L2.Indef.DefSide.abelian₂ (Formalism.LocalToGlobal.IndefiniteForms.IndefSide, lines 68–68)
Formalism.L2.Indef.IndefData.RR (Formalism.LocalToGlobal.IndefiniteForms.IndefSide, lines 281–282)
Formalism.DivisionValues.glueLat (Formalism.QuotientMetricDivisors.LatticeGlueUnique, lines 11–13)
18.4. Formalism.DivisionValues.lattice_gluing (Formalism.QuotientMetricDivisors.HermitianTwistDivisors.SecondStepGluing, lines 256–258)
Formalism.DivisionValues.GluingInclusion.glue_spec (Formalism.QuotientMetricDivisors.HermitianTwistDivisors.SecondStepGluing, lines 74–182)
18.5. Formalism.L2.Indef.IndefRows.RR_spec (Formalism.LocalToGlobal.IndefiniteForms.IndefUse, lines 219–229)
Formalism.L2.Indef.IndefColsR.RR_spec (Formalism.LocalToGlobal.IndefiniteForms.PairingConditionOfInverseMatrixIdentity, lines 332–342)
18.6. GlobalTheory.Mem.Top.TwFin.tw_fin (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ResCharConj, lines 58–78)
GlobalTheory.Mem.R3.tupL_of_twisted (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.MemHeightOneSpectrumIffAdditiveValuationPos, lines 82–188)
Formalism.IndefR.Links.RealDat3.height_change_of_D0 (Formalism.LocalToGlobal.IndefiniteReal.BaseLiftsPackageOfPrimitiveLinks, lines 105–108)
Formalism.L2.Indef.IndefData.H0 (Formalism.LocalToGlobal.IndefiniteForms.IndefSide, lines 294–295)
GlobalTheory.Mem.Col2.InEx.lg_colIn (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ExistsColumnInputAssembled, lines 328–373)
18.7. Formalism.L2.G.ServJoint.CutUp (Formalism.LocalToGlobal.GroupLattices.JointServices, lines 65–70)
Formalism.L2.G.ServJoint.cutUp_of_stable (Formalism.LocalToGlobal.GroupLattices.JointServices.Cut, lines 33–45)
18.8. Formalism.L2.G.ServJointF.ColBrF.Tied (Formalism.LocalToGlobal.GroupLattices.ServJointF, lines 30–40)
Formalism.L2.G.ServJointF.perToVal2F (Formalism.LocalToGlobal.GroupLattices.ServJointF, lines 42–82)
Formalism.L2.G.ServJointF.val2AllAtRF (Formalism.LocalToGlobal.GroupLattices.ServJointF, lines 84–102)
Formalism.L2.G.ServJoint.vAdd_eq_vw (Formalism.LocalToGlobal.GroupLattices.JointServices.VAddEqVw, lines 37–52)
Formalism.L2.G.ServJoint.perCol_lin (Formalism.LocalToGlobal.GroupLattices.JointServices.ValIntNonneg, lines 12–23)

Chapter 19: The pairing after division

19.1. Formalism.BasicNotions.DV.Iu (Formalism.LocalRingCardinality.DataDIVDefs, lines 17–20)
Formalism.BasicNotions.DV.NprimeDef (Formalism.LocalRingCardinality.DataDIVDefs, lines 22–24)
19.2. Formalism.DivisionValues.localizationAt_fractionalIdeal_smul_eq_smul (Formalism.QuotientMetricDivisors.HermitianTwistDivisors.TwistedLatticeNaturality, lines 547–572)
Formalism.DivisionValues.shift_existence (Formalism.QuotientMetricDivisors.HermitianTwistDivisors.TwistedLatticeNaturality, lines 649–662)
19.2. Formalism.DivisionValues.IsShiftAt (Formalism.QuotientMetricDivisors.DivIntoDualShift, lines 18–21)
Formalism.DivisionValues.IsShift (Formalism.QuotientMetricDivisors.DivIntoDualShift, lines 23–24)
19.3. Formalism.L2.DIV.sgRec (Formalism.LocalToGlobal.Divisibility.RecBridge, lines 387–388)
Formalism.L2.DIV.shiftZ (Formalism.LocalToGlobal.Divisibility.Inputs, lines 381–381)
Formalism.L2.DIV.shiftLat (Formalism.LocalToGlobal.Divisibility.Inputs, lines 383–385)
Formalism.L2.DIV.Cons.shift_nuOf (Formalism.LocalToGlobal.Divisibility.DivisibilityConstruction.ShiftOfColTie, lines 53–79)
19.4. Formalism.DivisionValues.shift_pairing_formula (Formalism.QuotientMetricDivisors.HermitianTwistDivisors.TwistedLatticeNaturality, lines 684–718)

19.5. Formalism.DivisionValues.reduction_first_step (Formalism.QuotientMetricDivisors.HermitianTwistDivisors.VwEqZeroOfNotMem, lines 34–60)
Formalism.DivisionValues.divisor_into_dual_lattice (Formalism.QuotientMetricDivisors.HermitianTwistDivisors.DivIntoDual, lines 49–61)
19.6. Formalism.DivisionValues.divisor_into_dual_shift (Formalism.QuotientMetricDivisors.HermitianTwistDivisors.TwistedLatticeNaturality, lines 720–730)
19.7. Formalism.L2.Indef.IndefColsR.intoDual (Formalism.LocalToGlobal.IndefiniteForms.PairingConditionOfInverseMatrixIdentity, lines 393–408)
Formalism.L2.DIV.oneMap_of_coord (Formalism.LocalToGlobal.Divisibility.RecBridge, lines 33–50)

Chapter 20: Slope change and the data for step 5

20.1. Formalism.MainNotions.J.mu_smul_ideal (Formalism.SharpRestriction.AdmissibleParameterJoins.SumLogPrimeFactorsLe, lines 30–71)
20.2. Formalism.MainNotions.J.mu_frac (Formalism.SharpRestriction.AdmissibleParameterJoins.SumLogPrimeFactorsLe, lines 73–170)
20.3. Formalism.MainNotions.J.Iu_ne_zero (Formalism.SharpRestriction.AdmissibleParameterJoins.SumLogPrimeFactorsLe, lines 352–356)
Formalism.MainNotions.J.log_absNorm_Iu (Formalism.SharpRestriction.AdmissibleParameterJoins.SumLogPrimeFactorsLe, lines 358–378)
20.4. Formalism.MainNotions.J.mu_diff (Formalism.SharpRestriction.AdmissibleParameterJoins.SumLogPrimeFactorsLe, lines 380–397)
Formalism.BasicNotions.SizeData₂.NuH (Formalism.LocalRingCardinality.DataParams₂, lines 236–236)
20.5. Formalism.DivisionValues.shift_index_formula (Formalism.QuotientMetricDivisors.HermitianTwistDivisors.TwistedLatticeNaturality, lines 90–97)
Formalism.DivisionValues.mu_log_conductor (Formalism.QuotientMetricDivisors.HermitianTwistDivisors.TwistedLatticeNaturality, lines 101–128)
Formalism.DivisionValues.shift_slope_mu (Formalism.QuotientMetricDivisors.HermitianTwistDivisors.TwistedLatticeNaturality, lines 145–182)
20.6. Formalism.MainNotions.J.round_le (Formalism.SharpRestriction.AdmissibleParameterJoins.SumLogPrimeFactorsLe, lines 401–429)
20.7. Formalism.MainNotions.J.sum_e_f (Formalism.SharpRestriction.AdmissibleParameterJoins.SumLogPrimeFactorsLe, lines 202–216)
Formalism.MainNotions.J.sum_lw_e (Formalism.SharpRestriction.AdmissibleParameterJoins.SumLogPrimeFactorsLe, lines 218–235)
20.8. Formalism.MainNotions.WeightDegreeSizes.gain_above (Formalism.SharpRestriction.Divisibility.WeightDegreeSizes, lines 294–303)
Formalism.BasicNotions.Eval_above (Formalism.LocalRingCardinality.DataErr, lines 174–187)
Formalism.MainNotions.WeightDegreeSizes.zero_gain_above_family (Formalism.SharpRestriction.Divisibility.WeightDegreeSizes, lines 399–400)
Formalism.L2.DIV.Wire.za_of (Formalism.LocalToGlobal.Divisibility.RecBridgeWire, lines 19–20)
20.9. Formalism.MainNotions.J.sg_eq_zero (Formalism.SharpRestriction.AdmissibleParameterJoins.SumLogPrimeFactorsLe, lines 431–446)
Formalism.MainNotions.J.nprimeMu_at (Formalism.SharpRestriction.AdmissibleParameterJoins.SumLogPrimeFactorsLe, lines 448–492)
20.10. Formalism.MainNotions.SizeQD (Formalism.SharpRestriction.ElevenLe, lines 56–63)
Formalism.MainNotions.SizeG (Formalism.SharpRestriction.ElevenLe, lines 123–125)
20.11. Formalism.MainNotions.J.nprimeMuR_at (Formalism.SharpRestriction.AdmissibleParameterJoins.SumLogPrimeFactorsLe, lines 694–709)
20.12. Formalism.L2.Indef.Tree.local_bound (Formalism.LocalToGlobal.IndefiniteForms.IndefiniteTree.ExampleSatisfiesConditionsAtFive, lines 409–452)
Formalism.EigenvalueLevelCeiling.lemmaGG_of (Formalism.MirabolicMackeyTheory.MirabolicMackeyProofs.LemmaGG, lines 218–365)
Formalism.EigenvalueLevelCeiling.idxPsiGood_of (Formalism.MirabolicMackeyTheory.MirabolicMackeyProofs.IdxPsiGood, lines 18–121)
20.13. Formalism.L2.Indef.IndefData (Formalism.LocalToGlobal.IndefiniteForms.IndefSide, lines 241–269)
Formalism.L2.Indef.IndefSide (Formalism.LocalToGlobal.IndefiniteForms.IndefSide, lines 363–367)
Formalism.L2.Indef.ColData.L1e (Formalism.LocalToGlobal.IndefiniteForms.IndefSide, lines 388–394)
Formalism.L2.Indef.IndefColsR (Formalism.LocalToGlobal.IndefiniteForms.PairingConditionOfInverseMatrixIdentity, lines 218–237)
Formalism.L2.Indef.IndefExtraR (Formalism.LocalToGlobal.IndefiniteForms.PairingConditionOfInverseMatrixIdentity, lines 239–248)
Formalism.L2.Indef.IndefCoreR (Formalism.LocalToGlobal.IndefiniteForms.PairingConditionOfInverseMatrixIdentity, lines 250–256)
20.14. Formalism.L2.DIV.DefFam3.Ok3 (Formalism.LocalToGlobal.Divisibility.InertRationalAdjoint, lines 139–145)
Formalism.L2.DIV.InertDat3.Ok3 (Formalism.LocalToGlobal.Divisibility.InertRationalAdjoint, lines 53–61)

Formalism.L2.DIV.DefFam.Ok (Formalism.LocalToGlobal.Divisibility.RowLat, lines 267–277)
Formalism.L2.DIV.BlockDat.Ok (Formalism.LocalToGlobal.Divisibility.RowLat, lines 176–198)
Formalism.L2.DIV.CSD.Good (Formalism.LocalToGlobal.Divisibility.ClassSum, lines 92–101)
Formalism.L2.DIV.CSD.ActGood (Formalism.LocalToGlobal.Divisibility.GoodPartHermitianSpec, lines 76–82)
Formalism.L2.DIV.CSD.ActRat (Formalism.LocalToGlobal.Divisibility.GoodPartHermitianSpec, lines 84–84)
Formalism.L2.Col.BlockOk (Formalism.LocalToGlobal.Columns.HeckeSp, lines 580–583)
20.15. Formalism.L2.DIV.DivInputs0R (Formalism.LocalToGlobal.Divisibility.ConstantGeTwentyEight, lines 26–103)
Formalism.L2.DIV.DivInputsDef3.toDivInputs0R (Formalism.LocalToGlobal.Divisibility.TwistedPlaceEtaSumLeExtended, lines 247–286)
Formalism.DivisorInequality.places_ell_weighted_sum_le_one (Formalism.DivisorInequality.DivisorInequalityProofs.BaseLevelBoundsWithCovolumeHolds, lines 35–45)
20.16. Formalism.L2.DIV.divClausesR_of (Formalism.LocalToGlobal.Divisibility.ConstantGeTwentyEight, lines 184–247)
Formalism.L2.DIV.DivClausesR (Formalism.LocalToGlobal.Divisibility.ConstantGeTwentyEight, lines 111–121)
Formalism.L2.DIV.HtNums.le_hu (Formalism.LocalToGlobal.Divisibility.DivisionHeightBoundStatement, lines 113–123)
20.17. Formalism.L2.DIV.two_sumTw_eq_Fo (Formalism.LocalToGlobal.Divisibility.SmallTies, lines 49–79)
Formalism.L2.DIV.two_FoOfR_eq_Fo (Formalism.LocalToGlobal.Divisibility.SmallTies, lines 86–89)
Formalism.L2.DIV.fixedScale_tied (Formalism.LocalToGlobal.Divisibility.EigenvalueSumLeClosedOf, lines 350–353)
Formalism.L2.DIV.ofRecord_Fval (Formalism.LocalToGlobal.Divisibility.NumbersLemmas, lines 780–782)
Formalism.L2.DIV.Fj_ofRecord_eq (Formalism.LocalToGlobal.Divisibility.EigenvalueSumLeClosedOf, lines 332–339)
20.18. Formalism.IndefR.LatticeTransfer.levelAmb_transfer (Formalism.LocalToGlobal.IndefiniteReal.LevelAmbTransfer, lines 13–54)
Formalism.BranchingDeterminantCap.meanTransferBound (Formalism.BranchSandwichDivisors.BranchIsometryProofs.TransferMean, lines 19–64)
Formalism.IndefR.CxPass.In3.passage_of_steps (Formalism.LocalToGlobal.IndefiniteReal.CutPassage3SOfSteps, lines 49–108)
20.19. Formalism.L2.DIV.DivisionPackageFace.wordFacts (Formalism.LocalToGlobal.Divisibility.RecBridge, lines 633–641)
Formalism.L2.DIV.WordFacts (Formalism.LocalToGlobal.Divisibility.RecBridge, lines 128–133)

Chapter 21: The lattice after division

21.1. Formalism.LFunctionChainParameters.characterSumData.ASK (Formalism.SizeWindowBrackets.FirstPrimeNeSecondPrime, lines 84–87)
Formalism.LFunctionChainParameters.characterSumData.askP (Formalism.SizeWindowBrackets.FirstPrimeNeSecondPrime, lines 75–77)
Formalism.LFunctionChainParameters.characterSumData.askD (Formalism.SizeWindowBrackets.FirstPrimeNeSecondPrime, lines 79–82)
Formalism.MainNotions.Lnk.Mod.data (Formalism.SharpRestriction.GlobalSchurBounds.QuadraticLogDerivCongr, lines 464–504)
21.2. Formalism.LFunctionChainParameters.characterSumData.askP (Formalism.SizeWindowBrackets.FirstPrimeNeSecondPrime, lines 75–77)
Formalism.LFunctionChainParameters.characterSumData.askD (Formalism.SizeWindowBrackets.FirstPrimeNeSecondPrime, lines 79–82)
21.3. Formalism.BasicNotions.SizeData₂.sg (Formalism.LocalRingCardinality.DataParams₂, lines 215–216)
Formalism.BasicNotions.SizeData₂.sz (Formalism.LocalRingCardinality.DataParams₂, lines 205–206)
Formalism.BasicNotions.SizeData₂.gainD (Formalism.LocalRingCardinality.DataParams₂, lines 212–213)
Formalism.BasicNotions.SizeData₂.Fo (Formalism.LocalRingCardinality.DataParams₂, lines 218–218)
Formalism.BasicNotions.SizeData₂.Ga (Formalism.LocalRingCardinality.DataParams₂, lines 220–220)
Formalism.BasicNotions.SizeData₂.Ee (Formalism.LocalRingCardinality.DataParams₂, lines 222–222)
Formalism.MainNotions.Lnk.IOsum (Formalism.SharpRestriction.GlobalSchurBounds.QuadraticLogDerivCongr, lines 163–163)
21.4. Formalism.BasicNotions.SizeData₂.Ga (Formalism.LocalRingCardinality.DataParams₂, lines 220–220)
Formalism.BasicNotions.SizeData₂.Fo (Formalism.LocalRingCardinality.DataParams₂, lines 218–218)
21.5. Formalism.MainNotions.LocalQuadraticParts.UnroundedValuationBound (Formalism.SharpRestriction.Divisibility.LocalQuadraticParts, lines 212–216)
Formalism.BasicNotions.DV.Iu (Formalism.LocalRingCardinality.DataDIVDefs, lines 17–20)
Formalism.BasicNotions.DV.NprimeDef (Formalism.LocalRingCardinality.DataDIVDefs, lines 22–24)
21.6. Formalism.L2.GlobalTheoryTheory.Places.mu_index_sum (Formalism.LocalToGlobal.GlobalTheory.PlacesP, lines 61–102)
Formalism.L2.GlobalTheoryTheory.Places.mu_eq (Formalism.LocalToGlobal.GlobalTheory.PlacesP, lines 129–169)
21.7. Formalism.L2.GlobalTheoryTheory.Places.muMax_le (Formalism.LocalToGlobal.GlobalTheory.PlacesP, lines 171–237)
21.8. Formalism.MainNotions.product_first_zeroth_parts.HidaSetting (Formalism.SharpRestriction.GlobalSchurBounds.ProductFirstZerothParts, lines 135–137)
Formalism.MainNotions.GlobalTheoryTheory.Alc.entry_bd (Formalism.SharpRestriction.GlobalSchurBounds.AlcoveOf, lines 65–121)
Formalism.MainNotions.GlobalTheoryTheory.Alc.alcove_of (Formalism.SharpRestriction.GlobalSchurBounds.AlcoveOf, lines 123–151)

Formalism.MainNotions.J.levelPrime_setting_of_sizes (Formalism.SharpRestriction.AdmissibleParameterJoins.DIVfromGLo, lines 249–292)
21.9. Formalism.L2.GlobalTheoryTheory.LocalLatticeBases.congruence_no_kernel_minimal (Formalism.LocalToGlobal.GlobalTheory.GL6.CongruenceNoKernelOfHeckeRamified, lines 75–158)
Formalism.MainNotions.J.division_rank_slopes_of_lines (Formalism.SharpRestriction.GlobalSchurBounds.LinkMin, lines 497–582)
Formalism.MainNotions.J.lemma_five_one_of_theorem_two (Formalism.SharpRestriction.GlobalSchurBounds.LinkMin, lines 587–612)
21.11. Formalism.MainNotions.Lnk.slopesN_at_of (Formalism.SharpRestriction.GlobalSchurBounds.TopDensityNGeSixteen, lines 1529–1539)
Formalism.LFunctionChainParameters.WeightedSlopeBounds.QuotientSlopeBoundsWith (Formalism.SizeWindowBrackets.QuotientSlopeBoundsWith, lines 19–22)
Formalism.L2.GlobalTheoryTheory.Places.places_at_of (Formalism.LocalToGlobal.GlobalTheory.PlacesP, lines 297–435)
21.12. Formalism.L2.DIV.height_chain_atR (Formalism.LocalToGlobal.Divisibility.ConstantGeTwentyEight, lines 123–182)
Formalism.BasicNotions.DV.hExpl6 (Formalism.LocalRingCardinality.KernelPlaceMeanBound, lines 25–27)
Formalism.BasicNotions.DV.logHook (Formalism.LocalRingCardinality.KernelPlaceMeanBound, lines 9–11)
Formalism.BasicNotions.DV.Lprime (Formalism.LocalRingCardinality.KernelPlaceMeanBound, lines 13–14)
Formalism.BasicNotions.DV.L6lo (Formalism.LocalRingCardinality.KernelPlaceMeanBound, lines 16–18)
Formalism.BasicNotions.DV.kernelPlaceMeanBound (Formalism.LocalRingCardinality.KernelPlaceMeanBound, lines 20–23)
21.13. GlobalTheory.BranchNorm.brNorm (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LastMapNorm, lines 336–393)
GlobalTheory.LastPeriod.coefOpLe_brAt (Development.AbelianVarietyTheory.Proofs.DivisorProofs.PeriodBoundLastLog, lines 26–43)
GlobalTheory.ColRowsTop.periodFun_le (Development.AbelianVarietyTheory.Proofs.DivisorProofs.IpAtNonnegOfDatFramed, lines 26–85)
21.14. GlobalTheory.LevelSupport.sum_heisO_along_eq_zero_level (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LevelFockVectorHeisenbergAverageIsotropicEqZero, lines 274–318)
GlobalTheory.DefLiftCusp.zcusp_lift (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ThetaLiftDefiniteConstantTermEqZero, lines 66–127)
GlobalTheory.TransvCusp.isCuspFun_of_zCusp (Development.AbelianVarietyTheory.Proofs.DivisorProofs.IsCuspFunOfTransvectionConstantTermsZero, lines 142–162)
21.15. GlobalTheory.VacCoef.vac_coef (Development.AbelianVarietyTheory.Proofs.TwistedOperators.WeilMatrixCoefficientVacuaUnramifiedPlace, lines 109–170)
GlobalTheory.LocZeta.hasSum_localZeta (Development.AbelianVarietyTheory.Proofs.TwistedOperators.HasSumLocalSphericalZetaUnramified, lines 101–196)
21.16. GlobalTheory.LowRow.scalar_of_form_key_lowC (Development.AbelianVarietyTheory.Proofs.DivisorProofs.AdelCOfInnerProductFormula, lines 214–242)
GlobalTheory.LowForm.exists_full_lift_six (Development.AbelianVarietyTheory.Proofs.DivisorProofs.RationalLiftLowerBoundOffBasePlace, lines 152–176)
GlobalTheory.LowForm.conjLift_lower_six (Development.AbelianVarietyTheory.Proofs.DivisorProofs.RationalLiftLowerBoundOffBasePlace, lines 196–206)
21.17. Formalism.DivisorInequality.height_bound_at_set (Formalism.DivisorInequality.DivisorInequalityProofs.EvenDegreeWeightBounds, lines 154–287)
Formalism.DivisorInequality.UniformCovolumeTail.stepDiSharp (Formalism.DivisorInequality.DivisorInequalityProofs.EvenDegreeWeightBounds, lines 27–148)
Formalism.L2.DIV.nK_of_cb (Formalism.LocalToGlobal.Divisibility.DivisionHeightBoundStatement, lines 565–583)
21.18. GlobalTheory.GalOne.ClassConj.class_scalar_conj_ofFive (Development.AbelianVarietyTheory.Proofs.DivisorProofs.CoefMapNormEquivariantBasisOfFive, lines 2202–2271)
21.19. GlobalTheory.ConjRatLiftFive.base_lift5 (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ConjugatesOfRationalThetaLiftFromCharacterisatio, lines 46–67)
GlobalTheory.ConjRatLiftFive.conj_lift5 (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ConjugatesOfRationalThetaLiftFromCharacterisatio, lines 71–138)
21.20. Formalism.IndefR.Links.RealDat3.PartAt (Formalism.LocalToGlobal.IndefiniteReal.ZeroPartEigenvalueConvention, lines 190–192)
Formalism.IndefR.Links.RealDat3.scal_off (Formalism.LocalToGlobal.IndefiniteReal.ZeroPartEigenvalueConvention, lines 202–208)
Formalism.IndefR.Links.RealDat3.scal_abs (Formalism.LocalToGlobal.IndefiniteReal.ZeroPartEigenvalueConvention, lines 212–213)
GlobalTheory.ConjScalarOff.galD_base (Development.AbelianVarietyTheory.Proofs.TwistedOperators.LiftScalarIdentityOffFixedPlaceBaseLevel, lines 40–72)
21.21. GlobalTheory.LowRow.lamLow_of_weak (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LiftEigenvalueLowerBoundOfWeakCharacterisation, lines 20–40)
GlobalTheory.LowRow.lamLow_closing_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.FinitenessInputsRealDataThree, lines 14–23)
Formalism.L2.Indef.IndefColsR.log_eigenvalue_sum_le_min (Formalism.LocalToGlobal.IndefiniteForms.PairingConditionOfInverseMatrixIdentity, lines 328–330)
21.22. GlobalTheory.MeanPeriodClosed.meanPeriodAt (Development.AbelianVarietyTheory.Proofs.DivisorProofs.MeanLogPeriodChowlaSelberg, lines 58–82)

`GlobalTheory.MeanPeriodGenerator.mean_logMet_eq` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.MeanLogMetricGeneratorEqMeanPeriod`, lines 45–57)
21.23. `GlobalTheory.EX.exists_hc_family` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsHcParamOfCode`, lines 128–241)
21.24. `Formalism.L2.DIV.NumDatum.NumFacts` (`Formalism.LocalToGlobal.Divisibility.NumbersLink`, lines 43–82)
`Formalism.L2.DIV.Arith4S.numFacts_rows` (`Formalism.LocalToGlobal.Divisibility.EigenvalueSumLeClosedOf`, lines 1462–1521)
`Formalism.L2.DIV.Arith4S.numFacts_rec` (`Formalism.LocalToGlobal.Divisibility.EigenvalueSumLeClosedOf`, lines 1969–1979)
`Formalism.L2.DIV.RecNums.ofNum` (`Formalism.LocalToGlobal.Divisibility.NumbersLink`, lines 541–649)
21.25. `Formalism.L2.DIV.NumDatum.eEN` (`Formalism.LocalToGlobal.Divisibility.NumbersLink`, lines 35–36)
`Formalism.L2.DIV.NumDatum.dN` (`Formalism.LocalToGlobal.Divisibility.NumbersLink`, lines 38–39)
`Formalism.DivisionValues.young_diagram_hook` (`Formalism.QuotientMetricDivisors.HermitianTwistDivisors.YoungDiagramHook`, lines 386–422)
`Formalism.DivisionValues.young_diagram_factorial` (`Formalism.QuotientMetricDivisors.HermitianTwistDivisors.YoungDiagramHook`, lines 250–331)
`Formalism.DivisionValues.young_diagram_period` (`Formalism.QuotientMetricDivisors.HermitianTwistDivisors.YoungDiagramHook`, lines 350–375)
`Formalism.DivisionValues.young_diagram_step_origin` (`Formalism.QuotientMetricDivisors.HermitianTwistDivisors.YoungDiagramTopExact`, lines 254–295)
`Formalism.L2.DIV.NumDatum.stepE_num` (`Formalism.LocalToGlobal.Divisibility.NumbersLink`, lines 388–537)
21.26. `Formalism.L2.DIV.RecNums.divIneq` (`Formalism.LocalToGlobal.Divisibility.DivisionHeightBoundStatement`, lines 131–180)
`Formalism.DivisorInequality.slope_bounds` (`Formalism.DivisorInequality.DivisorInequalityProofs.BaseLevelBoundsWithCovolumeHolds`, lines 47–69)
`Formalism.DivisionValues.lemmaRb_of` (`Formalism.QuotientMetricDivisors.HermitianTwistDivisors.ReductionLemmaSecond`, lines 163–286)
`Formalism.L2.Indef.IndefSide.boundsMu` (`Formalism.LocalToGlobal.IndefiniteForms.IndefUse`, lines 271–334)
21.27. `Formalism.DivisionValues.StepBAux.idx_change_img` (`Formalism.QuotientMetricDivisors.HermitianTwistDivisors.ReductionSecondStep`, lines 379–398)
`Formalism.DivisionValues.ConjugateWeightGram.local_idW` (`Formalism.QuotientMetricDivisors.HermitianTwistDivisors.ConjugateWeightLatticeOfGeneral`, lines 174–317)
`Formalism.DivisionValues.stepBW_of` (`Formalism.QuotientMetricDivisors.HermitianTwistDivisors.ConjugateWeightLatticeOfGeneral`, lines 324–498)
21.28. `Formalism.DivisionValues.ConjugateWeightGram.local_idW` (`Formalism.QuotientMetricDivisors.HermitianTwistDivisors.ConjugateWeightLatticeOfGeneral`, lines 174–317)
`Formalism.DivisionValues.second_step_refined_weak` (`Formalism.QuotientMetricDivisors.HermitianTwistDivisors.ConjugateWeightLatticeOfGeneral`, lines 506–516)
`Formalism.L2.Indef.IndefSide.boundsMu` (`Formalism.LocalToGlobal.IndefiniteForms.IndefUse`, lines 271–334)
`Formalism.L2.DIV.DivInputsDef3.toDivInputs0R` (`Formalism.LocalToGlobal.Divisibility.TwistedPlaceEtaSumLeExtended`, lines 247–286)
21.29. `Formalism.L2.DIV.DivClausesR` (`Formalism.LocalToGlobal.Divisibility.ConstantGeTwentyEight`, lines 111–121)
`Formalism.L2.DIV.divClausesR_of` (`Formalism.LocalToGlobal.Divisibility.ConstantGeTwentyEight`, lines 184–247)
21.30. `GlobalTheory.ColumnMultSum.columnMult` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ReordEqConj`, lines 35–77)
`Formalism.L2.DIV.Arith4S.q_rec` (`Formalism.LocalToGlobal.Divisibility.EigenvalueSumLeClosedOf`, lines 1885–1953)
`Formalism.L2.DIV.hg_of_sizeG` (`Formalism.LocalToGlobal.Divisibility.EigenvalueSumLeClosedOf`, lines 295–324)
`Formalism.L2.DIV.delta_mul_Q_le_one` (`Formalism.LocalToGlobal.Divisibility.EigenvalueSumLeClosedOf`, lines 126–138)
21.31. `Formalism.MainNotions.LinkDIV_at` (`Formalism.SharpRestriction.GlobalTheoremInputs`, lines 77–89)
21.32. `Formalism.MainNotions.LinkDIV_at` (`Formalism.SharpRestriction.GlobalTheoremInputs`, lines 77–89)
`Formalism.L2.DIV.wordFacts_of_face` (`Formalism.LocalToGlobal.Divisibility.RecBridge`, lines 204–225)
`Formalism.L2.DIV.linkDIV_at_words` (`Formalism.LocalToGlobal.Divisibility.RecBridge`, lines 327–339)
`Formalism.L2.DIV.DivisionPackageFace.wordFacts` (`Formalism.LocalToGlobal.Divisibility.RecBridge`, lines 633–641)
21.33. `Formalism.L2.DIV.FunSharpTop.closed_funSharp_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.HefunFunTop`, lines 88–116)
`Formalism.MainNotions.DIV.Proofs.height_bound_of_component_bounds` (`Formalism.SharpRestriction.Divisibility.SlopeExcess.HeightBoundOfComponentBounds`, lines 136–150)
21.34. `Formalism.MainNotions.StandardLattice.DataB` (`Formalism.SharpRestriction.GlobalSchurBounds.StandardLattice`, lines 500–516)
`Formalism.MainNotions.StandardLattice.RowsA` (`Formalism.SharpRestriction.GlobalSchurBounds.TheoremThreeOfExplicitAndCompact`, lines 115–121)
`Formalism.MainNotions.StandardLattice.DataA` (`Formalism.SharpRestriction.GlobalSchurBounds.TheoremThreeOfExplicitAndCompact`, lines 123–124)
`Formalism.MainNotions.GlobalTheoryTheory.JDB.dataA_of` (`Formalism.SharpRestriction.GlobalSchurBounds.GlobalRankLatticeProofs.FamilyPartFiveOfExplicitConstants`, lines 299–301)
21.35. `Formalism.MainNotions.GlobalTheoryTheory.count_at_of` (`Formalism.SharpRestriction.GlobalSchurBounds.JointsL`, lines 20–142)
`Formalism.MainNotions.StandardLattice.division_level_prime_family` (`Formalism.SharpRestriction.GlobalSchurBounds.LinkMin`, lines 425–453)

Formalism.MainNotions.GlobalTheoryTheory.countF_fam (Formalism.SharpRestriction.GlobalSchurBounds.JointsL, lines 1152–1155)
21.36. Formalism.L2.SG.Pf.SzHypFam.sizes3b_of_ranks (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.SzHypFam, lines 78–92)
Formalism.MainNotions.BridgeLemma.level_ratio_bound (Formalism.SharpRestriction.CuspSpanNormBounds.BruhatCellsKronecker.DegreeRatioFormulaStatement, lines 398–523)
21.37. Formalism.MainNotions.flipTwoThree.ColVal (Formalism.SharpRestriction.GlobalSchurBounds.FlipTwoThree, lines 403–440)
21.38. Formalism.L2.GlobalTheoryTheory.TraceIntegrality.bc_mu (Formalism.LocalToGlobal.GlobalTheory.GL4.ColumnLatticeExactSlope, lines 98–177)
Formalism.L2.GlobalTheoryTheory.TraceIntegrality.eLat_mu (Formalism.LocalToGlobal.GlobalTheory.GL4.MemDualCrdLat, lines 71–128)
Formalism.L2.GlobalTheoryTheory.TraceIntegrality.lower_slope_inequality_family (Formalism.LocalToGlobal.GlobalTheory.GL4.LowerSlopeInequalityFamily, lines 14–15)
Formalism.L2.GlobalTheoryTheory.TraceIntegrality.slope_inequality_constant_four (Formalism.LocalToGlobal.GlobalTheory.GL4.PStmtFree, lines 14–15)
21.39. Formalism.L2.GlobalTheoryTheory.TraceIntegrality.q_lattice_slope (Formalism.LocalToGlobal.GlobalTheory.GL4.ColumnLatticeExactSlope, lines 183–202)
Formalism.MainNotions.flipTwoThree.slopesMF_of_count_fam (Formalism.SharpRestriction.GlobalSchurBounds.GlobalRankLatticeProofs.SlopesMFOfCountFam, lines 25–32)
Formalism.MainNotions.GlobalTheoryTheory.JSL.slopesM_at_of (Formalism.SharpRestriction.GlobalSchurBounds.JointsL, lines 1518–1589)
21.40. Formalism.MainNotions.WeightDegreeSizes.SecondComplementInputs (Formalism.SharpRestriction.Divisibility.WeightDegreeSizes, lines 673–675)
Formalism.MainNotions.WeightDegreeSizes.FamilySecondComplementInputs (Formalism.SharpRestriction.Divisibility.WeightDegreeSizes, lines 683–685)
Formalism.MainNotions.WeightDegreeSizes.errG (Formalism.SharpRestriction.Divisibility.WeightDegreeSizes, lines 408–411)
Formalism.MainNotions.product_first_zeroth_parts.WeightBudgetBound (Formalism.SharpRestriction.GlobalSchurBounds.ProductFirstZerothParts, lines 324–328)
21.41. Formalism.MainNotions.LocalQuadraticParts.adm11 (Formalism.SharpRestriction.Divisibility.LocalQuadraticParts, lines 316–321)
Formalism.MainNotions.WeightDegreeSizes.secondComplement_of_inputs (Formalism.SharpRestriction.Divisibility.WeightDegreeSizes, lines 677–681)
Formalism.MainNotions.Lnk.Mod.data_EComp5 (Formalism.SharpRestriction.GlobalSchurBounds.QuadraticLogDerivCongr, lines 594–621)
21.42. Formalism.MainNotions.DIV.HookProducts.RBCompl3.compl3_diff (Formalism.SharpRestriction.Divisibility.HookProducts.Compl3Record, lines 15–21)
Formalism.MainNotions.DIV.HookProducts.RBCompl3.compl3_record (Formalism.SharpRestriction.Divisibility.HookProducts.Compl3Record, lines 217–280)
Formalism.MainNotions.Lnk.Mod.data_HComp (Formalism.SharpRestriction.GlobalSchurBounds.QuadraticLogDerivCongr, lines 624–638)

Chapter 22: Prime sums and net gain

22.1. Formalism.LFunctionChainParameters.DeficitSplitLemmas.splitEq_valid (Formalism.SizeWindowBrackets.ChainSlopeWindows.AdmissibleSchurLemma, lines 63–68)
Formalism.L2.GlobalTheoryTheory.Loss.Ga_at (Formalism.LocalToGlobal.GlobalTheory.PowFourLtCeilThresholdPrime, lines 210–245)
Formalism.L2.GlobalTheoryTheory.Loss.ASK_at (Formalism.LocalToGlobal.GlobalTheory.PowFourLtCeilThresholdPrime, lines 386–395)
22.2. Formalism.LFunctionChainParameters.DeficitSplitLemmas.split_classes_count (Formalism.SizeWindowBrackets.ChainSlopeWindows.AdmissibleSchurLemma, lines 110–139)
22.2. Formalism.L2.GlobalTheoryTheory.Loss.clAt (Formalism.LocalToGlobal.GlobalTheory.SuppSat, lines 43–47)
22.3. Formalism.LFunctionChainParameters.DeficitSplitLemmas.loss_bound (Formalism.SizeWindowBrackets.ChainSlopeWindows.AdmissibleSchurLemma, lines 762–918)
Formalism.LFunctionChainParameters.DeficitSplitLemmas.SecondKindLoss (Formalism.SizeWindowBrackets.SecondKindBucket, lines 261–293)
22.4. Formalism.L2.GlobalTheoryTheory.Loss.pd_fill (Formalism.LocalToGlobal.GlobalTheory.PowFourLtCeilThresholdPrime, lines 1025–1036)
Formalism.L2.GlobalTheoryTheory.Loss.pd_wz (Formalism.LocalToGlobal.GlobalTheory.PowFourLtCeilThresholdPrime, lines 1038–1044)
22.5. Formalism.LFunctionChainParameters.PrimeWindowSizeBounds.first_kind_bound (Formalism.SizeWindowBrackets.ChainSlopeWindows.OneLeThresholdMulPowers, lines 955–986)
Formalism.LFunctionChainParameters.PrimeWindowSizeBounds.first_le (Formalism.SizeWindowBrackets.ChainSlopeWindows.OneLeThresholdMulPowers, lines 1036–1100)

22.6. `Formalism.LFunctionChainParameters.SmallOrderPrimeLosses.out_discriminant_bound` (`Formalism.SizeWindowBrackets.ChainSlopeWindows.DiscriminantSettingSatisfied`, lines 32–74)
`Formalism.LFunctionChainParameters.SmallOrderPrimeLosses.second_discriminant_bound` (`Formalism.SizeWindowBrackets.ChainSlopeWindows.DiscriminantSettingSatisfied`, lines 76–94)
`Formalism.LFunctionChainParameters.SmallOrderPrimeLosses.io_discriminant_bound` (`Formalism.SizeWindowBrackets.ChainSlopeWindows.DiscriminantSettingSatisfied`, lines 96–112)
22.7. `Formalism.LFunctionChainParameters.SmallOrderPrimeLosses.joint_core` (`Formalism.SizeWindowBrackets.ChainSlopeWindows.DiscriminantSettingSatisfied`, lines 178–226)
`Formalism.LFunctionChainParameters.SmallOrderPrimeLosses.joint_discriminant_bound` (`Formalism.SizeWindowBrackets.ChainSlopeWindows.DiscriminantSettingSatisfied`, lines 228–232)
`Formalism.LFunctionChainParameters.SmallOrderPrimeLosses.joint_discriminant_eventually` (`Formalism.SizeWindowBrackets.ChainSlopeWindows.DiscriminantSettingSatisfied`, lines 234–241)
22.8. `Formalism.MainNotions.flipOneThree.Pf.KP.blockSplit` (`Formalism.SharpRestriction.GrowthBounds.ColumnSpanDensity.BlockSplit`, lines 85–134)
22.8. `Formalism.MainNotions.G.LPprimeAt` (`Formalism.SharpRestriction.GrowthBounds.UnramifiedQuadraticSetting`, lines 205–214)
`Formalism.MainNotions.G.LPprime` (`Formalism.SharpRestriction.GrowthBounds.UnramifiedQuadraticSetting`, lines 216–220)
22.9. `Formalism.CountingSumsFrames.Top.localizerVanishes` (`Formalism.PolynomialDimension.DetMulEqSumAdjugate`, lines 55–103)
`Formalism.MainNotions.G.WittFrameLocaliserOrders.localiser_vanishing_of_distinct_planes` (`Formalism.SharpRestriction.GrowthBounds.WittFrameLocaliserOrders`, lines 570–685)
22.10. `Formalism.MainNotions.Pf14.goodRank_of_prim` (`Formalism.SharpRestriction.GrowthBounds.ScaleEdges.GoodRankOfPrim`, lines 29–75)
`Formalism.MainNotions.Pf14.s7` (`Formalism.SharpRestriction.GrowthBounds.ScaleEdges.GoodRankOfPrim`, lines 102–170)
22.11. `Formalism.GM.Top.windowFullAP` (`Formalism.MahlerMeasure.CoarseLevelSlackInequalities`, lines 10–10)
`Formalism.MainNotions.G.GramWindowAlt.gramWindow2_proved` (`Formalism.SharpRestriction.GrowthBounds.FockGramWindow.GramWindow2Proved`, lines 138–236)
22.12. `Formalism.MainNotions.flipOneThree.Pf.avg_abs` (`Formalism.SharpRestriction.GrowthBounds.ColumnSpanDensity.DensityNumericalFacts`, lines 80–160)
`Formalism.MainNotions.flipOneThree.Pf.mean_of` (`Formalism.SharpRestriction.GrowthBounds.ColumnSpanDensity.ColumnSpanNumericalFacts`, lines 218–300)
22.13. `Formalism.MainNotions.G.PPath` (`Formalism.SharpRestriction.GrowthBounds.ObjPol`, lines 300–304)
`Formalism.MainNotions.G.IsSt` (`Formalism.SharpRestriction.GrowthBounds.ObjPol`, lines 460–463)
`Formalism.MainNotions.SelectionLemmas.Pf.Dat.Tset` (`Formalism.SharpRestriction.GrowthBounds.SegmentDegree.GoodPathWeight`, lines 41–43)
22.14. `Formalism.InvariantFactorPaths.BranchingDimensions.pathWeightBound_of` (`Formalism.ThetaLogCoordinates.ResidueClassEstimates.VandermondeRankEstimates.AtypicalWeightBoundOf`, lines 353–677)
`Formalism.MainNotions.SelectionLemmas.Pf.T.path_count_finite` (`Formalism.SharpRestriction.GrowthBounds.SegmentDegree.WindowSmall`, lines 87–178)
`Formalism.MainNotions.SelectionLemmas.Pf.T.lemT_of` (`Formalism.SharpRestriction.GrowthBounds.SegmentDegree.AtypicalPathSmallWeight`, lines 112–203)
22.15. `Formalism.MainNotions.SelectionLemmas.Pf.residue_class_count` (`Formalism.SharpRestriction.GrowthBounds.SegmentDegree.DminusLeBeS`, lines 10–42)
`Formalism.MainNotions.SelectionLemmas.Pf.restMon_of` (`Formalism.SharpRestriction.GrowthBounds.SegmentDegree.TauDegreeOnPathBound`, lines 94–265)
22.16. `Formalism.MainNotions.G.GLWeightTheory.unit_function_mahler_decay_of` (`Formalism.SharpRestriction.GrowthBounds.RootWeights.MemBoundedDegreeExponentsIff`, lines 314–351)
`Formalism.MainNotions.Proposition5Numerics.digQ` (`Formalism.SharpRestriction.GrowthBounds.DiagonalSupport.BinomCombFOnFinset`, lines 22–104)
22.17. `Formalism.MainNotions.CovariantDegrees.CanonicalTranslateJetIdentity` (`Formalism.SharpRestriction.GrowthBounds.StatementOfDegreeAndBound`, lines 417–435)
`Formalism.MainNotions.CovariantDegrees.CoefficientDependence` (`Formalism.SharpRestriction.GrowthBounds.StatementOfDegreeAndBound`, lines 407–415)
`Formalism.MainNotions.G.WittFrameLocaliserOrders.subst_twisted_row_even` (`Formalism.SharpRestriction.GrowthBounds.WittFrameLocaliserOrders`, lines 483–506)
`Formalism.MainNotions.G.WittFrameLocaliserOrders.residue_integral_primal_row` (`Formalism.SharpRestriction.GrowthBounds.WittFrameLocaliserOrders`, lines 508–531)
`Formalism.MainNotions.G.WittFrameLocaliserOrders.residue_zero_mem_order_one` (`Formalism.SharpRestriction.GrowthBounds.WittFrameLocaliserOrders`, lines 446–456)
22.18. `Formalism.MainNotions.G.WittFrameLocaliserOrders.localiser_vanishing_of_distinct_planes` (`Formalism.SharpRestriction.GrowthBounds.WittFrameLocaliserOrders`, lines 570–685)
`Formalism.MainNotions.G.second_theorem_closed_everywhere` (`Formalism.SharpRestriction.GrowthBounds.PlaneCountTheoremWithConstants`, lines 1121–1135)
`Formalism.MainNotions.G.second_theorem_absolute_constant` (`Formalism.SharpRestriction.GrowthBounds.PlaneCountTheoremWithConstants`, lines 1137–1142)
22.19. `Formalism.MainNotions.PairMembershipBlocks.restMEM_of_last` (`Formalism.SharpRestriction.GrowthBounds.PlaneCoefficients`, lines 34–40)

`Formalism.MainNotions.Proposition5Numerics.pick_core` (`Formalism.SharpRestriction.GrowthBounds.DiagonalSupport.PickCore`, lines 23–165)
`Formalism.MainNotions.PlaneGram.stepB_of` (`Formalism.SharpRestriction.GrowthBounds.MonomialOrderSegments.CardLinkIdxMulLe`, lines 127–233)
22.20. `Formalism.MainNotions.PlaneGram.segStep_of` (`Formalism.SharpRestriction.GrowthBounds.MonomialOrderSegments.RenamePullLCoefOut`, lines 150–197)
`Formalism.MainNotions.PlaneGram.stepC_of` (`Formalism.SharpRestriction.GrowthBounds.MonomialOrderSegments.ExistsOfMemSupportLin`, lines 409–426)
22.21. `Formalism.MainNotions.PlaneGram.stepE_of` (`Formalism.SharpRestriction.GrowthBounds.MonomialOrderSegments.SumOnLetterApplyInl`, lines 310–334)
`Formalism.MainNotions.PlaneGram.TwistData.value_monomial` (`Formalism.SharpRestriction.GrowthBounds.MonomialOrderSegments.SubstitutedExponentApplyLabel`, lines 237–326)
`Formalism.MainNotions.PlaneGram.omega_value` (`Formalism.SharpRestriction.GrowthBounds.MonomialOrderSegments.OmegaValue`, lines 12–106)
22.22. `Formalism.GM.M.MMwZ` (`Formalism.MahlerMeasure.CsSaturated`, lines 50–52)
`Formalism.GM.M.FilE` (`Formalism.MahlerMeasure.CsSaturated`, lines 54–57)
`Formalism.GM.M.grLat` (`Formalism.MahlerMeasure.CsSaturated`, lines 59–62)
22.23. `Formalism.InvariantFactorPaths.PW.Top.typeInt` (`Formalism.ThetaLogCoordinates.TraceIsometryWeights.FinalP`, lines 22–22)
`Formalism.InvariantFactorPaths.PW.Top.weylInt` (`Formalism.ThetaLogCoordinates.TraceIsometryWeights.DischargeHeightBound`, lines 8–8)
22.24. `Formalism.InvariantFactorPaths.L4.restriction_into_holds` (`Formalism.ThetaLogCoordinates.FockResolutionOnto.FockMonomialProofs.ResInto`, lines 430–468)
`Formalism.InvariantFactorPaths.L4.restriction_surjective` (`Formalism.ThetaLogCoordinates.FockResolutionOnto.FockMonomialProofs.DerivativeKeepsSpaceHolds`, lines 208–298)
`Formalism.InvariantFactorPaths.L4.restricted_saturation_holds` (`Formalism.ThetaLogCoordinates.FockResolutionOnto.FockMonomialProofs.RestrictedSaturationHolds`, lines 91–131)
22.25. `Formalism.InvariantFactorPaths.L4.grSpan_eq` (`Formalism.ThetaLogCoordinates.FockResolutionOnto.FockMonomialProofs.GramBoundOfRestrictionOfProj`, lines 200–235)
`Formalism.InvariantFactorPaths.L4.satAtL` (`Formalism.ThetaLogCoordinates.FockResolutionOnto.FockMonomialProofs.GramBoundUnconditionalOfSat`, lines 90–104)
`Formalism.InvariantFactorPaths.L4.graded_bound_pos_of_proj` (`Formalism.ThetaLogCoordinates.FockResolutionOnto.FockMonomialProofs.GramBoundPosOfProjSat`, lines 51–68)
22.26. `Formalism.InvariantFactorPaths.L4.graded_piece_bound_holds` (`Formalism.ThetaLogCoordinates.FockResolutionOnto.FockMonomialProofs.PartnerDeterminantHolds`, lines 114–198)
`Formalism.InvariantFactorPaths.L4.partner_determinant_holds` (`Formalism.ThetaLogCoordinates.FockResolutionOnto.FockMonomialProofs.PartnerDeterminantHolds`, lines 10–77)
22.27. `Formalism.GM.M.tieGam` (`Formalism.MahlerMeasure.HasOrder`, lines 14–19)
`Formalism.GM.gamma_degree_bound` (`Formalism.MahlerMeasure.MahlerMeasureProofs.GamDeg`, lines 1153–1171)
`Formalism.GM.gamma_leading_term` (`Formalism.MahlerMeasure.MahlerMeasureProofs.GamLead`, lines 1852–1947)
`Formalism.GM.MeanGammaLemmas.lead_part` (`Formalism.MahlerMeasure.MahlerMeasureProofs.MeanGam`, lines 400–539)
22.28. `Formalism.GM.DimensionMonotonicity.highest_weight_dim_monotone` (`Formalism.MahlerMeasure.MahlerMeasureProofs.WeightSpaceDimensionMonotone`, lines 2606–2856)
22.29. `Formalism.MainNotions.flipOneThree.Pf.contraction_of_pairing_lemmas` (`Formalism.SharpRestriction.GrowthBounds.HermitianTensor.ContractionOfPairingLemmas`, lines 9–173)
`Formalism.MainNotions.flipOneThree.Pf.cloudOne_of` (`Formalism.SharpRestriction.GrowthBounds.HermitianTensor.CloudA`, lines 9–96)
`Formalism.MainNotions.flipOneThree.Pf.iso_one` (`Formalism.SharpRestriction.GrowthBounds.HermitianTensor.CloudIso`, lines 33–59)
22.30. `Formalism.CountingSumsFrames.TO.tie_core` (`Formalism.PolynomialDimension.MultinomialDepth.TieOps`, lines 3104–3286)
`Formalism.InvariantFactorPaths.MC.partnerMX_of_parts` (`Formalism.ThetaLogCoordinates.PartnerMeanCoordinates.PartnerMeanProofs.GammaZeroIntegral`, lines 253–376)
`Formalism.InvariantFactorPaths.MC.pointMX_of_partner` (`Formalism.ThetaLogCoordinates.PartnerMeanCoordinates.PartnerMeanProofs.GammaZeroIntegral`, lines 378–394)
22.31. `Formalism.MainNotions.ColumnSystems.Pf.loc4One` (`Formalism.SharpRestriction.GrowthBounds.CoarseMultiplicity.KeyOne`, lines 101–112)
`Formalism.MainNotions.ColumnSystems.Pf.loc4Res_of` (`Formalism.SharpRestriction.GrowthBounds.CoarseMultiplicity.Loc4ResOfLine`, lines 33–49)
`Formalism.MainNotions.ColumnSystems.Pf.loc4Ord_of` (`Formalism.SharpRestriction.GrowthBounds.CoarseMultiplicity.LiftSubstStart`, lines 43–50)
22.32. `Formalism.MainNotions.ColumnSystems.Pf.SegHyp.seg_ord` (`Formalism.SharpRestriction.GrowthBounds.BinomialRatio.TopDegreeLeTwoB`, lines 436–514)
`Formalism.MainNotions.ColumnSystems.Pf.ord_all` (`Formalism.SharpRestriction.GrowthBounds.BinomialRatio.TopDegreeLeTwoB`, lines 570–628)
22.33. `Formalism.InvariantFactorPaths.MC.gamma_integral_holds` (`Formalism.ThetaLogCoordinates.PartnerMeanCoordinates.PartnerMeanProofs.GamIntC`, lines 302–364)
`Formalism.GM.gamma_degree_bound` (`Formalism.MahlerMeasure.MahlerMeasureProofs.GamDeg`, lines 1153–1171)

22.34. `Formalism.MainNotions.ColumnSystems.Pf.cth_SSt` (`Formalism.SharpRestriction.GrowthBounds.CoarseMultiplicity.OtherEFree`, lines 172–185)
`Formalism.MainNotions.ColumnSystems.Pf.op_SSt` (`Formalism.SharpRestriction.GrowthBounds.CoarseMultiplicity.GTheta`, lines 27–55)
`Formalism.MainNotions.ColumnSystems.Pf.sat_block` (`Formalism.SharpRestriction.GrowthBounds.CoarseMultiplicity.SatBlock`, lines 102–154)
22.35. `GlobalTheory.EX.RangeOne.rangeOne_abs` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.TensorOfPairLift`, lines 637–640)
`Formalism.MainNotions.flipOneThree.Conclusion2` (`Formalism.SharpRestriction.GrowthBounds.ProjectionLemmaBaseLevel`, lines 78–91)
22.36. `Formalism.MainNotions.AuxBlock8.Pf.St.starAt_D` (`Formalism.SharpRestriction.GrowthBounds.LogarithmicSums.PartnerCountOfCentral`, lines 597–688)
`Formalism.MainNotions.AuxBlock8.Pf.St.starAt_E` (`Formalism.SharpRestriction.GrowthBounds.LogarithmicSums.PartnerCountOfCentral`, lines 696–840)
`Formalism.MainNotions.AuxBlock8.Pf.St.star_fam` (`Formalism.SharpRestriction.GrowthBounds.LogarithmicSums.StarFam`, lines 11–36)
22.37. `GlobalTheory.Top.closed_linkG_of_duty` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LinkG`, lines 13–24)
`Formalism.MainNotions.LinkG_at` (`Formalism.SharpRestriction.GlobalTheoremInputs`, lines 122–136)
22.37. `Formalism.MainNotions.G.Sv` (`Formalism.SharpRestriction.GrowthBounds.IsResidueClassZero`, lines 336–337)
`Formalism.MainNotions.G.Joints.linear_programming_input_of_hypotheses` (`Formalism.SharpRestriction.GrowthBounds.Joints`, lines 2040–2127)
`Formalism.MainNotions.AuxBlock8.Pf.gain_bound_at_of_star` (`Formalism.SharpRestriction.GrowthBounds.LogarithmicSums.NotSingularOfCentral`, lines 619–728)
`Formalism.L2.GlobalTheoryTheory.Loss.loss_bound_at` (`Formalism.LocalToGlobal.GlobalTheory.PowFourLtCeilThresholdPrime`, lines 766–816)
22.38. `Formalism.MainNotions.G.first_theorem_closed_form` (`Formalism.SharpRestriction.GrowthBounds.PlaneCountTheoremWithConstants`, lines 1761–1781)
`Formalism.MainNotions.G.signs_closed` (`Formalism.SharpRestriction.GrowthBounds.PlaneCountTheoremWithConstants`, lines 1783–1788)
`Formalism.MainNotions.G.TopChoiceBounds.family_log_bound_closed` (`Formalism.SharpRestriction.GrowthBounds.TopChoiceBounds`, lines 4428–4432)
22.39. `Formalism.MainNotions.G.Joints.hypIIAt_ev` (`Formalism.SharpRestriction.GrowthBounds.SizesAtRecord`, lines 567–626)
`UpperTheory.even_addOrderOf_hP_of_pow_le` (`Development.AutomorphicArithmetic.OrbitLabellings.EvenAddOrderOfHPOfPowLe`, lines 8–28)
22.40. `Formalism.MainNotions.G.Joints.type_two_hypothesis_of_counts` (`Formalism.SharpRestriction.GrowthBounds.Joints`, lines 426–582)
22.41. `GlobalTheory.EX.RangeOne.rangeOne_abs` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.TensorOfPairLift`, lines 637–640)
`Formalism.MainNotions.G.Joints.type_one_hypothesis_of_segments` (`Formalism.SharpRestriction.GrowthBounds.Joints`, lines 1283–1325)
`Formalism.MainNotions.G.first_theorem_closed_form` (`Formalism.SharpRestriction.GrowthBounds.PlaneCountTheoremWithConstants`, lines 1761–1781)
22.42. `Formalism.L2.GlobalTheoryTheory.Loss7.losses_of_small_share_weight` (`Formalism.LocalToGlobal.GlobalTheory.LossRequirementsOfSmallShares`, lines 86–161)
22.43. `Formalism.MainNotions.Lnk.image_plMap` (`Formalism.SharpRestriction.GlobalSchurBounds.QuadraticLogDerivCongr`, lines 351–359)
`Formalism.MainNotions.Lnk.Mod.w_off` (`Formalism.SharpRestriction.GlobalSchurBounds.QuadraticLogDerivCongr`, lines 416–421)
`Formalism.MainNotions.Lnk.Mod.w_plMap` (`Formalism.SharpRestriction.GlobalSchurBounds.QuadraticLogDerivCongr`, lines 423–425)
`Formalism.MainNotions.Lnk.numN_at` (`Formalism.SharpRestriction.GlobalSchurBounds.QuadraticLogDerivCongr`, lines 1261–1312)
`Formalism.MainNotions.Lnk.nodes_at` (`Formalism.SharpRestriction.GlobalSchurBounds.QuadraticLogDerivCongr`, lines 1198–1248)
`Formalism.MainNotions.Lnk.nodesF` (`Formalism.SharpRestriction.GlobalSchurBounds.QuadraticLogDerivCongr`, lines 1314–1321)
22.43. `Formalism.MainNotions.FamCore.bTop_eq` (`Formalism.SharpRestriction.FamCore`, lines 124–125)
22.44. `Formalism.MainNotions.Lnk.lossF_of_7` (`Formalism.SharpRestriction.GlobalSchurBounds.LossesSmallShares`, lines 308–313)
`Formalism.MainNotions.ChosenWS` (`Formalism.SharpRestriction.GlobalSchurBounds.WeightVectorSmallSharesSeven`, lines 51–59)
22.44. `Formalism.MainNotions.FamCore.coverS_carry` (`Formalism.SharpRestriction.FamCore`, lines 187–205)
22.45. `GlobalTheory.EX.G4.wClosed` (`Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.WindowShareRequirementClosedFormProof`, lines 40–42)
22.45. `Formalism.MainNotions.AuxBlock8.Pf.sum_Sig_le` (`Formalism.SharpRestriction.GrowthBounds.LogarithmicSums.SlopeEstimateOfLemmaBSlopeLine`, lines 329–359)

Formalism.MainNotions.AuxBlock8.Pf.meanEx_of (Formalism.SharpRestriction.GrowthBounds.LogarithmicSums.SlopeEstimateOfLemmaBSlopeLine, lines 435–481)
Formalism.MainNotions.AuxBlock8.Pf.weight_condition_of_mean_excess (Formalism.SharpRestriction.GrowthBounds.LogarithmicSums.SlopeEstimateOfLemmaBSlopeLine, lines 483–501)
22.46. Formalism.MainNotions.chosenWS_of_wall (Formalism.SharpRestriction.GlobalSchurBounds.WeightVectorSmallSharesEight, lines 22–31)

Chapter 23: A slope bound for families of maps

23.1. Formalism.HN.slope_inequality_mean (Formalism.HarderNarasimhan.HarderNarasimhanProofs.SlopeIneqMean, lines 249–251)
Formalism.HN.SlopeIneqMean (Formalism.HarderNarasimhan.StatementPl, lines 148–159)
23.2. Formalism.HN.frameB_count_lt (Formalism.HarderNarasimhan.HarderNarasimhanProofs.FrameHermSharpB, lines 62–75)
Formalism.HN.frameB_sum_sl (Formalism.HarderNarasimhan.HarderNarasimhanProofs.FrameHermSharpB, lines 115–124)
Formalism.HN.frameB_hnStep (Formalism.HarderNarasimhan.HarderNarasimhanProofs.FrameHermSharpB, lines 77–78)
Formalism.HN.polygon_extreme_slopes (Formalism.HarderNarasimhan.HarderNarasimhanProofs.HNExtreme, lines 114–115)
23.3. Formalism.HN.thmF_share (Formalism.HarderNarasimhan.HarderNarasimhanProofs.TheoremFSharp, lines 11–33)
23.4. Formalism.HN.frame_linear_algebra_first (Formalism.HarderNarasimhan.HarderNarasimhanProofs.FrameLinearAlgebraFirst, lines 77–221)
23.5. Formalism.HN.thmF_place (Formalism.HarderNarasimhan.HarderNarasimhanProofs.TheoremFSharp, lines 35–159)
23.6. Formalism.CorankShareLayerCake.FR.layer_cake_formula (Formalism.CorankShare.CorankShareProofs.LayerCakeFormula, lines 57–97)
23.7. Formalism.HN.theoremFSharp_of (Formalism.HarderNarasimhan.HarderNarasimhanProofs.TheoremFSharp, lines 168–386)
Formalism.HN.TheoremFSharp (Formalism.HarderNarasimhan.StatementFrameF, lines 64–99)
Formalism.HN.frame_theorem_sharp (Formalism.HarderNarasimhan.HarderNarasimhanProofs.TheoremFSharp, lines 388–388)
Formalism.HN.theoremF_of_sharp (Formalism.HarderNarasimhan.HarderNarasimhanProofs.TheoremFSharp, lines 390–398)
Formalism.HN.frame_theorem_log_ideal (Formalism.HarderNarasimhan.HarderNarasimhanProofs.TheoremFSharp, lines 400–400)
23.8. Formalism.MainNotions.Lnk.frame_at_of (Formalism.SharpRestriction.GlobalSchurBounds.TopDensityNGeSixteen, lines 538–600)
Formalism.LFunctionChainParameters.MatrixLatticeEstimates.joint_a_ramified_holds (Formalism.SizeWindowBrackets.ChainSlopeWindows.JointDModelAppliesUnitLattice, lines 254–258)
Formalism.LFunctionChainParameters.MatrixLatticeEstimates.ramification_linear_case_holds (Formalism.SizeWindowBrackets.ChainSlopeWindows.JointDModelAppliesUnitLattice, lines 164–185)
Formalism.MainNotions.Lnk.FrameReal_at (Formalism.SharpRestriction.GlobalSchurBounds.TopDensityNGeSixteen, lines 519–531)
Formalism.MainNotions.GamL (Formalism.SharpRestriction.GlobalSchurBounds.Chosen, lines 14–14)
23.9. Formalism.L2.SG.RowsPos.kernel_diagonal_positive (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RowsPos, lines 202–208)
Formalism.MainNotions.S.linkS_concl (Formalism.SharpRestriction.CuspSpanNormBounds.StandingHypothesesHoldFamily, lines 186–204)
Formalism.MainNotions.Lnk.matK_at_of (Formalism.SharpRestriction.GlobalSchurBounds.TopDensityNGeSixteen, lines 1097–1107)
Formalism.MainNotions.product_first_zeroth_parts.rank_nat (Formalism.SharpRestriction.GlobalSchurBounds.FiniteModelOfAdmissibleTransports, lines 782–840)
Formalism.L2.SG.KappaOffDiagonal.cosetsC_holds (Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.KloostermanCosetSumLe, lines 186–212)
GlobalTheory.rowMainEqP_of_siegelWeil (Development.AbelianVarietyTheory.Proofs.FlagMeasures.RowMainEqPOfSiegelWeil, lines 10–16)
23.10. Formalism.MainNotions.Lnk.frameF_of (Formalism.SharpRestriction.GlobalSchurBounds.TopDensityNGeSixteen, lines 606–646)
Formalism.MainNotions.Lnk.Mod.data_HHBound (Formalism.SharpRestriction.GlobalSchurBounds.QuadraticLogDerivCongr, lines 714–728)
Formalism.MainNotions.Lnk.lat_at (Formalism.SharpRestriction.GlobalSchurBounds.QuadraticLogDerivCongr, lines 1323–1333)
Formalism.MainNotions.Lnk.Mod.data (Formalism.SharpRestriction.GlobalSchurBounds.QuadraticLogDerivCongr, lines 464–504)
23.11. Formalism.MainNotions.Ext.Lw.mu0Tie_of_sym (Formalism.SharpRestriction.GlobalSchurBounds.GlobalRankLatticeProofs.UnitTraceSumBounds.RankConditionsOfLayerTwo, lines 284–289)
Formalism.MainNotions.StandardLattice.HarishChandraReduction.chosenProviso_min (Formalism.SharpRestriction.GlobalSchurBounds.TheoremThreeFamilyOfFormalDegreeInput, lines 76–79)
23.12. UpperTheory.Top.designS (Development.AutomorphicArithmetic.CharacterBoxes.DesignS, lines 21–98)
Formalism.MainNotions.Lnk.matRealF_of (Formalism.SharpRestriction.GlobalSchurBounds.TopDensityNGeSixteen, lines 750–805)

Chapter 24: The pointwise bound and its eight error terms

Chapter introduction. Formalism.RankGapBounds.WeightedMeanData.Data.s (Formalism.RankGapBounds.MeanBoundSecondForm, lines 92–92)
Formalism.MainNotions.Lnk.Mod.data (Formalism.SharpRestriction.GlobalSchurBounds.QuadraticLogDerivCongr, lines 464–504)
24.3. Formalism.RankGapBounds.WeightedMeanData.Data.SDiv (Formalism.RankGapBounds.RankGapProofs.WeightedMeanData, lines 129–129)
Formalism.RankGapBounds.WeightedMeanData.Data.LoP (Formalism.RankGapBounds.MeanBoundSecondForm, lines 105–105)
Formalism.LFunctionChainParameters.WeightedSlopeBounds.Lo5 (Formalism.SizeWindowBrackets.QuotientSlopeBoundsWith, lines 15–17)
H11. Formalism.RankGapBounds.WeightedMeanData.Data.HidaSetting (Formalism.RankGapBounds.MeanBoundSecondForm, lines 111–116)
Formalism.LFunctionChainParameters.WeightedSlopeBounds.PointwiseBoundVariant (Formalism.SizeWindowBrackets.QuotientSlopeBoundsWith, lines 72–75)
24.1. Formalism.MainNotions.Lnk.linkCore (Formalism.SharpRestriction.GlobalSchurBounds.LossesSmallShares, lines 38–296)
Formalism.MainNotions.linkGLOBAL7_items (Formalism.SharpRestriction.GlobalSchurBounds.LinkMin, lines 693–721)
Formalism.MainNotions.ChainTied (Formalism.SharpRestriction.GlobalSchurBounds.TieGlobal, lines 97–114)
24.2. Formalism.MainNotions.LnkBr.prim_unique (Formalism.SharpRestriction.GlobalSchurBounds.TopDensityNGeSixteen, lines 133–157)
Formalism.MainNotions.Lnk.char_at (Formalism.SharpRestriction.GlobalSchurBounds.TopDensityNGeSixteen, lines 476–504)
Formalism.MainNotions.Lnk.charF (Formalism.SharpRestriction.GlobalSchurBounds.TopDensityNGeSixteen, lines 506–507)
24.3. Formalism.LFunctionChainParameters.WeightedSlopeBounds.Lam5 (Formalism.SizeWindowBrackets.ChainSlopeWindows.LevelOfCongruence, lines 17–19)
Formalism.RankGapBounds.WeightedMeanData.Data.Main (Formalism.RankGapBounds.MeanBoundSecondForm, lines 107–109)
24.4. Formalism.LFunctionChainParameters.WeightedSlopeBounds.key_identity5 (Formalism.SizeWindowBrackets.ChainSlopeWindows.LevelOfCongruence, lines 105–111)
24.4. Formalism.RankGapBounds.WeightedMeanData.Data.key_identity (Formalism.RankGapBounds.RankGapProofs.WeightedMeanData, lines 170–216)
24.5. Formalism.LFunctionChainParameters.WeightedSlopeBounds.s_lower5 (Formalism.SizeWindowBrackets.ChainSlopeWindows.LevelOfCongruence, lines 50–66)
24.6. Formalism.LFunctionChainParameters.WeightedSlopeBounds.Vvar_bound5 (Formalism.SizeWindowBrackets.ChainSlopeWindows.LevelOfCongruence, lines 68–79)
24.7. Formalism.LFunctionChainParameters.WeightedSlopeBounds.u_function_le_mean (Formalism.SizeWindowBrackets.ChainSlopeWindows.LevelOfCongruence, lines 81–103)
Formalism.RankGapBounds.WeightedMeanData.Data.uF_nonneg (Formalism.RankGapBounds.RankGapProofs.WeightedMeanData, lines 159–168)
24.8. Formalism.RankGapBounds.WeightedMeanData.phi_bound (Formalism.RankGapBounds.RankGapProofs.WeightedMeanData, lines 81–117)
24.9. Formalism.LFunctionChainParameters.WeightedSlopeBounds.Ga_bound5 (Formalism.SizeWindowBrackets.ChainSlopeWindows.LevelOfCongruence, lines 113–160)
24.10. Formalism.LFunctionChainParameters.WeightedSlopeBounds.weighted_mean_lower_bound_le (Formalism.SizeWindowBrackets.ChainSlopeWindows.LevelOfCongruence, lines 162–171)
Formalism.LFunctionChainParameters.WeightedSlopeBounds.each_unit_bound_statement (Formalism.SizeWindowBrackets.ChainSlopeWindows.LevelOfCongruence, lines 175–197)
24.11. Formalism.LFunctionChainParameters.WeightedSlopeBounds.ErrX5at (Formalism.SizeWindowBrackets.QuotientSlopeBoundsWith, lines 45–54)
Formalism.LFunctionChainParameters.WeightedSlopeBounds.UBar5 (Formalism.SizeWindowBrackets.QuotientSlopeBoundsWith, lines 41–43)
24.12. Formalism.LFunctionChainParameters.WeightedSlopeBounds.each_unit_bound_statement (Formalism.SizeWindowBrackets.ChainSlopeWindows.LevelOfCongruence, lines 175–197)
Formalism.LFunctionChainParameters.WeightedSlopeBounds.PointwiseBoundVariant (Formalism.SizeWindowBrackets.QuotientSlopeBoundsWith, lines 72–75)

Chapter 25: Four terms and the square-root bound

25.1. Formalism.LFunctionChainParameters.LErr.core_ineq (Formalism.SizeWindowBrackets.ChainSlopeWindows.SubThreeMulLogTwoLe, lines 13–207)
25.2. Formalism.LFunctionChainParameters.LChain.errX5at_le (Formalism.SizeWindowBrackets.ChainSlopeWindows.PChain, lines 613–672)
25.2. Formalism.MainNotions.Lnk.Mod.sum_places (Formalism.SharpRestriction.GlobalSchurBounds.QuadraticLogDerivCongr, lines 458–462)
Formalism.MainNotions.Lnk.Mod.ident (Formalism.SharpRestriction.GlobalSchurBounds.QuadraticLogDerivCongr, lines 792–797)

25.3. Deduction.ch_top_main_eq (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ChTop, lines 11–35)
25.4. Deduction.ch_top (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ChTop, lines 37–72)
25.5. Deduction.ch_balance (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ChBalance, lines 200–260)
25.6. Deduction.CharacterBalanceMoves.core (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ChCore, lines 283–338)
Deduction.CharacterBalanceMoves.th_le (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ChCore, lines 75–152)
25.7. Formalism.MainNotions.Lnk.winA (Formalism.SharpRestriction.GlobalSchurBounds.QuadraticLogDerivCongr, lines 1414–1415)
Formalism.MainNotions.Lnk.winL (Formalism.SharpRestriction.GlobalSchurBounds.QuadraticLogDerivCongr, lines 1417–1418)
25.8. Formalism.LFunctionChainParameters.ThickSetMeanBounds.SZ.T1 (Formalism.SizeWindowBrackets.ChainSlopeWindows.LipschitzWeightAndMinimum, lines 31–39)
25.9. Formalism.LFunctionChainParameters.ThickSetMeanBounds.SZ.T2 (Formalism.SizeWindowBrackets.ChainSlopeWindows.LipschitzWeightAndMinimum, lines 41–51)
25.10. Formalism.LFunctionChainParameters.ThickSetMeanBounds.SZ.T3 (Formalism.SizeWindowBrackets.ChainSlopeWindows.LipschitzWeightAndMinimum, lines 53–75)
25.11. Formalism.LFunctionChainParameters.ThickSetMeanBounds.SZ.T4 (Formalism.SizeWindowBrackets.ChainSlopeWindows.LipschitzWeightAndMinimum, lines 77–113)
25.12. Formalism.LFunctionChainParameters.ThickSetMeanBounds.proposition_size_bound (Formalism.SizeWindowBrackets.ChainSlopeWindows.LipschitzWeightAndMinimum, lines 507–592)
25.13. Deduction.ch_core (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ChCore, lines 343–442)
Deduction.CharacterBalanceMoves.one_char (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.ChCore, lines 202–281)
25.13. UpperTheory.Comparison₂.core_gm (Development.AutomorphicArithmetic.PrimitiveCosets.CoreGm, lines 12–74)

Chapter 26: The window bound and its inputs

26.1. Formalism.GaloisCohomology.Sha (Formalism.GaloisCohomology.Statement, lines 75–76)
Formalism.GaloisCohomology.shaLen (Formalism.GaloisCohomology.Statement, lines 78–78)
UpperTheory.Top.VVle (Development.AutomorphicArithmetic.CharacterBoxes.ExistsPageN, lines 12–17)
26.2. UpperTheory.Top.IsChiField (Development.AutomorphicArithmetic.CharacterBoxes.ExistsPageN, lines 19–25)
UpperTheory.Top.WinPair (Development.AutomorphicArithmetic.CharacterBoxes.ExistsPageN, lines 27–37)
UpperTheory.Top.PageN (Development.AutomorphicArithmetic.CharacterBoxes.ExistsPageN, lines 39–39)
26.3. UpperTheory.Top.VAt (Development.AutomorphicArithmetic.CharacterBoxes.ExistsPageN, lines 41–42)
UpperTheory.Top.VWinAt (Development.AutomorphicArithmetic.CharacterBoxes.ExistsPageN, lines 44–47)
UpperTheory.Top.VWin (Development.AutomorphicArithmetic.CharacterBoxes.ExistsPageN, lines 49–49)
26.4. UpperTheory.One.vwin_of_row (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.VwinOfRow, lines 4–5)
26.4. UpperTheory.LiteratureTheorems.OpenImageCondition (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TemperedProductOfSquareIntegrable, lines 207–208)
26.5. UpperTheory.ConvolutionTheory.G2.uSide (Development.AutomorphicArithmetic.Foundations.Convolution.WeightRankUnits.USide, lines 9–48)
UpperTheory.ConvolutionTheory.G2.g2s (Development.AutomorphicArithmetic.Foundations.Convolution.WeightRankUnits.NeZeroOfFinprod, lines 40–72)
26.5. UpperTheory.ConvolutionTheory.G2.matchSplit (Development.AutomorphicArithmetic.Foundations.Convolution.WeightRankUnits.MatchSplit, lines 12–21)
26.5. GlobalTheory.UnitLemma.InvAlg.kapSum (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.UnitLemmaGroup, lines 145–146)
GlobalTheory.UnitLemma.InvAlg.stSum (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.UnitLemmaGroup, lines 143–143)
26.6. GlobalTheory.CountId.countIdentity_of_unitMatch (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.RelativeFundamentalLemmaCountIdentityOfUnitMatch, lines 328–424)
26.7. GlobalTheory.StableLatCount.card_stable_lattices (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CardStableLattices, lines 16–33)
GlobalTheory.UnitLemma.FibSum.fib_card (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SumStableLatticesProductEqQPowMul, lines 15–52)
26.8. GlobalTheory.UnitLemma.unitLemma_lie_and_group (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.UnitLemmaLieAlgebraUnitaryGroupOfJacquetRallis, lines 14–17)
26.9. GlobalTheory.UnitLemma.unitLemma_identity_unconditional (Development.AbelianVarietyTheory.Proofs.DivisorProofs.UnitLemmaUnramifiedProductUnitaryGroups, lines 10–13)
26.10. PositivityTheory.Rg.Bc.ArchbcB.B.archimedean_base_change_first (Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.IwahoriFactorisation.MultiplicityOneBase.ArchimedeanBaseChangeFirst, lines 29–61)

26.11. `PositivityTheory.Rg.BcB.E1.growthJ_holds` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.ThetaCellCovers.GrowthOut`, lines 6–6)
26.11. `PositivityTheory.Rg.BcB.E1.wtCount_of` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.ThetaCellCovers.CutCountOf`, lines 56–94)
`PositivityTheory.Rg.BcB.E1.cutCount_of` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.ThetaCellCovers.CutCountOf`, lines 96–115)
`PositivityTheory.Rg.BcB.E1.growthJ_of` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.ThetaCellCovers.KernelDiagonalIntegrableOf`, lines 78–97)
26.12. `PositivityTheory.RgBcC.lefschetz_holds` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.ArchimedeanGelfandKernels.LeftEquivariantFunctions.LefschetzHolds`, lines 6–6)
26.13. `PositivityTheory.Rg.BcB.G3.g3` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.ThetaNormCommutation.FactorFields.Count1`, lines 35–36)
`PositivityTheory.Rg.BcB.G8.global_character_holds` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.ThetaNormCommutation.LocalComparisonSplitting.GlobalCharacterHolds`, lines 6–6)
26.14. `PositivityTheory.Rg.BcB.F.RunFrame.g` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.FrameBoundSpaces.InstIPSFB`, lines 56–59)
`PositivityTheory.Rg.BcB.F.FrameOk.comm_gen` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.AdelicHeckeBlocks.PStruct`, lines 134–142)
`PositivityTheory.Rg.BcB.F.FrameOk.comm_beta` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.AdelicHeckeBlocks.PStruct`, lines 144–152)
26.15. `PositivityTheory.Rg.BcB.F.Tail.sat_of_eig` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.FrameReading.RExpReadOf`, lines 71–109)
`PositivityTheory.Rg.BcB.N3.Final.n3` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.SphericalEmbeddings.Imports`, lines 12–14)
26.16. `PositivityTheory.Rg.BcB.SpectralAccumulation.FiniteDiscreteTypes.finDisc` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.InfinitesimalCharacterLimits.IsometrySymmetry.IsoVSymm`, lines 29–126)
26.17. `PositivityTheory.Rg.Bc.ArchbcB.B.M4.m4` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.IwahoriFactorisation.TwistedConjugationNorms.Development`, lines 1166–1176)
`GlobalTheory.CompanionTheory.MatchPow.pow_claim` (`Development.AutomorphicSpectrum.Proofs.BlockDecomposition.MatchingTransferOf`, lines 456–492)
26.18. `UpperTheory.BaseChangeIdentities.FrameWordIdentity.frameBound` (`Development.AutomorphicArithmetic.Foundations.BaseChangeIdentities.FrameMatching.HeckeWordBounds.FrameBound`, lines 42–43)
`PositivityTheory.Rg.BcB.F.B3.PTop.hW_frame` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.WordFrameBounds.FrameBoundOf`, lines 171–234)
26.19. `PositivityTheory.Rg.BcB.F.peak_trace_bound_holds` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.PeakTraceBoundHolds`, lines 368–412)
26.20. `PositivityTheory.Rg.BcB.F.Rd.B.realise_commuting_family` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.FrameReading.LimitbAlg`, lines 52–115)
26.21. `PositivityTheory.Rg.BcB.F.Rd.BlockOut` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.UniformizerValuations.FrameReading.ReadLimit`, lines 36–65)
26.21. `PositivityTheory.Rg.BcB.SpectralAccumulation.eFinal_of` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.InfinitesimalCharacterLimits.EFinalOf`, lines 11–13)
26.22. `PositivityTheory.Rg.BcN.IsBaseChangeN` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.BaseChangeIsobaricNOf`, lines 21–37)
`PositivityTheory.Rg.BcN.BaseChangeIsobaricN` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.BaseChangeIsobaricNOf`, lines 39–46)
`UpperTheory.BaseChangeIdentities.baseChangeN` (`Development.AutomorphicArithmetic.Foundations.BaseChangeIdentities.DefiniteNHolds`, lines 5–6)
26.22. `PositivityTheory.Rg.BcB.One.fieldN_of` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.FieldNOf`, lines 52–81)
`PositivityTheory.Rg.BcB.F.Rig.rCuspCsd` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.SatakeExponents.SatakeRigidity.LRCuspCsd`, lines 103–195)
26.23. `PositivityTheory.Rg.BcN.IsBaseChangeN` (`Development.WeilBound.RepresentationTheoryGL.ArchimedeanBaseChange.ArchimedeanBaseChangeProof.BaseChangeIsobaricNOf`, lines 21–37)
`GlobalTheory.UnitaryBC.infOfWeight` (`Development.AbelianVarietyTheory.PublishedTheorems.GaloisCohomology₂.BaseChangeDefiniteUnitary`, lines 142–143)
`PositivityTheory.Rg.Rs.AscEmb` (`Development.WeilBound.RepresentationTheoryGL.ResidualSpectrum.SliceTensors.IsSliceMap`, lines 53–55)
`UpperTheory.GalStar.Row0b` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.TypeConditions.WeightsConjBis`, lines 69–73)
`UpperTheory.Gal.Int.galoisLocalGlobalStrong_recParameter` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.RowSrc`, lines 6–6)

UpperTheory.GalStar.PropsB (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.TypeConditions.WeightsConjBis, lines 54–67)
26.24. UpperTheory.One.Join.local_inputs_from_literature (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TopMainZj, lines 18–37)
GlobalTheory.CompanionTheory.Hypotheses₂.totallyDisjointProductIrreducible_of_polynomialRing (Development.AutomorphicSpectrum.Proofs.GlobalProofs.TotallyDisjointProductIrreducibleOfPolynomialRin, lines 146–156)
26.25. GlobalTheory.CompanionTheory.MoyPrasadTop.moyPrasadJacquetIso (Development.AutomorphicSpectrum.Proofs.BlockDecomposition.MoyPrasadJacquetIso, lines 16–31)
GlobalTheory.DepthZeroPass.depthZeroInduction (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.FixedNeBotOfIsSubquotient, lines 64–81)
DiscreteGroups.sglob.LConv.localConverseGL (Development.DiscreteGroups.GlobalIrreducibility.LConvFinal, lines 18–20)
GlobalTheory.BrauerZero.brauerDegreeZero (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.BrauerDegreeZero, lines 203–215)
UpperTheory.LiteratureTheorems.chevalley_units_of_open (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TemperedProductOfSquareIntegrable, lines 253–254)
26.26. UpperTheory.One.Join.automorphic_inputs_from_literature (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TopMainZj, lines 38–54)
UpperTheory.LiteratureTheorems.hecke_ladic_character_above_of_open (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TemperedProductOfSquareIntegrable, lines 295–296)
UpperTheory.LiteratureTheorems.clozel_purity_of_open (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TemperedProductOfSquareIntegrable, lines 279–280)
26.27. UpperTheory.LiteratureTheorems.galois_cohomology_from_literature (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TemperedProductOfSquareIntegrable, lines 331–343)
ReciprocityTheory.Gal.UnramifiedOrthogonalityAt (ReciprocityTheory.Hypotheses.GalCoh, lines 36–52)
GlobalTheory.unramifiedOrthogonality_holds (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.UnramifiedOrthogonalityHolds, lines 7–8)
26.28. UpperTheory.One.Join.cm_abelian_inputs_from_literature (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TopMainZj, lines 59–70)
GlobalTheory.Published.grothendieckDeformation_of_lifting (Development.AbelianVarietyTheory.Proofs.DivisorProofs.GrothendieckDeformationOfLifting, lines 9–12)
GlobalTheory.AnPer.pTaylor (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ConstantCycleExtensionTaylorPeriodOfThickeningCo, lines 26–55)
26.29. UpperTheory.One.Join.cm_abelian_inputs_from_literature (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TopMainZj, lines 59–70)
26.30. GlobalTheory.Mem.LiteralAutomorphy.LitLocAlt (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.LitAut, lines 12–30)
GlobalTheory.Mem.LiteralAutomorphy.LiteratureAutomorphyHypothesis (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.LitAut, lines 31–46)
GlobalTheory.Mem.LiteralAutomorphy.LitCoh (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.Brauer, lines 105–117)
GlobalTheory.Mem.LiteralAutomorphy.LiteratureCMHypothesis (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.LitAut, lines 47–58)
GlobalTheory.Mem.LiteralAutomorphy.LiteratureRankNHypothesis (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.LitAut, lines 59–63)
26.31. UpperTheory.One.Join.automorphic_window_of_inputs (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TopMainZj, lines 71–74)
UpperTheory.Gal.Int.galoisLocalGlobalStrong_recParameter (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.RowSrc, lines 6–6)
26.32. UpperTheory.One.Join.automorphic_window_of_inputs (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TopMainZj, lines 71–74)
GlobalTheory.Mem.LiteralAutomorphy.LiteratureRankNHypothesis (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.LitAut, lines 59–63)
GlobalTheory.Mem.LiteralAutomorphy.LitCoh (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.Brauer, lines 105–117)
26.32. UpperTheory.One.Join.cm_abelian_inputs_from_literature (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TopMainZj, lines 59–70)
26.33. UpperTheory.V.TR.Cen.exists_cap_scalar (Development.AutomorphicArithmetic.AutomorphicVarieties.AutomorphicForms.CapCentral, lines 153–157)
UpperTheory.V.Af.cuspFormsFiniteReal (Development.AutomorphicArithmetic.AutomorphicVarieties.AutomorphicForms.TemperedLocalComponents.PCuspFin, lines 31–83)
UpperTheory.V.Af.admissibleReal (Development.AutomorphicArithmetic.AutomorphicVarieties.AutomorphicForms.TemperedLocalComponents.ExistsContRep, lines 39–63)
26.34. UpperTheory.V.Af.clozelCohC_alt (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TPCohC, lines 10–30)
UpperTheory.V.Z.Coh.clozelConjC (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClozelCohomology.ClozelConjC, lines 26–29)

UpperTheory.V.Lit.Geo.borelInjC (Development.AutomorphicArithmetic.AutomorphicVarieties.LiteratureResults.LieGeometryGrowth.BorelInjC, lines 74–96)
26.35. GlobalTheory.T.galois_rep_patching_infinite_family (PositivityTheory.AutomorphicForms.PublishedInputs.RealGaloisRepresentations.IwahoriSphericalTransfer.GaloisRepPatchingInfiniteFamily, lines 12–13)
GlobalTheory.IndexTwoPatch.main (PositivityTheory.AutomorphicForms.PublishedInputs.RealGaloisRepresentations.IwahoriSphericalTransfer.EqOneOrEqOfKer, lines 76–165)
26.36. UpperTheory.V.HV.DescentVanishingReductions.published_hypotheses_from_patching_alt (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.PublishedHypothesesFromPatching, lines 98–103)
UpperTheory.V.Af.newformRepOf (Development.AutomorphicArithmetic.AutomorphicVarieties.AutomorphicForms.TemperedLocalComponents.NewformRepOf, lines 114–138)
UpperTheory.ConvolutionTheory.RankTwoDescent.descS (Development.AutomorphicArithmetic.Foundations.Convolution.SplitEigenfunctions.AbsNormConj, lines 40–92)
26.37. UpperTheory.Gal.SV.galois_rep_exists_totally_real_rank_two_of_base_change (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.SVPatch, lines 103–165)
26.38. UpperTheory.V.Z.Bc.isAutR_of_isAutC_comp_galEmb_of_odd_quadratic_char (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.RankTwoDescentSplit, lines 88–114)
UpperTheory.V.Z.Bc.stepC_alt (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.CuspUnique, lines 24–65)
UpperTheory.V.Z.Bc.exDownC_sv_alt (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.SVRSol, lines 16–62)
UpperTheory.V.Z.Bc.automorphic_rank_two_of_automorphic_restrict_of_finite_det (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.DownQuadS, lines 70–122)
26.39. UpperTheory.V.HV.DescentVanishingReductions.published_hypotheses_from_patching_alt (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.PublishedHypothesesFromPatching, lines 98–103)
UpperTheory.V.HV.DescentVanishingReductions.vwin_of_hypotheses_H_E_alt (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.WindowBoundOfLiteratureLevel, lines 25–29)
26.39. UpperTheory.V.Z.Tf.SR.descWeak (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.TraceFormula.SplitRigidity.HasOfSplit, lines 29–30)
UpperTheory.V.Z.Bc.cycC_holds (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.BaseChangeUnits.CycFree, lines 87–88)
26.40. UpperTheory.V.HV.Core.of_four_alt (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.HVCore, lines 18–22)

Chapter 27: Three length bounds

27.1. UpperTheory.V.shaLen_eq_add (Development.AutomorphicArithmetic.AutomorphicVarieties.LengthEqAddEigen, lines 88–103)
UpperTheory.V.vvle_of_parts (Development.AutomorphicArithmetic.AutomorphicVarieties.LengthEqAddEigen, lines 127–189)
27.2. GlobalTheory.Published.PoitouTateDuality (Development.AbelianVarietyTheory.PublishedTheorems.GaloisCohomology₂.ExistsPrimitiveRootPow, lines 13–27)
UpperTheory.V.HV.U.pt_s (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.HVFinalU, lines 11–11)
27.3. UpperTheory.V.Bern.b_primmem (Development.AutomorphicArithmetic.AutomorphicVarieties.GaloisLibrary.BernoulliNumbers.Primmem, lines 17–54)
UpperTheory.V.Bern.b_nzC (Development.AutomorphicArithmetic.AutomorphicVarieties.GaloisLibrary.BernoulliNumbers.BNzC, lines 10–70)
UpperTheory.V.Bern.b_denupper (Development.AutomorphicArithmetic.AutomorphicVarieties.GaloisLibrary.BernoulliNumbers.Denupper, lines 28–38)
27.4. UpperTheory.V.ML.iv (Development.AutomorphicArithmetic.AutomorphicVarieties.PrimitiveFrobeniusFixed.Iv, lines 4–6)
27.5. UpperTheory.V.Stick.coef (Development.AutomorphicArithmetic.AutomorphicVarieties.GaloisLibrary.StickelbergerTheorem.ImportsAlt, lines 9–16)
UpperTheory.V.Odd.clKill (Development.AutomorphicArithmetic.AutomorphicVarieties.OddResidueBounds.ClKill, lines 17–55)
UpperTheory.V.Odd.homKill (Development.AutomorphicArithmetic.AutomorphicVarieties.OddResidueBounds.HomKill, lines 9–49)
27.6. UpperTheory.V.Odd.primeBound_of (Development.AutomorphicArithmetic.AutomorphicVarieties.OddResidueBounds.ThmBcOfPsiBound, lines 293–293)
27.7. UpperTheory.V.Odd.killPow (Development.AutomorphicArithmetic.AutomorphicVarieties.OddResidueBounds.KillPow, lines 31–104)
UpperTheory.V.Odd.stickBound (Development.AutomorphicArithmetic.AutomorphicVarieties.OddResidueBounds.ThmBcOfPsiBound, lines 291–291)

UpperTheory.V.Odd.deg (Development.AutomorphicArithmetic.AutomorphicVarieties.OddResidueBounds.DegExistsLift, lines 33–99)
27.8. UpperTheory.V.Odd.thmBcFree_of (Development.AutomorphicArithmetic.AutomorphicVarieties.OddResidueBounds.ThmBcOfPsiBound, lines 295–295)
UpperTheory.V.minusBound_le_exp_of_log_le (Development.AutomorphicArithmetic.AutomorphicVarieties.CMFieldExponentialBounds.minusle, lines 6–75)
27.9. UpperTheory.V.JointWin (Development.AutomorphicArithmetic.AutomorphicVarieties.PrimeSetBoundDataOfPrimitiveChar, lines 88–94)
27.10. UpperTheory.V.vwinAt_of_joint (Development.AutomorphicArithmetic.AutomorphicVarieties.CMFieldExponentialBounds.AdmissibleOfPageN, lines 41–129)
UpperTheory.V.log_conductor_mul_height_le_of_pageN (Development.AutomorphicArithmetic.AutomorphicVarieties.CMFieldExponentialBounds.LogConductorMulHeightLeOfPageN, lines 6–56)
UpperTheory.V.error_terms_le_sqrt_exp_rpow (Development.AutomorphicArithmetic.AutomorphicVarieties.CMFieldExponentialBounds.ErrorTermsLeSqrtExpRpow, lines 6–88)

Chapter 28: The cyclotomic tower and a common window

28.1. UpperTheory.V.ZK.sCount (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.SelmerOrderCounts, lines 16–111)
UpperTheory.V.ZK.sNum (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.NumThree, lines 211–378)
28.1. UpperTheory.V.ZK.lemmaK_of_steps (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.LemmaKOfSteps, lines 170–174)
UpperTheory.V.ZK.lemmaK (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.LemmaKOfLitN, lines 28–31)
28.2. UpperTheory.V.ZK.lemmaK_litN_alt (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.KFinal, lines 11–11)
UpperTheory.V.LemmaK (Development.AutomorphicArithmetic.AutomorphicVarieties.GeneratorCountTuples.LemmaK, lines 13–17)
28.2. UpperTheory.V.ZK.sFinal (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.SFinal, lines 100–163)
UpperTheory.V.LDataCardLeFinrank.vFormula_of_classicalTheorems (Development.AutomorphicArithmetic.AutomorphicVarieties.GaloisLibrary.GaloisCohomologyLength.Formula, lines 15–24)
28.3. GlobalTheory.HomStab.kStability_main (Development.AutomorphicArithmetic.AutomorphicVarieties.LiteratureResults.KTheoryStability.HomologicalStability.LinkExactnessStability.HomologicalStabilityTheorem, lines 10–27)
UpperTheory.V.KL.dedekindSR (Development.AutomorphicArithmetic.AutomorphicVarieties.TensorHomCubes.DedekindSR, lines 50–111)
UpperTheory.V.ZK.Up.torsStab (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.TorsionRankCounts.TorsStab, lines 16–39)
28.4. UpperTheory.V.KT.sx_exact (Development.AutomorphicArithmetic.AutomorphicVarieties.HermiteNormalForms.Assembly, lines 14–17)
UpperTheory.V.KT.gramdet (Development.AutomorphicArithmetic.AutomorphicVarieties.HermiteNormalForms.Gramdet, lines 49–100)
UpperTheory.V.KT.orb (Development.AutomorphicArithmetic.AutomorphicVarieties.HermiteNormalForms.ColsMinSmul, lines 34–75)
UpperTheory.V.KT.orbnum (Development.AutomorphicArithmetic.AutomorphicVarieties.HermiteNormalForms.Orbnum, lines 80–92)
UpperTheory.V.KT.stab (Development.AutomorphicArithmetic.AutomorphicVarieties.HermiteNormalForms.ImportsAlt, lines 8–26)
28.5. UpperTheory.V.KT.tors_le (Development.AutomorphicArithmetic.AutomorphicVarieties.HermiteNormalForms.Assembly, lines 25–30)
UpperTheory.V.KH.shift (Development.AutomorphicArithmetic.AutomorphicVarieties.TorsionBoundsChains.LogLeExp, lines 14–47)
UpperTheory.V.ZK.torsAll (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ClassCounting.ExponentLe, lines 70–117)
28.6. UpperTheory.V.Published.FerreroWashington (Development.AutomorphicArithmetic.AutomorphicVarieties.PublishedTheorems.FerreroWashington, lines 7–11)
UpperTheory.V.FW.ferreroWashington (Development.AutomorphicArithmetic.AutomorphicVarieties.GaloisLibrary.FerreroWashingtonIndex.FerreroWashington, lines 104–119)
28.7. UpperTheory.V.Two.towerFixed (Development.AutomorphicArithmetic.AutomorphicVarieties.QuadraticExtensions.IsAlgClosureRat, lines 98–110)
UpperTheory.V.Two.towerTwo (Development.AutomorphicArithmetic.AutomorphicVarieties.QuadraticExtensions.IsAlgClosureRat, lines 87–96)
UpperTheory.V.Two.baseRk (Development.AutomorphicArithmetic.AutomorphicVarieties.QuadraticExtensions.IsAlgClosureRat, lines 69–85)

28.8. UpperTheory.V.Two.shaTwo (Development.AutomorphicArithmetic.AutomorphicVarieties.QuadraticExtensions.ShaTwo, lines 11–28)
UpperTheory.V.Two.twoBase (Development.AutomorphicArithmetic.AutomorphicVarieties.QuadraticExtensions.IsAlgClosureRat, lines 112–128)
28.9. UpperTheory.V.PropA.descends (Development.AutomorphicArithmetic.AutomorphicVarieties.AdditiveValuationNorms.Descends, lines 8–15)
UpperTheory.V.PropA.Descends (Development.AutomorphicArithmetic.AutomorphicVarieties.AdditiveValuationNorms.ABoundOfDescends, lines 16–18)
UpperTheory.V.PropA.twLen (Development.AutomorphicArithmetic.AutomorphicVarieties.AdditiveValuationNorms.CokLenLeOfRangeLe, lines 26–26)
28.9. UpperTheory.V.Kum.resInfOnto (Development.AutomorphicArithmetic.AutomorphicVarieties.GaloisLibrary.KummerTheory.ResInfOnto, lines 102–166)
28.10. UpperTheory.V.PropA.exists_centres_level (Development.AutomorphicArithmetic.AutomorphicVarieties.AdditiveValuationNorms.ExistsCentresLevel, lines 23–32)
UpperTheory.V.PropA.exists_centres (Development.AutomorphicArithmetic.AutomorphicVarieties.AdditiveValuationNorms.ExistsCentresLevel, lines 34–49)
UpperTheory.V.PropA.exists_centres_norm (Development.AutomorphicArithmetic.AutomorphicVarieties.AdditiveValuationNorms.ExistsCentresLevel, lines 57–69)
28.11. UpperTheory.V.PropA.aBound_of_isTower (Development.AutomorphicArithmetic.AutomorphicVarieties.AdditiveValuationNorms.ABoundOfIsTower, lines 7–11)
UpperTheory.V.PropA.aBound_of_descends (Development.AutomorphicArithmetic.AutomorphicVarieties.AdditiveValuationNorms.ABoundOfDescends, lines 36–55)
UpperTheory.V.Win.SecondBound (Development.AutomorphicArithmetic.AutomorphicVarieties.LayeredIsomorphisms.SecondBound, lines 13–17)
28.11. UpperTheory.V.PropA.shaLeV (Development.AutomorphicArithmetic.AutomorphicVarieties.AdditiveValuationNorms.ShaLeV, lines 7–16)
28.12. UpperTheory.V.Win.exc (Development.AutomorphicArithmetic.AutomorphicVarieties.LayeredIsomorphisms.Exc, lines 7–42)
UpperTheory.V.Win.FirstBound (Development.AutomorphicArithmetic.AutomorphicVarieties.LayeredIsomorphisms.SecondBound, lines 27–30)
28.13. UpperTheory.V.Win.window1 (Development.AutomorphicArithmetic.AutomorphicVarieties.LayeredIsomorphisms.Window1, lines 6–58)
UpperTheory.V.Win.window (Development.AutomorphicArithmetic.AutomorphicVarieties.LayeredIsomorphisms.Window, lines 7–43)
28.14. UpperTheory.V.Win.point2 (Development.AutomorphicArithmetic.AutomorphicVarieties.LayeredIsomorphisms.Point2, lines 8–49)
UpperTheory.V.Win.final (Development.AutomorphicArithmetic.AutomorphicVarieties.LayeredIsomorphisms.Final, lines 6–68)
28.15. UpperTheory.V.Win.partA (Development.AutomorphicArithmetic.AutomorphicVarieties.LayeredIsomorphisms.PartA, lines 24–78)
UpperTheory.V.Win.partB (Development.AutomorphicArithmetic.AutomorphicVarieties.LayeredIsomorphisms.PartB, lines 50–95)
28.15. UpperTheory.V.Win.layeremb (Development.AutomorphicArithmetic.AutomorphicVarieties.LayeredIsomorphisms.Layeremb, lines 7–34)
UpperTheory.V.Win.disc (Development.AutomorphicArithmetic.AutomorphicVarieties.LayeredIsomorphisms.Imports, lines 6–44)
28.16. UpperTheory.V.Win.thmBab_of_lemmaK (Development.AutomorphicArithmetic.AutomorphicVarieties.LayeredIsomorphisms.ThmBabOfLemmaK, lines 9–12)
UpperTheory.V.Win.thmBab_of (Development.AutomorphicArithmetic.AutomorphicVarieties.LayeredIsomorphisms.ThmBabOf, lines 11–103)
28.16. UpperTheory.V.HV.DescentVanishingReductions.vwin_of_hypotheses_H_E_alt (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.WindowBoundOfLiteratureLevel, lines 25–29)
UpperTheory.V.HV.Core.of_four_alt (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.HVCore, lines 18–22)

Chapter 29: Summing the plus component by classes

29.1. UpperTheory.V.HV.Host.X (Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.DirichletPrimeToLPartMulLPowerPart, lines 188–189)
UpperTheory.V.HV.chiP (Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.DirichletPrimeToLPartMulLPowerPart, lines 86–86)
UpperTheory.V.Bchi (Development.AutomorphicArithmetic.AutomorphicVarieties.StmtFields, lines 66–67)
UpperTheory.V.HV.cases (Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.Cases, lines 8–48)
UpperTheory.V.HV.CountV (Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.Counts, lines 69–69)

`UpperTheory.V.HV.CountVs` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.Counts`, lines 71–71)
29.2. `UpperTheory.V.HV.hostIndex` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.HostIndex`, lines 67–164)
`UpperTheory.V.HV.hostTh1` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.OrdProjNDvd`, lines 37–117)
`UpperTheory.V.HV.hostComp` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.HostComp`, lines 21–81)
`UpperTheory.V.HV.hostOdd` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.HostOdd`, lines 70–111)
29.3. `UpperTheory.V.Chr.rho` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CharacterLevelBounds.ShaFamMono`, lines 17–18)
`UpperTheory.V.Chr.VS` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CharacterLevelBounds.ShaFamMono`, lines 30–30)
`UpperTheory.V.Chr.V'` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CharacterLevelBounds.ShaFamMono`, lines 34–34)
`UpperTheory.V.Chr.nrm` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CharacterLevelBounds.ShaFamMono`, lines 71–71)
`UpperTheory.V.HV.VLe` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.DirichletPrimeToLPartMulLPowerPart`, lines 120–120)
29.4. `UpperTheory.V.HV.coefIndep` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.CoefIndep`, lines 68–77)
29.5. `UpperTheory.V.Af.AttC` (`Development.AutomorphicArithmetic.AutomorphicVarieties.AutomorphicForms.BlochKatoSelmerVanishes`, lines 81–82)
`UpperTheory.V.Z.Win.WitA` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.AutomorphyLifting.WeakOfAtt`, lines 27–30)
`UpperTheory.V.Z.ImaginaryQuadraticEtaCharacter.WinAt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.LeviSmoothLocalization.WinUpDown`, lines 38–41)
`UpperTheory.V.Z.Win.IsAutQA` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.AutomorphyLifting.IsAutCsdA`, lines 16–20)
29.6. `UpperTheory.V.Pr.irrNormal` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.IrrNormal`, lines 62–179)
`UpperTheory.V.Th.Det.irrSmall` (`Development.AutomorphicArithmetic.AutomorphicVarieties.TheoremsMuLinked.CharacteristicPolynomials.IrrSmall`, lines 33–65)
29.7. `UpperTheory.V.Pr.matIrr_open_gd` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.MatIrrOpenGd`, lines 342–382)
`UpperTheory.V.U3.typeII_irr_cyc` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.ExistsUnitPowNe`, lines 72–142)
29.8. `UpperTheory.V.Z.J.rigIUE_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.ERigIU`, lines 80–146)
`UpperTheory.V.Z.Win.RR.rigII_a_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.RRRigII`, lines 25–102)
29.9. `UpperTheory.V.Z.Desc.V0.numJE_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TNumJE`, lines 171–198)
`UpperTheory.V.Z.Desc.V0.z_eq_dim_e_of_count_e_res_eq_inv_loc_kir` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TZEqDimE`, lines 57–105)
29.10. `UpperTheory.V.Z.Desc.V0.unramified_at_of_dense_purity_num_e_power_series` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.UnrE`, lines 21–54)
`UpperTheory.V.Z.Desc.V0.unrAlgW` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.CuspidalDescent.GLDenseRanges.InertiaPartMaps.UnrAlgW`, lines 141–204)
29.11. `UpperTheory.Gal.projectedFormsSystemPart.red2_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.EndPtOf`, lines 160–172)
`UpperTheory.V.J.redAbs` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.LinkedUnramifiedChains.ScalarIrreducibility.RedAbs`, lines 15–48)
29.12. `UpperTheory.V.Z.J.unitOf` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.LinkedUnramifiedChains.CrystallineCardinalities.UnitOf`, lines 245–305)
`UpperTheory.V.Z.J.locHigh` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.LinkedUnramifiedChains.CrystallineCardinalities.LocHigh`, lines 18–87)
`UpperTheory.V.Z.J.selLow` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.LinkedUnramifiedChains.CrystallineCardinalities.SelLow`, lines 60–98)
`UpperTheory.V.Z.J.tan` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.LinkedUnramifiedChains.CrystallineCardinalities.BoxUnit`, lines 128–160)
29.13. `UpperTheory.Gal.projectedFormsSystemPart.UnitaryRankTwoFlagsAlt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.UnitaryRankTwoFlags`, lines 36–60)
`UpperTheory.V.Z.J.unlEnd3E` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.LinkedUnramifiedChains.LogarithmicNormBounds.BlockValuationRep`, lines 234–346)
`UpperTheory.V.J.endJ` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.LinkedUnramifiedChains.DerivativeDivisibility.CoefInj`, lines 12–73)

`UpperTheory.V.J.tor` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.LinkedUnramifiedChains.DerivativeDivisibility.PowNeZero`, lines 124–143)
29.14. `UpperTheory.V.Z.J.admissible_typeI_even_level_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.AdmissibleTypeIEvenLevel`, lines 76–136)
`UpperTheory.V.Z.J.rigIUE_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.ERigIU`, lines 80–146)
`UpperTheory.V.Z.Win.RR.rigII_a_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.RRRigII`, lines 25–102)
29.15. `UpperTheory.V.U3.countI` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.CountI`, lines 22–136)
`UpperTheory.V.Z.J.count_i_of_printed_bound_k_kato_automorphy` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.LinkedUnlE`, lines 49–54)
29.16. `UpperTheory.V.U3.countI` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.CountI`, lines 22–136)
`UpperTheory.V.HV.countZ` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.CountZ`, lines 5–32)
29.17. `UpperTheory.V.GradeZeroInterpolatingElement.Raw.Good` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.DecompSigmaOrd`, lines 94–120)
`UpperTheory.V.Aut.OfChar` (`Development.AutomorphicArithmetic.AutomorphicVarieties.AutomorphicCases.SymplecticGSp4`, lines 12–17)
`UpperTheory.V.Aut.OfCase` (`Development.AutomorphicArithmetic.AutomorphicVarieties.AutomorphicCases.SymplecticGSp4`, lines 19–20)
`UpperTheory.V.Ws.Ctx2` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.NearSplit`, lines 55–57)
`UpperTheory.V.Ws.Ctx3` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.NearSplit`, lines 59–61)
29.18. `UpperTheory.V.GradeZeroInterpolatingElement.OnComp` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.DecompSigmaOrd`, lines 27–30)
`UpperTheory.V.GradeZeroInterpolatingElement.Ord` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.MapRepApply`, lines 99–102)
`UpperTheory.V.GradeZeroInterpolatingElement.Raw.IsPt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.DecompSigmaOrd`, lines 155–160)
`UpperTheory.V.GradeZeroInterpolatingElement.Raw.PtIn` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.DecompSigmaOrd`, lines 164–166)
`UpperTheory.V.GradeZeroInterpolatingElement.Raw.Fin0` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.DecompSigmaOrd`, lines 172–174)
`UpperTheory.V.GradeZeroInterpolatingElement.Raw.RigidAt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.DecompSigmaOrd`, lines 182–184)
29.19. `UpperTheory.V.X.soBar` (`Development.AutomorphicArithmetic.AutomorphicVarieties.LevelCohomology.FdAdjoinRange`, lines 78–96)
`UpperTheory.V.X.soVec` (`Development.AutomorphicArithmetic.AutomorphicVarieties.LevelCohomology.IntIffNorm`, lines 143–245)
29.20. `UpperTheory.V.GradeZeroInterpolatingElement.dataEx` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.DataEx`, lines 366–449)
29.21. `UpperTheory.V.GradeZeroInterpolatingElement.aSmooth` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.ASmooth`, lines 1126–1139)
`UpperTheory.V.GradeZeroInterpolatingElement.exModel` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.ExModel`, lines 425–520)
29.22. `UpperTheory.V.Th.sp4Main_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.ConductorAtMapInjectiveCoeff`, lines 131–171)
`UpperTheory.V.HV.DescentVanishingReductions.countII_and_countIII_of_printsH_of_sp4In` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.AdjointLiftingOfCohomology`, lines 33–39)
29.23. `UpperTheory.V.Base.kerResAb` (`Development.AutomorphicArithmetic.AutomorphicVarieties.GaloisLibrary.SelmerTransport.KerResAb`, lines 46–151)
`UpperTheory.V.Chr.upV` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CharacterLevelBounds.LiftGens`, lines 24–57)
29.24. `UpperTheory.V.HV.LayerLe` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.LayerLe`, lines 12–14)
`UpperTheory.V.HV.CountIV` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.LayerLe`, lines 16–20)
`UpperTheory.V.HV.countIV` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.CountIV`, lines 39–89)
29.25. `UpperTheory.V.HV.ivField` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.IvField`, lines 32–82)
29.26. `UpperTheory.V.Z.J.count_v_of_printed_bound_k_reduced_unitary_lift` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.LinkedUnlE`, lines 55–61)

UpperTheory.V.UnitaryThreeInertia.countV_of_finiteness_duality_euler_host (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.CountVOf, lines 33–60)
29.27. UpperTheory.V.HV.countVs (Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.CountVs, lines 86–171)
29.28. UpperTheory.V.X.propZ (Development.AutomorphicArithmetic.AutomorphicVarieties.LevelCohomology.PropZ, lines 11–16)
UpperTheory.V.X.zOne (Development.AutomorphicArithmetic.AutomorphicVarieties.LevelCohomology.TorsClass, lines 97–205)
UpperTheory.V.X.zQuad (Development.AutomorphicArithmetic.AutomorphicVarieties.LevelCohomology.ZqField, lines 228–233)
UpperTheory.V.HV.countZ (Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.CountZ, lines 5–32)
29.29. UpperTheory.V.HV.redAll (Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.RedAll, lines 6–21)
UpperTheory.V.HV.redPrime (Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.RedPrime, lines 11–57)
29.30. UpperTheory.V.HV.sumIn (Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.FinalLe, lines 86–263)
29.31. UpperTheory.V.HV.sumOut (Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.SumOut, lines 50–231)
29.32. UpperTheory.V.HV.pointHV_of_counts (Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.JointWinOfCounts, lines 16–18)
UpperTheory.V.HV.point (Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.PointArith, lines 36–92)
UpperTheory.V.BConst.J (Development.AutomorphicArithmetic.AutomorphicVarieties.PrimeSetBoundDataOfPrimitiveChar, lines 20–21)
UpperTheory.V.IsBWindow (Development.AutomorphicArithmetic.AutomorphicVarieties.PrimeSetBoundDataOfPrimitiveChar, lines 40–41)
UpperTheory.V.HV.thmHV (Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.ThmHV, lines 6–40)
UpperTheory.V.HV.jointWin_of_counts (Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.JointWinOfCounts, lines 25–28)
29.32. UpperTheory.V.HV.hostCond (Development.AutomorphicArithmetic.AutomorphicVarieties.HeightAndVanishing.HostCond, lines 256–323)

Chapter 30: Automorphic counts and deformation rigidity

Chapter introduction. UpperTheory.V.Af.Level (Development.AutomorphicArithmetic.AutomorphicVarieties.AutomorphicForms.BlochKatoSelmerVanishes, lines 111–114)
UpperTheory.V.Th.Unl (Development.AutomorphicArithmetic.AutomorphicVarieties.TheoremsMuLinked.AutP, lines 10–13)
30.1. UpperTheory.V.Z.Win.WitA (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.AutomorphyLifting.WeakOfAtt, lines 27–30)
UpperTheory.V.Z.Win.WitAQ (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.AutomorphyLifting.AAcrossOf, lines 17–20)
UpperTheory.V.Z.Win.IsAutQA (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.AutomorphyLifting.IsAutCsdA, lines 16–20)
30.1. UpperTheory.V.Z.J.Adm3U_s (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.NumberU3US, lines 27–29)
UpperTheory.V.Z.Win.Dn.Adm2A (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.AutomorphyLifting.AutomorphyDescent.AutNewformW, lines 14–20)
30.2. UpperTheory.V.U3.CohInp (Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.KatoUse, lines 19–24)
UpperTheory.V.Loc.LocalPublishedTheorems (Development.AutomorphicArithmetic.AutomorphicVarieties.PadicHodgeTheory.LocalPublishedTheorems, lines 12–19)
UpperTheory.V.U3.Dat (Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.RhoKApply, lines 16–33)
UpperTheory.V.U3.Marks (Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.BaseRepGL3NormalisedOfPsiEqOne, lines 68–73)
UpperTheory.V.U3.r0 (Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.UnitaryThreeReflectionRight, lines 291–297)
UpperTheory.V.U3.BoxCond (Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.GRepresents, lines 100–102)
UpperTheory.V.U3.NormCond (Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.GRepresents, lines 104–105)
UpperTheory.V.U3.SRing (Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.ConjStable, lines 49–49)
30.3. UpperTheory.V.U3.LocGood (Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.ConjStable, lines 34–37)

`UpperTheory.V.U3.ConjStable` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.ConjStable`, lines 18–20)
`UpperTheory.V.U3.LocChart` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.ConjStable`, lines 25–32)
`UpperTheory.V.U3.locDims` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.ConjStable`, lines 22–23)
`UpperTheory.V.Loc.rtChart` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PadicHodgeTheory.RtChart`, lines 41–74)
`UpperTheory.V.Loc.rtPoints` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PadicHodgeTheory.RtPoints`, lines 11–15)
`UpperTheory.V.Loc.lemma2c` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PadicHodgeTheory.ValuationRingClearLastRowEntry`, lines 219–246)
30.4. `UpperTheory.V.U3.TypeI` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.BaseRepGL3NormalisedOfPsiEqOne`, lines 20–20)
`UpperTheory.V.U3.TypeII` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.BaseRepGL3NormalisedOfPsiEqOne`, lines 30–31)
`UpperTheory.V.U3.typeII_facts` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.CoboundaryLemmaNoColumn`, lines 135–215)
`UpperTheory.V.U3.char_eq_one` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.PowEqOneOfUnr`, lines 103–129)
`UpperTheory.V.U3.pt_inertia_above` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.PtIneL`, lines 80–128)
`UpperTheory.V.U3.pt_inertia_away` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.PtInertiaAway`, lines 54–109)
30.5. `UpperTheory.V.Z.J.oneCharU_of_katoTwoW_rig_admI_automorphy` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.FOneCharU`, lines 75–95)
`UpperTheory.V.Z.J.OneCharU_s` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.NumberU3US`, lines 49–59)
30.6. `UpperTheory.V.U3.sha_inj_gen` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.ShaInjGen`, lines 338–361)
`UpperTheory.exists_base_pointDerivation_of_weight_homogeneous` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.ExistsBasePointDerivationOfWeightHomogeneous`, lines 48–134)
`UpperTheory.V.U3.model_lower` (`Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.ModelLower`, lines 373–401)
30.7. `UpperTheory.GalN.V.Win.N.numberU3U_of_prints` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.KeepNumU`, lines 79–80)
`UpperTheory.V.Z.J.NumberU3U_s` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.NumberU3US`, lines 30–45)
30.8. `UpperTheory.V.Keep.levelCore_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.KeepLevel`, lines 30–82)
`UpperTheory.V.Keep.levelKA_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.KeepLevel`, lines 95–101)
`UpperTheory.V.Keep.levelPA_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.KeepLevel`, lines 83–90)
`UpperTheory.V.Z.Win.A4.admissible_restriction_level_n_of_printed_results` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.LevelAdmissibleRestrictionHolds`, lines 16–23)
`UpperTheory.V.Lev.levelFRes` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.SemisimpleLevels.LevelFRes`, lines 46–73)
`UpperTheory.V.Z.Win.A4.safeUP'_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.SafeUP`, lines 16–19)
`UpperTheory.V.Z.Win.A4.potQ` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.AutomorphyLifting.AdmissibleLevelRestriction.PotQOf`, lines 63–72)
`UpperTheory.V.Z.Win.SplitIrr` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.AutomorphyLifting.SplitIrr`, lines 46–50)
`UpperTheory.V.Z.Win.A4.safeRes'` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.AutomorphyLifting.AdmissibleLevelRestriction.UnlAllOf`, lines 30–34)
`UpperTheory.V.Lev.unlGap` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.SemisimpleLevels.UnlGap`, lines 8–29)
`UpperTheory.V.Z.Bc.polarization_k_of_printed_results` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.UnderCPl`, lines 21–55)
30.9. `UpperTheory.V.Af.card_k_of_printed_results` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TCardK`, lines 18–36)
`UpperTheory.GalN.V.Win.N.countQ_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.FrobeniusTraceCountBoundOfCmEigensystemCount`, lines 255–335)
`UpperTheory.GalN.V.Win.traces_from_eigenvalues_rank3` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.AdmissibleResidues.FrobeniusTraceCountBoundOfRatEigensystemCount`, lines 78–145)
`UpperTheory.GalN.V.Win.traces_from_eigenvalues_rank2` (`Development.AutomorphicArithmetic.Foundations.`

GaloisRepresentations.AdmissibleResidues.FrobeniusTraceCountBoundOfRatEigensystemCount, lines 169–230)
30.10. UpperTheory.V.Z.J.propGK_of_numberU3U_of_oneCharU (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.FPropGKU, lines 110–172)
UpperTheory.V.U3.PropGK (Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.KatoUse, lines 36–42)
30.10. GlobalTheory.Ord2.Nf (Development.AbelianVarietyTheory.Definitions.Foundations.FinalTheorems, lines 53–66)
GlobalTheory.Ord2.Gr (Development.AbelianVarietyTheory.Definitions.Foundations.FinalTheorems, lines 72–73)
GlobalTheory.Ord2.Unr (Development.AbelianVarietyTheory.Definitions.Foundations.FinalTheorems, lines 68–70)
30.10. UpperTheory.V.Z.Kato.SphAt (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.KatoQuadraticCharacters.KatoTwoSOfTwo, lines 9–16)
30.10. UpperTheory.V.Af.IsBK (Development.AutomorphicArithmetic.AutomorphicVarieties.AutomorphicForms.BlochKatoSelmerVanishes, lines 135–140)
UpperTheory.V.Loc.finiteCocycles (Development.AutomorphicArithmetic.AutomorphicVarieties.PadicHodgeTheory.FiniteCohomology, lines 112–125)
UpperTheory.V.Loc.tiedP (Development.AutomorphicArithmetic.AutomorphicVarieties.PadicHodgeTheory.FiniteCohomology, lines 84–84)
30.11. UpperTheory.V.Af.Newform (Development.AutomorphicArithmetic.AutomorphicVarieties.AutomorphicForms.Newform, lines 21–33)
UpperTheory.V.Af.Newform.Att (Development.AutomorphicArithmetic.AutomorphicVarieties.AutomorphicForms.Newform, lines 43–47)
UpperTheory.V.Lit.KatoLit (Development.AutomorphicArithmetic.AutomorphicVarieties.LiteratureResults.CoefFieldEq, lines 58–66)
UpperTheory.V.U3.KatoUse (Development.AutomorphicArithmetic.AutomorphicVarieties.UnitaryThreeGroups.KatoUse, lines 13–17)
30.12. UpperTheory.V.Z.Kato.WeilSphericalRecords.KatoTwoW (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.KatoQuadraticCharacters.SphericalDecomposition.KatoTwoW, lines 10–14)
UpperTheory.V.Z.Kato.Used (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.KatoQuadraticCharacters.KatoTwoOfUseN, lines 11–41)
UpperTheory.V.Z.Kato.WeilSphericalRecords.katoTwoW_of_printsH (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.KatoTwoWOfPublishedH, lines 48–49)
UpperTheory.V.Z.Kato.WeilSphericalRecords.katoTwoW_of_printsH_of_rT_of_quadBC2 (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.KatoTwoWOfPublishedH, lines 38–47)
30.13. UpperTheory.V.Z.Kato.WeilSphericalRecords.srcS_alt (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TwistNewform, lines 116–118)
GlobalTheory.Ord2.exists_nf_srcUr (Development.AbelianVarietyTheory.Proofs.FoundationProofs.ExistsOrdinaryNormalFormUnramifiedSource, lines 36–139)
30.14. UpperTheory.V.Z.Kato.WeilSphericalRecords.twoS (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.KatoQuadraticCharacters.SphericalDecomposition.SphAtTw, lines 16–34)
GlobalTheory.Ord2.vanishOrdL_of_zetaUr (Development.AbelianVarietyTheory.Proofs.FoundationProofs.VanishOrdLOfZetaUr, lines 20–25)
30.15. GlobalTheory.EisM.lseries_ne_zero_edge (Development.AbelianVarietyTheory.Proofs.FoundationProofs.LseriesNeZeroReGeOneOfPositivity, lines 21–116)
GlobalTheory.EisM.twisted_lvalue_ne_zero_edge (Development.AbelianVarietyTheory.Proofs.FoundationProofs.TwistedLvalueNeZeroEdge, lines 167–219)
30.15. UpperTheory.V.GradeZeroInterpolatingElement.Raw.PtIn (Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.DecompSigmaOrd, lines 164–166)
UpperTheory.V.GradeZeroInterpolatingElement.Raw.EPt (Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.DecompSigmaOrd, lines 168–168)
30.16. UpperTheory.V.Loc.spPtL (Development.AutomorphicArithmetic.AutomorphicVarieties.PadicHodgeTheory.SplAdjUnit, lines 264–330)
30.17. UpperTheory.V.Loc.prC (Development.AutomorphicArithmetic.AutomorphicVarieties.PadicHodgeTheory.SnocZeroLastCoordinateLast, lines 345–352)
30.18. UpperTheory.V.Th.spPt (Development.AutomorphicArithmetic.AutomorphicVarieties.TheoremsMuLinked.GaloisRepresentationTypes.MapRepComp, lines 220–286)
UpperTheory.V.Ws.P1.condC (Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.ProjectiveLineFlags.CondC, lines 10–93)
UpperTheory.V.Th.nice0four_alt (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TNiceFour, lines 114–130)
adjoint_vanishing_gsp4_rigid_point (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.AdjointVanishingGsp4RigidPoint, lines 31–101)
30.19. UpperTheory.V.Th.rigSp4_alt (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.RigSp4, lines 181–194)
UpperTheory.V.GradeZeroInterpolatingElement.Raw.RigidAt (Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.DecompSigmaOrd, lines 182–184)
30.20. UpperTheory.V.AlgebraTools.hkDualEx (Development.AutomorphicArithmetic.AutomorphicVarieties.AlgebraTools.HkDualEx, lines 14–41)
UpperTheory.V.AlgebraTools.hkSandwich (Development.AutomorphicArithmetic.AutomorphicVarieties.AlgebraTools.HkSandwich, lines 92–131)

`UpperTheory.V.Pa.Lev.multiTwo` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.NilpotentLevelLowering.MultiTwo`, lines 174–314)
`UpperTheory.V.Pa.Lev.twoOf_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.LevTwoOf`, lines 103–117)
30.21. `UpperTheory.V.Loc.ordFlag_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.LocOrdFlag`, lines 158–168)
30.22. `UpperTheory.V.Hi.ordStruct` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.OrdinaryHeckeCharacters.OrdFree`, lines 295–437)
`UpperTheory.V.Hi.mem_ord_iff` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.OrdinaryHeckeCharacters.OrdFree`, lines 263–293)
`UpperTheory.exists_section_automorphicSpace_with_prescribed_values` (`Development.AutomorphicArithmetic.AutomorphicVarieties.BuildingsResidueChars.ExistsSectionAutomorphicSpaceWithPrescribedValue`, lines 8–103)
`UpperTheory.V.Hi.HSet` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.CoefficientRepTrivialOnUnramified`, lines 75–107)
30.23. `UpperTheory.V.CA.universalDeformationRing_nonempty_of_nonempty_pair` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebraHeight.UniversalDeformationRingNonemptyOfNonemptyPair`, lines 243–319)
`UpperTheory.V.Th.AnnihilatorQuotientFinite.levelX_of_isJ_unitaryHeckeData` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.HidaLevelX`, lines 192–198)
`UpperTheory.V.Th.has_operator_statement` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.THasOp`, lines 18–45)
`UpperTheory.V.W.HidaSetting` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeilGroupCharacters.ContinuousResP`, lines 73–92)
30.24. `UpperTheory.V.Hi.spec` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.OrdinaryHeckeCharacters.SpecIntIff`, lines 100–201)
`UpperTheory.V.Hi.ctrlOrd` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.OrdinaryHeckeCharacters.CtrlOrd`, lines 220–371)
`UpperTheory.V.Hi.ordBC` (`Development.AutomorphicArithmetic.AutomorphicVarieties.HeckeAlgebras.OrdinaryHeckeCharacters.OrdBCMapOrd`, lines 198–232)
30.25. `UpperTheory.Gal.projectedFormsSystemPart.irrArithPointsAutomorphicAway_of_isJ` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.EndoscopyRank2Ordinary`, lines 48–72)
30.26. `UpperTheory.V.Pa.linAlg` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.SelmerScaling.LinAlg`, lines 45–126)
30.27. `UpperTheory.V.Pa.Core.pmDim` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AnnihilatorBaseChange.PmDimQuot`, lines 99–149)
30.28. `UpperTheory.V.AlgebraTools.exists_least_prime_detIdeal` (`Development.AutomorphicArithmetic.AutomorphicVarieties.AlgebraTools.CaDetMin`, lines 11–50)
`UpperTheory.V.AlgebraTools.le_of_detIdeal_le` (`Development.AutomorphicArithmetic.AutomorphicVarieties.AlgebraTools.CaDetLocus`, lines 10–58)
30.29. `UpperTheory.V.Pa.locLemma` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.SelmerScaling.LocLemma`, lines 225–356)
30.30. `UpperTheory.V.Pa.Core.pmodEx` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AnnihilatorBaseChange.PmodEx`, lines 32–95)
`UpperTheory.V.Pa.Core.exists_ultraLimit_pair_of_eventual_quotient_maps` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AnnihilatorBaseChange.ExistsUltraLimitPairOfEventualQuotientMaps`, lines 48–105)
`UpperTheory.V.Pa.Core.quotientRegularAt_of_quotient_free_over_powerSeries` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AnnihilatorBaseChange.QuotientRegularAtOfQuotientFreeOverPowerSeries`, lines 9–55)
30.31. `UpperTheory.V.Pa.W1_bound_arithmetic_rank_two` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.EpsilonEtaMatrices.W1BoundArithmeticRankTwo`, lines 40–165)
`UpperTheory.V.Pa.B2.obstructionBound_of_duality` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.ArchimedeanH0Cards.ObstructionBoundOfDuality`, lines 145–216)
30.32. `UpperTheory.V.Pa.cohLim` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AutomorphicDetection.CohLim`, lines 12–19)
30.33. `UpperTheory.V.Pa.Core.pmodEx` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AnnihilatorBaseChange.PmodEx`, lines 32–95)
`UpperTheory.V.Pa.Core.pmDim` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AnnihilatorBaseChange.PmDimQuot`, lines 99–149)
30.34. `UpperTheory.V.W.modU_of_printsH` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.CoefIsCoef`, lines 16–22)
`UpperTheory.V.Pa.pFromAut_of` (`Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.ChebotarevFamilies.PFromAutOf`, lines 116–131)
30.35. `UpperTheory.V.Ws.Ch.nice` (`Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.ValuationCharacters.ConstOfFinite`, lines 101–124)
`UpperTheory.V.Pr.nicePrimeEx` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.NicePrimeEx`, lines 87–125)
30.36. `UpperTheory.V.GradeZeroInterpolatingElement.CohomologyHyp` (`Development.AutomorphicArithmetic.AutomorphicVarieties.SymplecticGSp4.EntryMem`, lines 161–167)

30.37. UpperTheory.V.Ws.x1 (Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.IncLoc, lines 97–117)
30.38. UpperTheory.V.Cu.chain_rank_four_of_red_a_y_coin_fib_y (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.ReductionTypeChainOfRedAYCoinFibY, lines 213–268)
UpperTheory.V.Cu.RankFourReductionTypeChain.ch_core4 (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.ReductionTypeChainOfRedAYCoinFibY, lines 78–211)
UpperTheory.V.Ws.residual_charpoly_rank_four (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.ClassicalFixedByIwahori, lines 90–126)
30.39. UpperTheory.V.Ws.redTan (Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.RedTan, lines 319–352)
UpperTheory.V.Cu.redTanA (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.CuspidalTwists.PRedTanA, lines 258–267)
UpperTheory.V.Cu.yCoin (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.CuspidalTwists.PYCoin, lines 117–152)
30.40. UpperTheory.V.Ws.off_diagonal_nilpotent_rank_four (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.OffDiagonalNilpotentRankFour, lines 10–100)
UpperTheory.V.Ws.fib (Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.FibLow, lines 68–110)
UpperTheory.V.Cu.fibY_of (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.CuspidalTwists.PFibY, lines 67–108)
30.41. UpperTheory.V.Sd.propE (Development.AutomorphicArithmetic.AutomorphicVarieties.IdeleGenerators.HolomorphicIrreducibility.SumDivHecke, lines 153–245)
UpperTheory.V.Z.Sym.Unl.propEC (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.SymmetricPowerTransfer.UnlinkedHalfIntegral.UnlinkedHeckeRoots.PropEC, lines 191–284)
30.42. UpperTheory.V.Pr.FinCrit (Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.IsUnitOrIsUnitOfAddOne, lines 114–122)
UpperTheory.V.Pr.finCrit (Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.FinCrit, lines 122–186)
UpperTheory.V.Ws.ptOver (Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.PtOver, lines 20–60)
UpperTheory.V.Alg.wPres_maximalIdeal_gr_zero_eq_sup_span_monomials (Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebra.WPresMaximalIdealGrZeroEqSupSpanMonomials, lines 14–81)
30.43. UpperTheory.Esk.pf_TrCong (PositivityTheory.AutomorphicForms.PublishedInputs.JacquetLanglands.TorusKernels.PfTrCong, lines 101–128)
UpperTheory.Esk.pf_HolWC (PositivityTheory.AutomorphicForms.PublishedInputs.JacquetLanglands.TorusKernels.PfHolWC, lines 174–304)
30.44. UpperTheory.V.Ws.eptV (Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.IsDvrCoef, lines 25–55)
UpperTheory.V.Ws.ptPair (Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.PtPair, lines 22–75)
UpperTheory.V.Ws.labels (Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.Labels, lines 22–98)
30.45. UpperTheory.V.Ws.lat (Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.LatSix, lines 24–61)
30.46. UpperTheory.V.Ws.restrQ (Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.RestrQ, lines 67–118)
UpperTheory.V.Ws.goodQ (Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.FlagFacts, lines 64–141)
30.47. UpperTheory.V.Ws.bl (Development.AutomorphicArithmetic.AutomorphicVarieties.WeakShimuraTheory.CompactIntegralFields.NotOver, lines 51–128)
30.48. UpperTheory.V.AlgebraTools.idCompMat (Development.AutomorphicArithmetic.AutomorphicVarieties.AlgebraTools.MonomialRepresentations.IdCompMat, lines 46–136)
UpperTheory.V.AlgebraTools.regCosetFour (Development.AutomorphicArithmetic.AutomorphicVarieties.AlgebraTools.MonomialRepresentations.RegCosetFour, lines 337–394)
UpperTheory.V.Pr.regCoset (Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.RegCoset, lines 143–161)
30.49. UpperTheory.V.Af.newformRepOf (Development.AutomorphicArithmetic.AutomorphicVarieties.AutomorphicForms.TemperedLocalComponents.NewformRepOf, lines 114–138)
UpperTheory.Gal.SV.galois_rep_exists_totally_real_rank_two_of_base_change (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.SVPatch, lines 103–165)
UpperTheory.V.Pr.kaOf (Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.ValIntLe, lines 102–139)
30.50. UpperTheory.Gal.projectedFormsSystemPart.cRealAv (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.PlainTorsionPairs.CRealAv, lines 150–332)
UpperTheory.V.Pr.p1Of (Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.FirstPrimeConditionTransfer, lines 5–19)

30.51. `UpperTheory.V.HV.DescentVanishingReductions.sp4In0_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.Sp4In0`, lines 18–19)
`UpperTheory.V.Aut.SymplecticFourImageDense` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.IsAut4`, lines 14–17)
30.52. `UpperTheory.GalN.V.numberSp4_n_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.VWLevel`, lines 18–20)
`UpperTheory.V.Af.NumberSp4Alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.AfStmt`, lines 34–43)
30.53. `UpperTheory.V.Pr.FinCrit` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.IsUnitOrIsUnitOfAddOne`, lines 114–122)
`UpperTheory.V.Pr.finCrit` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.FinCrit`, lines 122–186)
30.54. `UpperTheory.V.Aut.ThmS6Core_s` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.ThmS6CoreS`, lines 27–36)
`UpperTheory.V.Pr.symmetric_cube_theorem_of_endomorphisms_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.SymmetricCubeTheoremOfEndomorphisms`, lines 76–131)
30.55. `UpperTheory.V.Fr.Frame.Ok` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.ContinuousCoboundaries2`, lines 176–210)
`UpperTheory.V.Fr.H1C` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.ContinuousCoboundaries2`, lines 57–57)
`UpperTheory.V.Fr.Frame.EC` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.ContinuousCoboundaries2`, lines 167–174)
`UpperTheory.V.Pr.SeedClause` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.HostFrameEx`, lines 58–61)
`UpperTheory.V.Sd.FieldPairCharacterTripleConditions` (`Development.AutomorphicArithmetic.AutomorphicVarieties.IdeleGenerators.SlopeTestRemainingChoices`, lines 136–149)
30.56. `UpperTheory.V.GC.ec` (`Development.AutomorphicArithmetic.AutomorphicVarieties.GaloisCochains.ResH1SurjectiveOfIsomorphism`, lines 84–179)
30.57. `UpperTheory.V.Fr.detsec` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.Detsec`, lines 175–191)
30.58. `UpperTheory.V.Fr.epres` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.SumLe`, lines 42–115)
`UpperTheory.V.Fr.ep_pad` (`Development.AutomorphicArithmetic.AutomorphicVarieties.FrobeniusSmoothVectors.ExistsPaddedFamily`, lines 26–63)
30.59. `UpperTheory.Gal.projectedFormsSystemPart.linked_endomorphism_dense` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.LinkedEndE`, lines 18–67)
`UpperTheory.V.Pr.absMain` (`Development.AutomorphicArithmetic.AutomorphicVarieties.CurveDVRSpecialization.AbsMain`, lines 12–30)
`UpperTheory.Gal.projectedFormsSystemPart.twoRuns_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TwoRuns`, lines 80–109)
30.60. `UpperTheory.V.Z.Win.Dn.autEnd2_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.AutEnd2`, lines 39–168)
`UpperTheory.V.Z.Win.RR.autNewformA_of_exRS_of_compatT_of_repNewAE` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.AutNewformAOfEx`, lines 173–205)
`UpperTheory.V.Z.Win.Dn.across2'_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.Across2`, lines 15–21)
30.61. `UpperTheory.V.Z.Win.Dn.ends0_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.DnEndsTop`, lines 15–16)
`UpperTheory.V.Z.Win.Dn.ends_of_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.DnEnds`, lines 35–39)
`UpperTheory.V.Pr.GdEndAlt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.AutEnd3AltAlt`, lines 35–40)
30.62. `UpperTheory.V.Th.AdZero` (`Development.AutomorphicArithmetic.AutomorphicVarieties.TheoremsMuLinked.OrdZero`, lines 68–72)
`UpperTheory.V.Th.AdjointLiftingTheorem` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.AutomorphicPointGalois`, lines 35–39)
`UpperTheory.V.HV.DescentVanishingReductions.adjoint_lifting_of_cohomology` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.AdjointLiftingOfCohomology`, lines 31–31)
30.63. `UpperTheory.V.AlgebraTools.cocycle_trace_zero` (`Development.AutomorphicArithmetic.AutomorphicVarieties.AlgebraTools.CocycleTraceZero`, lines 6–93)
30.64. `UpperTheory.V.Z.Desc.W.jPT_of_alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.DualOfWeights`, lines 243–339)
30.65. `UpperTheory.V.Th.OrdVan2Alt` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.OrdVan2`, lines 13–20)
`UpperTheory.V.Th.ordinary_vanishing_two_of_printed_hida` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TOrdVan`, lines 159–192)

30.66. UpperTheory.V.Th.core_o_of_literal_automorphy_n (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.TCoreO, lines 129–159)
UpperTheory.V.Th.OrdinaryMainCase (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.ThDefs, lines 53–64)
30.67. UpperTheory.V.Pa.arPl (Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.AutomorphicDetection.ArPl, lines 13–179)
UpperTheory.V.Pa.ArPl (Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.TaylorWilesObstructionBound, lines 119–124)
30.68. UpperTheory.V.Th.b1O (Development.AutomorphicArithmetic.AutomorphicVarieties.TheoremsMuLinked.GaloisRepresentationTypes.OrdinaryHeckeAlgebraIntegral, lines 31–53)
UpperTheory.V.Th.b1absO (Development.AutomorphicArithmetic.AutomorphicVarieties.TheoremsMuLinked.CyclicReductionCohomology.ResidualRepOne, lines 44–129)
UpperTheory.V.Th.oneLevelO (Development.AutomorphicArithmetic.AutomorphicVarieties.TheoremsMuLinked.CyclicReductionCohomology.OneLevelO, lines 52–134)
UpperTheory.V.Th.dualNO (Development.AutomorphicArithmetic.AutomorphicVarieties.TheoremsMuLinked.CyclicReductionCohomology.DualNO, lines 50–144)
30.69. UpperTheory.V.Th.coc (Development.AutomorphicArithmetic.AutomorphicVarieties.TheoremsMuLinked.GaloisRepresentationTypes.LiftSmul, lines 39–118)
UpperTheory.V.Th.trPsi (Development.AutomorphicArithmetic.AutomorphicVarieties.TheoremsMuLinked.GaloisRepresentationTypes.TpSingle, lines 135–177)
30.70. UpperTheory.V.Th.count (Development.AutomorphicArithmetic.AutomorphicVarieties.TheoremsMuLinked.GaloisRepresentationTypes.MapRingHomSingleMulPartialCharpoly, lines 131–248)
UpperTheory.V.Th.Count (Development.AutomorphicArithmetic.AutomorphicVarieties.TheoremsMuLinked.OrdinaryHeckeAlgebraIntegral, lines 134–142)
UpperTheory.V.Pa.Lev.OpMod (Development.AutomorphicArithmetic.AutomorphicVarieties.PatchingModules.NilpotentLevelLowering.LevelModule, lines 40–73)
30.71. UpperTheory.V.Alg.M.fin_holds (Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebra.GradedModulesLength.FinHolds, lines 49–98)
UpperTheory.V.Alg.M.rk_holds (Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebra.GradedModulesLength.LinPartApply, lines 127–200)
UpperTheory.V.Alg.M.sm_holds (Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebra.GradedModulesLength.SmHolds, lines 55–112)
UpperTheory.V.Alg.M.cut_holds (Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebra.GradedModulesLength.CutHolds, lines 101–131)
UpperTheory.V.Alg.M.nz_holds (Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebra.GradedModulesLength.NzCoeffMul, lines 33–89)
30.72. UpperTheory.V.Alg.lemmaP (Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebra.LemmaP, lines 256–424)
UpperTheory.V.Alg.M.mprime_holds (Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebra.GradedModulesLength.Mprime, lines 38–80)
UpperTheory.V.Alg.M.main_holds (Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebra.GradedModulesLength.MainCotSpan, lines 32–70)
30.73. UpperTheory.V.Alg.M.low_holds (Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebra.GradedModulesLength.AdmissibleMonoIdeal, lines 191–322)
UpperTheory.V.Alg.M.hf_holds (Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebra.GradedModulesLength.ReesGradedPieceSurjection, lines 199–270)
UpperTheory.V.Alg.M.num_holds (Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebra.GradedModulesLength.FrobeniusInclusionXEqMulQ, lines 206–283)
30.74. UpperTheory.V.Alg.M.zii_holds (Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebra.GradedModulesLength.SameDegreeZeroOfRestriction, lines 15–42)
UpperTheory.V.Alg.M.up_holds (Development.AutomorphicArithmetic.AutomorphicVarieties.CommutativeAlgebra.GradedModulesLength.TwoSigned, lines 17–95)
30.75. UpperTheory.V.X.residual_bernoulli_nonvanishing (Development.AutomorphicArithmetic.AutomorphicVarieties.LevelCohomology.ResidualBernoulliNonvanishing, lines 174–263)
UpperTheory.V.X.corX (Development.AutomorphicArithmetic.AutomorphicVarieties.LevelCohomology.ChrInvNeg, lines 197–210)
30.76. UpperTheory.V.X.lx_partial (Development.AutomorphicArithmetic.AutomorphicVarieties.LevelCohomology.LxPartial, lines 15–114)
UpperTheory.V.X.lx_core (Development.AutomorphicArithmetic.AutomorphicVarieties.LevelCohomology.LxCore, lines 65–220)
UpperTheory.V.X.lemmaX (Development.AutomorphicArithmetic.AutomorphicVarieties.LevelCohomology.LemmaX, lines 78–303)
30.76. UpperTheory.V.Z.J.count_i_of_printed_bound_k_kato_automorphy (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.LinkedUnlE, lines 49–54)
30.76. UpperTheory.V.HV.DescentVanishingReductions.countII_and_countIII_of_printsH_of_sp4In (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.AdjointLiftingOfCohomology, lines 33–39)
30.76. UpperTheory.V.Z.J.count_v_of_printed_bound_k_reduced_unitary_lift (Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.ThresholdArguments.DenseThresholdCases.LinkedUnlE, lines 55–61)

Chapter 31: The exponent 401 and the downstream window

31.1. Formalism.MainNotions.PageFix (Formalism.SharpRestriction.PageFix, lines 9–18)
GlobalTheory.EX.TOv (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.PairRowIffQdRows, lines 17–17)
GlobalTheory.EX.nWv (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.PairRowIffQdRows, lines 19–20)
GlobalTheory.EX.RankVal (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.PairRowIffQdRows, lines 28–32)
UpperTheory.Top.eventually_rank_le_rpow_of_lt_one_div_fiftytwo (Development.AutomorphicArithmetic.CharacterBoxes.EventuallyRankLeRpowOfLtOneDivFiftytwo, lines 5–126)
31.2. UpperTheory.Top.WinDatum (Development.AutomorphicArithmetic.CharacterBoxes.TwistForm, lines 19–26)
UpperTheory.Top.winPairs (Development.AutomorphicArithmetic.CharacterBoxes.WinPairs, lines 5–81)

Chapter 32: Green's classification and its use

32.1. GlobalTheory.GLq.IsCusp (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.LabelDatum, lines 64–65)
32.2. GlobalTheory.GLq.IsPrimeLab (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.LabelDatum, lines 91–91)
GlobalTheory.Published.IsRegularChar (Development.AbelianVarietyTheory.PublishedTheorems.GaloisCohomology$_2$.GreenRegularCharacters, lines 10–11)
32.3. GlobalTheory.GLq.LabelDatum (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.LabelDatum, lines 81–86)
GlobalTheory.GLq.thetaOf (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.LabelDatum, lines 88–89)
32.4. GlobalTheory.GLq.IsCuspOf (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.LabelDatum, lines 93–96)
32.5. GlobalTheory.Published.GreenRegularCharacters (Development.AbelianVarietyTheory.PublishedTheorems.GaloisCohomology$_2$.GreenRegularCharacters, lines 13–20)
32.6. GlobalTheory.GLq.Green (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.LabelDatum, lines 108–112)
32.7. GlobalTheory.GreenExists.pow_eq_self_iff (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CuspidalCharExistsOfGreen, lines 60–66)
GlobalTheory.GreenExists.rootsBar_eq (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CuspidalCharExistsOfGreen, lines 68–94)
32.8. GlobalTheory.CuspGL.TeichEmb.exists_teichmuller (Development.AbelianVarietyTheory.Proofs.CharacterTheory.ExistsTeichmullerMapAlgebraicClosureFiniteField, lines 24–76)
32.9. GlobalTheory.CuspGL.rootSum (Development.AbelianVarietyTheory.Definitions.CommutativeAlgebra.InvariantSubspaceMobiusInverseRecEqOfFinrankLt, lines 65–66)
GlobalTheory.CuspGL.rootSum_isAdditive_and_pow_X_sub_one (Development.AbelianVarietyTheory.Proofs.CharacterTheory.RootPowerSumCharacterAdditiveAndXSubOnePow, lines 9–20)
32.10. GlobalTheory.CuspGL.psi (Development.AbelianVarietyTheory.Definitions.CommutativeAlgebra.InvariantSubspaceMobiusInverseRecEqOfFinrankLt, lines 35–37)
GlobalTheory.CuspGL.psi_eq (Development.AbelianVarietyTheory.Definitions.CommutativeAlgebra.InvariantSubspaceMobiusInverseRecEqOfFinrankLt, lines 55–63)
32.11. GlobalTheory.CuspGL.psi_conj (Development.AbelianVarietyTheory.Proofs.CharacterTheory.InvariantSubspaceMobiusInverseCharpolyLinearEqui, lines 43–51)
GlobalTheory.CuspGL.isPrimitive_psi (Development.AbelianVarietyTheory.Proofs.CharacterTheory.IsPrimitiveEndClassFunctionInvariantSubspaceMobi, lines 188–199)
32.12. GlobalTheory.CuspGL.psi_one (Development.AbelianVarietyTheory.Proofs.CharacterTheory.InvariantSubspaceMobiusInverseOneProdIco, lines 74–77)
GlobalTheory.CuspGL.psi_of_irreducible_charpoly (Development.AbelianVarietyTheory.Proofs.CharacterTheory.InvariantSubspaceMobiusInverseCharpolyOfIrreduci, lines 52–56)
32.13. GlobalTheory.CuspGL.rootSum_charpoly_mem_virt (Development.AbelianVarietyTheory.Proofs.CharacterTheory.RootPowerSumCharpolyMemVirtualCharactersGL, lines 213–233)
GlobalTheory.CuspGL.psi_rootSum_charpoly_mem_virt (Development.AbelianVarietyTheory.Proofs.CharacterTheory.IsCharacterCuspidalRootSumRegular, lines 55–88)
32.14. GlobalTheory.CuspGL.mass_psi_eq_glCard_mul_mangoldt_div (Development.AbelianVarietyTheory.Proofs.CharacterTheory.EndomorphismMassMobiusInverseMonicEq, lines 27–52)
32.15. GlobalTheory.CuspGL.ipGL_rootSum_eq_sum_torus (Development.AbelianVarietyTheory.Proofs.CharacterTheory.InvertibleEndInnerProductRootPowerSumEqTorusSum, lines 12–24)
GlobalTheory.CuspGL.torus_orthogonality (Development.AbelianVarietyTheory.Proofs.CharacterTheory.FrobeniusTorusCharacterOrthogonality, lines 66–92)
32.16. GlobalTheory.CuspGL.ipGL_psi_rootSum (Development.AbelianVarietyTheory.Proofs.CharacterTheory.InvertibleEndInnerProductRootPowerSumMobius, lines 30–96)
32.17. GlobalTheory.CuspGL.expand_primitive (Development.AbelianVarietyTheory.Proofs.CharacterTheory.PrimitiveClassFunctionExpansionRootSums, lines 156–220)
32.18. GlobalTheory.CuspGL.PrimOrth.ipGL_psi_eq (Development.AbelianVarietyTheory.Proofs.CharacterTheory.InvertibleEndomorphismInnerProductPrimitiveMobiu, lines 92–161)

32.19. GlobalTheory.CuspGL.isChar_psi_rootSum_of_regular (Development.AbelianVarietyTheory.Proofs.CharacterTheory.IsCharacterCuspidalRootSumRegular, lines 92–184)
32.20. GlobalTheory.ParamLabel.isCusp_psi (Development.AbelianVarietyTheory.Proofs.DivisorProofs.GreenCuspidalCharIsCuspidalOfMinpolyOfParam, lines 18–48)
32.21. GlobalTheory.GreenRegular.exists_cusp_of_regular (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsCuspidalCharacterOfRegularCharacterGLn, lines 65–225)
32.22. GlobalTheory.PrimeLabOrbit.primeLab_to_regularChar (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.MonicIrreducibleInjectRegularCharacterOrbits, lines 125–151)
32.23. GlobalTheory.GreenRegularCharactersInertiaEigenvalues.family (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.GreenCuspidalCharsByRegularCharsProof, lines 23–33)
32.24. GlobalTheory.GLq.GenericConstituent.exists_generic_constituent (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsGenericConstituentHarishChandraCuspidal, lines 98–125)
32.25. GlobalTheory.GLq.IpCuspProd.ip_hcL_cusp (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CharacterPairingHarishChandraCuspidalMackey, lines 259–263)
32.26. GlobalTheory.GLq.GGNorm.gg_norm (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.GelfandGraevCharacterGLnNormAndReciprocity, lines 88–91)
32.27. GlobalTheory.GLq.card_generic_irr_le (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.NcardGenericIrreducibleCharactersGLnLe, lines 49–88)
32.28. GlobalTheory.GLq.CuspMultisetCount.card_le (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CardCuspidalMultisetsTotalDegreeLe, lines 30–61)
32.29. GlobalTheory.GrPolys.card_monic_coeff_zero_ne_zero (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.NcardMonicNatDegreeCoeffZeroNeZero, lines 38–63)
32.30. GlobalTheory.GreenRegularCharactersInertiaEigenvalues.count (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.GreenCuspidalCharsByRegularCharsProof, lines 56–131)
32.31. GlobalTheory.greenRegularCharacters_holds (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.GreenCuspidalCharsByRegularCharsProof, lines 137–168)
32.32. GlobalTheory.GreenBridge.exists_emb_eq (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SumRangeEqRoots, lines 32–39)
GlobalTheory.GreenBridge.exists_unit_gen (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SumRangeEqRoots, lines 53–68)
32.33. GlobalTheory.GreenBridge.th_apply (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SumRangeEqRoots, lines 122–124)
GlobalTheory.GreenBridge.th_inj (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SumRangeEqRoots, lines 142–146)
GlobalTheory.GreenBridge.th_surj (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SumRangeEqRoots, lines 148–160)
32.34. GlobalTheory.GreenBridge.th_pow (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SumRangeEqRoots, lines 137–140)
GlobalTheory.GreenBridge.regular_iff (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SumRangeEqRoots, lines 207–211)
32.35. GlobalTheory.GreenBridge.rootOf_spec (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SumRangeEqRoots, lines 189–195)
GlobalTheory.GreenBridge.root_unit (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SumRangeEqRoots, lines 197–205)
GlobalTheory.GreenBridge.pow_inj (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SumRangeEqRoots, lines 180–185)
32.36. GlobalTheory.GreenBridge.sum_range_eq_roots (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SumRangeEqRoots, lines 213–216)
GlobalTheory.GreenBridge.thetaOf_pow (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SumRangeEqRoots, lines 218–223)
32.37. GlobalTheory.GreenBridge.green_of_regular (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SumRangeEqRoots, lines 225–304)
32.37. UpperTheory.LiteratureTheorems.green_regular_characters_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 399–400)
UpperTheory.LiteratureTheorems.published_results_eigen (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 666–726)
UpperTheory.Lit.every_character_of_theta_integrals_local_matching (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.SiegelEisensteinSecondTermHypotheses, lines 69–73)
32.38. GlobalTheory.GLm.card_proper_subfield_elements_lt (Development.AbelianVarietyTheory.Proofs.FlagMeasures.CardProperSubfieldElementsLt, lines 67–102)
32.39. GlobalTheory.GLq.LabelUnique.exists_eq_of_sum_eq (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CuspidalCharacterLabelUniqueGLn, lines 28–109)
32.40. GlobalTheory.GLq.LabelUnique.label_eq (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CuspidalCharacterLabelUniqueGLn, lines 174–257)
GlobalTheory.GLq.isCuspOf_label_unique (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CuspidalCharacterLabelUniqueGLn, lines 263–273)
32.41. GlobalTheory.GLq.HasLabel (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.LabelDatum, lines 98–100)

GlobalTheory.GLq.eSerC (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.LabelDatum, lines 102–103)
GlobalTheory.GLq.wtC (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.LabelDatum, lines 105–106)
32.42. GlobalTheory.GLq.Label.label_main (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.HarishChandraSeriesLabelExistsUniqueGLn, lines 271–278)
GlobalTheory.gl_list_product_perm_char (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.HarishChandraProductListIsCharacterAndPermInvari, lines 74–77)
32.43. GlobalTheory.Top.Lq.exists_perm (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.AdmissibleLabelWeightsOfGreenCuspidalParametrisa, lines 89–100)
GlobalTheory.Top.Lq.ext_back (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.AdmissibleLabelWeightsOfGreenCuspidalParametrisa, lines 129–133)
GlobalTheory.Top.Lq.eSer_eq (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.AdmissibleLabelWeightsOfGreenCuspidalParametrisa, lines 233–249)
32.44. GlobalTheory.T.whittakerJacquetIdempotents_properties_primeField (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WhittakerJacquetIdempotentsPropertiesPrimeFieldStatement, lines 8–9)
GlobalTheory.T.harishChandraSeriesWeight_nonneg_of_idempotent_glnPrime (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.HarishChandraSeriesWeightNonnegOfIdempotentGlnPrStatement, lines 8–9)
32.45. GlobalTheory.GLq.wh_cusp_eq_one (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WhittakerDimensionCuspidalGreenDegreeEqOne, lines 66–79)
32.46. GlobalTheory.GLq.ExactMult.wh_hc (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TraceUnitriangularColumnVanishingIdempotentEqSum, lines 54–61)
32.47. ProductIntervalSplit.CompanionTheory.Member.harishChandraSeriesWeight_squarefree_label_unipotent (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.HarishChandraSeriesWeightSquarefreeLabelGelfandG, lines 238–293)
32.48. GlobalTheory.GLq.tr_eJ_eq_tr_eJ' (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.GroupAlgebraTracePatternIdempotentExchange, lines 13–27)
GlobalTheory.RootExch.step (Development.AbelianVarietyTheory.Proofs.FlagMeasures.PhIUnionTrans, lines 332–354)
32.49. GlobalTheory.GLq.ExactMult.tr_eJ'_eq_sum_wh_der (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TraceUnitriangularColumnVanishingIdempotentEqSum, lines 69–92)
GlobalTheory.GLm.tr_eJ'_eq_wh_tail (Development.AbelianVarietyTheory.Proofs.FlagMeasures.IsUnitOfIsUniUp, lines 225–276)
GlobalTheory.GLm.MirSum.eq_sum_indR_der (Development.AbelianVarietyTheory.Proofs.FlagMeasures.MirabolicConjInvariantEqSumInducedDerivatives, lines 110–133)
32.50. GlobalTheory.GLq.lowerWtComparison (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.HarishChandraSeriesWeightLowerComparison, lines 49–98)
32.51. GlobalTheory.GLq.Leibniz.main (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.BernsteinZelevinskyDerivativeHarishChandraProduct, lines 102–142)
32.52. GlobalTheory.GLq.UpperWeight.comb_bound (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.HarishChandraSeriesWeightLeSquarefree, lines 30–80)
GlobalTheory.GLq.UpperWeight.wtC_eJ_le (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.HarishChandraSeriesWeightLeSquarefree, lines 88–139)
32.53. GlobalTheory.Top.Lq.admissibleLabelWeights_of_greenCuspidalParametrisation (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.AdmissibleLabelWeightsOfGreenCuspidalParametrisa, lines 255–299)
Formalism.MainNotions.ArithmeticCylinders.admLq (Formalism.SharpRestriction.LevelLetterCocycles.ArithmeticCylinders, lines 340–351)
Formalism.L2.GlobalTheoryTheory.ConcNWeight.sum_wtOf (Formalism.LocalToGlobal.GlobalTheory.ConcNWeight, lines 74–85)
32.54. Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.type_set_weight_bound (Formalism.PadicSubsetSums.StableGradedCounts.LocalEstimateIVOfTypeSetWeight, lines 1851–1892)
Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.type_weight_exact (Formalism.PadicSubsetSums.StableGradedCounts.LocalEstimateIVOfTypeSetWeight, lines 1282–1314)
Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.divisor_square_weight_sum_le (Formalism.PadicSubsetSums.StableGradedCounts.LocalEstimateIVOfTypeSetWeight, lines 1908–1952)
Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.square_mass_lower_bound (Formalism.PadicSubsetSums.StableGradedCounts.LocalEstimateIVOfTypeSetWeight, lines 1243–1267)
32.55. Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.local_estimate_fourth (Formalism.PadicSubsetSums.StableGradedCounts.LocalEstimateIVOfTypeSetWeight, lines 2451–2452)
Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.divisor_degree_subsum (Formalism.PadicSubsetSums.StableGradedCounts.IRRDivisorDegreeSubSum, lines 14–34)
Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.prodMaxInvLeSq (Formalism.PadicSubsetSums.StableGradedCounts.IRRLemmaP, lines 133–213)
32.56. Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.prime_reciprocal_sum_bound (Formalism.PadicSubsetSums.StableGradedCounts.IRRPrimeRecipSum, lines 333–339)
Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.PrimeCountingLemmas.S_small (Formalism.PadicSubsetSums.StableGradedCounts.IRRPrimeRecipSum, lines 122–127)
Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.PrimeCountingLemmas.upper_card (Formalism.PadicSubsetSums.StableGradedCounts.IRRPrimeRecipSum, lines 211–269)
32.57. Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.local_estimate_fifth (Formalism.PadicSubsetSums.StableGradedCounts.LocalEstimateIVOfTypeSetWeight, lines 2454–2455)

Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.prime_cycle_perm_count (Formalism.PadicSubsetSums.StableGradedCounts.IRRCycleProbPrime, lines 32–111)
Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.prime_cycle_probability_sum (Formalism.PadicSubsetSums.StableGradedCounts.IRRCycleProbSum, lines 6–6)
32.58. Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.local_estimate_second (Formalism.PadicSubsetSums.StableGradedCounts.SquareSetMemExceptionalUnion, lines 429–583)
Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.same_factor_count (Formalism.PadicSubsetSums.StableGradedCounts.CoeffZeroNeZeroOfIrreducible, lines 152–287)
Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.cross_factor_count (Formalism.PadicSubsetSums.StableGradedCounts.CrossFactorCount, lines 51–102)
Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.local_estimate_third (Formalism.PadicSubsetSums.StableGradedCounts.LocalEstimateThird, lines 354–383)
32.59. Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.square_mass_lower_bound (Formalism.PadicSubsetSums.StableGradedCounts.LocalEstimateIVOfTypeSetWeight, lines 1243–1267)
Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.local_estimate_second (Formalism.PadicSubsetSums.StableGradedCounts.SquareSetMemExceptionalUnion, lines 429–583)
Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.local_estimate_third (Formalism.PadicSubsetSums.StableGradedCounts.LocalEstimateThird, lines 354–383)
Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.localEstimateIV_of (Formalism.PadicSubsetSums.StableGradedCounts.LocalEstimateIVOfTypeSetWeight, lines 1590–1607)
Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.localEstimateV_of (Formalism.PadicSubsetSums.StableGradedCounts.LocalEstimateIVOfTypeSetWeight, lines 1609–1679)
Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.prodMaxInvLeSq (Formalism.PadicSubsetSums.StableGradedCounts.IRRLemmaP, lines 133–213)
Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.prime_reciprocal_sum_bound (Formalism.PadicSubsetSums.StableGradedCounts.IRRPrimeRecipSum, lines 333–339)
32.59. Formalism.CyclotomicNorms.IsGood (Formalism.CyclotomicNorms.StatementT, lines 19–29)
Formalism.MainNotions.ArithmeticCylinders.goodAt (Formalism.SharpRestriction.LevelLetterCocycles.ArithmeticCylinders, lines 355–358)
32.60. Formalism.L2.GlobalTheoryTheory.ConcNWeight.weightAllRows_of (Formalism.LocalToGlobal.GlobalTheory.ConcNWeight, lines 345–355)
Formalism.L2.GlobalTheoryTheory.AllJoin.weight_of_join (Formalism.LocalToGlobal.HilbertRowConditions.AllJoinP, lines 22–24)
Formalism.IrreducibleCountingLemmas.GaloisIrreducibility.property_W_of_estimates (Formalism.PadicSubsetSums.StableGradedCounts.PropertyWOfEstimates, lines 7–7)
Formalism.MainNotions.N.PlaceLabelTeichmuller.good_of (Formalism.SharpRestriction.LevelLetterCocycles.PlaceLabelTeichmuller, lines 660–681)
32.61. Formalism.L2.GlobalTheoryTheory.AllJoin.AllJoinA (Formalism.LocalToGlobal.HilbertRowConditions.AllJoin, lines 18–22)
Formalism.MainNotions.PlaceDegreeData.RowsTie (Formalism.SharpRestriction.CuspSpanNormBounds.PlaceDegreeData, lines 380–382)
Formalism.MainNotions.S.AutTie (Formalism.SharpRestriction.CuspSpanNormBounds.DstarEq, lines 10–22)
Formalism.L2.GlobalTheoryTheory.AllJoin.LevelSharp (Formalism.LocalToGlobal.HilbertRowConditions.AllJoin, lines 14–16)
32.62. GlobalTheory.Top.Irr.linkIRR_of_join_ex (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.LinkIRROfJoinEx, lines 17–32)
Formalism.L2.GlobalTheoryTheory.ConcN.CountAllRows (Formalism.LocalToGlobal.GlobalTheory.IrreducibleOfCountSizes, lines 183–186)
Formalism.MainNotions.ArithmeticCylinders.AllRows (Formalism.SharpRestriction.LevelLetterCocycles.ArithmeticCylinders, lines 244–260)
Formalism.MainNotions.ArithmeticCylinders.LabelCountGoodInjective (Formalism.SharpRestriction.LevelLetterCocycles.ArithmeticCylinders, lines 271–274)
32.63. GlobalTheory.Top.Irr.closed_linkIRR_of_join_count (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LinkIrr, lines 22–43)
GlobalTheory.Top.Irr.sizesRec_at (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.LinkIRROfJoinEx, lines 39–137)
32.64. UpperTheory.Top.shape_closed_link_n_of_green (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.TopShape, lines 33–55)

Chapter 33: A common witness for the 46 conditions

Chapter introduction. GlobalTheory.Cp.D3 (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.CountsIrreducibilityConstants, lines 35–35)
Formalism.MainNotions.Des (Formalism.SharpRestriction.SmoothDesignExistence.DesignExistsDes, lines 10–11)
Formalism.MainNotions.Pk.DGamSmooth (Formalism.SharpRestriction.SmoothDesignExistence.DGamSmooth, lines 23–25)

33.1. `GlobalTheory.Mem.Top.Bl.anyGeometry` (`Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.OverIx`, lines 19–19)
`GlobalTheory.Mem.Top.Bl.GeoCl` (`Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.NotGeoClFalse`, lines 21–24)
`GlobalTheory.Mem.Top.Bl.prm0` (`Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.ParMem`, lines 137–137)
`GlobalTheory.Mem.Top.Bl.prm1` (`Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.MultW`, lines 44–45)
`GlobalTheory.Mem.Top.Bl.parametersOfDependence` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.Prm1`, lines 20–21)
`GlobalTheory.Mem.Top.Bl.parametersOfExtension` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CoreAltAltBis`, lines 61–61)
33.1. `UpperTheory.Lit.nonempty_hyps_definite_side_of_closed_bound` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.SiegelEisensteinSecondTermHypotheses`, lines 44–68)
33.2. `Formalism.BasicNotions.NTw` (`Formalism.SharpRestriction.LevelLetterCocycles.DataN`, lines 256–427)
`Formalism.MainNotions.rPrAv` (`Formalism.SharpRestriction.LevelLetterCocycles.TwGood`, lines 114–116)
`Formalism.MainNotions.q0Av` (`Formalism.SharpRestriction.LevelLetterCocycles.TwGood`, lines 118–118)
`Formalism.MainNotions.avPrimes` (`Formalism.SharpRestriction.LevelLetterCocycles.TwGood`, lines 120–124)
`Formalism.MainNotions.mem_avRec` (`Formalism.SharpRestriction.LevelLetterCocycles.TwGood`, lines 146–147)
33.2. `Formalism.BasicNotions.G0g` (`Formalism.SharpRestriction.LevelLetterCocycles.DataN`, lines 761–762)
`Formalism.BasicNotions.NExp` (`Formalism.SharpRestriction.LevelLetterCocycles.DataN`, lines 769–853)
`Formalism.BasicNotions.ThData` (`Formalism.LocalRingCardinality.DataNChTie`, lines 37–40)
`Formalism.MainNotions.IntegralHeckeForms.A0` (`Formalism.SharpRestriction.LevelLetterCocycles.IntegralHeckeForms`, lines 23–24)
`Formalism.MainNotions.IntegralHeckeForms.V0` (`Formalism.SharpRestriction.LevelLetterCocycles.IntegralHeckeForms`, lines 36–37)
`Formalism.MainNotions.IntegralHeckeForms.eTh` (`Formalism.SharpRestriction.LevelLetterCocycles.IntegralHeckeForms`, lines 52–53)
`Formalism.MainNotions.IntegralHeckeForms.eThPsi` (`Formalism.SharpRestriction.LevelLetterCocycles.IntegralHeckeForms`, lines 55–58)
`Formalism.MainNotions.IntegralHeckeForms.eThPMt` (`Formalism.SharpRestriction.LevelLetterCocycles.IntegralHeckeForms`, lines 60–62)
33.2. `Formalism.MainNotions.IntegralHeckeForms.sIdx` (`Formalism.SharpRestriction.LevelLetterCocycles.IntegralHeckeForms`, lines 68–70)
`Formalism.MainNotions.IntegralHeckeForms.IdxAt` (`Formalism.SharpRestriction.LevelLetterCocycles.IntegralHeckeForms`, lines 201–211)
33.2. `Formalism.MainNotions.BlochKatoReduction.Gen.thEval` (`Formalism.SharpRestriction.LevelLetterCocycles.BlochKatoPhases`, lines 211–214)
`Formalism.MainNotions.BlochKatoReduction.Gen.c0` (`Formalism.SharpRestriction.LevelLetterCocycles.BlochKatoPhases`, lines 224–224)
`Formalism.MainNotions.BlochKatoReduction.Gen.tameVal` (`Formalism.SharpRestriction.LevelLetterCocycles.BlochKatoPhases`, lines 226–228)
`GlobalTheory.LamTie.sum_th_empty_ne_zero` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SumThEmptyNeZero`, lines 10–23)
33.3. `GlobalTheory.Cp.Duty` (`Development.AbelianVarietyTheory.Definitions.GlobalTwisting.CountsIrreducibilityConstants`, lines 48–49)
`Formalism.MainNotions.PageFix` (`Formalism.SharpRestriction.PageFix`, lines 9–18)
`Formalism.MainNotions.PageWin` (`Formalism.SharpRestriction.PageFix`, lines 20–25)
`Formalism.MainNotions.TameDistinct` (`Formalism.SharpRestriction.LevelLetterCocycles.AlignN`, lines 14–19)
`Formalism.MainNotions.TwGoodAtAll` (`Formalism.SharpRestriction.LevelLetterCocycles.TwAllF`, lines 22–23)
`Formalism.MainNotions.IntegralHeckeForms.GoodTw` (`Formalism.SharpRestriction.LevelLetterCocycles.IntegralHeckeForms`, lines 219–231)
`Formalism.MainNotions.BlochKatoReduction.Gen.goodE` (`Formalism.SharpRestriction.LevelLetterCocycles.BlochKatoPhases`, lines 100–108)
`Formalism.MainNotions.BlochKatoReduction.Gen.bk` (`Formalism.SharpRestriction.LevelLetterCocycles.BlochKatoPhases`, lines 167–179)
`Formalism.MainNotions.BlochKatoReduction.Gen.selT` (`Formalism.SharpRestriction.LevelLetterCocycles.BlochKatoPhases`, lines 184–187)
`Formalism.MainNotions.BlochKatoReduction.Gen.sLen` (`Formalism.SharpRestriction.LevelLetterCocycles.BlochKatoPhases`, lines 193–195)
`Formalism.MainNotions.BlochKatoReduction.Gen.partAT` (`Formalism.SharpRestriction.LevelLetterCocycles.BlochKatoPhases`, lines 304–308)
`Formalism.MainNotions.BlochKatoReduction.Gen.partC` (`Formalism.SharpRestriction.LevelLetterCocycles.BlochKatoPhases`, lines 317–321)
`Formalism.MainNotions.BlochKatoReduction.Gen.partD` (`Formalism.SharpRestriction.LevelLetterCocycles.BlochKatoPhases`, lines 323–325)
33.4. `Formalism.MainNotions.ThickSetEventually` (`Formalism.SharpRestriction.GlobalSchurBounds.Chosen`, lines 65–65)
`Formalism.BasicNotions.Host.secondSet_card_lower_bound` (`Formalism.LocalRingCardinality.DataHost`, lines 165–166)
`Formalism.MainNotions.ChTT_at` (`Formalism.SharpRestriction.GlobalSchurBounds.Chosen`, lines 18–19)

Formalism.MainNotions.ChBad_at (Formalism.SharpRestriction.GlobalSchurBounds.Chosen, lines 34–36)
Formalism.MainNotions.ChQD_at (Formalism.SharpRestriction.GlobalSchurBounds.Chosen, lines 42–45)
Formalism.MainNotions.ChCode_at (Formalism.SharpRestriction.GlobalSchurBounds.Chosen, lines 56–57)
Formalism.MainNotions.CodeUpF (Formalism.SharpRestriction.DesignExistsOfChoices, lines 18–20)
33.5. Formalism.CyclotomicNorms.IsGood (Formalism.CyclotomicNorms.StatementT, lines 19–29)
Formalism.BasicNotions.NData.goodRow (Formalism.SharpRestriction.LevelLetterCocycles.DataN, lines 234–237)
Formalism.BasicNotions.NTypes.goodTuple (Formalism.SharpRestriction.LevelLetterCocycles.DataN, lines 996–999)
33.6. GlobalTheory.Fc.BundleCondition (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ColumnChoiceWithTies, lines 47–54)
GlobalTheory.Fc.familySelection (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ColumnChoiceWithTies, lines 56–56)
Formalism.MainNotions.Closed (Formalism.SharpRestriction.ParamsHeightRankConstant, lines 159–161)
33.6. GlobalTheory.Fc.R1.SelectionDefiniteRows (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CountsIrreducibilityConstants, lines 91–93)
33.6. GlobalTheory.Fc.HypsAlt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.Hyps, lines 22–90)
UpperTheory.Top.FReq.cls (Development.AutomorphicArithmetic.CharacterBoxes.Supply, lines 22–23)
UpperTheory.Top.ExS (Development.AutomorphicArithmetic.CharacterBoxes.ExS, lines 10–12)
GlobalTheory.EX.HostExW (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.ThinScalesIffHostBound, lines 120–123)
Formalism.MainNotions.MinkMult (Formalism.SharpRestriction.GlobalSchurBounds.MainIn, lines 51–52)
Formalism.MainNotions.DesignChoices (Formalism.SharpRestriction.DesignExistsOfChoices, lines 22–26)
Formalism.MainNotions.TwGoodE (Formalism.SharpRestriction.LevelLetterCocycles.TwGood, lines 219–223)
Formalism.MainNotions.TwGoodAll (Formalism.SharpRestriction.LevelLetterCocycles.ExistsGoodPlacesEveryConjugate, lines 38–40)
Formalism.IndefR.Links.PrimRows3 (Formalism.LocalToGlobal.IndefiniteReal.BaseLiftsPackageOfPrimitiveLinks, lines 199–222)
Formalism.L2.Rows.RowsHoldLam3 (Formalism.LocalToGlobal.HilbertRowConditions.ColumnsFunctionAtSubspaces, lines 129–139)
Formalism.MainNotions.TuSize (Formalism.SharpRestriction.ParamsHeightRankConstant, lines 180–182)
Formalism.BasicNotions.SizeData₂.TuGe (Formalism.LocalRingCardinality.DataParams₂, lines 252–252)
Formalism.MainNotions.StandardLattice.ClassNumberFormula₂ (Formalism.SharpRestriction.GlobalSchurBounds.StandardLattice, lines 453–456)
Formalism.L2.SG.JoinSI.IndefiniteRealisationJoin (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ColJoinSOfSTiesR, lines 67–81)
Formalism.L2.SG.JoinSI.ColNormT (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.JoinSI, lines 62–63)
GlobalTheory.genusMassAt_two (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.GenusMassAtTwo, lines 7–8)
Formalism.MainNotions.FamP.Large (Formalism.SharpRestriction.ParamsHeightRankConstant, lines 138–140)
GlobalTheory.Cp.SelectionServed (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CountsIrreducibilityConstants, lines 48–62)
Formalism.L2.DIV.lPrj (Formalism.LocalToGlobal.Divisibility.RowSideTie, lines 96–98)
Formalism.L2.DIV.BlockDat.lPr (Formalism.LocalToGlobal.Divisibility.RowLat, lines 148–149)
Formalism.MainNotions.StandardLattice.VolTie (Formalism.SharpRestriction.GlobalSchurBounds.StandardLattice, lines 258–263)
33.7. Formalism.MainNotions.S.StandingNumericalData (Formalism.SharpRestriction.CuspSpanNormBounds.StandingNumericalData, lines 84–104)
Formalism.MainNotions.S.StandingHypotheses (Formalism.SharpRestriction.CuspSpanNormBounds.StandingNumericalData, lines 106–136)
Formalism.MainNotions.S.Cnt.Hyp (Formalism.SharpRestriction.CuspSpanNormBounds.CntPartners, lines 36–48)
33.8. Formalism.MainNotions.count_hypotheses_of_sizes (Formalism.SharpRestriction.CuspSpanNormBounds.ThresholdCubicInequalitySignPattern, lines 230–236)
Formalism.MainNotions.count_hypotheses_diagonal_words (Formalism.SharpRestriction.CuspSpanNormBounds.ThresholdCubicInequalitySignPattern, lines 1254–1263)
33.9. GlobalTheory.Mass.sublattice_count (Development.AbelianVarietyTheory.Proofs.FlagMeasures.EncardSublatticesOfIndexLe, lines 10–40)
GlobalTheory.Mass.SpaceMass.main (Development.AbelianVarietyTheory.Proofs.FlagMeasures.MassIntegralLatticesDiscriminantOrderLe, lines 79–156)
GlobalTheory.Mass.MaxFormula.main (Development.AbelianVarietyTheory.Proofs.FlagMeasures.MaximalHermitianLatticeMassRatio, lines 111–117)
GlobalTheory.MaxLatStruct.main (Development.AbelianVarietyTheory.Proofs.FlagMeasures.MaximalHermitianLatticeLocalStructureDiscriminan, lines 511–550)
GlobalTheory.MaxLatLam.main (Development.AbelianVarietyTheory.Proofs.FlagMeasures.MaximalLatticeLocalFactorProductPosAndLe, lines 25–35)
33.10. GlobalTheory.Mass.PrimMean.main (Development.AbelianVarietyTheory.Proofs.FlagMeasures.PrimitiveSublatticeMeanCountBound, lines 139–214)
GlobalTheory.Mass.FewOrbits.main (Development.AbelianVarietyTheory.Proofs.FlagMeasures.FewOrbitsSubspacesHermitianLatticeDiscriminant, lines 40–110)

GlobalTheory.Mass.PrimDisc.main (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
PrimitiveHermitianSublatticeDiscriminantGenerato, lines 150–190)
GlobalTheory.Mass.Glue.main (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
CardHermitianLatticesGluedSubspaceComplement, lines 210–276)
GlobalTheory.Mass.FixedSub.orbMass_stabilizer_le (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
OrbitStabilizerMassSubspaceCyclicDiscriminantLe, lines 100–210)
33.11. GlobalTheory.SmallRankMass.mass_small (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
ExistsBoundedBasis, lines 205–254)
GlobalTheory.ClassCount.exists_finset_classes (Development.AbelianVarietyTheory.Proofs.CuspFormEvaluations.
ExistsFinsetSelfDualClassesHerm, lines 399–441)
GlobalTheory.EtaValueBound.lValEta_bounds (Development.AbelianVarietyTheory.Proofs.FlagMeasures.LValEtaBounds, lines
61–70)
GlobalTheory.MassRatioBound.mass_mul_mass_le (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
UnimodularHermitianMassMulLeMass, lines 131–184)
33.12. GlobalTheory.Mass.CyclicDisc.main (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
MassGenusCyclicDiscriminantSplitPrime, lines 119–181)
GlobalTheory.DegCnt.PrimGenus.main (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
MeanPrimitiveSublatticesGenusDefiniteLattice, lines 158–198)
33.13. GlobalTheory.HermLat.IsModIso (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.
FlagPairDefinitePartLattice, lines 115–119)
GlobalTheory.DegCnt.IsDat (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.
NormalFormDualOrthogonalLast, lines 127–130)
GlobalTheory.DegCnt.Trip (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.NormalFormDualOrthogonalLast,
lines 132–135)
33.14. GlobalTheory.DegCnt.FixedSpaceCount.main (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
CountIsometryPairsWithGivenFixedSpace, lines 232–313)
GlobalTheory.DegCnt.FixDisc.main (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
DiscriminantOrderCommonFixedLatticeEmbeddedData, lines 372–408)
GlobalTheory.TraceCount.main (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
EncardSublatticesIndexFixedTraceLe, lines 153–216)
GlobalTheory.DegCnt.DatCount.main (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
CardEmbeddedDataWithGivenFixedLattice, lines 207–299)
GlobalTheory.IsoCount.isoCnt (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
CardIsometriesFixingSubspaceDefiniteLatticesLe, lines 115–151)
GlobalTheory.PairNoFix.main (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
PairsIsometryNoFixedVectorCountBound, lines 58–110)
33.15. GlobalTheory.Amp.dyadic_summation (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
DyadicSummationWithDecay, lines 104–239)
33.16. GlobalTheory.TwoSidedMomentsGap.moment_of_bessel_gap (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
MajorantLengthTwoSidedMomentsLargeGap, lines 57–127)
GlobalTheory.TwoSidedMomentsGap.inv_moment_of_bessel_gap (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
MajorantLengthTwoSidedMomentsLargeGap, lines 129–198)
GlobalTheory.InvLenMomentGap.inv_len_moment_gap (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
InverseMajorantLengthMomentLargeGap, lines 24–76)
33.17. GlobalTheory.DegCnt.Absorb.exists_absorbing_ideal (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
ExistsStarStableIdealAbsorbingMinors, lines 471–598)
GlobalTheory.DegCnt.Absorb.exists_ideal (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
ExistsStarStableIdealAbsorbingMinors, lines 355–402)
33.18. GlobalTheory.rowsPos_typeP (Development.AbelianVarietyTheory.Proofs.FlagMeasures.RowsPosTypeP, lines 9–10)
Formalism.L2.SG.RowsPos.TypeP (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RowsPos, lines
18–22)
Formalism.L2.SG.KappaOffDiagonal.TypeAt (Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.
RankPointCounts.DimFormC, lines 40–45)
GlobalTheory.TypeAsm.typeBound_of_amplitude_pairs (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
TermAmplitudeFactorNonneg, lines 22–66)
GlobalTheory.T.bessel_multiplicity_one_large_gap_and_whittaker_bessel (Development.AbelianVarietyTheory.Proofs.
FlagMeasures.BesselMultiplicityOneLargeGapAndWhittakerBesselStatement, lines 6–7)
GlobalTheory.GapIn.hyp_of_uniqGap (Development.AbelianVarietyTheory.Proofs.FlagMeasures.HypOfUniqGap, lines 9–11)
33.19. GlobalTheory.DegAsm.sum_weights_per_datum (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
FlagFrameVectorMemCommonSubspace, lines 204–270)
GlobalTheory.TypeAsm.perRowC (Development.AbelianVarietyTheory.Proofs.FlagMeasures.
TermPairsRowLatticeCountedViaEmbeddings, lines 114–211)
33.20. GlobalTheory.DegCnt.PairMean.main (Development.AbelianVarietyTheory.Proofs.FlagMeasures.DiscNResNeZero, lines
312–428)
33.21. WlocSk.ANz.aNonzero_of (Formalism.LocalToGlobal.SiegelLatticeTheory.HaarTransversals.UnitaryHaarProducts.
LocalWeightCharts.ANonzeroOf, lines 34–147)
WlocSk.AQ.wave_packet_l2_finite (Formalism.LocalToGlobal.SiegelLatticeTheory.HaarTransversals.UnitaryHaarProducts.
LocalWeightCharts.WavePacketL1L2OfHypotheses, lines 397–478)

WlocSk.scWlocShape_of (Formalism.LocalToGlobal.SiegelLatticeTheory.HaarTransversals.UnitaryHaarProducts.LocalWeightCharts.L2SGWfEqOne, lines 87–124)
33.22. Formalism.MainNotions.BridgeCS3dBOW.bow_core (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.BesselOfWPf, lines 132–221)
Formalism.FourierDecayAlt.ThreeDimData.DDForm (Formalism.HermitianCurvature.BesselNonvanishingReduction, lines 60–74)
33.23. CompanionTheory.Lime.BM.UniqGap (CompanionTheory.AutomorphicTheory.LocalBesselUniqueness.BesselModels.AtFormAdmOneOfSteps, lines 15–26)
CompanionTheory.Bmu.besselMapUniqGap (CompanionTheory.AutomorphicTheory.BesselOrbitBounds.BesselMapUniqueness.BesselMapUniqGapAlt, lines 33–34)
33.24. ReciprocityTheory.ChartRestriction.DS.Lap.TopClosed.besselNVG (ReciprocityTheory.ReductiveGroups.DiscreteSeries.LaplaceOperators.BesselVectors.Closed, lines 10–10)
Formalism.L2.SG.OwnFacts.factBesselNVGap (Formalism.LocalToGlobal.SiegelLatticeTheory.WeightedGaugeFactors.FactsDS, lines 18–18)
Formalism.L2.SG.Pf.wuDecayOfGap_of_uniqGap (CompanionTheory.AutomorphicTheory.LocalBesselUniqueness.BesselModels.WuFinal, lines 9–11)
CompanionTheory.Lime.BM.uniqGap_holds (CompanionTheory.AutomorphicTheory.LocalBesselUniqueness.BesselModels.MultiplicityOneUnitaryBranchingOfGapHyp, lines 8–8)
CompanionTheory.Lime.BM.wuDecayOfGap_holds' (CompanionTheory.AutomorphicTheory.LocalBesselUniqueness.BesselModels.MultiplicityOneUnitaryBranchingOfGapHyp, lines 12–12)
Formalism.L2.SG.Pf.WuDec.LocDecayStep (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.WuDecayAll, lines 433–435)
33.25. GlobalTheory.wuDecayOfGap (Development.AbelianVarietyTheory.Proofs.FlagMeasures.WuDecayOfGap, lines 7–7)
Formalism.L2.SG.Pf.WuDec.wuDecayOfGap_of_steps (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.WuDecayAll, lines 727–733)
33.26. GlobalTheory.Fc.sevenDuties_of_class (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.ValidForConstructedPointClassesEventuallyWorking, lines 245–246)
GlobalTheory.EX.DQ.duty_seven_library (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.DutySevenLibrary, lines 15–23)
33.26. GlobalTheory.EX.DC.cls (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.AuxiliaryPrimeMemAvoidedPrimes, lines 106–107)
GlobalTheory.EX.DQ.weightAdmissible_smallShares_at_point (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.WeightAdmissibleSmallSharesAtPoint, lines 23–29)
33.27. GlobalTheory.EX.sig_eq_loc (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.WeightRequirementOfValid, lines 20–30)
GlobalTheory.EX.weightWindowRequirement_host_of_point (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.WeightWindowRequirementHostOfPoint, lines 17–54)
GlobalTheory.EX.wreq_of_wok (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.WeightRequirementOfValid, lines 32–69)
33.27. UpperTheory.Top.hostWW (Development.AutomorphicArithmetic.CharacterBoxes.HostWW, lines 27–87)
UpperTheory.Top.kitS (Development.AutomorphicArithmetic.CharacterBoxes.KitS, lines 14–62)
UpperTheory.Top.designS (Development.AutomorphicArithmetic.CharacterBoxes.DesignS, lines 21–98)
33.28. GlobalTheory.rowsFinDim_link (Development.AbelianVarietyTheory.Proofs.FlagMeasures.RowsFinDimLink, lines 7–8)
CompanionTheory.D.gk.RF.automorphic_forms_finite_dimensional (CompanionTheory.HarishChandraModules.KFiniteAnalyticVectors.FundamentalDomainCusps.AutomorphicFormsFiniteDimensional, lines 25–27)
33.29. GlobalTheory.Fc.S0.StageU.hypotheses_nonempty_of_unramified_exponents_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.HypothesesNonemptyOfUnramifiedExponents, lines 41–61)
33.30. GlobalTheory.Fc.nonempty_hyps_of_allRows_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.NonemptyHypsOfAllRows, lines 19–101)
33.30. GlobalTheory.EX.DQ.cls7 (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.FamilyClassSevenDutiesAndStableLattices, lines 53–53)
GlobalTheory.EX.DQ.QD (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.FamilyClassSevenDutiesAndStableLattices, lines 32–32)
GlobalTheory.EX.DC.cls (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.AuxiliaryPrimeMemAvoidedPrimes, lines 106–107)
GlobalTheory.EX.DC.withObj (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.AuxiliaryPrimeMemAvoidedPrimes, lines 104–104)
GlobalTheory.EX.withDuty (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.ThinScalesIffHostBound, lines 244–248)
Formalism.MainNotions.DGamChoice_at (Formalism.SharpRestriction.GlobalSchurBounds.MainIn, lines 57–58)
GlobalTheory.IndefiniteRowSideFullHecke.TameGeneratorWindow.lJ (Development.AbelianVarietyTheory.BlockTheory.GlobalSignature.TameGeneratorWindow, lines 50–51)
33.31. GlobalTheory.Mem.Top.DsP (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.CmQuadraticFormEqAlternatingForm, lines 23–26)
GlobalTheory.Mem.Top.DsLow.trueDsig_eq_dimV (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.DsPPosAndLower, lines 15–43)
GlobalTheory.EX.WinLow.window_lower (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.WindowLower, lines 177–295)

33.32. `UpperTheory.Top.classSel` (`Development.AutomorphicArithmetic.CharacterBoxes.ClassSel`, lines 14–28)
33.32. `GlobalTheory.Fc.RowL.and` (`Development.AbelianVarietyTheory.Definitions.GlobalTwisting.EventuallyAllScaleContextsAndOfBoth`, lines 76–80)
`GlobalTheory.Fc.Row.elim` (`Development.AbelianVarietyTheory.Definitions.GlobalTwisting.ValidForConstructedPointClassesEventuallyWorking`, lines 364–366)
33.33. `GlobalTheory.T.rowFormsMatchPeriodMap_of_furtherProperties` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.RowFormsMatchPeriodMapOfFurtherPropertiesStatement`, lines 7–8)
33.33. `GlobalTheory.AtG.RX.lat_of_rx` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.RowFormsMatchPeriodMapOfFurtherProperties`, lines 71–101)
`GlobalTheory.AtG.RX.rowPin_core` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.RowFormsMatchPeriodMapOfFurtherProperties`, lines 111–127)
`GlobalTheory.AtG.RX.rowPin_of_rx` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.RowFormsMatchPeriodMapOfFurtherProperties`, lines 131–140)
33.34. `GlobalTheory.Fc.AllAtMemberStage0.allRows_of_open0_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LinkROfIsLinked`, lines 50–86)
`GlobalTheory.Fc.AllRowsAlt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.AllRows`, lines 12–39)
33.35. `GlobalTheory.Mem.R3.Rw.Bare.bare_form_saturated` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.ExistsBareFormOfSaturation`, lines 19–57)
33.36. `GlobalTheory.OrderEval.order_eval` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsLevelOrderExponentsSmallValuationDetermina`, lines 278–376)
`GlobalTheory.TwistChoice.exists_level` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.IsAuxiliaryPlaceFamilyAtLevelMono`, lines 321–353)
33.37. `GlobalTheory.SelDetLength.twisted_selmer_lengths_eq_determinant_valuation` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TwistedSelmerLengthsEqDeterminantValuation`, lines 375–385)
`GlobalTheory.TwistChoiceAll.admissibleIndexBoundAllPowers_of_conditions` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleIndexBoundAllPowersOfConditions`, lines 92–222)
33.38. `GlobalTheory.Mem.Top.ExportAt` (`Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.AmbientBaseHeq`, lines 36–40)
`GlobalTheory.TwAll.goodTwistExistsAllCoprimePowers` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.AdmissibleIndexBoundAllPowers`, lines 23–30)
`Formalism.MainNotions.BlochKatoReduction.Gen.SetHyp` (`Formalism.SharpRestriction.LevelLetterCocycles.BlochKatoPhases`, lines 327–330)
`Formalism.MainNotions.IntegralHeckeForms.GoodTw` (`Formalism.SharpRestriction.LevelLetterCocycles.IntegralHeckeForms`, lines 219–231)
33.39. `GlobalTheory.TripleFrobDual.frobDual_of_tripleAtt_n` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.StringRepresentations.FrobDualOfTripleAttN`, lines 139–183)
`GlobalTheory.LatticeSelfDual.selfDual_of_charDual` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SelfDualLatticeOfConjDualCharpoly`, lines 76–88)
`GlobalTheory.ChooseForms.nu_pos_of_resIso` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsElementsSelmerFormsSmallCorank`, lines 410–425)
`UpperTheory.KStar.exists_eps_eq_pow_nu` (`Development.AutomorphicArithmetic.CyclotomicBounds.CyclotomicOrders.ExistsCycZPowNeOne`, lines 21–29)
33.40. `GlobalTheory.AdmChars.exists_ray_chars` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.NotValuationNatCastLtOne`, lines 329–387)
`GlobalTheory.AdmChars.exists_admissible_family` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.NotValuationNatCastLtOne`, lines 453–467)
`GlobalTheory.ChosenPrimes.stepAB` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsAuxiliaryPrimesPrescribedFrobenius`, lines 274–418)
33.41. `GlobalTheory.SelFun.oneMatrix_of_galoisCohomology` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SelmerPresentationMatrixOfGaloisCohomology`, lines 20–35)
`GlobalTheory.LocalConditionCount.locCount` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalCountOfSetup`, lines 491–516)
`GlobalTheory.SelObstruction.selObs` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SelmerObstructionLeResidualSelmerSimpleExtension`, lines 521–561)
`GlobalTheory.KRep.kernelRep` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TorsionLeftExactFunctorKernelSquareMatrix`, lines 340–464)
33.42. `GlobalTheory.ResOrder.core` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ResidualSelmerDeterminantOrderBound`, lines 118–322)
`GlobalTheory.ResOrder.resBound` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ResidualSelmerDeterminantOrderBound`, lines 324–350)
`GlobalTheory.ResRecip.reciprocity` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ReciprocityAuxiliaryPlacesSecondHalf`, lines 326–428)
`GlobalTheory.ResSel.exists_unit_coeff_of_not_dvd_line` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsUnitCoeffOfNotXPowDvdLine`, lines 41–80)
33.43. `KStar.Tree.twist_mul_coord_eq_zero` (`Development.AutomorphicArithmetic.CyclotomicBounds.ClassNumberBounds.CyclotomicTotient.PolarTraceMulVecMulVecEq`, lines 28–89)

KStar.Tree.twist_smul_eq_zero (Development.AutomorphicArithmetic.CyclotomicBounds.ClassNumberBounds.CyclotomicTotient.TwistTrace, lines 78–111)
GlobalTheory.NewPrime.FormVanish.two_mul_values_eq_zero (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TwoMulCocycleValuesZeroOnSmallSubgroup, lines 49–122)
33.44. GlobalTheory.NewPrime.LocalRing.exists_finite_local_ring (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsFiniteLocalRingPolyDividesVarPow, lines 282–430)
33.45. GlobalTheory.TwGalInputs.galois_inputs_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.GaloisInputs, lines 19–29)
GlobalTheory.TripleLocal.triple_att_n_of_is_triple_with (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.TripleAttOfIsTripleWith, lines 54–116)
GlobalTheory.TripleAdm.irredOn_rbar_and_inertia_of_residual (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.SGalRead, lines 199–246)
GlobalTheory.TripleGal.adm1_adm8_selfDual_noJump_of_tripleAttN (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.GaloisHypothesesOfAttached, lines 23–34)
GlobalTheory.SettingHypotheses.setHyp_of_isTriple_of_nonzeroIndexPlace (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.SetHypOfIsTripleOfGalHodge, lines 14–24)
33.46. GlobalTheory.ResDual.eqDim (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ResidualSelmerFinrankEqDual, lines 257–270)
GlobalTheory.ResSel.SpanOddPf.spanOdd (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CocycleFormsSpanOddResidueChar, lines 231–361)
GlobalTheory.ResSel.AltSpan.spanTwo_of_galHyp (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SpanTwoOfGaloisHypothesis, lines 196–274)
GlobalTheory.ChooseForms.single_field (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsElementsSelmerFormsSmallCorank, lines 427–465)
GlobalTheory.ChooseForms.chooseH (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsElementsSelmerFormsSmallCorank, lines 557–577)
33.47. GlobalTheory.Growth.compare (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ChebotarevInductionGrowBoundOfPresentation, lines 315–451)
GlobalTheory.Growth.grow (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ChebotarevInductionGrowBoundOfPresentation, lines 462–502)
GlobalTheory.OrdMul.coeff_mem_of_pow_mul (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CoeffMemOfPowMul, lines 81–92)
33.48. GlobalTheory.NewPrime.step (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AntiOfKer, lines 153–352)
GlobalTheory.NewPrime.DualClass.exists_not_cob (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsDualSelmerClassNotKilled, lines 169–197)
GlobalTheory.DualSub.dualSel_of_smul_cob (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.DualSelmerClassFromCoeffsKilledByMul, lines 48–124)
GlobalTheory.NewPrime.FormNonzero.exists_bform_ne_zero (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsSmallSubgroupCocycleFormNeZero, lines 39–95)
33.49. GlobalTheory.NewPrime.step (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AntiOfKer, lines 153–352)
GlobalTheory.NewPrime.det_ne_zero (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AntiOfKer, lines 85–151)
GlobalTheory.LowDet.det_expand (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.DetNeZeroOfLowestFormNeZeroPresentation, lines 188–230)
GlobalTheory.LowDet.det_ne_zero (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.DetNeZeroOfLowestFormNeZeroPresentation, lines 234–372)
33.50. GlobalTheory.ResSel.OddBound.residual_bound_of_parts (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ResidualBoundOfComponents, lines 28–57)
GlobalTheory.StepTwo.StartInd.start_of_galHyp (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ContinuousCocycleChangeScalarsBack, lines 452–495)
GlobalTheory.StepTwo.TwoBound.induct (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AddValLeOfNotMem, lines 60–85)
GlobalTheory.StepTwo.TwoBound.thmC_of_parts (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AddValLeOfNotMem, lines 87–144)
33.51. GlobalTheory.TwInputs.admT4_of_full_compatibility (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmT4OfRootsInField, lines 110–126)
GlobalTheory.TwTraceCx.trace_cx (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TwistedThetaGeneratorEqClassSumIntegralForm, lines 217–241)
33.52. GlobalTheory.XiBound.main (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.UniversalSeriesCoeffBoundedInSupOfTwistSeriesSid, lines 40–94)
GlobalTheory.TwInputs.univSup_of_xi (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.UnivSupOfXi, lines 13–44)
33.53. GlobalTheory.ExportFirst.exportF_isBuiltE_alt_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.SExport, lines 20–32)
GlobalTheory.ExportClose.exportAt_of_bundles_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ExportAtOfBundles, lines 17–35)

`GlobalTheory.TwistChoiceAll.admissibleIndexBoundAllPowers_of_conditions` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleIndexBoundAllPowersOfConditions`, lines 92–222)
`GlobalTheory.ExportFromInputs.twistModule_all_character_powers_of_admissible` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TwistModuleAllCharacterPowersOfAdmissible`, lines 17–53)
33.54. `Formalism.L2.Fam.NPC.LocT` (`Formalism.LocalToGlobal.NumberFieldFamilies.NPartCNd`, lines 135–137)
`GlobalTheory.HeckeLocN.ndSpec_exists_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.NdSpecExists`, lines 21–103)
`GlobalTheory.HeckeLocN.mkX` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.UnitsLiftSpec`, lines 143–212)
`GlobalTheory.HeightCounting.Fin3.lv3_of_third_alt_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.FinitenessOfHyp`, lines 61–83)
33.55. `GlobalTheory.T.large_unramified_exponent_bundles_exist_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.MemberBundlesLargeUnramifiedExponents`, lines 69–70)
`GlobalTheory.Fc.StageOneU.member_bundles_large_unramified_exponents_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.MemberBundlesLargeUnramifiedExponents`, lines 34–63)
33.55. `GlobalTheory.Mem.LiteralAutomorphy.memberE_extensionWithAtProperty_of_memberE` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.IsBuiltEPrmAIff`, lines 35–53)
`GlobalTheory.Mem.LiteralAutomorphy.exists_weilBlock_ne_at_member_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ExistsWeilBlockNeAtMember`, lines 24–33)
33.56. `GlobalTheory.MiTop.rel_of_links_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RelOfLinks`, lines 22–90)
`GlobalTheory.MiTop.served'` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.NotMemQBase`, lines 47–80)
33.57. `GlobalTheory.Mem.SRe.exists_sRest_top` (`Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.ExistsSpectralLevelObjectsReplacedStages`, lines 24–37)
`Formalism.L2.SG.RowDict.range_sEE` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RowDict`, lines 148–163)
33.58. `GlobalTheory.Chain.ChainPar` (`Development.AbelianVarietyTheory.Definitions.TwistedInnerProducts.IsStepWeil`, lines 32–36)
`GlobalTheory.StepRule.rulePar_step` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.FirstOccurrenceRuleThetaChainStepBothDirections`, lines 58–79)
`GlobalTheory.LowerMembersRule.lowFock` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.LowerTowerLiftsVanishOnFock`, lines 91–213)
`GlobalTheory.PairLift.cuspAll_of_lowAll` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.ThetaLiftsCuspidalOfLowerTowerLiftsVanish`, lines 142–152)
33.59. `GlobalTheory.ChainOnto.chainLW_holds` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.ThetaChainLValuesNonvanishing`, lines 51–117)
`GlobalTheory.ChainOnto.QuotL.quotL_entire_ne_zero` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.ThetaChainCharQuotientLFunctionEntireNeZero`, lines 45–81)
33.60. `GlobalTheory.Fam.MemberCount.rankMu_of_bundle_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RankMuOfBundle`, lines 17–19)
`GlobalTheory.Diag.PG.duWin_three_of_count` (`Development.AbelianVarietyTheory.Proofs.FlagMeasures.DuWinThreeOfCount`, lines 60–93)
`GlobalTheory.Core.Rk.ramShare_of_trueD` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.RamShareOfFreeT`, lines 36–38)
33.61. `GlobalTheory.Core.Deg.rowsND_one_record` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.NondegenerateCollisionRowBoundAtParameters`, lines 18–84)
`GlobalTheory.Core.Deg.rowsDeg_at_record` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.DegenerateCollisionRowBoundAtParams`, lines 15–91)
`GlobalTheory.Core.Deg.corankSums_expLoss` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.CorankSumsBoundWithExpLoss`, lines 184–220)
`GlobalTheory.Core.Deg.collS_of_rowBounds` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.CollisionAbsSumBoundOfTwoRowBounds`, lines 16–91)
33.62. `GlobalTheory.diag_abstract_collisions_once` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.AbstractDiagonalBoundCollisionsCountedOnce`, lines 21–87)
`Formalism.FourierDecayAlt.P.GrowthPolynomialOperator.t_PriceOfIota` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.GrowthOfBasic`, lines 384–392)
`GlobalTheory.Core.Deg.pgC_le_six_fifths` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.EqualIdentificationPairPriceLeSixFifths`, lines 55–74)
`GlobalTheory.Core.Diag.diag_arith` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.DiagonalTheoremOfFourPointBounds`, lines 36–47)
`GlobalTheory.Core.Rk.diagonalRow_bound_from_windowWidthThree` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.DiagonalRowBoundFromWindowWidthThree`, lines 38–97)
33.63. `Formalism.L2.SG.TermOp.vec` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.TermOp`, lines 343–345)
`Formalism.L2.SG.TermOp.rowOrth` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.TermOpStmt`, lines 83–99)

33.64. `GlobalTheory.Core.One.stepClass1_of` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.StepClass1Of`, lines 9–39)
`GlobalTheory.Core.One.scT_sc_genSet` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.RowGen1POfSteps`, lines 16–21)
`GlobalTheory.Core.One.genericTerms_points_agree_groupSums_corankOne` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.GenericTermsPointsAgreeGroupSumsCorankOne`, lines 8–58)
`Formalism.L2.SG.Pf.MainEqCut.meanAmplitudeGeneric_holds` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.MainEqCut`, lines 389–460)
33.65. `kappa_off_diagonal_dim_form_C` (`Formalism.PeriodIntegrals.OrbitalBoundsConclusion.UnitaryDimensionFormula`, lines 6–7)
`Formalism.L2.SG.KappaOffDiagonal.DimFormC` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.DimFormC`, lines 80–82)
`PI.UnitaryDimensionFormula.dimFormC_of_three` (`Formalism.PeriodIntegrals.UnitaryGroupMassFormula.RankTwoDimensionFormula.DimFormCOfThree`, lines 78–79)
`PI.UnitaryDimensionFormula.dimFormC_of_steps` (`Formalism.PeriodIntegrals.UnitaryGroupMassFormula.RankTwoDimensionFormula.DimFormCOfSteps`, lines 117–119)
`PI.UnitaryDimensionFormula.dimFormC_of` (`Formalism.PeriodIntegrals.UnitaryGroupMassFormula.RankTwoDimensionFormula.GammaOneEqProd`, lines 106–120)
`PI.UnitaryDimensionFormula.fibreBound_of` (`Formalism.PeriodIntegrals.UnitaryGroupMassFormula.RankTwoDimensionFormula.DimFormCOfSteps`, lines 79–115)
`PI.UnitaryDimensionFormula.orbitCount` (`Formalism.PeriodIntegrals.UnitaryGroupMassFormula.RankTwoDimensionFormula.DimFormCOfThree`, lines 13–58)
`PI.UnitaryDimensionFormula.kuStab` (`Formalism.PeriodIntegrals.UnitaryGroupMassFormula.RankTwoDimensionFormula.DimFormCOfThree`, lines 60–76)
`PI.UnitaryDimensionFormula.transport_holds` (`Formalism.PeriodIntegrals.UnitaryGroupMassFormula.RankTwoDimensionFormula.Transp`, lines 101–151)
`PI.UnitaryDimensionFormula.stabBound_holds` (`Formalism.PeriodIntegrals.UnitaryGroupMassFormula.RankTwoDimensionFormula.StabBoundHolds`, lines 261–318)
`PI.UnitaryDimensionFormula.indexGe` (`Formalism.PeriodIntegrals.UnitaryGroupMassFormula.RankTwoDimensionFormula.IndexGeFull`, lines 10–10)
`PI.UnitaryDimensionFormula.indexGe_of` (`Formalism.PeriodIntegrals.UnitaryGroupMassFormula.RankTwoDimensionFormula.IndexGe`, lines 117–165)
`PI.UnitaryDimensionFormula.gamma_one_eq_card` (`Formalism.PeriodIntegrals.UnitaryGroupMassFormula.RankTwoDimensionFormula.IndexGe`, lines 78–115)
`PI.UnitaryDimensionFormula.idxHom` (`Formalism.PeriodIntegrals.UnitaryGroupMassFormula.RankTwoDimensionFormula.IdxKHom`, lines 25–31)
`PI.UnitaryDimensionFormula.idxSurj_of_frame` (`Formalism.PeriodIntegrals.UnitaryGroupMassFormula.RankTwoDimensionFormula.IdxDChk`, lines 12–13)
`PI.UnitaryDimensionFormula.idxDFrame` (`Formalism.PeriodIntegrals.UnitaryGroupMassFormula.RankTwoDimensionFormula.IdxDcvFrameAsm`, lines 83–83)
33.66. `Formalism.L2.SG.KappaOffDiagonal.irrVec_nohyp` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.CovolumeSpans`, lines 28–33)
`Formalism.L2.SG.KappaOffDiagonal.IrrVecC` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.DimFormC`, lines 47–51)
`Formalism.L2.SG.KappaOffDiagonal.vector_eq_zero_of_orbital_zero` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.CovolumeSpans`, lines 17–26)
`Formalism.L2.SG.TermOp.Vanish.ooVanish_of` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.Vanish`, lines 512–569)
`Formalism.L2.SG.NonRelPf.nonRelCharC_holds` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.LieAveraging.NonRelCharCHolds`, lines 118–261)
`Formalism.L2.SG.TermOp.Vanish.rational_cocompact_holds` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.RCocompactPfFinal`, lines 7–7)
`Formalism.L2.SG.TermOp.Vanish.rCocompactC_of` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.RCocompactPfMain`, lines 235–331)
`Formalism.L2.SG.Pf.ginfUnimod_holds` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.NilradicalLatticeZMemSpan`, lines 335–335)
`Formalism.L2.SG.Pf.nLamNinf_holds` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.NilradicalLatticeZMemSpan`, lines 294–314)
`Formalism.L2.SG.Pf.psiInfChar_holds` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.NilradicalLatticeZMemSpan`, lines 471–487)
`Formalism.L2.SG.Pf.rClosedC_holds` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.NilradicalLatticeZMemSpan`, lines 489–495)
`Formalism.L2.SG.Pf.roDiscreteC_holds` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.NilradicalLatticeZMemSpan`, lines 337–344)
`Formalism.L2.SG.Closed.ChiTrivialC_holds` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedVolumeStatements.UnipotentInLevelSubgroup`, lines 55–55)
`Formalism.L2.SG.Closed.AvgCore_holds` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedVolumeStatements.UnipotentInLevelSubgroup`, lines 56–56)

33.67. `Formalism.L2.SG.NormCut.normExpansionOn_tOf` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.NormCut`, lines 202–266)
`Formalism.L2.SG.NormCut.normExp_abs` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.NormAsm`, lines 184–342)
`Formalism.FourierDecayAlt.PB.Terms.NormExpansionOn` (`Formalism.HermitianCurvature.StatementPB`, lines 86–87)
`hilbert_schmidt_blocks` (`Formalism.PeriodIntegrals.OrbitalBoundsConclusion.NormHSBlocks`, lines 5–6)
`NormHilbertSchmidtBlocks.hSBlocks_main` (`Formalism.PeriodIntegrals.UnitaryGroupMassFormula.HilbertSchmidtBlocks.HSBlocksMain`, lines 13–35)
`chi_n_bilinear` (`Formalism.PeriodIntegrals.OrbitalBoundsConclusion.NormChiNBi`, lines 3–4)
`PIW.NormChiNBi.chiNBi_holds` (`Formalism.PeriodIntegrals.UnitaryGroupMassFormula.CharacterNormMatrixInverse.FmatMulInv`, lines 265–275)
`Formalism.L2.SG.NormCut.colFree_steps` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.NormLinks`, lines 34–37)
`Formalism.L2.SG.NormCut.colFreeBij_of` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.ColFreePf`, lines 58–148)
`Formalism.L2.SG.Pf.colLatK` (`Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.LatDictK`, lines 59–65)
`Formalism.L2.SG.Pf.NormSteps.levi_onto_reduction` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.NormStepsLeviOntoSEnd`, lines 23–24)
`Formalism.L2.SG.Pf.NormSteps.LeviOntoSPf.exact_dual_frame_exists` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.NormStepsLeviFrame`, lines 563–579)
`Formalism.L2.SG.Pf.NormSteps.column_lattice_representatives` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.NormStepsColLatRepS`, lines 203–250)
`Formalism.L2.SG.Pf.massAC_holds` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.ActMassC`, lines 7–8)
`Formalism.L2.SG.Pf.SatGram.satGram_of` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.SatGram`, lines 529–542)
`Formalism.L2.SG.Pf.SatGram.zs_gram` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.SatGram`, lines 486–494)
`Formalism.L2.SG.NormCut.leftActC'_of_PD` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.NormExpSatHolds`, lines 23–35)
`Formalism.L2.SG.NormCut.rightActC'_holds` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GaugeLatticeShifts.NormExpSatHolds`, lines 37–42)
33.68. `GlobalTheory.Core.Deg.PairsDegOfType` (`Development.AbelianVarietyTheory.Definitions.LatticeFrames.TermSumBoundOfPairsAmplitude`, lines 32–36)
`GlobalTheory.Core.Deg.pairsDegOfType_holds` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.PairsAmplitudeBoundOfTypeBound`, lines 20–91)
`GlobalTheory.Core.Deg.sum_labels_one_record` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.SumLabelsOneRecord`, lines 12–37)
`Formalism.L2.SG.KappaOffDiagonal.kloostermanCosetSum_le` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.KloostermanCosetSumLe`, lines 102–142)
`Formalism.L2.SG.KappaOffDiagonal.delta_pos_le_height_sq` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.RelevantSetZeroGNotIdentity`, lines 60–110)
`Formalism.MainNotions.KappaEndpoints.rank_contribution_le` (`Formalism.SharpRestriction.CuspSpanNormBounds.KappaEndpoints`, lines 3161–3223)
`Formalism.MainNotions.KappaEndpoints.final_bound_closes` (`Formalism.SharpRestriction.CuspSpanNormBounds.KappaEndpoints`, lines 2593–2601)
`GlobalTheory.Core.Deg.pgC_le_two_at_record` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.PgCLeTwoAtRecord`, lines 10–29)
33.69. `GlobalTheory.Core.Deg.TermSumBoundOfPairsAmplitude` (`Development.AbelianVarietyTheory.Definitions.LatticeFrames.TermSumBoundOfPairsAmplitude`, lines 38–41)
`GlobalTheory.Core.Deg.termSumBound_of_pairsAmplitude` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.TermSumBoundOfPairsAmplitude`, lines 11–26)
`Formalism.L2.SG.RowsOther.kppDiag_of_pairs` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RowsOther`, lines 35–46)
`Formalism.L2.SG.RowsOther.pairsDeg_of_amp` (`Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RowsOther`, lines 122–124)
`Formalism.FourierDecayAlt.PB.Terms.NormExpansionOn` (`Formalism.HermitianCurvature.StatementPB`, lines 86–87)
`Formalism.L2.SG.TermOp.ggBound` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.TermOpStmt`, lines 135–147)
`Formalism.L2.SG.TermOp.rowOrth` (`Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.TermOpStmt`, lines 83–99)
33.70. `GlobalTheory.Core.Deg.vanishingGramBoundWindow_of_termPairTypeBound` (`Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.VanishingGramBoundWindowOfTermPairTypeBound`, lines 9–20)
`GlobalTheory.Core.Deg.VanishingGramTermsBound` (`Development.AbelianVarietyTheory.Definitions.LatticeFrames.VanishingGramTermsBoundParams`, lines 65–67)
`GlobalTheory.Core.Deg.VanishingGramTermsBoundWindow` (`Development.AbelianVarietyTheory.Definitions.LatticeFrames.VanishingGramTermsBoundParams`, lines 104–104)

GlobalTheory.TypeAsm.TypeP (Development.AbelianVarietyTheory.Definitions.SemilocalFrames.PerRowC, lines 30–34)
Formalism.MainNotions.NonGenericAndKTwoTheorems.Hyp_of (Formalism.SharpRestriction.CuspSpanNormBounds.ThresholdCubicInequalitySignPattern, lines 238–259)
33.71. GlobalTheory.Core.Diag.sDiagAt_of_points (Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.DiagonalTheoremOfFourPointBounds, lines 49–106)
GlobalTheory.Core.Rk.levelPrime_window (Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.LevelPrimeWindow, lines 11–91)
GlobalTheory.MiTop.puNZ_of_sDiag (Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.NonvanishingFromTruePack, lines 48–64)
33.72. GlobalTheory.Core.Rk.finrank_ker_Pu_le_hsDev (Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.FinrankKerPuLeHsDev, lines 9–16)
GlobalTheory.MiTop.puNZ_at (Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.NonvanishingFromTruePack, lines 26–46)
GlobalTheory.MiTop.anchor_pairs_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.AnchorPairs, lines 24–71)
33.73. GlobalTheory.MiTop.rel_of_links_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RelOfLinks, lines 22–90)
GlobalTheory.MiTop.AtReal.anchor_word_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.MiStar2AtRealisation, lines 48–79)
33.74. GlobalTheory.T.open0_of_unitary_group_hypotheses (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.OpenOrbitOfUnramifiedExponents, lines 75–76)
GlobalTheory.Fc.S0.Open0ProofU.open_orbit_of_unramified_exponents_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.OpenOrbitOfUnramifiedExponents, lines 39–69)
Formalism.IndefR.Cx.CxData.MIid (Formalism.LocalToGlobal.IndefiniteReal.IsometricPeriodsOfHypotheses, lines 99–103)
Formalism.IndefR.Cx.CxData.MIstar2 (Formalism.LocalToGlobal.IndefiniteReal.IsometricPeriodsOfUnitHypotheses, lines 27–28)
GlobalTheory.Glob.open_and_linked_of_stronger_extension_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.OpenAndLinkedOfStrongerExtension, lines 28–33)
33.75. GlobalTheory.BigTop.lamLow_of_hypotheses_alt_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LamLowOfPublished, lines 17–35)
33.76. GlobalTheory.BigOn.canonical_error_real (Development.AbelianVarietyTheory.Proofs.DivisorProofs.OnOfOnWithCanonical, lines 8–61)
GlobalTheory.BigOn.OntoPath.onto_path_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.OntoPath, lines 18–76)
33.76. GlobalTheory.Glob.open_and_linked_of_stronger_extension_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.OpenAndLinkedOfStrongerExtension, lines 28–33)
33.77. GlobalTheory.T.period_entries_tie_and_level_valuations_at_built (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.SixRowsAtMemberStatement, lines 12–13)
GlobalTheory.Fc.six_rows_at_member_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.SixRowsAtMember, lines 11–24)
33.78. GlobalTheory.LocalRows.lg1_sLp_sO_hTF (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.AtChoice, lines 27–28)
Formalism.MainNotions.S.sC' (Formalism.SharpRestriction.CuspSpanNormBounds.LeSCSCl, lines 11–11)
Formalism.MainNotions.S.sCl (Formalism.SharpRestriction.CuspSpanNormBounds.LeSCSCl, lines 13–14)
Formalism.MainNotions.FamP.cLast (Formalism.SharpRestriction.ParamsHeightRankConstant, lines 114–117)
Formalism.MainNotions.S.le_sC'_sCl (Formalism.SharpRestriction.CuspSpanNormBounds.LeSCSCl, lines 56–61)
Formalism.L2.SG.ThmSOwn.LoopRowS (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.ThmSOwn, lines 1015–1024)
33.78. GlobalTheory.LocalRows.at_choice_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.AtChoice, lines 16–23)
Formalism.L2.SG.ThmSOwn.offDiagonalK0_of_rows (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.LinkSFinalP, lines 58–114)
33.79. Formalism.L2.SG.DataCut.dom_points (Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.DataCut, lines 761–894)
Formalism.L2.SG.DataCut.rowsDomAtP_of (Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.RowsInputPOfTermsFiniteSig, lines 67–162)
Formalism.L2.SG.DataCut.rowsInputP_of (Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.RowsInputPOfTermsFiniteSig, lines 164–170)
33.80. CompanionTheory.WaveGrid.grids_PStar (CompanionTheory.AutomorphicTheory.BesselWaveGrids.GridsText, lines 52–63)
Formalism.FourierDecayAlt.P.TBBx5.GridOK (Formalism.HermitianCurvature.HermitianCurvatureProofs.OnePointBasisFundamentalDomainVolume, lines 162–169)
33.81. Formalism.FourierDecayAlt.P.CutBP.family_expansion_of_parts (Formalism.HermitianCurvature.HermitianCurvatureProofs.NormSqUnitNormalizationLe, lines 200–338)
Formalism.FourierDecayAlt.P.CutBP.coefficient_growth_le_power (Formalism.HermitianCurvature.HermitianCurvatureProofs.NormSqUnitNormalizationLe, lines 340–366)
33.82. Formalism.FourierDecayAlt.D.positionted (Formalism.HermitianCurvature.StatementD, lines 193–194)
Formalism.FourierDecayAlt.M.sgn (Formalism.HermitianCurvature.StatementM, lines 225–225)

`Formalism.FourierDecayAlt.Wd.Letter` (`Formalism.HermitianCurvature.StatementWd`, lines 13–19)
33.82. `Formalism.FourierDecayAlt.Q.Glev` (`Formalism.HermitianCurvature.StatementQ`, lines 208–210)
`Formalism.FourierDecayAlt.Q.GGl` (`Formalism.HermitianCurvature.StatementQ`, lines 557–558)
`Formalism.FourierDecayAlt.Q.DomR` (`Formalism.HermitianCurvature.StatementQ`, lines 922–923)
`Formalism.FourierDecayAlt.Q.RadSp` (`Formalism.HermitianCurvature.StatementQ`, lines 925–954)
`Formalism.FourierDecayAlt.Q.Pr` (`Formalism.HermitianCurvature.StatementQ`, lines 889–889)
`Formalism.FourierDecayAlt.Q.mkP` (`Formalism.HermitianCurvature.StatementQ`, lines 918–918)
33.82. `Formalism.FourierDecayAlt.Q.ActualOps.QQ` (`Formalism.HermitianCurvature.StatementQ`, lines 1004–1005)
33.82. `Formalism.FourierDecayAlt.Q.IsPlace` (`Formalism.HermitianCurvature.StatementQ`, lines 687–691)
`Formalism.FourierDecayAlt.Q.ActualOps` (`Formalism.HermitianCurvature.StatementQ`, lines 967–978)
`Formalism.FourierDecayAlt.Q.IsActualLevel` (`Formalism.HermitianCurvature.StatementQ`, lines 1007–1013)
`Formalism.FourierDecayAlt.R.LevelData` (`Formalism.HermitianCurvature.StatementR`, lines 7–84)
`Formalism.FourierDecayAlt.Fr.Start` (`Formalism.HermitianCurvature.StatementFr`, lines 180–186)
`Formalism.FourierDecayAlt.Fr.IsStarLevel` (`Formalism.HermitianCurvature.StatementFr`, lines 47–48)
`Formalism.FourierDecayAlt.Fr.IsLevelStart` (`Formalism.HermitianCurvature.StatementFr`, lines 188–198)
`Formalism.FourierDecayAlt.R.LevelData.l0` (`Formalism.HermitianCurvature.StatementR`, lines 117–117)
`Formalism.FourierDecayAlt.P5.IsData` (`Formalism.HermitianCurvature.CompositeFormulaLevelZero`, lines 103–109)
33.82. `Formalism.FourierDecayAlt.M.Par.lo` (`Formalism.HermitianCurvature.StatementM`, lines 102–102)
`Formalism.FourierDecayAlt.M.Par.hi` (`Formalism.HermitianCurvature.StatementM`, lines 104–104)
`Formalism.FourierDecayAlt.M.Par.wid` (`Formalism.HermitianCurvature.StatementM`, lines 106–106)
`Formalism.FourierDecayAlt.M.Ty` (`Formalism.HermitianCurvature.StatementM`, lines 118–118)
`Formalism.FourierDecayAlt.M.rho` (`Formalism.HermitianCurvature.StatementM`, lines 122–123)
`Formalism.FourierDecayAlt.M.xC` (`Formalism.HermitianCurvature.StatementM`, lines 130–130)
`Formalism.FourierDecayAlt.M.qC` (`Formalism.HermitianCurvature.StatementM`, lines 134–135)
`Formalism.FourierDecayAlt.Fr.admitted` (`Formalism.HermitianCurvature.StatementFr`, lines 219–222)
`Formalism.FourierDecayAlt.I.OnCut` (`Formalism.HermitianCurvature.StatementI`, lines 13–14)
`Formalism.FourierDecayAlt.P5.WordOK` (`Formalism.HermitianCurvature.CompositeFormulaLevelZero`, lines 116–118)
33.82. `Formalism.FourierDecayAlt.Wd.P0` (`Formalism.HermitianCurvature.StatementWd`, lines 49–49)
33.82. `Formalism.YangianSpanCharacter.T.Tm` (`Formalism.YangianSpanCharacter.StatementT`, lines 9–11)
`Formalism.FourierDecayAlt.Wd.eU` (`Formalism.HermitianCurvature.StatementWd`, lines 34–35)
`Formalism.FourierDecayAlt.Wd.ins` (`Formalism.HermitianCurvature.StatementWd`, lines 41–41)
`Formalism.FourierDecayAlt.Wd.lift0` (`Formalism.HermitianCurvature.StatementWd`, lines 45–45)
`Formalism.FourierDecayAlt.Wd.Q0` (`Formalism.HermitianCurvature.StatementWd`, lines 51–51)
`Formalism.FourierDecayAlt.Wd.Rh0` (`Formalism.HermitianCurvature.StatementWd`, lines 53–53)
`Formalism.FourierDecayAlt.Wd.Rht0` (`Formalism.HermitianCurvature.StatementWd`, lines 55–55)
`Formalism.FourierDecayAlt.Wd.TTw` (`Formalism.HermitianCurvature.StatementWd`, lines 89–92)
`Formalism.FourierDecayAlt.Wd.RRw` (`Formalism.HermitianCurvature.StatementWd`, lines 94–96)
`Formalism.FourierDecayAlt.Wd.vlet` (`Formalism.HermitianCurvature.StatementWd`, lines 132–133)
`Formalism.FourierDecayAlt.Wd.bT` (`Formalism.HermitianCurvature.StatementWd`, lines 135–137)
`Formalism.FourierDecayAlt.P5.tprod` (`Formalism.HermitianCurvature.CompositeFormulaLevelZero`, lines 47–49)
`Formalism.FourierDecayAlt.P5.QQi` (`Formalism.HermitianCurvature.CompositeFormulaLevelZero`, lines 51–53)
`Formalism.FourierDecayAlt.P5.RRi` (`Formalism.HermitianCurvature.CompositeFormulaLevelZero`, lines 55–57)
`Formalism.FourierDecayAlt.P5.bTl` (`Formalism.HermitianCurvature.CompositeFormulaLevelZero`, lines 59–61)
`Formalism.FourierDecayAlt.P5.act` (`Formalism.HermitianCurvature.CompositeFormulaLevelZero`, lines 63–64)
33.82. `Formalism.FourierDecayAlt.B.hsInner` (`Formalism.HermitianCurvature.StatementB`, lines 168–168)
`Formalism.FourierDecayAlt.P5.InK` (`Formalism.HermitianCurvature.CompositeFormulaLevelZero`, lines 66–66)
`Formalism.FourierDecayAlt.P5.pairAt` (`Formalism.HermitianCurvature.CompositeFormulaLevelZero`, lines 68–69)
`Formalism.FourierDecayAlt.P5.Cf` (`Formalism.HermitianCurvature.CompositeFormulaLevelZero`, lines 71–71)
`Formalism.FourierDecayAlt.P5.glue` (`Formalism.HermitianCurvature.CompositeFormulaLevelZero`, lines 73–74)
`Formalism.FourierDecayAlt.P5.form` (`Formalism.HermitianCurvature.CompositeFormulaLevelZero`, lines 76–77)
`Formalism.FourierDecayAlt.P5.RRform` (`Formalism.HermitianCurvature.CompositeFormulaLevelZero`, lines 79–82)
`Formalism.FourierDecayAlt.P5.PosP` (`Formalism.HermitianCurvature.CompositeFormulaLevelZero`, lines 138–141)
`Formalism.FourierDecayAlt.P5.GramPSDOfThmP` (`Formalism.HermitianCurvature.CompositeFormulaLevelZero`, lines 216–227)
33.82. `Formalism.FourierDecayAlt.C3.AAop` (`Formalism.HermitianCurvature.OperatorsCommuteTranslations`, lines 241–248)
`Formalism.FourierDecayAlt.C3.AAvop` (`Formalism.HermitianCurvature.OperatorsCommuteTranslations`, lines 252–259)
`Formalism.QuantumDeterminants.Gf` (`Formalism.FormalDegreeBessel.StatementG`, lines 85–88)
`Formalism.FourierDecayAlt.D.vLet` (`Formalism.HermitianCurvature.StatementD`, lines 222–230)
`Formalism.FourierDecayAlt.P.PBG.letter_level_zero_eq` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.PBGWaveMain`, lines 13–26)
33.82. `Formalism.FourierDecayAlt.TBB.NP` (`Formalism.HermitianCurvature.StatementTBB`, lines 38–40)
`Formalism.FourierDecayAlt.TBB.Ew` (`Formalism.HermitianCurvature.StatementTBB`, lines 42–44)
33.82. `Formalism.FourierDecayAlt.Q.CV` (`Formalism.HermitianCurvature.StatementQ`, lines 196–196)
`Formalism.FourierDecayAlt.Q.tL` (`Formalism.HermitianCurvature.StatementQ`, lines 198–206)
`Formalism.FourierDecayAlt.D.vecV` (`Formalism.HermitianCurvature.StatementD`, lines 114–120)
`Formalism.FourierDecayAlt.D.actOp` (`Formalism.HermitianCurvature.StatementD`, lines 189–189)
`Formalism.FourierDecayAlt.D.QQlev` (`Formalism.HermitianCurvature.StatementD`, lines 201–204)
`Formalism.FourierDecayAlt.D.RRlev` (`Formalism.HermitianCurvature.StatementD`, lines 206–208)
`Formalism.FourierDecayAlt.D.formLev` (`Formalism.HermitianCurvature.StatementD`, lines 212–214)

`Formalism.FourierDecayAlt.D.uwG` (`Formalism.HermitianCurvature.StatementD`, lines 232–235)
`Formalism.FourierDecayAlt.D.RRplace` (`Formalism.HermitianCurvature.StatementD`, lines 259–270)
`Formalism.FourierDecayAlt.Wd.Good` (`Formalism.HermitianCurvature.StatementWd`, lines 108–109)
`Formalism.FourierDecayAlt.C7.principalHermitianLattice` (`Formalism.HermitianCurvature.LetterProductDoubleDegreeCount`, lines 230–233)
`Formalism.FourierDecayAlt.P.Dx1.LZ.levelZero` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.ComponentFamilySettingOfSetting`, lines 739–740)
33.82. `Formalism.FourierDecayAlt.KB.plusW` (`Formalism.HermitianCurvature.RealInputsClosedAdmissibleSettings`, lines 18–20)
`Formalism.FourierDecayAlt.KB.minusW` (`Formalism.HermitianCurvature.RealInputsClosedAdmissibleSettings`, lines 22–24)
`Formalism.FourierDecayAlt.KB.pairJ` (`Formalism.HermitianCurvature.RealInputsClosedAdmissibleSettings`, lines 26–31)
33.82. `Formalism.MainNotions.S.LocBessel` (`Formalism.SharpRestriction.CuspSpanNormBounds.RelevantZeroGramTerms`, lines 460–493)
`Formalism.MainNotions.S.Point` (`Formalism.SharpRestriction.CuspSpanNormBounds.Point`, lines 11–12)
`Formalism.MainNotions.S.ptWeight` (`Formalism.SharpRestriction.CuspSpanNormBounds.RelevantZeroGramTerms`, lines 246–246)
`Formalism.MainNotions.BesselShellMomentBounds.GPoint_nonneg_of_rep` (`Formalism.SharpRestriction.CuspSpanNormBounds.BesselShellMomentBounds`, lines 690–696)
`Formalism.MainNotions.S.OrbInt` (`Formalism.SharpRestriction.CuspSpanNormBounds.RootMeasBig`, lines 61–64)
`Formalism.MainNotions.S.ThB.orbSq` (`Formalism.SharpRestriction.CuspSpanNormBounds.LeadMinorK`, lines 103–106)
33.83. `GlobalTheory.WigSh.WigMap` (`Development.AbelianVarietyTheory.Definitions.Jets.WigMap`, lines 12–20)
`GlobalTheory.WigSh.exists_shifted_types_lower_bound` (`Development.AbelianVarietyTheory.Proofs.JetHarmonics.ExistsShiftedTypesLowerBound`, lines 19–65)
33.84. `Formalism.FourierDecayAlt.P.PBK.memLp_bz_of_sup` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.MemLpBzOfSup`, lines 16–79)
`Formalism.FourierDecayAlt.P.PBK.GermD.germLoc_tail` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.GermLocTail`, lines 15–47)
`Formalism.FourierDecayAlt.P.latticeHerglotzBound_statement` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.PairingTriangular`, lines 245–349)
`Formalism.FourierDecayAlt.P.PBC.herglotz_germ` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.PBCStep`, lines 97–161)
33.85. `Formalism.FourierDecayAlt.F7.PolyStep` (`Formalism.HermitianCurvature.LevelRecursionCondNeeded`, lines 130–133)
`Formalism.FourierDecayAlt.P.extract_K_bound` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.LagrangeExtraction`, lines 151–151)
33.86. `Formalism.FourierDecayAlt.KB.pairJ` (`Formalism.HermitianCurvature.RealInputsClosedAdmissibleSettings`, lines 26–31)
`Formalism.FourierDecayAlt.P.PBG.letter_level_zero_eq` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.PBGWaveMain`, lines 13–26)
`Formalism.FourierDecayAlt.P.PBG.wv_step` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.PBGDegW`, lines 76–236)
33.87. `Formalism.FourierDecayAlt.P5.ThmPActualCpt` (`Formalism.HermitianCurvature.HerGenC`, lines 63–78)
`Formalism.FourierDecayAlt.P5.Auser` (`Formalism.HermitianCurvature.CompositeFormulaLevelZero`, lines 340–340)
`Formalism.FourierDecayAlt.P5.mmUser` (`Formalism.HermitianCurvature.CompositeFormulaLevelZero`, lines 342–342)
`Formalism.FourierDecayAlt.P.CutSE.actual_compact_theorem` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.CutSETopFinal`, lines 14–14)
33.88. `Formalism.FourierDecayAlt.KB.Pl` (`Formalism.HermitianCurvature.RealInputsClosedAdmissibleSettings`, lines 97–109)
`Formalism.FourierDecayAlt.KB.IsTopLift2` (`Formalism.HermitianCurvature.RealInputsClosedAdmissibleSettings`, lines 83–88)
`Formalism.FourierDecayAlt.D.IsBaseFrame` (`Formalism.HermitianCurvature.StatementD`, lines 104–106)
`Formalism.FourierDecayAlt.D.TowerNormalised` (`Formalism.HermitianCurvature.StatementD`, lines 152–154)
`Formalism.FourierDecayAlt.P.slice_compact_support_coefficient` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.SliceCompactSupportCoefficient`, lines 268–274)
`Formalism.FourierDecayAlt.P.radial_module_form` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.LevelFormSesquilinearFiniteClass`, lines 3203–3205)
`Formalism.FourierDecayAlt.P.module_form_pos_compact_support` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.ModuleFormPosCompactSupport`, lines 21–45)
`Formalism.FourierDecayAlt.P.vector_isomorphism_compact_support` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.VectorIsomorphismCompactSupport`, lines 117–117)
`Formalism.FourierDecayAlt.P.size_standard_compact` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.SizeStandardCompact`, lines 189–189)
`Formalism.FourierDecayAlt.P.PBASM.rr_top_rep` (`Formalism.HermitianCurvature.HermitianCurvatureProofs.PositivityBoundOfActual`, lines 45–77)
33.89. `Formalism.QuantumDeterminants.IsSection` (`Formalism.FormalDegreeBessel.StatementRP`, lines 74–78)
`Formalism.MainNotions.S.ThB.jNorm2` (`Formalism.SharpRestriction.CuspSpanNormBounds.LeadMinorK`, lines 86–88)
`Formalism.QuantumDeterminants.jNormSq` (`Formalism.FormalDegreeBessel.StatementBF`, lines 151–153)
`Formalism.QuantumDeterminants.bcWeight` (`Formalism.FormalDegreeBessel.StatementBF`, lines 146–149)
33.89. `PI.BesselPeriodGGP.eps48` (`Formalism.PeriodIntegrals.BesselPeriodGGP`, lines 24–25)
`CompanionTheory.WaveGrid.WEnd.WBp` (`CompanionTheory.AutomorphicTheory.BesselWaveGrids.PowFourLeRpowNineHalves`, lines 85–103)

Formalism.L2.SG.Pf.BesselExpansion.CoefCC1 (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.TBBTie, lines 30–34)
CompanionTheory.WaveGrid.WGlue.wLoc1N_of (CompanionTheory.AutomorphicTheory.BesselWaveGrids.WeightedUniformRotatedOfExpansion, lines 30–159)
CompanionTheory.WaveGrid.weighted_bound_rotated_uniform_of_hypsG (CompanionTheory.AutomorphicTheory.BesselWaveGrids.WeightedBoundRotatedUniformOfHypsG, lines 7–8)
CompanionTheory.PieriExpansion.hypsG_of_zuck (CompanionTheory.AutomorphicTheory.PieriRules.InfinitesimalCharacterDiscreteSeries, lines 51–53)
CompanionTheory.Zuck.zuckerman_gap (CompanionTheory.AutomorphicTheory.ZuckermanCommutation.ZuckermanFunctors.ZuckermanGapSmall, lines 9–9)
33.90. Formalism.MainNotions.FrameBoundFromBesselTorus.OneBesselTor (Formalism.SharpRestriction.CuspSpanNormBounds.IsPtRepTrDat, lines 14–29)
Formalism.MainNotions.OnePlaceExpansion.frame_bound_of_bessel_torus (Formalism.SharpRestriction.CuspSpanNormBounds.OnePlaceExpansion, lines 71–115)
CompanionTheory.WaveGrid.WGlue.wLoc1_of_N (CompanionTheory.AutomorphicTheory.BesselWaveGrids.WeightedUniformRotatedOfExpansion, lines 161–186)
CompanionTheory.WaveGrid.WGlue.w_in_frame (CompanionTheory.AutomorphicTheory.BesselWaveGrids.WeightedUniformRotatedOfExpansion, lines 218–273)
Formalism.MainNotions.TheoremBConstants.num_Fr (Formalism.SharpRestriction.CuspSpanNormBounds.TheoremBConstants, lines 796–826)
Formalism.MainNotions.TheoremBConstants.GsC (Formalism.SharpRestriction.CuspSpanNormBounds.TheoremBConstants, lines 133–133)
Formalism.MainNotions.TheoremBConstants.FrC (Formalism.SharpRestriction.CuspSpanNormBounds.TheoremBConstants, lines 129–131)
33.91. CompanionTheory.WaveGrid.weighted_bound_rotated_uniform (CompanionTheory.AutomorphicTheory.BesselWaveGrids.ThmBFinal, lines 7–8)
CompanionTheory.WaveGrid.WGlue.weighted_uniform_rotated_of_expansion_alt_alt (CompanionTheory.AutomorphicTheory.BesselWaveGrids.WeightedUniformRotatedOfExpansion, lines 295–430)
Formalism.MainNotions.S.ThB.orbSq (Formalism.SharpRestriction.CuspSpanNormBounds.LeadMinorK, lines 103–106)
33.92. GlobalTheory.HL.QuadricShare.shares (Development.AbelianVarietyTheory.Proofs.FlagMeasures.QuadricCharacterLeakSharesFinite, lines 285–372)
GlobalTheory.HL.GaussSummand.norm_gauss_sum_le (Development.AbelianVarietyTheory.Proofs.FlagMeasures.NormGaussSumLe, lines 48–190)
GlobalTheory.two_geometric_sums_index_bound (Development.AbelianVarietyTheory.Proofs.FlagMeasures.TwoGeometricSumsIndexBound, lines 10–106)
33.93. Formalism.L2.SG.NGCut.TorsionTotientBound.card_mul_conj_eq_le_of_fractionalIdeal (Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.CardMulConjEqLeOfFractionalIdeal, lines 10–102)
Formalism.L2.SG.NGCut.NormalInvariants.count_bound (Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.NormalInvariants, lines 97–125)
33.94. Formalism.FourierDecayAlt.PB.Terms.allPairs (Formalism.HermitianCurvature.StatementPB, lines 65–65)
Formalism.MainNotions.S.Point (Formalism.SharpRestriction.CuspSpanNormBounds.Point, lines 11–12)
Formalism.MainNotions.S.ThB.leadMinorK (Formalism.SharpRestriction.CuspSpanNormBounds.LeadMinorK, lines 135–136)
Formalism.MainNotions.S.ThB.Gglob (Formalism.SharpRestriction.CuspSpanNormBounds.LeadMinorK, lines 145–148)
Formalism.L2.SG.RowsPos.RowNGP (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RowsPos, lines 37–41)
Formalism.L2.SG.RowsPos.RowMainEqP (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RowsPos, lines 43–47)
33.95. HeckeCharacterZeroLocus.SiegelWeilU.IsContDensityOn (Formalism.UnipotentAdelicQuotients.SWXSDefs, lines 167–171)
GlobalTheory.CompanionTheory.SW.TopDens.siegelDensities (CompanionTheory.SpectralGeometry.LocalGlobalProofs.AdelicAnalysis.SiegelDensitiesOfHermitianForm, lines 47–106)
33.96. GlobalTheory.CompanionTheory.SW.siegelWeilUnitaryConvergent (CompanionTheory.SpectralGeometry.LocalGlobalProofs.AdelicAnalysis.Imports, lines 10–12)
GlobalTheory.CompanionTheory.SW.finVol_unitaryGroup (CompanionTheory.SpectralGeometry.LocalGlobalProofs.AdelicAnalysis.UnitaryGroupFiniteVolume, lines 8–9)
GlobalTheory.T.siegel_densities_of_hermitian_form (CompanionTheory.SpectralGeometry.LocalGlobalProofs.AdelicAnalysis.SiegelDensitiesOfHermitianFormStatement, lines 7–8)
Formalism.L2.SG.RealObj.EnergyConvJ.BM.realObj_inInfinityX_jsz (CompanionTheory.AutomorphicTheory.LocalBesselUniqueness.BesselModels.RealObjInInfinityXJsz, lines 17–20)
GlobalTheory.St.stepChoiceDataC_of_siegelWeil_inInfinity (Development.AbelianVarietyTheory.Proofs.FlagMeasures.StepChoiceConstantsCOfSiegelWeilAtInfinity, lines 9–15)
GlobalTheory.localMass_le_genericMass_nine (Development.AbelianVarietyTheory.Proofs.FlagMeasures.LocalMassLeGenericMassConstNine, lines 7–9)
GlobalTheory.T.familyGenericMassPositiveBounded_proof (Development.AbelianVarietyTheory.Proofs.FlagMeasures.FamilyGenericMassPositiveBoundedProofStatement, lines 13–14)
33.97. Formalism.L2.SG.Pf.BDatR.jointAmplitudeFactorizationRotated_of (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.BDataOfR, lines 117–172)
Formalism.L2.SG.Pf.ThmBChainG.jointAmplitudeRotated_generic (Formalism.LocalToGlobal.SiegelLatticeTheory.

FlagStabilizerTensors.RootLatticeThresholds.ThmBChainG, lines 20–71)
Formalism.MainNotions.S.RMeas (Formalism.SharpRestriction.CuspSpanNormBounds.NormalizedLocalScale, lines 163–166)
Formalism.MainNotions.S.GoodW (Formalism.SharpRestriction.CuspSpanNormBounds.NormalizedLocalScale, lines 127–135)
Formalism.MainNotions.S.IsQuotMeas (Formalism.SharpRestriction.CuspSpanNormBounds.RelevantZeroGramTerms, lines 208–210)
Formalism.L2.SG.NGData.a2C (Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.NGData, lines 43–44)
Formalism.L2.SG.Pf.MainEqCut.a2GC (Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.MainEqCut, lines 122–124)
33.98. Formalism.MainStatements.RowRest.analytic_pinnedTheoremB_uniform (CompanionTheory.AutomorphicTheory.UniformLocalFormulas.Closed, lines 20–22)
Formalism.MainStatements.RowRest.bDat_theoremB_global (CompanionTheory.AutomorphicTheory.UniformLocalFormulas.Closed, lines 24–26)
33.99. GlobalTheory.rowNGP_of_siegelWeil (Development.AbelianVarietyTheory.Proofs.FlagMeasures.RowPositivityOfSiegelWeil, lines 10–15)
GlobalTheory.rowMainEqP_of_siegelWeil (Development.AbelianVarietyTheory.Proofs.FlagMeasures.RowMainEqPOfSiegelWeil, lines 10–16)
33.100. Formalism.HermitianLatticeCounting.P.nondegenerateRowBound (Formalism.FlagUnipotentLattices.MinkowskiFactorialBounds.LatticeSumBounds, lines 914–967)
Formalism.HermitianLatticeCounting.P.degenerateRowBound (Formalism.FlagUnipotentLattices.MinkowskiFactorialBounds.LatticeSumBounds, lines 969–1028)
33.101. Formalism.HermitianLatticeCounting.P.prodMinLeOfProdRatio (Formalism.FlagUnipotentLattices.MinkowskiFactorialBounds.LatticeSumBounds, lines 1150–1217)
Formalism.MainNotions.BridgeLemma₂.rowsT_small_one (Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsTrm, lines 839–893)
Formalism.MainNotions.BridgeLemma₂.rowsT_Mt_small (Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsTrm, lines 1057–1127)
Formalism.MainNotions.BridgeLemma₂.small_term_pow_le (Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsOneC, lines 152–195)
33.102. Formalism.FourierDecayAlt.PB.Terms.allPairs (Formalism.HermitianCurvature.StatementPB, lines 65–65)
Formalism.L2.SG.TermOp.wt (Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.TermOp, lines 341–341)
Formalism.L2.SG.TermOp.vec (Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.TermOp, lines 343–345)
Formalism.MainNotions.S.ThB.leadMinorK (Formalism.SharpRestriction.CuspSpanNormBounds.LeadMinorK, lines 135–136)
Formalism.MainNotions.S.ThB.Gglob (Formalism.SharpRestriction.CuspSpanNormBounds.LeadMinorK, lines 145–148)
33.103. GlobalTheory.RowsFive.rowsND_of (Development.AbelianVarietyTheory.Proofs.FlagMeasures.AdmissibleFamilySatisfiesRowEstimates, lines 29–131)
GlobalTheory.RowsFive.rowsDeg_of (Development.AbelianVarietyTheory.Proofs.FlagMeasures.AdmissibleFamilySatisfiesRowEstimates, lines 133–251)
Formalism.MainNotions.BridgeLemma₂.rows_ND_record (Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsND, lines 324–365)
Formalism.MainNotions.BridgeLemma₂.rows_nd_prod (Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsND, lines 191–276)
Formalism.MainNotions.BridgeLemma₂.rows_deg_count (Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsDeg, lines 185–231)
Formalism.MainNotions.BridgeLemma₂.rows_deg_prod (Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsDeg, lines 233–329)
Formalism.MainNotions.BridgeLemma₂.rows_cF_le (Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsNDn, lines 61–79)
Formalism.MainNotions.BridgeLemma₂.rows_q_bound (Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsNDF, lines 11–39)
33.104. Formalism.MainNotions.BridgeLemma₂.rowsT_small_one (Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsTrm, lines 839–893)
33.105. Formalism.MainNotions.S.Cnt.Hyp (Formalism.SharpRestriction.CuspSpanNormBounds.CntPartners, lines 36–48)
Formalism.MainNotions.BridgeLemma₂.small_term_pow_le (Formalism.SharpRestriction.CuspSpanNormBounds.CayleyDegreeNondegeneracy.RowsOneC, lines 152–195)
Formalism.MainNotions.BridgeLemma.RowsOne (Formalism.SharpRestriction.CuspSpanNormBounds.BruhatCellsKronecker.FamilyPacketOfEquivalentStatement, lines 389–389)
33.106. GlobalTheory.LoopRowBoundsGap.rows_along_family (Development.AbelianVarietyTheory.Proofs.FlagMeasures.LoopRowBoundsAlongFamilyLargeGaps, lines 25–43)
Formalism.L2.SG.LoopsTies.loop_evaluation_threshold_zero (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.LoopEvaluationThresholdZero, lines 44–119)
Formalism.FourierDecayAlt.P.GrowthPolynomialOperator.t_LoopsJoint (Formalism.HermitianCurvature.HermitianCurvatureProofs.GrowthOfBasic, lines 1191–1228)
33.107. Formalism.FourierDecayAlt.P.Jx2.t_LatticeSumsJoint (Formalism.HermitianCurvature.HermitianCurvatureProofs.LevelSizesJointMean, lines 124–166)
Formalism.FourierDecayAlt.S6.perLat (Formalism.HermitianCurvature.PrimeToDifferent, lines 51–52)
Formalism.FourierDecayAlt.S6.liftLat (Formalism.HermitianCurvature.PrimeToDifferent, lines 54–55)
33.108. Formalism.FourierDecayAlt.P.change_of_variables_xy (Formalism.HermitianCurvature.HermitianCurvatureProofs.ChangeOfVariablesXy, lines 22–23)

Formalism.FourierDecayAlt.P.germ_rate (Formalism.HermitianCurvature.HermitianCurvatureProofs.ChangeOfVariablesXy, lines 34–37)
33.109. Formalism.FourierDecayAlt.P.PBK.memLp_bz_of_sup (Formalism.HermitianCurvature.HermitianCurvatureProofs.MemLpBzOfSup, lines 16–79)
Formalism.FourierDecayAlt.P.PBK.GermD.tendsto_cross (Formalism.HermitianCurvature.HermitianCurvatureProofs.TendstoRatioPow, lines 77–121)
Formalism.FourierDecayAlt.P.PBK.GermD.germLoc_tail (Formalism.HermitianCurvature.HermitianCurvatureProofs.GermLocTail, lines 15–47)
33.110. Formalism.FourierDecayAlt.P.FormNormSumIdentities.cutN_smooth (Formalism.HermitianCurvature.HermitianCurvatureProofs.PbGermRateLowBOfParts, lines 132–140)
Formalism.FourierDecayAlt.P.FormNormSumIdentities.cutN_supp (Formalism.HermitianCurvature.HermitianCurvatureProofs.PbGermRateLowBOfParts, lines 142–175)
Formalism.FourierDecayAlt.P.JFxG.cptInCC_G (Formalism.HermitianCurvature.HermitianCurvatureProofs.JfBetaAdjCOfLemmaBNorm, lines 24–43)
33.111. Formalism.FourierDecayAlt.P.GrowthPolynomialOperator.t_MainRow (Formalism.HermitianCurvature.HermitianCurvatureProofs.GrowthOfBasic, lines 1070–1183)
33.112. Formalism.L2.SG.GGMirabolic.termsGGBound_of_matrix (Formalism.LocalToGlobal.SiegelLatticeTheory.ClosedLocalOrbits.GGMirabolic, lines 136–144)
33.112. Formalism.FourierDecayAlt.PB.Terms.NormExpansionOn (Formalism.HermitianCurvature.StatementPB, lines 86–87)
Formalism.FourierDecayAlt.PB.Terms.RowOrth (Formalism.HermitianCurvature.StatementPB, lines 92–94)
33.113. Formalism.FourierDecayAlt.P.GrowthPolynomialOperator.t_PairBoundFromLocal (Formalism.HermitianCurvature.HermitianCurvatureProofs.GrowthOfBasic, lines 1236–1267)
33.113. Formalism.FourierDecayAlt.P.GrowthPolynomialOperator.t_MainRow (Formalism.HermitianCurvature.HermitianCurvatureProofs.GrowthOfBasic, lines 1070–1183)
Formalism.FourierDecayAlt.P.GrowthPolynomialOperator.t_PairBound (Formalism.HermitianCurvature.HermitianCurvatureProofs.GrowthOfBasic, lines 525–550)
Formalism.L2.SG.TheoremSConclusion.theoremS_of_conj (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.TheoremSConclusion, lines 68–82)
Formalism.L2.SG.FixedDecomposition.kppOffC_of_steps0 (Formalism.LocalToGlobal.SiegelLatticeTheory.CokernelGLCounts.RankPointCounts.FixedDecomposition, lines 755–775)

Chapter 34: Theta correspondence, test vectors and evaluation

34.2. GlobalTheory.SplitFlip.bigP (Development.AbelianVarietyTheory.Definitions.Divisors.RankEqCardIffExistsFrameMatrixFirstColumnBlock, lines 716–718)
GlobalTheory.SplitFlip.smallP (Development.AbelianVarietyTheory.Definitions.Divisors.RankEqCardIffExistsFrameMatrixFirstColumnBlock, lines 720–721)
34.1. GlobalTheory.ThK.Dbl.heckeL_ne_zero (Development.WeilBound.ThetaLifts.LatticeBasics.ExistsIsHeckeLOfCompleted, lines 134–160)
GlobalTheory.HeckePartial.removed_ne_zero (Development.AbelianVarietyTheory.PublishedTheorems.TwistedTraceFormula.HeckeLPartialContinuationHolds, lines 40–42)
34.1. Formalism.BasicNotions.TH.fockAct1 (Formalism.LocalRingCardinality.DataTheta, lines 209–210)
Formalism.BasicNotions.TH.fockAct2 (Formalism.LocalRingCardinality.DataTheta, lines 212–213)
34.2. PositivityTheory.KReal.nontrivial_of_pair (Development.WeilBound.CoreStatements.GaussianSeminormBounds.RealSeminormFiltration.CoreNonv, lines 160–238)
PositivityTheory.KReal.pair_pos (Development.WeilBound.CoreStatements.GaussianSeminormBounds.RealSeminormFiltration.CorePair, lines 101–113)
PositivityTheory.KReal.pair_neg (Development.WeilBound.CoreStatements.GaussianSeminormBounds.RealSeminormFiltration.CorePair, lines 115–127)
PositivityTheory.KReal.side_of (Development.WeilBound.CoreStatements.GaussianSeminormBounds.RealSeminormFiltration.CoreSide, lines 159–211)
PositivityTheory.KReal.la_eq (Development.WeilBound.CoreStatements.GaussianSeminormBounds.RealSeminormFiltration.CoreComb, lines 355–369)
34.3. PositivityTheory.KReal.star_th (Development.WeilBound.CoreStatements.GaussianSeminormBounds.RealSeminormFiltration.CoreStar, lines 77–166)
PositivityTheory.KReal.root_of_hw (Development.WeilBound.CoreStatements.GaussianSeminormBounds.RealSeminormFiltration.CoreRoot, lines 18–135)
PositivityTheory.KReal.comb_pos (Development.WeilBound.CoreStatements.GaussianSeminormBounds.RealSeminormFiltration.CoreComb, lines 119–207)
PositivityTheory.KReal.comb_neg (Development.WeilBound.CoreStatements.GaussianSeminormBounds.RealSeminormFiltration.CoreComb, lines 209–303)
34.4. PositivityTheory.KReal.cas_bound_gl (Development.WeilBound.CoreStatements.GaussianSeminormBounds.RealSeminormFiltration.CoreCasGl, lines 158–228)
PositivityTheory.KReal.common_quot (Development.WeilBound.CoreStatements.GaussianSeminormBounds.RealSeminormFiltration.CoreMain, lines 583–713)

`PositivityTheory.KReal.coreLM` (`Development.WeilBound.CoreStatements.GaussianSeminormBounds.RealSeminormFiltration.CoreLM`, lines 14–30)
`PositivityTheory.KRealA.onePlace` (`Development.WeilBound.CoreStatements.GaussianSeminormBounds.RealSeminormFiltration.KRealAsmOne`, lines 143–154)
34.5. `GlobalTheory.ThK.Dbl.bump` (`Development.WeilBound.ThetaLifts.LatticeBasics.NearUnits`, lines 113–185)
`GlobalTheory.ThK.Dbl.nonZeroBottom_of_doubling_integral` (`Development.WeilBound.ThetaLifts.LatticeBasics.BotAsm`, lines 117–146)
`GlobalTheory.ThK.Dbl.pair_ne_zero` (`Development.WeilBound.ThetaLifts.LatticeBasics.BotAsm`, lines 58–115)
`GlobalTheory.ThK.Dbl.badFactor_of` (`Development.WeilBound.ThetaLifts.LatticeBasics.BadFactor`, lines 96–191)
34.6. `GlobalTheory.ThK.Dbl.LD.eq_case` (`Development.WeilBound.ThetaLifts.LatticeBasics.LocDistE`, lines 106–143)
`GlobalTheory.ThK.Dbl.TC.inv_loc` (`Development.WeilBound.ThetaLifts.LatticeBasics.ThConstInv`, lines 86–152)
`GlobalTheory.ThK.Dbl.thConst_of'` (`Development.WeilBound.ThetaLifts.LatticeBasics.ThConstFin`, lines 13–15)
`GlobalTheory.ThK.Dbl.normOne_mem` (`Development.WeilBound.ThetaLifts.LatticeBasics.ConstIdele`, lines 134–181)
34.7. `UpperTheory.LiteratureTheorems.theta_integrals_agree_and_local_matching_of_literature` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin`, lines 565–566)
`PositivityTheory.ThKR.theoremK_head2` (`Development.WeilBound.ThetaLifts.Reductions.FiniteClosure.TheoremKOfMod`, lines 26–27)
`GlobalTheory.IndefiniteRowSideFullHecke.WeilTensorModels.ThetaIntegralsAgree` (`Development.AbelianVarietyTheory.BlockTheory.DivisorBase.WeilTensorModels`, lines 585–592)
`GlobalTheory.IndefiniteRowSideFullHecke.WeilTensorModels.LocalMatching` (`Development.AbelianVarietyTheory.BlockTheory.DivisorBase.WeilTensorModels`, lines 581–583)
`GlobalTheory.IndefiniteRowSideFullHecke.WeilTensorModels.StepBeta` (`Development.AbelianVarietyTheory.BlockTheory.DivisorBase.WeilTensorModels`, lines 559–567)
`GlobalTheory.IndefiniteRowSideFullHecke.WeilTensorModels.StepAlpha` (`Development.AbelianVarietyTheory.BlockTheory.DivisorBase.WeilTensorModels`, lines 571–579)
34.7. `PositivityTheory.ThKG.Leaves.const_of` (`Development.WeilBound.ThetaLifts.ConeHypotheses.CtSmallCone`, lines 88–136)
`PositivityTheory.ThKG.Leaves.isConst_of` (`Development.WeilBound.ThetaLifts.ConeHypotheses.CtSmallCone`, lines 179–189)
`PositivityTheory.ThKG.Leaves.kConst` (`Development.WeilBound.ThetaLifts.ConeHypotheses.CtSmallCone`, lines 191–198)
`PositivityTheory.ThKG.Leaves.kBeta` (`Development.WeilBound.ThetaLifts.ConeHypotheses.CtSmallCone`, lines 200–218)
`PositivityTheory.ThKG.Leaves.kAlpha` (`Development.WeilBound.ThetaLifts.ConeHypotheses.CtSmallCone`, lines 220–231)
`PositivityTheory.ThKEnd.Two.twoStep_of'` (`Development.WeilBound.ThetaLifts.EigenvalueEndgame.SignatureSplitting.TwoStepOf`, lines 22–102)
34.8. `PositivityTheory.ThKEst.sqInt` (`Development.WeilBound.ThetaLifts.FrameEstimates.WeilConv`, lines 68–69)
`PositivityTheory.ThKEst.iCont` (`Development.WeilBound.ThetaLifts.FrameEstimates.WeilConv`, lines 71–72)
34.9. `PositivityTheory.Sph.SplitPl.irrAt_ne` (`Development.WeilBound.ThetaLifts.SphericalSplitting.SphSplitGeo`, lines 21–45)
`GlobalTheory.CompanionTheory.SoCoinvNarrow.soCoinv_of` (`Development.AutomorphicSpectrum.Proofs.BlockDecomposition.SoCoinvOfNarrow`, lines 197–245)
34.10. `PositivityTheory.ThKF.factor_of` (`Development.WeilBound.ThetaLifts.CartanFactorisation.Factor`, lines 117–178)
`PositivityTheory.Sph.theta_kills_cusp_forms_spherical` (`Development.WeilBound.ThetaLifts.SphericalLattices.ThetaKillsCuspFormsSpherical`, lines 154–203)
34.11. `PositivityTheory.ThKEnd.Comp.IZero` (`Development.WeilBound.ThetaLifts.EigenvalueEndgame.EigenKilling.ConeIntegralFactorisation`, lines 76–80)
`PositivityTheory.ThKEnd.Comp.realVanish_of` (`Development.WeilBound.ThetaLifts.EigenvalueEndgame.EigenKilling.RealVanishOf`, lines 60–112)
34.12. `GlobalTheory.ThK.Dbl.coefLine_of` (`Development.WeilBound.ThetaLifts.LatticeBasics.CoefLine`, lines 13–61)
34.13. `PositivityTheory.ThKG.Tow.Pln.exists_plane` (`Development.WeilBound.ThetaLifts.TowerModels.PlaneMain`, lines 26–136)
`PositivityTheory.ThKG.Tow.towerB_of` (`Development.WeilBound.ThetaLifts.TowerModels.TowerCodimensionTwoOf`, lines 32–73)
`PositivityTheory.ThKG.Tow.towerA_of` (`Development.WeilBound.ThetaLifts.TowerModels.TowerCodimensionTwoOf`, lines 75–126)
`PositivityTheory.ThKG.Tow.towerCodimensionTwo_of` (`Development.WeilBound.ThetaLifts.TowerModels.TowerCodimensionTwoOf`, lines 128–172)
34.14. `PositivityTheory.ThKCT.genSmall_of` (`Development.WeilBound.ThetaLifts.SmallCones.GenSmallOf`, lines 18–128)
`PositivityTheory.ThKG.Loc.Cone.Id.idealId'` (`Development.WeilBound.ThetaLifts.LocalTheory.ConeShifts.IdealId0`, lines 49–73)
`PositivityTheory.ThKG.Std.eps_eq` (`Development.WeilBound.ThetaLifts.LocalTheory.StandardValuations.ExistsVol`, lines 85–99)
`PositivityTheory.ThKG.Loc.locB_of` (`Development.WeilBound.ThetaLifts.LocalTheory.LocBOf`, lines 17–249)
`PositivityTheory.ThKG.Loc.locA_of` (`Development.WeilBound.ThetaLifts.LocalTheory.LocAOf`, lines 17–173)
`PositivityTheory.ThKG.keyB_of` (`Development.WeilBound.ThetaLifts.ConeHypotheses.KeyAOf`, lines 37–70)
`PositivityTheory.ThKG.keyA_of` (`Development.WeilBound.ThetaLifts.ConeHypotheses.KeyAOf`, lines 88–126)
34.15. `GlobalTheory.ThK.Dbl.ct_zero` (`Development.WeilBound.ThetaLifts.LatticeBasics.CuspLift`, lines 52–162)
`GlobalTheory.ThK.Dbl.cusp_lift` (`Development.WeilBound.ThetaLifts.LatticeBasics.OrthAn`, lines 73–201)
`GlobalTheory.ThK.Dbl.orthAn_of` (`Development.WeilBound.ThetaLifts.LatticeBasics.OrthAn`, lines 333–409)

PositivityTheory.ThKEnd.Cusp.Key.keyThree_of_locThree (Development.WeilBound.ThetaLifts.EigenvalueEndgame.CuspidalAverages.ActsOfReads, lines 495–505)
PositivityTheory.ThKEnd.Cusp.Key.keyTwo_of_locTwo_vacBar (Development.WeilBound.ThetaLifts.EigenvalueEndgame.CuspidalAverages.ActsOfReads, lines 629–704)
PositivityTheory.ThKEnd.Cusp.kill_coef (Development.WeilBound.ThetaLifts.EigenvalueEndgame.CuspidalAverages.KillCoef, lines 14–60)
PositivityTheory.ThKEnd.Cusp.cuspOfLine_of_keys (Development.WeilBound.ThetaLifts.EigenvalueEndgame.CuspidalAverages.CuspOfLineOfKeys, lines 169–198)
PositivityTheory.ThKEnd.zero_of_line (Development.WeilBound.ThetaLifts.EigenvalueEndgame.TowerCodimensionTwo, lines 73–85)
34.16. GlobalTheory.ThK.Dbl.EisBL (Development.WeilBound.ThetaLifts.LatticeBasics.HypothesesE, lines 46–53)
GlobalTheory.ThK.Dbl.SiegelWeilStd (Development.WeilBound.ThetaLifts.LatticeBasics.HypothesesS, lines 11–21)
UpperTheory.LiteratureTheorems.eis_bl_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 561–562)
UpperTheory.LiteratureTheorems.siegel_weil_std_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 563–564)
34.17. GlobalTheory.Published.SiegelWeilUnitaryConvergent (Development.AbelianVarietyTheory.PublishedTheorems.SiegelWeil₂.SiegelWeilUnitaryConvergent, lines 18–32)
HeckeCharacterZeroLocus.SiegelWeilU.SiegelWeilUnitaryConvergent (Formalism.UnipotentAdelicQuotients.SiegelWeilUnitaryConvergent, lines 46–49)
HeckeCharacterZeroLocus.SiegelWeilU.SiegelDensities (Formalism.UnipotentAdelicQuotients.SiegelWeilUnitaryConvergent, lines 18–40)
GlobalTheory.Published.RallisInnerProductConvergent (Development.AbelianVarietyTheory.PublishedTheorems.TwistedTraceFormula.LowerLiftsZeroOfWittIdxEqZero, lines 41–59)
UpperTheory.LiteratureTheorems.siegel_weil_unitary_convergent_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 481–482)
UpperTheory.LiteratureTheorems.siegelWeilU_unitary_convergent_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 581–582)
UpperTheory.LiteratureTheorems.rallis_inner_product_convergent_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 485–486)
34.18. GlobalTheory.Hk3.howeUnram_holds (Development.AbelianVarietyTheory.Proofs.SatakeParameters.RhoMemOfMkEq, lines 250–256)
GlobalTheory.Hk3.HoweUnram (Development.AbelianVarietyTheory.PublishedTheorems.GaloisGeometry₂.HoweUnram, lines 8–19)
34.18. GlobalTheory.Hk3.locVac_span_classes_indep (Development.AbelianVarietyTheory.Proofs.SatakeParameters.LatticeModelVacuumTranslateClassesIndependent, lines 8–98)
GlobalTheory.Hk3.HoweG.gramClass_orbit_mod_lower (Development.AbelianVarietyTheory.Proofs.SatakeParameters.ExistsConjugateModelExtremalDualParameter, lines 16–129)
GlobalTheory.Hk3.HoweDuality.fixed_mem_span (Development.AbelianVarietyTheory.Proofs.SatakeParameters.RhoMemOfMkEq, lines 145–240)
34.19. Formalism.MainNotions.SizesAt (Formalism.SharpRestriction.ElevenLe, lines 134–138)
34.20. PositivityTheory.Sph.isVacSph_of (Development.WeilBound.ThetaLifts.SphericalLattices.ExistsFixedSum, lines 90–133)
PositivityTheory.Sph.howeInert (Development.WeilBound.ThetaLifts.SphericalLattices.HoweDuality, lines 13–28)
34.21. GlobalTheory.MoyalTheta.orthogonality (Development.AbelianVarietyTheory.Proofs.TwistedOperators.MoyalOrthogonalityHeisenbergThetaFunctions, lines 25–75)
GlobalTheory.Asc.annVac (Development.AbelianVarietyTheory.Proofs.TwistedOperators.HeisenbergVacuumCommutantPreservesLatticeVacua, lines 36–77)
GlobalTheory.Irr.irr (Development.AbelianVarietyTheory.Proofs.TwistedOperators.TameFunctionalVacuumIrreducibility, lines 57–76)
34.22. GlobalTheory.ConjDataTotal.fockTot (Development.AbelianVarietyTheory.Proofs.TwistedOperators.FockConjugatesSpanDepthData, lines 190–270)
GlobalTheory.FirstRowDepth.datumDep (Development.AbelianVarietyTheory.Proofs.TwistedOperators.FirstRowDataHaveDepth, lines 19–29)
34.23. GlobalTheory.Asc.ascent_of_depFixed (Development.AbelianVarietyTheory.Proofs.TwistedOperators.BesselPeriodNonvanishingAscentOfDepthData, lines 21–25)
GlobalTheory.Asc.ascent_of_rungs (Development.AbelianVarietyTheory.Proofs.TwistedOperators.BesselPeriodAscentOfTowerRungs, lines 15–41)
34.24. GlobalTheory.GLq.ESerTie (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.LabelDatum, lines 114–116)
GlobalTheory.GLq.eSerC (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.LabelDatum, lines 102–103)
34.25. UpperTheory.Lit.functionalsSeparate_of_howeUnram_of_siegelWeil (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.FunctionalsSeparateOfHoweUnramOfSiegelWeil, lines 17–19)
UpperTheory.Lit.real_functionals_separate_of_hypothesis (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.RealFunctionalsSeparateOfHypothesis, lines 15–30)
GlobalTheory.IndefiniteRowSideFullHecke.LevelPlusSubgroup.FunctionalsSeparate (Development.AbelianVarietyTheory.BlockTheory.DivisorBase.LevelPlusSubgroup, lines 83–91)
34.25. GlobalTheory.SepRows.weakSep_of_steps (Development.AbelianVarietyTheory.Proofs.TwistedOperators.RowSeparationByRealFunctionalsOfChainSteps, lines 71–85)

UpperTheory.Lit.Bur.separate_of_burGapPos (Development.AutomorphicArithmetic.ThetaCorrespondence.BesselModels.SepGap, lines 222–256)
34.26. PositivityTheory.UnitaryDecomposition.DecB.dec_U (Development.WeilBound.CoreStatements.BranchingTraces.CasimirDecomposition.UnitaryDecompositionExists, lines 45–176)
PositivityTheory.UnitaryDecomposition.Com.comat_of (Development.WeilBound.CoreStatements.BranchingTraces.BranchingComatrices.OneLeDiff, lines 136–276)
CompanionTheory.Lime.T3.kFiniteWordBound_indefUnitary (Development.WeilBound.CoreStatements.OrbitDecayEstimates.CurveDerivativeDecay.KFiniteWordBoundIndefUnitary, lines 245–358)
PositivityTheory.UnitaryDecomposition.fdeg_shift (Development.WeilBound.CoreStatements.BranchingTraces.ContinuousEmb, lines 190–238)
34.27. PositivityTheory.RgHe.A.Pf.bigProjP_of (Development.WeilBound.RepresentationTheoryGL.HolomorphicDiscreteSeries.SchurEigenProjections.SchurFiniteDimensionality.IsFdOfSchurK, lines 53–275)
34.28. PositivityTheory.UnitaryDecomposition.momentHeA_of (Development.WeilBound.RepresentationTheoryGL.HolomorphicDiscreteSeries.SchurEigenProjections.HolomorphicMoments.MomentHigherEstimateHolds, lines 28–179)
PositivityTheory.UnitaryDecomposition.moment_higher_estimate_holds (Development.WeilBound.RepresentationTheoryGL.HolomorphicDiscreteSeries.SchurEigenProjections.HolomorphicMoments.MomentHigherEstimateHolds, lines 181–181)
34.29. UpperTheory.Lit.Branch.packEx_of_eigen (Development.AutomorphicArithmetic.ThetaCorrespondence.RealBands.BranchPack, lines 205–220)
UpperTheory.Lit.Cusp.exists_cusp_form_theta_period_ne_zero (Development.AutomorphicArithmetic.ThetaCorrespondence.RealBands.Period, lines 18–107)
34.30. UpperTheory.Lit.Branch.periodNe_of_eigen (Development.AutomorphicArithmetic.ThetaCorrespondence.RealBands.BranchPeriod, lines 14–23)
34.31. GlobalTheory.ChainOnto.heckeVec_howe (Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.UnramifiedHeckeVacuumLargerEqSmaller, lines 23–51)
GlobalTheory.SphTheta.satU_eq_satU_liftPar_of_heckeVec_eq (Development.AbelianVarietyTheory.Proofs.TwistedOperators.SatakeTransformEqAtThetaLiftOfVacuumEq, lines 81–177)
34.32. GlobalTheory.EvenOnto.parBound_of_rapidDecay (Development.AbelianVarietyTheory.Proofs.TwistedOperators.EvenRankChainParamsTrivialBoundRapidDecay, lines 28–62)
GlobalTheory.FinZeta.ParamBound.exists_param_at (Development.AbelianVarietyTheory.Proofs.TwistedOperators.ExistsSatakeParametersBoundedEigenfunction, lines 79–132)
GlobalTheory.ChainOnto.transferOn_all (Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.SphericalParameterTransferAllFinitePlaces, lines 22–26)
GlobalTheory.EvenOnto.upPar_of_transfer (Development.AbelianVarietyTheory.Proofs.TwistedOperators.EvenThetaChainUpwardLiftHeckeParamTransferred, lines 48–72)
34.33. GlobalTheory.CoreOnto.coreOnto_of_chain (Development.AbelianVarietyTheory.Proofs.TwistedOperators.CoreChainLiftsSurjectiveOfEvenRank, lines 28–54)
GlobalTheory.FunPr.ThetaLocalInputs (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.BernsteinUnitaryProductIrreducible, lines 6–59)
GlobalTheory.Published.WeilSiegelFormulaUnitary (Development.AbelianVarietyTheory.PublishedTheorems.TwistedTraceFormula.WeilSiegelFormulaUnitary, lines 56–69)
GlobalTheory.Published.DoublingMethodLocalZetaIntegrals (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.DiscreteSpectrumResidualClassification, lines 6–6)
GlobalTheory.Published.UnramifiedDoublingGL (Development.AbelianVarietyTheory.PublishedTheorems.TwistedTraceFormula.PairwiseDisjointCell, lines 314–324)
GlobalTheory.Published.HarishChandraFinite (Development.AbelianVarietyTheory.PublishedTheorems.AutomorphicForms$_2$.HarishChandraFinite, lines 50–58)
34.34. GlobalTheory.SquareComm.IsTriple (Development.AbelianVarietyTheory.Proofs.TwistedOperators.BlkGLMem, lines 106–128)
GlobalTheory.SquareComm.IsMatch (Development.AbelianVarietyTheory.Proofs.TwistedOperators.BlkGLMem, lines 130–143)
Formalism.IndefR.Datum.IsWeilDatumO (Formalism.LocalToGlobal.IndefiniteReal.LocalNewformFunction, lines 53–75)
GlobalTheory.SquareAgree.kapX (Development.AbelianVarietyTheory.Proofs.TwistedOperators.ThetaEqB, lines 40–44)
34.35. GlobalTheory.T.exists_placewise_seesaw_triple_over_hermitian (Development.AbelianVarietyTheory.Proofs.TwistedOperators.ExistsPlacewiseSeesawTripleOverHermitianStatement, lines 16–27)
GlobalTheory.SquareAgree.kapX (Development.AbelianVarietyTheory.Proofs.TwistedOperators.ThetaEqB, lines 40–44)
GlobalTheory.SquareAgree.thetaInt_eq_modelI (Development.AbelianVarietyTheory.Proofs.TwistedOperators.ConjUMeasurableEmbedding, lines 59–71)
34.36. GlobalTheory.T.twisted_theta_kernels_agree_up_to_scalar (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.ThetaKernelMatchOnOppositeParabolic, lines 189–203)
34.37. GlobalTheory.WeilConv.twisted_weil_representation_convergence (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.TwistedWeilRepresentationConvergence, lines 16–17)
GlobalTheory.T.twisted_local_factors_match (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.TwistedSquareLocalMatching, lines 88–89)
GlobalTheory.T.twisted_square_theta_integrals_eq (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.TwistedSquareThetaIntegralsEq, lines 48–49)
34.37. GlobalTheory.WeilConv.weilConvergence_of_siegelSet (Development.AbelianVarietyTheory.Proofs.TwistedOperators.WeilConvergenceOfSiegelSet, lines 24–82)
GlobalTheory.SiegelSet.lintegral_cone_modulus_lt_top (Development.AbelianVarietyTheory.Proofs.TwistedOperators.LintegralSiegelConeInvModulusLtTopWeil, lines 160–207)

GlobalTheory.SquareLM.twisted_square_local_matching (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.TwistedSquareLocalMatching, lines 18–81)
GlobalTheory.SquareAgree.twisted_theta_integrals_agree (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.TwistedSquareThetaIntegralsEq, lines 17–41)
GlobalTheory.SquareAgree.CoinvPass.coinv_passage (Development.AbelianVarietyTheory.Proofs.TwistedOperators.ConjUInvMemGfC, lines 233–279)
34.37. GlobalTheory.RowTest.NumbersIn (Development.AbelianVarietyTheory.Definitions.TwistedInnerProducts.MemHalfCosetSelf, lines 245–261)
34.38. GlobalTheory.RowEx.BotField.sums_mem_bottomField (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.BottomFieldIsNumberField, lines 18–28)
34.39. GlobalTheory.T.theta_lift_extension_field_exists (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.ThetaLiftExtensionFieldExists, lines 50–51)
GlobalTheory.RowEx.FieldFor (Development.AbelianVarietyTheory.Definitions.TwistedInnerProducts.TwistPlacesOffPrimeQZero, lines 231–236)
GlobalTheory.RowEx.ExField (Development.AbelianVarietyTheory.Definitions.TwistedInnerProducts.TwistPlacesOffPrimeQZero, lines 241–242)
34.39. GlobalTheory.RowEx.exKey_of (Development.AbelianVarietyTheory.Proofs.TwistedOperators.ExKeyOf, lines 23–89)
GlobalTheory.RowEx.exField_of_keyFin_exKey (Development.AbelianVarietyTheory.Proofs.TwistedOperators.ExFieldOfKeyFinExKey, lines 22–36)
34.40. GlobalTheory.Mem.Top.Bl.KPx (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.PositionOfIntro, lines 21–21)
GlobalTheory.Mem.Fld.listField (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.FixedCoefficientFieldInComplex, lines 501–502)
GlobalTheory.Mem.Top.TypesTrue (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.TypeIdempotentCoeffAcrossLayers, lines 32–36)
GlobalTheory.Mem.R3.GrpAt (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.IsAbsIrrCharCompIff, lines 248–255)
Formalism.IndefR.Links.RealDat.ExAt (Formalism.LocalToGlobal.IndefiniteReal.LinksW, lines 130–138)
GlobalTheory.RealGeo.InGeo (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.JEvExists, lines 119–127)
34.41. GlobalTheory.T.test_vector_assembly_holds (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.TestVectorAtDatumOfJoin, lines 39–40)
GlobalTheory.RowTestAssembly.test_vector_at_datum_of_join (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.TestVectorAtDatumOfJoin, lines 20–31)
Formalism.IndefR.Rat.RowTest (Formalism.LocalToGlobal.IndefiniteReal.RationalAdjoints.RatZOfTests, lines 18–20)
34.41. GlobalTheory.RowTestFromClaims.rowTest_of_choice (Development.AbelianVarietyTheory.Proofs.TwistedOperators.RowTestHilbertModelsOfChoiceClaims, lines 104–133)
GlobalTheory.Mem.R3.Rw.RowAt.atDat_rowTest (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.AtDatRowTest, lines 44–64)
GlobalTheory.Mem.LiteralAutomorphy.rowRatW_of_rowTest (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.RowRatWOfRowTest, lines 9–12)
34.42. GlobalTheory.Mem.LiteralAutomorphy.Full (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.GKeyFin, lines 43–47)
GlobalTheory.Mem.LiteralAutomorphy.GridAt (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.GKeyFin, lines 49–54)
GlobalTheory.Mem.LiteralAutomorphy.ExGrid (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.GKeyFin, lines 56–59)
GlobalTheory.Mem.LiteralAutomorphy.GridA (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.GKeyFin, lines 61–62)
GlobalTheory.Mem.LiteralAutomorphy.exGrid_of (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.GridExistenceOfPieces, lines 34–143)
34.43. GlobalTheory.SqTop.SqField.sqIn_of (Development.AbelianVarietyTheory.Proofs.DivisorProofs.IsAlgebraicDivOfBeta, lines 73–127)
GlobalTheory.ToricSumValue.toricSum_eq (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ToricSumEqHeckeLValueMulAuxiliaryFactors, lines 95–130)
34.44. GlobalTheory.Mem.LiteralAutomorphy.evaluation_grid_of_join (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.EvaluationGridOfJoin, lines 23–44)
GlobalTheory.Mem.LiteralAutomorphy.evLgG_gridA_of_exGrid (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.EventuallyLargeGrid, lines 14–41)
34.45. GlobalTheory.FunObj.SqNum.IsTorVal (Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder, lines 274–279)
GlobalTheory.SqTop.thbSt_of_inputs (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ToricValueHeightBoundOfDenominatorsMeans, lines 158–241)
GlobalTheory.BigTop.Fun.thb_tie_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ThbTie, lines 16–60)
34.45. GlobalTheory.DenSumPf.denSum (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LinearCombinationLogBoundRealInequality, lines 266–391)

GlobalTheory.SqTop.lBound_of (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
HeckeLValueUpperBoundPermutedWeightsOfFE, lines 16–91)
GlobalTheory.SqTop.betaArch_of_transport (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
BetaConjugateMeanLogAbsBoundOfTransport, lines 43–144)
GlobalTheory.HeightMeanValueBounds.height_meanVal (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
ToricValueHeightBoundOfDenominatorsMeans, lines 36–119)
34.46. Formalism.BasicNotions.Ranks (Formalism.LocalRingCardinality.Params, lines 6–22)
Formalism.HC.Frame (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.
ZsmulGeneratorEqZeroOfProjection, lines 15–23)
Formalism.HC.Frame.A (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.
ZsmulGeneratorEqZeroOfProjection, lines 29–29)
Formalism.BasicNotions.HCParam (Formalism.LocalRingCardinality.Params, lines 50–69)
Formalism.BasicNotions.sExp (Formalism.LocalRingCardinality.Params, lines 71–71)
Formalism.BasicNotions.sExp' (Formalism.LocalRingCardinality.Params, lines 73–73)
Formalism.BasicNotions.wG (Formalism.LocalRingCardinality.Params, lines 75–75)
Formalism.BasicNotions.Host (Formalism.LocalRingCardinality.DataHost, lines 39–109)
Formalism.BasicNotions.HSpace (Formalism.LocalRingCardinality.DataSpaces, lines 26–46)
Formalism.BasicNotions.Spaces (Formalism.LocalRingCardinality.DataSpaces, lines 77–139)
Formalism.BasicNotions.sigYD (Formalism.LocalRingCardinality.DataSpaces, lines 75–75)
Formalism.BasicNotions.Rows.Data (Formalism.LocalRingCardinality.DataRows, lines 36–168)
Formalism.BasicNotions.HLat (Formalism.LocalRingCardinality.DataArith, lines 61–72)
Formalism.BasicNotions.DivData (Formalism.LocalRingCardinality.DataArith, lines 139–159)
Formalism.BasicNotions.D7 (Formalism.LocalRingCardinality.RhoDetPrimeEqMuTildeMulChiPow, lines 76–334)
Formalism.BasicNotions.Member (Formalism.LocalRingCardinality.RhoDetPrimeEqMuTildeMulChiPow, lines 71–74)
Formalism.BasicNotions.ColumnChain (Formalism.LocalRingCardinality.DataKernels, lines 14–47)
Formalism.BasicNotions.ColumnsAtT (Formalism.LocalRingCardinality.DataKernels, lines 104–137)
Formalism.BasicNotions.PeriodMap (Formalism.LocalRingCardinality.DataKernels, lines 163–182)
Formalism.BasicNotions.MemberForms (Formalism.LocalRingCardinality.DataForms, lines 107–215)
Formalism.BasicNotions.MemberForms.ipC (Formalism.LocalRingCardinality.DataForms, lines 227–227)
Formalism.BasicNotions.MemberForms.ramSub (Formalism.LocalRingCardinality.DataForms, lines 239–239)
Formalism.MainNotions.S.Draft.SData (Formalism.SharpRestriction.CuspSpanNormBounds.DataS, lines 13–84)
Formalism.BasicNotions.SizeData₂ (Formalism.LocalRingCardinality.DataParams₂, lines 76–182)
GlobalTheory.Mem.Stages (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.AssembleLattice, lines
188–195)
GlobalTheory.Mem.Top.Ctx (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.
CmQuadraticFormEqAlternatingForm, lines 28–43)
GlobalTheory.Mem.Top.TwDat (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.
CmQuadraticFormEqAlternatingForm, lines 79–89)
GlobalTheory.Mem.Top.V4.Core (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.RationalRootsDef,
lines 38–61)
GlobalTheory.EX.OB.Lay (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.SetUnipotentTranslate, lines
83–119)
GlobalTheory.EX.OB.LedPart (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.SetUnipotentTranslate, lines
148–159)
GlobalTheory.Fc.BPt (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.
ValidForConstructedPointClassesEventuallyWorking, lines 67–83)
GlobalTheory.Fc.Built (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.
ValidForConstructedPointClassesEventuallyWorking, lines 250–252)
GlobalTheory.BigTop.IndefiniteRowSideFullHecke (Development.AbelianVarietyTheory.Definitions.Divisors.
TupleReadBaseTupleEqFamily, lines 61–109)
Formalism.IndefR.Links.RealDat3 (Formalism.LocalToGlobal.IndefiniteReal.ZeroPartEigenvalueConvention, lines 89–97)
Formalism.IndefR.FWT.FWTie (Formalism.LocalToGlobal.IndefiniteReal.FWTie, lines 17–59)
Formalism.IndefR.FWT.OMDat (Formalism.LocalToGlobal.IndefiniteReal.OneMapT, lines 32–59)
GlobalTheory.BigTop.PtAt (Development.AbelianVarietyTheory.Definitions.Divisors.TupleReadBaseTupleEqFamily, lines
145–258)
Formalism.L2.DIV.frameOf (Formalism.LocalToGlobal.Divisibility.Frames, lines 22–22)
Formalism.L2.DIV.frameOfSpec (Formalism.LocalToGlobal.Divisibility.FramesLemmas, lines 77–89)
Formalism.L2.DIV.quotFrame (Formalism.LocalToGlobal.Divisibility.Frames, lines 41–43)
Formalism.L2.DIV.BlockDat.XQ (Formalism.LocalToGlobal.Divisibility.RowLat, lines 144–144)
34.47. Formalism.L2.Host.PiBar (Formalism.LocalToGlobal.UnitaryNorms.CharHost, lines 44–46)
Formalism.L2.Host.Omg (Formalism.LocalToGlobal.UnitaryNorms.CharHost, lines 51–53)
Formalism.L2.Host.IsMu0 (Formalism.LocalToGlobal.UnitaryNorms.CharHost, lines 57–64)
Formalism.L2.Host.IsEtaT (Formalism.LocalToGlobal.UnitaryNorms.CharHost, lines 26–30)
Formalism.L2.Host.IsChiPsi (Formalism.LocalToGlobal.UnitaryNorms.CharHost, lines 38–42)
Formalism.L2.Host.IsDelta (Formalism.LocalToGlobal.UnitaryNorms.CharHost, lines 68–73)
Formalism.L2.Host.CharFrame (Formalism.LocalToGlobal.UnitaryNorms.CharTied, lines 16–31)
Formalism.IndefR.Links.Yc (Formalism.LocalToGlobal.IndefiniteReal.LinksDefs, lines 358–358)
Formalism.IndefR.Links.Vc (Formalism.LocalToGlobal.IndefiniteReal.LinksDefs, lines 366–366)

Formalism.IndefR.Links.Wc (Formalism.LocalToGlobal.IndefiniteReal.LinksDefs, lines 362–362)
Formalism.IndefR.Links.LevCh (Formalism.LocalToGlobal.IndefiniteReal.LinksDefs, lines 388–419)
GlobalTheory.ChiGrid.Consts (Development.AbelianVarietyTheory.Definitions.Divisors.BasePlaceNotMemChiGridBadPlaces, lines 22–28)
GlobalTheory.ChiGrid.S1 (Development.AbelianVarietyTheory.Definitions.Divisors.LeastSplitAuxiliaryPrimeThetaChainSpec, lines 262–263)
Formalism.L2.Theta.IsPlaceIdem (Formalism.LocalToGlobal.BesselHermitianTheta.TiedWeil, lines 68–69)
Formalism.L2.Theta.IsSplitIdem (Formalism.LocalToGlobal.BesselHermitianTheta.TiedWeil, lines 71–72)
Formalism.L2.NTheta.IsIdemAt (Formalism.LocalToGlobal.HeckeThetaElements.IsHeckeElt, lines 68–70)
GlobalTheory.ChiGrid.Places (Development.AbelianVarietyTheory.Definitions.Divisors.BasePlaceNotMemChiGridBadPlaces, lines 43–62)
GlobalTheory.FunObj.SqNum.condExp (Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder, lines 212–212)
GlobalTheory.FunObj.SqNum.condF (Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder, lines 214–215)
GlobalTheory.FunObj.SqNum.condE (Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder, lines 217–217)
GlobalTheory.FunObj.SqNum.levQ (Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder, lines 219–219)
GlobalTheory.FunObj.SqNum.log_levQ (Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder, lines 223–227)
Formalism.L2.Host.HChar.TrivOn (Formalism.LocalToGlobal.UnitaryNorms.IsUnitaryNormOf, lines 194–195)
Formalism.L2.Host.HChar.HasInf (Formalism.LocalToGlobal.UnitaryNorms.IsUnitaryNormOf, lines 90–91)
Formalism.L2.Host.HChar.UnrAt (Formalism.LocalToGlobal.UnitaryNorms.IsUnitaryNormOf, lines 71–71)
Formalism.L2.Host.HChar.LevelLe (Formalism.LocalToGlobal.UnitaryNorms.IsUnitaryNormOf, lines 67–69)
GlobalTheory.ChiGrid.cen (Development.AbelianVarietyTheory.Definitions.Divisors.LeastSplitAuxiliaryPrimeThetaChainSpec, lines 150–150)
GlobalTheory.ChiGrid.Chars (Development.AbelianVarietyTheory.Definitions.Divisors.BasePlaceNotMemChiGridBadPlaces, lines 117–139)
Formalism.L2.Theta.frameMat (Formalism.LocalToGlobal.BesselHermitianTheta.IsAutFunDef, lines 79–80)
GlobalTheory.ChiGrid.antiDiagonalSignGramMatrix (Development.AbelianVarietyTheory.Definitions.Divisors.CmHostImaginaryGeneratorNeZero, lines 102–103)
GlobalTheory.ChiGrid.HermitianOrthogonalExtensionData (Development.AbelianVarietyTheory.Definitions.Divisors.CmHostImaginaryGeneratorNeZero, lines 105–117)
GlobalTheory.ChiGrid.stairId (Development.AbelianVarietyTheory.Definitions.Divisors.CmHostImaginaryGeneratorNeZero, lines 133–134)
GlobalTheory.ChiGrid.Vertices (Development.AbelianVarietyTheory.Definitions.Divisors.BasePlaceNotMemChiGridBadPlaces, lines 168–197)
GlobalTheory.ChiGrid.Node (Development.AbelianVarietyTheory.Definitions.Divisors.BasePlaceNotMemChiGridBadPlaces, lines 199–206)
GlobalTheory.ChiGrid.Node.rank (Development.AbelianVarietyTheory.Definitions.Divisors.BasePlaceNotMemChiGridBadPlaces, lines 215–218)
GlobalTheory.ChiGrid.GridData (Development.AbelianVarietyTheory.Definitions.Divisors.BasePlaceNotMemChiGridBadPlaces, lines 335–341)
GlobalTheory.ChiGrid.sLen (Development.AbelianVarietyTheory.Definitions.Divisors.LeastSplitAuxiliaryPrimeThetaChainSpec, lines 121–121)
GlobalTheory.ChiGrid.rankV (Development.AbelianVarietyTheory.Definitions.Divisors.LeastSplitAuxiliaryPrimeThetaChainSpec, lines 127–127)
GlobalTheory.ChiGrid.rankP (Development.AbelianVarietyTheory.Definitions.Divisors.LeastSplitAuxiliaryPrimeThetaChainSpec, lines 129–129)
GlobalTheory.ChiGrid.rankX (Development.AbelianVarietyTheory.Definitions.Divisors.LeastSplitAuxiliaryPrimeThetaChainSpec, lines 131–131)
GlobalTheory.BigTop.Fun.Grid (Development.AbelianVarietyTheory.Definitions.Divisors.TwistedPlaceChoiceLiesOverPair, lines 32–32)
34.48. Formalism.MainNotions.TaylorLogMetric.IsCurveIdeal (Formalism.SharpRestriction.GlobalSchurBounds.GeneralLinearActions.IsDiscriminantTwelfthRootOfIdealLattice, lines 56–57)
Formalism.MainNotions.TaylorLogMetric.IsDiscriminantTwelfthRootOfIdealLattice (Formalism.SharpRestriction.GlobalSchurBounds.GeneralLinearActions.IsDiscriminantTwelfthRootOfIdealLattice, lines 59–60)
Formalism.MainNotions.StandardLatticesGL3.Crv (Formalism.SharpRestriction.GlobalSchurBounds.StandardLatticesGL3, lines 28–28)
Formalism.MainNotions.TaylorLogMetric.lamOf (Formalism.SharpRestriction.GlobalSchurBounds.GeneralLinearActions.IsDiscriminantTwelfthRootOfIdealLattice, lines 53–54)
Formalism.MainNotions.TaylorLogMetric.area (Formalism.SharpRestriction.GlobalSchurBounds.GeneralLinearActions.IsDiscriminantTwelfthRootOfIdealLattice, lines 50–51)
Formalism.MainNotions.GL3Theta.IsTwoTorsionPointUnkilled (Formalism.SharpRestriction.GlobalSchurBounds.GL3Theta, lines 198–200)
GlobalTheory.NormSum.SplitIn (Development.AbelianVarietyTheory.Definitions.Foundations.NormSumPairRankZeroOfCurveCoordNorm, lines 112–112)

`GlobalTheory.NormSum.GoodLevel` (`Development.AbelianVarietyTheory.Definitions.Foundations.NormSumPairRankZeroOfCurveCoordNorm`, lines 166–166)
`Formalism.MainNotions.StandardLatticesGL3.GA` (`Formalism.SharpRestriction.GlobalSchurBounds.StandardLatticesGL3`, lines 374–376)
`GlobalTheory.NormSum.IntegralOver` (`Development.AbelianVarietyTheory.Statements.CuspidalUnitaryTwists.ExistsSmallSpherical`, lines 18–19)
`Formalism.MainNotions.TaylorLogMetric.VC` (`Formalism.SharpRestriction.GlobalSchurBounds.GeneralLinearActions.IsDiscriminantTwelfthRootOfIdealLattice`, lines 66–66)
`Formalism.MainNotions.StandardLatticesGL3.oddE0` (`Formalism.SharpRestriction.GlobalSchurBounds.StandardLatticesGL3`, lines 496–500)
`Formalism.MainNotions.StandardLatticesGL3.WeilO` (`Formalism.SharpRestriction.GlobalSchurBounds.StandardLatticesGL3`, lines 443–470)
`GlobalTheory.NormSum.Pres` (`Development.AbelianVarietyTheory.Definitions.Foundations.NormSumPairRankZeroOfCurveCoordNorm`, lines 114–119)
`GlobalTheory.NormSum.Cls` (`Development.AbelianVarietyTheory.Definitions.Foundations.NormSumPairRankZeroOfCurveCoordNorm`, lines 16–17)
`GlobalTheory.NormSum.Pair` (`Development.AbelianVarietyTheory.Definitions.Foundations.NormSumPairRankZeroOfCurveCoordNorm`, lines 121–149)
`GlobalTheory.NormSum.Pair.sq` (`Development.AbelianVarietyTheory.Definitions.Foundations.NormSumPairRankZeroOfCurveCoordNorm`, lines 159–159)
`GlobalTheory.NormSum.Pair.norm` (`Development.AbelianVarietyTheory.Definitions.Foundations.NormSumPairRankZeroOfCurveCoordNorm`, lines 161–162)
`GlobalTheory.NormSum.Pair.Small` (`Development.AbelianVarietyTheory.Definitions.Foundations.NormSumPairRankZeroOfCurveCoordNorm`, lines 164–164)
`GlobalTheory.FunObj.SqNum.botBig` (`Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder`, lines 171–171)
`GlobalTheory.FunObj.SqNum.botSmall` (`Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder`, lines 173–173)
`GlobalTheory.FunObj.SqNum.alphaH` (`Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder`, lines 235–235)
`GlobalTheory.FunObj.SqNum.betaH` (`Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder`, lines 237–237)
`GlobalTheory.FunObj.SqNum.HasChars` (`Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder`, lines 57–59)
`GlobalTheory.FunObj.SqNum.IsStdDatum` (`Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder`, lines 49–51)
`GlobalTheory.FunObj.SqNum.IsBlockLevel` (`Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder`, lines 53–55)
`GlobalTheory.FunObj.SqNum.IsBottom` (`Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder`, lines 256–268)
`GlobalTheory.FunObj.SqNum.wtA` (`Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder`, lines 192–193)
`GlobalTheory.FunObj.SqNum.wtA'` (`Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder`, lines 195–196)
`GlobalTheory.FunObj.SqNum.maxWt` (`Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder`, lines 204–206)
`GlobalTheory.NormSum.intBound` (`Development.AbelianVarietyTheory.Statements.CuspidalUnitaryTwists.ExistsSmallSpherical`, lines 21–23)
`GlobalTheory.NormSum.Exponents` (`Development.AbelianVarietyTheory.Statements.CuspidalUnitaryTwists.ExistsSmallSpherical`, lines 25–26)
`GlobalTheory.NormSum.CoordInt` (`Development.AbelianVarietyTheory.Statements.CuspidalUnitaryTwists.ExistsSmallSpherical`, lines 28–34)
`GlobalTheory.NormSum.NormInt` (`Development.AbelianVarietyTheory.Statements.CuspidalUnitaryTwists.ExistsSmallSpherical`, lines 36–42)
`Formalism.PairingDivisionLocalBalance.wt` (`Formalism.PairingDivisors.StatementH`, lines 16–17)
`GlobalTheory.FunObj.SqNum.IsIntExp` (`Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder`, lines 285–292)
`GlobalTheory.FunObj.SqNum.exists_isIntExp` (`Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder`, lines 303–310)
34.49. `GlobalTheory.RowTest.vK` (`Development.AbelianVarietyTheory.Definitions.TwistedInnerProducts.ExistsMatchOver`, lines 184–184)
`GlobalTheory.FunObj.vSum` (`Development.AbelianVarietyTheory.Definitions.Divisors.InvVolConjLevelOne`, lines 407–408)
34.49. `Formalism.BasicNotions.SizeData₂` (`Formalism.LocalRingCardinality.DataParams₂`, lines 76–182)
`GlobalTheory.IndefiniteRowSideFullHecke.LevelPlusSubgroup.SizeRows` (`Development.AbelianVarietyTheory.BlockTheory.DivisorBase.LevelPlusSubgroup`, lines 20–31)
`Formalism.IndefR.Links.HostHyp` (`Formalism.LocalToGlobal.IndefiniteReal.LinksLemmas`, lines 496–504)
`GlobalTheory.SqTop.Rec` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.NormSmall`, lines 24–32)
`Formalism.IndefR.Links.Face3.lv2` (`Formalism.LocalToGlobal.IndefiniteReal.ZeroPartEigenvalueConvention`, lines 55–62)

`Formalism.L2.Theta.IsStdChar` (`Formalism.LocalToGlobal.BesselHermitianTheta.TiedWeil`, lines 59–62)
`Formalism.IndefR.Mi.SignsAtG` (`Formalism.LocalToGlobal.IndefiniteReal.RealSeparation.SignsG`, lines 23–31)
`GlobalTheory.BigTop.Fun.SettingAtPoint` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.SettingAtPoint`, lines 15–16)
`GlobalTheory.SqTop.GBig` (`Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.NormSmall`, lines 34–36)
`Formalism.IndefR.Datum.IsWeilDatumO` (`Formalism.LocalToGlobal.IndefiniteReal.LocalNewformFunction`, lines 53–75)
`GlobalTheory.FunObj.locPl` (`Development.AbelianVarietyTheory.Definitions.Divisors.ExistsGivenStageEqOne`, lines 114–114)
`GlobalTheory.FunObj.cosExp` (`Development.AbelianVarietyTheory.Definitions.Divisors.ExistsGivenStageEqOne`, lines 118–118)
`GlobalTheory.FunObj.depthAt` (`Development.AbelianVarietyTheory.Definitions.Divisors.ExistsGivenStageEqOne`, lines 120–120)
`Formalism.L2.DIV.IsFrame` (`Formalism.LocalToGlobal.Divisibility.DefGroup`, lines 467–469)
`GlobalTheory.FunObj.IsSumFrame` (`Development.AbelianVarietyTheory.Definitions.Divisors.InvVolConjLevelOne`, lines 72–76)
`GlobalTheory.FunObj.Std` (`Development.AbelianVarietyTheory.Definitions.Divisors.KernelNormConstantSpec`, lines 203–235)
`GlobalTheory.FunObj.Normal` (`Development.AbelianVarietyTheory.Definitions.Divisors.KernelNormConstantSpec`, lines 257–282)
`GlobalTheory.FunObj.GridChoice` (`Development.AbelianVarietyTheory.Definitions.Divisors.InvVolConjLevelOne`, lines 185–207)
`Formalism.IndefR.Bridge.FamTie` (`Formalism.LocalToGlobal.IndefiniteReal.BridgeTie`, lines 79–103)
`GlobalTheory.FunObj.SumChoice` (`Development.AbelianVarietyTheory.Definitions.Divisors.InvVolConjLevelOne`, lines 388–395)
`GlobalTheory.RowTest.NumbersIn` (`Development.AbelianVarietyTheory.Definitions.TwistedInnerProducts.MemHalfCosetSelf`, lines 245–261)
`GlobalTheory.FunObj.LaP` (`Development.AbelianVarietyTheory.Definitions.Divisors.GridTestLatticeLargestSlopeBound`, lines 68–70)
`GlobalTheory.FunObj.TestIx` (`Development.AbelianVarietyTheory.Definitions.Divisors.GridTestLatticeLargestSlopeBound`, lines 80–80)
`GlobalTheory.FunObj.ClassReps` (`Development.AbelianVarietyTheory.Definitions.Divisors.GridTestLatticeLargestSlopeBound`, lines 82–86)
`Formalism.GradedWeylDivisibility.ssIdx` (`Formalism.GradedWeylDivisibility.StatementD`, lines 9–10)
`Formalism.L2.DIV.VVIdx` (`Formalism.LocalToGlobal.Divisibility.CoefMod`, lines 256–256)
`GlobalTheory.FunObj.Vol` (`Development.AbelianVarietyTheory.Definitions.Divisors.BplusOfEquivariant`, lines 45–53)
`GlobalTheory.RowTest.ChainVec` (`Development.AbelianVarietyTheory.Definitions.TwistedInnerProducts.ExistsMatchOver`, lines 67–73)
`GlobalTheory.FunObj.RealMatch` (`Development.AbelianVarietyTheory.Definitions.Divisors.ExistsNotRelZero`, lines 65–84)
`GlobalTheory.BigTop.Fun.SlopeTestRemainingChoices` (`Development.AbelianVarietyTheory.Definitions.Divisors.SlopeTestRemainingChoicesAlt`, lines 45–63)
`GlobalTheory.BigTop.Fun.Fld` (`Development.AbelianVarietyTheory.Definitions.Divisors.SlopeTestRemainingChoicesAlt`, lines 81–81)
`Formalism.L2.Host.ValLL0` (`Formalism.LocalToGlobal.UnitaryNorms.CharTied`, lines 35–40)
`GlobalTheory.BigTop.Fun.FieldHas` (`Development.AbelianVarietyTheory.Definitions.Divisors.TwistedPlaceChoiceLiesOverPair`, lines 148–152)
`GlobalTheory.BigTop.Fun.TwistedPlaceChoiceLiesOverPair` (`Development.AbelianVarietyTheory.Definitions.Divisors.TwistedPlaceChoiceLiesOverPair`, lines 195–198)
`Formalism.MainNotions.BlochKatoReduction.Gen.QOk` (`Formalism.SharpRestriction.LevelLetterCocycles.BlochKatoPhases`, lines 250–252)
`Formalism.MainNotions.BlochKatoReduction.Gen.partC` (`Formalism.SharpRestriction.LevelLetterCocycles.BlochKatoPhases`, lines 317–321)
`Formalism.L2.Host.PsiChoiceCompatible` (`Formalism.LocalToGlobal.UnitaryNorms.CharGal`, lines 129–134)
`GlobalTheory.LamTie.IsLamTieFull` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.NotTeichTrivOfIsDelta`, lines 106–113)
`GlobalTheory.BigTop.Fun.GridFit` (`Development.AbelianVarietyTheory.Definitions.Divisors.TwistedPlaceChoiceLiesOverPair`, lines 204–219)
`GlobalTheory.BigTop.Fun.HasGrid` (`Development.AbelianVarietyTheory.Definitions.Divisors.TwistedPlaceChoiceLiesOverPair`, lines 247–248)
`GlobalTheory.BigTop.Fun.Fitted` (`Development.AbelianVarietyTheory.Definitions.Divisors.TwistedPlaceChoiceLiesOverPair`, lines 250–250)
`GlobalTheory.BigTop.Fun.FunIn` (`Development.AbelianVarietyTheory.Definitions.Divisors.TwistedPlaceChoiceLiesOverPair`, lines 37–75)
`GlobalTheory.FunObj.IsTestLat` (`Development.AbelianVarietyTheory.Definitions.Divisors.GridTestLatticeLargestSlopeBound`, lines 116–118)
`GlobalTheory.BigTop.Fun.eV0` (`Development.AbelianVarietyTheory.Definitions.Divisors.TieTest`, lines 16–19)
`GlobalTheory.FunObj.SquareNonzero` (`Development.AbelianVarietyTheory.Definitions.Divisors.ExistsNotRelZero`, lines 217–219)
`GlobalTheory.FunObj.IsSquareConst` (`Development.AbelianVarietyTheory.Definitions.Divisors.ExistsNotRelZero`, lines 213–215)

GlobalTheory.IndefiniteRowSideFullHecke.ClassZetaValuationBounds.IsClassZetaVals (Development.AbelianVarietyTheory.BlockTheory.DivisorBase.VpBounds, lines 23–24)
GlobalTheory.IndefiniteRowSideFullHecke.ClassZetaValuationBounds.betaBound_vp (Development.AbelianVarietyTheory.BlockTheory.DivisorBase.VpBounds, lines 66–67)
GlobalTheory.FunObj.SqNum.IsDenom (Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder, lines 281–283)
GlobalTheory.FunObj.IsSqOf (Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder, lines 318–346)
GlobalTheory.BigTop.Fun.TieTest (Development.AbelianVarietyTheory.Definitions.Divisors.TieTest, lines 25–25)
GlobalTheory.BigTop.Fun.TieV (Development.AbelianVarietyTheory.Definitions.Divisors.TieTest, lines 27–28)
GlobalTheory.BigTop.Fun.TieSq (Development.AbelianVarietyTheory.Definitions.Divisors.TieTest, lines 30–31)
GlobalTheory.BigTop.Fun.TieAt (Development.AbelianVarietyTheory.Definitions.Divisors.TieTest, lines 33–41)
GlobalTheory.BigTop.Fun.Comparison₂ (Development.AbelianVarietyTheory.Definitions.Divisors.TieTest, lines 43–43)
34.50. Formalism.L2.Indef.Tree.FunData (Formalism.LocalToGlobal.IndefiniteForms.IndefiniteTree.FunctionalLowerBoundOfConditionsNumeric, lines 20–92)
Formalism.L2.Indef.Tree.FunData.s (Formalism.LocalToGlobal.IndefiniteForms.IndefiniteTree.FunctionalLowerBoundOfConditionsNumeric, lines 102–102)
Formalism.L2.Indef.Tree.FunData.bDel (Formalism.LocalToGlobal.IndefiniteForms.IndefiniteTree.FunctionalLowerBoundOfConditionsNumeric, lines 139–141)
Formalism.L2.Indef.Tree.FunData.bArch (Formalism.LocalToGlobal.IndefiniteForms.IndefiniteTree.FunctionalLowerBoundOfConditionsNumeric, lines 143–145)
Formalism.L2.Indef.Tree.FunData.bTH (Formalism.LocalToGlobal.IndefiniteForms.IndefiniteTree.FunctionalLowerBoundOfConditionsNumeric, lines 147–147)
GlobalTheory.BigTop.Fun.Close.cloudIcc (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ZetaDenominatorsFitFunction, lines 65–80)
GlobalTheory.BigTop.Fun.Lc_le_of_cloud_le (Development.AbelianVarietyTheory.Proofs.DivisorProofs.RamifiedSumsFifthEstimate, lines 30–44)
GlobalTheory.BigTop.Fun.RamSum.leaves_of_sum (Development.AbelianVarietyTheory.Proofs.DivisorProofs.RamifiedSumsSeventhEstimate, lines 39–68)
GlobalTheory.BigTop.Fun.LeafMMTie.leafMM_of_test (Development.AbelianVarietyTheory.Proofs.DivisorProofs.MultiplicityAtFittedPoints, lines 94–138)
GlobalTheory.SizeLeafCheck.tree_leaf_parameter_arithmetic_of_bounds (Development.AbelianVarietyTheory.Counterexamples.DegenerateExamples.TreeLeafParameterArithmeticOfBounds, lines 66–201)
Formalism.L2.Indef.Tree.PrimePowerModuliSumBound (Formalism.LocalToGlobal.IndefiniteForms.IndefiniteTree.FunctionalLowerBoundOfConditionsNumeric, lines 219–220)
Formalism.L2.Indef.Tree.DegreeAndSizeBounds (Formalism.LocalToGlobal.IndefiniteForms.IndefiniteTree.FunctionalLowerBoundOfConditionsNumeric, lines 215–217)
34.51. GlobalTheory.SumSize.Loss (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt, lines 20–21)
GlobalTheory.SumSize.LossLe (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt, lines 23–23)
GlobalTheory.SumSize.InLat (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt, lines 29–31)
DenominatorLossAdditivity.GlobalTheory.CompanionTheory.FinDen.core_loss (Development.AbelianVarietyTheory.Proofs.DivisorProofs.FiniteSumDenominatorBoundOfFactorClaims, lines 280–338)
DenominatorLossAdditivity.GlobalTheory.CompanionTheory.FinDen.loss_comb (Development.AbelianVarietyTheory.Proofs.DivisorProofs.FiniteSumDenominatorBoundOfFactorClaims, lines 459–496)
DenominatorLossAdditivity.GlobalTheory.CompanionTheory.FinDen.fin_at (Development.AbelianVarietyTheory.Proofs.DivisorProofs.FiniteSumDenominatorBoundOfFactorClaims, lines 507–531)
34.52. GlobalTheory.T.sum_size_bound_all_places (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.SumSizeFiniteAwayArchimedeanCombined, lines 50–51)
GlobalTheory.SumSize.Join.sum_size_finite_away_archimedean_combined (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.SumSizeFiniteAwayArchimedeanCombined, lines 26–44)
GlobalTheory.SumSize.BigC (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt, lines 83–87)
GlobalTheory.SumSize.FinAt (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt, lines 135–137)
GlobalTheory.SumSize.AwayAt (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt, lines 139–139)
GlobalTheory.SumSize.FinK (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt, lines 203–203)
GlobalTheory.SumSize.AwayK (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt, lines 205–205)
34.53. GlobalTheory.T.leafFixedPoint_of_finK_awayK (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LeafFixedPointOfFinKAwayKStatement, lines 12–13)
GlobalTheory.Adopt.leafFixedPoint_of_finK_awayK (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LeafFixedPointOfFinKAwayK, lines 53–54)
GlobalTheory.BigTop.Fun.LeafFixedPoint (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LeafInfinitePlaceBound, lines 28–28)
Formalism.L2.Indef.Tree.LeafFP (Formalism.LocalToGlobal.IndefiniteForms.IndefiniteTree.FunctionalLowerBoundOfConditionsNumeric, lines 168–173)

GlobalTheory.finitePlaceBound_inv_rootOfUnity_smul (Development.AbelianVarietyTheory.Proofs.DivisorProofs.MatLatSmulLeOfIsIntegral, lines 24–32)
34.54. GlobalTheory.QuotientNormCoisometry.htMap_quot (Development.AbelianVarietyTheory.Proofs.DivisorProofs.InfinitePartIdentity, lines 64–83)
Formalism.HN.htMap (Formalism.HarderNarasimhan.StatementPl, lines 60–62)
34.55. GlobalTheory.T.leafHeightMapBound_of_infK (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.IpaTieStatement, lines 12–13)
GlobalTheory.QuotientNormCoisometry.leafHeightMapBound_tie_of_infK (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.IpaTie, lines 22–50)
GlobalTheory.BigTop.Fun.LeafHeightMapBound (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LeafInfinitePlaceBound, lines 29–33)
GlobalTheory.SumSize.InfAt (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt, lines 176–179)
GlobalTheory.SumSize.InfK (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.InfClassAt, lines 225–225)
Formalism.HN.htMap (Formalism.HarderNarasimhan.StatementPl, lines 60–62)
34.56. GlobalTheory.T.leafMinimalMass_of_setAt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.MmTieStatement, lines 11–12)
GlobalTheory.BigTop.Fun.LeafMMTie.mm_tie_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.MmTie, lines 23–26)
GlobalTheory.BigTop.Fun.LeafMinimalMass (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LeafInfinitePlaceBound, lines 23–23)
Formalism.L2.Indef.Tree.LeafMM (Formalism.LocalToGlobal.IndefiniteForms.IndefiniteTree.FunctionalLowerBoundOfConditionsNumeric, lines 158–159)
34.57. GlobalTheory.SqTop.OneSqProof.vw_ratio_le (Development.AbelianVarietyTheory.Proofs.DivisorProofs.OneSqStOfClaims, lines 16–71)
34.58. GlobalTheory.SqTop.OneSqSt (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.NormSmall, lines 159–167)
GlobalTheory.SqTop.ThA (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.NormSmall, lines 66–70)
GlobalTheory.SqTop.DenSum (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.NormSmall, lines 89–90)
GlobalTheory.SqTop.square_top_term_joined_estimates (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.SquareTopTermJoinedEstimates, lines 13–35)
GlobalTheory.ToricTransport.toricRatioConjugates_of_transport (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ToricRatioConjugatesOfTransport, lines 57–86)
GlobalTheory.ThmAUpper.thmA_upper_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ThmAUpper, lines 17–29)
GlobalTheory.BigTop.Fun.oneSq_of_log_derivative_bound_thA (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.OneSqTie, lines 23–79)
GlobalTheory.BigTop.Fun.den_tie_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.DenTie, lines 14–26)
GlobalTheory.T.oneSq_of_log_derivative_bound_and_oneSqSt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.OneSqTieStatement, lines 13–14)
GlobalTheory.T.leafDensity_of_logDeriv_bound_and_denSum (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.DenTieStatement, lines 13–14)
34.59. GlobalTheory.NormSum.DenomBound.exists_exponents (Development.AbelianVarietyTheory.Proofs.FoundationProofs.ClassSumDenominatorBoundOfCmThetaHypotheses, lines 48–110)
GlobalTheory.NormSum.intBound (Development.AbelianVarietyTheory.Statements.CuspidalUnitaryTwists.ExistsSmallSpherical, lines 21–23)
34.60. GlobalTheory.T.cmThetaNormBoundHypotheses_satisfied (Development.AbelianVarietyTheory.Proofs.FoundationProofs.CmThetaNormBoundHypothesesSatisfiedStatement, lines 8–9)
GlobalTheory.NormSum.inp_all (Development.AbelianVarietyTheory.Proofs.FoundationProofs.CmThetaNormBoundHypothesesSatisfied, lines 29–51)
GlobalTheory.T.classSumDenominatorBound_of_cmThetaHypotheses (Development.AbelianVarietyTheory.Proofs.FoundationProofs.ClassSumDenominatorBoundOfCmThetaHypothesesStatement, lines 9–10)
GlobalTheory.NormSum.DenomBound.intSt_of_forall_inp (Development.AbelianVarietyTheory.Proofs.FoundationProofs.ClassSumDenominatorBoundOfCmThetaHypotheses, lines 118–148)
GlobalTheory.NormSum.IntSt (Development.AbelianVarietyTheory.Statements.CuspidalUnitaryTwists.ExistsSmallSpherical, lines 44–47)
GlobalTheory.FunObj.SqNum.exists_isIntExp (Development.AbelianVarietyTheory.Definitions.Divisors.FinitePrimesUnder, lines 303–310)
GlobalTheory.BigTop.Fun.TieModel.isIntExp_mono (Development.AbelianVarietyTheory.Definitions.Divisors.MulVecEqOfBasis, lines 38–50)
34.61. GlobalTheory.SqTop.BotSmall (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.NormSmall, lines 110–112)
GlobalTheory.SqTop.BotNe (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.NormSmall, lines 122–123)
GlobalTheory.BigTop.Fun.int_tie_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.IntTie, lines 17–76)
GlobalTheory.T.leafIntegrality_of_logDeriv_bound (Development.AutomorphicArithmetic.ThetaCorrespondence.

ThetaPairLifts.ThirdPlaceSeries.IntTieStatement, lines 13–14)
Formalism.L2.Indef.Tree.LeafINT (Formalism.LocalToGlobal.IndefiniteForms.IndefiniteTree.
FunctionalLowerBoundOfConditionsNumeric, lines 197–200)
34.62. GlobalTheory.PositiveMeans.means_le (Development.AbelianVarietyTheory.Proofs.FoundationProofs.
ExistsBoundMeansPosLogAreaNormalisedTheta, lines 233–296)
GlobalTheory.ArchFree.means.means_le (Development.AbelianVarietyTheory.Proofs.FoundationProofs.
ConjugateLatticeMeansPosLogAreaThetaBounded, lines 55–118)
GlobalTheory.LatticeAreaBound.area_bounded_of_discr_eq_one (Development.AbelianVarietyTheory.Proofs.
FoundationProofs.AreaBoundedOfDiscrEqOne, lines 59–82)
34.63. GlobalTheory.ArchNorm.norm_meanInf_posLog_le (Development.AbelianVarietyTheory.Proofs.FoundationProofs.
LatticeUnitNormSqNonneg, lines 225–342)
GlobalTheory.ArchTop.archSt_of_inp (Development.AbelianVarietyTheory.Proofs.FoundationProofs.
ThetaJetClassSumArchimedeanBoundOfCmTheta, lines 12–18)
GlobalTheory.NormSum.ArchSt (Development.AbelianVarietyTheory.Statements.CuspidalUnitaryTwists.ExistsSmallSpherical,
lines 49–59)
GlobalTheory.BigTop.Fun.arch_tie_of_logDeriv_bound (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.ArchTie, lines 14–47)
GlobalTheory.T.arch_tie_of_setAt_logDeriv (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.
ThirdPlaceSeries.ArchTieStatement, lines 13–14)
Formalism.L2.Indef.Tree.LeafARCH (Formalism.LocalToGlobal.IndefiniteForms.IndefiniteTree.
FunctionalLowerBoundOfConditionsNumeric, lines 202–204)
34.64. GlobalTheory.T.existsFunctionIn_of_sqRatio_range (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.ExOfClaimsStatement, lines 13–14)
GlobalTheory.BigTop.Fun.ex_of_claims_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.
ThirdPlaceSeries.ExOfClaims, lines 16–74)
GlobalTheory.BigTop.Fun.square_ratio_at_point_of_leading_term_alt (Development.AutomorphicArithmetic.
ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.SquareRatioAtPointOfLeadingTerm, lines 19–45)
34.65. GlobalTheory.BigTop.Fun.oneMap_of_bundle_setAt_kernel_conditions (Development.AutomorphicArithmetic.
ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.OneMapOfRows, lines 128–133)
GlobalTheory.BigTop.Fun.leafKernel_of_linK_injK_constK_formula2K (Development.AutomorphicArithmetic.
ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.KerEqOfFormula, lines 87–101)
GlobalTheory.RowClaim.JoinF.joined_local_sum_estimates (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.LocalTypeDictionary.JoinedLocalSumEstimates, lines 12–16)
GlobalTheory.RowTest.injK_of_separates (Development.AbelianVarietyTheory.Proofs.TwistedOperators.InjKOfSeparates,
lines 91–104)
GlobalTheory.BigTop.Fun.bundle_of_rows_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.
ThirdPlaceSeries.BundleOfRows, lines 13–17)
34.66. GlobalTheory.BigTop.Fun.existsIntegralMultiple_holds_alt (Development.AutomorphicArithmetic.
ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ExistsIntegralMultipleHolds, lines 12–26)
GlobalTheory.BigTop.Fun.Close.desHLeaf_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.
ThirdPlaceSeries.DesHLeaf, lines 13–55)
GlobalTheory.BigTop.Fun.rankGapBound_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.
ThirdPlaceSeries.RamifiedSumsFifthBound, lines 16–127)
GlobalTheory.BigTop.Fun.ramified_sums_seventh_bound_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.RamifiedSumsSeventhBound, lines 18–38)
GlobalTheory.BigTop.Fun.wt_tie_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.
ThirdPlaceSeries.WtTie, lines 20–41)
GlobalTheory.BigTop.Fun.desPol_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.
ThirdPlaceSeries.DesPol, lines 17–58)
34.67. GlobalTheory.BigTop.Fun.Close.fittedFunction_at_zero_close_alt (Development.AutomorphicArithmetic.
ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.FittedFunctionAtZeroClose, lines 42–67)
GlobalTheory.BigTop.Fun.fun0Fit_of_leaves_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.LeafInfinitePlaceBound, lines 83–120)
Formalism.L2.Indef.Tree.LeafOneSquare (Formalism.LocalToGlobal.IndefiniteForms.IndefiniteTree.
FunctionalLowerBoundOfConditionsNumeric, lines 182–191)
Formalism.L2.Indef.Tree.PrimeCountingSumBound (Formalism.LocalToGlobal.IndefiniteForms.IndefiniteTree.
FunctionalLowerBoundOfConditionsNumeric, lines 193–195)
34.67. GlobalTheory.T.leafIntegrality_of_logDeriv_bound (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.IntTieStatement, lines 13–14)
GlobalTheory.T.arch_tie_of_setAt_logDeriv (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.
ThirdPlaceSeries.ArchTieStatement, lines 13–14)
34.68. GlobalTheory.BigTop.Fun.Close.fourier_coefficient_fit_of_join_alt (Development.AutomorphicArithmetic.
ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.FourierCoefficientFitOfJoin, lines 12–17)
GlobalTheory.BigTop.Fun.ConstantFunctionFits (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.
ThirdPlaceSeries.LeafInfinitePlaceBound, lines 52–53)
Formalism.L2.Indef.IndefData.FUN (Formalism.LocalToGlobal.IndefiniteForms.IndefSide, lines 331–332)
34.68. Formalism.L2.Indef.Tree.functional_lower_bound_of_conditions (Formalism.LocalToGlobal.IndefiniteForms.
IndefiniteTree.FunctionalLowerBoundOfConditionsNumeric, lines 940–1033)

Formalism.L2.Indef.Tree.size_arith (Formalism.LocalToGlobal.IndefiniteForms.IndefiniteTree.
FunctionalLowerBoundOfConditionsNumeric, lines 243–397)
GlobalTheory.T.polynomial_descent_estimate' (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.
ThirdPlaceSeries.DesPolStatement, lines 11–12)
GlobalTheory.BigTop.Fun.desPol_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.
ThirdPlaceSeries.DesPol, lines 17–58)
34.69. GlobalTheory.Fc.StageOne.rowFun_at_gridA_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.RowFunAtGridA, lines 32–51)
GlobalTheory.Fc.S0.FunctionalCondition (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.
ThirdPlaceSeries.GeometricWeakTies, lines 34–35)
34.69. GlobalTheory.Fc.Ind.evWC_fun_of_fun0Fit_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.EvWCFunOfFun0Fit, lines 14–21)
34.69. UpperTheory.Lit.nonempty_hyps_definite_side_of_closed_bound (Development.AutomorphicArithmetic.
ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.SiegelEisensteinSecondTermHypotheses, lines 44–68)

Chapter 35: Covolume bounds

35.1. Formalism.HN.ht (Formalism.HarderNarasimhan.StatementPl, lines 29–31)
Formalism.HN.htMap (Formalism.HarderNarasimhan.StatementPl, lines 60–62)
GlobalTheory.DescentFamily.htMap_le_of_points (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
HtMapLeOfPoints, lines 34–56)
Formalism.DivisionValues.lcv (Formalism.QuotientMetricDivisors.Statement, lines 34–36)
35.2. Formalism.L2.DIV.ZscOfNum (Formalism.LocalToGlobal.Divisibility.ZscOfNum, lines 59–72)
Formalism.L2.DIV.ofRecord_logOm (Formalism.LocalToGlobal.Divisibility.NumbersLemmas, lines 786–789)
GlobalTheory.CovTop.unitE (Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol, lines 22–22)
GlobalTheory.CovTop.mainT (Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol, lines 24–24)
Formalism.L2.Indef.IndefScale.num (Formalism.LocalToGlobal.IndefiniteForms.IndefSide, lines 224–232)
35.2. GlobalTheory.BigTop.FamHyp (Development.AbelianVarietyTheory.Definitions.Divisors.TupleReadBaseTupleEqFamily,
lines 503–507)
GlobalTheory.CovTop.RowsTAlt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.
ThirdPlaceSeries.LiftLambdaBound, lines 17–17)
GlobalTheory.CovTop.RecTied (Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol, lines 161–161)
35.1. GlobalTheory.BigTop.PtAt.PointConditions' (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.Realised, lines 63–80)
GlobalTheory.BigTop.PtAt.CompatibilityConditions (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.Realised, lines 50–62)
GlobalTheory.FinalJoin2.FieldSmall (Development.AbelianVarietyTheory.Definitions.HeightFunctions.
NormComparisonConstants, lines 32–35)
Formalism.MainNotions.WeightAlgebra.P0 (Formalism.SharpRestriction.Divisibility.WeightAlgebra, lines 271–271)
GlobalTheory.GLq.ESerTie (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.LabelDatum, lines 114–116)
GlobalTheory.GLq.eSerC (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.LabelDatum, lines 102–103)
35.2. GlobalTheory.CovTop.CovolumeBoundOfX (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.
ThirdPlaceSeries.LiftLambdaBound, lines 60–61)
GlobalTheory.CovTop.EventuallyAtAllPoints (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.
ThirdPlaceSeries.LiftLambdaBound, lines 18–19)
Formalism.L2.Indef.IndefData.L3 (Formalism.LocalToGlobal.IndefiniteForms.IndefSide, lines 315–316)
Formalism.MainNotions.FamP.Large (Formalism.SharpRestriction.ParamsHeightRankConstant, lines 138–140)
35.2. GlobalTheory.CovTop.LiftsCondition (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.
ThirdPlaceSeries.LiftLambdaBound, lines 26–28)
GlobalTheory.CovTop.CoversCondition (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.
ThirdPlaceSeries.LiftLambdaBound, lines 29–32)
Formalism.MainNotions.Closed (Formalism.SharpRestriction.ParamsHeightRankConstant, lines 159–161)
GlobalTheory.CovTop.EventuallyAtAllPoints.and (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.LiftLambdaBound, lines 20–25)
35.2. GlobalTheory.CovTop.uM (Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol, lines 34–34)
GlobalTheory.CovTop.cZeroN (Development.AbelianVarietyTheory.Definitions.Divisors.
NormNegCmConstantEqAtAllEmbeddings, lines 83–84)
GlobalTheory.CovTop.cZero (Development.AbelianVarietyTheory.Definitions.Divisors.NormNegCmConstantEqAtAllEmbeddings,
lines 86–86)
35.3. GlobalTheory.AllWt.GoodW (Development.AbelianVarietyTheory.Definitions.Foundations.
IsProbabilityMeasureMuPart, lines 806–807)
GlobalTheory.AllWt.GoodOn (Development.AbelianVarietyTheory.Definitions.Foundations.IsProbabilityMeasureMuPart, lines
809–810)
GlobalTheory.CovTop.DeepGoodAt (Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol, lines 111–114)
GlobalTheory.FlatLine.FlatExistsAt (Development.AbelianVarietyTheory.Definitions.Foundations.ExistsFlatSecRankZero,
lines 746–747)

GlobalTheory.FlatLine.FlatDegree (Development.AbelianVarietyTheory.Definitions.Foundations.ExistsFlatSecRankZero, lines 754–754)
35.4. GlobalTheory.FlatLine.FlatDegree.c1_eq (Development.AbelianVarietyTheory.Definitions.Foundations.ExistsFlatSecRankZero, lines 788–790)
GlobalTheory.FlatCore.flatDegreeSt_of_prismaticDieudonne (Development.AbelianVarietyTheory.Proofs.DivisorProofs.FlatDegreeFromPrismatic, lines 23–35)
GlobalTheory.FlatSplit.twice_twist_ge (Development.AbelianVarietyTheory.Proofs.DivisorProofs.FlatBoundOfAnchor, lines 76–102)
GlobalTheory.CovTop.Period.period_eq_kronecker (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LogPeriodEqKroneckerCharacterLogPeriodOfHost, lines 13–17)
GlobalTheory.CovTop.AnchorRd.anchorAt_of (Development.AbelianVarietyTheory.Proofs.DivisorProofs.AnchorBoundStatement, lines 77–209)
GlobalTheory.FlatSplit.flatAt_of (Development.AbelianVarietyTheory.Proofs.DivisorProofs.FlatBoundOfAnchor, lines 125–187)
35.5. ReciprocityTheory.ChartRestriction.G.Geo.Tor.Model.SplitFibres (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.BirationalGeometry.ToroidalCompactification.UniformisationConclusion, lines 113–117)
UpperTheory.LiteratureTheorems.toroidal_compactification_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 521–522)
35.5. UpperTheory.LiteratureTheorems.vol_ge_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 531–532)
35.5. UpperTheory.LiteratureTheorems.vol_nef_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 533–534)
35.5. UpperTheory.LiteratureTheorems.ar_calc_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 249–250)
35.5. UpperTheory.LiteratureTheorems.abelian_de_rham_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 239–240)
35.5. UpperTheory.LiteratureTheorems.ray_class_field_st_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 489–490)
35.5. UpperTheory.LiteratureTheorems.semistable_reduction_st_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 501–502)
35.5. UpperTheory.LiteratureTheorems.strong_approx_su_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 517–518)
35.5. UpperTheory.LiteratureTheorems.green_series_st_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 401–402)
35.5. UpperTheory.LiteratureTheorems.good_chern_weil_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 395–396)
35.5. UpperTheory.LiteratureTheorems.landherr_st_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 453–454)
35.5. UpperTheory.LiteratureTheorems.hasse_norm_st_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 417–418)
35.5. UpperTheory.LiteratureTheorems.semi_abelian_hom_extension_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 497–498)
35.5. UpperTheory.LiteratureTheorems.induced_simple_lowest_alcove_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 441–442)
35.5. UpperTheory.LiteratureTheorems.principal_polarisation_one_section_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 409–410)
35.5. UpperTheory.LiteratureTheorems.ab_mul_n_st_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 237–238)
35.5. UpperTheory.LiteratureTheorems.grothendieck_messing_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 407–408)
35.5. UpperTheory.LiteratureTheorems.prismatic_dieudonne_hodge_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 475–476)
35.5. UpperTheory.LiteratureTheorems.kawamata_viehweg_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 447–448)
35.5. UpperTheory.LiteratureTheorems.nef_numerics_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 469–470)
35.5. UpperTheory.LiteratureTheorems.l2_extension_log_canonical_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 451–452)
35.5. UpperTheory.LiteratureTheorems.arith_hilbert_samuel_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 255–256)
35.5. UpperTheory.LiteratureTheorems.chow_form_exists_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 325–326)
35.5. UpperTheory.LiteratureTheorems.cycle_heights_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 347–348)
35.5. UpperTheory.LiteratureTheorems.serre_holomorphic_sections_of_literature (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 503–504)

35.5. `UpperTheory.LiteratureTheorems.nadel_vanishing_and_non_klt_divisor_of_literature` (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 467–468)
35.5. `UpperTheory.LiteratureTheorems.principalisation_of_literature` (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 253–254)
35.5. `UpperTheory.LiteratureTheorems.formal_algebraisation_of_literature` (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 381–382)
35.5. `GlobalTheory.CovTop.PublishedHodgeBundleTheorems` (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.HodgeBundleWeights.ToKnownTheoremsOf, lines 7–35)
`UpperTheory.LiteratureTheorems.published_results_arithmetic` (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 781–809)
35.6. `GlobalTheory.FlatCore.archEx_always` (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ArchimedeanExistenceEverywhere, lines 38–105)
`GlobalTheory.CompanionTheory.Ar.Flat.A0.flat_zero_of_resolution_gaga` (Development.AutomorphicSpectrum.Proofs.GeometryProofs.FlatZeroFromResolutionAndGaga, lines 39–56)
`GlobalTheory.CompanionTheory.Ar.Flat.FlatFirstTheorem.flat_first_of_resolution_gaga` (Development.AutomorphicSpectrum.Proofs.GeometryProofs.FlatFirstFromResolutionAndGaga, lines 103–122)
35.7. `GlobalTheory.CompanionTheory.Ar.Flat.det_adapted_eq_unit_mul_const` (Development.AutomorphicSpectrum.Proofs.AlgebraProofs.CmTypeValuationEmbedding, lines 301–302)
`GlobalTheory.CompanionTheory.Ar.Flat.CPoint.value_at_point` (Development.AutomorphicSpectrum.Proofs.GeometryProofs.EigenLinePAdicFrame, lines 72–338)
35.8. `GlobalTheory.CompanionTheory.Ar.Flat.hodgeLat_model` (Development.AutomorphicSpectrum.Proofs.GeometryProofs.EigenLinePAdicBound, lines 170–255)
`GlobalTheory.CompanionTheory.Ar.Flat.GermSys` (Development.AutomorphicSpectrum.Definitions.GlobalGeometry.GermSys, lines 19–45)
`ReciprocityTheory.ChartRestriction.G.Geo.Tor.SAB` (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.BirationalGeometry.ToroidalCompactification.TwoWeightEqOneAddRoot, lines 72–82)
`ReciprocityTheory.ChartRestriction.G.Geo.Tor.Germ.FB` (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.BirationalGeometry.ToroidalCompactification.CocycleRightZeroThd, lines 86–158)
`GlobalTheory.CompanionTheory.Ar.Conj.GeomPF` (Development.AutomorphicSpectrum.Definitions.GlobalGeometry.ToFirst, lines 36–39)
`GlobalTheory.CompanionTheory.Ar.Flat.HodgeLat` (Development.AutomorphicSpectrum.Definitions.GlobalGeometry.LinearIndependentEigProj, lines 21–32)
`GlobalTheory.CompanionTheory.Ar.Conj.transH` (Development.AutomorphicSpectrum.Definitions.CommutativeAlgebra.GermComp1ZeroPointLeft, lines 99–102)
`GlobalTheory.CompanionTheory.Ar.Flat.DegZ.vmap_rhoH` (Development.AutomorphicSpectrum.Proofs.GeometryProofs.ExistsSubMemEigW, lines 87–101)
35.9. `GlobalTheory.CompanionTheory.Ar.Flat.DegZ.GermFam.germ_family` (Development.AutomorphicSpectrum.Proofs.GeometryProofs.FibreCurveGermPowers, lines 215–270)
`GlobalTheory.CompanionTheory.Ar.Flat.DegProd.curve_deg` (Development.AutomorphicSpectrum.Proofs.GeometryProofs.ValuationBoundedFrame, lines 326–363)
`GlobalTheory.CompanionTheory.Ar.Flat.germ_degZero` (Development.AutomorphicSpectrum.Proofs.GeometryProofs.HodgeLatticeDvrDegree, lines 117–237)
`GlobalTheory.CompanionTheory.Ar.Flat.exists_trans` (Development.AutomorphicSpectrum.Proofs.GeometryProofs.HodgeLatticeDvrDegree, lines 72–109)
`GlobalTheory.CompanionTheory.Ar.Flat.DegZ.Frame.exists_basis_frame` (Development.AutomorphicSpectrum.Proofs.GeometryProofs.HodgeLatticeGermEigen, lines 130–148)
`GlobalTheory.CompanionTheory.Ar.Flat.DegZ.charpoly_lmul_splits_of_conjugates` (Development.AutomorphicSpectrum.Proofs.GeometryProofs.CharpolySplitsOfEmbeddings, lines 34–63)
`GlobalTheory.CompanionTheory.Ar.Flat.DegCurve.step` (Development.AutomorphicSpectrum.Proofs.AlgebraProofs.SplitHermitianGeneratorsExist, lines 241–250)
`GlobalTheory.CompanionTheory.Ar.Flat.DegCurve.deg_count` (Development.AutomorphicSpectrum.Proofs.AlgebraProofs.SplitHermitianGeneratorsExist, lines 318–351)
35.10. `GlobalTheory.CompanionTheory.Ar.Flat.germ_degZero` (Development.AutomorphicSpectrum.Proofs.GeometryProofs.HodgeLatticeDvrDegree, lines 117–237)
`GlobalTheory.CompanionTheory.Ar.Flat.COrder.order_on_component` (Development.AutomorphicSpectrum.Proofs.GeometryProofs.HodgeLatticeValuationAgree, lines 12–43)
`GlobalTheory.CompanionTheory.Ar.Flat.FinAll.finConstAll` (Development.AutomorphicSpectrum.Proofs.GeometryProofs.FiniteConstantAllHolds, lines 137–195)
`GlobalTheory.CompanionTheory.Ar.Br.FinGood.finGood` (Development.AutomorphicSpectrum.Proofs.GeometryProofs.FrameFunctionFiniteNorm, lines 23–40)
35.11. `SelfIntersectionVolumeEstimates.topCovolSlim2_use_alt_alt` (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.TopCovolSlim2Use, lines 16–22)
`PositivityTheory.MbHsUse.HSUse` (Development.WeilBound.CoreStatements.AutomorphicKernels.MoriwakiBigness.HilbertSamuelApplications.SmallGeneratedCountUpperOfUse, lines 12–13)
35.11. `SelfIntersectionVolumeEstimates.topCovolG_alt_alt` (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.TopCovolG, lines 39–58)

35.12. ReciprocityTheory.ChartRestriction.G.Geo.chartLat (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.BirationalGeometry.ArithmeticHilbertSamuelLowerBound, lines 55–60)
ReciprocityTheory.ChartRestriction.G.Geo.chartLatNm (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.BirationalGeometry.ArithmeticHilbertSamuelLowerBound, lines 62–67)
GroundTheory.Heights₂.Heights.Model.FiniteIntegralClosure.proof (GroundTheory.HeightsAndTheta.Heights.IntegralModels.PFiniteIntegralClosure, lines 9–16)
GroundTheory.Heights₂.Heights.HilbertSamuel.Conductor.exists_chart_conductor (GroundTheory.HeightsAndTheta.Heights.HilbertSamuel.PConductor, lines 54–88)
GroundTheory.Heights₂.Heights.HilbertSamuel.Conductor.proof (GroundTheory.HeightsAndTheta.Heights.HilbertSamuel.PConductor, lines 102–159)
GroundTheory.Heights₂.Heights.HilbertSamuel.LeviCompHol.proof (GroundTheory.HeightsAndTheta.Heights.HilbertSamuel.PLeviCompHol, lines 11–60)
35.13. GlobalTheory.Izumi.exists_dominating_eq_mul_of_not_mem_span (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsDominatingLocalDomainEqMulMaximal, lines 70–99)
GlobalTheory.CurvePoint.exists_ringHom_padicAlgCl_ne_zero (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsRingHomPadicAlgClNeZero, lines 10–101)
GlobalTheory.CplDiv.dvd_of_norm_le_at_classical (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsNormLeOfMemMapPow, lines 142–207)
35.14. GlobalTheory.ZH.exists_normal_proj_hasDiv_family (Development.AbelianVarietyTheory.Proofs.DegenerationProofs.ExistsAdmissibleAffineModelDividingTuple, lines 21–47)
GlobalTheory.Poinc.LabConst.main (Development.AbelianVarietyTheory.Proofs.DivisorProofs.TorsionLabelsConstantCompletedLocalRingClosedPoi, lines 64–143)
GlobalTheory.ThetaSpan.eq_sum_translates (Development.AbelianVarietyTheory.Proofs.UniformizationProofs.ValuationNonnegIffSpanMultiplicationTranslates, lines 32–90)
GlobalTheory.UT.nuGe_zero_iff_span_translates (Development.AbelianVarietyTheory.Proofs.UniformizationProofs.ValuationNonnegIffSpanMultiplicationTranslates, lines 219–330)
GlobalTheory.UT.CoefUnit.exists_fam_coef_val (Development.AbelianVarietyTheory.Proofs.UniformizationProofs.ExistsFamCoefVal, lines 130–160)
GlobalTheory.ZH.WeightBound.main (Development.AbelianVarietyTheory.Proofs.DegenerationProofs.WeightSpaceValuesIntegralTorsionBoundShortestVec, lines 319–340)
GlobalTheory.TestFibre.exists_norm_FF_eq (Development.AbelianVarietyTheory.Proofs.DegenerationProofs.ExistsNormTorsionSumShortestVectorEq, lines 88–190)
GlobalTheory.Poinc.CNorm.main (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ThetaConstantFourierPartsBoundedClassMinima, lines 100–162)
GlobalTheory.QUnit.main (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsMinimalPsiCoordinateAssociatedFourierPower, lines 25–144)
35.15. GlobalTheory.TraitLift.exists_lift_adicCompletion (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsLiftAdicCompletion, lines 11–35)
GlobalTheory.TorusOntoSpread.restrict_surjective (Development.AbelianVarietyTheory.Proofs.DivisorProofs.RestrictSurjectiveAlt, lines 18–89)
GlobalTheory.Poinc.TorCoord.coordMap_of_torMatch (Development.AbelianVarietyTheory.Proofs.DivisorProofs.CoordMapOfTorMatch, lines 22–75)
GlobalTheory.UPoinc.TraitAtClosed.exists_trait_lattice_map (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsTraitLatticeMapAtClosedPoint, lines 24–86)
GlobalTheory.UPoinc.TraitPull.exists_pull_unif (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsLinearUniformisationPulledBackTrait, lines 18–34)
GlobalTheory.UPoinc.uOrdEq_stalkHom_of_associated (Development.AbelianVarietyTheory.Proofs.DivisorProofs.UniformiserOrderEqOfAssociatedStalkImage, lines 17–44)
35.16. GlobalTheory.UPoinc.CellCones.exists_finite_cones (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsFiniteConesPolarizedFanLevelStructure, lines 99–215)
GlobalTheory.Poinc.MinSysLat.finite_minimiser_systems_lat (Development.AbelianVarietyTheory.Proofs.DivisorProofs.FiniteMinimiserSystemsPosCombinationDefiniteSum, lines 24–69)
35.17. GlobalTheory.Poinc.exists_idealSheaf_comap_eq_of_approx (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsIdealSheafPullbackEqDivisorOfApprox, lines 16–40)
35.18. GlobalTheory.NormSheaf.normDescentSheaf (Development.AbelianVarietyTheory.Proofs.DivisorProofs.NormDescentIdealSheafFiniteOpenClosedImage, lines 16–41)
GlobalTheory.NormSheaf.normIdeal_map_eq_prod_lifts (Development.AbelianVarietyTheory.Proofs.DivisorProofs.NormCoefficientIdealMapCompleteDvrEqProdLifts, lines 13–89)
35.19. UpperTheory.LiteratureTheorems.published_results_arithmetic (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 781–809)
UpperTheory.LiteratureTheorems.every_character_of_literature_of_vWin (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 810–813)
35.20. GlobalTheory.T.cov_size_bound (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.SizeAltBisStatement, lines 11–12)
35.20. GlobalTheory.CovTop.SizeRows.exists_isDiv (Development.AbelianVarietyTheory.Proofs.DivisorProofs.SizeBoundStatement, lines 38–50)
GlobalTheory.CovTop.SizeRows.rho_bounds (Development.AbelianVarietyTheory.Proofs.DivisorProofs.SizeBoundStatement, lines 52–106)

GlobalTheory.CovTop.SizeRows.sizeAt_of (Development.AbelianVarietyTheory.Proofs.DivisorProofs.SizeBoundStatement, lines 260–358)
GlobalTheory.CovTop.SizeRows.size_of_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.SizeAltBis, lines 18–23)
35.21. GlobalTheory.CovTop.DeepExist.exists_deep_covering_level_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ExistsDeepCoveringLevel, lines 16–24)
35.22. GlobalTheory.CovTop.covering_gram_matrix_of_local_traces_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CoveringGramMatrixOfLocalTraces, lines 17–20)
GlobalTheory.CovTop.MeasRd.covering_measure_reduction_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CoveringMeasureReductionBound, lines 18–27)
GlobalTheory.CovTop.LamAll.covering_lambda_all_places_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CoveringLambdaInvariantBound, lines 21–37)
35.22. GlobalTheory.CovHt.WtGap (Development.AbelianVarietyTheory.Definitions.Divisors.SemiposAtOfPositive, lines 88–89)
GlobalTheory.CovHt.bndB (Development.AbelianVarietyTheory.Definitions.Divisors.SemiposAtOfPositive, lines 91–92)
GlobalTheory.CovHt.errX (Development.AbelianVarietyTheory.Definitions.Divisors.SemiposAtOfPositive, lines 94–95)
GlobalTheory.CompHt.ValFam (Development.AbelianVarietyTheory.Definitions.Divisors.IsHomogeneousCompIdeal, lines 245–246)
GlobalTheory.CompHt.recVals (Development.AbelianVarietyTheory.Definitions.Divisors.IsHomogeneousCompIdeal, lines 250–250)
GlobalTheory.CompHt.ValFam.patch (Development.AbelianVarietyTheory.Definitions.Divisors.IsHomogeneousCompIdeal, lines 254–258)
GlobalTheory.CompHt.htVals (Development.AbelianVarietyTheory.Definitions.Divisors.IsLatBasisHtLatIffSpanEq, lines 485–489)
GlobalTheory.CompHt.htPrimes (Development.AbelianVarietyTheory.Definitions.Divisors.IsLatBasisHtLatIffSpanEq, lines 515–515)
GlobalTheory.CompHt.mixVals (Development.AbelianVarietyTheory.Definitions.Divisors.IsLatBasisHtLatIffSpanEq, lines 534–534)
GlobalTheory.CompHt.compVol (Development.AbelianVarietyTheory.Definitions.Divisors.IsHomogeneousCompIdeal, lines 273–274)
GlobalTheory.CompHt.compVolSlope (Development.AbelianVarietyTheory.Definitions.Divisors.IsHomogeneousCompIdeal, lines 280–281)
35.23. GroundTheory.Heights$_2$.Heights.LowerBoundFree.ZhangLowerX.Statement (GroundTheory.HeightsAndTheta.Heights.HodgeLowerBound.StatementsZhangLowerX, lines 11–17)
GroundTheory.Heights$_2$.Heights.IndefiniteRowSideFullHecke.HeightsSet.height (GroundTheory.HeightsAndTheta.Heights.HeightsOfSets.IsSaturatedFirstCoordinateLine, lines 19–21)
GlobalTheory.CompanionTheory.Ar.Giv.lineOfLargeDegree_of_use (Development.WeilBound.CoreStatements.AutomorphicKernels.MoriwakiBigness.HilbertSamuelApplications.LineOfLargeDegreeOfUse, lines 10–88)
35.24. PositivityTheory.HtPt.Ah.ais (Development.WeilBound.CoreStatements.HodgeTatePoints.AhSlim, lines 22–24)
GlobalTheory.CompanionTheory.Ar.Tw.Main.poincareLelong_of_inv (Development.AutomorphicSpectrum.Proofs.MetricProofs.PoincareLelongOfInv, lines 262–331)
GlobalTheory.CompanionTheory.Ar.Pub.arithIntersectionSystem_of_poincareLelong_resK (Development.AutomorphicSpectrum.Proofs.MetricProofs.ArithIntersectionSystemOfPoincareLelong, lines 42–48)
GlobalTheory.CompanionTheory.SAH.semiAbelianHomExtension (Development.AutomorphicSpectrum.Proofs.GlobalProofs.SemiAbelianHomExtension, lines 13–19)
GlobalTheory.HomExtNormal.existsUnique_hom_extension_of_dvr (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.IsOpenImmersionQuasiFiniteLocusOfIsIntegral, lines 263–556)
GlobalTheory.existsUnique_hom_extension_semiAbelian (Development.AbelianVarietyTheory.Proofs.CoherentCohomologyProofs.ExistsUniqueHomExtensionSemiAbelian, lines 11–14)
PositivityTheory.Xgn.farguesHodgeTate_of_farguesExact (Development.WeilBound.CoreStatements.TorusUniformisation.ChowMahlerFamilies.XXgzFarOfExact, lines 50–60)
GlobalTheory.CompanionTheory.Gaga.gagaSections (Development.AutomorphicSpectrum.Proofs.GlobalProofs.GagaSections, lines 104–138)
GlobalTheory.CompanionTheory.Ar.GR.QuotientByFiniteFlatStatement (Development.AutomorphicSpectrum.PublishedTheorems.Algebra$_2$.QuotientByFiniteFlatStatement, lines 21–29)
GlobalTheory.CompanionTheory.Ar.GR.ZarhinStatement (Development.AutomorphicSpectrum.PublishedTheorems.Algebra$_2$.ZarhinStatement, lines 21–27)
35.25. GlobalTheory.CompanionTheory.Ar.Link.good_tower_and_volume_final (Development.AutomorphicSpectrum.Proofs.GeometryProofs.GoodTowerVolumeTheoremAlt, lines 32–42)
GlobalTheory.CompanionTheory.Ar.Top.goodTower_and_volume (Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.GoodTowerVolumeTheorem, lines 36–50)
GlobalTheory.CompanionTheory.Ar.Top.GoodTowerExistsX (Development.AutomorphicSpectrum.Statements.TowerExistence.GoodTowerExistsX, lines 22–26)
GlobalTheory.T.exists_good_root_tower (Development.AutomorphicSpectrum.Proofs.GeometryProofs.ExistsGoodRootTowerStatement, lines 12–13)
35.26. PositivityTheory.HtPt.Ah.deep_slim_use (Development.WeilBound.CoreStatements.AutomorphicKernels.MoriwakiBigness.HilbertSamuelApplications.AhSlimU, lines 9–14)
GlobalTheory.CompanionTheory.Ar.Plug.DeepStR (Development.AutomorphicSpectrum.Definitions.GlobalGeometry.ModelSmall,

lines 63–66)
GlobalTheory.CovHt.MeanGe (Development.AbelianVarietyTheory.Definitions.Divisors.SemiposAtOfPositive, lines 97–99)
GlobalTheory.CovHt.MeanAll (Development.AbelianVarietyTheory.Definitions.Divisors.SemiposAtOfPositive, lines 101–101)
35.26. GlobalTheory.CompanionTheory.Ar.Plug.deepStR_of_use (Development.WeilBound.CoreStatements.
AutomorphicKernels.MoriwakiBigness.HilbertSamuelApplications.DeepStROfUse, lines 12–23)
GlobalTheory.CompanionTheory.Ar.Link.good_tower_volume_theorem_bis (Development.WeilBound.CoreStatements.
AutomorphicKernels.MoriwakiBigness.HilbertSamuelApplications.GoodTowerVolumeTheoremBis, lines 9–19)
GlobalTheory.CompanionTheory.Ar.Plug.top_meanAll (Development.AutomorphicSpectrum.Proofs.GeometryProofs.
AbelianCmCompLawsBound, lines 54–137)
GlobalTheory.CompanionTheory.Ar.Plug.deepStR_of_volume (Development.AutomorphicSpectrum.Proofs.GeometryProofs.
DeepStROfVolume, lines 21–58)
GlobalTheory.CompanionTheory.Ar.hostLval (Development.AutomorphicSpectrum.Definitions.GlobalGeometry.TowerOrigin,
lines 123–126)
GlobalTheory.CompanionTheory.Ar.Plug.costTwo (Development.AutomorphicSpectrum.Definitions.GlobalGeometry.ModelSmall,
lines 54–54)
35.26. GlobalTheory.CompanionTheory.Ar.Top.GoodOrigin.Extra (Development.AutomorphicSpectrum.Statements.
TowerExistence.GoodTowerExistsX, lines 11–20)
35.27. GlobalTheory.CompanionTheory.Ar.Top.RootModel (Development.AutomorphicSpectrum.Definitions.
DivisibilityTowers.Birational, lines 11–41)
GlobalTheory.CompanionTheory.Ar.Top.RootTower (Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.
Proportional, lines 10–26)
GlobalTheory.CompanionTheory.Ar.Top.RootTower.PPc (Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.
Proportional, lines 45–45)
GlobalTheory.CompanionTheory.Ar.Top.RootTower.PPt (Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.
Proportional, lines 47–47)
GlobalTheory.CompanionTheory.Ar.Top.RootTower.volT (Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.
Proportional, lines 43–43)
GlobalTheory.CompanionTheory.Ar.Top.RootTower.slope (Development.AutomorphicSpectrum.Definitions.
DivisibilityTowers.Proportional, lines 62–63)
GlobalTheory.CompanionTheory.Ar.Top.degFl (Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.
CompactGapsGE, lines 22–24)
GlobalTheory.CompanionTheory.Ar.Top.linRoot (Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.
CompactGapsGE, lines 26–26)
GlobalTheory.CompanionTheory.Ar.Top.flagDeg (Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.
CompactGapsGE, lines 28–28)
GlobalTheory.CompanionTheory.Ar.Top.Dominant (Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.
CompactGapsGE, lines 52–52)
GlobalTheory.CompanionTheory.Ar.Top.RegDominant (Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.
CompactGapsGE, lines 54–54)
GlobalTheory.CompanionTheory.Ar.Top.I9 (Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.Proportional,
lines 133–137)
GlobalTheory.CompanionTheory.Ar.Top.FrobShape (Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.
HeckeSectionsIsotropicCommutatorPairingPower, lines 15–15)
GlobalTheory.CompanionTheory.Ar.Top.deltaTerm (Development.AutomorphicSpectrum.Definitions.DivisibilityTowers.
CompactGapsGE, lines 83–85)
35.28. GlobalTheory.CompanionTheory.Ar.Top.flagDeg_eq_weyl (Development.AutomorphicSpectrum.Proofs.
DivisibilityProofs.FlagDegreeWeightFormula, lines 9–32)
GlobalTheory.CompanionTheory.Ar.Top.form_top_succ_eq_sum_pk (Development.AutomorphicSpectrum.Proofs.
DivisibilityProofs.HomogeneousPowerSumDecomposition, lines 9–10)
GlobalTheory.CompanionTheory.Ar.Top.sh_tot_pk_mul (Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.
FlagShiftTotalPowerSum, lines 8–9)
GlobalTheory.CompanionTheory.Ar.Top.testForms_sh_tot (Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.
ExistsFlagDegreeOnePolys, lines 9–10)
GlobalTheory.T.power_sum_delta_decomposition (Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.
PowerSumDeltaDecompositionStatement, lines 11–12)
GlobalTheory.CompanionTheory.Ar.Top.restTerm_bound_and_tendsto (Development.AutomorphicSpectrum.Proofs.
DivisibilityProofs.RemainderTermSummable, lines 14–15)
35.29. GlobalTheory.CompanionTheory.Ar.Top.slope_lower_of_base_settings (Development.AutomorphicSpectrum.Proofs.
DivisibilityProofs.SlopeLowerBaseSettings, lines 110–118)
35.30. GroundTheory.Heights₂.Heights.RestTerms.SameFactorBound.proof (GroundTheory.HeightsAndTheta.Heights.
RemainderTerms.PSameFactorBound, lines 43–151)
GroundTheory.Heights₂.Heights.RestTerms.OtherFactorBound.proof (GroundTheory.HeightsAndTheta.Heights.
RemainderTerms.POtherFactorBound, lines 83–170)
35.31. GlobalTheory.CompanionTheory.Ar.Top.exists_identAt (Development.AutomorphicSpectrum.Proofs.
DivisibilityProofs.TowerIdentityAtCutoff, lines 155–190)
GlobalTheory.CompanionTheory.Ar.Top.test_height_bound_slope_flat (Development.AutomorphicSpectrum.Proofs.
DivisibilityProofs.TestHeightBound, lines 32–142)
35.32. RootFieldNumerics.mean_slope_of_proportional_volume (Development.WeilBound.CoreStatements.FlagImmersions.

MeanSlopeOfProportionalVolume, lines 13–39)
35.32. GlobalTheory.CompanionTheory.Ar.Cusp.volumeTheoremX_of_inputs (Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.VolumeTheoremOfInputs, lines 72–121)
GlobalTheory.CompanionTheory.Ar.Top.tower_T2_of_inputs (Development.AutomorphicSpectrum.Proofs.DivisibilityProofs.TowerT2, lines 69–164)
RootFieldNumerics.rest_sum_total_bound (Development.WeilBound.CoreStatements.FlagImmersions.TwoMulThreePowLeFactorial, lines 112–163)
SheetDecomposition.volEq_of (Development.WeilBound.CoreStatements.VolumeEquality.VolEqOf, lines 57–97)
35.33. ReciprocityTheory.ChartRestriction.G.ArakelovGeometry.MoriwakiBigStatement (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.ArakelovGeometry.MoriwakiSt, lines 29–36)
PositivityTheory.Pf1.HilbertSamuelMoriwaki.moriwakiBig_outright (Development.WeilBound.CoreStatements.AutomorphicKernels.MoriwakiBigness.MoriwakiBigTheorem.Outright, lines 9–11)
GroundTheory.Heights₂.Heights.HilbertSamuel.ArithmeticVolumeDegreeZero.proof (GroundTheory.HeightsAndTheta.Heights.HilbertSamuel.CoeffLeOfEvalLeMul, lines 79–94)
GroundTheory.Heights₂.Heights.Targets.MoriwakiBig.of_found (GroundTheory.HeightsAndTheta.Heights.OrbitLimits.MoriwakiBig, lines 11–16)
UpperTheory.LiteratureTheorems.every_character_of_literature_of_vWin (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 810–813)
35.33. GlobalTheory.CovTop.lenT (Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol, lines 42–42)
GlobalTheory.CovTop.zed (Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol, lines 44–45)
GlobalTheory.CovTop.Quo (Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol, lines 49–49)
GlobalTheory.CovTop.SizeAt (Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol, lines 75–79)
35.34. GlobalTheory.HtLowProof.htLow (Development.AbelianVarietyTheory.Proofs.DivisorProofs.MultiformHeightLowerBound, lines 107–156)
GlobalTheory.HSShort.deg_ge_ht_det (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LatticeDegreeGeHeightPolynomialVectorsDet, lines 91–121)
35.35. GlobalTheory.CovTop.DegProof.covering_degree_eq_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CoveringDegreeMultiplicative, lines 25–55)
GlobalTheory.CovTop.ShareRowsNef.share_alt_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ShareStatementOfKnownTheorems, lines 30–66)
GlobalTheory.CovTop.DegAt (Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol, lines 140–141)
GlobalTheory.CovTop.ShareAt (Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol, lines 143–144)
GlobalTheory.CovTop.HSAt (Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol, lines 153–156)
GlobalTheory.CovTop.SumAt (Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol, lines 158–159)
35.35. GlobalTheory.CovTop.ShortRoad.hilbertSamuel_of_component_estimates_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.HilbertSamuelLowerBoundOfComponents, lines 25–113)
GlobalTheory.CovHSNum.deg_bound_of_claim_numbers (Development.AbelianVarietyTheory.Proofs.DivisorProofs.DegBoundOfClaimNumbers, lines 9–129)
GlobalTheory.CovSum.sum_deg_div_le_mu (Development.AbelianVarietyTheory.Proofs.DivisorProofs.SumComponentDegreesDivLeLatticeSlope, lines 61–77)
35.36. GlobalTheory.exists_isShortGen_of_short_span (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsShortGeneratorsOfShortSpanningFamily, lines 13–152)
35.37. ReciprocityTheory.ChartRestriction.G.I2.exists_maximal (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.SquareIntegrability.ExistsMaximal, lines 196–222)
ReciprocityTheory.ChartRestriction.G.I2.split_pos (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.SquareIntegrability.SplitG, lines 109–211)
ReciprocityTheory.ChartRestriction.G.I2.cs_bound (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.SquareIntegrability.SplitVB, lines 125–244)
ReciprocityTheory.ChartRestriction.G.I2.sum_sq_norm_ge_of_span (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.SquareIntegrability.SplitV, lines 32–159)
35.38. GlobalTheory.LeviZero.levi_phla_zero (Development.AbelianVarietyTheory.Proofs.TorusHaarMeasure.LeviFormWeightMetricOrthonormalDiagonal, lines 417–470)
GlobalTheory.LeviOriginMargin.levi_margin_origin (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LeviMarginOrthonormalChartOrigin, lines 99–115)
GlobalTheory.frobenius_lower_comparison (Development.AbelianVarietyTheory.Proofs.DivisorProofs.FrobeniusStrictLowerComparisonNearIdentity, lines 33–135)
GlobalTheory.PeakWeight.levi_lower (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsHormanderPeakCutoffSingularWeights, lines 160–232)
GlobalTheory.HormanderPeak.hormander_peak (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsHormanderPeakOrthonormalFrameL2Bound, lines 34–316)
35.39. GlobalTheory.BallSum.ballSum_le (Development.AbelianVarietyTheory.Proofs.DivisorProofs.OrbitSumKernelBallMeasureLeIntegral, lines 192–253)
GlobalTheory.KernelIntegral.lintegral_kaF_le (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LintegralUnitaryKernelLeExpPowEight, lines 279–301)
GlobalTheory.BallVol.ballvol (Development.AbelianVarietyTheory.Proofs.DivisorProofs.InvariantMeasureChartBallOrthonormalLowerBound, lines 91–143)
GlobalTheory.PoincareL2.poincare_l2_of_orthogonal (Development.AbelianVarietyTheory.Proofs.DivisorProofs.PoincareL2InequalityFundamentalSet, lines 87–215)

35.40. `GlobalTheory.Dict.thInCusp_cusp` (Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.ThInCuspCusp, lines 24–61)
`GlobalTheory.ThFr.IsAdapted.flowCoeff_eq_zero` (Development.AbelianVarietyTheory.Proofs.CuspFormEvaluations.IsHarmonicAndMemMinorSpan, lines 591–630)
`GlobalTheory.Dict.CuspAny.mem_SS_of_eq_sum_thFr` (Development.AbelianVarietyTheory.Proofs.SiegelThetaTransformation.ExistsExpansionOfRowHighest, lines 68–96)
35.41. `ReciprocityTheory.ChartRestriction.G.I2.PT` (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.SquareIntegrability.GCauchyStatement, lines 29–46)
`ReciprocityTheory.ChartRestriction.G.I2.PT.la1` (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.SquareIntegrability.GCauchyStatement, lines 64–64)
`ReciprocityTheory.ChartRestriction.G.I2.th` (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.SquareIntegrability.GCauchyStatement, lines 144–144)
`ReciprocityTheory.ChartRestriction.G.I2.Par` (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.SquareIntegrability.GCauchyStatement, lines 156–176)
`ReciprocityTheory.ChartRestriction.G.I2.phiFD` (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.SquareIntegrability.TorusAt, lines 20–21)
`GlobalTheory.EtaAn.RunOK` (Development.AbelianVarietyTheory.Definitions.TorusSections.IsTorusOf, lines 56–57)
`GlobalTheory.EtaAn.p3` (Development.AbelianVarietyTheory.Definitions.TorusSections.IsTorusOf, lines 61–63)
`GlobalTheory.EtaAn.p16` (Development.AbelianVarietyTheory.Definitions.TorusSections.IsTorusOf, lines 67–70)
35.42. `GlobalTheory.RunParametersOfLower.real_bounds` (Development.AbelianVarietyTheory.Proofs.CuspFormEvaluations.RealBounds, lines 18–110)
`GlobalTheory.RunParametersOfLower.runOK_of_le` (Development.AbelianVarietyTheory.Proofs.CuspFormEvaluations.RealBounds, lines 112–161)
`GlobalTheory.ThetaSizeBounds.archimedeanBoundTotalConstant_nonneg_le_bound` (Development.AbelianVarietyTheory.Proofs.DivisorProofs.EtaBoundLengthUnit, lines 26–70)
`GlobalTheory.EtaAn.archimedeanBoundTotalConstant_size` (Development.AbelianVarietyTheory.Proofs.TorusSplitting.ArchimedeanBoundTotalConstantSize, lines 15–166)
`GlobalTheory.T.exists_borel_orthonormal_flag_frame` (Development.AbelianVarietyTheory.Proofs.TorusSplitting.ExistsBorelOrthonormalFlagFrameStatement, lines 11–12)
`GlobalTheory.EtaAn.exists_unit_frame` (Development.AbelianVarietyTheory.Proofs.TorusSplitting.ExistsBorelOrthonormalFlagFrame, lines 15–35)
`GlobalTheory.EtaAn.AHFrame` (Development.AbelianVarietyTheory.Definitions.TorusSections.IsTorusOf, lines 23–37)
`GlobalTheory.EtaAn.IsFnOf` (Development.AbelianVarietyTheory.Definitions.TorusSections.IsTorusOf, lines 39–43)
`GlobalTheory.T.exists_torus_of_period_hermitian` (Development.AbelianVarietyTheory.Proofs.TorusSplitting.ExistsTorusOfPeriodHermitianStatement, lines 11–12)
`GlobalTheory.T.periodTorus_isTorusAt_frame` (Development.AbelianVarietyTheory.Proofs.TorusSplitting.PeriodTorusIsTorusAtFrameStatement, lines 11–12)
`GlobalTheory.T.theta_function_torus_l2_norm` (Development.AbelianVarietyTheory.Proofs.TorusSplitting.ThetaFunctionTorusL2NormStatement, lines 11–12)
`GlobalTheory.T.fullLinearLevel_eq_exp_periods_complexPoint` (Development.AbelianVarietyTheory.Proofs.TorusSplitting.FullLinearLevelEqExpPeriodsComplexPointStatement, lines 11–12)
`GlobalTheory.T.norm_nullVal_eq_norm_theta_alt` (Development.AbelianVarietyTheory.Proofs.TorusSplitting.NormNullValEqNormThetaStatement, lines 11–12)
`GlobalTheory.EtaAn.ArchPt.gen_mod` (Development.AbelianVarietyTheory.Proofs.TorusSplitting.NormEtaProductFormLeAtUnitFrame, lines 36–53)
`GlobalTheory.nullValue_existsUnique_natural_linear` (Development.AbelianVarietyTheory.Proofs.CuspFormEvaluations.ThetaNullValueExistsUniqueNaturalLinear, lines 159–168)
`GlobalTheory.ThetaConst.tauValAt_self` (Development.AbelianVarietyTheory.Definitions.Divisors.ExistsIsTauZeroNonzeroAll, lines 114–119)
`GlobalTheory.ThetaConst.ArchBound.norm_pow_le_at_unit_frame` (Development.AbelianVarietyTheory.Proofs.DivisorProofs.NormSqThetaConstantSpanningClassesLeExp, lines 46–147)
`GlobalTheory.ThetaConst.normSq_thetaConst_le_exp` (Development.AbelianVarietyTheory.Proofs.DivisorProofs.NormSqThetaConstantSpanningClassesLeExp, lines 155–190)
35.43. `GlobalTheory.AllWt.bergE` (Development.AbelianVarietyTheory.Definitions.Foundations.IsProbabilityMeasureMuPart, lines 648–649)
`GlobalTheory.AllWt.bergOn` (Development.AbelianVarietyTheory.Definitions.Foundations.IsProbabilityMeasureMuPart, lines 753–753)
`GlobalTheory.ThinBerg.pnorm_pos'` (Development.AbelianVarietyTheory.Proofs.DivisorProofs.FrameExponentialVolumeLarge, lines 22–30)
`GlobalTheory.ThinBerg.le_bergOn_of_ratio` (Development.AbelianVarietyTheory.Proofs.DivisorProofs.FrameExponentialVolumeLarge, lines 32–59)
`GlobalTheory.BergLower.le_bergOn` (Development.AbelianVarietyTheory.Proofs.DivisorProofs.CuspFormLowerBoundBergmanFunctionOnComponent, lines 59–152)
`GlobalTheory.T.cuspForm_lower_bound_bergmanFunction_on_component` (Development.AbelianVarietyTheory.Proofs.DivisorProofs.CuspFormLowerBoundBergmanFunctionOnComponentStatement, lines 15–16)
35.44. `GlobalTheory.LevGeo.Laws` (Development.AbelianVarietyTheory.Definitions.Divisors.UnifOfSpaceEq, lines 20–41)
`UpperTheory.Dv.Thin.theta_sup_le` (Development.AutomorphicArithmetic.LocalEmbeddings.ThinCore, lines 26–105)
`GlobalTheory.Dict.ThetaPt.theta_bound_ortho` (Development.AbelianVarietyTheory.Proofs.CuspFormEvaluations.

ThetaGeneratorPointwiseBoundOrthonormalFrame, lines 46–101)
35.45. GlobalTheory.LevGeo.Laws (Development.AbelianVarietyTheory.Definitions.Divisors.UnifOfSpaceEq, lines 20–41)
Formalism.L2.Pj.FromTest (Formalism.LocalToGlobal.ProjectiveLattices.AlgebraMapCAt, lines 51–52)
GlobalTheory.ThetaLower.thetaRule_sum_lower (Development.AbelianVarietyTheory.Proofs.CuspFormEvaluations.
SumLeFinsumOfSurjectsOntoFinset, lines 240–337)
ReciprocityTheory.ChartRestriction.G.I2.Par.Jbound (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.
SquareIntegrability.GCauchyStatement, lines 182–185)
GlobalTheory.BlockTh.p14 (Development.AbelianVarietyTheory.Definitions.ThetaValues.IsBlockProd, lines 41–43)
GlobalTheory.AnVal.neg_log_sqrt_le_of_jbound (Development.AbelianVarietyTheory.Proofs.CuspFormEvaluations.
NegLogSqrtLeOfJbound, lines 21–75)
Formalism.L2.Pj.I2FramePointLemmas.jbound_pos (Formalism.LocalToGlobal.ProjectiveLattices.ContDiffAtOfDiff, lines
140–148)
UpperTheory.Dv.Thin.theta_lower (Development.AutomorphicArithmetic.LocalEmbeddings.ThinCore, lines 134–199)
35.46. GlobalTheory.ThetaSizeBounds.run_params_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.RunParams, lines 15–166)
GlobalTheory.ThetaConst.normSq_thetaConst_le_exp (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
NormSqThetaConstantSpanningClassesLeExp, lines 155–190)
GlobalTheory.T.bergman_kernel_thin_part_estimate_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.BergmanKernelThinPartEstimate, lines 58–59)
GlobalTheory.ThickBerg.exp_neg_B_le_bergOn_of_mem_framesOn (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.BergmanKernelThickPartEstimate, lines 16–81)
35.47. GlobalTheory.ThetaSteep.tau_bound (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
ExistsNonzeroSectionWeightBound, lines 109–127)
GlobalTheory.ThetaSteep.phiC_le_valuation (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
ExistsNonzeroSectionWeightBound, lines 36–107)
35.48. GlobalTheory.SumBound.primeSum_eOf (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
WeightedPrimeLogSumOscillationBoundLe, lines 145–267)
GlobalTheory.RefSum.sumAt_of_sizes (Development.AbelianVarietyTheory.Proofs.DivisorProofs.SumPointOfWeightNumerical,
lines 55–97)
GlobalTheory.RefSum.ToWeight.primeSum_add (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
ReferenceWeightOfTheta, lines 51–70)
35.49. GlobalTheory.ShortPhi.exists_shortBasis (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
ExistsShortPeriodLatticeBasisBalancedSlope, lines 18–51)
Formalism.MainNotions.ModuliIntegralCharts.H1.gram_form_phi_holds (Formalism.SharpRestriction.Divisibility.
GeneratorPowers.GramSpace, lines 55–194)
35.50. GlobalTheory.SmallJets.small_characters_span_jets (Development.AbelianVarietyTheory.Proofs.JetHarmonics.
SmallCharactersSpanJets, lines 425–505)
GlobalTheory.UnifJets.jets (Development.AbelianVarietyTheory.Proofs.JetHarmonics.ExistsPowFrame, lines 92–160)
35.51. GlobalTheory.RefAway.refExp_le_of_oscLe (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
BaseLocusExponentLeOscillationNonTypePrime, lines 125–229)
GlobalTheory.RefSum.iInf_thRule_le_nuLocOf (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
IInfThRuleLeNuLocOf, lines 25–93)
GlobalTheory.RefSum.refExp_vp_cover_le (Development.AbelianVarietyTheory.Proofs.DivisorProofs.MemSecOfMemCuspWWtZ,
lines 30–64)
35.52. GlobalTheory.exists_coset_minimiser (Development.AbelianVarietyTheory.Proofs.JetHarmonics.
ExistsPosZsmulMemRange, lines 117–185)
GlobalTheory.DepthBound.exists_eigen_column (Development.AbelianVarietyTheory.Proofs.JetHarmonics.ExistsEigenColumn,
lines 123–246)
35.53. GlobalTheory.FactorDepth.nuFac (Development.AbelianVarietyTheory.Definitions.Divisors.
NonemptyValExtOfOverLa, lines 276–277)
GlobalTheory.FGDepth.detDepth (Development.AbelianVarietyTheory.Definitions.Foundations.IsSatFrame, lines 49–50)
GlobalTheory.RefSum.OscLe (Development.AbelianVarietyTheory.Definitions.Divisors.EightDvdNCol, lines 33–34)
Formalism.L2.Pj.harmA (Formalism.LocalToGlobal.ProjectiveLattices.IsThRule, lines 86–87)
GlobalTheory.FactorDepth.nuLocOf (Development.AbelianVarietyTheory.Definitions.Divisors.NonemptyValExtOfOverLa, lines
260–264)
Formalism.L2.Pj.nabValN (Formalism.LocalToGlobal.ProjectiveLattices.PlaceBelow, lines 30–34)
GlobalTheory.SlopeRec.refExp_vp (Development.AbelianVarietyTheory.Definitions.Foundations.NonnegOfMemCStarSet, lines
272–274)
35.54. GlobalTheory.FGDepth.detDepth_bounds (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
FormalGroupDeterminantPowDepthBounds, lines 29–44)
GlobalTheory.FGDepth.detDepth_eq_zero_of_lt (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
FormalGroupDeterminantPowDepthEqZeroOfLtPrime, lines 55–84)
GlobalTheory.RefSum.facBetween_zero_boundAll (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
FactorDepthBetweenZeroAndBound, lines 14–41)
35.55. GlobalTheory.RefSum.nuBetween_sum_and_eLoc_le (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
ProductDepthSumRoundingOscillation, lines 16–46)
GlobalTheory.RefSum.oscLe_zero_above_and_degree (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
DepthOscillationZeroAboveAndDegreeBound, lines 78–99)

35.56. GlobalTheory.RefSum.Clear.clear (Development.AbelianVarietyTheory.Proofs.DivisorProofs.CuspFormClearDenominatorsAwayFromPlace, lines 16–47)
GlobalTheory.RefSum.ScalApprox.exists_scal (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsScal, lines 35–115)
35.57. GlobalTheory.IsoCols.exists_isIsoCols (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsNormEqNegOne, lines 194–242)
GlobalTheory.IsoCols.exists_pair (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsNormEqNegOne, lines 97–153)
GlobalTheory.IsoCols.exists_norm_eq_neg_one (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsNormEqNegOne, lines 11–40)
GlobalTheory.RefGlob.exists_harmA_congr_polyT (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsGlobalHarmonicCongrColumnPolynomial, lines 97–118)
GlobalTheory.totalDegree_le_and_exists_combination_of_mem_FFV (Development.AbelianVarietyTheory.Proofs.JetHarmonics.TotalDegreeLeAndExistsCombinationOfMemFFV, lines 21–50)
35.58. GlobalTheory.RefSum.refExp_vp_cover_le (Development.AbelianVarietyTheory.Proofs.DivisorProofs.MemSecOfMemCuspWWtZ, lines 30–64)
GlobalTheory.CoverDown.exists_intFr_below (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsIntFrBelow, lines 94–105)
35.59. GlobalTheory.RefSum.LowP.thRule_lower (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ThRuleLower, lines 26–80)
GlobalTheory.RefSum.TermLoc.harmonic_value_ge_local_depth (Development.AbelianVarietyTheory.Proofs.DivisorProofs.HarmonicValueGeLocalDepthPrimeToP, lines 24–62)
GlobalTheory.RefSum.iInf_thRule_le_nuLocOf (Development.AbelianVarietyTheory.Proofs.DivisorProofs.IInfThRuleLeNuLocOf, lines 25–93)
35.60. GlobalTheory.RefAway.refExp_le_of_oscLe (Development.AbelianVarietyTheory.Proofs.DivisorProofs.BaseLocusExponentLeOscillationNonTypePrime, lines 125–229)
GlobalTheory.thetaRules_exist_and_cusp (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ThetaRulesExistAndCusp, lines 18–27)
35.61. GlobalTheory.LimInv.limSub (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LimitEntriesMatrixInverseClosedProof, lines 9–27)
GlobalTheory.LimInv.inv_mem_limSub (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LimitEntriesMatrixInverseClosedProof, lines 29–70)
GlobalTheory.IsoTriv.limitEntriesMatrixInverseClosed_proof (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LimitEntriesMatrixInverseClosedProof, lines 75–98)
35.62. GlobalTheory.ColumnGerm.exists_nabX_eq (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsIntegralGermOnCopiesSameValue, lines 47–86)
35.63. GlobalTheory.IsoTriv.PrimeIdempotent.epsSt (Development.AbelianVarietyTheory.Proofs.DivisorProofs.PrimeIdempotentApproximationsDVRProof, lines 40–81)
GlobalTheory.IsoTriv.LimAct.cauchy_eps_mul (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LimitActionLieAlgebraOfPrimeIdempotentApprox, lines 107–134)
35.64. GlobalTheory.IsoTriv.limActSt_of_epsSt (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LimitActionLieAlgebraOfPrimeIdempotentApprox, lines 144–193)
GlobalTheory.IsoTriv.limPt_of_steps (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LimitActionIntegralPointOfSteps, lines 13–52)
35.65. GlobalTheory.IsoTriv.limFrame (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LimitIsotropicFrameIntegralOfSteps, lines 44–113)
35.66. GlobalTheory.FactorDepth.nonempty_valExt_of_overLa (Development.AbelianVarietyTheory.Definitions.Divisors.NonemptyValExtOfOverLa, lines 320–321)
GlobalTheory.ColApprox.exists_germ_cols (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsIntegralGermOnColumnsSameValue, lines 90–166)
GlobalTheory.ColApprox.exists_germ_close (Development.AbelianVarietyTheory.Proofs.DivisorProofs.SubstTwoPointsPowerSeriesZeroOfVanishFirstAxis, lines 331–428)
GlobalTheory.ColumnGerm.exists_germ_on_copies (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsIntegralGermOnCopiesSameValue, lines 97–115)
GlobalTheory.ColumnGerm.exists_nabX_eq (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsIntegralGermOnCopiesSameValue, lines 47–86)
35.67. GlobalTheory.ColDepth.nuLocOf_eq_nuP (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ColumnPolynomialLocalDepthEqPlaceDepth, lines 60–76)
GlobalTheory.ColDepth.iInf_eq_of_approx (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ColumnPolynomialLocalDepthEqPlaceDepth, lines 49–56)
GlobalTheory.DepthDescent.nuLocOf_eq_nuLocX_vp (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LocalDepthOverDvrEqOverClosedField, lines 19–95)
GlobalTheory.ColApprox.exists_germ_cols (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsIntegralGermOnColumnsSameValue, lines 90–166)
GlobalTheory.ColumnGerm.exists_germ_on_copies (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsIntegralGermOnCopiesSameValue, lines 97–115)
GlobalTheory.IsoFactorDepth.nuH_vp_eq_nuP (Development.AbelianVarietyTheory.Proofs.DivisorProofs.IsotropicFactorDepthEqSumFactorDepths, lines 47–83)

35.68. `ReciprocityTheory.ChartRestriction.G.TB.harmSecond_holds` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.WickVanishing.HarmSecond`, lines 23–59)
`ReciprocityTheory.ChartRestriction.G.TB.pairMin` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.GermSC`, lines 22–24)
`ReciprocityTheory.ChartRestriction.G.PullbackSections.Pf.sum_pairMin_le` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.PullbackSections.HolomorphicFamilies.ChartValueRamifiedOfLifts`, lines 110–126)
`ReciprocityTheory.ChartRestriction.G.PullbackSections.Pf.Jv` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.PullbackSections.HolomorphicFamilies.ChartValueRamifiedOfLifts`, lines 35–45)
`ReciprocityTheory.ChartRestriction.G.PullbackSections.Pf.mem_Jv_pow` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.PullbackSections.HolomorphicFamilies.ChartValueRamifiedOfLifts`, lines 56–80)
35.69. `ReciprocityTheory.ChartRestriction.G.TB.ordNabla` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.GermSC`, lines 14–20)
`ReciprocityTheory.ChartRestriction.G.TB.LemmaHColsSt` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.GermSCCols`, lines 24–37)
`ReciprocityTheory.ChartRestriction.G.TB.SCColsSt` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.GermSCCols`, lines 10–22)
`ReciprocityTheory.ChartRestriction.G.TB.SCPf.ordNabla_value_expand` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.WickVanishing.LemmaH`, lines 58–67)
`ReciprocityTheory.ChartRestriction.G.TB.WickVanishing.HCols.colAlg` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.WickVanishing.LemmaHCols`, lines 153–207)
`ReciprocityTheory.ChartRestriction.G.TB.WickVanishing.HCols.lemmaHCols_of` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.WickVanishing.LemmaHCols`, lines 223–238)
`ReciprocityTheory.ChartRestriction.G.TB.SCPf.sc_of` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.WickVanishing.SCMain`, lines 28–83)
`ReciprocityTheory.ChartRestriction.G.TB.WickVanishing.Closed.scaled_connection_columns` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.WickVanishing.Closed`, lines 47–62)
35.70. `ReciprocityTheory.ChartRestriction.G.TB.sgOf` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.ConnectionKillsSeries`, lines 18–19)
`ReciprocityTheory.ChartRestriction.G.TB.sg1` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.ConnectionKillsSeries`, lines 21–22)
`ReciprocityTheory.ChartRestriction.G.TB.Hset` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.ConnectionKillsSeries`, lines 41–43)
`ReciprocityTheory.ChartRestriction.G.TB.WickVanishing.Nab.nabla_eq` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.WickVanishing.ConnectionKillsSeries`, lines 101–106)
`ReciprocityTheory.ChartRestriction.G.TB.WickVanishing.Nab.connection_second_of_commute` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.WickVanishing.ConnectionSecondOfCommute`, lines 38–48)
`ReciprocityTheory.ChartRestriction.G.TB.WickVanishing.Nab.connection_kills_series` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.WickVanishing.ConnectionKillsSeries`, lines 117–120)
`ReciprocityTheory.ChartRestriction.G.TB.WickVanishing.Val.harmonic_value_integral` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.WickVanishing.HarmonicValueIntegral`, lines 144–147)
`ReciprocityTheory.ChartRestriction.G.TB.WickVanishing.Nab.connection_first_identity` (`ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.TaylorDiscriminants.WickVanishing.ExpConjConnectionIntegral`, lines 84–90)
35.71. `GlobalTheory.PlaceInt.chBound_zero_of_isUnit_discr` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.HarmonicValueIntegralOfUnitDiscriminant`, lines 168–234)
`GlobalTheory.PlaceInt.ChartRam.chBound_ramified` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.HarmonicValueBoundIntegralChartRamified`, lines 55–89)
`GlobalTheory.PlaceInt.host_galois_discr_moves` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.HostFieldGaloisDegreeDiscrConjMovesPrimes`, lines 26–54)
`GlobalTheory.PlaceInt.HostChart.chBound_host` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.HarmonicValuationLowerBoundHostFieldAllPlaces`, lines 36–69)
35.72. `GlobalTheory.PlaceInt.HostChart.chBound_host` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.HarmonicValuationLowerBoundHostFieldAllPlaces`, lines 36–69)
`GlobalTheory.PlaceInt.host_galois_discr_moves` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.HostFieldGaloisDegreeDiscrConjMovesPrimes`, lines 26–54)
`GlobalTheory.PlaceInt.chBound_zero_of_isUnit_discr` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.HarmonicValueIntegralOfUnitDiscriminant`, lines 168–234)
`GlobalTheory.PlaceInt.ChartRam.chBound_ramified` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.HarmonicValueBoundIntegralChartRamified`, lines 55–89)
`GlobalTheory.RefSum.GlobalPoly.exists_polyBetween` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsHarmonicPolynomialLocalDepthBetween`, lines 62–92)
`GlobalTheory.RefSum.GlobalPoly.vV_prime_le` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsHarmonicPolynomialLocalDepthBetween`, lines 35–58)
`GlobalTheory.nuLocOf_pert` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LocalDepthStableUnderPrimePowPerturbation`, lines 63–83)
35.73. `GlobalTheory.PlaceInt.uu_eq_pull_reach` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.MixedTranslatePullbackEqReachedSection`, lines 17–64)
`GlobalTheory.ReachRegular.regular_of_reach` (`Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.ReachedSectionRegularOnIntegralModel`, lines 71–100)

GlobalTheory.RegularGerm.regular_germ (Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.
RegularSectionIntegralGermUniqueScale, lines 250–252)
GlobalTheory.PlaceInt.hv_eq_nabValN (Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.
HarmonicValueIntegralPointEqValueOnBiextensionCo, lines 76–93)
GlobalTheory.PlaceInt.hv_pulls (Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.
HarmonicValueChainRuleNormalizedPullback, lines 15–54)
GlobalTheory.PlaceInt.TermBound.term_bound (Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.
HarmonicValueTranslatedProductValuationBound, lines 76–123)
35.74. GlobalTheory.ThetaBound.bound_of_chBound (Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.
ThetaGeneratorPlaceValuationBound, lines 88–148)
GlobalTheory.TorsFac.factorisation (Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.DvdOfCastDvd, lines
91–171)
GlobalTheory.T.harmonicValue_translated_product_valuation_bound (Development.AbelianVarietyTheory.Proofs.
RingsOfIntegers.HarmonicValueTranslatedProductValuationBoundStatement, lines 11–12)
35.75. GlobalTheory.ThetaLow.thetaRule_bounded_below (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
ThetaRuleBoundedBelow, lines 43–121)
GlobalTheory.ThetaBound.bound_of_chBound (Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.
ThetaGeneratorPlaceValuationBound, lines 88–148)
GlobalTheory.ThetaBound.Prep.exists_good_chart_allRat (Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.
ExistsExtAllModelsUniformised, lines 318–335)
GlobalTheory.TorsFac.factorisation (Development.AbelianVarietyTheory.Proofs.RingsOfIntegers.DvdOfCastDvd, lines
91–171)
GlobalTheory.SplitLevel.level_halfSplit (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LevelHalfSplit, lines
15–23)
GlobalTheory.RefLevel.coprime_level (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
SupLatticeReferenceExponentThetaWeightLevelLeOne, lines 66–69)
35.75. ReciprocityTheory.ChartRestriction.G.Geo.Tor.Datum.mult (ReciprocityTheory.ReductiveGroups.
ArithmeticGeometry.BirationalGeometry.ToroidalCompactification.FlagBasis, lines 76–76)
35.76. UpperTheory.Depth.SplitPlaces (Development.AutomorphicArithmetic.CMTraceForms.Places, lines 19–29)
GlobalTheory.RefSum.oscLe_eOf (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
DepthOscillationLeBoundEveryPrime, lines 17–61)
GlobalTheory.T.depth_oscillation_le_at_two_cmField (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
DepthOscillationLeAtTwoCmFieldStatement, lines 11–12)
GlobalTheory.RefSum.oscLe_two (Development.AbelianVarietyTheory.Proofs.DivisorProofs.DepthOscillationLeAtTwoCmField,
lines 27–96)
GlobalTheory.OscSplit.oscLe_split (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
DepthOscillationLeOddSplitUnramifiedPrime, lines 57–96)
GlobalTheory.OscSplit.sumBig_le (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
DepthOscillationLeOddSplitUnramifiedPrime, lines 39–55)
35.77. GlobalTheory.T.frobenius_orbit_of_unramified_prime (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
FrobeniusOrbitOfUnramifiedPrimeStatement, lines 10–11)
GlobalTheory.ChF.unramified_companion_chart_characters (Development.AbelianVarietyTheory.Proofs.UnramifiedDepth.
UnramifiedCompanionChartCharacters, lines 32–75)
GlobalTheory.CharMinor.colength_bound (Development.AbelianVarietyTheory.Proofs.JetHarmonics.
ExistsCharactersBoundedMinorEigenspace, lines 82–151)
GlobalTheory.RefSum.hasChars_frame_of_hasAF (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
FlagStepCharactersOfColength, lines 20–66)
35.78. GlobalTheory.T.factorDepth_upperBound_oddUnramifiedPrime (Development.AbelianVarietyTheory.Proofs.
DivisorProofs.FactorDepthUpperBoundOddUnramifiedPrimeStatement, lines 10–11)
GlobalTheory.RefSum.facBetween_zero_boundAll (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
FactorDepthBetweenZeroAndBound, lines 14–41)
35.79. GlobalTheory.T.formalGroup_rootsOfUnity_typed_lieRanks (Development.AbelianVarietyTheory.Proofs.
DivisorProofs.FormalGroupRootsOfUnityTypedLieRanksStatement, lines 5–6)
GlobalTheory.PrimePart.formalGroupTypes (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
FormalGroupRootsOfUnityTypedLieRanks, lines 221–222)
35.80. GlobalTheory.RefSum.facBetween_fine (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
FactorDepthBoundedOddInertPrimeCmType, lines 28–83)
35.81. GlobalTheory.IndefiniteRowSideFullHecke.FactorDepthInertBounds.boundFine (Development.AbelianVarietyTheory.
BlockTheory.DivisorBase.FactorDepthInertBounds, lines 38–40)
GlobalTheory.RefSum.oscLe_eSmall_of_inert (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
DepthOscillationLeAtSmallOddInertPrime, lines 180–194)
GlobalTheory.RefSum.FacBetween (Development.AbelianVarietyTheory.Definitions.Divisors.EightDvdNCol, lines 21–24)
35.82. GlobalTheory.detPow_content_const (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
DividedPowerContentDetPowDependsOnPIK, lines 51–76)
GlobalTheory.T.formalGroupDeterminantPowDepth_bounds (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
FormalGroupDeterminantPowDepthBoundsStatement, lines 8–9)
35.83. GlobalTheory.FGDepth.InertBound.content_le_detDepth_add (Development.AbelianVarietyTheory.Proofs.
DivisorProofs.ContentLeDetDepthAdd, lines 151–299)

GlobalTheory.T.exists_typicalCurve_basis_adapted_to_saturated_frame (Development.AbelianVarietyTheory.Proofs.
DivisorProofs.ExistsTypicalCurveBasisAdaptedToSaturatedFrameStatement, lines 8–9)
ReciprocityTheory.ChartRestriction.G.Sh.pt5v' (ReciprocityTheory.ReductiveGroups.ArithmeticGeometry.
WeightedSumBounds.WtSubOneLe, lines 99–189)
35.84. GlobalTheory.T.factorDepth_bounds_inert_prime (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
FactorDepthBoundsInertPrimeStatement, lines 10–11)
GlobalTheory.IndefiniteRowSideFullHecke.FactorDepthInertBounds.boundInert (Development.AbelianVarietyTheory.
BlockTheory.DivisorBase.FactorDepthInertBounds, lines 28–28)
GlobalTheory.RefSum.InertOsc.big_arith (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
DepthOscillationLargeInertPrime, lines 32–36)
GlobalTheory.RefSum.InertOsc.big_bound (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
DepthOscillationLargeInertPrime, lines 38–54)
35.85. GlobalTheory.RefSum.oscLe_eOf (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
DepthOscillationLeBoundEveryPrime, lines 17–61)
GlobalTheory.OscSplit.oscLe_split (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
DepthOscillationLeOddSplitUnramifiedPrime, lines 57–96)
GlobalTheory.T.factorDepth_upperBound_oddUnramifiedPrime (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
FactorDepthUpperBoundOddUnramifiedPrimeStatement, lines 10–11)
GlobalTheory.RefSum.oscLe_eSmall_of_inert (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
DepthOscillationLeAtSmallOddInertPrime, lines 180–194)
GlobalTheory.RefSum.OscInert.sum_bound_le (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
DepthOscillationLeAtSmallOddInertPrime, lines 112–174)
GlobalTheory.RefSum.InertOsc.oscLe_inert_large (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
DepthOscillationLargeInertPrime, lines 62–117)
GlobalTheory.T.factorDepth_bounds_inert_prime (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
FactorDepthBoundsInertPrimeStatement, lines 10–11)
GlobalTheory.RefSum.InertOsc.big_bound (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
DepthOscillationLargeInertPrime, lines 38–54)
35.86. GlobalTheory.RefSum.ThetaExp.expBd_thA (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
ThetaExponentsBigSmallFactorBounds, lines 234–251)
GlobalTheory.RefSum.oscLe_eOf (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
DepthOscillationLeBoundEveryPrime, lines 17–61)
GlobalTheory.RefSum.sumAt_of_sizes (Development.AbelianVarietyTheory.Proofs.DivisorProofs.SumPointOfWeightNumerical,
lines 55–97)
GlobalTheory.CharDepth.twoPlaces_host (Development.AbelianVarietyTheory.Proofs.JetHarmonics.
HostFieldTwoConjugatePlacesOverTwo, lines 16–26)
UpperTheory.KR.small_vd_sum (Development.AutomorphicArithmetic.CyclotomicBounds.RamifiedDiscriminants.SmallVdSum,
lines 7–55)
UpperTheory.splitPlaces_discriminantDepth_of_splitAt (Development.AutomorphicArithmetic.CMTraceForms.
SplitPlacesDiscriminantDepthOfSplitAt, lines 59–149)
GlobalTheory.OscSplit.oscLe_split (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
DepthOscillationLeOddSplitUnramifiedPrime, lines 57–96)
GlobalTheory.RefSum.oscLe_eSmall_of_inert (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
DepthOscillationLeAtSmallOddInertPrime, lines 180–194)
GlobalTheory.RefSum.InertOsc.oscLe_inert_large (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
DepthOscillationLargeInertPrime, lines 62–117)
35.87. GlobalTheory.LevelPrimePoint.exists_minor_harm (Development.AbelianVarietyTheory.Proofs.JetHarmonics.
ThetaGeneratorsLeastValueOverLevelPrime, lines 57–130)
GlobalTheory.LevelPrimePoint.crP_mem_harmK_of_disjoint (Development.AbelianVarietyTheory.Proofs.JetHarmonics.
ThetaGeneratorsLeastValueOverLevelPrime, lines 31–53)
Formalism.L2.Pj.Crude.crP_rowInv (Formalism.LocalToGlobal.ProjectiveLattices.CrudeMin, lines 143–193)
Formalism.L2.Pj.Crude.crP_rowDeg (Formalism.LocalToGlobal.ProjectiveLattices.CrudeMin, lines 344–349)
35.88. GlobalTheory.PowGerm.lemmaBT_of_characters (Development.AbelianVarietyTheory.Proofs.JetHarmonics.
IsUnitIffNotIsUnitOfCoprime, lines 50–139)
GlobalTheory.T.exists_characters_boundedMinors_of_hodgeTateChartBound (Development.AbelianVarietyTheory.Proofs.
JetHarmonics.ExistsCharactersBoundedMinorsOfHodgeTateChartBouStatement, lines 19–20)
GlobalTheory.T.depth_bound_from_characters_pointDatum (Development.AbelianVarietyTheory.Proofs.JetHarmonics.
DepthBoundFromCharactersPointDatumStatement, lines 13–14)
35.89. GlobalTheory.T.pointStatements_on_copies_hold (Development.AbelianVarietyTheory.Proofs.JetHarmonics.
PointStatementsOnCopiesHoldStatement, lines 15–16)
GlobalTheory.PowGerm.GenPointT (Development.AbelianVarietyTheory.Definitions.Jets.JetsOnCopiesCompatibleFrames, lines
341–367)
GlobalTheory.PtTie.harmonicValue_translate (Development.AbelianVarietyTheory.Proofs.JetHarmonics.
HarmonicValueTranslateEqUnitMulNormaliserThetaVa, lines 62–130)
GlobalTheory.LevelPrimePoint.exists_val_le (Development.AbelianVarietyTheory.Proofs.JetHarmonics.
ThetaGeneratorsLeastValueOverLevelPrime, lines 156–164)
35.90. GlobalTheory.LevelPrimePoint.least_value_le (Development.AbelianVarietyTheory.Proofs.JetHarmonics.
ThetaGeneratorsLeastValueOverLevelPrime, lines 196–302)

GlobalTheory.ChF.hodgeTate_inclusion_chart (Development.AbelianVarietyTheory.Proofs.UnramifiedDepth.
HodgeTateInclusionChartCharacters, lines 49–92)
35.91. GlobalTheory.RefLevel.refExp_level_le_one (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
SupLatticeReferenceExponentThetaWeightLevelLeOne, lines 163–171)
GlobalTheory.RefLevel.refExp_le_of_halfSplit (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
SupLatticeReferenceExponentThetaWeightLevelLeOne, lines 71–161)
GlobalTheory.RefLevel.lattice_terms_le (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
SupLatticeReferenceExponentThetaWeightLevelLeOne, lines 27–34)
GlobalTheory.LevelPrimePoint.least_value_le (Development.AbelianVarietyTheory.Proofs.JetHarmonics.
ThetaGeneratorsLeastValueOverLevelPrime, lines 196–302)
GlobalTheory.ChF.hodgeTate_inclusion_chart (Development.AbelianVarietyTheory.Proofs.UnramifiedDepth.
HodgeTateInclusionChartCharacters, lines 49–92)
35.92. GlobalTheory.T.refW_of_prints_size_flatDeg_refTheta (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.RefinedSumWeightOfRefinedTheta, lines 48–49)
GlobalTheory.RefSum.ToWeight.primeSum_add (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
ReferenceWeightOfTheta, lines 51–70)
GlobalTheory.RefSum.ToWeight.sum_different_le (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
ReferenceWeightOfTheta, lines 32–44)
GlobalTheory.RefSum.refinedSum_weight_of_size_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.RefinedSumWeightOfSizeLe, lines 16–17)
35.93. GlobalTheory.RefSum.refCore_of_size_sumPt (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.RefCoreOfSumPt, lines 15–37)
GlobalTheory.RefSum.primeSum_add_pair (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
ReferenceCoreOfSumPoint, lines 15–37)
GlobalTheory.RefSum.ToWeight.primeSum_add (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
ReferenceWeightOfTheta, lines 51–70)
GlobalTheory.RefSum.ToWeight.sum_different_le (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
ReferenceWeightOfTheta, lines 32–44)
GlobalTheory.RefSum.refinedSum_weight_of_size_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.RefinedSumWeightOfSizeLe, lines 16–17)
35.94. SelfIntersectionVolumeEstimates.height_lower_bound_general_alt (Development.AutomorphicArithmetic.
ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.HeightLowerBoundGeneral, lines 21–22)
GlobalTheory.CovTop.HeightAt (Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol, lines 146–151)
35.94. GlobalTheory.CovHt.HtTopG.slope_le_height (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.HodgeBundleWeights.HeightAddLogNonneg, lines 29–65)
GlobalTheory.CovHt.HtTopG.heightAt_eq_of_cover (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.HodgeBundleWeights.HeightAddLogNonneg, lines 66–105)
GlobalTheory.CovHt.HtTopG.height_of_meanAll_bound (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.HeightOfSteps, lines 19–38)
GlobalTheory.CompHt.le_compVolSlope_iff (Development.AbelianVarietyTheory.Definitions.Divisors.
IsHomogeneousCompIdeal, lines 283–286)
35.95. GlobalTheory.T.uniform_lower_bound_of_sums' (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.ULowOfStatement, lines 11–12)
GlobalTheory.T.liftLowBound_of_cover_estimates (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.LowOfCoverHypothesesStatement, lines 11–12)
35.95. GlobalTheory.CovTop.DefLow.covering_low_defect_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.CoveringLowDefectBound, lines 17–46)
GlobalTheory.T.two_definiteBound_ge_anchoredMainTerm_sub_crude_errors (Development.AbelianVarietyTheory.Proofs.
DivisorProofs.TwoDefiniteBoundGeAnchoredMainTermSubCrudeErrorsStatement, lines 12–13)
GlobalTheory.T.weight_dimension_log_bound' (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.
ThirdPlaceSeries.ParametersScaleRankGapBoundsStatement, lines 13–14)
GlobalTheory.CovTop.DefSizes.parameters_scale_rank_gap_bounds (Development.AutomorphicArithmetic.
ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ParametersScaleRankGapBounds, lines 24–41)
GlobalTheory.CovTop.DefSizes.sizes_at (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
WeightDimensionLogBound, lines 188–240)
GlobalTheory.CovTop.FromComp.core (Development.AbelianVarietyTheory.Proofs.DivisorProofs.UniformLowerBoundOfSums,
lines 15–32)
Formalism.ExtensionFiltration.sumComponents' (Formalism.MeanHeightDescent.AverageDegreeBounds.SumComponents, lines
8–118)
GlobalTheory.DescentFamily.descent (Development.AbelianVarietyTheory.Proofs.DivisorProofs.HtMapLeOfPoints, lines
69–143)
GlobalTheory.CovTop.LowCover.lowAt_of_cover (Development.AbelianVarietyTheory.Proofs.DivisorProofs.
LowerBoundOfAllEstimates, lines 65–105)
GlobalTheory.CovTop.LowCover.low_of_cover_hypotheses (Development.AutomorphicArithmetic.ThetaCorrespondence.
ThetaPairLifts.ThirdPlaceSeries.LowOfCoverHypotheses, lines 16–32)
35.95. GlobalTheory.CovTop.latGood (Development.AbelianVarietyTheory.Definitions.Divisors.
NormNegCmConstantEqAtAllEmbeddings, lines 44–45)
GlobalTheory.CovTop.latBad (Development.AbelianVarietyTheory.Definitions.Divisors.

NormNegCmConstantEqAtAllEmbeddings, lines 47–48)
35.96. GlobalTheory.T.liftBadSet_of_deep_size_flatDeg_share (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CoveringBadLocusOfDescentSteps, lines 54–55)
GlobalTheory.CovTop.BadAt (Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol, lines 69–71)
35.96. GlobalTheory.CovTop.LocShareNum.local_share_count_le_exp_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LocalShareCountLeExp, lines 31–56)
GlobalTheory.CovTop.LocShareNum.wg_sum_le (Development.AbelianVarietyTheory.Proofs.DivisorProofs.LocalShareSumBound, lines 271–282)
GlobalTheory.BadEmb.embReg_deep (Development.AbelianVarietyTheory.Proofs.DivisorProofs.CoveringRepresentationTypeFrame, lines 185–208)
GlobalTheory.BadEmb.regular_embeddings_cover_bad_embedding_locus_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RegularEmbeddingsCoverBadEmbeddingLocus, lines 38–86)
GlobalTheory.BadIndef.bad_components_indefinite_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.BadComponentsIndefinite, lines 24–54)
GlobalTheory.CovTop.definite_bad_place_lift_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ExistsLiftAtDefiniteBadPlaces, lines 21–53)
GlobalTheory.CovTop.covering_bad_locus_of_descent_steps_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CoveringBadLocusOfDescentSteps, lines 21–40)
35.97. GlobalTheory.CutNum.slope_good_ge (Development.AbelianVarietyTheory.Proofs.DivisorProofs.MajAtAddRight, lines 111–142)
35.97. GlobalTheory.CutNum.numeric (Development.AbelianVarietyTheory.Proofs.DivisorProofs.MajAtAddRight, lines 93–109)
Formalism.ExtensionFiltration.goodBadCut (Formalism.MeanHeightDescent.DivisionInequalityWithCovolume, lines 23–23)
35.97. GlobalTheory.CovTop.GramAt (Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol, lines 53–55)
GlobalTheory.CovTop.RecAt (Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol, lines 57–59)
GlobalTheory.CovTop.CutAt (Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol, lines 61–67)
GlobalTheory.CovTop.CoreAt (Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol, lines 92–94)
GlobalTheory.CovTop.LowAt (Development.AbelianVarietyTheory.Definitions.Divisors.CovolXOfCovol, lines 96–96)
GlobalTheory.CovTop.recAt_of_gramAt_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RecAtOfGramAt, lines 17–46)
GlobalTheory.T.group_idempotent_coefficients_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.TiedGoodIdemStatement, lines 12–18)
GlobalTheory.CovTop.cutAt_of_rows_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CutAtOfRows, lines 18–26)
GlobalTheory.CoreSlope.coreLe_of_gram_meas_rat_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CoreLeOfGramMeasRat, lines 16–24)
35.98. GlobalTheory.CovTop.Assembly.lcv_le (Development.AbelianVarietyTheory.Proofs.DivisorProofs.CovolumeFromGramAndBounds, lines 39–84)
35.99. GlobalTheory.CovTop.Assembly.covolX_of_lift_bounds_and_coreLe (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CovolXOfSteps, lines 16–40)
35.99. GlobalTheory.CovTop.Assembly.exists_loss_le (Development.AbelianVarietyTheory.Proofs.DivisorProofs.CovolumeFromGramAndBounds, lines 25–34)
35.100. GlobalTheory.AtMember.fieldTies_of_ksExact (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.IndexBoundCoefficientFieldOfExactSmallField, lines 28–77)
GlobalTheory.FieldSize.sz_Om_poR (Development.AbelianVarietyTheory.Proofs.IwasawaDecomposition.OmegaFieldDegreeRootDiscrBoundedByRingPolynomial, lines 52–141)
35.101. GlobalTheory.Glob.CovolRow.linked_bundles_l3_of_open_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LinkedBundlesL3OfOpen, lines 31–46)
GlobalTheory.Fc.S0.CovolumeBoundThirdPlaceSeries (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.GeometricWeakTies, lines 32–33)
GlobalTheory.Fc.Row (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.ValidForConstructedPointClassesEventuallyWorking, lines 268–269)
GlobalTheory.Fc.W3 (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.ValidForConstructedPointClassesEventuallyWorking, lines 371–372)
GlobalTheory.Fc.Ind.EvWC (Development.AbelianVarietyTheory.Definitions.Divisors.BaseRankDimension, lines 90–91)
35.101. GlobalTheory.Fc.Ind.eventual_L3_of_covolume_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EventualL3OfCovolume, lines 14–20)
GlobalTheory.Fc.Ind.evWC_onPt_of_guarded_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EvWCOnPtOfGuarded, lines 12–32)
GlobalTheory.Fc.Ind.summandL3Bound (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.SummandFunctionalBound, lines 54–54)
35.101. UpperTheory.Lit.nonempty_hyps_definite_side_of_closed_bound (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.SiegelEisensteinSecondTermHypotheses, lines 44–68)
Formalism.L2.Indef.IndefSide.boundsMu (Formalism.LocalToGlobal.IndefiniteForms.IndefUse, lines 271–334)
Formalism.L2.DIV.wordFacts_of_face (Formalism.LocalToGlobal.Divisibility.RecBridge, lines 204–225)

Chapter 36: The definite side and local errors

36.1. Formalism.L2.DIV.NumDatum.NumFacts (Formalism.LocalToGlobal.Divisibility.NumbersLink, lines 43–82)
Formalism.L2.DIV.HtNums (Formalism.LocalToGlobal.Divisibility.Inputs, lines 210–310)
Formalism.L2.DIV.GrpTie (Formalism.LocalToGlobal.Divisibility.RowSideTie, lines 156–176)
Formalism.L2.DIV.LvlTie (Formalism.LocalToGlobal.Divisibility.RowSideTie, lines 128–154)
Formalism.DimensionVariation.liftedMapLogSize (Formalism.ThetaSupremumBounds.DivIneqExplicit6, lines 15–22)
GlobalTheory.Fc.DefiniteSideCondition (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.DefiniteSideCondition, lines 25–35)
Formalism.L2.DIV.DivTied3 (Formalism.LocalToGlobal.Divisibility.DivisionCompatibleToSub2Indef2, lines 172–196)
Formalism.L2.DIV.ColTie (Formalism.LocalToGlobal.Divisibility.RowSideTie, lines 247–257)
Formalism.L2.DIV.Cons.nuOf (Formalism.LocalToGlobal.Divisibility.DivisibilityConstruction.ShiftOfColTie, lines 14–16)
Formalism.L2.RowDataTransport.RowsU (Formalism.LocalToGlobal.PtKOfReal, lines 141–159)
Formalism.L2.DIV.ColChain (Formalism.LocalToGlobal.Divisibility.Inputs, lines 319–340)
Formalism.L2.DIV.FirstCols (Formalism.LocalToGlobal.Divisibility.DivisionHeightBoundStatement, lines 504–506)
36.2. GlobalTheory.CoreHt.HtRowsS (Development.AbelianVarietyTheory.Definitions.LatticeFrames.RecHeightSWith, lines 11–18)
GlobalTheory.CoreHt.RowsOfDesign.rows_of_design (Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.TwoLeOfLogPos, lines 24–103)
GlobalTheory.EX.szRow_sizeNodes (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExplicitChoiceSizeNodes, lines 11–72)
GlobalTheory.CoreHt.famRowsSizes (Development.AbelianVarietyTheory.Proofs.DivisorProofs.FamRowsSizes, lines 13–68)
36.3. Formalism.L2.Theta.RealTied (Formalism.LocalToGlobal.BesselHermitianTheta.MinfDensity, lines 82–90)
Formalism.MainNotions.RT.ChiFin0 (Formalism.SharpRestriction.GlobalSchurBounds.IntegralityOfRationalityLocal, lines 15–18)
Formalism.L2.DIV.ColumnCharactersCompatible (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.DIVColPins, lines 26–72)
Formalism.L2.Ker.KerEntTied.WtTie (Formalism.LocalToGlobal.KernelLattices.JetFieldLeKR, lines 108–111)
GlobalTheory.ColRows.KerTies (Development.AbelianVarietyTheory.Definitions.Divisors.KerSizes, lines 17–25)
GlobalTheory.ColRows.KerSizes (Development.AbelianVarietyTheory.Definitions.Divisors.KerSizes, lines 27–30)
Formalism.L2.Col.Sizes (Formalism.LocalToGlobal.Columns.AbelianHostFieldConditions, lines 486–494)
GlobalTheory.Core.Ht2.SizeRowsS (Development.AbelianVarietyTheory.Definitions.LatticeFrames.ColHeightSharpWith, lines 12–26)
36.4. GlobalTheory.Fc.RL.DefSide.definiteSideCondition_of_nprimeDef_of_nuFull (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.PointEvaluationSide, lines 49–64)
GlobalTheory.ColRows.rowsU_of_colPins_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RowsUOfColPins, lines 17–84)
36.5. GlobalTheory.Fc.RL.DefSide.linkedAt_df_of_parameters (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.PointEvaluationSide, lines 65–83)
GlobalTheory.Fc.RL.DefSide.full_rank (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.WeilUnitaryEntries, lines 25–32)
36.6. GlobalTheory.BigTop.PtAt.PointConditions' (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.Realised, lines 63–80)
GlobalTheory.BigTop.PtAt.CompatibilityConditions (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.Realised, lines 50–62)
GlobalTheory.BigOn.More (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.SingleIndexPlacesAreTwisted, lines 82–82)
GlobalTheory.BigOn.CoreTies (Development.AbelianVarietyTheory.Definitions.Divisors.IsDeltaDelAtOwnAt, lines 19–32)
GlobalTheory.BigOn.RecTies (Development.AbelianVarietyTheory.Statements.PlaceShiftStatements.SingleIndexPlacesAreTwisted, lines 64–80)
36.7. GlobalTheory.Fc.StageZeroRow.row_rRows_stage0_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.TupAllUsedAtPart, lines 46–67)
GlobalTheory.Fc.row_rRows_of_parts_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RowRRowsOfParts, lines 21–63)
GlobalTheory.Fc.R1.SelectionDefiniteRows (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CountsIrreducibilityConstants, lines 91–93)
36.8. Formalism.L2.Indef.IndefData.ON (Formalism.LocalToGlobal.IndefiniteForms.IndefSide, lines 318–323)
Formalism.L2.Indef.Tree.on_core (Formalism.LocalToGlobal.IndefiniteForms.IndefiniteTree.ExampleSatisfiesConditionsAtFive, lines 603–871)
GlobalTheory.BigOn.canonical_error_real (Development.AbelianVarietyTheory.Proofs.DivisorProofs.OnOfOnWithCanonical, lines 8–61)
GlobalTheory.BigOn.OntoPath.onto_path_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.OntoPath, lines 18–76)
36.9. GlobalTheory.Published.CasselmanWallach (Development.AbelianVarietyTheory.PublishedTheorems.AutomorphicForms₂.HasDRepIffHasDerivAt, lines 98–111)
PositivityTheory.CWB.casselmanWallach (Development.WeilBound.CoreStatements.PublishedTheorems.CasselmanWallach.FinalB, lines 8–8)

36.9. `PositivityTheory.CW.e3d` (`Development.WeilBound.CoreStatements.PublishedTheorems.CasselmanWallach.E3D`, lines 10–56)
`GlobalTheory.CompanionTheory.SmoothClosed.isClosed_image_of_sobolev_lower` (`Development.AutomorphicSpectrum.Proofs.BlockDecomposition.IsClosedImageOfSobolevLower`, lines 26–60)
`PositivityTheory.CW.cw_of_steps` (`Development.WeilBound.CoreStatements.PublishedTheorems.CasselmanWallach.CasselmanWallachAdmissible`, lines 61–108)
36.10. `Formalism.MainNotions.Closed` (`Formalism.SharpRestriction.ParamsHeightRankConstant`, lines 159–161)
`GlobalTheory.Fc.S0.NL` (`Development.AbelianVarietyTheory.Definitions.GlobalTwisting.GeometricWeakTiesOfStrong`, lines 36–36)
36.11. `Formalism.MainNotions.LinkN` (`Formalism.SharpRestriction.GlobalTheoremInputs`, lines 45–47)
`Formalism.MainNotions.ShaFinOdd` (`Formalism.SharpRestriction.GlobalTheoremInputs`, lines 25–26)
`Formalism.MainNotions.vvTerm` (`Formalism.SharpRestriction.GlobalTheoremInputs`, lines 31–32)
`Formalism.MainNotions.etaSum` (`Formalism.SharpRestriction.GlobalTheoremInputs`, lines 36–38)
`Formalism.MainNotions.LinkN_at` (`Formalism.SharpRestriction.GlobalTheoremInputs`, lines 40–43)
36.12. `Formalism.L2.DIV.sum_le_etaSum_num` (`Formalism.LocalToGlobal.Divisibility.SumLeEtaSumNum`, lines 28–58)
36.13. `UpperTheory.Top.vvRecord` (`Development.AutomorphicArithmetic.CharacterBoxes.DivinNOfHypV`, lines 17–57)
`UpperTheory.Top.divinN_at` (`Development.AutomorphicArithmetic.CharacterBoxes.DivinNOfHypV`, lines 76–115)
`UpperTheory.Top.divinN_of_hypV` (`Development.AutomorphicArithmetic.CharacterBoxes.DivinNOfHypV`, lines 117–130)
`Formalism.L2.DIV.sum_le_etaSum_num` (`Formalism.LocalToGlobal.Divisibility.SumLeEtaSumNum`, lines 28–58)
`Formalism.L2.DIV.inEta3_num` (`Formalism.LocalToGlobal.Divisibility.SumLeEtaSumNum`, lines 78–84)
36.14. `ReciprocityTheory.Gal.LocalFiniteness` (`ReciprocityTheory.Hypotheses.GalCoh`, lines 195–199)
`ReciprocityTheory.Gal.LocalTateDuality` (`ReciprocityTheory.Hypotheses.GalCoh`, lines 201–205)
`ReciprocityTheory.Gal.LocalEulerChar` (`ReciprocityTheory.Hypotheses.GalCoh`, lines 213–217)
36.15. `GlobalTheory.NLink.Hypotheses₂` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.BoundAtTwoAndThreeForFamily`, lines 18–25)
36.15. `GlobalTheory.NLink.SetupAlt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.BoundAtTwoThreeFamilyFon`, lines 13–21)
`GlobalTheory.HubX.Canon` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.DecompositionInvariantsCondition`, lines 40–46)
`GlobalTheory.HubLoc.cBnum` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.FinrankMaxUnramifiedSubfieldPadicFinExtPos`, lines 376–377)
`Formalism.MainNotions.N.KillingConditions.DataAD` (`Formalism.SharpRestriction.LevelLetterCocycles.JointConditionsZpowers.CommutingEndomorphismCondition`, lines 167–178)
`GlobalTheory.NLink.rowC` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.BoundAtTwoAndThreeForFamily`, lines 29–29)
36.16. `UpperTheory.N.at23_family_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.At23OfAll`, lines 19–20)
`UpperTheory.N.boundAtTwoThree_linked_of_recT` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.At23OfAll`, lines 24–26)
`GlobalTheory.NLink.BoundAtTwoThree` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.BoundAtTwoThreeFamily`, lines 52–53)
36.17. `GlobalTheory.PrismS.exists_prism` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsWachPrismOverIntegers`, lines 331–370)
`GlobalTheory.Prism.LocalValue.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.PrismaticFCrystalValueLocalPrismAinfWittTensor`, lines 141–151)
`GlobalTheory.WachLattice.build_prism` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachLatticesExistAllLayersOfBhattScholze`, lines 26–168)
`GlobalTheory.HWL.lattice_free` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachLatticeFreeOfCrystalline`, lines 43–74)
`GlobalTheory.NL.LatLow.lat_low` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachLatticeExpansionsStdBelowLevel`, lines 186–219)
`GlobalTheory.NL.GamModX.sub_mem_X_of_val` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsSmulOfDvdRepr`, lines 70–96)
36.18. `GlobalTheory.NL.InvX.sand_of` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachFreeLatticeEqWachLatticeInvertX`, lines 74–205)
`GlobalTheory.NL.InvX.compare` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachFreeLatticeEqWachLatticeInvertX`, lines 207–229)
`GlobalTheory.NL.FrobeniusMatrixUniqueness.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachModuleUniquenessCharacterisation`, lines 247–269)
36.19. `GlobalTheory.NL.CokerKill.X_pow_kills_coker` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CokernelCrystallineSurjectionKilledByXPow`, lines 411–437)
`Formalism.MainNotions.LiftingTheory.coreWB_of` (`Formalism.SharpRestriction.LevelLetterCocycles.FrobeniusPowUniformizerFactorisation`, lines 3449–3457)
36.20. `GlobalTheory.Cyc.TwistLift.lift_cocycle_of_duality` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ContinuousFixingSubgroupEquiv`, lines 307–382)
36.21. `GlobalTheory.NL.WCont.letterwise` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsCocycleSolutionConv`, lines 238–322)

`GlobalTheory.NL.exists_cocycle_solution_conv` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsCocycleSolutionConv`, lines 405–422)
36.22. `UpperTheory.N.locAll` (`Development.AutomorphicArithmetic.ResidueCharacters.LocAll`, lines 9–13)
`UpperTheory.N.LiftK` (`Development.AutomorphicArithmetic.ResidueCharacters.InvariantsKilledOverCyclotomicAlt`, lines 31–34)
`UpperTheory.N.InvariantsKilledOverCyclotomic` (`Development.AutomorphicArithmetic.ResidueCharacters.InvariantsKilledOverCyclotomicAlt`, lines 38–41)
`UpperTheory.N.TBK` (`Development.AutomorphicArithmetic.ResidueCharacters.InvariantsKilledOverCyclotomicAlt`, lines 43–45)
`Formalism.MainNotions.N.TBAt` (`Formalism.SharpRestriction.LevelLetterCocycles.QuotientHomNonvanishingBound`, lines 171–187)
36.23. `UpperTheory.N.conclusion_split_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ConclusionSplitOfHub`, lines 21–82)
`Formalism.MainNotions.EmbeddingTypes.boundSplit` (`Formalism.SharpRestriction.LevelLetterCocycles.EmbeddingTypes`, lines 293–304)
36.24. `GlobalTheory.Published.cheb_holds` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ChebotarevDegreeOne`, lines 66–67)
36.25. `Formalism.MainNotions.AdmissibleThreePrime.residual_irreducible_of_theorem_three_prime_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ResidualIrreducibleOfTheoremThreePrime`, lines 12–14)
`Formalism.MainNotions.N.GlobalDatumResidualRepCondition` (`Formalism.SharpRestriction.LevelLetterCocycles.JointConditionsZpowers.ResidualIrreducibleOfTheorem`, lines 13–19)
`Formalism.MainNotions.N.IsRhoBar` (`Formalism.SharpRestriction.LevelLetterCocycles.DataN`, lines 1324–1330)
36.25. `UpperTheory.GalC.GaloisPreconditions` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.GaloisPreconditions`, lines 13–31)
`Formalism.BasicNotions.NTw.NonzeroIndexPlace` (`Formalism.SharpRestriction.LevelLetterCocycles.DataN`, lines 455–459)
`Formalism.MainNotions.BlochKatoReduction.Gen.goodE` (`Formalism.SharpRestriction.LevelLetterCocycles.BlochKatoPhases`, lines 100–108)
36.26. `UpperTheory.GalC.exists_good_cut_facts_tie4` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CFacts`, lines 58–62)
`UpperTheory.GalC.linkN_of_twGood_galoisPreconditions` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LinkNFam`, lines 37–53)
36.27. `GlobalTheory.NLink.twist_kills_selmer` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ChosenTwistAnnihilatesSelmerGroups`, lines 8–32)
`UpperTheory.KStar.sum_nuL_le` (`Development.AutomorphicArithmetic.CyclotomicBounds.CyclotomicOrders.NuHost`, lines 37–39)
`GlobalTheory.NLink.multLe_of_rank` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.GoodPointMultiplicityLeRankProof`, lines 70–113)
36.28. `Formalism.RepresentationBoundsCohomology.N2.second_descent_statement_holds` (`Formalism.WachLatticeTwists.LagrangeExtensionCocycles.ExtensionCocycleProofs.SecondDescentStatementHolds`, lines 7–76)
`Formalism.RepresentationBoundsCohomology.N2.unit_lemma_prime_case_holds` (`Formalism.WachLatticeTwists.LagrangeExtensionCocycles.ExtensionCocycleProofs.Letters`, lines 10–95)
`Formalism.MainNotions.WachCoordinates.rF_le` (`Formalism.SharpRestriction.LevelLetterCocycles.BoundedNumeratorPowerSummability`, lines 100–147)
36.29. `Formalism.ExponentWindows.E.Full.consts_DD` (`Formalism.DistinguishedUnitIdeals.DistinguishedOrderProofs.ThmEFull`, lines 52–66)
`Formalism.ExponentWindows.levels_mod_const_iff_of_down_down` (`Formalism.DistinguishedUnitIdeals.DistinguishedOrderProofs.LevelsModConstIffOfDownDown`, lines 477–633)
`Formalism.ExponentWindows.E.CL.lifts_mod_const_of_down_down` (`Formalism.DistinguishedUnitIdeals.DistinguishedOrderProofs.ArcLiftOfLocalizationData`, lines 395–407)
36.30. `Formalism.MainNotions.ElementDenominator.CongruenceModules.Ynum` (`Formalism.SharpRestriction.LevelLetterCocycles.LocalTheoremOfGoodExtensions`, lines 42–137)
`Formalism.IrreducibleCountingLemmas.WindowSums.valuationSumBound` (`Formalism.PadicSubsetSums.StableGradedCounts.LiftingTheExponent`, lines 34–67)
`Formalism.MainNotions.ElementDenominator.CongruenceModules.thmA_of_claim` (`Formalism.SharpRestriction.LevelLetterCocycles.LocalTheoremOfGoodExtensions`, lines 171–201)
36.31. `GlobalTheory.LocalFieldsQD.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalFieldsAtAuxiliaryPrimes`, lines 19–28)
`GlobalTheory.CrysFromGTwo.crys_loc_alt_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CrysLoc`, lines 15–21)
`GlobalTheory.HubFon.main_alt_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.MainAltAltMore`, lines 11–41)
36.32. `Formalism.WachCohomologyFacts.V2.sgN` (`Formalism.FrobeniusContraction.HighestWeightResidueAvoids`, lines 40–41)
`Formalism.MainNotions.tableLineDegreeTwo.PositionHyps` (`Formalism.SharpRestriction.LevelLetterCocycles.TableLineDegreeTwo`, lines 25–31)
`Formalism.WachCohomologyFacts.V2.ResidueAvoidance` (`Formalism.FrobeniusContraction.HighestWeightResidueAvoids`, lines 72–74)

36.33. `GlobalTheory.PositionData.position_conditions` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.JumpPositionConditionsAtLetter`, lines 57–215)
`GlobalTheory.PositionData.far_from_block` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.JumpPositionConditionsAtLetter`, lines 42–51)
`GlobalTheory.T.shiftedJumpUpperBound_le_twist_bound` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ShiftedJumpUpperBoundLeTwistBoundStatement`, lines 7–13)
`GlobalTheory.PositionData.count_US` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.MemIntPts`, lines 40–46)
36.34. `Formalism.MainNotions.tableLineDegreeTwo.TopCore` (`Formalism.SharpRestriction.LevelLetterCocycles.TableLineDegreeTwo`, lines 193–212)
`Formalism.MainNotions.tableLineDegreeTwo.TopHyp` (`Formalism.SharpRestriction.LevelLetterCocycles.TableLineDegreeTwo`, lines 180–191)
`Formalism.MainNotions.tableLineDegreeTwo.IsAdmCoc` (`Formalism.SharpRestriction.LevelLetterCocycles.TableLineDegreeTwo`, lines 172–178)
`Formalism.MainNotions.tableLineDegreeTwo.AdmQ` (`Formalism.SharpRestriction.LevelLetterCocycles.TableLineDegreeTwo`, lines 93–98)
`Formalism.MainNotions.tableLineDegreeTwo.IsFull` (`Formalism.SharpRestriction.LevelLetterCocycles.TableLineDegreeTwo`, lines 59–63)
`Formalism.MainNotions.tableLineDegreeTwo.TopTheoremLocal` (`Formalism.SharpRestriction.LevelLetterCocycles.TableLineDegreeTwo`, lines 398–400)
`GlobalTheory.TopClosing.blochKatoTop_of_localTateDuality_eulerChar_alt` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.BlochKatoTopOfLocalTateDualityEulerChar`, lines 25–42)
36.35. `GlobalTheory.TopSteps.TopSit` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.ResBound`, lines 36–45)
`GlobalTheory.AdmLoc.Ea` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.MkMemAdmissibleAtIffCocycle`, lines 308–311)
`GlobalTheory.FinExtIntegers.dvr_complete` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.DvrComplete`, lines 10–29)
`GlobalTheory.T.exists_pow_smul_eq_zero_of_finite_of_not_isUnit` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.FiniteModuleKilledByPowOfNotIsUnitStatement`, lines 21–23)
36.36. `GlobalTheory.PhiGam.RingPG` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.PowerSeriesOneAddConstMulXMulInv`, lines 9–23)
`GlobalTheory.PhiGam.IsLat` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.PowerSeriesOneAddConstMulXMulInv`, lines 29–29)
`GlobalTheory.IndefiniteRowSideFullHecke.OnePlusPowerCharacters.IsCyclotomic` (`Development.AbelianVarietyTheory.BlockTheory.NonGaloisCusps.OnePlusPowerAssembly`, lines 28–34)
`GlobalTheory.IndefiniteRowSideFullHecke.FrobeniusBinomialReduction.IsCyclotomic` (`Development.AbelianVarietyTheory.BlockTheory.NonGaloisCusps.KernelCharacterDelta`, lines 31–36)
`GlobalTheory.IndefiniteRowSideFullHecke.FrobeniusBinomialReduction.frobπ` (`Development.AbelianVarietyTheory.BlockTheory.NonGaloisCusps.KernelCharacterDelta`, lines 21–21)
`GlobalTheory.IndefiniteRowSideFullHecke.FrobeniusBinomialReduction.gamπ` (`Development.AbelianVarietyTheory.BlockTheory.NonGaloisCusps.KernelCharacterDelta`, lines 23–23)
`GlobalTheory.IndefiniteRowSideFullHecke.FrobeniusBinomialReduction.kerPsi` (`Development.AbelianVarietyTheory.BlockTheory.NonGaloisCusps.KernelCharacterDelta`, lines 53–53)
36.38. `GlobalTheory.PsiShiftProof.lemmaPsi` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsPullBasis`, lines 239–294)
`GlobalTheory.IndefiniteRowSideFullHecke.OnePlusPowerCharacters.PsiMod` (`Development.AbelianVarietyTheory.BlockTheory.NonGaloisCusps.OnePlusPowerAssembly`, lines 52–58)
36.39. `GlobalTheory.PhiGam.ModPG` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.PowerSeriesOneAddConstMulXMulInv`, lines 54–72)
`GlobalTheory.PhiGam.CondC` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.PowerSeriesOneAddConstMulXMulInv`, lines 43–43)
`GlobalTheory.Models.DFun` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.TwistedModelAdmissibleCocycles`, lines 46–89)
`GlobalTheory.Models.Lats` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.TwistedModelAdmissibleCocycles`, lines 19–19)
`GlobalTheory.Models.IsModel` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.TwistedModelAdmissibleCocycles`, lines 99–102)
`GlobalTheory.BasePairs.RedN` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsModThOfPres`, lines 21–26)
`GlobalTheory.Present.Pres` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.JSmul`, lines 28–51)
`GlobalTheory.Models.resCount` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.TwistedModelAdmissibleCocycles`, lines 110–111)
`GlobalTheory.Present.RedLat.red_isLat` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.WachDeltaInvariantsReductionIsLattice`, lines 277–286)
36.41. `Formalism.IrreducibleCountingLemmas.TOP.carryFloorLeGeom` (`Formalism.PadicSubsetSums.StableGradedCounts.TOPSgLe`, lines 19–40)
36.42. `GlobalTheory.BasePairs.eq_of_dvd` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.`

NonemptyEquivOfSemilinear, lines 322–340)
36.43. GlobalTheory.TopSteps.mem_cob_of_repr_mem_herrB (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.MemCoboundariesOfHerrPairCoboundary, lines 9–103)
GlobalTheory.ResidueBound.exists_independent_admissible_cocycles_residue (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsIndependentAdmissibleCocyclesResidue, lines 175–207)
GlobalTheory.Models.DFun (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.TwistedModelAdmissibleCocycles, lines 46–89)
36.44. GlobalTheory.TopVanish.vanish_of_lattice (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.VanishOfLattice, lines 191–216)
GlobalTheory.TopVanish.vanish_of_real (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.VanishOfLattice, lines 234–255)
36.45. GlobalTheory.UnramAdm.unr_subset (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleQuotientOfFullRepresentation, lines 480–527)
GlobalTheory.UnramAdm.unramified_admissible (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleQuotientOfFullRepresentation, lines 537–544)
GlobalTheory.TopSteps.ChiOne.eq_hom_of_res (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ClsEqHomModInertia, lines 146–252)
GlobalTheory.TopSteps.ChiOne.cls_eq_homModInertia (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ClsEqHomModInertia, lines 278–289)
36.46. GlobalTheory.T.admissibleLetterData_window_arithmetic (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleLetterDataWindowArithmeticStatement, lines 8–9)
36.47. GlobalTheory.Models.AdmBound.exists_ne_zero_smul_mem_of_mem_span' (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleCocyclesCardLeSumDims, lines 60–71)
GlobalTheory.T.card_cocycles_independent_mod_coboundaries_le (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CardCocyclesIndependentModCoboundariesLeStatement, lines 8–9)
GlobalTheory.Models.InZZ (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.TwistedModelAdmissibleCocycles, lines 173–177)
GlobalTheory.PhiGam.sen (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.PowerSeriesOneAddConstMulXMulInv, lines 52–52)
GlobalTheory.Models.herrB (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.TwistedModelAdmissibleCocycles, lines 179–183)
36.48. GlobalTheory.BasePairs.exists_modTh_of_pres (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsModThOfPres, lines 28–70)
GlobalTheory.BasePairs.real_of_presented (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsModThOfPres, lines 72–81)
GlobalTheory.Models.ModTh (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.TwistedModelAdmissibleCocycles, lines 117–158)
GlobalTheory.T.obj_and_abc_of_model (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ModelObjAndAbcStatement, lines 8–9)
GlobalTheory.Models.EPolyModel.model_ePoly (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.MapAevalNab, lines 50–60)
GlobalTheory.T.twistedPhiGammaModel_character_eq_standard (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TwistedPhiGammaModelCharacterEqStandardStatement, lines 8–9)
GlobalTheory.T.model_preimage_of_injective_finDimRepModelHom (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ModelPreimageOfInjectiveFinDimRepModelHomStatement, lines 8–9)
GlobalTheory.Models.ModTh.model_push_of_surjective (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ModelPushOfSurjective, lines 10–24)
GlobalTheory.T.exists_largest_model' (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsLargestModelStatement, lines 8–9)
GlobalTheory.T.exists_phiGammaModel_extension_cocycle_pair (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsPhiGammaModelExtensionCocyclePairStatement, lines 8–9)
GlobalTheory.T.twistedPhiGammaModel_sen_weight_count (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TwistedPhiGammaModelSenWeightCountStatement, lines 8–9)
36.49. GlobalTheory.Models.AdmBound.main (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleCocyclesCardLeSumDims, lines 73–81)
GlobalTheory.T.exists_largest_model' (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsLargestModelStatement, lines 8–9)
GlobalTheory.Models.EPolyModel.model_ePoly (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.MapAevalNab, lines 50–60)
36.50. GlobalTheory.Models.ModTh.hom_chr_eq_zero (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.HomToCharacterLineEqZero, lines 25–63)
GlobalTheory.T.senWeights_close_of_height_gamma_conditions (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SenWeightsCloseOfHeightGammaConditionsStatement, lines 8–9)
GlobalTheory.T.senMultiplicity_subquotient_lattice_le (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SenMultiplicitySubquotientLatticeLeStatement, lines 8–9)
36.51. GlobalTheory.CharSplit.admissibleCocycle_negCharPower_mem_coboundaries (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleCocycleNegCharPowerMemCoboundaries, lines 14–81)
GlobalTheory.TopSteps.TrivLine.main (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.

`TrivialLineAdmissibleCocyclesNotIndependent`, lines 141–174)
36.52. `GlobalTheory.AdmClosure.admissible_closed` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleOfSurjective`, lines 176–179)
`GlobalTheory.TopSteps.CrysRules.crysCalc` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CrystallineSumsTwistsCharactersExpectedJumps`, lines 205–234)
`GlobalTheory.TopSteps.StdAdm.stdAdm` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleStd`, lines 380–384)
`GlobalTheory.AdmClasses.admissibleLocalSituation_admissible_classes` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleClassesPropsOfFlatCoefficient`, lines 360–365)
`GlobalTheory.T.admissible_quotientExtension_iff_admissible_ext` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleQuotientExtensionIffAdmissibleExtStatement`, lines 21–28)
`GlobalTheory.T.cocycle_class_admissible` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CocycleClassAdmissibleStatement`, lines 21–29)
`GlobalTheory.RedFun.realD` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsPresTensor`, lines 556–570)
`GlobalTheory.BasePairs.real_of_presented` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsModThOfPres`, lines 72–81)
`GlobalTheory.TopBridge.stmtAt_of_real` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TwistedLatticeCohomologyTheoremTOP`, lines 44–65)
`GlobalTheory.TopSteps.CharVanish.stmt_of_real` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.StmtOfReal`, lines 24–29)
`GlobalTheory.TopSteps.admissibleUntwistedCharUnramified_of_residueRealisation` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.AdmissibleUntwistedCharUnramifiedOfResidueRealis`, lines 16–29)
36.53. `Formalism.MainNotions.tableLineDegreeTwo.TopTheoremLocal` (`Formalism.SharpRestriction.LevelLetterCocycles.TableLineDegreeTwo`, lines 398–400)
`GlobalTheory.TopBridge.twistedLatticeCohomology_theoremTOP` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TwistedLatticeCohomologyTheoremTOP`, lines 67–82)
`GlobalTheory.TopDict.topCore_of_typings` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExactLocalConditionsTopCore`, lines 75–171)
36.54. `GlobalTheory.LocalPositionData.position_data_exist` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsLocalSlotDataOffExceptionalPrimes`, lines 21–116)
`GlobalTheory.IndefiniteRowSideFullHecke.BaseSizes.Exc` (`Development.AbelianVarietyTheory.BlockTheory.NonGaloisCusps.BaseSizes`, lines 31–32)
`Formalism.BasicNotions.LevelStarExponents.lStar` (`Formalism.LocalRingCardinality.DataParams`, lines 38–40)
`GlobalTheory.Pass.cutCoc` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.CentralIdemCoeff`, lines 122–133)
`GlobalTheory.UseLoc.Cases.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.UnramifiedEnlargementCutCocyclesBlochKato`, lines 126–146)
36.56. `Formalism.MainNotions.N.Ln34.killA` (`Formalism.SharpRestriction.LevelLetterCocycles.LocalFormOfThreeInputs`, lines 28–38)
`Formalism.MainNotions.ElementDenominator.CongruenceModules.YA` (`Formalism.SharpRestriction.LevelLetterCocycles.LocalTheoremOfGoodExtensions`, lines 150–151)
`Formalism.MainNotions.ElementDenominator.thmA` (`Formalism.SharpRestriction.LevelLetterCocycles.ElementDenominator`, lines 36–47)
`Formalism.WeaklyAdmissibleFiltrations.LatHom.ImF` (`Formalism.WeaklyAdmissibleFiltrations.Statement`, lines 260–260)
`GlobalTheory.theoremA_closing` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalLiftingTheoremAOfBlochKatoTateDualityEulerC`, lines 14–16)
36.57. `Formalism.MainNotions.N.Ln34.DvdPf.dvdCondition` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.DvdCondition`, lines 85–101)
`Formalism.MainNotions.N.Ln34.quotient_jumps_subset` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.QuotPf`, lines 62–93)
`Formalism.MainNotions.N.Ln34.normR_holds` (`Formalism.SharpRestriction.LevelLetterCocycles.NormAHolds`, lines 55–57)
`Formalism.MainNotions.N.Ln34.uniformiser_ring_of_hypotheses` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.UniformiserRingOfHypotheses`, lines 9–25)
`Formalism.MainNotions.N.Ln34.ClassMapPf.classmap_pf` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.ClassMapPf`, lines 248–250)
`Formalism.MainNotions.N.Ln34.LamPS` (`Formalism.SharpRestriction.LevelLetterCocycles.LamPS`, lines 33–35)
`Formalism.MainNotions.N.Ln34.JZ` (`Formalism.SharpRestriction.LevelLetterCocycles.LamPS`, lines 57–58)
`Formalism.MainNotions.DeltaInvariantLattices.p2aK` (`Formalism.SharpRestriction.LevelLetterCocycles.DeltaInvariantLattices`, lines 308–312)
`Formalism.MainNotions.N.Ln34.UnivLinkPf.univLinkPf_local` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.UnivLinkPfB`, lines 186–276)
`Formalism.MainNotions.N.Ln34.splitR` (`Formalism.SharpRestriction.LevelLetterCocycles.MixedPositionHypothesis`, lines 44–53)
36.58. `Formalism.BasicNotions.LocWach` (`Formalism.SharpRestriction.LevelLetterCocycles.DataNLoc`, lines 164–311)
`Formalism.MainNotions.DeltaInvariantLattices.ptK` (`Formalism.SharpRestriction.LevelLetterCocycles.DeltaInvariantLattices`, lines 504–508)
`Formalism.MainNotions.N.Ln34.valAt` (`Formalism.SharpRestriction.LevelLetterCocycles.DownSlotProposition`, lines 22–26)

`Formalism.MainNotions.DeltaInvariantLattices.FilC` (`Formalism.SharpRestriction.LevelLetterCocycles.DeltaInvariantLattices`, lines 156–157)
`Formalism.MainNotions.N.Ln34.LcQ` (`Formalism.SharpRestriction.LevelLetterCocycles.LamPS`, lines 49–50)
`Formalism.MainNotions.N.Ln34.LA` (`Formalism.SharpRestriction.LevelLetterCocycles.DownSlotProposition`, lines 36–36)
`Formalism.MainNotions.N.Ln34.LamLA` (`Formalism.SharpRestriction.LevelLetterCocycles.DownSlotProposition`, lines 50–52)
`Formalism.MainNotions.N.Ln34.QLamLA` (`Formalism.SharpRestriction.LevelLetterCocycles.DownSlotProposition`, lines 96–97)
`Formalism.MainNotions.N.Ln34.QLamP` (`Formalism.SharpRestriction.LevelLetterCocycles.DownSlotProposition`, lines 99–100)
36.60. `Formalism.MainNotions.N.Ln34.IsRepL` (`Formalism.SharpRestriction.LevelLetterCocycles.DownSlotProposition`, lines 54–57)
`Formalism.MainNotions.N.Ln34.expS` (`Formalism.SharpRestriction.LevelLetterCocycles.DownSlotProposition`, lines 65–67)
`Formalism.MainNotions.N.Ln34.NLset` (`Formalism.SharpRestriction.LevelLetterCocycles.DownSlotProposition`, lines 69–72)
`Formalism.MainNotions.N.Ln34.IsFrame` (`Formalism.SharpRestriction.LevelLetterCocycles.DownSlotProposition`, lines 122–151)
`Formalism.MainNotions.N.Ln34.InvL` (`Formalism.SharpRestriction.LevelLetterCocycles.DownSlotProposition`, lines 82–84)
`Formalism.MainNotions.N.Ln34.IIrest` (`Formalism.SharpRestriction.LevelLetterCocycles.DownSlotProposition`, lines 86–88)
`Formalism.MainNotions.N.Ln34.IsCoord` (`Formalism.SharpRestriction.LevelLetterCocycles.DownSlotProposition`, lines 163–170)
`Formalism.MainNotions.N.Ln34.FilL` (`Formalism.SharpRestriction.LevelLetterCocycles.DownSlotProposition`, lines 28–30)
36.61. `Formalism.MainNotions.N.Ln34.ModelL` (`Formalism.SharpRestriction.LevelLetterCocycles.ModelKStep`, lines 29–40)
`Formalism.MainNotions.N.Ln34.modell_pf` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.ModelLPf`, lines 11–34)
`Formalism.MainNotions.N.Ln34.ModelF` (`Formalism.SharpRestriction.LevelLetterCocycles.DownSlotProposition`, lines 198–211)
`Formalism.MainNotions.N.Ln34.modelflink_pf` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.ModelFPf`, lines 9–67)
`Formalism.MainNotions.N.Ln34.modelqlink_pf` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.ModelQPf`, lines 14–92)
36.62. `Formalism.MainNotions.N.Ln34.LevelExpansions.dv` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.ExpansionLowCoeffsIntegral`, lines 199–219)
`Formalism.MainNotions.N.Ln34.LevelExpansions.simple` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.ExpansionLowCoeffsIntegral`, lines 234–248)
36.63. `Formalism.MainNotions.N.Ln34.LevelExpansions.fact2` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.ExpansionLowCoeffsIntegral`, lines 252–327)
`Formalism.MainNotions.N.Ln34.LevelExpansions.caseC` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.PolyExpand`, lines 417–451)
36.64. `Formalism.MainNotions.N.Ln34.u1link_pf` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.U1linkPf`, lines 16–67)
`Formalism.MainNotions.N.Ln34.FirstPropositionFrame` (`Formalism.SharpRestriction.LevelLetterCocycles.DownSlotProposition`, lines 216–231)
`Formalism.MainNotions.N.Ln34.SecondPropositionFree` (`Formalism.SharpRestriction.LevelLetterCocycles.DownSlotProposition`, lines 233–237)
`Formalism.MainNotions.N.Ln34.ThirdPropositionSlot` (`Formalism.SharpRestriction.LevelLetterCocycles.DownSlotProposition`, lines 244–250)
`Formalism.MainNotions.N.Ln34.kapPiPf_main` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.KapPiPf`, lines 217–283)
`Formalism.MainNotions.N.Ln34.UnivLinkPf.univLinkPf_cong` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.UnivLinkPfB`, lines 39–109)
36.65. `Formalism.MainNotions.N.Ln34.killA` (`Formalism.SharpRestriction.LevelLetterCocycles.LocalFormOfThreeInputs`, lines 28–38)
`Formalism.MainNotions.N.Ln34.killR_of_steps` (`Formalism.SharpRestriction.LevelLetterCocycles.MixedPositionHypothesis`, lines 105–170)
`Formalism.MainNotions.N.Ln34.lifting_theorem_of_uniform_link` (`Formalism.SharpRestriction.LevelLetterCocycles.DvrLatticeRank.MainPf`, lines 38–46)
`GlobalTheory.TheoremAFinal.integralityFromMetric_of_le` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalLiftingOfEtaleExactDeRhamCrystallineConst`, lines 16–20)
`GlobalTheory.theoremA_closing` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalLiftingTheoremAOfBlochKatoTateDualityEulerC`, lines 14–16)
36.66. `Formalism.MainNotions.ElementDenominator.CongruenceModules.thmA_of_claim` (`Formalism.SharpRestriction.LevelLetterCocycles.LocalTheoremOfGoodExtensions`, lines 171–201)
`Formalism.MainNotions.N.Ln34.claimA_of_steps` (`Formalism.SharpRestriction.LevelLetterCocycles.LocalFormOfThreeInputs`, lines 40–54)
`Formalism.MainNotions.ElementDenominator.CongruenceModules.claimA` (`Formalism.SharpRestriction.LevelLetterCocycles.LocalTheoremOfGoodExtensions`, lines 157–166)
36.67. `Formalism.MainNotions.PairedCharacterCondition.PairedCharacterConditionAlt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LocalGlobalCompatibility`, lines 15–27)
`Formalism.MainNotions.PairedCharacterCondition.rhs` (`Formalism.SharpRestriction.LevelLetterCocycles.PairedCharacterCondition`, lines 151–157)
`Formalism.MainNotions.PairedCharacterCondition.HostJ` (`Formalism.SharpRestriction.LevelLetterCocycles.`

PairedCharacterCondition, lines 132–142)
GlobalTheory.ClosingLocalInputs.main_alt_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.AdmG14OfAux, lines 15–32)
36.68. GlobalTheory.EtaFromCut.lenUnit_le_of_family (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LenUnitLeOfFamily, lines 64–93)
36.69. GlobalTheory.CutFam.perPoint (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsOfMemCcAt, lines 114–153)
36.70. GlobalTheory.ClosingStatement.main (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalAbsorptionLengthBoundOfBlochKato, lines 15–25)
Formalism.MainNotions.BlochKatoDuality.second_proposition_every_localisation (Formalism.SharpRestriction.LevelLetterCocycles.RamificationTwoAndConjugationAtQAndD, lines 1405–1407)
36.71. GlobalTheory.InertiaEigenvalues.theoremT_all_records (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CrystallinePeriodBoundAllConstantGeForty, lines 13–19)
Formalism.MainNotions.ElementDenominator.thmT (Formalism.SharpRestriction.LevelLetterCocycles.ElementDenominator, lines 178–197)
Formalism.BasicNotions.pc (Formalism.SharpRestriction.LevelLetterCocycles.DataNLoc, lines 657–693)
Formalism.WeaklyAdmissibleFiltrations.PotCrysData.InertiaThrough (Formalism.WeaklyAdmissibleFiltrations.StatementPC, lines 73–75)
36.72. GlobalTheory.StepLiftPf.liftFam_of_eq (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LiftFMonoLoc, lines 156–198)
GlobalTheory.StepLiftPf.liftFam_core (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LiftFMonoLoc, lines 111–144)
36.73. GlobalTheory.Pass.LocDirect.main (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalCountDirectLiftingProof, lines 20–76)
GlobalTheory.Pass.LocDual.main (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalCountDualLiftingProof, lines 30–55)
36.74. GlobalTheory.ClosingLocalInputs.main_alt_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.AdmG14OfAux, lines 15–32)
GlobalTheory.T.admT_of_artin_reciprocity_and_local_conditions (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ConclusionOfAdmissiblesStatement, lines 11–12)
GlobalTheory.Pass.cutBound_of_steps_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CutBoundOfSteps, lines 15–78)
GlobalTheory.EtaFromCut.etaBound_of_cut (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LenUnitLeOfFamily, lines 107–108)
36.75. GlobalTheory.NLinkTyp.etaLoc_le_rhs_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EtaLocLeRhs, lines 21–57)
36.76. Formalism.MainNotions.RamifiedCoefficientChains.MassData (Formalism.SharpRestriction.LevelLetterCocycles.ExistsPosConstLagrangeCondition, lines 269–282)
Formalism.MainNotions.RamifiedCoefficientChains.SetCS (Formalism.SharpRestriction.LevelLetterCocycles.RamifiedCoefficientChains, lines 60–71)
Formalism.MainNotions.RamifiedCoefficientChains.eBd_parts (Formalism.SharpRestriction.LevelLetterCocycles.ExistsPosConstLagrangeCondition, lines 153–166)
GlobalTheory.ramifiedCoefficientChains_prv_twenty_five (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.RamifiedCoefficientChainsPrvTwentyFive, lines 17–28)
36.77. Formalism.MainNotions.RamifiedCoefficientChains.S9.AdB.refine (Formalism.SharpRestriction.LevelLetterCocycles.RingClausesAtUniformizer, lines 29–116)
Formalism.MainNotions.RamifiedCoefficientChains.S9.AdB.canon (Formalism.SharpRestriction.LevelLetterCocycles.RingClausesAtUniformizer, lines 19–27)
36.78. Formalism.MainNotions.CrystallineLiftingInputs.DL (Formalism.SharpRestriction.LevelLetterCocycles.CrystallineLiftingInputs, lines 29–30)
Formalism.MainNotions.CrystallineLiftingInputs.InPN (Formalism.SharpRestriction.LevelLetterCocycles.CrystallineLiftingInputs, lines 32–33)
Formalism.MainNotions.CrystallineLiftingInputs.PolesFin (Formalism.SharpRestriction.LevelLetterCocycles.CrystallineLiftingInputs, lines 35–36)
Formalism.MainNotions.CrystallineLiftingInputs.InI (Formalism.SharpRestriction.LevelLetterCocycles.CrystallineLiftingInputs, lines 119–120)
Formalism.MainNotions.CrystallineLiftingInputs.Dlat (Formalism.SharpRestriction.LevelLetterCocycles.CrystallineLiftingInputs, lines 38–39)
Formalism.MainNotions.CrystallineLiftingInputs.DlatImg (Formalism.SharpRestriction.LevelLetterCocycles.CrystallineLiftingInputs, lines 41–43)
Formalism.MainNotions.CrystallineLiftingInputs.pullJ (Formalism.SharpRestriction.LevelLetterCocycles.CrystallineLiftingInputs, lines 143–145)
Formalism.MainNotions.LiftingTheory.qp (Formalism.SharpRestriction.LevelLetterCocycles.LiftingTheory, lines 328–328)
GlobalTheory.AdmHerrLocal.main (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ImportsBisRest, lines 9–13)
Formalism.MainNotions.RamifiedCoefficientChains.S9.PStep (Formalism.SharpRestriction.LevelLetterCocycles.LatticeBoundsOfThreeParts, lines 15–19)

`Formalism.MainNotions.RamifiedCoefficientChains.S9.PChain` (`Formalism.SharpRestriction.LevelLetterCocycles.LatticeBoundsOfThreeParts`, lines 26–29)
`Formalism.MainNotions.RamifiedCoefficientChains.S9.cocyclesToWachVectorsExponent` (`Formalism.SharpRestriction.LevelLetterCocycles.LatticeBoundsOfThreeParts`, lines 172–174)
`Formalism.MainNotions.RamifiedCoefficientChains.S9.EG` (`Formalism.SharpRestriction.LevelLetterCocycles.LatticeBoundsOfThreeParts`, lines 199–200)
`Formalism.MainNotions.RamifiedCoefficientChains.S9.wachVectorLiftingExponent` (`Formalism.SharpRestriction.LevelLetterCocycles.LatticeBoundsOfThreeParts`, lines 202–204)
`Formalism.MainNotions.RamifiedCoefficientChains.S9.B6` (`Formalism.SharpRestriction.LevelLetterCocycles.LatticeBoundsOfThreeParts`, lines 206–206)
36.80. `Formalism.MainNotions.RamifiedCoefficientChains.S9.CocyclesToWachVectorsBound` (`Formalism.SharpRestriction.LevelLetterCocycles.LatticeBoundsOfThreeParts`, lines 250–264)
`Formalism.MainNotions.RamifiedCoefficientChains.S9.ExtensionLemmaAlt.cocyclesToWachVectorsBound_of_extTool` (`Formalism.SharpRestriction.LevelLetterCocycles.CocyclesToWachVectorsBoundOfExtTool`, lines 75–93)
`Formalism.MainNotions.RamifiedCoefficientChains.S9.LocalLemmasFromLiterature.local_lemmas_of_literature` (`Formalism.SharpRestriction.LevelLetterCocycles.UniformizerClauseOfAdmittedAlt`, lines 53–113)
36.81. `Formalism.MainNotions.RamifiedCoefficientChains.S9.lemGA_la_of_adm` (`Formalism.SharpRestriction.LevelLetterCocycles.LiftRingOfUniformizerClause`, lines 31–37)
`Formalism.MainNotions.RamifiedCoefficientChains.S9.lemGA_x_of` (`Formalism.SharpRestriction.LevelLetterCocycles.QPowSmulFrobeniusMem`, lines 70–241)
`Formalism.MainNotions.RamifiedCoefficientChains.S9.lemGA0_of_tools` (`Formalism.SharpRestriction.LevelLetterCocycles.LemGA0ConvVanish`, lines 106–176)
`Formalism.MainNotions.RamifiedCoefficientChains.S9.lemGB_of` (`Formalism.SharpRestriction.LevelLetterCocycles.DerivativeProdSubEval`, lines 58–239)
36.82. `Formalism.MainNotions.RamifiedCoefficientChains.S9.AdM.adaptBas_of` (`Formalism.SharpRestriction.LevelLetterCocycles.RingClausesAtUniformizer`, lines 393–601)
`Formalism.MainNotions.RamifiedCoefficientChains.S9.ShB.AdaptData` (`Formalism.SharpRestriction.LevelLetterCocycles.DualLemmaOfDeterminant`, lines 48–56)
36.83. `Formalism.MainNotions.RamifiedCoefficientChains.S9.WachVectorLiftingBound` (`Formalism.SharpRestriction.LevelLetterCocycles.LatticeBoundsOfThreeParts`, lines 386–398)
`Formalism.MainNotions.RamifiedCoefficientChains.Asm.frobenius_lift_lemmas_of_admitted` (`Formalism.SharpRestriction.LevelLetterCocycles.FrobeniusLiftLemmasOfAdmitted`, lines 251–268)
`Formalism.MainNotions.N.Pv.LocalLengthBound.length_bound_of_hypotheses` (`Formalism.SharpRestriction.LevelLetterCocycles.QuotientImageDivisibility.LengthBoundOfHypotheses`, lines 101–313)
36.84. `Formalism.MainNotions.N.Pv.LocalLengthBound.pchain_DN` (`Formalism.SharpRestriction.LevelLetterCocycles.QuotientImageDivisibility.PchainDNAlt`, lines 69–141)
36.85. `Formalism.MainNotions.N.Pv.LocalLengthBound.exists_inI_inPN_pow_smul_sub_map` (`Formalism.SharpRestriction.LevelLetterCocycles.QuotientImageDivisibility.ExistsInIInPNPowSmulSubMap`, lines 44–108)
36.86. `Formalism.MainNotions.RamifiedCoefficientChains.S9.LatticeBoundSix` (`Formalism.SharpRestriction.LevelLetterCocycles.LatticeBoundsOfThreeParts`, lines 400–412)
`Formalism.MainNotions.N.Pv.LocalLengthBound.frobenius_sq_bound_of_local` (`Formalism.SharpRestriction.LevelLetterCocycles.QuotientImageDivisibility.FrobeniusSqBoundOfLocal`, lines 55–165)
`Formalism.MainNotions.N.Pv.LocalLengthBound.rhoLift_of` (`Formalism.SharpRestriction.LevelLetterCocycles.QuotientImageDivisibility.RhoLiftOf`, lines 20–149)
`Formalism.MainNotions.N.Pv.LocalLengthBound.rhoLift_D` (`Formalism.SharpRestriction.LevelLetterCocycles.QuotientImageDivisibility.RhoLiftOf`, lines 151–165)
36.87. `Formalism.MainNotions.UnusedSlot80.UnivF.Sat` (`Formalism.SharpRestriction.LevelLetterCocycles.UniversalGeneralOfRestriction`, lines 17–18)
`Formalism.MainNotions.UnusedSlot80.UnivF.adapted` (`Formalism.SharpRestriction.LevelLetterCocycles.UniversalGeneralOfRestriction`, lines 20–46)
36.88. `GlobalTheory.ramifiedCoefficientChains_prv_twenty_five` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.RamifiedCoefficientChainsPrvTwentyFive`, lines 17–28)
`GlobalTheory.T.ramified_coefficient_chains_of_hypotheses` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TheoremCConst25OfHypothesesStatement`, lines 12–13)
`Formalism.MainNotions.RamifiedCoefficientChains.Asm.theorem_of_literature_hypotheses` (`Formalism.SharpRestriction.LevelLetterCocycles.FrobeniusLiftLemmasOfAdmitted`, lines 292–297)
36.89. `GlobalTheory.T.theoremE_corollaryA_closing_all_data` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TheoremECorollaryAClosingAllDataStatement`, lines 8–9)
`GlobalTheory.NLink.localBoundTableFromData_rowB_rowEntries` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LocalBoundTableFromDataRowBRowEntries`, lines 90–113)
36.90. `Formalism.MainNotions.LiftingTheory.arith_sizeA` (`Formalism.SharpRestriction.LevelLetterCocycles.FrobeniusPowUniformizerFactorisation`, lines 1799–1840)
`Formalism.MainNotions.N.Ln17.sum_alt` (`Formalism.SharpRestriction.LevelLetterCocycles.RepresentationCongruences.LogbHBNonneg`, lines 386–406)
`Formalism.MainNotions.N.Ln17.frobenius_statement_of_fontaine` (`Formalism.SharpRestriction.LevelLetterCocycles.FrobeniusStatementOfFontaine`, lines 218–221)
36.91. `GlobalTheory.NLink.BBSize.blockBoundDeclaration_le_middleRange` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.NonPlacePartLocalBoundLeMiddleRange`, lines 100–117)

`GlobalTheory.NLink.BBSize.blockBoundAtZero_le_middleRange` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.NonPlacePartLocalBoundLeMiddleRange`, lines 119–149)
`GlobalTheory.NLink.BBSize.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.NonPlacePartLocalBoundLeMiddleRange`, lines 151–173)
36.92. `GlobalTheory.HTW.RecS.fam_jointBCD_closed_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.FamJointBCDClosed`, lines 13–19)
`GlobalTheory.T.star_ad_and_bb_le_of_joint_cond` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RowsADInSettingStatement`, lines 11–12)
`GlobalTheory.NLink.RowA.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.TwoEALe`, lines 49–78)
36.93. `GlobalTheory.NLinkClose.main_alt_alt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ImportsFurtherTail`, lines 22–109)
`Formalism.MainNotions.TwistedPlaces.etaSum_le` (`Formalism.SharpRestriction.LevelLetterCocycles.MemPrimesOverOfMem`, lines 558–647)
`Formalism.MainNotions.TwistedPlaces.point_bound` (`Formalism.SharpRestriction.LevelLetterCocycles.ExportBoundOfAdmissible`, lines 893–996)
36.94. `GlobalTheory.ToricTransport.toricRatioConjugates_of_transport` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ToricRatioConjugatesOfTransport`, lines 57–86)
`GlobalTheory.ToricRatioBridge.ratioConj_of_toricRatioConjugates` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.RatioConjOfToricRatioConjugates`, lines 84–136)
36.95. `GlobalTheory.LevelFourier.gens` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.VacuumPairUniformLevelBound`, lines 44–110)
`GlobalTheory.LevelFourier.smul_mem_trSp` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.VacuumPairUniformLevelBound`, lines 131–300)
36.96. `GlobalTheory.LevelSeries.exists_denom` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SerIntAtOfSeriesRat`, lines 90–133)
`GlobalTheory.ConstBound.const_dvd` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.IsTorsionFreeIntOf`, lines 99–149)
36.97. `GlobalTheory.TowerPlace.placeIn_unique` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.PlaceInUnique`, lines 9–117)
`GlobalTheory.InvDiffLocal.valuation_mul_different_le_one` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ValuationMulLeOneOfTraceApproxIdempotentIntegral`, lines 117–183)
`GlobalTheory.TowerDifferent.relDifferent_order_bounded` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.RelativeDifferentOrderBoundedAbsoluteRamificatio`, lines 56–60)
36.98. `GlobalTheory.GoodThetaBound.theta_value_bounded` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ShortThetaValueBoundedGoodReduction`, lines 191–230)
`GlobalTheory.GaugeBoundedDenom.exists_canonGauge_nabVal_bounded_denom` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsCanonGaugeNabValBoundedDenom`, lines 239–255)
36.99. `GlobalTheory.GoodTransfer.bddAll_of_bddGood` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.CuspFormBoundGoodPointsExtendsAllPoints`, lines 497–501)
36.100. `GlobalTheory.LevelSeries.series_bound` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SerIntAtOfSeriesRat`, lines 137–239)
`GlobalTheory.LevelSeries.serIntAt_record` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SerIntAtOfSeriesRat`, lines 274–376)
36.101. `GlobalTheory.DescIntRoad.descIntAt_of_laws` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.UnramifiedDescentIntegralFormsTowerCover`, lines 21–30)
`GlobalTheory.XiIntSteps.intForm_descent_below` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.IntFormDescentBelow`, lines 95–175)
36.102. `GlobalTheory.ThetaTransport.existsUnique_trace` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.HeckeIsogenyThetaGeneratorTransport`, lines 668–704)
`GlobalTheory.HeckePointLaw.exists_hkRelFr` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.HeckePointLawsOfTransport`, lines 201–212)
36.103. `GlobalTheory.HkIntegral.heckeFrameInt` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.IsUnitNatCastUp`, lines 327–374)
`GlobalTheory.HkIntegral.heckeValueInt` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.IsUnitNatCastUp`, lines 376–401)
36.104. `GlobalTheory.HeckeEx.secTransform` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.HeckeTransformOfPointLaws`, lines 225–254)
`GlobalTheory.HeckeEx.cuspTransform` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.HeckeTransformOfPointLaws`, lines 404–417)
36.105. `GlobalTheory.HeckeReadProof.heckeReadTie` (`Development.AbelianVarietyTheory.Proofs.TwistedOperators.HeckeReadTie`, lines 34–78)
36.106. `GlobalTheory.HeightCounting.Up.PublishedInputsThirdLayerAlt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.PublishedInputsThirdLayer`, lines 14–28)
`GlobalTheory.NRows.ThetaLocalInputsAlt` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ThetaLocalInputs`, lines 13–28)
`GlobalTheory.NRows.GeometricGaloisRepCompatibility` (`Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.LocalTypeDictionary.GeometricGaloisRepCompatibility`, lines 6–18)

GlobalTheory.HeightCounting.PhtOn (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.NonemptyAllIff, lines 97–101)
GlobalTheory.HeightCounting.Up.SerN (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.CenUsedOfCenReal, lines 56–62)
GlobalTheory.MemberClose.exists_hyp_n_of_geometric_galois_compat (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.ThetaLiftHypothesesOfNonvanishing, lines 24–46)
36.107. UpperTheory.GalC.final_alt_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CFinal, lines 19–22)
UpperTheory.GalC.final_of_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LinkNFam, lines 75–86)
UpperTheory.GalC.linkN_of_boundAtTwoThree_slot (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.LinkNFam, lines 54–63)

Chapter 37: Primes in short intervals, masses and counts

37.1. Formalism.MainNotions.LitThin (Formalism.SharpRestriction.GlobalSchurBounds.MainSup, lines 88–90)
GlobalTheory.ThinScaleLiterature.litThin (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.LitThin, lines 8–9)
37.2. GlobalTheory.ZerosNearOne.density_and_regions (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.DensityAndRegions, lines 10–16)
Formalism.MainNotions.ShortIntervalPrimes.adm6 (Formalism.SharpRestriction.GlobalSchurBounds.ShortIntervalPrimes, lines 128–139)
37.3. GlobalTheory.ZeroBounds.explicit_formula_and_window_count (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExplicitFormulaAndWindowCount, lines 9–13)
Formalism.MainNotions.ShortIntervalPrimes.adm7 (Formalism.SharpRestriction.GlobalSchurBounds.ShortIntervalPrimes, lines 141–147)
Formalism.MainNotions.ShortIntervalPrimes.adm11 (Formalism.SharpRestriction.GlobalSchurBounds.ShortIntervalPrimes, lines 158–161)
37.4. Formalism.MainNotions.chosenThin_of_boxesN (Formalism.SharpRestriction.GlobalSchurBounds.MainSup, lines 105–114)
Formalism.MainNotions.chosenThin_closed (Formalism.SharpRestriction.GlobalSchurBounds.MainSup, lines 124–127)
Formalism.HC.Frame.o (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.ZsmulGeneratorEqZeroOfProjection, lines 70–70)
Formalism.HC.Frame.n0 (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.ZsmulGeneratorEqZeroOfProjection, lines 68–68)
Formalism.BasicNotions.card_HGrp (Formalism.LocalRingCardinality.RankBoundMonotoneInDimension, lines 466–467)
Formalism.MainNotions.FamCore (Formalism.SharpRestriction.FamCore, lines 71–136)
37.5. GlobalTheory.Mass.massRatio_all_unramified_cmFields (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.MassRatioAllUnramifiedCmFields, lines 8–9)
Formalism.HermitianLattices.SiegelMassBounds.siegel_mass_ratio (Formalism.GL3MeanSquares.HookFormulaTorsion.MassUpperBoundClosed, lines 12–12)
Formalism.MainNotions.StandardLattice.SiegelMassRatioCM (Formalism.SharpRestriction.GlobalSchurBounds.StandardLattice, lines 281–284)
37.6. GlobalTheory.VolRatioFromSiegelWeil.volumeRatio (Development.AbelianVarietyTheory.Proofs.FlagMeasures.UnitaryHyperplaneVolumeRatioSiegelWeil, lines 10–14)
Formalism.MainNotions.StandardLattice.VolumeRatio₂ (Formalism.SharpRestriction.GlobalSchurBounds.ThickPlacesCorollaryOfAnyTranslate, lines 2330–2340)
37.7. GlobalTheory.DefiniteMass.mass_eq_std (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.UnramifiedOfIsCMUnram, lines 147–166)
GlobalTheory.genusMassAt_two (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.GenusMassAtTwo, lines 7–8)
37.8. PI.UnitaryGroupMassFormula.HCFdegRGap.hCFdegRGap_holds (Formalism.PeriodIntegrals.UnitaryGroupMassFormula.FormalDegreeGap.HCFdegRGapHolds, lines 108–111)
Formalism.MainNotions.GappedMultiplicityBounds.theorem_three_rows_of_real_place (Formalism.SharpRestriction.GlobalSchurBounds.TheoremThreeFamilyOfGappedMultiplicity, lines 131–193)
Formalism.MainNotions.GappedMultiplicityBounds.explicit_bound_family (Formalism.SharpRestriction.GlobalSchurBounds.TheoremThreeFamilyOfGappedMultiplicity, lines 200–201)
37.9. Formalism.L2.GlobalTheoryTheory.GenusMassVolumes.vol_indef (Formalism.LocalToGlobal.GlobalTheory.GL5.DiscreteSeriesOfFormalDegree, lines 57–84)
Formalism.L2.GlobalTheoryTheory.GenusMassVolumes.volume_eq_mass_rank_two (Formalism.LocalToGlobal.GlobalTheory.GL5.DiscreteSeriesOfFormalDegree, lines 104–110)
37.10. GlobalTheory.Fc.row_hTu_cTmin_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RowHTuCTmin, lines 11–21)
Formalism.L2.GlobalTheoryTheory.GenusMassVolumes.volume_eq_mass_rank_two (Formalism.LocalToGlobal.GlobalTheory.GL5.DiscreteSeriesOfFormalDegree, lines 104–110)
37.11. UpperTheory.T3.levelPrimeN (Development.AutomorphicArithmetic.ShortNorms.levelPrimeN, lines 36–161)

GlobalTheory.EX.OB.exists_levCh_of_levelPrime (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsLevChOfLevelPrime, lines 20–112)
37.12. Formalism.L2.SG.Pf.AllRowsLab.rowsLab_holds (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.AllRowsLab, lines 19–24)
Formalism.L2.SG.Pf.AllRowsAddFin.eHom_fin (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.AllRowsAddFin, lines 15–24)
Formalism.L2.SG.Pf.RowsGoodRed.rowsGood_of (Formalism.LocalToGlobal.SiegelLatticeTheory.FlagStabilizerTensors.RootLatticeThresholds.RowsGoodRed, lines 39–105)
37.13. GlobalTheory.CountRowByName.row_counts_of_labels_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RowCountsOfLabels, lines 13–14)
GlobalTheory.CountRow.row_counts_of_ties_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RowCountsOfTies, lines 12–21)
37.14. GlobalTheory.CountBridge.count_unbundled (Development.AbelianVarietyTheory.Proofs.FlagMeasures.UnitaryAutomorphicCountingUnbundled, lines 40–74)
GlobalTheory.AllRowsAtRecord.countAllRows_of (Development.AbelianVarietyTheory.Proofs.FlagMeasures.AllRowsAtRecord, lines 37–65)
Formalism.MainNotions.S.CountE.CountIdempotentsClass (Formalism.SharpRestriction.CuspSpanNormBounds.BridgeCountE, lines 45–59)
Formalism.L2.GlobalTheoryTheory.ConcN.CountAllRows (Formalism.LocalToGlobal.GlobalTheory.IrreducibleOfCountSizes, lines 183–186)
37.15. GlobalTheory.Jet.Germ.jets_vanish (Development.AbelianVarietyTheory.Proofs.FlagMeasures.HighestWeightGapBound, lines 244–284)
GlobalTheory.Jet.UniqAlg.uniqAlg (Development.AbelianVarietyTheory.Proofs.FlagMeasures.JetUniquenessAlgebraic, lines 249–319)
37.16. GlobalTheory.DecGapBound.decGap_of_formulaGap (Development.AbelianVarietyTheory.Proofs.FlagMeasures.LowestKTypeDecayGapOfCompressionFormulaGap, lines 11–32)
GlobalTheory.Jet.FormGapFree.formulaGap (Development.AbelianVarietyTheory.Proofs.FlagMeasures.LowestKTypeCompressionFormulaOfStrictLowestKType, lines 22–99)
GlobalTheory.AutBall.autBallCount (Development.AbelianVarietyTheory.Proofs.FlagMeasures.AutBallCount, lines 28–48)
37.17. GlobalTheory.FlagMeas.AbsVol.absoluteVolume_of_volumeStep (Development.AbelianVarietyTheory.Proofs.FlagMeasures.HermAbsoluteVolumeBoundOfVolumeRatioInduction, lines 37–69)
GlobalTheory.FlagMeas.constantBound (Development.AbelianVarietyTheory.Proofs.FlagMeasures.FlagExplicitConstantLeExpNegHalf, lines 82–228)
GlobalTheory.NonzeroFlag.nonzeroFlagBound (Development.AbelianVarietyTheory.Proofs.FlagMeasures.NonzeroFlagProbabilityBoundOfVolumeRatioInductio, lines 84–148)
37.18. GlobalTheory.deep_region_exponents (Development.AbelianVarietyTheory.Proofs.FlagMeasures.DeepRegionExponents, lines 54–204)
37.19. GlobalTheory.DeepFamilyGap.deep_bound_family (Development.AbelianVarietyTheory.Proofs.FlagMeasures.DeepPointBoundCuspidalIsometricIntertwinerFamily, lines 118–183)
37.20. GlobalTheory.RowsFinMult.rows_finiteDimensional (Development.AbelianVarietyTheory.Proofs.FlagMeasures.FiniteDimensionalRowsSpaceDiscreteSeriesCount, lines 238–344)
GlobalTheory.DeepKernel.deep_kernel_bound_of_ptBound (Development.AbelianVarietyTheory.Proofs.FlagMeasures.DeepPointKernelBoundOfPointwiseBound, lines 22–48)
GlobalTheory.DeepRegion.deepBound (Development.AbelianVarietyTheory.Proofs.FlagMeasures.DeepRegionKernelDiagonalBound, lines 28–150)
37.21. GlobalTheory.Fc.R0.classCountBound_paramsLambda_constructedPoints (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ClassCountBoundParamsLambdaConstructedPoints, lines 17–25)
Formalism.MainNotions.StandardLattice.ClassNumberFormula₂ (Formalism.SharpRestriction.GlobalSchurBounds.StandardLattice, lines 453–456)
Formalism.MainNotions.J.level_prime_splits_totally_real (Formalism.SharpRestriction.AdmissibleParameterJoins.DIVfromGLz, lines 13–18)
37.22. GlobalTheory.ClassSumSpan.mem_span_of_map_mem (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ClassSumsSpan, lines 32–53)
37.23. GlobalTheory.ClassSumSpan.atDict_indep (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ClassSumsSpan, lines 165–195)
GlobalTheory.reading_linearIndependent_and_injective (Development.AbelianVarietyTheory.Proofs.UnitaryExistence.ReadingLinearIndependentAndInjective, lines 22–54)
37.24. GlobalTheory.Published.DiscreteSeriesCover (Development.AbelianVarietyTheory.PublishedTheorems.TwistedTraceFormula.AdmissibleDiscreteSeriesOfHasParam, lines 35–38)
Formalism.BasicNotions.TH.GrpData (Formalism.LocalRingCardinality.DataTheta, lines 93–119)
Formalism.BasicNotions.TH.DSRep (Formalism.LocalRingCardinality.DataTheta, lines 165–178)
Formalism.BasicNotions.TH.IsParam (Formalism.LocalRingCardinality.DataTheta, lines 382–384)
GlobalTheory.CoverDS.HasTypes (Development.AbelianVarietyTheory.PublishedTheorems.TwistedTraceFormula.AdmissibleDiscreteSeriesOfHasParam, lines 24–27)
GlobalTheory.CoverDS.UnitEquiv (Development.AbelianVarietyTheory.PublishedTheorems.TwistedTraceFormula.AdmissibleDiscreteSeriesOfHasParam, lines 16–17)
37.24. GlobalTheory.Published.HarishChandraModules (Development.AbelianVarietyTheory.PublishedTheorems.TwistedTraceFormula.HarishChandraModulesOn, lines 43–44)

GlobalTheory.Published.HarishChandraModulesOn (Development.AbelianVarietyTheory.PublishedTheorems.TwistedTraceFormula.HarishChandraModulesOn, lines 24–35)
GlobalTheory.Published.KFiniteAnalyticUJ (Development.AbelianVarietyTheory.PublishedTheorems.TwistedTraceFormula.HarishChandraModulesOn, lines 37–41)
37.24. GlobalTheory.Published.CuspRapidDecay (Development.AbelianVarietyTheory.PublishedTheorems.AutomorphicForms₂.ExistsNormLeHeightPow, lines 28–32)
Formalism.L2.Theta.IsAutFun (Formalism.LocalToGlobal.BesselHermitianTheta.IsAutFunDef, lines 43–46)
Formalism.L2.Theta.IsCuspFun (Formalism.LocalToGlobal.BesselHermitianTheta.IsAutFunDef, lines 67–70)
GlobalTheory.IndefiniteRowSideFullHecke.LieAlgebraExponentialFamilies.IsUnifModerate (Development.AbelianVarietyTheory.BlockTheory.HeightDefect.WordDerivatives, lines 122–124)
GlobalTheory.CuspDecay.IsRapidDecay (Development.AbelianVarietyTheory.PublishedTheorems.AutomorphicForms₂.ExistsNormLeHeightPow, lines 21–22)
37.24. GlobalTheory.Published.CasselmanWallach (Development.AbelianVarietyTheory.PublishedTheorems.AutomorphicForms₂.HasDRepIffHasDerivAt, lines 98–111)
GlobalTheory.SmoothGlob.IsSmoothModRep (Development.AbelianVarietyTheory.PublishedTheorems.AutomorphicForms₂.HasDRepIffHasDerivAt, lines 22–30)
GlobalTheory.SmoothGlob.sobNorm (Development.AbelianVarietyTheory.PublishedTheorems.AutomorphicForms₂.HasDRepIffHasDerivAt, lines 34–35)
GlobalTheory.SmoothGlob.IsGKMap (Development.AbelianVarietyTheory.PublishedTheorems.AutomorphicForms₂.HasDRepIffHasDerivAt, lines 37–40)
GlobalTheory.SmoothGlob.IsSmoothGMap (Development.AbelianVarietyTheory.PublishedTheorems.AutomorphicForms₂.HasDRepIffHasDerivAt, lines 42–46)
Formalism.REAL.hs (Formalism.RealPlaceAnalysis.Complexify, lines 27–27)
37.24. GlobalTheory.Fc.R1.row_hRB_univ_alt (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RowHRBUniv, lines 19–24)
Formalism.IndefR.IndefiniteRealisationBoundedRows (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.IndefRRealTB, lines 39–41)
37.25. GlobalTheory.Corev.CmTwoClosing.cmGoodModel_of_globalArtinReciprocity (Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.CmGoodModelOfGlobalArtinReciprocity, lines 10–11)
GlobalTheory.HondaCM3.cm3 (Development.AbelianVarietyTheory.Proofs.DecoratedLatticeProofs.OneSubTwoOmegaSq, lines 77–158)
37.26. GlobalTheory.T.hServ_of_linkR (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.RowHServOfLinkStatement, lines 14–15)
GlobalTheory.Cp.SelectionServed (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.CountsIrreducibilityConstants, lines 48–62)
37.27. Formalism.L2.KerInert.beProp_ofA (Formalism.LocalToGlobal.InertKernelTheory.FinalA, lines 18–21)
Formalism.L2.KerInert.BeCob.coboundary_exact (Formalism.LocalToGlobal.InertKernelTheory.InertCoprimeLemmas.ExistsWittSeriesInverseFormalGroupIso, lines 111–205)
Formalism.L2.Th1.C.norm_adjoint_all_places (Formalism.LocalToGlobal.TorsionIntegrality.SecondCoefficientNorms.BinomialSeriesCoefficients.UnitaryInverseIdentity, lines 424–460)
37.28. GlobalTheory.OrbitBoundStdRec.orbit_bound_rec (Development.AbelianVarietyTheory.Proofs.SatakeParameters.OrbitBoundStandardSeriesInertPlaces, lines 22–42)
37.29. Formalism.L2.KerInert.kernels_integral_inert (Formalism.LocalToGlobal.InertKernelTheory.KernelsIntegralInert, lines 9–9)
37.30. Formalism.L2.KerInert.Rep.k1gen_of_orth (Formalism.LocalToGlobal.InertKernelTheory.InertCoprimeLemmas.RepAsm, lines 17–24)
37.31. GlobalTheory.TwV.site_cmpLite_siteAdm (Development.AbelianVarietyTheory.Proofs.SatakeParameters.JointSiteChainComparedCarriesAdmittedSystem, lines 77–89)
GlobalTheory.TwV.Lt.AdmAtEnt2 (Development.AbelianVarietyTheory.Definitions.GaloisGeometry.ThetaSeriesDataHypothesesBelowMaximalLevel, lines 106–129)

Chapter 38: The chain length

38.1. GlobalTheory.Mem.Num.evenCeil (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.EvenCeil, lines 11–11)
GlobalTheory.Mem.Num.mAt (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.EvenCeil, lines 21–21)
Formalism.BasicNotions.Scale.delta1 (Formalism.LocalRingCardinality.DataParams₂, lines 45–45)
38.2. GlobalTheory.Mem.Num.RChosen (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.EventuallyThickCardLeCyclicOrder, lines 60–62)
Formalism.HC.Frame.o (Formalism.SharpRestriction.RamificationFrobenius.TotientGaloisClasses.ZsmulGeneratorEqZeroOfProjection, lines 70–70)
GlobalTheory.Mem.Num.kAt (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.EvenCeil, lines 23–23)
GlobalTheory.Mem.Num.nWAt (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.EvenCeil, lines 27–27)
GlobalTheory.Mem.Num.tAt (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.EvenCeil, lines 33–33)
GlobalTheory.Mem.Num.RanksOK (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.EvenCeil, lines 41–43)

GlobalTheory.Mem.Num.ranksOf (Development.AbelianVarietyTheory.Definitions.AutomorphicMembership.EvenCeil, lines 45–59)
Formalism.BasicNotions.Ranks (Formalism.LocalRingCardinality.Params, lines 6–22)
38.3. GlobalTheory.EX.exists_ranks_rankVal_frame (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsRanksRankValFrame, lines 45–99)
GlobalTheory.EX.TOv (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.PairRowIffQdRows, lines 17–17)
GlobalTheory.EX.nWv (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.PairRowIffQdRows, lines 19–20)
GlobalTheory.EX.RankVal (Development.AbelianVarietyTheory.Definitions.GlobalTwisting.PairRowIffQdRows, lines 28–32)
38.3. GlobalTheory.EX.exists_even_between (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsRanksRankValFrame, lines 10–15)
GlobalTheory.EX.exists_ranks_of_frame (Development.AbelianVarietyTheory.Proofs.GlobalInterpolation.ExistsRanksRankValFrame, lines 24–43)

Chapter 39: Galois representations attached to polarized cuspidal representations

39.1. UpperTheory.LiteratureTheorems.OpenLiteratureStatements (Development.AutomorphicArithmetic.ThetaCorrespondence.ThetaPairLifts.ThirdPlaceSeries.EveryCharacterOfLiteratureOfVWin, lines 215–218)
UpperTheory.FinalTheorems.everyCharacter_of_row (Development.AutomorphicArithmetic.CharacterBoxes.MainConclusion, lines 7–8)
UpperTheory.FinalTheorems.everyCharacter (Development.AutomorphicArithmetic.CharacterBoxes.MainConclusion, lines 10–11)
39.2. UpperTheory.Gal.Int.galoisLocalGlobalStrong_recParameter (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.RowSrc, lines 6–6)
39.3. UpperTheory.Gal.Int.galoisLocalGlobalStrong_of_geometry_lift (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.Development, lines 30–31)
UpperTheory.Gal.Int.galoisRepresentations_of_thetaLift (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GaloisRepresentationsOfThetaLift, lines 6–6)
39.4. UpperTheory.GalStar.PropsB (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.TypeConditions.WeightsConjBis, lines 54–67)
UpperTheory.GalStar.Row0b (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.TypeConditions.WeightsConjBis, lines 69–73)
39.5. UpperTheory.GalStar.AwayB (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.TypeConditions.WeightsConjBis, lines 26–31)
UpperTheory.GalStar.Sqf (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.TypeConditions.WeightsConjAlt, lines 29–33)
39.6. Formalism.BasicNotions.LocTied (Formalism.LocalRingCardinality.AdicCompletionLocal.LocTied, lines 11–36)
39.6. UpperTheory.GalStar.AboveB (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.TypeConditions.WeightsConjBis, lines 39–52)
39.7. GlobalTheory.RegAlgCharAlg.isAlgPos_of_polarizedPair (Development.AbelianVarietyTheory.Proofs.SatakeParameters.IsAlgebraicHeckeCharOfRegularAlgebraicPolarizedP, lines 37–62)
UpperTheory.heckeChar_parallel_weight (Development.AutomorphicArithmetic.AutomorphicVarieties.BuildingsResidueChars.HeckeCharParallelWeight, lines 234–289)
UpperTheory.Gal.Int.Row.exists_weights (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.InertCharpoly.GaloisLocalGlobalOfLocalGlobalCompatibility, lines 127–148)
39.8. UpperTheory.Gal.Int.Row.galoisLocalGlobal_of_placeClassCompatibility (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.InertCharpoly.GaloisLocalGlobalOfLocalGlobalCompatibility, lines 157–171)
UpperTheory.Gal.Int.Row.away (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.InertCharpoly.GaloisLocalGlobalOfLocalGlobalCompatibility, lines 96–106)
UpperTheory.Gal.Int.Row.above (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.InertCharpoly.GaloisLocalGlobalOfLocalGlobalCompatibility, lines 108–125)
UpperTheory.Gal.Int.rest (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.Development, lines 24–25)
39.9. UpperTheory.Gal.Int.Geo.GeoOne (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.HodgeTrAt, lines 79–90)
UpperTheory.Gal.Int.Geo.geoOne_holds (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.WeightOneGeometricRealization, lines 11–11)
39.10. UpperTheory.Gal.CertificateRingHomMap.seedGU_holds (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ProjectiveAutomorphicHecke.NhdsBasisLevelSubgroup, lines 82–83)
UpperTheory.Gal.Int.Geo.Pl.vecEx_of (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.PrimeLevelLocalization.PVecWt, lines 13–24)
39.11. UpperTheory.Gal.Int.Geo.Pl.master (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.PrimeLevelLocalization.SameKTrans, lines 86–205)
UpperTheory.Gal.Spc.eigEx (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ScalarCommutation.FrameTh, lines 52–91)

UpperTheory.Gal.W2.Seed.pHasEigOf (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SphericalCoordinates.AlgebraEquivSetup.PHasEigOf, lines 45–86)
39.12. UpperTheory.Gal.Spc.specTopG (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ScalarCommutation.SpecTopG, lines 9–21)
UpperTheory.Gal.Spc.specOfF (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ScalarCommutation.SpecOfF, lines 179–227)
39.13. UpperTheory.Gal.Int.Geo.PlanEx (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.PackExOf, lines 82–91)
UpperTheory.Gal.Int.Geo.Pl.planEx_holds (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.PrimeLevelLocalization.SpecTopHolds, lines 9–9)
UpperTheory.Gal.Int.Geo.Plan.pack (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.PackExOf, lines 101–125)
39.14. GlobalTheory.UPel.HT.Gen.GObj (Development.AbelianVarietyTheory.Definitions.Divisors.ExistsGDatumOneOneOne, lines 139–167)
GlobalTheory.UPel.HT.Gen.GenModel (Development.AbelianVarietyTheory.Definitions.Divisors.ExistsGDatumOneOneOne, lines 181–185)
39.15. GlobalTheory.UPel.HT.model_nonempty (Development.AbelianVarietyTheory.Proofs.DivisorProofs.HarrisTaylorIntegralModelNonempty, lines 11–15)
GlobalTheory.UPel.HT.exists_repFam_qproj (Development.AbelianVarietyTheory.Proofs.DivisorProofs.ExistsQuasiProjectiveHarrisTaylorFamily, lines 13–36)
GlobalTheory.UPel.HT.RepFam.smoothOfRelativeDimension (Development.AbelianVarietyTheory.Proofs.DivisorProofs.SmoothOfRelativeDimension, lines 15–37)
GlobalTheory.UPel.HT.Model.isProjOver_of_compactB (Development.AbelianVarietyTheory.Proofs.DivisorProofs.IsProjOverOfCompactB, lines 10–14)
39.16. GlobalTheory.T.unitaryCharZeroGenericModel_nonempty (Development.AbelianVarietyTheory.Proofs.DivisorProofs.UnitaryCharZeroGenericModelNonemptyStatement, lines 8–9)
GlobalTheory.UPel.HT.Gen.genModel_nonempty (Development.AbelianVarietyTheory.Proofs.DivisorProofs.UnitaryCharZeroGenericModelNonempty, lines 118–122)
39.17. UpperTheory.Gal.Int.Geo.KS (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.IntegOfMem, lines 48–59)
UpperTheory.Gal.G3.pKSEx (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.PAdicWittLattices.ExistsDeeperNormalLevel, lines 100–172)
UpperTheory.Gal.G3.kSNice_holds (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.PAdicWittLattices.FibrePowerTowerAdmissibleLevel, lines 152–152)
UpperTheory.Gal.G3.kSEx_holds (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.PAdicWittLattices.FibrePowerTowerAdmissibleLevel, lines 172–172)
39.18. UpperTheory.baseExtension_holds (Development.AutomorphicArithmetic.CommutativeAlgebra.BaseExtensionHolds, lines 76–138)
UpperTheory.site_exists_of_splitting_and_power_projector (Development.AutomorphicArithmetic.CommutativeAlgebra.SiteExistsOfSplittingAndPowerProjector, lines 11–38)
UpperTheory.Gal.crystallineComparison.Mod.esNorm_of (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.AnnihilatorModels.MEsNorm, lines 16–43)
UpperTheory.Gal.crystallineComparison.Mod.ordKSE_of (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.AnnihilatorModels.MOrdKSE, lines 67–149)
UpperTheory.Gal.FiniteEtaleModels.ksGood_holds (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.FiniteEtaleModels.KsGoodHolds, lines 45–47)
39.19. GlobalTheory.EtC.HlSub (Development.AbelianVarietyTheory.Definitions.Divisors.PullHlComp, lines 216–223)
GlobalTheory.EtC.HQ (Development.AbelianVarietyTheory.Definitions.Divisors.PullHlComp, lines 239–239)
GlobalTheory.GalV.Htot (Development.AbelianVarietyTheory.Definitions.Divisors.IsCorrOpOfAvg, lines 11–11)
GlobalTheory.GalV.galTotRep (Development.AbelianVarietyTheory.Definitions.Divisors.IsCorrOpOfAvg, lines 58–62)
39.20. UpperTheory.Gal.G3.pHopSpec (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.PAdicWittLattices.PHopSpec, lines 17–50)
UpperTheory.Gal.Int.Geo.hopSpec (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.RepMain, lines 56–57)
39.21. UpperTheory.Gal.Int.Geo.repEx_of (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.RepAsm, lines 13–59)
UpperTheory.Gal.Int.Geo.repEx_core (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.RepMain, lines 74–75)
39.22. UpperTheory.Gal.crystallineComparison.ordDecomp_of (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.AnnihilatorModels.POrdDecomp, lines 98–169)
UpperTheory.Gal.crystallineComparison.telescope_of (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.AnnihilatorModels.PTelescope, lines 47–76)
UpperTheory.Gal.crystallineComparison.Mod.congKS_link (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.AnnihilatorModels.QuotObjLink, lines 13–13)
UpperTheory.Gal.crystallineComparison.ordCount_holds (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.AnnihilatorModels.GenericMulOfProdHolds, lines 51–51)
39.23. UpperTheory.Gal.Mod.esStep (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.BasicOpenSections.PEsStep, lines 49–94)

`UpperTheory.Gal.Mod.towES` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.BasicOpenSections.PTowES`, lines 147–220)
`UpperTheory.Gal.Int.Geo.CongFor` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.HodgeTrAt`, lines 54–56)
`UpperTheory.Gal.Int.Geo.Pack.rep` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.MapInvMul`, lines 43–87)
39.24. `UpperTheory.Gal.LocalSmoothExtension.HolomorphicLefschetzFormula` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.LefschetzSigmaSums.HodgeTraceFromLefschetzAndGroupSide`, lines 102–111)
`UpperTheory.Gal.Int.Geo.weight_one_geometric_realization` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.WeightOneGeometricRealization`, lines 7–9)
39.25. `UpperTheory.Gal.Cnt.weighted_count_definite_unitary` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.DefiniteTraceForms.WeightedCountDefiniteUnitary`, lines 10–11)
`UpperTheory.Cb.complex_count_regular_marked_points` (`Development.AutomorphicArithmetic.Foundations.EtaleClopenDescent.ComplexCountRegularMarkedPoints`, lines 552–571)
`GlobalTheory.LKDef.cls_equiv_dcls_irr` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.MarkedFieldPairsEquivDefiniteFieldPairsOdd`, lines 81–186)
39.26. `UpperTheory.Gal.LocalSmoothExtension.groupTop_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.LefschetzSigmaSums.PGroupTop`, lines 12–65)
`UpperTheory.Gal.LocalSmoothExtension.hodgeEig_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.LefschetzSigmaSums.PHodgeEig`, lines 6–67)
`UpperTheory.Gal.LocalSmoothExtension.endPol` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.LefschetzSigmaSums.InvPol`, lines 43–43)
`UpperTheory.Gal.LocalSmoothExtension.markOf` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.LefschetzSigmaSums.InvPol`, lines 45–46)
39.27. `UpperTheory.Gal.Int.Geo.PhiK` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.HodgeTrAt`, lines 32–36)
`UpperTheory.Gal.Int.Geo.CrysStep` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.HodgeTrAt`, lines 112–115)
`UpperTheory.Gal.Int.Geo.JumpStep` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.HodgeTrAt`, lines 117–120)
`UpperTheory.Gal.Int.Geo.PhiStep` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.HodgeTrAt`, lines 122–125)
`UpperTheory.Gal.Int.Geo.jumpStep_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.GeometricFlagBases.GeometricRealizationOfWeightOne`, lines 18–18)
`UpperTheory.Ru.crystallineComparison.phiStep` (`Development.AutomorphicArithmetic.CommutativeAlgebra.EtaleCohomology.SmoothModelCrystalline`, lines 13–13)
39.28. `UpperTheory.Gal.KunnethLefschetz.core_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.KunnethLefschetz.CoreHolds`, lines 5–6)
39.29. `GlobalTheory.EtC.etC_c2` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.SmoothAffineCurveEtaleCohomologicalDimensionLeOn`, lines 9–14)
39.30. `UpperTheory.Cb.etale_cohomology_ind_etale_base_change` (`Development.AutomorphicArithmetic.Foundations.EtaleClopenDescent.EtaleCohomologyIndEtaleBaseChange`, lines 272–308)
`UpperTheory.Cb.C3.local_eqv` (`Development.AutomorphicArithmetic.Foundations.EtaleClopenDescent.EtaleCohomologyIndEtaleBaseChange`, lines 210–246)
39.31. `GlobalTheory.EtC.HomotopyInv.homotopy_invariance_etale` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.HomotopyInvarianceEtale`, lines 36–45)
39.32. `GlobalTheory.EtC.laurentPolynomial_cohomology_strictlyHenselian_vanishing` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.LaurentPolynomialCohomologyVanishing`, lines 8–8)
39.33. `GlobalTheory.EtC.projectiveBundleFormula_vectorBundle_any_space` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ProjectiveBundleFormulaVectorBundleAnySpace`, lines 11–12)
`GlobalTheory.T.projectiveBundleFormula_discrete_space` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ProjectiveBundleFormulaDiscreteSpaceStatement`, lines 5–6)
`GlobalTheory.PBFg.pbf_of_disc` (`Development.AbelianVarietyTheory.Proofs.DivisorProofs.ProjectiveBundleFormulaOfDiscreteSpaces`, lines 65–76)
39.34. `UpperTheory.Gal.KunnethLefschetz.QDual` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.KunnethLefschetz.OneMul`, lines 191–192)
`UpperTheory.Gal.KunnethLefschetz.Core.qDual` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.KunnethLefschetz.EtSideGood`, lines 45–45)
39.35. `UpperTheory.Gal.Mod.spSpecIf` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.BasicOpenSections.PSpSpecIf`, lines 18–107)
`UpperTheory.Gal.Mod.specGood` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.BasicOpenSections.EsGenAll`, lines 8–8)
39.36. `GlobalTheory.CongL.SeedWU4` (`Development.AbelianVarietyTheory.Definitions.GaloisGeometry.LimWU4OfLimWU3`, lines 15–35)
`UpperTheory.Gal.Int.Div.seedFinOdd_of_geoOne` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.ZetaCharacterDivisibility.MainFin`, lines 13–15)
`UpperTheory.Gal.Int.Div.seedFinOdd_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.ZetaCharacterDivisibility.SeedFinOddOf`, lines 44–78)

`UpperTheory.Gal.Int.Div.divOne_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.ZetaCharacterDivisibility.PDivOne`, lines 35–72)
39.37. `UpperTheory.Gal.Int.Div.DivSet` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.ZetaCharacterDivisibility.PhiOfDivOf`, lines 121–129)
`LateralTheory.DivDb.Key` (`Development.EtaleCohomology.DivisorCycles.CycleCount`, lines 51–55)
`LateralTheory.DivDb.plus_of` (`Development.EtaleCohomology.DivisorCycles.DivPlus`, lines 37–69)
`UpperTheory.Gal.Int.Div.divSet_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.ZetaCharacterDivisibility.DivisionOneGeometricRepresentation`, lines 52–52)
`LateralTheory.DivDb.KeyPf.Key_of` (`Development.EtaleCohomology.DivisorCycles.SumMulSum`, lines 90–176)
39.38. `UpperTheory.Gal.Int.Div.rootPin_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.ZetaCharacterDivisibility.PRootPin`, lines 16–37)
`UpperTheory.Gal.Int.Div.divAll_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.ZetaCharacterDivisibility.DivisionOneGeometricRepresentation`, lines 54–57)
`UpperTheory.Gal.Int.Div.detEx_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.ZetaCharacterDivisibility.DivisionOneGeometricRepresentation`, lines 49–49)
39.39. `UpperTheory.Gal.Gen.DenseGen` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SeparatingAlgebras.GenDataEmpty`, lines 65–69)
`UpperTheory.Gal.Gen.denseGen_pf` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SeparatingAlgebras.DenseGen`, lines 14–51)
`UpperTheory.Gal.Gen.step_pf` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SeparatingAlgebras.StepPf`, lines 8–64)
39.40. `GlobalTheory.CongL.CongUW4` (`Development.AbelianVarietyTheory.Definitions.GaloisGeometry.LimWU4OfLimWU3`, lines 37–55)
`GlobalTheory.CongL.PowSatU` (`Development.AbelianVarietyTheory.Definitions.GaloisGeometry.HasLevelOffOf`, lines 51–52)
`UpperTheory.Gal.Int.Cong.congFinOdd_of_ex` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.CongruenceOddLevel.CongFinOddOfLvLac`, lines 15–15)
`UpperTheory.Gal.Int.Cong.congUWG_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.CongruenceOddLevel.NotApartOfOverPair`, lines 111–250)
`UpperTheory.Gal.Int.Cong.CongUWG.congUW4` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.CongruenceOddLevel.NotApartOfOverPair`, lines 252–263)
39.41. `GlobalTheory.Cong.PolyApprox` (`Development.AbelianVarietyTheory.Definitions.GaloisGeometry.PolyApproxZeroPadic`, lines 10–12)
39.42. `GlobalTheory.Cong.exists_limit_rep` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsLimitRep`, lines 14–28)
39.43. `GlobalTheory.PsChar.exists_ssRep` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.FiniteSemisimple`, lines 64–145)
`GlobalTheory.SSCont.continuous_of_continuous_trace` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ContinuousOfContinuousTrace`, lines 60–69)
39.44. `UpperTheory.Gal.Lim.limFinOdd_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.TraceLimits.LimFinOddOf`, lines 197–197)
`UpperTheory.Gal.Lim.exists_trace_limit` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.TraceLimits.ExistsTraceLimit`, lines 15–169)
`UpperTheory.Gal.Lim.limWU4_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.TraceLimits.LimFinOddOf`, lines 16–195)
`GlobalTheory.CongL.crysDRU_of_linApprox` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.BlowUpFibreSemilocal`, lines 67–97)
39.45. `GlobalTheory.BKLim.exists_frame_sst` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsBreuilKisinFrameSemistableBoundedJumps`, lines 109–134)
`GlobalTheory.BKLim.Frame.isCr_of_wSeed` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.DvdGalSub`, lines 55–111)
`GlobalTheory.BKLim.WDR.frame_isDR` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.SemistablePeriodFrameIsDeRham`, lines 18–41)
`GlobalTheory.BKLim.exists_beta_of_disc_cocycle` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsBetaOfDiscCocycle`, lines 103–243)
39.46. `GlobalTheory.BKLim.beta_bdd_denominators` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.BreuilKisinBetaBoundedDenominators`, lines 41–105)
`GlobalTheory.BKLim.exists_isZ_not_root` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsGaloisElemCyclotomicPeriodNotRootOfUnity`, lines 108–162)
`GlobalTheory.BKLim.beta_step` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.BreuilKisinBetaNextCoeffThetaMem`, lines 63–138)
`GlobalTheory.LiftDiv.dvd_mod_E` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.ExistsPrimePowDvdModEisensteinOfTheta`, lines 14–47)
`GlobalTheory.BKLim.BetaDiv.isBeta_div` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.BreuilKisinIsBetaSubOfLpowSmul`, lines 14–73)
39.47. `UpperTheory.Gal.ClosedPhiFromClosedTrace.coverJ_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ClosedPhiFunctoriality.CoverJ`, lines 23–124)
`UpperTheory.Gal.ClosedPhiFromClosedTrace.frEx_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ClosedPhiFunctoriality.FrExHolds`, lines 11–37)

`UpperTheory.Gal.ClosedPhiFromClosedTrace.opEx_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ClosedPhiFunctoriality.OpExHolds`, lines 72–102)
`GlobalTheory.AcrisPow.main` (`Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.PrimePowDvdTiltUniformizerPowAcrisEventually`, lines 70–83)
`UpperTheory.Gal.ClosedPhiFromClosedTrace.secU_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ClosedPhiFunctoriality.SecUHolds`, lines 12–86)
39.48. `GlobalTheory.CongPhi.ClosedPhi` (`Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.JointApproxOfMem`, lines 79–93)
`GlobalTheory.LimitTop.closedAtL` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.ClosedAtL`, lines 25–117)
`UpperTheory.Gal.ClosedPhiFromClosedTrace.trStep_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ClosedPhiFunctoriality.TrStep`, lines 83–161)
`UpperTheory.Gal.ClosedPhiFromClosedTrace.trLim_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ClosedPhiFunctoriality.TrLimHolds`, lines 8–57)
`UpperTheory.Gal.ClosedPhiFromClosedTrace.congPhi_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ClosedPhiFunctoriality.ClosedPhiHolds`, lines 44–44)
39.49. `UpperTheory.ConvolutionTheory.D.muEx` (`Development.AutomorphicArithmetic.Foundations.Convolution.HermitianDescent.OffUnitsLeUnitIdl`, lines 357–401)
39.50. `UpperTheory.V.Z.Tf.GlobalTraceComparison.Asm.rUEval` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.TraceFormula.GlobalTraceComparison.TraceAsymptotics.RUEval`, lines 197–331)
`UpperTheory.Zero.ConvolutionTheory.EB.rEndo_of` (`Development.AutomorphicArithmetic.Foundations.Convolution.EndomorphismSlices.REndoOf`, lines 53–101)
`UpperTheory.Zero.ConvolutionTheory.EB.rStable` (`Development.AutomorphicArithmetic.Foundations.Convolution.EndomorphismSlices.FiniteAdelicEstimates.RStable`, lines 75–133)
`UpperTheory.V.Z.Tf.GlobalTraceComparison.Asm.ends` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.TraceFormula.GlobalTraceComparison.TraceAsymptotics.EvaluationCombinedTestFunction`, lines 322–347)
39.51. `UpperTheory.V.Z.Tf.GaloisFromTraceFormula.asmTh_of` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.TraceFormula.GaloisFromTraceFormula.AsmThOf`, lines 10–112)
`UpperTheory.Gal.Desc.weakAll` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.BaseChangeDescent.BcBackOffT`, lines 26–26)
39.52. `GlobalTheory.Cong.DescOffT` (`Development.AbelianVarietyTheory.Definitions.GaloisGeometry.MatchOfMatchOffTEmpty`, lines 32–36)
`UpperTheory.Gal.Desc.descOffT` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.BaseChangeDescent.BcBackOffT`, lines 31–31)
39.53. `GlobalTheory.QuadLiftBridge.quadLift` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.QuadLift`, lines 14–23)
39.54. `UpperTheory.Gal.Int.Core.toCoreK` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.CrystallineDescent.ToCoreK`, lines 7–9)
`UpperTheory.Gal.Int.Inp.coreAll_of_core` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.CrystallineInputs.CoreAllOfRowStarCore`, lines 4–5)
`UpperTheory.Gal.Int.Core.patchOne_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.CrystallineDescent.PatchOne`, lines 83–127)
`UpperTheory.Gal.Int.Core.stepDown_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.CrystallineDescent.JoinEndHolds`, lines 92–134)
`UpperTheory.Gal.Int.Inp.coreAll_of_core_inp` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.CrystallineInputs.CoreAllOfRowStarCoreInp`, lines 538–550)
`UpperTheory.Gal.Int.Core.joinU_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.CrystallineDescent.JoinEndHolds`, lines 19–62)
`GlobalTheory.GoodCsdTwist.exists_goodCsd_twist` (`Development.AbelianVarietyTheory.Proofs.SatakeParameters.ExistsGoodConjSelfDualTwistSplitUnramified`, lines 65–110)
`UpperTheory.Gal.Int.Core.joinUp_of'` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.CrystallineDescent.BadFinite`, lines 46–93)
39.55. `GalTw.GlobalTheory.RoadU.window` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SphericalGaloisCompatibility.ToralDeRham.NormEqOneOfIsUnitarisable`, lines 100–118)
39.56. `UpperTheory.V.Z.ImaginaryQuadraticEtaCharacter.gamLocX` (`Development.AutomorphicArithmetic.AutomorphicVarieties.ModularityArgument.LeviSmoothLocalization.PrincipalSeriesMatching.GamLocX`, lines 56–86)
39.57. `GalTw.GlobalTheory.RoadU.exists_sph_lift_of_toral` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SphericalGaloisCompatibility.ToralDeRham.FrobeniusCharpolyOfUnlinkedUnramifiedSupport`, lines 32–45)
`GalTw.GlobalTheory.RoadU.TorLiftC.exists_cm_lift_fix_charKer` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SphericalGaloisCompatibility.ToralDeRham.WeaklyAttachedToralOfCmLift`, lines 29–91)
`UpperTheory.Gal.GamX.charSupp_of` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.MixedUnlinkedExponents.CharLiftPrimeOf`, lines 93–114)
`GalTw.GlobalTheory.RoadU.TorLift.hasSat_charPol_above` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SphericalGaloisCompatibility.ToralDeRham.FrobeniusCharpolyEqSatakeOfPlaceAbove`, lines 28–92)
39.58. `UpperTheory.Gal.ToralGalois.galoisCompat_unlinkedSupport_of_spherical` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SphericalGaloisCompatibility.GaloisCompatUnlinkedSupportOfSpherical`, lines 10–10)
`GalTw.GlobalTheory.RoadU.toralGal_of_holds` (`Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SphericalGaloisCompatibility.ToralDeRham.ToralGalois2`, lines 22–56)

GalTw.GlobalTheory.RoadU.toralAbove_of_holds (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SphericalGaloisCompatibility.ToralDeRham.ToralAboveOfHolds, lines 31–71)
GalTw.GlobalTheory.DeRhamOfPlainTorus.deRham_of_plainTor (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SphericalGaloisCompatibility.ToralDeRham.GaloisCompatUnlinkedSupportOfSpherical, lines 27–67)
39.59. UpperTheory.Gal.ToralGalois.R2.toTor (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.SphericalGaloisCompatibility.PAdicSplitting.PToTor, lines 61–99)
UpperTheory.Gal.InertialTypesAlt.suppAny (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.RegularSemisimpleFibres.PSuppAny, lines 30–102)
UpperTheory.Gal.InertialTypesAlt.mixUnlinked (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.RegularSemisimpleFibres.PMixUnlinked, lines 10–16)
39.60. UpperTheory.TP.TLiftK (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.HalfIndexLattices.TLiftK, lines 12–23)
UpperTheory.TP.W.GoodK (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.HalfIndexLattices.SplitStab, lines 18–24)
UpperTheory.TP.LocPair (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.SchurStep, lines 93–101)
UpperTheory.TP.IsLiftWt (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.SchurStep, lines 103–104)
UpperTheory.TP.IsLift (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.SchurStep, lines 106–112)
UpperTheory.TP.K.tLiftK (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.ThetaKernels.TLiftK, lines 4–5)
39.61. UpperTheory.TP.Rallis_k (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.ThetaKernels.GalTPDefs, lines 61–68)
UpperTheory.TP.NonVan_k (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.ThetaKernels.GalTPDefs, lines 114–121)
UpperTheory.TP.rallis_k (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.ThetaKernels.TPGRallis, lines 99–225)
UpperTheory.TP.nonVan_k (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.ThetaKernels.TPGNonVan, lines 26–154)
UpperTheory.TP.IsBump (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.SchurStep, lines 44–50)
39.62. UpperTheory.Gal.TP.Loc.liftIso_of (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.SatakeLifts.IsUnitaryCharInv, lines 22–28)
UpperTheory.Gal.TP.Loc.pairSat_holds (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.SatakeLifts.LiftSatUHolds, lines 31–32)
39.63. UpperTheory.Gal.TP.twoUpOK_of_steps (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.TwoUpOKOfSteps, lines 13–31)
UpperTheory.Gal.TP.TwoUpOK (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.AllCoreOf, lines 18–22)
UpperTheory.Gal.TP.InertOK (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.AllCoreOf, lines 12–16)
UpperTheory.Gal.TP.Sub.IsUp (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.SplitLineSubtraction.RowDefOfOdd, lines 85–94)
UpperTheory.Gal.TP.twoUpOK_of_k (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.TwoUpOKOfSteps, lines 33–33)
UpperTheory.Gal.TP.Chr.chrPairs (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.IdeleNormCharacters.ChrPairsPf, lines 26–92)
39.64. UpperTheory.Gal.TP.frobIdQ (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.CoreAtQIffRowQ, lines 52–86)
39.65. UpperTheory.Gal.TP.Sub.peelAll_holds (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.SplitLineSubtraction.PPeelAll, lines 17–51)
UpperTheory.Gal.TP.Sub.eigTransfer_holds (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.SplitLineSubtraction.PEigTransfer, lines 13–34)
UpperTheory.Gal.TP.Sub.splitLine_holds (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.SplitLineSubtraction.PSplitLine, lines 100–132)
39.66. UpperTheory.Gal.TP.atPlaceQ (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.CoreAtQIffRowQ, lines 103–174)
39.67. UpperTheory.Gal.TP.subtractQ_of (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.CoreAtQIffRowQ, lines 200–209)
UpperTheory.Gal.TP.weakIsoQ (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.ThetaParseval.CoreAtQIffRowQ, lines 88–101)
UpperTheory.Gal.Int.Core.subtractQ_holds (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.CrystallineDescent.LimOddOfLimFinOdd, lines 31–33)

Chapter 40: A weak local Langlands correspondence

40.1. UpperTheory.Zero.LLC.recTExists (Development.AutomorphicArithmetic.Foundations.LocalLanglands.LocalCentralUnits.RecTExists, lines 20–66)
40.2. UpperTheory.Zero.LLC.LawsT (Development.AutomorphicArithmetic.Foundations.LocalLanglands.IsRecTOfIsRec, lines 55–72)
UpperTheory.Zero.LLC.IsRecT (Development.AutomorphicArithmetic.Foundations.LocalLanglands.IsRecTOfIsRec, lines 76–76)
40.3. UpperTheory.Zero.LLC.GoodCusp (Development.AutomorphicArithmetic.Foundations.LocalLanglands.IsRecTOfIsRec, lines 20–22)
UpperTheory.Zero.LLC.Good (Development.AutomorphicArithmetic.Foundations.LocalLanglands.IsRecTOfIsRec, lines 24–24)
40.4. UpperTheory.Zero.LLC.TypeLaw (Development.AutomorphicArithmetic.Foundations.LocalLanglands.IsRecTOfIsRec, lines 35–53)
40.5. UpperTheory.Zero.LLC.artEx (Development.AutomorphicArithmetic.Foundations.LocalLanglands.LocalCentralUnits.ArtEx, lines 7–15)
GlobalTheory.LocLanglands.IsLocalArtin (Development.AbelianVarietyTheory.PublishedTheorems.AutomorphicForms$_2$.HasExponentZeroIff, lines 145–152)
40.6. UpperTheory.Zero.LLC.parEx (Development.AutomorphicArithmetic.Foundations.LocalLanglands.LocalCentralUnits.CentralUnit, lines 39–200)
UpperTheory.Zero.LLC.parUniq (Development.AutomorphicArithmetic.Foundations.LocalLanglands.LocalCentralUnits.ParUniq, lines 15–54)
UpperTheory.Zero.LLC.goodMap (Development.AutomorphicArithmetic.Foundations.LocalLanglands.LocalCentralUnits.GoodMap, lines 144–164)
UpperTheory.Zero.LLC.twTame (Development.AutomorphicArithmetic.Foundations.LocalLanglands.LocalCentralUnits.TwTame, lines 190–194)
40.7. UpperTheory.Zero.LLC.junkEx (Development.AutomorphicArithmetic.Foundations.LocalLanglands.LocalCentralUnits.JunkEx, lines 257–262)
UpperTheory.tameInertiaCharacter_frobenius_conj_eq_pow (Development.AutomorphicArithmetic.WhittakerHecke.TameInertiaCharacterFrobeniusConjEqPow, lines 111–177)
40.8. UpperTheory.Zero.LLC.cuspEx (Development.AutomorphicArithmetic.Foundations.LocalLanglands.LocalCentralUnits.CuspEx, lines 168–266)
UpperTheory.Zero.LLC.expo (Development.AutomorphicArithmetic.Foundations.LocalLanglands.LocalCentralUnits.ExponentMulAbsPow, lines 63–96)
40.9. UpperTheory.Zero.LLC.Pr.exh (Development.AutomorphicArithmetic.Foundations.LocalLanglands.Multi, lines 11–11)
UpperTheory.Zero.LLC.Pr.multi (Development.AutomorphicArithmetic.Foundations.LocalLanglands.Multi, lines 12–12)
GlobalTheory.LocLanglands.segSum (Development.AbelianVarietyTheory.Definitions.NonGaloisDescent.Langlands, lines 17–18)
GlobalTheory.LocLanglands.exists_isLExt (Development.AbelianVarietyTheory.Proofs.InertiaEigenvalues.LanglandsExtensionUniqueOnCuspidal, lines 34–56)
40.10. UpperTheory.Zero.LLC.recTUnique (Development.AutomorphicArithmetic.Foundations.LocalLanglands.LocalCentralUnits.RecTUnique, lines 28–53)
40.11. UpperTheory.Gal.Int.Row.plainTor_of_tor (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.InertCharpoly.CharPolOfInert, lines 97–106)
UpperTheory.Gal.Int.Row.dz_of_cusp (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.InertCharpoly.CharPolOfInert, lines 114–139)
UpperTheory.Gal.Int.Row.mixed_of_tame (Development.AutomorphicArithmetic.Foundations.GaloisRepresentations.IntegralStructures.InertCharpoly.CharPolOfInert, lines 151–180)

Chapter 41: Laplace derivatives exclude poles

41.1. Deduction.landau_window_key (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.landauWindow, lines 40–73)
41.2. Deduction.iteratedDeriv_laplace_hasDerivAt (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.IteratedDerivLaplace, lines 48–82)
Deduction.iteratedDeriv_laplace (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.IteratedDerivLaplace, lines 84–105)
41.3. Deduction.landau_coeff (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.LandauCoeff, lines 10–28)
41.4. Deduction.kernel_integral (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.kernelIntegral, lines 9–20)
Deduction.landau_kernel_max (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.LandauKernelMax, lines 9–19)
Deduction.landau_tails_integrable (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.landauTails, lines 10–16)
41.5. Deduction.landau_tails_chernoff (Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.landauTails, lines 18–60)

`Deduction.landau_tails_pow` (`Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.landauTails`, lines 62–78)
41.6. `Deduction.landau_tails` (`Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.landauTails`, lines 80–131)
41.7. `Deduction.landau_window_ineq` (`Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.landauWindowIneq`, lines 27–111)
`Deduction.landau_window_ineq_int` (`Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.landauWindowIneq`, lines 10–25)
41.8. `Deduction.landau_dichotomy` (`Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.LandauDichotomy`, lines 24–87)
41.8. `Deduction.eq_ofReal_sub_of_dist_le_of_le_sub_re` (`Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.LandauDichotomy`, lines 10–22)
41.9. `Deduction.landau_nearest` (`Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.LandauNearest`, lines 36–109)
41.9. `Deduction.landau_nearest_equi` (`Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.LandauNearest`, lines 10–34)
41.10. `Deduction.landau_cauchy` (`Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.LandauCauchy`, lines 9–33)
41.11. `Deduction.landau_pole_asym` (`Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.LandauPoleAsym`, lines 19–79)
41.11. `Deduction.landau_pole_asym_deriv` (`Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.LandauPoleAsym`, lines 10–17)
41.12. `Deduction.landau_growth_contra` (`Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.LandauGrowthContra`, lines 9–40)
`Deduction.landau_window_contra` (`Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.landauWindow`, lines 20–38)
41.13. `Deduction.landau_window` (`Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.landauWindow`, lines 83–166)
41.14. `Deduction.landau_window` (`Development.AutomorphicArithmetic.DedekindLFunctions.LandauTheorems.landauWindow`, lines 83–166)

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